FACTORING QUADRATICS 8.1.1 and 8.1.2

Chapter 8 introduces students to quadratic equations. These equations can be written in the form of $y = ax^2 + bx + c$ and, when graphed, produce a curve called a parabola. There are multiple methods that can be used to solve quadratic equations. One of them requires students to factor.

Students have used algebra tiles to build rectangles of quadratic expressions. Later, they used the formula for the area of a rectangle, $(base)(height) = area$, to create generic rectangles and write equations expressing the area as both a sum and a product. In the figure below, the length and width of the rectangle are *factors* since they multiply together to produce the quadratic $x^2 + 6x + 8$. Notice the $4x$ and $2x$ are located diagonally from each other. They are like terms and can be combined and written as $6x$.

The dimensions of the rectangle are $(x+2)$ and $(x+4)$. The area is $x^2 + 6x + 8$.

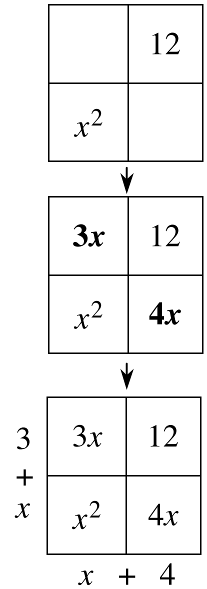
4	$4x$	8
+		
x	x^2	$2x$
	x +	2

The factors of $x^2 + 6x + 8$ are $(x+2)\,(x+4)$.

The ax^2 term $(1x^2)$ and the constant (8) are always diagonal to one another in a generic rectangle. In this example, the diagonal product is $8x^2$ and the c term is the *product*: $2(4) = 8$. The two x-terms are the other diagonal and can be combined into a sum when they are like terms. The b term of the quadratic is the *sum* of the factors: $2x + 4x = 6x$. Their diagonal product is $(2x)(4x) = 8x^2$, the same as the other diagonal. (See the Math Notes box on page 332 for why the products of the two diagonals are always equal.) To factor, students need to think about *sums and products* simultaneously. Students can Guess & Check or use a "diamond problem" from Chapter 1 to help organize their sums and products. See the Math Notes box on page 338.

Example 1

Factor $x^2 + 7x + 12$.



Sketch a generic rectangle with 4 sections.

Write the x^2 and the 12 along one diagonal.

Find two terms whose product is $12x^2$ and whose sum is $7x$ ($3x$ and $4x$). Students are familiar with this situation as a "diamond problem" from Chapter 1.

Write these terms as the other diagonal.

Find the base and height of the rectangle by using the partial areas.

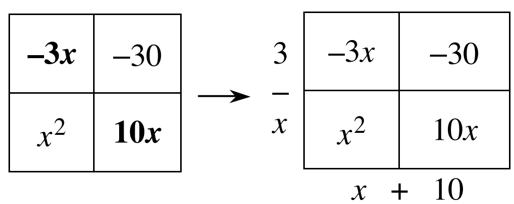
Write the complete equation. $(x+3)(x+4) = x^2 + 7x + 12$

Example 2

The same process works with negative values. Factor $x^2 + 7x - 30$.

Sketch a generic rectangle with 4 sections.

Write the x^2 and the -30 along one diagonal.



Find two terms whose product is $-30x^2$ and whose sum is $7x$.

Write these terms as the other diagonal.

Find the base and height of the rectangle. Pay particular attention to the signs so the sum is $7x$, not $-7x$.

Write the complete equation. $(x-3)(x+10) = x^2 + 7x - 30$

Example 3

Factor $x^2 - 15x + 56$.

Sketch a generic rectangle with 4 sections.

Write the x^2 and the 56 along one diagonal.

Find two terms whose product is $56x^2$ and whose sum is -15x.

Write these terms as the other diagonal.

$\mathbf{-8x}$	56
x^2	$\mathbf{-7x}$

$\longrightarrow$

8	$-8x$	56
$-$		
x	x^2	$-7x$
	x $-$	7

Find the base and height of the rectangle. Mentally multiply the factors to be sure they are correct. Pay close attention to the signs of the factors.

Write the complete equation. $(x-7)(x-8) = x^2 - 15x + 56$

Example 4

Factor $2x^2 + 7x + 6$.

Sketch a generic rectangle with 4 sections.

Write the $2x^2$ and the 6 along one diagonal.

Find two terms whose product is $12x^2$ and whose sum is 7x.

Write these terms as the other diagonal.

$\mathbf{3x}$	6
$2x^2$	$\mathbf{4x}$

$\longrightarrow$

3	$3x$	6
+		
$2x$	$2x^2$	$4x$
	x +	2

Find the base and height of the rectangle.

Write the complete equation. $(2x+3)(x+2) = 2x^2 + 7x + 6$

Example 5

Factor $12x^2 - 19x + 5$.

Sketch a generic rectangle with 4 sections.

Write the $12x^2$ and the 5 along one diagonal.

Find two terms whose product is $60x^2$ and whose sum is $-19x$.

Write these terms as the other diagonal.

$\mathbf{-15x}$	5
$12x^2$	$\mathbf{-4x}$

→

5	$-15x$	5
$-$ $4x$	$12x^2$	$-4x$
	$3x$ $-$	1

Find the base and height of the rectangle. Check the signs of the factors.

Write the complete equation. $(3x-1)(4x-5) = 12x^2 - 19x + 5$

Problems

1.	$x^2 + 5x + 6$	2.	$2x^2 + 5x + 3$	3.	$3x^2 + 4x + 1$	4.	$x^2 + 10x + 25$
5.	$x^2 + 15x + 44$	6.	$x^2 + 7x + 6$	7.	$x^2 + 11x + 24$	8.	$x^2 + 4x - 32$
9.	$4x^2 + 12x + 9$	10.	$12x^2 + 11x - 5$	11.	$x^2 + x - 72$	12.	$3x^2 - 20x - 7$
13.	$x^2 - 11x + 28$	14	$2x^2 + 11x - 6$	15.	$2x^2 + 5x - 3$	16.	$x^2 - 3x - 10$
17.	$4x^2 - 12x + 9$	18.	$3x^2 + 2x - 5$	19.	$6x^2 - x - 2$	20.	$9x^2 - 18x + 8$

Answers

1.	$(x+2)(x+3)$	2.	$(x+1)(2x+3)$	3.	$(3x+1)(x+1)$	4.	$(x+5)(x+5)$
5.	$(x+11)(x+4)$	6.	$(x+6)(x+1)$	7.	$(x+8)(x+3)$	8.	$(x+8)(x-4)$
9.	$(2x+3)(2x+3)$	10.	$(3x-1)(4x+5)$	11.	$(x-8)(x+9)$	12.	$(x-7)(3x+1)$
13.	$(x-4)(x-7)$	14.	$(x+6)(2x-1)$	15.	$(x+3)(2x-1)$	16.	$(x-5)(x+2)$
17.	$(2x-3)(2x-3)$	18.	$(3x+5)(x-1)$	19.	$(2x+1)(3x-2)$	20.	$(3x-4)(3x-2)$

USING THE ZERO PRODUCT PROPERTY 8.2.3 and 8.2.4

A parabola is a symmetrical curve. Its highest or lowest point is called the vertex. It is formed using the equation $y = ax^2 + bx + c$. Students have been graphing parabolas by substituting values for *x* and solving for *y*. This can be a tedious process if the range of *x* values is unknown. One possible shortcut is to find the *x*-intercepts first, then find the vertex and other convenient points. To find the *x*-intercepts, substitute 0 for *y* and solve the quadratic equation, $0 = ax^2 + bx + c$. Students will learn multiple methods to solve quadratic equations in this course.

The method described below utilizes two ideas:

(1) When the product of two or more numbers is zero, then one of the numbers must be zero.
(2) Some quadratics can be factored into the product of two binomials, where coefficients and constants are integers.

This procedure is called the **Zero Product Method** or **Solving by Factoring**. See the Math Notes boxes on pages 349 and 361.

Example 1

Find the *x*-intercepts of $y = x^2 + 6x + 8$.

The *x*-intercepts are located on the graph where $y = 0$, so write the quadratic expression equal to zero, then solve for *x*. $\quad x^2 + 6x + 8 = 0$

Factor the quadratic. $\quad (x+4)(x+2) = 0$

Set each factor equal to 0. $\quad (x+4) = 0$ or $(x+2) = 0$

Solve each equation for *x*. $\quad x = -4$ or $x = -2$

The *x*-intercepts are (-4, 0) and (-2, 0).

You can check your answers by substituting them into the original equation.

$(-4)^2 + 6(-4) + 8 \Rightarrow 16 - 24 + 8 \Rightarrow 0$

$(-2)^2 + 6(-2) + 8 \Rightarrow 4 - 12 + 8 \Rightarrow 0$

Example 2

Solve $2x^2 + 7x - 15 = 0$.

Factor the quadratic. $(2x-3)(x+5) = 0$

Set each factor equal to 0. $(2x-3) = 0$ or $(x+5) = 0$

Solve for each *x*.

$$2x = 3$$
$$x = \tfrac{3}{2} \quad \text{or} \quad x = -5$$

Example 3

If the quadratic does not equal 0, rewrite it algebraically so that it does, then use the zero product property.

Solve $2 = 6x^2 - x$.

Set the equation equal to 0.

$$2 = 6x^2 - x$$
$$0 = 6x^2 - x - 2$$

Factor the quadratic.

$$0 = (2x+1)(3x-2)$$

Solve each equation for *x*.

$$(2x+1) = 0 \quad \textit{or} \quad (3x-2) = 0$$
$$2x = -1 \quad \textit{or} \quad 3x = 2$$
$$x = -\tfrac{1}{2} \qquad x = \tfrac{2}{3}$$

Example 4

Solve $9x^2 - 6x + 1 = 0$.

$$9x^2 - 6x + 1 = 0$$

Factor the quadratic.

$$(3x-1)(3x-1) = 0$$

Solve each equation for *x*. Notice the factors are the same so there will be only one *x* value. For this parabola, the *x*-intercept is the vertex.

$$(3x-1) = 0$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

Problems

Solve for x.

1. $x^2 - x - 12 = 0$	2. $3x^2 - 7x - 6 = 0$	3. $x^2 + x - 20 = 0$
4. $3x^2 + 11x + 10 = 0$	5. $x^2 + 5x = -4$	6. $6x - 9 = x^2$
7. $6x^2 + 5x - 4 = 0$	8. $x^2 - 6x + 8 = 0$	9. $6x^2 - x - 15 = 0$
10. $4x^2 + 12x + 9 = 0$	11. $x^2 - 12x = 28$	12. $2x^2 + 8x + 6 = 0$
13. $2 + 9x = 5x^2$	14. $2x^2 - 5x = 3$	15. $x^2 = 45 - 4x$

Answers

1. 4 or -3	2. $-\frac{2}{3}$ or 3	3. -5 or 4
4. $-\frac{5}{3}$ or -2	5. -4 or -1	6. 3
7. $-\frac{4}{3}$ or $\frac{1}{2}$	8. 4 or 2	9. $-\frac{3}{2}$ or $\frac{5}{3}$
10. $-\frac{3}{2}$	11. 14 or -2	12. -1 or -3
13. $-\frac{1}{5}$ or 2	14. $-\frac{1}{2}$ or 3	15. 5 or -9

USING THE QUADRATIC FORMULA 8.3.1

When a quadratic equation is not factorable, another method is needed to solve for x. The Quadratic Formula can be used to calculate the roots of the equation, that is, the x-intercepts of the quadratic. The Quadratic Formula can be used with any quadratic equation, factorable or not. There may be two, one or no solutions, depending on whether the parabola intersects the x-axis twice, once, or not at all.

For any quadratic $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The $\pm$ symbol is read as "plus or minus." It is shorthand notation that tells you to calculate the formula twice, once with + and again with – to get both x-values.

To use the formula, the quadratic equation must be written in *standard form*: $ax^2 + bx + c = 0$. This is necessary to correctly identify the values of a, b, and c. Once the equation is in standard form and equal to 0, a is the coefficient of the x^2 term, b is the coefficient of the x term and c is the constant. See the Math Notes boxes on pages 358 and 361 and problem 12-52 on page 510.

Example 1

Solve $2x^2 - 5x - 3 = 0$.

Identify a, b, and c. Watch your signs carefully. $a = 2$, $b = -5$, $c = -3$

Write the quadratic formula.
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute a, b, and c in the formula and do the initial calculations.
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$
$$x = \frac{5 \pm \sqrt{25 - (-24)}}{4}$$

Simplify the $\sqrt{\ }$.
$$x = \frac{5 \pm \sqrt{49}}{4}$$

Calculate both values of x.
$$x = \frac{5+7}{4} = \frac{12}{4} = 3 \text{ or } x = \frac{5-7}{4} = \frac{-2}{4} = -\frac{1}{2}$$

The solution is $x = 3$ or $x = -\frac{1}{2}$. The x-intercepts are (3, 0) and $(-\frac{1}{2}, 0)$.

Example 2

Solve $3x^2 + 5x + 1 = 0$.

Identify *a, b*, and *c*.

$a = 3,\ b = 5,\ c = 1$

Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute *a, b*, and *c* in the formula and do the initial calculations.

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

Simplify the $\sqrt{\ }$.

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

The solution is $x = \frac{-5 + \sqrt{13}}{6}$ or $x = \frac{-5 - \sqrt{13}}{6}$. These are the exact values of *x*.
The *x* values are the *x*-intercepts of the parabola. To graph these points, use a calculator to find the decimal approximation of each one. The *x*-intercepts are $(\approx -0.23, 0)$ and $(\approx -1.43, 0)$.

Example 3

Solve $25x^2 - 20x + 4 = 0$.

Identify *a, b*, and *c*.

$a = 25,\ b = -20,\ c = 4$

Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute *a, b*, and *c* in the formula and do the initial calculations.

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(25)(4)}}{2(25)}$$

$$x = \frac{20 \pm \sqrt{400 - 400}}{50}$$

Simplify the $\sqrt{\ }$.

$$x = \frac{20 \pm \sqrt{0}}{50}$$

This quadratic has only one solution: $x = \frac{2}{5}$.

Example 4

Solve $x^2 + 4x + 10 = 0$.

Identify *a, b*, and *c*. $a = 1,\ b = 4,\ c = 10$

Write the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute *a, b*, and *c* in the formula and do the initial calculations. $x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(10)}}{2(1)}$

$$x = \frac{-4 \pm \sqrt{16 - 40}}{2}$$

Simplify the $\sqrt{\ }$. $x = \frac{-4 \pm \sqrt{-24}}{2}$

It is impossible to take the square root of a negative number; therefore this quadratic has **no** real solution. The graph would still be a parabola, but there would be no *x*-intercepts. This parabola is above the *x*-axis.

Example 5

Solve $(3x + 1)(x + 2) = 1$.

Rewrite in standard form.

$$(3x + 1)(x + 2) = 1$$
$$3x^2 + 7x + 2 = 1$$
$$3x^2 + 7x + 1 = 0$$

Identify *a, b*, and *c*. $a = 3,\ b = 7,\ c = 1$

Write the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute *a, b*, and *c* in the formula and do the initial calculations. $x = \frac{-(7) \pm \sqrt{(7)^2 - 4(3)(1)}}{2(3)}$

$$x = \frac{-7 \pm \sqrt{49 - 12}}{6}$$

Simplify . $x = \frac{-7 \pm \sqrt{37}}{6}$

The *x*-intercepts are $(\approx -0.15,\ 0)$ and $(\approx -2.18,\ 0)$.

Example 6

Solve $3x^2 + 6x + 1 = 0$.

Identify *a, b*, and *c*. $a = 3,\ b = 6,\ c = 1$

Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute *a, b*, and *c* in the formula and do the initial calculations.

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 12}}{6}$$

Simplify .

$$x = \frac{-6 \pm \sqrt{24}}{6}$$

The *x*-intercepts are (≈ -1.82 , 0) and (≈ -0.18 , 0).

Math Note 8.3.3 describes another form of this expression that can be written by simplifying the square root. The result is equivalent to the exact values above.

Factor the $\sqrt{24}$, then simplify by taking the square root of 4.

$$\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

$$x = \frac{-6 \pm 2\sqrt{6}}{6}$$

Simplify the fraction by dividing every term by 2.

$$x = \frac{-3 \pm \sqrt{6}}{3}$$

Problems

Solve each of these using the Quadratic Formula.

1. $x^2 - x - 2 = 0$	2. $x^2 - x - 3 = 0$	3. $-3x^2 + 2x + 1 = 0$
4. $-2 - 2x^2 = 4x$	5. $7x = 10 - 2x^2$	6. $-6x^2 - x + 6 = 0$
7. $6 - 4x + 3x^2 = 8$	8. $4x^2 + x - 1 = 0$	9. $x^2 - 5x + 3 = 0$
10. $0 = 10x^2 - 2x + 3$	11. $x(-3x + 5) = 7x - 10$	12. $(5x + 5)(x - 5) = 7x$

Answers

1. $x = 2$ or $x = -1$	2. $x = \frac{1 \pm \sqrt{13}}{2}$ ≈ 2.30 *or* -1.30	3. $x = -\frac{1}{3}$ or $x = 1$
4. $x = -1$	5. $x = \frac{-7 \pm \sqrt{129}}{4}$ ≈ 1.09 *or* -4.59	6. $x = \frac{1 \pm \sqrt{145}}{-12}$ ≈ -1.09 *or* 0.92
7. $x = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$ ≈ 1.72 *or* -0.39	8. $x = \frac{-1 \pm \sqrt{17}}{8}$ ≈ 0.39 *or* -0.64	9. $x = \frac{5 \pm \sqrt{13}}{2}$ ≈ 4.30 *or* 0.70
10. No solution	11. $x = \frac{2 \pm \sqrt{124}}{-6} = \frac{1 \pm \sqrt{31}}{-3}$ ≈ -2.19 *or* 1.52	12. $x = \frac{27 \pm \sqrt{1229}}{10}$ ≈ 6.21 *or* -0.81