February 4 Math 2335 sec 51 Spring 2016

Section 3.2: Newton's Method

We wish to find a number α that is a zero of the function $f(x)$

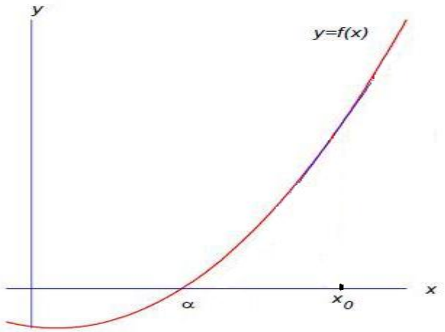

Figure: We begin by making a guess x_0 with the hope that $\alpha \approx x_0$.

Newton's Method

Next, we obtain a better approximation x_1 to the true root α.

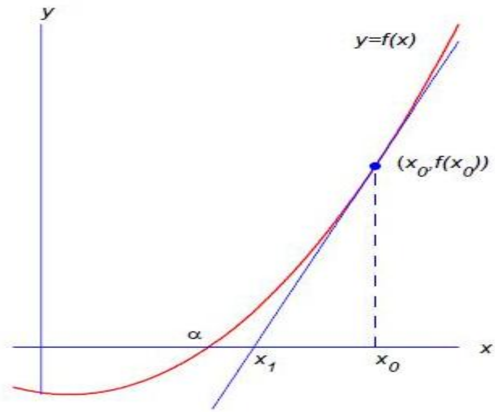

Figure: We choose x_1 to be the zero of $p_1(x)$, the tangent line approximation to f at x_0.

Formula for x_1:

We assume that $f(x)$ is differentiable on an interval containing α.

To find $p_1(x)$, we need a point and slope.

point : $(x_0, f(x_0))$.   slope : $m = f'(x_0)$

$$p_1(x) - f(x_0) = f'(x_0)(x - x_0)$$

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

x_1 is the x-intercept so $p_1(x_1) = 0$

$$P_1(x_1) = 0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

Suppose $f'(x_0) \neq 0$

$$f'(x_0)(x_1 - x_0) = -f(x_0)$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Iterative Scheme for Newton's Method

We start with a *guess* x_0. Then set

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, we can find a tangent to the graph of f at $(x_1, f(x_1))$ and update again

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's Iteration Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \dots$$

The sequence begins with a starting *guess* x_0 expected to be near the desired root.

Exit Strategy for Newton's Method

Newton's method may or may not converge on the solution α. [1] Since we hope that x_n is getting closer and closer to α, we generally stop when either

$$|x_{n+1} - x_n| < \epsilon$$

or when

$$n \geq N$$

where ϵ is some error tolerance and N is some predetermined maximum number of iterations.

If the latter condition is used to stop the process, the method is probably not working.

[1] More on this important issue later!

Example

Consider finding the real solution α of the equation

$$x^3 = x^2 + x + 1.$$

(a) Define an appropriate function $f(x)$ that has α as a root.

Let $f(x) = x^3 - x^2 - x - 1$

If $f(\alpha) = 0$, then $\alpha^3 = \alpha^2 + \alpha + 1$

Example: $x^3 = x^2 + x + 1$

(b) Determine the Newton Iteration formula for this problem.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n \geq 0$$

$$f(x) = x^3 - x^2 - x - 1 \quad , \quad f'(x) = 3x^2 - 2x - 1$$

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - x_n - 1}{3x_n^2 - 2x_n - 1}$$

$$= \frac{x_n(3x_n^2 - 2x_n - 1) - (x_n^3 - x_n^2 - x_n - 1)}{3x_n^2 - 2x_n - 1}$$

$$x_{n+1} = \frac{2x_n^3 - x_n^2 + 1}{3x_n^2 - 2x_n - 1}$$

Example: $x^3 = x^2 + x + 1$

(c) Take $x_0 = 2$ and compute x_1 and x_2.

$$x_{n+1} = \frac{2x_n^3 - x_n^2 + 1}{3x_n^2 - 2x_n - 1}$$

$$x_1 = \frac{2(2)^3 - 2^2 + 1}{3(2)^2 - 2 \cdot 2 - 1} = \frac{13}{7}$$

$$x_2 = \frac{2\left(\frac{13}{7}\right)^3 - \left(\frac{13}{7}\right)^2 + 1}{3\left(\frac{13}{7}\right)^2 - 2\cdot\frac{13}{7} - 1} = \frac{1777}{966}$$

$$\doteq 1.839544513$$

Example: $x^3 = x^2 + x + 1$ TI-89

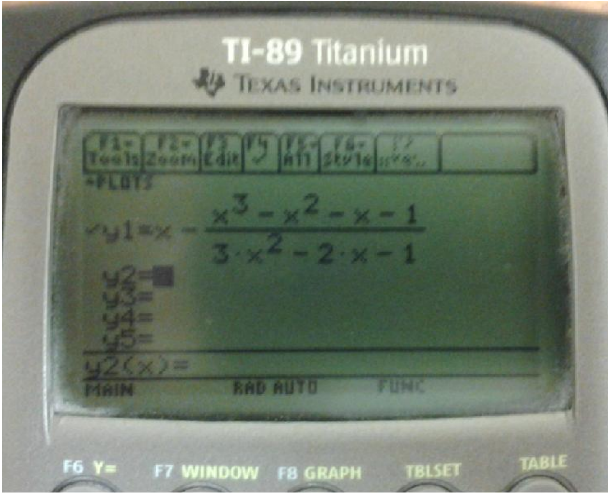

Figure: From the home window 2 [sto] x [enter], y1(x) [sto] x [enter], repeat.

Example: $x^3 = x^2 + x + 1$ TI-84

Use [Y=]. To access variables Y_i, hit [vars], select [Y-VARS], select [Function..], select desired Y_i.

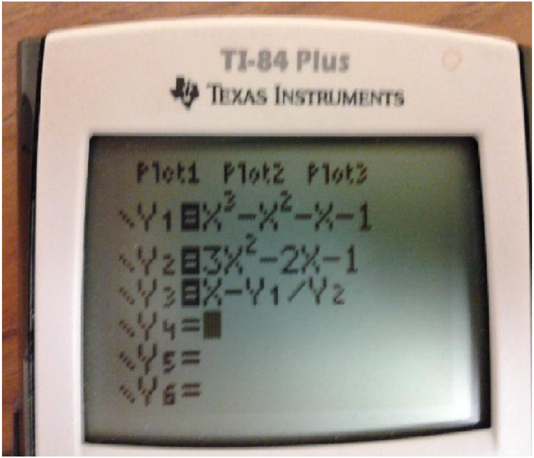

Figure: Set up $Y_1 = x^3 - x^2 - x - 1$, $Y_2 = 3x^2 - 2x - 1$ and $Y_3 = x - Y_1/Y_2$.

Example: $x^3 = x^2 + x + 1$ TI-84

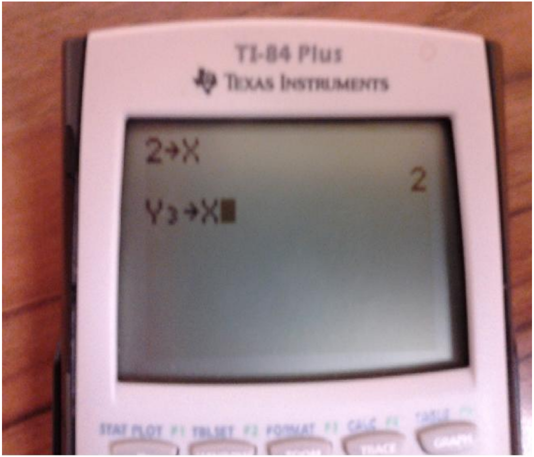

Figure: From the home screen 2 [sto] X [enter], then Y3 [sto] X [enter]. Keep hitting [enter].

Example: $x^3 = x^2 + x + 1$

Produced with Matlab with a tolerance of $\epsilon = 10^{-8}$.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	2.0000000000	0.1428571428	1.0000000000
1	1.8571428571	0.0175983436	0.0991253644
2	1.8395445134	0.0002577038	0.0014103289
3	1.8392868100	0.0000000548	0.0000003000
4	1.8392867552	0.0000000000	0.0000000000
5	1.8392867552		0.0000000000

Newton's method finds the root to within 10^{-8} in 5 full iterations.
Compare this to the 27 iterates needed for the bisection method!

Computing Reciprocals without Division

Early computers (and even some supercomputers used today) did not compute with the operation $\div$. We consider a method for producing a reciprocal

$$\frac{1}{b} \quad \text{for a known nonzero number} \quad b$$

that relies only on the operations $+$, $-$, and $\times$.

Let $f(x) = b - \frac{1}{x}$. Then f is continuously differentiable for $x > 0$ and

$$f\left(\frac{1}{b}\right) = 0 \quad \text{i.e.} \quad \alpha = \frac{1}{b}$$

is the unique zero of f.

Example: Computing Reciprocal

Find the Newton's method iteration formula for solving $f(x) = 0$ where $f(x) = b - \frac{1}{x}$ and $b > 0$ is some constant. Reduce the formula so that it only entails the operations $+$, $-$, and $\times$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

$$f(x) = b - \frac{1}{x}, \qquad f'(x) = \frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{b - \frac{1}{x_n}}{\frac{1}{x_n^2}}$$

$$x_{n+1} = x_n - \frac{b - \frac{1}{x_n}}{\frac{1}{x_n^2}} \cdot \frac{x_n^2}{x_n^2} = x_n - \frac{bx_n^2 - x_n}{1}$$

$$x_{n+1} = x_n - (bx_n^2 - x_n) = 2x_n - bx_n^2$$

$$\boxed{x_{n+1} = 2x_n - bx_n^2}$$

This formula requires only the operations $+$, $-$, and $\times$.

Example: Computing Reciprocal

From the iteration formula $x_{n+1} = 2x_n - bx_n^2$ show that the relative error[2] satisfies

$$\mathrm{Rel}(x_{n+1}) = [\mathrm{Rel}(x_n)]^2.$$

Here $\alpha = \frac{1}{b}$

$$\mathrm{Rel}(x_{n+1}) = \frac{\frac{1}{b} - x_{n+1}}{\frac{1}{b}} = 1 - bx_{n+1}$$

similarly $\mathrm{Rel}(x_n) = 1 - bx_n$

[2] Recall that the relative error in x_k is

$$\mathrm{Rel}(x_k) = \frac{\alpha - x_k}{\alpha}.$$

$$\mathrm{Rel}(x_{n+1}) = 1 - b x_{n+1}$$

$$= 1 - b(2x_n - b x_n^2)$$

$$= 1 - 2bx_n + b^2 x_n^2$$

a perfect square

$$= (1 - bx_n)^2$$

$$= (\mathrm{Rel}(x_n))^2$$

Example: Computing Reciprocal

Use this result to conclude that Newton's method will only converge to the true root (with any given tolerance) if

$$0 < x_0 < \frac{2}{b}.$$

Recall If $a > 0$

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & 0 < a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$$

$$\mathrm{Rel}(x_1) = \left(\mathrm{Rel}(x_0)\right)^2$$

$$\mathrm{Rel}(x_2) = \left(\mathrm{Rel}(x_1)\right)^2$$

$$= \left(\mathrm{Rel}(x_0)\right)^4$$

$$\text{Rel}(x_3) = \left(\text{Rel}(x_2)\right)^2 = \left(\text{Rel}(x_0)\right)^8$$

$$\vdots$$

$$\text{Rel}(x_n) = \left(\text{Rel}(x_0)\right)^{2^n}$$

If $x_n \to \alpha$, the error must $\to 0$.

This requires $\quad |\text{Rel}(x_0)| < 1$

$$\text{Rel}(x_0) = 1 - bx_0$$

We require

$$-1 < 1 - bx_0 < 1$$

i.e. $$-1 < bx_0 - 1 < 1$$

$$0 < bx_0 < 2$$

$$\Rightarrow \quad 0 < x_0 < \frac{2}{b}$$

Example: Computing Reciprocal

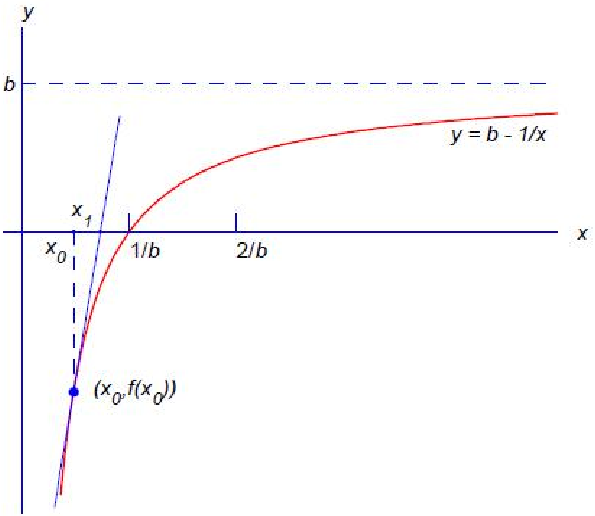

Figure: Illustration of using Newton's method to compute the reciprocal $1/b$.

Example: Computing Reciprocal

Computing the reciprocal of the number e.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	0.5000	0.1796	0.7183
1	0.3204	0.0413	-0.4025
2	0.3618	0.0060	-0.0460
3	0.3678	0.0001	-0.0008
4	0.3679	0.0000	-0.0000
5	0.3679	0.0000	-0.0000
6	0.3679		0.0000

Six iterations are required with an initial guess of $x_0 = 0.5$ and a tolerance of $\epsilon = 10^{-8}$.

Example: Computing Reciprocal

Computing the reciprocal of the number e.

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$f(x_n)$
0	0.7500	0.7790	1.3849
1	−0.0290	0.0313	37.1612
2	−0.0604	0.0703	19.2860
3	−0.1306	0.1770	10.3741
4	−0.3076	0.5648	5.9691
5	−0.8725	2.9416	3.8645
6	−3.8141	43.3572	2.9805

The same six iterations with an initial guess of $x_0 = 0.75$ produces garbage results.

Error Analysis: Newton's Method

Suppose that f has at least two derivatives on an interval containing α and that

$$f'(\alpha) \neq 0.$$

By Taylor's Theorem, we can write

$$f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_n)$$

where c_n is some number between α and x_n.

Error Analysis: Newton's Method

$$f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_n)$$

c_n is some number between α and x_n

From this, let's show that $\mathrm{Err}(x_{n+1})$ is proportional to $[\mathrm{Err}(x_n)]^2$.

As $f(\alpha) = 0$

$$0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_n)$$

$$\frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = -\frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)}$$

$$\underbrace{\alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)} \right)}_{x_{n+1} \text{ from Newton's formula}} = -\frac{1}{2} (\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)}$$

$$\alpha - x_{n+1} = -\frac{1}{2} \frac{f''(c_n)}{f'(x_n)} (\alpha - x_n)^2$$

$$\mathrm{Err}(x_{n+1}) = K_n \left[\mathrm{Err}(x_n) \right]^2$$

where $k_n = -\dfrac{f''(c_n)}{2f'(x_n)}$

Error Analysis: Newton's Method

$$f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_n)$$

Recalling that $f(\alpha) = 0$, divide both sides by $f'(x_n)$ to get

$$0 = \frac{f(x_n)}{f'(x_n)} + \alpha - x_n + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)} \quad \Longrightarrow$$

$$\alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) = -\frac{f''(c_n)}{2f'(x_n)}(\alpha - x_n)^2 \quad \Longrightarrow$$

$$\alpha - x_{n+1} = -\frac{f''(c_n)}{2f'(x_n)}(\alpha - x_n)^2$$

Error Analysis: Newton's Method

$$\text{Err}(x_{n+1}) = \alpha - x_{n+1} = K_n(\alpha - x_n)^2 = K_n[\text{Err}(x_n)]^2$$

where

$$K_n = -\frac{f''(c_n)}{2f'(x_n)}.$$

If α and x_n are very close together, then

$$-\frac{f''(c_n)}{2f'(x_n)} \approx -\frac{f''(\alpha)}{2f'(\alpha)} \equiv M.$$

Thus

$$\alpha - x_{n+1} \approx M(\alpha - x_n)^2 \quad \Longrightarrow \quad M(\alpha - x_{n+1}) \approx [M(\alpha - x_n)]^2.$$

Error Analysis: Newton's Method

$$M(\alpha - x_{n+1}) \approx [M(\alpha - x_n)]^2$$

Note what condition this gives on the error at the n^{th} step:

$$\begin{aligned}
M(\alpha - x_1) &\approx [M(\alpha - x_0)]^2 \\
M(\alpha - x_2) &\approx [M(\alpha - x_1)]^2 \approx [M(\alpha - x_0)]^4 \\
M(\alpha - x_3) &\approx [M(\alpha - x_2)]^2 \approx [M(\alpha - x_0)]^8 \\
&\vdots \\
M(\alpha - x_n) &\approx [M(\alpha - x_0)]^{2^n}
\end{aligned}$$

Error Analysis: Newton's Method

The error is only expected to go to zero (meaning x_n is converging to α) if

$$|M(\alpha - x_0)| < 1 \quad \text{i.e. provided} \quad |\alpha - x_0| < \frac{1}{|M|} = \frac{2|f'(\alpha)|}{|f''(\alpha)|}.$$

If $|M|$ is very large, Newton's method may be impractical. Or another method such as bisection may be needed to get a starting value x_0 close enough for convergence.

Example

We wish to find the root of $\tan^{-1}(x) - \frac{\pi}{4}$. (The exact solution is $\alpha = 1$.) Use the error bound formula

$$|\alpha - x_0| < \frac{2|f'(\alpha)|}{|f''(\alpha)|}$$

to determine a suitable interval for the initial guess x_0.

$$f(x) = \tan^{-1} x - \frac{\pi}{4}, \quad f'(x) = \frac{1}{1+x^2}, \quad f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$f'(\alpha) = f'(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(\alpha) = f''(1) = \frac{-2}{(1+1)^2} = \frac{-2}{4} = \frac{-1}{2}$$

We require

$$|1 - x_0| < \frac{2|f'(1)|}{|f''(1)|} = \frac{2\left(\frac{1}{2}\right)}{\frac{1}{2}} = 2$$

$$-2 < x_0 - 1 < 2$$

$$\boxed{-1 < x_0 < 3}$$

For x_0 in the interval $(-1, 3)$, Newton's method will converge.

Example

(a) Write an iteration formula for finding the cube root of 4 based on Newton's method. Give the formula in simplified form.

We need a function whose true root $\alpha = \sqrt[3]{4}$

(without a priori knowledge of $\sqrt[3]{4}$.)

Take $f(x) = x^3 - 4$, $f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 - 4}{3x_n^2} = x_n - \frac{1}{3}x_n + \frac{4}{3x_n^2}$$

$$= \frac{2}{3}x_n + \frac{4}{3x_n^2} = \frac{2x_n^3 + 4}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3+4}{3x_n^2}}$$

Example Continued...

(b) Use the quantity M defined previously to show that the error and relative error satisfy

$$\alpha - x_{n+1} \approx -\frac{1}{\alpha}(\alpha - x_n)^2, \quad \text{and} \quad |\mathrm{Rel}(x_{n+1})| \approx [\mathrm{Rel}(x_n)]^2$$

$$\alpha - x_{n+1} \approx M(\alpha - x_n)^2$$

$$f(x) = x^3 - 4$$

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$M = -\frac{f''(\alpha)}{2f'(\alpha)} = \frac{-6\alpha}{2(3\alpha^2)}$$

$$= \frac{-1}{\alpha}$$

So $\quad \alpha - x_{n+1} \approx \frac{-1}{\alpha}(\alpha - x_n)^2$ as expected.

$$|\mathrm{Rel}(x_{n+1})| = \frac{|\mathrm{Err}(x_{n+1})|}{\alpha} \approx \frac{\left|\frac{-1}{\alpha}\left(\mathrm{Err}(x_n)\right)^2\right|}{\alpha}$$

$$= \frac{|(\mathrm{Err}(x_n))^2|}{\alpha^2}$$

$$= \left[\frac{\mathrm{Err}(x_n)}{\alpha}\right]^2 = \left[\mathrm{Rel}(x_n)\right]^2$$