SOME COMBINATORIAL ASPECTS IN ALGEBRAIC TOPOLOGY AND GEOMETRIC GROUP THEORY

YANGA BAVUMA

Department of Mathematics and Applied Mathematics
University of Cape Town
Private Bag X1, 7701, Rondebosch,Cape Town, South Africa
E-mail: BVMYAN001@myuct.ac.za

Abstract. The present Msc thesis deals with classical topics of topology and it has been written, referring to [C. Kosniowski, Introduction to Algebraic Topology, Cambridge University Press, 1980, Cambridge], which is a well known textbook of algebraic topology. It has been selected a list of main exercises from this reference, whose solutions were not directly available, or subject to differerent methods. In fact combinatorial methods have been preferred and the result is a self-contained dissertation on the theory of the fundamental group and of the coverings. Finally, there are some recent problems in geometric group theory which are related to the presence of finitely presented groups which appear naturally as fundamental groups.

List of contents

Introduction

The present thesis describes a recent problem in algebraic topology and geometric group theory, regarding the structure of groups with quadratic Dehn function. This problem was originally investigated by Olshanskij and Sapir [12] and is reported in Question 12.3. In order to understand the context in which it is placed and the different mathematical notions which are involved, we spend nine chapters bringing up and building basic topological concepts through exercises from a well known textbook of algebraic topology:

C. Kosniowski, *Introduction to algebraic topology*, Cambridge University Press, 1980, Cambridge.

Most of them has no answer in the book itself and, when hints are given, it is possible to use different methods of solution. In fact we offer elementary methods and combinatorial techniques.

After establishing a solid basis in the topological concepts, we use these to introduce loops and homotopical equivalences. More precisely, the first six chapters deal with classical notions of general topology like continuity, metric spaces, induced topologies, quotients topologies, product topologies and actions of groups.

We use a "deductive approach", that is, we solve an exercise, then we find analogies with the same method of solution. In this way, we justify a theorem as a rule that allows us to describe a series of examples that fit into a prescribed situation. When it was impossible (or very hard) to solve some exercises in a direct way, classical results have been reported. The presence of Heine-Borel's Theorem (see Chapter 7), Tychonoff's Theorem (see Chapter 6) and Urysohn's Theorem (see Chapter 8) with proofs must be understood in this perspective.

On the other hand, we mention briefly some constructions in general topology, that have their independent interest of research (see [3, 4, 5, 6, 9, 10, 11] in Chapter 8) and this has been done for a precise reason: these constructions of general topology involve metric spaces and metric spaces are involved in the theory of finitely presented groups via the word metric. This means that there is an important connection with Dehn functions. In addition, it has been reported a very recent contribution on fundamental groups (see[1]) when we want to look at fundamental groups, not from the point of view of the word metric, but via covering spaces and weak forms of connectivity.

Chapters 10, 11, and 12 deal with fundamental groups. We introduce (without proof) the theorem of Seifert-Van Kampen, which is used to define finitely presented groups, corresponding word problems, and so the Dehn function.

`Acknowledements.` I'd like to thank the National Research Foundation of South Africa for the "2018 freestanding, scarce skills master's and doctoral scholarship", the University of Cape Town for "KW Johnston Bequest fellowship" and for "Master's research need scholarship". This provided me with the necessary resources to complete the present thesis. I'd like to thank Prof. Hans-Peter Kunzi (University of Cape Town) for co-supervising me, and most of all I'd like to thank Dr. Francesco Russo (University of Cape Town) for his patience, passion and diligence in his supervision. Best supervisor ever!

1. Classical notions on metric spaces

We refer to [8] for the usual definitions of sets and groups. Here we will illustrate some exercises whose details are not mentioned in [8, Chapter 0].

Given the classical notion of metric in [8, Definition 1.1] it is interesting to see the properties which follow from the axioms. This motivates to show the following fact.

Exercise 1.1 (See [8], Exercise 1.2)**.** If d is a metric for a topological space A, then $d(a,b) \geq 0$ and $d(a,b) = d(b,a)$ for all $a, b \in A$.

Solution. The first part follows from

$$d(b,b) \leq d(a,b) + d(a,b) \Rightarrow 0 \leq 2d(a,b) \Rightarrow 0 \leq d(a,b).$$

The second part follows from

$$d(a,b) \leq d(b,a) + d(b,b) \Rightarrow d(a,b) \leq d(b,a),$$
$$d(b,a) \leq d(a,b) + d(a,a) \Rightarrow d(b,a) \leq d(a,b).$$

Another classical fact is the presence of different positive non-degenerate functions which are not necessarily metrics. The following exercise illustrates some well known examples of metrics on $\mathbb{R}^n$.

Exercise 1.2 (See [8], Exercise 1.3(a))**.** Show that the following functions satisfy the axioms of a metric for $\mathbb{R}^n$ for any positive integer n:

$$d(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}} = \|x - y\|; \; d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{if } x \neq y. \end{cases}$$

$$d(x,y) = \sum_{i=1}^n |x - y|; \quad d(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Solution. Of course, the first example is the well known Euclidean metric on $\mathbb{R}^n$ and the proof is routine. The second example is the well known discrete metric on $\mathbb{R}^n$. Again this can be checked easily. Let's give details on the third example.

(i) We claim that $d(x,y) = 0 \Leftrightarrow x = y$. In fact,

$$d(x,y) = 0 \Rightarrow \max_{1 \leq i \leq n} |x_i - y_i| = 0 \Rightarrow |x_i - y_i| = 0 \; \forall \; 1 \leq i \leq n$$
$$\Rightarrow x_i = y_i \; \forall \; 1 \leq i \leq n \Rightarrow x = y.$$
$$x = y \Rightarrow x_i = y_i \; \forall \; 1 \leq i \leq n \Rightarrow |x_i - y_i| = 0 \; \forall \; i \leq n$$
$$\Rightarrow \max_{1 \leq i \leq n} |x_i - y_i| = 0 \Rightarrow d(x,y) = 0.$$

(ii) We claim that $d(x,y) + d(x,z) \geq d(y,z) \;\; \forall \; x, y, z \in \mathbb{R}^n$. In fact,

$$\forall \; 1 \leq i \leq n, \quad |y_i - z_i| \leq |x_i - y_i| + |x_i - z_i|$$
$$\Rightarrow \forall \; 1 \leq i \leq n, \quad |y_i - z_i| \leq |x_i - y_i| + \max_{1 \leq i \leq n} |x_i - z_i|$$
$$\Rightarrow \forall \; 1 \leq i \leq n, \quad |y_i - z_i| \leq \max_{1 \leq i \leq n} |x_i - y_i| + \max_{1 \leq i \leq n} |x_i - z_i|$$
$$\Rightarrow \max_{1 \leq i \leq n} |y_i - z_i| \leq \max_{1 \leq i \leq n} |x_i - y_i| + \max_{1 \leq i \leq n} |x_i - z_i|$$
$$\Rightarrow d(y,z) \leq d(x,y) + d(y,z).$$

An example of positive non-degenerate function which is not a metric is reported below. Here we stress the role of the triangular inequality among all the non-degenerate positive functions.

Exercise 1.3 (See [8], Exercise 1.3(b))**.** The function $d(x,y)=(x-y)^2$ does not define a metric on $\mathbb{R}$.

Solution. Let $x=2, y=1, z=1.5$ Then $d(x,y)=(2-1)^2=1^2=1$ and $d(x,z)=(2-1.5)^2=(0.5)^2=0.25$ and $d(y,z)=(1-1.5)^2=(0.5)^2=0.25$. Here $d(x,y)>d(x,z)+d(y,z)$ which violates the triangular inequality.

Further examples of positive functions which are not metric may be produced when we violate the axiom of symmetry instead of the axiom which is related to the triangular inequality.

Exercise 1.4 (See [8], Exercise 1.3(c))**.** The function $d(x,y)=\min\limits_{1\le i\le n}|x_i-y_i|$ does not define a metric on $\mathbb{R}^n$.

Solution. Let $x=(0,1,...,0)$ and $y=(0,...,0)\in\mathbb{R}^n$. Clearly $x\neq y$, but

$$\min_{1\le i\le n}|x_i-y_i|=0=d(x,y).$$

Now we want to see how to produce metrics, when we have a prescribed metric in our hands. A possible way is via deformation via a multiplicative factor. More details are given by the exercise below.

Exercise 1.5 (See [8], Exercise 1.3(d))**.** Let X be a metric space with metric d and let r be a positive real number. Then d_r defined by $d_r(x,y)=rd(x,y)$ is also a metric on X.

Solution. We must check the axioms.

(i) Of course, one is satisfied by looking at $d_r(x,x)=rd(x,x)=r\cdot 0=0$. About the symmetry, we note that

$$d_r(x,y)=0\Leftrightarrow rd(x,y)=0\Leftrightarrow d(x,y)=\frac{0}{r}\Leftrightarrow d(x,y)=0\Leftrightarrow x=y.$$

(ii) Finally, for all $x,y,z\in X$

$$d_r(y,z)=rd(y,z)\le r(d(x,y)+d(x,z))=d_r(x,y)+d_r(x,z).$$

Essentially the exercise above shows that we can "compress" or "dilate" a metric by a factor of r and we still have a metric. Another way to deform a prescribed metric is offered by following formula.

Exercise 1.6 (See [8], Exercise 1.3(e))**.** Let d be a metric on a metric space X. Then the following map

$$d' \ :(x,y)\in X\times X\mapsto d'(x,y)=\frac{d(x,y)}{1+d(x,y)}\in\mathbb{R}$$

is also a metric.

Solution. The symmetry of d' is reduced to use the symmetry of d. In fact

$$d'(x,y)=0 \iff \frac{d(x,y)}{1+d(x,y)}=0 \Rightarrow d(x,y)=0 \iff x=y$$

$$d'(x,x)=\frac{d(x,x)}{1+d(x,x)}=\frac{0}{1+0}=0.$$

About the triangular inequality,

$$d(y,z)\le d(x,y)+d(x,z) \Rightarrow d(y,z)\le d(x,y)+d(x,z)+d(x,y)d(x,z),$$

adding $d(x,y)d(y,z)$ becomes

$$d(y,z)+d(x,y)d(y,z)\le d(x,y)+d(x,z)+d(x,y)d(x,z)+d(x,y)d(y,z)$$

adding $d(x,z)d(y,z)$ becomes

$$d(y,z)+d(x,y)d(y,z)+d(x,z)d(y,z)$$
$$\le d(x,y)+d(x,z)+d(x,y)d(x,z)+d(x,y)d(y,z)+d(x,z)d(y,z)$$

adding $d(x,y)d(x,z)d(y,z)$ becomes

$$d(y,z)+d(x,y)d(y,z)+d(x,z)d(y,z)+d(x,y)d(x,z)d(y,z)$$
$$\le d(x,y)+d(x,z)+d(x,y)d(x,z)+d(x,y)d(y,z)+d(x,z)d(y,z)+d(x,y)d(x,z)d(y,z),$$

or, equivalently,

$$d(y,z)\big(1+d(x,y)+d(x,z)+d(x,y)d(x,z)\big)$$
$$\le (1+d(y,z))\big(d(x,y)+d(x,z)+d(x,y)d(x,z)\big).$$

Therefore

$$d'(y,z)=\frac{d(y,z)}{1+d(y,z)}\le\frac{d(x,y)+d(x,z)+d(x,y)d(x,z)}{1+d(x,y)+d(x,z)+d(x,y)d(x,z)}$$

so

$$\begin{aligned}\frac{d(y,z)}{1+d(y,z)} &\le \frac{d(x,y)+d(x,z)+2d(x,y)d(x,z)}{1+d(x,y)+d(x,z)+d(x,y)d(x,z)}\\ &=\frac{(d(x,y)+d(x,y)d(x,z))+(d(x,z)+d(x,y)d(x,z))}{(1+d(x,y))(1+d(x,z))}\\ &\quad+\frac{d(x,y)(1+d(x,z))+d(x,z)(1+d(x,y))}{(1+d(x,y))(1+d(x,z))}\\ &=\frac{d(x,y)(1+d(x,z))}{(1+d(x,y))(1+d(x,z))}+\frac{d(x,z)(1+d(x,y))}{(1+d(x,y))(1+d(x,z))}\\ &=\frac{d(x,y)}{1+d(x,y)}+\frac{d(x,z)}{1+d(x,z)}.\end{aligned}$$

It is appropriate to recall here that a metric d on a space X is a positive non-degenerate symmetric function from $X\times X$ to the subset $[0,+\infty)$ of $\mathbb{R}$. Of course, the set of natural numbers $\mathbb{N}$ is contained in $[0,+\infty)$, so one may wonder whether it is possible to find a metric d with values only in $\mathbb{N}$. An example is reported below.

Exercise 1.7 (See [8], Exercise 1.3(f))**.** In $\mathbb{R}$, the usual distance between two points $x,y\in\mathbb{R}$ is given by $|x-y|$. The function

$$d\ :(x,y)\in\mathbb{R}^2\mapsto d(x,y)=\lfloor|x-y|\rfloor=\text{smallest integer}\ge|x-y|$$

defines a metric on $\mathbb{R}$.

Solution. Let $d(x,y) = \lfloor |x-y| \rfloor$. Of course, $d(x,y) = \lfloor |x-y| \rfloor = \lfloor |y-x| \rfloor = d(y,x)$. Moreover $x = y \Leftrightarrow d(x,y) = \lfloor |x-y| \rfloor = \lfloor 0 \rfloor = 0$ and

$$d(x,z) = \lfloor |x-z| \rfloor \leq \lfloor |x-y| + |y-z| \rfloor \leq \lfloor |x-y| \rfloor + \lfloor |y-z| \rfloor = d(x,y) + d(y,z).$$

Now we pass to study the problem of the continuity of a metric.

Exercise 1.8 (See [8], Exercise 1.5(a))**.** Let A be a metric space with metric d and fix $y \in A$. The function $f : x \in A \mapsto f(x) \in \mathbb{R}$ defined by $f(x) = d(x,y)$ is continuous, where $\mathbb{R}$ has the usual metric.

Solution. We check that for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in B_\delta(x) = \{t \in A \mid d(x,t) < \delta\}$, then $f(z) \in B_\varepsilon(f(x))$. In fact, if $d(x,z) < \delta_1$ and $d(x,y) < \delta_2$, then we consider $\delta = \max\{\delta_1, \delta_2\}$ and $y, z \in B_\delta(x)$, so

$$\begin{aligned}|f(x) - f(z)| &= |d(x,y) - d(z,y)| \leq d(x,y) + d(z,y) \leq d(x,z) + 2d(z,y) \\ &\leq d(x,z) + 2d(z,x) + 2d(x,y) = 3d(x,z) + 2d(x,y) < 3\delta_1 + 2\delta_2 \leq 5\delta.\end{aligned}$$

In other words, for all $\varepsilon > 0$, we may choose $\delta < \varepsilon/5$ and we get $|f(x) - f(z)| < \varepsilon$, which means that f is continuous.

Of course, $\mathbb{R}$ is a metric space with respect to the usual metric, but it is a metric space even if we consider the discrete metric on it. These two topological spaces are different, even if both of them are metric. The following fact shows that there are no embeddings of one space in the other in this situation.

Exercise 1.9 (See [8], Exercise 1.5(b))**.** Let M be the metric space $(\mathbb{R}, d)$ where d is the usual metric and M_0 be the discrete metric space $(\mathbb{R}, d_0)$, where

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{if } x \neq y. \end{cases}$$

Then all functions $f : M_0 \to M$ are continuous. On the other hand, there does not exist any injective continuous function from M to M_0.

Solution. We begin to show that f is continuous. The definition of continuity (in ε and δ) is equivalent to $f^{-1}(V)$ is open for all V open in M. Since M_0 is discrete, each subset of M_0 is open and so is $f^{-1}(V)$.

Assume f is injective, $x \in M$ and $B^0_\varepsilon(f(x)) = \{f(x)\}$ the open ball in M_0. Here $\forall\ \delta > 0$, $\exists\ y \in M$, $y \neq\ x$ such that $y \in B_\delta(x)$, where B is an open ball in M. So $B_\delta(x) \not\subset f^{-1}(B^0_\varepsilon(f(x)))$ and f is not continuous.

Now assume f is continuous. This means that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that $B_\delta(x) \subset f^{-1}(B^0_\varepsilon(f(x)))$. But $\forall\ \delta > 0$, $\exists\ y \in M$, $y \neq x$ such that $y \in B_\delta(x)$ and, for $0 < \varepsilon < 1$, we get $B^0_\varepsilon(f(x)) = \{f(x)\}$, so $f(x) = f(y)$, which means that f is not injective.

As we have seen, one can formulate the notion of continuity in Exercise 1.9 in a more abstract way. In fact the above argument works not only from $\mathbb{R}$ with the discrete metric to $\mathbb{R}$ with the usual metric, but from any discrete metric space to any metric space. The following example shows how open and closed may be produced in metric spaces.

Exercise 1.10 (See [8], Exercise 1.8(d))**.** Give an example of an infinite collection of open sets of $\mathbb{R}$ (with the usual metric) whose intersection is not open.

Solution. First of all we note that family of open sets $\mathscr{F}$ arising from a metric space A with metric d satisfies the following conditions.

(i) The empty set $\emptyset$ and the whole set belong to $\mathscr{F}$,
(ii) The intersection of two members of $\mathscr{F}$ belongs to $\mathscr{F}$,
(iii) The union of any number of members of $\mathscr{F}$ belongs to $\mathscr{F}$.

Of course, (i) is true. About (ii), if $U_1, U_2 \in \mathscr{F}$, then $\forall\, x \in U_i,\ \ i \in \{1,2\},\ \ \exists\, \varepsilon_x^i > 0$ such that $B_{\varepsilon_x^i}(x) = \{y \in A : d(y,x) < \varepsilon_x^i\} \subset U_i$. Let $\varepsilon_x = \min\{\varepsilon_x^1, \varepsilon_x^2\}$. Then $B_{\varepsilon_x}(x) \subset U_1 \cap U_2$. Finally, about (iii), if $U_i \in \mathscr{F},\ \ i \in I$ and $x \in \bigcup\limits_{i\in I}$, then $\exists k \in I$ such that $x \in U_k$ and $\exists\, \varepsilon_x$ so that $B_{\varepsilon_x}(x) \subseteq U_k \subseteq \bigcup\limits_{i\in I} U_i$. One might wonder whether the condition (ii) can be extended to arbitrary intersections and not only to finite intersections. The answer is negative, because for all $n \in \mathbb{N}$ the set $(-1/n, 1/n)$ shows that

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

At this point, we may ask whether two different metrics give rise to equivalent topologies on the same space. This is shown by the following exercise.

Exercise 1.11 (See [8], Exercise 1.10)**.** The metrics $d'(x,y) = \sum |x_i - y_i|$ and $d''(x,y) = \max |x_i - y_i|$ on $\mathbb{R}^n$ give rise to the same family of open sets as that arising from the usual metric on $\mathbb{R}^n$.

Solution. We check that the two metrics are equivalent to the usual metric.

Let $d'(x,y) = \sum |x_i - y_i|$ and take $r > 0$,

$y \in B'_r(x) \Rightarrow d'(x,y) < r \Rightarrow \sum |x_i - y_i| < r \Rightarrow (\sum |x_i - y_i|)^2 < r^2$

$\Rightarrow (|x_1 - y_1| + \cdots + |x_n - y_n|)^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 + \xi(x_i, y_j)$,

where $\xi : (x_i, y_j) \in \mathbb{R}^2 \to \xi(x_i, y_j) \in \mathbb{R}$ and $\xi(x_i, y_i)$ is a positive polynomial in which x_i and y_j appear with powers of $k = 0, 1$.

$$\Rightarrow \sum (x_i - y_i)^2 + \xi(x_i, y_j) < r^2 \Rightarrow \sum (x_i - y_i)^2 < r^2 - \xi(x_i, y_j) < r^2 \Rightarrow y \in B_r(x)$$

So for $x' \in B_\epsilon(x)$, let $\delta < \min\{\epsilon - d(x,x'), d(x,x')\} < \varepsilon$ then

$$B'_\delta(x') \subseteq B_\delta(x') \subseteq B_\varepsilon(x).$$

Let $y \in B_r(x) \Rightarrow d(x,y) < r \Rightarrow \sqrt{\sum (x_i - y_i)^2} < r \Rightarrow \sum (x_i - y_i)^2 < r^2$
$\Rightarrow (\sum |x_i - y_i|)^2 - \xi(x_i, y_i) < r^2 \Rightarrow (\sum |x_i - y_i|)^2 < \xi(x_i, y_i) + r^2$
$\Rightarrow \sum |x_i - y_i| < r^2 + r^2 \Rightarrow y \in B'_{2r^2}(x)$

So for $x' \in B'_\varepsilon(x)$, let $0 < \delta' < \min\{\varepsilon - d'(x,x'), d'(x,x')\} < \varepsilon$ and let $\delta = \sqrt{\frac{\delta'}{2}} \Rightarrow 2\delta^2 = \delta' < \varepsilon$ then $B_\delta(x') \subseteq B'_{2\delta^2}(x') \subset B'_\varepsilon(x)$.

Now let $d''(x,y) = \max |x_i - y_i|$ and take $r > 0$

$y \in B''_{\sqrt{\frac{r}{n}}}(x) \Rightarrow d''(x,y) = \max |x_i - y_i| < \sqrt{\frac{r}{n}}$

$\Rightarrow |x_i - y_i| < \sqrt{\frac{r}{n}} \Rightarrow (x_i - y_i)^2 < \sqrt{\frac{r}{n}} \Rightarrow \sum (x_i - y_i)^2 < \frac{r}{n} n$

$\Rightarrow \sum (x_i - y_i)^2 < r \Rightarrow y \in B_r(x)$

So for $x' \in B_\varepsilon(x)$, let $0 < \delta' < \min\{\varepsilon - d(x,x'), d(x,x')\} < \varepsilon$, $\delta = \sqrt{\frac{\delta'}{n}}$ then

$$B''_\delta(x') \subseteq B'_\delta(x') \subseteq B_\varepsilon(x).$$

And $y \in B_r(x) \Rightarrow d(x,y) < r \Rightarrow \sum(x_i - y_i)^2 < r \Rightarrow (x_i - y_i)^2 < r \Rightarrow |x_i - y_i| < \sqrt{r} \Rightarrow \max|x_i - y_i| < \sqrt{r} \Rightarrow y \in B''_{\sqrt{r}}(x)$

So for $x' \in B''_\varepsilon(x)$, let $0 < \delta' < \min\{\varepsilon - d''(x,x'), d''(x,x')\} < \varepsilon$, and $\delta = \delta'^2 \Rightarrow \sqrt{\delta} = \delta' < \varepsilon$, then $B_\delta(x') \subseteq B''_{\sqrt{\delta}}(x') \subseteq B''_\varepsilon(x)$.

2. Topological spaces

The present chapter illustrates some examples of topological spaces which differ from the classical idea of $\mathbb{R}$ with the usual topology. This helps to clarify that certain properties of $\mathbb{R}$ are not elementary. A first example is a metric space of arbitrary cardinality.

Exercise 2.1 (See [8], Exercise 2.2(a))**.** Show that there are metrizable topological spaces of arbitrary cardinality.

Solution. Let d be the discrete metric on a space X, where $d(x, y) = 0$ if $x = y$ and 1 otherwise for $x, y \in X$. In this space, X may be of arbitrary cardinality and it is always metrizable.

Let's give a condition that allow us to detect non-metrizable spaces.

Exercise 2.2 (See [8], Exercise 2.2(b))**.** Let X be a topological space that is metrizable. Prove that for every pair a, b of distinct points of X there are open sets U_a and U_b containing a and b respectively, such that $U_a \cap U_b = \emptyset$. In addition, if X has at least two points and has the trivial topology then it is not metrizable.

Solution. Let d be a metric on X and $d(a, b) = \varepsilon$. If we let $0 < \delta < \frac{\varepsilon}{2}$ then $B_\delta(a) \cap B_\delta(b) = \emptyset$. If X has the trivial topology and a, b are distinct elements of this X, then if U is open either $a, b \in U$ or $a, b \notin U$.

This is an example in which one can find the so- called *axioms of separation.* We will come later on these notions in Chapter 8. Another important example of non-metrizable space is given by $\mathbb{R}$ with the *topology of the half lines.* Details are given by the next example.

Exercise 2.3 (See [8], Exercise 2.3(a))**.** Define on $\mathbb{R}$ the subset $\mathscr{U} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{(-\infty, x) \mid x \in \mathbb{R}\}$ of the power set $\mathcal{P}(\mathbb{R})$. Check that $\mathscr{U}$ is a topology on $\mathbb{R}$ and that $\mathbb{R}$ is not metrizable with respect to $\mathscr{U}$.

Solution. We check the axioms of topology.

(i) Clearly the empty set and the whole set are elements of $\mathscr{U}$

(ii) Let $U, V \in \mathscr{U}$, if $U = \emptyset$, then $U \cap V = U \in \mathscr{U}$. If $U = \mathbb{R}$ then $U \cap V = V \in \mathscr{U}$. If $U, V \in \{(-\infty, x) \mid x \in \mathbb{R}\}$, then $\exists\ x_u, x_v \in \mathbb{R}$ such that $U = (-\infty, x_u)$ and $V = (-\infty, x_v)$. Then $U \cap V = (-\infty, x_u) \cap (-\infty, x_v) = (-\infty, \min\{x_u, x_v\}) \in \{(-\infty, x) \mid x \in \mathbb{R}\} \subset \mathscr{U}$.

(iii) Let $U_i \in \mathscr{U}$ and $U_i \in \{(-\infty, x) \mid x \in \mathbb{R}\}$. Then $\exists\ x_i \in \mathbb{R}$ such that $U_i = (-\infty, x_i)$ and either the x_i are bounded from above or not. If not, then $\bigcup_{i \in I} U_i = \mathbb{R} \in \mathscr{U}$. Otherwise, put $t = \inf\{x_i \mid i \in I\}$ (existing by the axiom of completeness in $\mathbb{R}$), and $\bigcup_{i \in I} U_i = (-\infty, t) \in \mathscr{U}$. Of course, the case in which U_j are empty for some $j \in I$ implies that $\bigcup_{i \neq j} U_i = U_j \cup (\bigcup_{i \neq j} U_i) = \bigcup_{i \neq j} U_i \in \mathscr{U}$ as before.

In a metrizable space we can always separate two points with two different and disjoint neighbourhoods. If we take $x, y \in \mathbb{R}$ with $x < y$, a neighbourhood U_x of x and U_y of y cannot be disjoint, since U_x would be the half line contained in the half line U_y. Therefore this space cannot be metrizable.

Another example of topology on $\mathbb{R}$ which endows $\mathbb{R}$ of the structure of nonmetrizable space is the following.

Exercise 2.4 (See [8], Exercise 2.3(b))**.** The space $\mathbb{N}$ with the topology $\mathscr{U} = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{O_n \mid n \geq 1\}$, where $O_n = \{n, n+1, n+2, ...\}$, is a nonmetrizable space.

Solution. We begin to check the axioms of topology.

(i) Clearly the empty set and the whole set are elements of $\mathscr{U}$.

(ii) Let $U, V \in \mathscr{U}$, if $U = \mathbb{N}$, then $U \cap V = V \in \mathscr{U}$. If $U, V \in \{O_n \mid n \geq 1\}$, then $\exists\ u, v \in \mathbb{N}$ such that $U = O_u$ and $V = O_v$. Then $U \cap V = O_u \cap O_v = O_{\max\{u,v\}} \in \{O_n \mid n \geq 1\} \subset \mathscr{U}$.

(iii) If $U_i \in \mathscr{U}$ and $U_k = \mathbb{N}$ for some $k \in I$, then $\bigcup\limits_{i \in I} U_i = U_k \in \mathscr{U}$. If $U_i \in \mathscr{U}$ and $U_i \in \{O_n \mid n \geq 1\}$, then $\exists\ n_i \in \mathbb{N}$ such that $U_i = O_{n_i}$ and, writing $m = \min\{n_i \mid i \in I\}$, we find that $\bigcup\limits_{i \in I} U_i = O_m \in \{O_n \mid n \geq 1\} \in \mathscr{U}$. Of course, the case in which U_j are empty for some $j \in I$ implies that $\bigcup\limits_{i \neq j} U_i = U_j \cup (\bigcup\limits_{i \neq j} U_i) = \bigcup\limits_{i \neq j} U_i \in \mathscr{U}$ as before.

If we take $m, n \in \mathbb{N}$ with $n < m$, a neighbourhood O_n of n and O_m of m cannot be disjoint since $O_n \subseteq O_m$. Therefore this space cannot be metrizable.

One can generalise the topology $\mathscr{U}$ of Example 3.4 and define $A \in \mathscr{V}$ if and only if $|\mathbb{N} - A| \leq \mathbb{N}|$. This is the so-called *cocountable topology* on $\mathbb{N}$. One way to check that $\mathscr{V}$ is a topology is by using the steps in the previous exercise An alternative approach may be via the laws of De Morgan.

Another very similar topological structure, generalizing Example 3.4 and the notion of cocountable topology, is presented below.

Exercise 2.5 (See [8], Exercise 2.6(a))**.** Let X be a set and let $\mathscr{V}$ be a family of subsets of X such that:

(i) $\emptyset, X \in \mathscr{V}$;

(ii) the union of any pair of elements of $\mathscr{V}$ belongs to $\mathscr{V}$;

(iii) the intersection of any number of elements of $\mathscr{V}$ belongs to $\mathscr{V}$.

Then $\mathscr{U} = \{X - V \mid V \in \mathscr{V}\}$ is a topology for X.

Solution. As usual, we check the axioms of topology.

(i) $\emptyset = X - X \in \mathscr{U}$ and $X = X - \emptyset \in \mathscr{U}$

(ii) Let $V_1, V_2 \in \mathscr{V}$, then $V_1 \cup V_2 \in \mathscr{V}$ and $X - V_1, X - V_2 \in \mathscr{U}$, so $X - (V_1 \cup V_2) = (X - V_1) \cap (X - V_2) \in \mathscr{U}$ by one of the laws of De Morgan.

(iii) If $V_i \in \mathscr{V}$ for $i \in I$, then $\bigcap\limits_{i \in I} V_i \in \mathscr{V}$ and $X - V_i \in \mathscr{U}$, so $X - \left(\bigcap\limits_{\forall i \in I} V_i\right) = \bigcup\limits_{\forall i \in I} (X - V_i) \in \mathscr{U}$.

In particular, if $\mathscr{V}$ is the family of all the countable subsets of X and X is countable, then $\mathscr{U}$ is the cocountable topology on X.

Another interesting example which can be given on $\mathbb{R}$ is the so called *line of Sorgenfrey.* There are several important properties for this space.

Exercise 2.6 (See [8], Exercise 2.3(c))**.** Define on $\mathbb{R}$ the topology in which $U \in \mathscr{U}$ if and only if U is a subset of $\mathbb{R}$ and for each $s \in U$ there is a $t > s$ such that

$[s,t) \subset U$, where $[s,t) = \{x \in \mathbb{R} \mid s \le x < t\}$. Any subset $[s,t)$ of $\mathbb{R}$ (with the topology $\mathscr{U}$) is both open and closed.

Solution. Again we check with an argument of routine the axioms of topology

(i) Clearly the empty set and the whole set are elements of $\mathscr{U}$.

(ii) Let $U, V \in \mathscr{U}$. If $U = \mathbb{R}$, then $U \cap V = V \in \mathscr{U}$. If $U, V \neq \emptyset$ or $\mathbb{R}$ and $s \in U \cap V$, then $\exists t_u > s$ such that $[s, t_u) \subset U$ and $\exists t_v > s$ such that $[s, t_v) \subset V$. Then $[s, \min\{t_u, t_v\}) \subset [s, t_u) \subset U$ and $[s, \min\{t_u, t_v\}) \subset [s, t_v) \subset V$, meaning $[s, \min\{t_u, t_v\}) \subset U \cap V$. So $U \cap V \in \mathscr{U}$.

(iii) Let $U_i \in \mathscr{U}$ and $U_k = \mathbb{R}$ for some $k \in I$. Then $\bigcup\limits_{i\in I} U_i = U_k \in \mathscr{U}$. If $U_i \neq \mathbb{R}$ or $\emptyset$ $\forall i \in I$ and $s \in \bigcup\limits_{i\in I} U_i$, then $\exists\ h \in I$ such that $s \in U_h$. Since $s \in U_h$, $\exists t_h > s$ such that $[s, t_h) \subset U_h \subset \bigcup\limits_{i\in I} U_i \in \mathscr{U}$. Of course, the case in which U_j are empty for some $j \in I$ implies that $\bigcup\limits_{i\neq j} U_i = U_j \cup (\bigcup\limits_{i\neq j} U_i) = \bigcup\limits_{i\neq j} U_i \in \mathscr{U}$ as before.

It is clear that $[s,t)$ is open. $\mathbb{R} - [s,t) = (-\infty, s) \cup [t, \infty) \in \mathscr{U}$ since it is the union of two open sets in this topology.

We will see in Chapter 9 the notion of connected space and the line of Sorgenfrey is an example of totally disconnected space, since each of its open is always open and closed at the same time. In addition, it is not elementary to see whether the line of Sorgenfrey is metrizable or not. In fact here we can separate two points with two disjoint neighbourhoods, so further considerations must be done, in order to discuss its metrizability. A simpler example of totally disconnected space is the following.

Exercise 2.7 (See [8], Exercise 2.6(b))**.** Any subset of a discrete topological space is simultaneously open and closed.

Solution. Let X be a discrete topological space and $U \subset X$. U is open in X, so $X - U$ is closed in X. But $X - U \subset X$ so $X - U$ is open in X, this means that U is closed in X as well.

We give just one example of non-topological structure on $\mathbb{R}$.

Exercise 2.8 (See [8], Exercise 2.3(e))**.** The set $\mathscr{S} = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{(-\infty, x] \mid x \in \mathbb{R}\}$ is not a topology on $\mathbb{R}$.

Solution The axiom of stability under unions is not satisfied when $I = \mathbb{N}$ and $U_i = (-\infty, x - \frac{1}{i}] \in \mathscr{S}$. In fact $\bigcup\limits_{i\in I} U_i = (-\infty, x) \notin \mathscr{S}$.

We end here with miscellaneous examples of topologies on $\mathbb{R}$, different from the usual one, and pass to describe some topologies which are possible on arbitrary finite sets. The general description involves sophisticated techniques of combinatorics, but for small finite set we can work without computational machineries.

Exercise 2.9 (See [8], Exercise 2.3(d))**.** The number of distinct topologies on a set with three elements.

Solution. There are 29 distinct topologies on $X = \{a, b, c\}$. Below there are 9 basic structures of the topologies and the number of iterations of each generated by permutations of the points.
1. $\{\emptyset, X\}$
2. $\{\emptyset, X, \{c\}\}$
3. $\{\emptyset, X, \{a, b\}\}$
4. $\{\emptyset, X, \{c\}, \{a, b\}\}$
5. $\{\emptyset, X, \{c\}, \{b, c\}\}$
6. $\{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$
7. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$
8. $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$
9. $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

The last 5 of these are all T_0 (which we define in Chapter 8). The first one is trivial, while in 2, 3, and 4 the points a and b are topologically indistinguishable.

n	Distinct Topologies	Distinct T_0 Topologies	Inequivalent Topologies	Inequivalent T_0 Topologies
0	1	1	1	1
1	1	1	1	1
2	4	3	3	2
3	29	19	9	5
4	355	219	33	16
5	6942	4231	139	63
6	209527	130023	718	318
7	9535241	6129859	4535	2045
8	642779354	431723379	35979	16999
9	63260289423	44511042511	363083	183231
10	8977053873043	6611065248783	4717687	2567284

Let $T(n)$ denote the number of distinct topologies on a set with n points. There is no known simple formula to compute $T(n)$ for arbitrary n. The Online Encyclopedia of Integer Sequences presently lists $T(n)$ for $n \leq 18$. The number of distinct T_0 topologies on a set with n points, denoted $T_0(n)$, is related to $T(n)$ by the formula

$$T(n) = \sum_{k=0}^{n} S(n, k) T_0(k)$$

where $S(n, k)$ denotes the *Stirling number of the second kind* (in combinatorics, a Stirling number of the second kind, or *Stirling partition number*, is the number of ways to partition a set of n objects into k non-empty subsets).

We want to list another property of discrete spaces here, focussing on the presence of closed subspaces which are singletons.

Exercise 2.10 (See [8], Exercise 2.6(c))**.** If a topological space has only a finite number of points each of which is closed, then it has the discrete topology.

Solution. Let X be this space and U a subset. Since X is finite, $X - U$ is also finite and hence closed because it is a finite union of closed points. So U is open. This is enough to conclude that the space is discrete.

Some explicit descriptions may be appropriate here.

Exercise 2.11 (See [8], Exercise 2.8(a)). In $\mathbb{R}$ with the usual topology, the closure $\overline{S}$ of a subspace S of $\mathbb{R}$ is the smallest closed of $\mathbb{R}$ containing S. We have that $A = \{1, 2, 3, ...\}$ has $\overline{A} = A$; $\mathbb{Q}$ has $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{R} - \mathbb{Q}$ has $\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$

A subspace Y of a space X such that $\overline{Y} = X$ is called *dense* in X, so both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$. Let's see what happens for the intervals of $\mathbb{R}$.

Exercise 2.12 (See [8], Exercise 2.8(b)). In $\mathbb{R}$ with the usual topology, the closure of each of the following subsets of $(a, b), [a, b), (a, b], [a, b]$ is equal to $[a, b]$.

One of the main observations, when we deal with a closure $\overline{Y}$ of a subspace Y of a space X, is to realize that $\overline{Y}$ may be always written as the union

$$\overline{Y} = Y \cup \partial Y \qquad (\dagger)$$

where ∂Y denotes the points having a neighbourhood which intersects both $\overline{Y}$ and $\overline{X - Y}$. These last points belong to the so called *boundary* ∂Y of Y. In fact $\partial Y = \overline{Y} \cap \overline{X - Y}$. On the other hand, there is a corresponding notion for the largest open $\overset{\circ}{Y}$ contained in Y (called *interior* of Y) and one can see that $\overset{\circ}{Y} \subseteq Y$ and that $\partial Y = \partial(X - Y)$. Some classical properties are reported below.

Exercise 2.13 (See [8], Exercise 2.9(a) and (b)). If Y is a subset of a topological space X with $Y \subset F \subset X$ and F closed, then $\overline{Y} \subset F$. Moreover, Y is closed if and only if $Y = \overline{Y}$.

Solution. Since F is closed and contains Y, $\overline{Y} = F \cap \left(\bigcap_{j \in J} F_j \right) \subset F$ where $F_j's$ are all the closed subsets that contain Y.

Assume Y is closed. Then

$$\overline{Y} = Y \cap \left(\bigcap_{j \in J} F_j \right) \subset Y, \quad \text{but} \quad Y \subset \overline{Y}, \quad \text{so} \quad Y = \overline{Y},$$

where F_j are closed subsets containing Y. The viceversa is clear.

We don't mention the corresponding variation of the above result for the interior, since it is very similar to what we have seen above. On the other hand, we must mention that the closure operator is involutory, that is, if applied twice on a closed set, gives the closed set again, or, equivalently:

Exercise 2.14 (See [8], Exercise 2.9(c)). $\overline{\overline{Y}} = \overline{Y}$

Solution. Since $\overline{Y}$ is closed, we have $\overline{\overline{Y}} = \overline{Y}$.

Similarly, this happens for the interior operator. Let's now see what happens for the closure operator with respect to the usual set theoretical operations.

Exercise 2.15 (See [8], Exercise 2.9(d)). In a topological space X, given two subsets A and B we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$, that is, the closure operator respects the union, but $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, that is, the closure operator is monotone on the intersections.

Solution. $x \in \overline{A \cup B} \iff$ for every open set U containing x, $U \cap (A \cup B) \neq \emptyset$ $\Rightarrow$ for every open set U containing x, $(U \cap A) \cup (U \cap B) \neq \emptyset \Rightarrow$ for every open set U containing x, $U \cap A \neq \emptyset$ or $U \cap B \neq \emptyset \Rightarrow \quad x \in \overline{A}$ or $x \in \overline{B} \Rightarrow \quad x \in \overline{A} \cup \overline{B} \quad \Rightarrow$ $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Viceversa, $x \in \overline{A} \cup \overline{B} \Rightarrow$ for every open set U containing x, $U \cap A \neq \emptyset$ or $U \cap A \neq \emptyset$ $\Rightarrow$ for every open set U containing x, $(U \cap A) \cup (U \cap B) \neq \emptyset \Rightarrow$ for every open set U containing x, $U \cap (A \cup B) \neq \emptyset \Rightarrow x \in \overline{A \cup B} \Rightarrow \quad \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Therefore $\overline{A \cup B} = \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B} \iff$ for every open set U containing x, $U \cap (A \cap B) \neq \emptyset \iff$ for every open set U containing x, $(U \cap A) \cap (U \cap B) \neq \emptyset \iff$ for every open set U containing x, $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset \Rightarrow x \in \overline{A}$ and $x \in \overline{B} \Rightarrow x \in \overline{A} \cap \overline{B}$.

Therefore $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Again we do not report the corresponding property for the interior operator, but we mention now some conditions in which the two operators are involved simultaneously.

Exercise 2.16 (See [8], Exercise 2.9(e))**.** Given a subspace Y of a topological space X, we have $X - \mathring{Y} = \overline{(X - Y)}$.

Solution. We have $x \in X - \mathring{Y} \iff x \notin \mathring{Y} \iff$ for every open set $U \not\subseteq Y$, $x \in U \iff$ for every open set U containing x, $U \cap (X - Y) \neq \emptyset \iff x \in \overline{(X - Y)}$.

We will give more details on (†) in the next statement.

Exercise 2.17 (See [8], Exercise 2.9(f))**.** The equality $\overline{Y} = Y \cup \partial Y$ is always true.

Solution. We have

$$Y \cup \partial Y = Y \cup [\overline{Y} \cap \overline{(X - Y)}] = Y \cup [\overline{Y} \cap (X - \mathring{Y})] = (Y \cup \overline{Y}) \cap [Y \cup (X - \mathring{Y})]$$
$$= \overline{Y} \cap X = \overline{Y}.$$

The behaviour of the boundary may be used, in order to detect whether a given set is closed or not.

Exercise 2.18 (See [8], Exercise 2.9(g))**.** Y is closed if and only if $\partial Y \subset Y$. Moreover, $\partial Y = \emptyset$ if and only if Y is both open and closed.

Solution. Assume Y is closed. Then $\overline{Y} = Y$, so $\partial Y \subseteq Y$. Viceversa $\partial Y \subseteq Y$ implies $\overline{Y} = Y \cup \partial Y = Y \Rightarrow Y = \overline{Y}$. The first part is proved.

Assume now $\partial Y = \emptyset$. Then $\overline{Y} = Y \cup \partial Y = Y \cup \emptyset = Y$ so Y is closed. In addition $\partial Y = \overline{Y} \cap \overline{(X - Y)}$, hence $\emptyset = Y \cap (X - \mathring{Y})$. Since $\mathring{Y} \subseteq Y$ then $\mathring{Y} = Y$ and so Y is open. Viceversa assume that Y is both open and closed. This means $\overline{Y} = Y = \mathring{Y}$. Therefore $\partial Y = \overline{Y} \cap \overline{(X - Y)} = \mathring{Y} \cap (X - \mathring{Y}) = \emptyset$.

The above result will be useful to determine whether a given subspace of topological group will be connected or not, as we will see in the following chapters. Now we will show some explicit boundaries of well known subsets of real numbers.

Exercise 2.19 (See [8], Exercise 2.9(i))**.** We have $\partial((a, b)) = \partial([a, b]) = \{a, b\}$.

Solution. Noting that (a,b) is open, $\overline{(-\infty,a)} = (-\infty,a]$, $\overline{(b,+\infty)} = [b,\infty)$ and $\overline{(a,b)} = [a,b]$, we get

$$\partial((a,b)) = \overline{(a,b)} \cap \overline{\mathbb{R}-(a,b)} = [a,b] \cap \overline{(-\infty,a)\cup(b,+\infty)}$$
$$= [a,b] \cap (\overline{(-\infty,a)} \cup \overline{(b,+\infty)}) = [a,b] \cap ((-\infty,a] \cup [b,+\infty)) = \{a,b\}.$$

The rest works by analogy.

We pass now to study the notion of neighbourhood of point in a topological space: this is nothing else than a set containing an open passing through the point. The first property that we study is the so called *cofinality*:

Exercise 2.20 (See [8], Exercise 2.11 (ii))**.** If N is a neighbourhood of x in a topological space X and $N \subseteq M$, then M is also a neighbourhood of x.

Solution. Since N is a neighbourhood of x, $\exists U = U^{\circ}$ such that $U \subset N \subset M$, so also M contains an open passing through x.

Another property is the stability under intersections.

Exercise 2.21 (See [8], Exercise 2.11 (iii))**.** If M and N are neighbourhoods of x then so is $N \cap M$.

Solution. There are open sets U_1, U_2, $U_1 \subset N$ and $U_2 \subset M$. Then $U_1 \cap U_2 \subset U_1 \subset N$ and $U_1 \cap U_2 \subset U_2 \subset M$, so $x \in U_1 \cap U_2 \subset N \cap M$.

We reformulate more properly the cofiniality between neighbourhoods.

Exercise 2.22 (See [8], Exercise 2.11 (iv))**.** For each $x \in X$ and each neighbourhood N of x there exists a neighbourhood U of x such that $U \subset N$ and U is a neighbourhood of each of its points.

Solution. By definition, a neighbourhood N of x has an open set U such that $x \in U \subset N$, but this makes U a neighbourhood of x and open sets are neighbourhoods of all their points.

We take the opportunity to recall the notion of a local basis for a topology $\mathcal{T}$ on a space X. Denote by $\mathcal{I}(x)$ the set of all the neighbourhoods of x and $\mathcal{U}(x)$ be a subset of $\mathcal{I}(x)$. We say that $\mathcal{U}(x)$ is a *local basis* at a point x if

$$\forall\ U_x \in \mathcal{I}(x)\ \exists\ V_x \in \mathcal{U}(x)\ :\ V_x \subset U_x$$

We say that a space X satisfies the axiom N_1, or satisfies the axiom of countability of the first type, if X admits at least one local basis which is countable. We say that $\mathcal{B}$ is basis for the topology $\mathcal{T}$ if $\mathcal{B} \subseteq \mathcal{T}$ and each open of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

We say that a space X satisfies the axiom N_2, or satisfies the axiom of countability of the second type, if X admits at least one basis which is countable.

Now it is clear that $N_2 \Rightarrow N_1$ because each basis is of course a local basis. Viceversa the notion of local basis is weaker than the notion of basis so $N_1 \not\Rightarrow N_2$. This can be seen with the line of Sorgenfrey (see Exercise 2.3 (c)).

3. Continuous functions

In the present chapter we deal with the appropriate functions that come together with topological spaces. These are the continuous functions and are defined as those functions whose counterimages of open (of the codomain) are open (of the domain). We will discuss some of their classical properties.

Exercise 3.1 (See [8], Exercise 3.2(a))**.** Let X be an arbitrary set and let $\mathscr{U}, \mathscr{U}'$ be topologies on X. The identity mapping $(X, \mathscr{U}) \to (X, \mathscr{U}')$ is continuous if and only if $\mathscr{U}' \subseteq \mathscr{U}$.

Solution. Let $\iota : (X, \mathscr{U}) \to (X, \mathscr{U}')$ be the identity map. If ι is continuous then for all $U \in \mathscr{U}'$, $\iota^{-1}(U) \in \mathscr{U} \Rightarrow U \in \mathscr{U}$ so $\mathscr{U}' \subseteq \mathscr{U}$. Viceversa, if $\mathscr{U}' \subseteq \mathscr{U}$ then for all $U \in \mathscr{U}'$, $\iota^{-1}(U) = U$ then $U \in \mathscr{U}$. So $\iota^{-1}(U) \in \mathscr{U}$.

What happens if all the functions are continuous ?

Exercise 3.2 (See [8], Exercise 3.2(b))**.** If X is a topological space with the property that, for every topological space Y, every function $f : X \to Y$ is continuous, then X has the discrete topology.

Solution. In particular, we can choose $Y = X$ and f as the identity map and $\mathscr{U}'$ (topology on Y) as the discrete topology. Then we know that the topology $\mathscr{U}$ is at least as big as the discrete topology, i.e. $\mathcal{P}(X) \subseteq \mathscr{U}$. But since $\mathcal{P}(X)$ is the largest possible topology on X, $\mathscr{U} = \mathcal{P}(X)$.

Replacing the role of the domain with that of the codomain in the previous example, we get an alternative description.

Exercise 3.3 (See [8], Exercise 3.2(c))**.** If Y is a topological space with the property that, for every topological space X, every function $f : X \to Y$ is continuous, then Y has the trivial topology.

Solution. Let X be the space Y and f the identity map, but let $\mathscr{U}$ be the trivial topology on X. Then we know that the topology $\mathscr{U}'$ is at most as large as the trivial topology i.e., $\mathscr{U} \subseteq \{\emptyset, Y\}$. But since $\{\emptyset, Y\}$ is the smallest possible topology on Y, $\mathscr{U}' = \{\emptyset, Y\}$.

Treated apart from these extremal situations, we begin to see when a given function is continuous with respect to a topology but it is no longer continuous when we consider a different topology on the same space.

Exercise 3.4 (See [8], Exercise 3.2(d))**.** Consider $\mathbb{R}$ with the topology of the left half-lines. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is non-decreasing (i.e. if $x > x'$ then $f(x) \geq f(x')$) and continuous on the right in the classical sense (i.e. for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \leq x' < x + \delta$ then $|f(x) - f(x')| < \varepsilon$).

Solution. Assume f is continuous. Let $x > x'$. $f^{-1}((-\infty, f(x))) = (-\infty, x)$ and $f^{-1}((-\infty, f(x'))) = (-\infty, x')$. Since $(-\infty, x') \subset (-\infty, x)$, it follows that $f(x) \geq f(x')$. Also for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x' - x| < \delta$ then $|f(x) - f(x')| < \varepsilon$ since f is continuous. Assume $x' > x$, then $|x' - x| < \delta \Rightarrow 0 \leq x' - x < \delta \Rightarrow x \leq x' < x + \delta$ such that $|f(x) - f(x')| < \varepsilon$. That is continuity implies continuity on the right side.

Viceversa, assume f is non-decreasing and continuous on the right side. Let $(-\infty, y)$ be an open subset in $\mathbb{R}$. Since f is nondecreasing and continuous, we have that $f^{-1}((-\infty, y)) = (-\infty, f^{-1}(y))$ which is open in $\mathbb{R}$ and so f is continuous.

The above example shows that the function $-x$, defined on $\mathbb{R}$, is continuous with respect to the usual topology on $\mathbb{R}$ but is not continuous with respect to the topology of the left half-lines on $\mathbb{R}$.

Another fundamental notion is encountered when we ask whether a function sends open sets in open sets. This gives the notion of *open map* and it is possible to see that there are open maps which are not continuous, but there are also continuous maps which are not open. Similarly, one can formulate the notion of *closed map*, requiring that a function sends closed in closed and see that the notion of being a closed map is different from that of being continuous. Below there are some examples which help to understand these concepts.

Exercise 3.5 (See [8], Exercise 3.4)**.** Let f be a continuous function $f : X \to Y$ between the topological spaces X and Y. Then various options may arise between open, closed, injective, surjective, bijective functions.

Solution. If f is injective, then f may be neither open nor closed, may be open but not closed and may be closed but not open. If f is surjective, then f may be open but not closed, closed but not open. Finally if f is bijective, then f may be both open and closed.

One way to define homeomorphisms is by requiring a continuous bijective map along with its inverse. In fact one may wonder whether there are bijective continuous maps such that their inverse is not continuous. The example below illustrates this circumstance.

Exercise 3.6 (See [8], Exercise 3.7(a))**.** Example of spaces X, Y and a continuous bijection $f : X \to Y$ such that f^{-1} is not continuous.

Solution. Let $X = Y = \{a, b\}$ with topologies $\mathscr{U}_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathscr{U}_Y = \{\emptyset, \{a\}, \{a, b\}\}$. Define the functions

$$f : x \in X \longmapsto f(x) = x \in Y \quad ; \quad f^{-1} : x \in Y \longmapsto f^{-1}(x) = x \in X.$$

The function is clearly invertible and bijective but in the first case $f^{-1}(\{a\}) = \{a\}$, $f^{-1}(Y) = X$, so it is continuous. But in the second case, $(f^{-1})^{-1}(\{b\}) = f(\{b\})$ is not an open of Y.

Open bijective maps are homeomorphisms, as indicated below.

Exercise 3.7 (See [8], Exercise 3.7(b))**.** Let X and Y be topological spaces. Then X and Y are homeomorphic if and only if there exists a function $f : X \to Y$ such that (i) f is bijective and (ii) a subset U of X is open if and only if $f(U)$ is open.

Solution. Assume X and Y are homeomorphic. Then there exists inverse continuous functions $f : X \to Y$, $g : Y \to X$, i.e. $f \circ g = 1_Y$ and $g \circ f = 1_X$. This implies that f and g are bijective. Assume U is open in X. Then $1_X(U) = gf(U) = U$ is open, but since $f = g^{-1}$, g^{-1} is continuous. So $g^{-1}(U) = g^{-1}1_X(U) = g^{-1}gf(U) = f(U)$ is open. Assume $f(U)$ is open. Since f is continuous $f^{-1}(f(U)) = U$ is open.

Viceversa, assume there is $f : X \to Y$ and f is bijective and in addition a subset U of X is open if and only if $f(U)$ is open. Since f is bijective, $f \circ f^{-1} = 1_X$ and $f(U)$ is open so f^{-1} is continuous. Then f, f^{-1} are both continuous functions with their inverses and so X, Y are homeomorphic.

We want to apply what we have seen until now to describe homeomorphisms of metric spaces.

Exercise 3.8 (See [8], Exercise 3.7(c))**.** If two metrics d_1 and d_2 on a set Y are such that, for some positive m and M,

$$m\ d_1(y, y') \leq d_2(y, y') \leq M\ d_1(y, y')$$

for all $y, y' \in Y$, then the two topological spaces (on Y) arising from these metrics are homeomorphic.

Solution. Let (Y, d_1) be the metric space Y with the metric d_1 , and (Y, d_2) be the metric space Y with the metric d_2. Let $1_1 : (Y, d_1) \to (Y, d_2)$ be the identity map $x \mapsto x$. Of course, 1_1 is bijective and invertible with an inverse $1_Y^2 : (Y, d_2) \to (Y, d_1)$, defined by $x \mapsto x$. Of course, both $1_1 \circ 1_2$ and $1_2 \circ 1_1$ give again the identity map of Y. To see that $1_1 \circ 1_2$ is continuous it is enough to show that an open set U is open in both metric spaces, since $1_1^{-1}(U) = 1_2(U) = U$. In fact $1_2^{-1} = 1_1(U) = U$. Note that U being open in (Y, d_2) means that $\forall\ x \in U\ \ \exists\ \varepsilon > 0$ such that $B^2_\varepsilon = \{y \in X : d_2(y, x) < \varepsilon\} \subseteq U$, but since $md_1(y, y') \leq d_2(y, y') < \varepsilon \Rightarrow B^1_{\frac{1}{m}\varepsilon}(x) = \{y \in X : d_1(y, x) < \frac{1}{m}\varepsilon\} \subseteq B^2_\varepsilon(x)$. The same can be said viceversa, noting that $B^2_{M\varepsilon}(x) \subseteq B^1_\varepsilon(x)$.

The above result is very useful when we deal with equivalent metrics in functional analysis. The idea is to find suitable lower and upper bounds, as we have seen. Another useful fact, concerning homeomorphisms, is that they form an algebraic structure with respect to the composition of functions.

Exercise 3.9 (See [8], Exercise 3.7(d))**.** Let X be a topological space and let $G(X)$ denote the set of homeomorphisms $f : X \to X$. The set $G(X)$ is group with respect to the operation of composition of two functions. For $x \in X$, the set of all the fixed points $G_x(X) = \{f \in G(X) \mid f(x) = x\}$ is a subgroup of $G(X)$.

Solution. We are going to check the axioms of groups. If $f \in G(X)$, then f has an inverse f^{-1}, and, being bijective and continuous it is also an homeomorphism so is an element of $G(X)$. This means that the symmetric of each element of $G(X)$ is contained in $G(x)$. If $f, g \in G(X)$, then $g \circ f$ is continuous since g and f are continuous and the composition of two continuous functions is continuous, but also $(g \circ f)^{-1} = f^{-1} \circ g^{-1} \in G(X)$ since the inverses are in $G(X)$. This means that the product of two elements of $G(X)$ is again in $G(X)$. Finally, the identity map 1_X is obviously in $G(X)$. So we may conclude that $G(X)$ is a group.

Now if $f, g \in G_x(X)$, then $g(x) = x$ implies $f(x) = f(g^{-1}(x))$, hence $x = f(g^{-1}(x))$, so $x = f \circ g^{-1}(x)$, and this means that $f \circ g^{-1} \in G_x(X)$, thus $G_x(X)$ is a subgroup.

4. Induced topology

In the present section we will illustrate another classical notion on general topology. Given a subset Y of a topological space X with topology $\mathscr{U}_X$, we may consider the family $\mathscr{U}_Y = \{Y \cap A \mid A \in \mathscr{U}_X\}$ of subsets of Y, which turns out to be a topology on Y. The set Y is said to have the *induced topology* by X, if it comes with the topology $\mathscr{U}_Y$.

Exercise 4.1 (See [8], Exercise 4.5(a))**.** If Y is a subspace of X, and Z is a subspace of Y, then Z is a subspace of X.

Solution. Since Z is a subspace of Y, every $K \in \mathscr{U}_Z$ has the form $K = V \cap Z$, where $V \in \mathscr{U}_Y$, but Y is a subspace of X, so $V = U \cap Y$, where $U \in \mathscr{U}_X$, hence every open of Z has the form $K = (U \cap Y) \cap Z = U \cap (Y \cap Z) = U \cap Z$ with $U \in \mathscr{U}_X$, that is, Z is subspace of X.

Exercise 4.2 (See [8], Exercise 4.5(b))**.** The metrizability is invariant via subspaces. In other words, a subspace of a metrizable space is metrizable.

Solution. Let d be the metric defined on X and S a subspace of X. Then define $d_S = d|_{(S\times S)}$, the restriction of d to the subset S. Note that $\forall\ s \in S$ and $\varepsilon > 0$ so $B_{d_S}(s,\varepsilon) = B_d(s,\varepsilon) \cap S$. The left hand side of this equation will give us the form of elements in the base of $\mathscr{U}_{d_S}$, the topology on S induced by the metric d_S, and the right hand side will give an element in the basis of $\mathscr{U}_S$ since $B_d(s,\varepsilon)$ is open in X. So the topologies coincide since they have the same base.

Now we come with an important result, in which we begin to see the notion of *strong topology* and of *weak topology*. Given two topologies $\mathscr{U}$ and $\mathscr{V}$ on the same set X, we say that $\mathscr{U}$ is *stronger* than $\mathscr{V}$, if $\mathscr{U} \subseteq \mathscr{V}$. Viceversa, $\mathscr{U}$ is *weaker* than $\mathscr{V}$, if $\mathscr{U} \supseteq \mathscr{V}$.

Exercise 4.3 (See [8], Exercise 4.5(c))**.** Suppose that S is a subspace of X. Then the inclusion map $S \to X$ is continuous. Furthermore, S has the weakest topology (i.e. the least number of open sets) such that the inclusion $S \to X$ is continuous.

Solution. Let U be an open in X and f the inclusion map. Then $f^{-1}(U) = U \cap S$ which is an open set in the induced topology of S. Assume $\mathscr{U}_S$ is not the weakest topology, so $\exists\ \mathscr{U} \subset \mathscr{U}_S$ such that for an open $V = U \cap S \in \mathscr{U}_S$, $V = U \cap S \notin \mathscr{U}$, U is open in X and f is continous with respect to this topology. But then $f^{-1}(U) = U \cap S \notin \mathscr{U}$, and f can't be continuous, which is a contradiction. So $\mathscr{U}$ is the weakest topology such that the inclusion map is continuous.

We now pass to describe the closure operator and its behaviour in the induced topology. Let Y be a subspace of a topological space X and let A be a subset of Y. Denote by $\mathrm{Cl}_X(A)$ the closure of A in X and by $\mathrm{Cl}_Y(A)$ the closure of A in Y. In general it is clear that $\mathrm{Cl}_Y(A) \subseteq \mathrm{Cl}_X(A)$, but the inclusion may be strict.

Exercise 4.4 (See [8], Exercise 4.5(e))**.** We describe a general condition for which $\mathrm{Cl}_Y(A)$ turns out to be properly contained in $\mathrm{Cl}_X(A)$.

Solution. Let $Y \subset X$, $A = Y$, Y open in X and $u \in \overline{Y} \cap (X - A)$. Then $\mathrm{Cl}_Y(A) = A \cup \partial A$, where $\partial A = \mathrm{Cl}_Y(A) \cap \overline{(Y-A)} = \mathrm{Cl}_Y(A) \cap \overline{(Y-Y)} = \mathrm{Cl}_Y(A) \cap \emptyset = \emptyset$, so that $\mathrm{Cl}_Y(A) = A$. On the other hand, $\partial A = \mathrm{Cl}_X(A) \cap \overline{(X-A)}$ and so $\mathrm{Cl}_X(A) = A \cup \partial A = \mathrm{Cl}_X(A) \cap \overline{(X-Y)} = \overline{A} \cap \overline{(X-Y)} = \overline{Y} \cap \overline{(X-Y)} = \overline{Y} \cap (X-A) \neq \emptyset$. Therefore $\mathrm{Cl}_X(A) = A \cup \partial A \supset A = \mathrm{Cl}_Y(A)$ and the inclusion is strict.

The intuitive idea that a "subspace is smaller than the space in which it is placed" is evidently wrong, and this is explained in the following situation.

Exercise 4.5 (See [8], Exercise 4.5(f))**.** The interval (a,b) of $\mathbb{R}$ with the induced topology is homeomorphic to $\mathbb{R}$.

Solution. There are various ways to justify this fact. We offer one possible way. Consider the functions:

$$f : x \in (a,b) \longmapsto f(x) = \tan\left[\pi\left(\frac{b+a}{2(a-b)} - \frac{x}{a-b}\right)\right] \in \mathbb{R};$$
$$g : x' \in \mathbb{R} \longmapsto g(x') = \left(\frac{b-a}{\pi}\right)\arctan(x') + \frac{b+a}{2} \in (a,b)$$

We will show that f and g are one the inverse of the other and that they are both continuous. This will be enough to conclude that f is homeomorphism. To see that $f \circ g = 1_{\mathbb{R}}$, let $x \in \mathbb{R}$, then we have

$$(f\circ g)(x) = f(g(x)) = \tan\left[\pi\left(\frac{b+a}{2(a-b)} - \frac{\frac{b-a}{\pi}\arctan(x) + \frac{b+a}{2}}{a-b}\right)\right]$$
$$= \tan\left[\pi\left(\frac{(b+a) - 2(\frac{b-a}{\pi}\arctan(x) + \frac{b+a}{2})}{2(a-b)}\right)\right]$$
$$= \tan\left[\pi\left(\frac{(b+a) - \frac{2(b-a)}{\pi}\arctan(x) - (b+a)}{2(a-b)}\right)\right] = \tan\left[\pi\left(\frac{-\frac{2(b-a)}{\pi}\arctan(x)}{2(a-b)}\right)\right]$$
$$= \tan\left[\pi\left(\frac{\frac{2(a-b)}{\pi}\arctan(x)}{2(a-b)}\right)\right] = \tan\left[\frac{\pi 2(a-b)\arctan(x)}{\pi 2(a-b)}\right] = \tan(\arctan(x)) = x$$

and to see that $g \circ f = 1_(a,b)$ let $x \in (a,b)$, we have

$$(g\circ f)(x) = g(f(x)) = \frac{b-a}{\pi}\arctan\left[\tan\left(\pi\left(\frac{b+a}{2(a-b)} - \frac{x}{a-b}\right)\right)\right] + \frac{b-a}{2}$$
$$= \frac{b-a}{\pi}\left[\pi\left(\frac{b+a}{2(a-b)} - \frac{x}{(a-b)}\right)\right] + \frac{b+a}{2} = -(a-b)\left[\frac{b+a}{2(a-b)} - \frac{x}{(a-b)}\right] + \frac{b+a}{2}$$
$$= -\frac{(b-a)}{2} + x + \frac{(b+a)}{2} = x.$$

We already know that $\tan(x)$ is continuous for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since it is an elementary function, so f is continuous on (a,b) and we know also that $\arctan(x)$ is continuous on $\mathbb{R}$, so g is continuous on $\mathbb{R}$. Therefore our claim $(a,b) \cong \mathbb{R}$ follows.

Of course, any other continuous bijective map from a bounded interval (a,b) to $\mathbb{R}$ would have served for the argument which we have just described.

Let's continue to find more intervals, not necessarily bounded and see whether we have homeomorphisms with $\mathbb{R}$ or not.

Exercise 4.6 (See [8], Exercise 4.5(h))**.** The half-line $(1,\infty)$ (with the induced topology of $\mathbb{R}$) is homeomorphic to $(0,1)$ (with the induced topology of $\mathbb{R}$) and so both of them are homeomorphic to $\mathbb{R}$ (with the usual topology).

Solution. The functions $f : x \in (1,\infty) \longmapsto f(x) = 1/x \in (0,1)$ and $g : x \in (0,1) \longmapsto g(x) = 1/x \in (1,\infty)$ are one the inverse of the other, both are continuous, and realize the required homeomorphism between $(1,\infty)$ and $(0,1)$. Of course, Example 5.5 will give the remaining homeomorphism with $\mathbb{R}$.

Basically, the previous two results show that all the open intervals of $\mathbb{R}$ are homeomorphic between themselves. The methods of proofs overlap the arguments which we have just seen. A completely different nature of homeomorphism is illustrated in higher dimensions between punctured sphere and $\mathbb{R}^n$.

Exercise 4.7 (See [8], Exercise 4.5(i))**.** The punctured n-sphere $S^n - \{(0,0,...,0,1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology (via the *stereographic projection*).

Solution. Let $x \in \mathbb{R}^n$ and consider the functions

$$\psi : x \in \mathbb{R}^n \mapsto \psi(x) = \frac{1}{1+\|x\|^2}(2x_1, 2x_2, ..., 2x_n, \|x\|^2 - 1) \in S^n - \{(0,0,...,0,1)\};$$

$$\varphi : x \in S^n - \{(0,0,...,0,1)\} \mapsto \varphi(x) = \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}}\right) \in \mathbb{R}^n.$$

Then

$$(\varphi \circ \psi)(x) = \varphi\left(\frac{1}{1+\|x\|^2}(2x_1, 2x_2, ..., 2x_n, \|x\|^2 - 1)\right)$$

$$= \left(\frac{\frac{2x_1}{1+\|x\|^2}}{1 - \frac{\|x\|^2-1}{1+\|x\|^2}}, \frac{\frac{2x_2}{1+\|x\|^2}}{1 - \frac{\|x\|^2-1}{1+\|x\|^2}}, ..., \frac{\frac{2x_n}{1+\|x\|^2}}{1 - \frac{\|x\|^2-1}{1+\|x\|^2}}\right) = \left(\frac{\frac{2x_1}{1+\|x\|^2}}{\frac{2}{1+\|x\|^2}}, \frac{\frac{2x_2}{1+\|x\|^2}}{\frac{2}{1+\|x\|^2}}, ..., \frac{\frac{2x_n}{1+\|x\|^2}}{\frac{2}{1+\|x\|^2}}\right)$$

$$= (x_1, x_2, ..., x_n)$$

and

$$(\psi \circ \varphi)(x) = \psi\left(\left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}}\right)\right)$$

$$= \frac{1}{1 + \frac{x_1^2+x_2^2+\cdots+x_n^2}{(1-x_{n+1})^2}}\left(\frac{2x_1}{1-x_{n+1}}, \frac{2x_2}{1-x_{n+1}}, ..., \frac{2x_n}{1-x_{n+1}}, \frac{x_1^2+x_2^2+\cdots+x_n^2}{(1-x_{n+1})^2} - 1\right)$$

$$= \frac{(1-x_{n+1})^2}{2-2x_{n+1}}\left(\frac{2x_1}{1-x_{n+1}}, \frac{2x_2}{1-x_{n+1}}, ..., \frac{2x_n}{1-x_{n+1}}, \frac{2x_{n+1}-2x_{n+1}^2}{(1-x_{n+1})^2}\right)$$

$$= \frac{(1-x_{n+1})^2}{2(1-x_{n+1})}\left(\frac{2x_1}{1-x_{n+1}}, \frac{2x_2}{1-x_{n+1}}, ..., \frac{2x_n}{1-x_{n+1}}, \frac{2x_{n+1}(1-x_{n+1})}{(1-x_{n+1})^2}\right)$$

$$= (x_1, x_2, ..., x_n, x_{n+1}).$$

Therefore φ and ψ are one the inverse of the another, and so bijective. Moreover φ and ψ are compositions of continuous functions, so they are continuous and realize the required homeomorphism.

The function φ, introduced in the previous argument, is called stereographic projection. In fact, if we replace its domain with S^n, that is, we add the point $(0,0,...,0,1)$ and remove the multiplicative factor $1/(1-x_{n+1})$, then φ becomes just a projection of $(x_1,\ldots,x_n,x_{n+1}) \in S^n$ onto its first n components $(x_1,\ldots,x_n) \in \mathbb{R}^n$. In order to justify with the intuition this map, we may fix $n = 1$ and consider the usual circle and the usual line. We may remove the northern pole $(0,1)$ from S^1, consider a line l passing trough $(0,1)$ and intersecting S^1 in P and shift P to the corresponindg intersection of l with the line $x = 0$, representing $\mathbb{R}$. The points of S^1 between $(-1,0)$ and $(0,1)$ maps onto the interval $[-1,1]$ bijectively.

Those between $(0,1)$ and $(-1,0)$ onto the left half line $(-\infty,-1)$ and, finally, those between$(0,1)$ and $(-1,0)$ onto right half line $(1,+\infty)$. Globally we have constructed the homeomorphism φ above. This argument, of geometric nature, is generalized in n-dimension with the notations we have seen before. The idea of extending φ, keeping this function as homeomorphism, would imply that we add the point $\{\infty\}$ to $\mathbb{R}$. In fact we will see later on that $\mathbb{R}^\infty \simeq S^1$, that is, the compactification that we form adding ∞ to $\mathbb{R}$ determines a topological space which is homeomorphic to S^1. The homeomorphism is in fact an extension of the stereographic projection.

It might be useful to note that we have also homeomorphisms between the punctured plane and the sphere in any dimension bigger than three. In fact $\mathbb{R}^{n+1} - \{0\}$ turns out to be homeomorphic to S^n via the function $f(x) = x/\|x\|$.

5. Quotient topology and groups acting on spaces

In every algebraic and geometric structure, the first step is to look for substructures which are stable under prescribed properties and the second step is to look for the stability for structures which are obtained by imposing equivalence relations on the original structure. These are the so called quotients. We begin with the formal notion.

Definition 5.1 (See [8], Definition 5.1)**.** Suppose that $f : X \to Y$ is a surjective mapping from a topological space X onto a set Y. The *quotient topology* on Y with respect to f is the family

$$\mathscr{U}_f = \{U : f^{-1}(U) \text{ is open in } X\}$$

While the induced topology on a subspace involved the notion of weak topology, here we have the extremal case when we deal with the quotient topology.

Exercise 5.2 (See [8], Exercise 5.3(a))**.** Suppose that Y is given the quotient topology with respect to the mapping $f : X \to Y$. Then Y has the strongest topology such that f is continuous.

Solution. Let $\mathscr{U}_f$ be the quotient topology of Y w.r.t. $f : X \to Y$, and $\mathcal{V}$ be a topology on Y such that f is continuous. Let $U \in \mathcal{V} \Rightarrow f^{-1}(U)$ is open in $X \Rightarrow U \in \mathscr{U}_f$. So $\mathcal{V} \subseteq \mathscr{U}_f$

Of course, one can interchange open sets with closed sets in the definition of a quotient topology.

Exercise 5.3 (See [8], Exercise 5.3(b))**.** Suppose that Y has the quotient topology with respect to the mapping $f : X \to Y$. Then a subset A of Y is closed if and only if $f^{-1}(A)$ is closed X.

Solution. If A is closed in Y, then $f^{-1}(A)$ is closed in X since f is continuous. Viceversa, $f^{-1}(A)$ closed in X implies $X - f^{-1}(A)$ open, so $X - f^{-1}(A) = f^{-1}(Y - A)$ and $Y - A$ is open in Y, by definition of the quotient topology, thus A is closed in Y.

There are cases where the quotient topology and induced topology of a space coincide.

Exercise 5.4 (See [8], Exercise 5.3(c))**.** Let $f : \mathbb{R} \to S^1$ be a parametrization of the unit circle S^1 by $f(t) = (\cos(2\pi t), \sin(2\pi t)) \in \mathbb{R}^2$ with $t \in \mathbb{R}$. Then the quotient topology $\mathscr{U}_f$ on S^1 determined by f is the same as the topology $\mathscr{U}_{S^1}$ induced from the natural topology on $\mathbb{R}^2$.

Solution. Since the basis for the topology on $\mathbb{R}^2$ consists of open balls, the intersection of these open balls with S^1 results in open arcs as a basis for the topology $\mathscr{U}_{S^1}$. Arcs for S^1 can be captured by their start and end points by varying the degrees in radians between $2\pi a$ and $2\pi b$, where $0 \leq a < b \leq 1$. So an arc would be represented by $f((a, b))$. This gives the basis $\{f((a, b)) : 0 \leq a < b \leq 1\}$ for $\mathscr{U}_{S^1}$.

The basis for an open in $\mathbb{R}$ is $\{(a, b) : a, b \in \mathbb{R}, a < b\}$. But note that $f(t) = f(t + n)$, $\forall\ n \in \mathbb{Z}$. So $f((a, b)) = f((a + n, b + n))$, $\forall\ n \in \mathbb{Z}$. So $f^{-1}(f(a, b)) = \bigcup_{n\in\mathbb{Z}} (a + n, b + n)$, which is open in $\mathbb{R}$. So $f((a, b))$ is open in S^1 and forms a basis for $\mathscr{U}_f$. This means that a basis for $\mathscr{U}_f$ is the same basis of $\mathscr{U}_{S^1}$ so the two topologies are equal. This detail explains why the construction $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ where $(n, x) \mapsto n \cdot x = n + x$ leads to the quotient space $\mathbb{R}/\mathbb{Z} \cong S^1$ which we will see later on in this chapter.

We might wonder what quotient topology would arise from a composition of functions. An answer is here:

Exercise 5.5 (See [8], Exercise 5.3(d))**.** Let X, Y, Z be topological spaces and let $f : X \to Y, g : Y \to Z$ be surjections. If the topologies of Y and Z are the quotient topologies determined by f and g respectively then the topology of Z is the quotient topology determined by $g \circ f : X \to Z$.

Solution. Let $\mathscr{U}_f, \mathscr{U}_g$ be the quotient topologies of Y and Z w.r.t. functions f and g respectively, and $\mathscr{U}_h$ the quotient topology of Z w.r.t. the function $g \circ f = h$. If $U \in \mathscr{U}_g$, then $g^{-1}(U)$ is open in Y, $f^{-1}(g^{-1}(U))$ is open in X and $h^{-1}(U)$ is open in X. This means that $U \in \mathscr{U}_h$ so $\mathscr{U}_g \subseteq \mathscr{U}_h$. Viceversa, $h^{-1}(U)$ is open in X and $f^{-1}(g^{-1}(U))$ is open in X which means that $g^{-1}(U)$ is open in Y, so $U \in \mathscr{U}_g$. So $\mathscr{U}_h \subseteq \mathscr{U}_g$ and $\mathscr{U}_h = \mathscr{U}_g$.

We may embedd the real projective plane of dimension 2 in terms of $\mathbb{R}^4$.

Exercise 5.6 (See [8], Exercise 5.3(f))**.** The function $f : \mathbb{RP}^2 \to \mathbb{R}^4$, defined by $\{x, -x\} \to (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$, is continuous and injective.

Solution. We may show the injectivity, distinguishing three cases. The reason will be apparent later on. Let $f(x, -x) = f(y, -y)$ then we have these five constraints, (we count the first constraint as one)

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2 (= 1) \\ x_1^2 - x_2^2 &= y_1^2 - y_2^2 \\ x_1x_2 &= y_1y_2 \\ x_1x_3 &= y_1y_3 \\ x_2x_3 &= y_2y_3 \end{aligned}$$

Case 1. Assume: $\sqrt{2}y_1 + y_3 = 0$. Then

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2.$$

Adding the second constraint to the first,

$$\Rightarrow 2x_1^2 + x_3^2 = 2y_1^2 + y_3^2 \Rightarrow (\sqrt{2}x_1 + x_3)^2 - 2\sqrt{2}x_1x_3$$
$$= (\sqrt{2}y_1 + y_3)^2 - 2\sqrt{2}y_1x_3 \Rightarrow (\sqrt{2}x_1 + x_3)^2 = (\sqrt{2}y_1 + y_3)^2$$
$$\Rightarrow \sqrt{2}x_1 + x_3 = \pm(\sqrt{2}y_1 + y_3) \Rightarrow \sqrt{2}x_1x_2 + x_2x_3 = \pm x_2(\sqrt{2}y_1 + y_3)$$

$\Rightarrow \sqrt{2}y_1y_2 + y_2y_3 = \pm x_2(\sqrt{2}y_1 + y_3) \Rightarrow y_2(\sqrt{2}y_1 + y_3 = \pm x_2(\sqrt{2}y_1 + y_3) \Rightarrow y_2 = \pm x_2$. So we get $x_1^2 = y_1^2$ and $x_1 = \pm y_1$. This implies $x_3 = \pm y_3$.

Case 2. Assume $\sqrt{2}y_2 + y_3 = 0$. Then we get

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2.$$

Subtracting the second constraint to the first,

$$\Rightarrow 2x_2^2 + x_3^2 = 2y_2^2 + y_3^2 \Rightarrow (\sqrt{2}x_2 + x_3)^2 - 2\sqrt{2}x_2x_3 = (\sqrt{2}y_2 + y_3)^2 - 2\sqrt{2}y_2x_3$$
$$\Rightarrow (\sqrt{2}x_2 + x_3)^2 = (\sqrt{2}y_2 + y_3)^2 \Rightarrow \sqrt{2}x_2 + x_3 = \pm(\sqrt{2}y_2 + y_3)$$

$$\Rightarrow \sqrt{2}x_1x_2 + x_1x_3 = \pm x_1(\sqrt{2}y_2 + y_3) \Rightarrow \sqrt{2}y_1y_2 + y_1y_3 = \pm x_1(\sqrt{2}y_1 + y_3)$$
$$\Rightarrow y_1(\sqrt{2}y_2 + y_3 = \pm x_1(\sqrt{2}y_1 + y_3) \Rightarrow y_1 = \pm x_1$$

so we get $x_2 = \pm y_2$ and $x_3 = \pm y_3$.

Case 3. Assume $\sqrt{2}y_1 + y_3 = 0$ and $\sqrt{2}y_2 + y_3 = 0$. Then clearly

$$y_1 = y_2 \text{ and } y_1 = -\frac{1}{2}y_3$$

And so

$$y_1^2 + y_2^2 + y_3^2 = 1 \Rightarrow \frac{1}{2}y_3^2 + \frac{1}{2}y_3^2 + y_3^2 = 1 \Rightarrow y_3^2 = \frac{1}{2} \Rightarrow y_3 = \pm\frac{1}{\sqrt{2}}, \quad y_1, y_2 = \mp\frac{1}{2}$$

And since

$$x_1^2 - x_2^2 = y_1^2 - y_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$$

but $y_1 = y_2$ and

$$x_1x_2 = \pm y_1y_2 \Rightarrow x_1^2 = \pm\frac{1}{4} \Rightarrow x_1 = x_2$$

since x_1^2 must be positive and we conclude that

$$x_1, x_2 = \pm\frac{1}{2}, \quad x_3 = \pm\frac{1}{\sqrt{2}}.$$

So again $x_1 = \pm y_1, x_2 = \pm y_2, x_3 = \pm y_3$.

Now it remains to prove that indeed the points are anti-podal. For that we consider that the last three constraints. If any of the coordinates had a sign different from the other three than there would be a negative sign in at least one of the equalities. So $x = \pm y$, that is $\{x, -x\} = \{y, -y\}$. So f is injective, and is continuous because it is the composition of continuous maps.

We come back to discuss the properties of the quotient topology in a more general setting in presence of open, or closed, maps. In fact if we have an open or closed mapping, then image is homeomorphic to the quotient space.

Exercise 5.7 (See [8], Exercise 5.3(g))**.** Let X be a topological space and let $f : X \to Y$ be a surjective map. Let $\mathscr{U}_f$ denote the quotient topology on Y. Suppose that $\mathscr{U}$ is a topology on Y so that $f : X \to Y$ is continuous with respect to this topology. If f is a closed (or an open) mapping, then $(Y, \mathscr{U})$ is homeomorphic to $(Y, \mathscr{U}_f)$. Furthermore, there are examples for which if f is neither open nor closed, then $(Y, \mathscr{U}_f) \ncong (Y, \mathscr{U})$.

Solution. We already know that $\mathscr{U} \subseteq \mathscr{U}_f$ by Exercise 6.2. If f is closed and $U \in \mathscr{U}_f$, then $f^{-1}(Y - U)$ is closed in X and $f(f^{-1}(Y - U)) = Y - U$ is closed in Y with the topology $\mathscr{U}$. So $U \in \mathscr{U}$, and $\mathscr{U} = \mathscr{U}_f$. In this case, the identity map would be a homeomorphism from $(Y, \mathscr{U})$ to $(Y, \mathscr{U}_f)$. A similar argument applies if f is an open mapping.

Finally, we may consider the map $f : t \in \mathbb{R} \mapsto f(t) = (\cos t, \sin t) \in S^1$ with $Y = S^1$ with the trivial topology on Y and the usual topology on $\mathbb{R}$. Of course, f is surjective and $f^{-1}(S^1) = \mathbb{R}$ so it is continuous because there are no more opens in Y, but f is neither closed nor open and in fact Y with the trivial topology is not homeomorphic to Y with $\mathscr{U}_f$, which is the unit circle with the topology induced by the usual plane.

Just like the spaces they are derived from, subspaces can have different topologies that vary in 'strength' and we have seen that this motivates the notion of strong topology with respect to another one. The following exercise deals with the role of strong topologies in quotient spaces.

Exercise 5.8 (See [8], Exercise 5.3(h))**.** Suppose that $f : X \to Y$ is a surjective map from a topological space X to a set Y. Let Y have the quotient topology determined by f and let A be a subspace of X. Let $\mathscr{U}_1$ denote the topology on $B = f(A) \subseteq Y$ induced by Y and let $\mathscr{U}_2$ denote the quotient topology determined by the map $f|A : A \to B$. Then $\mathscr{U}_1 \subseteq \mathscr{U}_2$. Moreover there are examples such that the inclusion is proper. Finally, if either A is a closed subset of X and f is a closed map, or A is an open subset of X and f is an open map, then $\mathscr{U}_1 = \mathscr{U}_2$.

Solution. Let $\mathscr{U}$ be the topology $\{V \cap A \ : \ V \text{ is open in } X\}$, $\mathscr{U}_1 = \{O \cap B \ : \ O \text{ is open in } Y\}$ and $\mathscr{U}_2 = \{O \ : \ f|A^{-1}(O) \text{ is open in } A\}$. If $U \in \mathscr{U}_1$, then

$$f^{-1}(U) = f^{-1}(O \cap B) = f^{-1}(O) \cap f^{-1}(B) = f^{-1}(O) \cap f^{-1}(f(A)) = f^{-1}(O) \cap A \in \mathscr{U}_2$$

and $\mathscr{U}_1 \subseteq \mathscr{U}_2$.

Again the example of the function $f : t \in \mathbb{R} \mapsto f(t) = (\cos t, \sin t) \in S^1$ with the trivial topology on S^1 shows that $\mathscr{U}_1$ may be in general properly included in $\mathscr{U}_2$.

Finally, if A is a closed subset of X and f is a closed map, then the closed sets of subspace A are closed in X and $B = f(A)$ is closed. Let $U \in \mathscr{U}_2 \Rightarrow B - U$ is closed in $Y \Rightarrow f|A^{-1}(U)$ is open in $A \Rightarrow f|A^{-1}(U)$ is open in $X \Rightarrow X - f|A^{-1}(U)$ is closed in $X \Rightarrow f(X - f|A^{-1}(U)) = Y - f(f|A^{-1}(U))$ is closed in $Y \Rightarrow Y - U$ is closed in $Y \Rightarrow U$ is open in $Y \Rightarrow U \cap f(A) = B$ is open in $B \Rightarrow U \subseteq \mathscr{U}_1$. Of course a similar argument applies when we deal with an open map.

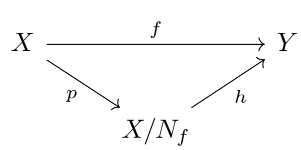
Intuitively, we may join the poin 0 and 1 of the interval $[0, 1]$ getting the circle. This geometric transformation is illustrated at the level of quotient topologies in the following exercise. There are various ways to approach this solution. We will invoke a classical result on the quotient topologies, called *theorem of representation*, and will give a proof for it in the course of the exercise itself.

Exercise 5.9 (See [8], Exercise 5.4(a))**.** If $I = [0, 1] \subseteq \mathbb{R}$ and $\sim$ is the equivalence relation $x \sim x'$, defined by $\{x, x'\} = \{0, 1\}$ or $x = x'$, then $I/\sim$ is homeomorphic to S^1.

Solution. If $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$ are two topological spaces and $f : X \to Y$ is a continuous and open map, then define x in relation N_f with y, briefly $x \ N_f \ y$, if and only if $f(x) = f(y)$, that is, if they are mapped into the same element via f. This relation turns out to be an equivalence relation on X and implies the existence of a unique homeomorphism

$$h : [x]_{N_f} \in (X/_{N_f}, \mathscr{U}_{X/_{N_f}}) \longmapsto h([x]_{N_f}) = f(x) \in (f(X), \mathscr{U}_{f(X)})$$

such that the following diagram is commutative



that is, $h \circ p = f$, where

$$p : x \in X \to p(x) = [x]_{N_f} \in X/_{N_f}$$

is the canonical projection of X onto $X/_{N_f}$. Note that h is well defined and injective because,

$$h([x]_{N_f}) = h([y]_{N_f}) \Leftrightarrow f(x) = f(y) \Leftrightarrow [x]_{N_f} = [y]_{N_f}.$$

In addition, if $y \in f(X)$, then there exists some $x \in X$ such that $f(x) = y$, and since p is onto $[x]_{N_f} \in X/_{N_f}$ such that $h([x]_{N_f}) = f(x) = y$. So h is also surjective, hence bijective. It remains to see that h is continuous and open, in order to conclude that it is homeomorphism.

Now $f^{-1} = (h \circ p)^{-1} = p^{-1} \circ h^{-1}$, so for all $V \in \mathscr{U}_Y$ we mus get $f^{-1}(V) = p^{-1}(h^{-1}(V)) \in \mathscr{U}_X$ because f is continuous. If $W = h^{-1}(V) \notin \mathscr{U}_{X/_{N_f}}$, then $f^{-1}(V) = p^{-1}(W) \notin \mathscr{U}_X$, which would be in contradiction with the continuity of f if $h^{-1}(V) \notin \mathscr{U}_{X/_{N_f}}$. Therefore h is a continuous. On the other hand, if f is open and p is open, then for all $U \in \mathscr{U}_X$, the set $f(U) = h(p(U)) = h([U]_{N_f}) \in \mathscr{U}_{f(X)}$ must be open, because if not, as before, we would get a contradiction. Therefore h is a continuous, open, bijective map, as claimed.

In order to apply this to our exercise, consider $t \in [0,1] \mapsto f(t) = e^{2\pi i t} \in S^1$. One can see easily that $\sim$ is the relation N_f for this specific choice of f. Moreove, f is continuous and open along with the natural projection onto $I/\sim$. This allows us to conclude that there is a homeomorphism h between $I/\sim$ and S^1.

We begin to explore an important connection between quotient topologies and abstract algebra, given by the notion of action of a group on a topological space.

Definition 5.10 (See [8], Definition 5.6)**.** Let X be a set and let G be a group. We say that G *acts* on X and that X is a G-set if there is a function from $G \times X$ to X, denoted by $(g, x) \to g \cdot x$, such that

(i) $1 \cdot x = x$ for all $x \in X$, where 1 is the identity element of G,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

If in addition the action is continuous and the map $\theta_g : x \in X \mapsto g \cdot x \in X$ is homeomorphism for all $g \in G$, we say that X is a G-*space*.

The previous definition of a G-action is strictly speaking that of a *left* G-action and one can see easily that the above axioms are not always satisfied. A first fact that we can observe is that it is possible to extend, or restrict, naturally the actions of groups. The following result describes the actions, restricted to subgroups.

Exercise 5.11 (See [8], Exercise 5.7(b))**.** Let H be a subgroup of a group G. For $h \in H$, $g \in G$ define $h \cdot g$ to be hg. This defines an action of H on G.

Solution. Let $h \in H$ and $g \in G$ then $hg \in G$. So define a function from $H \times G \to G$ such that $(h, g) \to h \cdot g = hg$. Then the first axiom is trivially satisfied and about the second $k \cdot (h \cdot g) = k \cdot (hg) = k \cdot hg = khg = (kh)g = (kh) \cdot g$.

Before to describe the role of the actions when we form group quotients, we need to make some preliminaries. First of all, some actions are well known in literature, because they play an important role in the structure of several algebraic structures. One of these is the action of the group on its subgroups lattice.

Exercise 5.12 (See [8], Exercise 5.7(c))**.** Let G be a group and $\mathscr{L}(G)$ the set of all subgroups of G. Then $g \cdot H = \{gh \mid h \in H\}$, where $g \in G$ and $H \in \mathscr{L}(G)$, defines an action of G on $\mathscr{L}(G)$.

Solution. We have the function

$$(g, H) \in G \times \mathscr{L}(G) \to g \cdot H = gH \in \mathscr{L}(G)$$

and it is easy to check that the axioms (i) and (ii) above are satisfied.

Let's now study the orbits and the stabilizers in the actions.

Definition 5.13. Let G be a group that acts on the set X (from the left). For $x \in X$, we define the orbit of x under the action of G by

$$\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\}$$

Of course, $\mathrm{Orb}_G(x)$ is a subset of X and we note that the orbits induce a partition of X, because of the relation

$$y \approx x \iff y \in \mathrm{Orb}_G(x).$$

This means that two elements $x, y \in X$ are related by $\approx$ if and only if $y = g \cdot x$ for some $g \in G$. More precisely,

- **Reflexive: :** $x = 1 \cdot x$ shows that $x \approx x$ for all $x \in X$.
- **Symmetric::** If $y \approx x$, then there is $g \in G$ such that $g \cdot x = y$. If we multiply by g^{-1} from the left, then $x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot y$, and so $x \approx y$.
- **Transitive::** If $y \approx x$ and $x \approx z$, then there are $g_1, g_2 \in G$ such that $g_1 \cdot x = y$ and $g_2 \cdot z = x$. By substituting, we have $y = g_1 \cdot (g_2 \cdot z) = (g_1 g_2) \cdot z$ and so $y \sim z$. Thus $\approx$ is transitive.

In particular, if $X = G$ is a finite group with the discrete topology, then the map

$$(g, x) \in G \times G \mapsto g \cdot x = g^{-1} x g \in G$$

defines the *conjugation* between elements of G and the map

$$(g, H) \in G \times \mathscr{L}(G) \mapsto g \cdot H = g^{-1} H g \in \mathscr{L}(G)$$

defines the conjugation between subgroups of G.

In the first case,

$$\mathrm{Orb}_G(x) = \{g x g^{-1} \mid g \in G\}$$

and this allows us to write an important equation

$$|G| = |Z(G)| + \sum_{x \in G - Z(G)} |\mathrm{Orb}_G(x)|,$$

known as *class equation* of G, where $Z(G)$ denotes the *center* of G, that is, the set of elements permuting with all the elements of G. The center is in fact the smallest subgroup which is defined via the notion of stabiliser. In fact we may give the following general notion.

Definition 5.14. Given a left action of a group G on a nonempty set X, the *stabilizer* of x in G is given by

$$G_x = \mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$$

The stabiliser of x in G is the set of all elements of G fixing x under the given action. In particular, if the action is by conjugation, we introduce a special notation for $\mathrm{Stab}_G(x)$:

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\}$$

and this set turns out to be a subgroup of G and is called *centralizer* of x in G. Of course, $C_G(x)$ may be described even as the subgroup of all elements of G commuting with x. For instance, if $A = \langle x \rangle = \{1, x, x^2, x^3, \ldots\}$ is the subgroup generated by all the powers of x, then $C_G(x) = C_G(\langle x \rangle)$. More generally, if A is an arbitrary subgroup of G,

$$C_G(A) = \bigcap_{a \in A} C_G(a),$$

and, in particular, $C_G(G) = Z(G)$. The following exercise shows the aforementioned class equation in G.

Exercise 5.15 (See [8], Exercise 5.9(d))**.** Let X be a G-set and G an aribtrary group. For each $x \in X$ the stabilizer G_x acts on G and so the quotient G/G_x is defined. Moreover G/G_x is the set of left cosets of G_x in G and there is a G-equivariant bijection between the orbit of x and G/G_x.

Solution. It is enough to note that $(g, x) \in G_x \times G \mapsto g \cdot x = g^{-1}xg \in G$ defines an action (via conjugation) of G_x on G. Here $\mathrm{Orb}_{G_x}(x)$ is given by the conjugacy classes of x in G and G/G_x is the union of all the orbits so $G/G_x = \{aG_x \mid a \in G\}$ which induced the well defined continuous G-invariant bijection $g^{-1}ag \in \mathrm{Orb}_{G_x}(a) \mapsto aG_x \in G/G_x$. Note that here G_x is not necessarily a normal subgroup of G.

We have all that we need in order to describe actions of groups when group quotients are involved.

Exercise 5.16 (See [8], Exercise 5.13(c))**.** Suppose X is a G-space and H is a normal subgroup of G. Then X/H is a (G/H)-space and $(X/H)/(G/H)$ is homeomorphic to X/G.

Solution. Note that $G/H = \{gH \mid g \in H\}$ is a group and X/H is by definition the disjoint union of all $\mathrm{Orb}_H(x)$ for $x \in X$. The continuous function $(g, x) \in G \times X \mapsto g \cdot x \in X$ allows us to consider the continuous function $(gH, \mathrm{Orb}_H(x)) \in G/H \times X/H \mapsto gH \cdot \mathrm{Orb}_H(x) \in X/H$. Note also that an element of $(X/H)/(G/H)$ has the form $\mathrm{Orb}_{G/H}(\mathrm{Orb}_H(x))$. Then we may define the map

$$f : \mathrm{Orb}_{G/H}(\mathrm{Orb}_H(x)) \in \frac{X/H}{G/H} \mapsto \mathrm{Orb}_G(x) \in \frac{X}{G}$$

which turns out to be continuous, bijective and open, so it is the required homeomorphism.

An important application of what we have seen until here is the following.

Exercise 5.17 (See [8], Exercise 5.9(a))**.** Let X be the infinite strip

$$\left\{(x, y) \in \mathbb{R}^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\right\}$$

in $\mathbb{R}^2$ with $\mathbb{Z}$ acting on it by $m \cdot (x, y) = (m + x, (-1)^m y)$. Then the quotient space $X/\mathbb{Z}$ is homeomorphic to the Moebius strip.

Solution. Let $M/_\sim$ be the equivalence classes of the unit square with the relation $\sim$, such that: $(x,y)_\sim(x',y')$ if and only if $(x,y)=(x',y')$ or $\{x,x'\}=\{0,1\}$ and $y=1-y'$. Then $M/_\sim$ is the Moebius strip. Consider the following two functions:

$$f:[(x,y)]_{X/\mathbb{Z}}\in X/\mathbb{Z}\longmapsto\left[\left(x-\lfloor x\rfloor,(-1)^{\lfloor x\rfloor}y+\frac{1}{2}\right)\right]_\sim\in M/_\sim$$

$$g:[(x,y)]_\sim\in M/_\sim\longmapsto\left[\left(x,y-\frac{1}{2}\right)\right]_{X/\mathbb{Z}}\in X/\mathbb{Z}$$

We note that f is well defined: if $(x,y)\in X$ and $n\in\mathbb{Z}$, then $[(x,y)]_{X/\mathbb{Z}}=[(n+x,(-1)^ny)]_{X/\mathbb{Z}}$ and we get

$$f([(n+x,(-1)^ny)]_{X/\mathbb{Z}})=[(n+x-(\lfloor n+x\rfloor),(-1)^{\lfloor n+x\rfloor}(-1)^ny+\frac{1}{2})]_\sim$$
$$=[(x-\lfloor x\rfloor,(-1)^{2n+\lfloor x\rfloor}y+\frac{1}{2})]_\sim=[(x-\lfloor x\rfloor,(-1)^2n(-1)^{\lfloor x\rfloor}y+\frac{1}{2})]_\sim$$
$$=[(x-\lfloor x\rfloor,(-1)^{\lfloor x\rfloor}y+\frac{1}{2})]_\sim$$

Also g is well defined: since the equivalence classes that do not contain points from the left and right edges only contain one point, we will consider only the points on the left and right edges. If $[(0,y)]_\sim=[(1,1-y)]_\sim$, then $g([(0,y)]_\sim)=[(0,y-\frac{1}{2})]_{X/\mathbb{Z}}$ and $g([(1,1-y)]_\sim)=[(1,1-y-\frac{1}{2})]_{X/\mathbb{Z}}=[(1,\frac{1}{2}-y)]_{X/\mathbb{Z}}$.

Since $(1,\frac{1}{2}-y)=(1+0,(-1)^1(y-\frac{1}{2}))=1\cdot(0,y-\frac{1}{2})$, we have $[(1,\frac{1}{2}-y)]_{X/\mathbb{Z}}=[(0,y-\frac{1}{2})]_{X/\mathbb{Z}}$. Now an argument of routine allows us to conclude that f and g are one the inverses of the other and so they are bijective. Moreover they are continuous since they involve elementary functions in their definition.

Due to the importance of the spaces that we get as result of a group acting on it, we may ask when two group actions determine homeomorphic quotient spaces. There is functional property that can help.

Exercise 5.18 (See [8], Exercise 5.9(b))**.** Let X and Y be G-sets. We say that the function $f:X\to Y$ is G-*equivariant* if $f(g\cdot x)=g\cdot f(x)$ for all $x\in X$ and all $g\in G$. If X and Y are topological spaces and f is a G-equivariant homeomorphism (i.e. both G-equivariant and a homeomorphism), then X/G and Y/G are homeomorphic.

Solution. Let $x,x'\in X$ such that $x\ _Xx$ then $\exists\ g\in G$ such that $gx=x'$ so $f(gx)=f(x')$ and $gf(x)=f(x')$. This implies that $f(x)\ f(x')$ so $X/G\cong Y/G$.

It is interesting to note that groups that form G-spaces can be seen as appropriate images of the group of homeomorphisms of the original space.

Exercise 5.19 (See [8], Exercise 5.11)**.** Suppose that X is a G-space. The function $\theta_g:x\in X\mapsto g\cdot x\in X$ is a homeomorphism from X to itself for all $g\in G$ and there is a homomorphism from G to the group of homeomorphisms $G(X)$ of X.

Solution. The first part is clear from the definitions. Now define $\pi:g\in G\mapsto\pi(g)=\theta_g\in G(X)$. This function is a group homomorphism because $\pi(g\cdot h)=\theta_{g\circ h}=\theta_g\circ\theta_h$ for all $g,h\in G$. Then $(G/\ker\pi)\simeq\pi(G)$.

Now we pass to describe an important aspect in algebraic topology: actions of groups on topological spaces induce what it is called a *covering map* and covering maps allow us to describe the *fundamental group* of a given space. We will see the formalization of these notions in the following chapters, but for the moment we describe an important property.

Exercise 5.20 (See [8], Exercise 5.13(b))**.** Let X be a G-space with G finite group. Then the natural projection $\pi : x \in X \mapsto \pi(x) = \mathrm{Orb}_G(x) \in X/G$ is both a closed and open mapping.

Solution. Let U be a closed set in X. Consider $\pi^{-1}(\pi(U))$. Then

$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{x \in X \ : \ \pi(x) \in \pi(U)\}\\ &= \{x \in X \ : \ x = g \cdot y \text{ for some } y \in U, g \in G\}\\ &= \{x \in X \ : \ x \in g \cdot U, \ g \in G\} = \bigcup_{g \in G} g \cdot U.\end{aligned}$$

The action of each $g \in G$ is an homeomorphism, so if U is closed then so is $\pi^{-1}(\pi(U))$ (since G is finite) and hence $\pi(U)$ is closed in X/G. Now one can apply exactly the same steps, concluding that p is also open.

To generalise the above result, we can say that

Proposition 5.21. If X is a G-space, where G is an infinite group. Then the canonical projection $\pi : X \to X/G$ is always an open mapping, but is not necessarily a closed mapping.

Proof. In the solution of the exercise above, if U is open, then so is $\pi^{-1}(\pi(U))$, regardless the cardinality of G, since the union of the $g \cdot U$ is a union of opens anyway. In general, if U is closed, then this union is no longer closed if it is not made by finitely many $g \cdot U$, so here the cardinality of G must be finite. □

6. Product Spaces

After we have seen the behaviour of topological subspaces and topological quotients, we discuss another important operation in general topology: the formation of products.

Definition 6.1 (See [8], Definition 6.1)**.** Let X and Y be topological spaces. The (*topological*) *product* $X \times Y$ is the set $X \times Y$ with topology $\mathscr{U}_{X\times Y}$ consisting of the family of sets that are unions of products of open sets of $X \times Y$.

We show that being homeomorphic is a property that is preserved by product spaces.

Exercise 6.2 (See [8], Exercise 6.2(a))**.** If $X_1 \cong X_2$ and $Y_1 \cong Y_2$, then $X_1 \times Y_1 \cong X_2 \times Y_2$.

Solution.
Since $X_1 \cong X_2$, there are f_X and g_X that are continuous and inverses of one another and similarly $Y_1 \cong Y_2$ shows that there are f_Y and g_Y that are continuous and inverses of one another. Define $f : X_1 \times Y_1 \to X_2 \times Y_2$ by $(x_1, y_1) \mapsto f(x_1, y_1) = (f_X(x_1), f_Y(y_1))$ and $g : X_2 \times Y_2 \to X_1 \times Y_1$ by $(x_2, y_2) \mapsto g(x_2, y_2) = (g_X(x_2), g_Y(y_2))$. If $(x_1, y_1) \in X_1 \times Y_1$, then $(g \circ f)(x_1, y_1) = g(f(x_1, y_1)) = g((f_X(x_1), f_Y(y_1))) = (g_X(f_X(x_1)), g_Y(f_Y(y_1))) = (x_1, y_1)$. This shows that $f \circ g = 1_{X_1\times Y_1}$. Similarly one can prove $g \circ f = 1_{X_2\times Y_2}$. If $\bigcup_{j\in I} U_j \times V_j \in \mathscr{U}_{X_2\times Y_2}$, then$f^{-1}\left(\bigcup_{j\in I}(U_j \times V_j)\right) = \bigcup_{j\in J} f^{-1}(U_j \times V_j) = \bigcup_{j\in I}(f_X^{-1}(U_j) \times f_Y^{-1}(V_j)) \in \mathscr{U}_{X_1\times Y_1}$. The continuity of g is proved analogously.

If a product topology was made from two metrizable spaces, we can form a metric on the product space from the metrics of the individual spaces.

Exercise 6.3 (See [8], Exercise 6.2(b))**.** Let X, Y be metrizable spaces and suppose that they arise from metrics d_X, d_Y respectively. Then

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

is a metric on $X \times Y$ which produces the product space topology on $X \times Y$.

Solution. We have already seen in the proof of Exercise 2.2 that the maximum between finitely many metrics is a metric as well. So we don't need to check again that the axioms of metric are satisfied for d. On the other hand, we need to check that d actually induces the product topology on $X \times Y$. Let $B_\varepsilon((x_1, y_1))$ be an open ball of $X \times Y$ of center (x_1, y_1) and radius ε. Then

$$B_\varepsilon((x_1, y_1)) = \{(x_2, y_2) \in X \in Y \ : \ d((x_1, x_2), (x_2, y_2)) < \varepsilon\}$$
$$= \{(x_2, y_2) \in X \times Y \ : \ \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} < \varepsilon\}$$
$$= \{(x_2, y_2) \in X \times Y : d_X(x_1, x_2) < \varepsilon \text{ and } d_Y(y_1, y_2) < \varepsilon\} = B_\varepsilon(x_1) \times B_\varepsilon(y_1).$$

In particular, we may deduce that the product topology on $\mathbb{R}^n \times \mathbb{R}^m$ ($\mathbb{R}^n, \mathbb{R}^m$ with the usual topology), is the same as the usual topology on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. Now we can recognize in the product space $\mathbb{R}^2$ a particular subspace,

$$\mathrm{diag}(\mathbb{R}^2) = \{(x, x) \mid x \in \mathbb{R}\},$$

called *diagonal subspace*, and, of course, one can introduce the notion of diagonal subspace $\operatorname{diag}(X \times X)$ in the same way for an arbitrary topological space X. This subspace is homeomorphic via the map

$$x \in X \longmapsto (x, x) \in \operatorname{diag}(X \times X) = \{(y, y) \mid y \in X\}$$

to X, and is different (as set) from the factors of the product $X \times X$. This shows that (in general) the structure of a product of two copies of a given topological space is richer than one may expect after a first look. Another interesting subspace that we may find in a product space of two copies of a prescribed space is the graph of a function. The graph of a function is indeed a generalisation of the geometric structures we are familiar with from $\mathbb{R}^2$ and $\mathbb{R}^3$. The *graph of a function* $f : X \to Y$ is defined by

$$\operatorname{graph}(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$$

and of course $\operatorname{diag}(X \times X)$ is a special case of $\operatorname{graph}(f)$ when $Y = X$ and $f = 1_X$. We find another copy of X in $X \times Y$ when f is continuous.

Exercise 6.4 (See [8], Exercise 6.2(c))**.** If f is a continuous function between topological spaces, then $\operatorname{graph}(f)$ is homeomorphic to X.

Solution. Consider the function $g : x \in X \mapsto (x, f(x)) \in \operatorname{graph}(f) \subseteq X \times Y$. If $x, y \in X$ and $x \neq y$, then $(x, f(x)) \neq (y, f(y))$, so g is injective, and trivially it is also surjective. Let $\bigcup_{j \in I} U_j \times V_j$ be open in $\operatorname{graph}(f)$. Then

$$g^{-1}\left(\bigcup_{j \in I} U_j \times V_j\right) = \bigcup_{j \in I} g^{-1}(U_j \times V_j) = \bigcup_{j \in I} U_j$$

is open in X, so g is continuous. The inverse of g is the projection on the first component $p : (x, f(x)) \in \operatorname{graph}(f) \subseteq X \times Y \mapsto p(x, f(x)) = x \in X$. Similarly one can check that p is injective and surjective. An open U in Y is such that $f^{-1}(U)$ is open in X, then

$$p^{-1}(f^{-1}(U)) = f^{-1}(U) \times f(f^{-1}(U)) = f^{-1}(U) \times U$$

is open in $\operatorname{graph}(f)$. We may conclude that the two spaces $\operatorname{graph}(f)$ and X are homeomorphic.

We are able to provide a useful homeomorphism between the punctured plane and a product space consisting of two familiar spaces.

Exercise 6.5 (See [8], Exercise 6.2(d))**.** $\mathbb{R}^2 - \{0\}$ is homeomorphic to $\mathbb{R} \times S^1$.

Solution. Thinking at $\mathbb{R}^2$ as $\mathbb{C}$ and choosing $\theta \in [0, 2\pi]$, the function

$$f : (x, \cos\theta, \sin\theta) \in \mathbb{R} \times S^1 \longmapsto f(x, \cos\theta, \sin\theta) = 2^x e^{\theta i} \in \mathbb{C} - \{0\}$$

is injective because if $2^x e^{\theta i} = 2^y e^{\phi i}$ then $2^x = 2^y \Rightarrow x = y$, and $e^{\theta i} = e^{\phi i} \Rightarrow (\cos\theta, \sin\theta) = (\cos\phi, \sin\phi)$, so $(x, \cos\theta, \sin\theta) = (y, \cos\phi, \sin\phi)$. It is surjective because $a + bi = \|a + bi\| e^{\psi i} = f(\log_2(\|a + bi\|), \cos\psi, \sin\psi)$ for all $a + bi \in \mathbb{C} - \{0\}$. Now we define the inverse of f like

$$g : z = \|z\| e^{\theta i} \in \mathbb{C} - \{0\} \longmapsto g(z) = \left(\|z\| - \frac{1}{\|z\|}, \cos\theta, \sin\theta\right) \in \mathbb{R} \times S^1.$$

Again g is injective because if $(\|z\|-\frac{1}{\|z\|},\cos\theta,\sin\theta)=(\|v\|-\frac{1}{\|v\|},\cos\phi,\sin\phi)$, then $\theta=\phi+2\pi k, k\in\mathbb{Z}$ and

$$\|z\|-\frac{1}{\|z\|}=\|v\|-\frac{1}{\|v\|}\Rightarrow\|v\|\|z\|^2-\|v\|=\|z\|\|v\|^2-\|z\|$$

$\Rightarrow\|v\|\|z\|^2+\|z\|=\|z\|\|v\|^2+\|v\|\Rightarrow\|z\|(\|v\|\|z\|+1)=\|v\|(\|z\|\|v\|+1)\Rightarrow\|z\|=\|v\|$. So $\|z\|e^{\theta i}=\|v\|e^{\psi i}$. Finally f and g are continuous since they are constructed with continuous functions.

We have already seen the notion of weak topology, characterizing the continuity of the embeddings, and that of strongest topology, characterizing the continuity of the projections. Of course, having projections from $X\times Y$ on the factors, and, viceversa, embeddings of the factors in $X\times Y$, there is a result of characterization of the product topologies in terms of projections and embeddings of the factors. The following exercise gives us the relationship between the product topology and the projection maps.

Exercise 6.6 (See [8], Exercise 6.6(a))**.** The product topology on $X\times Y$ is the weakest topology such that the projections π_X and π_Y are continuous.

Solution. Let $\mathscr{U}_1$ be the product topology and $\mathscr{U}_2$ a topology such that π_X and π_Y are continuous. Let U_j be open in X for $j\in J$, then $\pi_X^{-1}(U_j)=U_j\times Y\in\mathscr{U}_2$; let V_j be open in Y for $j\in J$, then $\pi_Y^{-1}(V_j)=X\times U_j\in\mathscr{U}_2$. So

$$(U_j\times Y)\cap(X\times V_j)=U_j\times V_j\in\mathscr{U}_2$$

for all $j\in J$. So $\bigcup\limits_{j\in J}U_j\times V_j\in\mathscr{U}_2$. This means that $\mathscr{U}_1\subseteq\mathscr{U}_2$.

Here we prove a useful result in which actions of products groups are involved in the products of topological spaces.

Exercise 6.7 (See [8], Exercise 6.6(b))**.** Let X be a G-space and let Y be an H-space. Then the space $(X\times Y)/(G\times H)$ is homeomorphic to $(X/G)\times(Y/H)$.

Solution. The idea is to prove that there is a homeomorphism

$$\mathrm{Orb}_{G\times H}((x,y))\in\frac{X\times Y}{G\times H}\mapsto\mathrm{Orb}_G(x)\times\mathrm{Orb}_H(y)\in\frac{X}{G}\times\frac{Y}{H}.$$

In order to justify this, we consider the function

$$f:[(x,y)]\in\frac{X\times Y}{G\times H}\mapsto f([(x,y)])=([x],[y])\in\frac{X}{G}\times\frac{Y}{H}$$

which is well defined since for $[(g_1,h_1)\cdot(x,y)]=[(g_2,h_2)\cdot(x,y)]$, we have

$$\begin{aligned}f([(g_1,h_1)\cdot(x,y)])&=f([(g_1\cdot x,h_1\cdot y)])=f([(g_1x,h_1y)])\\&=([g_1x],[h_1y])=([g_1\cdot x],[h_1\cdot y])=([x],[y])\end{aligned}$$

and

$$\begin{aligned}f([(g_2,h_2)\cdot(x,y)])&=f([(g_2\cdot x,h_2\cdot y)])=f([(g_2x,h_2y)])\\&=([g_2x],[h_2y])=([g_2\cdot x],[h_2\cdot y])=([x],[y])\end{aligned}$$

So $f([(g_1,h_1)\cdot(x,y)])=f([(g_2,h_2)\cdot(x,y)])$.

Similarly, we define

$$g:([x],[y])\in\frac{X}{G}\times\frac{Y}{H}\mapsto g([x],[y])=[(x,y)]\in\frac{X\times Y}{G\times H}$$

and check with the same strategy that it is well defined. Now to show that these functions are bijective inverses of one another we take $[(x,y)] \in (X \times Y)/(G \times H)$ and note that$(g \circ f)([(x,y)]) = g(f([(x,y)])) = g([x],[y]) = [(x,y)]$ and by analogy that $(f \circ g)([x],[y]) = ([x],[y])$. The assumptions of having X a G-space and Y an H-space show that $X \times Y$ is in fact a $(G \times H)$-space so that both f and g are continuous and we have constructed the required homeomorphism.

An application of the above fact allows us to find the torus as quotient space of $\mathbb{R}^2$ under the action of $\mathbb{Z} \times \mathbb{Z}$.

Exercise 6.8 (See [8], Exercise 6.6(c))**.** For $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ and $(x,y) \in \mathbb{R}^2$, the map $(n,m) \cdot (x,y) = (n+x, m+y)$ defines an action that makes $\mathbb{R}^2$ into a $(\mathbb{Z} \times \mathbb{Z})$-space. Moreover $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$ is homeomorphic to $S^1 \times S^1$.

Solution. The fact that the given map is an action is easy to see checking the axioms of action. On the other hand, we have seen previously that $\mathbb{R}/\mathbb{Z} \simeq S^1$, so the result is an application of Exercise 6.7.

We provide an alternative description for the torus.

Exercise 6.9 (See [8], Exercise 6.6(d))**.** The torus, represented by the unit square $M = [0,1] \times [0,1]$ with the relation $\sim$ given by $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ is homeomorphic to $S^1 \times S^1$.

Solution. Consider the functions

$$f : [(x,y)]_\sim \in M/_\sim \mapsto f([(x,y)]_\sim) = [(x,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} \in \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$$

and

$$g : [(x,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} \in \mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z}) \mapsto g([(x,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}) = [(x-\lfloor x \rfloor, y-\lfloor y \rfloor)]_\sim \in M/_\sim.$$

One can see that f is well defined since for points not on the boundaries of M, the equivalence classes only contain one element and since $[(0,y)]_\sim = [(1,y)]_\sim$, we have $f([(0,y)]_\sim) = [(0,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$ and $f([(1,y)]_\sim) = [(1,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} = [(1,0) \cdot (0,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} = [(0,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$, or since $[(x,0)]_\sim = [(x,1)]_\sim$, we have $f([(x,0)]_\sim) = [(x,0)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$ and $f([(x,1)]_\sim) = [(0,1) \cdot (x,0)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} = [(x,0)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$.

Now we pass to see that g is well defined. Let $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. We have $[(x,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})} = [(n+x, m+y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$ which gives $g([(x,y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}) = [(x - \lfloor x \rfloor, y - \lfloor y \rfloor)]_\sim$ and $g([(n+x, m+y)]_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}) = [(n+x-\lfloor n+x \rfloor, m+y-\lfloor m+y \rfloor)]_\sim = [(n+x-n-\lfloor x \rfloor, m+y-m-\lfloor y \rfloor)]_\sim = [(x-\lfloor x \rfloor, y-\lfloor y \rfloor)]_\sim$ so g is also well defined. An argument that we have seen several times until now is to check that $f \circ g = 1_{\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})}$ and that $g \circ f = 1_{M/\sim}$ in order to conclude that f and g are bijective and one the inverse of the other. This, plus the fact that they are defined involving continuous functions, allows us to conclude that we have homeomorphisms.

It is possible to provide still another construction for the torus.

Exercise 6.10 (See [8], Exercise 6.6(e))**.** For $n \in \mathbb{C} - \{\mathbf{0}\}$ the map

$$(n,z) \in \mathbb{Z} \times (\mathbb{C} - \{\mathbf{0}\}) \mapsto n \cdot z = 2^n z \in \mathbb{C} - \{\mathbf{0}\}$$

defines an action that makes $\mathbb{C} - \{\mathbf{0}\}$ into $\mathbb{Z}$-space. Moreover $(\mathbb{C} - \{\mathbf{0}\})/\mathbb{Z}$ is homeomorphic to $S^1 \times S^1$.

Solution. It is elementary to check that the above map is indeed an action of $\mathbb{Z}$ on $\mathbb{C}-\{\mathbf{0}\}$. Looking at $\mathbb{C}$ as $\mathbb{R}^2$, we have seen in Exercise 6.5 that $\mathbb{C}-\{\mathbf{0}\} \cong \mathbb{R}\times S^1$. On the other hand, $\mathbb{Z} \cong \mathbb{Z} \times \{\mathbf{1}\}$ as groups, so Exercise 6.7 implies that

$$\mathbb{C} - \{\mathbf{0}\}/\mathbb{Z} \cong (\mathbb{R} \times S^1)/(\mathbb{Z} \times \{\mathbf{1}\}) \cong (\mathbb{R}/\mathbb{Z}) \times (S^1/\{\mathbf{1}\}) \cong S^1 \times S^1.$$

We've seen that $\mathbb{R}^2 - \{\mathbf{0}\}$ is homeomorphic to $S^1 \times \mathbb{R}$ in Exercise 6.5, and so we may generalise this result.

Exercise 6.11 (See [8], Exercise 6.6(g))**.** For all $n \geq 2$, $\mathbb{R}^n - \{\mathbf{0}\}$ and $S^{n-1} \times \mathbb{R}$ are homeomorphic spaces.

Solution. We consider the functions

$$f : (x,t) \in S^{n-1} \times \mathbb{R} \longmapsto f(x,t) = 2^t x \in \mathbb{R}^n - \{\mathbf{0}\}$$

and

$$g : x \in \mathbb{R}^n - \{\mathbf{0}\} \longmapsto g(x) = \left(\frac{x}{\|x\|}, \log_2(\|x\|)\right) \in S^{n-1} \times \mathbb{R}$$

which are one the inverse of the other and both continuous, because they involve continuous functions. The logic of proof is always the same, noting that

$$(g \circ f)(x,t) = g(f(x,t)) = g(2^t x) = \left(\frac{2^t x}{\|2^t x\|}, \log_2(\|2^t x\|)\right) = \left(\frac{2^t x}{2^t\|x\|}, \log_2(2^t\|x\|)\right)$$
$$= \left(\frac{x}{1}, \log_2(2^t 1)\right) = (x, \log_2(2^t)) = (x,t);$$
$$(f \circ g)(x) = f\left(\frac{x}{\|x\|}, \log_2(\|x\|)\right) = 2^{\log_2(\|x\|)}\frac{x}{\|x\|} = x.$$

In a more general way, it is possible to consider the following situation.

Exercise 6.12 (See [8], Exercise 6.6(h))**.** The subset $S_{p,q} \subseteq \mathbb{R}^n$ defined by

$$S_{p,q} = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 = 1\},$$

where p and q are positive integers such that $p + q = n$, is homeomorphic to $S^{p-1} \times \mathbb{R}^q$.

Solution. Consider the function

$$f : (x_1, ..., x_p, y_1, ..., y_q) \in S^{p-1} \times \mathbb{R}^q \longmapsto (x_1 z, x_2 z, ..., x_p z, y_1, y_2, ..., y_q) \in S_{p,q},$$

where $z = \sqrt{(1 + y_1^2 + y_2^2 + \cdots + y_p^2)}$. Here $(x_1 z, x_2 z, ..., x_p z, y_1, y_2, ..., y_q)$ is an element of $S_{p,q}$ and

$$(x_1 z)^2 + (x_2 z)^2 + \cdots (x_p z)^2 - y_1^2 - y_2^2 - \cdots - y_q^2 = x_1^2 z^2 + x_2^2 z^2 + \cdots x_p^2 z^2 - y_1^2 - y_2^2 - \cdots - y_q^2$$
$$= z^2(x_1^2 + x_2^2 + \cdots + x_p^2) - (y_1^2 + y_2^2 + \cdots + y_q^2) = z^2(1) - (y_1^2 + y_2^2 + \cdots + y_q^2)$$
$$= (1 + y_1^2 + y_2^2 + \cdots + y_q^2) - (y_1^2 + y_2^2 + \cdots + y_q^2) = 1.$$

Now consider the function

$$g : (x_1, ..., x_p, y_1, ..., y_q) \in S_{p,q} \longmapsto \left(\frac{x_1}{z}, \frac{x_2}{z}, ..., \frac{x_p}{z}, y_1, y_2, ..., y_q\right) \in S^{p-1} \times \mathbb{R}^q$$

and note that

$$x_1^2 + x_2^2 + \cdots + x_p^2 - y_1^2 - y_2^2 - \cdots - y_q^2 = 1 \Rightarrow x_1^2 + x_2^2 + \cdots x_p^2 = 1 + y_1^2 + y_2^2 + \cdots y_q^2.$$

Then

$$\left(\frac{x_1}{z}, \frac{x_2}{z}, ..., \frac{x_p}{z}, y_1, y_2, ..., y_q\right) \in S^{p-1} \times \mathbb{R}^q$$

$$\Rightarrow \left(\frac{x_1}{z}\right)^2 + \left(\frac{x_2}{z}\right)^2 + \cdots + \left(\frac{x_p}{z}\right)^2 = \frac{x_1^2}{z^2} + \frac{x_2^2}{z^2} + \cdots + \frac{x_p^2}{z^2}$$

$$= \frac{x_1^2 + x_2^2 + \cdots + x_p^2}{z^2} = \frac{x_1^2 + x_2^2 + \cdots + x_p^2}{1 + y_1^2 + y_2^2 + \cdots + y_q^2} = \frac{x_1^2 + x_2^2 + \cdots + x_p^2}{x_1^2 + x_2^2 + \cdots + x_p^2} = 1.$$

$$\Rightarrow \left(\frac{x_1}{z}, \frac{x_2}{z}, ..., \frac{x_p}{z}\right) \in S^{p-1}.$$

Of course, g, f are continuous since they are compositions of continuous functions. Now all that is left to show is that they are bijective inverses of one another. With obvious meaning of symbols, we get

$$(g \circ f)(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q) = g(f(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q))$$

$$= g(x_1 z, x_2 z, ..., x_p z, y_1, y_2, ..., y_q) = \left(\frac{x_1 z}{z}, \frac{x_2 z}{z}, ..., \frac{x_p z}{z}, y_1, y_2, ..., y_q\right)$$

$$= (x_1, x_2, ..., x_p, y_1, y_2, ..., y_q)$$

and

$$(f \circ g)(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q) = f(g(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q))$$

$$= f\left(\frac{x_1}{z}, \frac{x_2}{z}, ..., \frac{x_p}{z}, y_1, y_2, ..., y_q\right) = \left(\left(\frac{x_1}{z}\right) z, \left(\frac{x_2}{z}\right) z, ..., \left(\frac{x_p}{z}\right) z, y_1, y_2, ..., y_q\right)$$

$$= (x_1, x_2, ..., x_p, y_1, y_2, ..., y_q).$$

Our last construction for the torus is realized in terms of group homeomorphisms, acting on $\mathbb{R}^n - \{\mathbf{0}\}$ when $n \geq 2$. In fact, this is not evident in the arguments we have seen until now.

Exercise 6.13 (See [8], Exercise 6.6(i))**.** Let G be the group of homeomorphisms $\{T^i \ : \ i \in \mathbb{Z}\}$, where

$$T : x \in \mathbb{R}^n - \{\mathbf{0}\} \mapsto T(x) = 2x \in \mathbb{R}^n - \{\mathbf{0}\}.$$

Then $(\mathbb{R}^n - \{\mathbf{0}\})/G$ is homeomorphic to $S^{n-1} \times S^1$ for all $n \geq 2$.

Solution. Since G is isomorphic (as group) to $\mathbb{Z}$, Exercise 6.7 implies

$$\mathbb{R}^n - \{\mathbf{0}\}/G \cong S^{n-1} \times \mathbb{R}/\{\mathbf{1}\} \times \mathbb{Z} \cong (S^{n-1}/\{\mathbf{1}\}) \times (\mathbb{R}/\mathbb{Z}) \cong S^{n-1} \times S^1.$$

The next question shows the existence of a homeomorphism between the n-cube and n-disc.

Exercise 6.14 (See [8], Exercise 6.2(j))**.** The sets $I^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, i = 1, 2, ..., n\}$ and $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ are homeomorphic.

Solution. Note that $[0, 1] \cong [-1, 1]$, intuitively by stretching the 0 end to -1 or formally by the homeomorphism $f : x \in [0, 1] \longmapsto f(x) = 2x - 1 \in [-1, 1]$. On the other hand, what we have seen until now on the product topology allows us to conclude that $I^n \cong ([-1, 1])^n$. Therefore we may consider the functions

$$\varphi \in x \in ([-1, 1])^n \mapsto \varphi(x) = \frac{\max\{|x_1|, |x_2|, ..., |x_n|\}}{\|x\|}(x_1, x_2, ..., x_n) \in D^n$$

and

$$\psi : y \in D^n \mapsto \psi(y) = \frac{\|y\|}{\max\{|y_1|, |y_2|, ..., |y_n|\}}(y_1, y_2, ..., y_n) \in ([-1, 1])^n$$

and apply the logic that we have used until now. Firstly we check that one is the inverse of the other, so they are bijective, then we note that their construction involves continuous functions, so they allow us to conclude that the given spaces are homeomorphic.

Intuitively we can get $\mathbb{R}^n$ by stretching out $\mathring{D}^n$, but this exercise makes that intuition more rigorous.

Exercise 6.15 (See [8], Exercise 6.6(k))**.** We have $\mathring{D}^n \cong \mathbb{R}^n$ for all $n \geq 1$.

Solution. We already know that $D^n \cong I^n$ from the previous exercise, and this induces a homeomorphism on the boundaries and on the interiors, so that $\mathring{D}^n \cong \mathring{I}^n$. Since $\mathring{I}^n \cong (\mathring{I})^n$ and $\mathring{I} \cong \mathbb{R}$, we may conclude that $(\mathring{I})^n \cong \mathbb{R}^n$ because product of the same number of homeomorphic spaces, so, replacing $(\mathring{I})^n$ by the homeomorphic space $\mathring{D}^n$ we get the required homeomorphism $\mathring{D}^n \cong \mathbb{R}^n$.

7. Compact spaces

We begin to discuss the presence of special cover in a topological space from which it is possible to extract at least one of prescribed nature.

Definition 7.1 (See [8], Definitions 7.1, 7.2, 7.3, 7.4)**.** A *cover* (or *covering*) of a subset S of a set X is a collection of subsets $\{U_j : j \in J\}$ of X such that $S \subseteq \bigcup_{j\in J} U_j$. If in addition the indexing set J is finite then $\{U_j : j \in J\}$ is said to be a *finite cover.*

Suppose that $\{U_j : j \in J\}$ and $\{V_k : k \in K\}$ are covers of the subset S of X. If for all $j \in J$ there is a $k \in K$ such that $U_j = V_k$ then we say that $\{U_j : j \in J\}$ is a *subcover* of the cover $\{V_k : k \in K\}$,

We say that the cover $\{U_j : j \in J\}$ is an *open cover* of S if each U_j, is an open subset of X. S is said to be *compact* if every open cover of S has a finite subcover. S is said to be *Lindelöff* if every open cover of S has a countable subcover.

Of course, a compact space is Lindelöff, but $\mathbb{R}$ is an example of a non-compact space which is Lindelöff. Therefore the notion of being Lindelöff generalizes the notion of being compact.

There are several compact topologies which we have already encountered.

Exercise 7.2 (See [8], Exercise 7.5(a))**.** If X has the cofinite topology, then X is compact. Actually each subset of X is compact.

Solution. Recall that $\mathscr{U} = \{A \subseteq X \ : \ X - A \text{ is finite}\}$. If $\{U_i \ : \ i \in I\}$ is an open cover of X, then for some $k \in I$ the set $X - U_k$ is finite. Since $\bigcup_{i\in I} U_i = X$, $X - U_k \subseteq \bigcup_{i\in I, i\neq k} U_i$ and denoted by U_x an open containing $x \in X - U_k$

$$X = \left(\bigcup_{x\in X-U_k} U_x \right) \cup U_k$$

is a finite subcover, so X is compact. We can apply the same logic to each subset of S with the induced topology concluding that it is compact as well.

There is no change of argument in the above proof if we replace the cofinite topology on X with the co-countable topology on X. In fact these spaces are also compact along with their subspaces. We will see that all the compact subsets of $\mathbb{R}$ are characterized to be closed and bounded, and we will see that $[0, 1]$ is compact as subset of $\mathbb{R}$ with the usual metric, but if we change the topology on $\mathbb{R}$ then $[0, 1]$ is not necessarily compact. This happens in the line of Sorgenfrey.

Exercise 7.3 (See [8], Exercise 7.5(c))**.** The subset $[0, 1]$ of $\mathbb{R}$ is not compact, if we consider in $\mathbb{R}$ the topology of Sorgenfrey.

Solution. We argue by contradiction. Let $\mathscr{F}$ be the topology on $\mathbb{R}$ defined by: $U \in \mathscr{F}$ if and only if for each $s \in U$ there is a $t > s$ such that $[s, t) \subseteq U$. Consider the collection of open sets $\{[\frac{1}{n}, 1 - \frac{1}{n}) : n \in \mathbb{N}\}$. This is an open cover for $[0, 1]$. A finite subcover must be of the form $\{[\frac{1}{k}, 1 - \frac{1}{k}) : k \in K\}$, where $|K| = \{k_1, \dots, k_p\}$ is finite. Denoting by $T = \max\{k_1, \dots, k_p\}$, we have

$$\bigcup_{k\in K} \left[\frac{1}{k}, 1 - \frac{1}{k}\right) \subseteq \left[\frac{1}{T}, 1 - \frac{1}{T}\right) \subset [0, 1]$$

and so $[\frac{1}{T}, 1-\frac{1}{T})$ is strictly contained in $[0,1]$. Then we have shown that there is at least one open which is not contained in the original open covering. Here is the contradiction.

Examples of compact subspaces in $\mathbb{R}^n$ are the following

$$D^n, \text{ and } R = \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq 1, 0 \leq t \leq 4\} = [0,1] \times [0,4].$$

Now the following two subsets of $\mathbb{R}^3$

$A = \{(s,t,u) \in \mathbb{R}^3 : s^2+t^2 \leq 1\} =$ Cylinder with z-axis of symmetry and radius 1;

$B = \{(s,t,u) \in \mathbb{R}^3 : t^2+u^2 \leq 1\} =$ Cylinder with x-axis of symmetry and radius 1;

are not compact, because they are not bounded, but their intersection $A \cap B$ is compact, since it is closed and bounded. The following exercise formalizes this intuition and illustrate a part of the famous Heine-Borel's Theorem.

Exercise 7.4 (See [8], Exercise 7.13(b))**.** A compact subset of $\mathbb{R}^n$ is bounded.

Solution. We will show this by contradiction. Let U be a compact unbounded subset of $\mathbb{R}$. Then there is $x \in U$ such that $|x| > K, \forall K \in \mathbb{N}$ by the Archimedean property of the real numbers. Then $\{(-n,n) : n \in \mathbb{N}\}$ is an open cover for U, but it does not have a finite subcover, since if P is a finite subset of positive integers, then $\{(-n,n) : n \in P\} \subseteq (-m,m)$ where $m = \max P$. This gives contradiction and shows that U cannot be unbounded. We may argue in the same way when $n \geq 2$. If a subset S of $\mathbb{R}^n$ is compact and unbounded, then there exists an x_i in $x = (x_1, x_2, ..., x_n) \in S$ such that $|x_i| > K$ for all $K \in \mathbb{N}$ and the cover $\{(-n,n) \times (-n,n) \times \cdots \times (-n,n) : n \in \mathbb{N}\}$ does not have a finite subcover, getting to a contradiction again. Therefore a compact subset of $\mathbb{R}^n$ must be bounded.

We repeat a fundamental step of the proof of the Heine-Borel's Theorem, just to emphasize the role of the axiom of completeness when we deal with the usual metric spaces.

Proposition 7.5 (See [8], Theorem 7.7)**.** The unit interval $[0,1] \subseteq \mathbb{R}$ is compact.

Proof. Let $\{U_j : j \in J\}$ be an open cover of $[0,1]$ and suppose that there is no finite subcover. This means that at least one of the intervals $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ cannot be covered by a finite subcollection of $\{U_j : j \in J\}$. Denote by $[a,b]$ one of those intervals, that is $[a,b]$ cannot be covered by a inite subcollection of $\{U_j : j \in J\}$. Again at least one of the intervals $[a_1, \frac{1}{2}(a_1+b_1)]$ or $[\frac{1}{2}(a_1+b_1), b_1]$ cannot be covered by a finite subcollection of $\{U_j : j \in J\}$; denote one such by $[a_2,b_2]$. Continuing in this manner we get a sequence of interval $[a_1,b_1], [a_2,b_2], ..., [a_n,b_n]...$, such that no finite subcollection of $\{U_j : j \in J\}$ covers any of the intervals. Furthermore $b_n - a_n = 2^{-n}$ and $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all n. This last condition implies that $a_m \leq b_n$ for every pair of integers m and n so that b_n is an upper bound for the set $\{a_1, a_2, ...\}$. Let a be the least upper bound of the set $\{a_1, a_2, ...\}$. Since $a \leq b_n$ for each n, a is a lower bound of $\{b_1, b_2, ...\}$. Let b be the greatest lower bound of the set $\{b_1, b_2, ...\}$. By definition we have $a_n \leq a \leq b \leq b_n$ for each n. But since $b_n - a_n = 2^{-n}$ we have $b - a \leq 2^{-n}$ for each n and so $a = b$.
Since $\{U_j : j \in J\}$ covers $[0,1]$ and $a = b \in [0,1]$ we have $a \in U_j$ for some $j \in J$. Since U_j is open there is an open interval $(a-\varepsilon, a+\varepsilon) \subseteq U_j$ for some $\varepsilon > 0$. Choose a positive integer N so that $2^{-N} < \varepsilon$ and hence $b_N - a_N < \epsilon$. However $a \in [a_N, b_N]$

and $a - a_N < 2^{-N} < \varepsilon$, $b - b_N < 2^N < \varepsilon$ so that $[a_N, b_N] \subseteq (a-\varepsilon, a+\varepsilon) \subseteq U_j$ which is a contradiction to $[a_N, b_N]$ not being covered by a finite subcollection of $\{U_j : j \in J\}$. □

Note that, instead of 2, we can choose any prime p and repeat the same argument. Another classical result deals with the fact that continuous images of compact sets are compact. This result is somehow hidden when one studies the well known Theorem of Karl Weierstrass in real analysis, in fact it is originally due to Bernard Bolzano, but in a different form.

Proposition 7.6 (See [8], Exercise 7.13(b))**.** Let $f : X \to Y$ be a continuous map. If $S \subseteq X$ is a compact subspace, then $f(S)$ is compact.

Proof. Suppose that $\{U_j : j \in J\}$ is an open cover of $f(S)$; then $\{f^{-1}(U_j); j \in J\}$ is an open cover of S. Since S is compact there is a finite subcover $\{f^{-1}(U_k) : k \in K\}$, K finite. But $f(f^{-1}(U_k)) \subseteq U_k$ and so $\{U_k : k \in K\}$ is a cover of $f(S)$ which is a finite subcover of $\{U_j : j \in J\}$. □

From the previous two results, it follows that each interval $[a, b] \subseteq \mathbb{R}$ is compact. Now we show another property of closure of the compactness via subspaces. This property will involve the notion of *Hausdorff space* in Definition 8.1.

Exercise 7.7 (See [8], Exercise 7.13(b))**.** A closed subset of a compact space is compact. Viceversa, in a Hausdorff space a compact subset is closed.

Solution. Let $\{U_j : j \in J\}$ be an open cover of the subset $S \subseteq X$ where each U_j is an open subset of X. Since $S \subseteq \bigcup_{j \in J} U_j$ we see that $\{U_j : j \in J\} \cup \{X - S\}$ is an open cover of X and as X is compact it has a finite subcover. This finite subcovering of X is of the form $\{U_k : k \in K\}$ or $\{U_k : k \in K\} \cup \{X - S\}$ where K is finite. Hence $\{U_k : k \in K\}$ is a finite subcover of $\{U_j : j \in J\}$ which covers S.

Let A be a compact subset of X. We may assume that $A \neq \emptyset$ and $A \neq X$ since otherwise it is already closed and there is nothing left to prove. Choose a point $x \in X - A$. For each $a \in A$ there is a pair of disjoint open sets U_a, V_a with $x \in U_a$ and $a \in V_a$. The set $\{V_a : a \in A\}$ covers A and since A is compact there is a finite subcover say $\{V_{a(1)}, V_{a(2)}, ..., V_{a(n)}\}$ which covers A. The set $U = U_{a(1)} \cap U_{a(2)} \cap ... \cap U_{a(n)}$ is an open set containing x which is disjoint from each of the $V_{a(i)}$ and hence $U \subseteq X - A$. Thus each point $x \in X - A$ has an open set containing it which is contained in $X - A$, which means that $X - A$ is open and A is closed.

Using the theorems above we can prove one part of the Heine-Borel's Theorem.

Exercise 7.8 (See [8], Exercise 8.14(n))**.** A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Solution. Let H be a compact subset of $\mathbb{R}^n$, then H is bounded from what we have seen in Exercise 7.4. Also since $\mathbb{R}^n$ is a Hausdorff space, H is closed. Viceversa, let H be a closed and bounded set of $\mathbb{R}^n$. Since H is bounded there exists an interval $[a, b]$ such that $H \subseteq [a, b]^n$ and $[a, b]$ is compact by Proposition 7.5. H is closed and so it is a closed subset of a compact space, so H is compact.

A way of ensuring the compactness of a graph is given by the continuity of the underlying function.

Exercise 7.9 (See [8], Exercise 7.13(c))**.** Given the function $f : x \in I = [0,1] \mapsto f(x) = x \in \mathbb{R}$, the subset graph$(f)$ of $\mathbb{R}^2$ is compact if and only if f is continuous.

Solution. Consider a closed subset C of $\mathbb{R}$ and the projection $\pi : (x,c) \in I \times C \mapsto \pi(x,c) = x \in I$ onto I. Then $f^{-1}(C) = \pi((I \times C) \cap \mathrm{graph}(f))$. Since $x \in f^{-1}(C)$ if and only if $f(x) \in C$, we have $(x, f(x)) \in (I \times C) \cap \mathrm{graph}(f)$, and $\pi(x, f(x)) = x$, so $x \in \pi((I \times C) \cap \mathrm{graph}(f))$. Also if $x \in \pi((I \times C) \cap \mathrm{graph}(f))$, then $(x,y) \in (I \times C) \cap \mathrm{graph}(f)$ for some $y \in \mathrm{graph}(f)$ and so $y = f(x)$, thus $x \in f^{-1}(C)$. This means that f is continuous.

Since $I \times C$ is closed and $\Gamma(f)$ is compact their intersection is compact. So a continuous image, $\pi((I \times C) \cap \Gamma(f))$ will be compact in I, and therefore closed. To prove the converse we need only to note that I is compact and since $\Gamma(f)$ is homeomorphic to I by Exercise 6.4, then it must be compact as well.

Using the previous example, one can offer examples in which it is possible to note that being closed isn't enough to get compactness. There are in fact examples of discontinuous functions $g : I = [0,1] \to \mathbb{R}$ with a graph which is closed but not compact. One of them is

$$g : x \in I = [0,1] \longmapsto g(x) = \begin{cases} 1/x, & x \neq 0 \\ \\ 0, & x = 0. \end{cases}$$

which is discontinuous, with a closed graph that isn't compact.

It is useful here to note that the compactness may be deduced only looking at a basis of the topology. In fact:

Proposition 7.10. Given a basis $\mathscr{B}$ for a topological space S, the space S is compact if and only if any open covering of S, made by elements of $\mathscr{B}$, contains a finite subcovering.

Proof. Let $\mathscr{B}$ be a basis for $\mathscr{U}_S$, topology on S. Let $\mathscr{R}$ be an open covering of S, where $\mathscr{R} \subseteq \mathscr{B}$. It is clear that we are able to extract from R a finite subcover since S is compact. Viceversa, assume that $\mathscr{R} = \{A_i : i \in I\}$ is an arbitrary covering of S, where $A_i \in \mathscr{U}_S$. Then $A_i = \bigcup\limits_{j \in J} B^i_j$ for some $B^i_j \in \mathscr{B}$, and $\mathscr{B}$ contains a finite subcover $\{B^{i_1}_{j_1}, ..., B^{i_k}_{j_k}\}$. Then $\{A_{i_1}, ..., A_{i_k}\}$ is a finite subcovering, extracted from $\mathscr{R}$. So S is compact. □

The proposition above shows that it is enough to check whether $\mathscr{B}$ has a finite subcovering or not in order to conclude that S is compact, so this illustrates why the notion of basis is important for a topology. It is possible to present a more general result.

Theorem 7.11. *Given a basis $\mathscr{B}$ for a topological space S, the space S is Lindelöff if and only if any open covering of S, made by elements of $\mathscr{B}$, contains a countable subcovering.*

Proof Let $\mathscr{B}$ be a basis for $\mathscr{U}_S$, topology on S. Let R be an open covering of S, where $\mathscr{R} \subseteq \mathscr{B}$. It is clear that we are able to extract from $\mathscr{R}$ a countable subcover since S is Lindelöff. Viceversa, assume that $\mathscr{R} = \{A_i : i \in I\}$ is an arbitrary covering of S, where $A_i \in \mathscr{U}_S$. Then $A_i = \bigcup_{j \in J} B^i_j$ for some $B^i_j \in \mathscr{B}$, and $\mathscr{B}$ contains a countable subcover $\{B^{i_1}_{j_k} : k \in \mathbb{N}\}$. Then $\{A_{i_k} : k \in \mathbb{N}\}$ is a countable subcovering, extracted from $\mathscr{R}$. So S is Lindelöff.

A fundamental result for compact spaces in products of topological spaces is presented below. We report a classical proof, that uses the so called Lemma of Alexander.

Theorem 7.12 (Theorem of Tychonoff)**.** *For all $i \in I$ the product space $\Pi_{i \in I} S_i$ is compact if and only if each factor S_i is compact.*

The idea is to sketch the main steps of the following result:

Lemma 7.13 (Lemma of Alexander)**.** Given $\mathscr{S}$ subbasis of S, if each covering of S, made by elements of $\mathscr{S}$, contains a finite subcovering, then S is compact.

First recall that $\mathscr{S}$ is a subbasis of S if $\mathscr{S} \subseteq \mathscr{U}_S$ and $\mathscr{S}$ with the set of finite intersection of elements of $\mathscr{S}$ is a basis for $\mathscr{U}_S$.

Sketch of the Lemma of Alexander. Assume S is not compact. There is a covering, say $\mathscr{R}_0$, of S that does not contain a finite subcovering and is minimal with respect to this property. If $A \in \mathscr{R}_0$ and $A_1 \cap A_2 \cap \cdots \cap A_h \subseteq A$ for some $A_1, A_2, ..., A_h \in \mathscr{S}$, then at least one of the $A_1, ..., A_h$ must belong to $\mathscr{R}_0$. Then one can show that $\mathscr{S} \cap \mathscr{R}_0$ is an open covering for S and so $\mathscr{S} \cap \mathscr{R}_0$ should contain a finite subcovering contained in $\mathscr{R}_0$ and this is impossible. So there is a contradiction. □

We may hence proof the result of invariance of the compactness for products.

Proof of Tychonoff's Theorem. Let $S = \{U_j : j \in J\}$ be an open cover for S_t, where J is arbitrary and $t \in I$. Then $\{S_1 \times S_2 \times \cdots \times S_{t-1} \times U_j \times S_{t+1} \times \cdots : j \in J\}$ is an open cover of $\Pi_{i \in I} S_i$. This means that there exists a finite set $K \subset J$ such that $\{S_1 \times S_2 \times \cdots \times S_{t-1} \times U_k \times S_{t+1} \times \cdots : k \in K\}$ is a subcover of $\Pi_{i \in I} S_i$. Therefore $\{U_k : k \in K\}$ is a finite subcover of S_t.

Viceversa, assume S_i is compact and consider $\{p^{-1}(A_i) : A_i \in \mathscr{U}_{S_i}\}$ which is a subbasis for the product topology of $\Pi_{i \in I} S_i$, realised as the strongest topology making $p : \Pi_{i \in I} S_i \to S_i$ a continuous map. Consider now an arbitrary covering $\mathscr{R}$ of $S = \Pi_{i \in I} S_i$ which is contained in the subbasis $\{p^{-1}(A_i) : A_i \in \mathscr{U}_{S_i}\}$. If we are able to extract from this $\mathscr{R}$ a finite subcovering, then it is enough to apply the lemma of Alexander in order to conclude that S is compact.

In fact it can be checked easily that $\mathscr{P}_i = \{U_i \in \mathscr{U}_{S_i} : p^{-1}(U_i) \in R\}$ is a covering of S_i. Therefore $\exists\ i \in I$ such that $\mathscr{P}_i$ is a covering of S_i and S_i is compact so $\exists\ U^1_i, ..., U^h_i$ finitely many opens of $\mathscr{P}_i$ such that $S_i = \bigcup_{j=1}^{h} U^j_i$. Then $\bigcup_{j=1}^{h} p_i^{-1}(U^j_i) = \Pi_{i \in I} S_i$ is a finite subcovering extracted from $\mathscr{R}$, as claimed.

□

We now deal with an example of functional spaces. The following exercise is an example of a topology on a set of functions and involves the so called *compact-open topology.*

Exercise 7.14 (See [8], Exercise 7.13(d) and (e))**.** Let X, Y be topological spaces and $\mathscr{F}(X,Y)$ the set of all continuous functions from X to Y. If $A \subseteq X$ and $B \subseteq Y$, then write

$$F(A,B) = \{f \in \mathscr{F}(X,Y) \ : \ f(A) \subseteq B\}$$

and

$$\mathscr{L} = \{F(A,B) \ : \ A \text{ is a compact subset of } X \text{ and } B \text{ is an open set of } Y\}.$$

Define

$$\mathscr{U} = \{U \subset \mathscr{F}(X,Y) \ : \ f \in U \Rightarrow \exists F_1, ..., F_n \in \mathscr{L} \text{ such that } f \in F_1 \cap ... \cap F_n \subseteq U\}.$$

The set $\mathscr{U}$ is a topology for $\mathscr{F}(X,Y)$, called the *compact-open* topology. Furthermore, if X is compact and metrizable and Y is metrizable with metric d, then

$$d^*(f,g) = \sup_{x \in X} d(f(x), g(x))$$

defines a metric on $\mathscr{F}(X,Y)$.

Solution. We begin to check the axioms of topology.

- $F(A, \emptyset) = \emptyset$ so if $f \in \emptyset$ then $f \in F(A,\emptyset) \subset \emptyset$. So $\emptyset \in \mathscr{U}$. $\mathscr{F}(X,Y)$ is trivially in $\mathscr{U}$.
- If $f \in \bigcup_{i \in I} U_i$ where $U_i \in \mathscr{U}$ then $\exists k \in I$ such that $f \in U_k$ and so $\exists\ F_1, F_2, ..., F_n \in \mathscr{L}$ such that $f \in F_1 \cap F_2 \cdots \cap F_n \subseteq U_k \subseteq \bigcup_{i \in I} U_i$. So $\bigcup_{i \in I} U_i \in \mathscr{U}$.
- If $f \in U_1 \cap U_2$ where $U_1, U_2 \in \mathscr{U}$ then $\exists\ F_1^1, F_2^1, ..., F_n^1, F_1^2, F_2^2, ..., F_m^2 \in \mathscr{L}$ such that $f \in F_1^1 \cap F_2^1 \cap ... F_n^1 \subseteq U_1$ and $f \in F_1^2 \cap F_2^2 \cap \cdots \cap F_m^2 \subseteq U_2$ so $f \in F_1^1 \cap F_2^1 \cap \cdots \cap F_n^1 \cap \cdots F_m^2 \subseteq U_1 \cap U_2$. So $U_1 \cap U_2 \in \mathscr{U}$.

In order to check that d^* is a metric, we need to check the axioms with an argument of routine that we have already seen in previous proofs so we omit the details.

The topological space, which is mentioned in the previous exercise, is of fundamental importance in functional analysis, see [7].

We end the present chapter, noting that the property of being compact need not necessarily apply to the entire space, in fact it is possible to formulate a local notion of compactness. A space X is said to be *locally compact* if for all $x \in X$ every neighbourhood of x contains a compact neighbourhood of x.

Exercise 7.15 (See [8], Exercise 7.13(f))**.** If X is locally compact, then the *evaluation* map $e : \mathscr{F}(X,Y) \times X \to Y$, given by $e(f,x) = f(x)$, is continuous.

Solution. Let U be an open set in Y such that $f(x) \in U$. Then $x \in f^{-1}(U)$ where $f^{-1}(U)$ is open in X. So there is a neighbourhood $C_x \subseteq f^{-1}(U)$ that is compact with $x \in C_x$. Since $f(C_x) \subseteq U$, then $F(C_x, U)$ is open in $\mathscr{F}(X,Y)$. This means that $(F(C_x,U), C_x^\circ)$ is open in $\mathscr{F}(X,Y) \times X$. So $\bigcup_{x \in f^{-1}(U)} (F(C_x,U), C_x^\circ)$ is also open in $\mathscr{F}(X,Y) \times X$. We claim that

$$\bigcup_{x \in f^{-1}(U)} (F(C_x,U), C_x^\circ) = e^{-1}(U).$$

In fact

$$e\Big(\bigcup_{x\in f^{-1}(U)}(F(C_x,U),C_x^{\circ})\Big)\subseteq U\Rightarrow \bigcup_{x\in f^{-1}(U)}(F(C_x,U),C_x^{\circ})\subseteq e^{-1}(U).$$

Conversely,

$$(f,x)\in e^{-1}(U)\Rightarrow f(x)\in U\Rightarrow f\in F(C_x,U)$$
$$\Rightarrow x\in f^{-1}(U)\Rightarrow (f,x)\in (F(C_x,U),C_x^{\circ})\Rightarrow (f,x)\in \bigcup_{x\in f^{-1}(U)}(F(C_x,U),C_x^{\circ}).$$

So $e^{-1}(U)\subseteq \bigcup_{x\in f^{-1}(U)}(F(C_x,U),C_x^{\circ})$ and $e^{-1}(U)$ is open.

If a metric space has a finite subcover, then we can expect to have a measure of the size of a subset which must be contained in a finite cover. This remark was originally due to Lebesgue.

Exercise 7.16 (See [8], Exercise 7.13(g))**.** If X is a compact topological space arising from some metric space with metric d and if $\{U_j : j\in J\}$ is an open cover of X, then there exists a real number $\delta>0$ (called the *Lebesgue number* of $\{U_j : j\in J\}$) such that any subset of X of diameter less than δ is contained in one of the sets $U_j, j\in J$.

Solution. Let $U=\{u_j : j\in J\}$. For every point $x\in X$, select an element $U(x)$ of U so that $x\in U(x)$. Since $U(x)$ is open we can find some radius $r(x)>0$ such that $B_{r(x)}(x)\subseteq U(x)$. The collection of open balls $\mathscr{B}=\{B_{\frac{r(x)}{2}}(x) : x\in X\}$ is clearly an open cover of X. Since X is compact there is a finite subcover, say $\mathscr{B}'=\{B_{\frac{r(x_i)}{2}}(x_i) : i=1,2,...,n\}$. Let

$$\delta=\min\left\{\frac{r(x_i)}{2} : i=1,2,...,n\right\}$$

and suppose that E is a subset of X with

$$\operatorname{diam}(E)=\sup\{d(a,b)\mid a,b\in E\}<\delta.$$

Fix a point $p\in E$. The point p must be in some element of the cover $\mathscr{B}'$, say $B_{\frac{r(x_i)}{2}}(x_i)$. Then let q be any point in E. We have $d(p,q)\leq \operatorname{diam}(E)<\delta\leq \frac{r(x_i)}{2}$ and so we get

$$d(x_i,q)\leq d(x_i,p)+d(p,q)<\frac{r(x_i)}{2}+\frac{r(x_i)}{2}=r(x_i).$$

This shows that $E\subseteq B_{r(x_i)}(x_i)$. But then $E\subset B_{r(x_i)}(x_i)\subset U(x_i)$ so E is contained in the element $U(x_i)$ of U.

We end this chapter with a notion of compactification of a point at infinity. This is what one is forced to do in absence of a compact space at the origin. Let X be a topological space and define X^{∞} to be $X\cup\{\infty\}$ where ∞ is an element not contained in X. If $\mathscr{U}$ is the topology for X, then define $\mathscr{U}^{\infty}$ to be $\mathscr{U}$ together with all sets of the form $V\cup\{\infty\}$ where $V\subseteq X$ and $X-V$ is both compact and closed in X. The space X^{∞}, endowed of such topology , is called the *one-point compactification* of X. It is not difficult to see that $S^1=\mathbb{R}^{\infty}$, that is, the unit circle is obtained as compactification of the real line, adding the point ∞.

Exercise 7.17 (See [8], Exercise 7.13(h))**.** The set $\mathscr{U}^{\infty}$ is a topology for X^{∞} and X^{∞} is compact. Moreover, X is a subspace of X^{∞}.

Solution. We check the axioms of topology. Of course, the first axiom is satisfied.

Assume now $U, V \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$. Then $U \cap V \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$. If $U \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$ and $N \cup \{\infty\} \in \{V \cup \{\infty\} \ : \ X - V$ is compact and closed$\}$ then

$$U \cap (N \cup \{\infty\}) = (U \cap N) \cup (U \cap \{\infty\}) = (U \cap N) \cup \emptyset = U \cap V \in \mathscr{U} \subseteq \mathscr{U}^{\infty}.$$

If $U \cup \{\infty\}$ and $N \cup \{\infty\} \in \{V \cup \{\infty\} : \ X - V$ is compact and closed$\}$ then

$$(U \cup \{\infty\}) \cap (N \cup \{\infty\}) = (U \cap N) \cup \{\infty\}.$$

We conclude that $X - (U \cap N) = (X - U) \cup (X - N)$ is compact and closed, since $(X - U)$ and $(X - V)$ are compact and closed.

Finally, if $U_i \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$, then $\bigcup\limits_{i \in I} U_i \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$. If $U_i \in \mathscr{U} \subseteq \mathscr{U}^{\infty}$ and $V_i \cup \{\infty\} \in \{V \cup \{\infty\} : X - V$ is compact and closed$\}$, then $(\bigcup\limits_{i \in I} U_i) \cup (\bigcup\limits_{i \in I} V_i \cup \{\infty\}) = \left(\bigcup\limits_{i \in I} (U_i \cup V_i)\right) \cup \{\infty\}$. Therefore

$$X - \bigcup_{i \in I} (U_i \cup V_i) = X - \left(\bigcup_{i \in I} U_i \cup \bigcup_{i \in I} V_i\right)$$

$$= \left(X - \bigcup_{i \in I} U_i\right) \cap \left(X - \bigcup_{i \in I} V_i\right) = \left(\bigcap_{i \in I} (X - U_i)\right) \cap \left(\bigcup_{i \in I} (X - V_i)\right) \subseteq X - V_k, k \in I.$$

Since $X - (\bigcup\limits_{i \in I} (U_i \cup V_i))$ is a closed subset of a compact set, it is compact as well. So $(\bigcup\limits_{i \in I} U_i) \cup (\bigcup\limits_{i \in I} V_i \cup \{\infty\}) \in \{V \cup \{\infty\} \ \ X - V$ is compact and closed$\}$. The missing case $U_i \cup \{\infty\}$ and $V_i \cup \{\infty\}$ can be checked in analogy with the previous case.

The fact that X is a subspace of X^{∞} is straightforward from the definitions.

8. Hausdorff spaces and axioms of separation

In the present chapter we deal with some classical notions, related to the axioms of separations of Tychonoff.

Definition 8.1 (See [8], Definition 8.1)**.** A space X is *Hausdorff* if for every pair of distinct points x, y there are open sets U_x, U_y containing x, y respectively such that $U_x \cap U_y = \emptyset$

The cofinite topology is not Hausdorff in general, but when the entire space is finite, then it is Hausdorff.

Exercise 8.2 (See [8], Exercise 8.2(a))**.** A space X with the cofinite topology is Hausdorff if and only if X is finite.

Solution. Let $x, y \in X$ be two distinct points and X Hausdorff. Then $\exists\, U_x, U_y$ open sets where $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. So $X - U_x$ is finite, and since $U_y \subseteq X - U_x$, U_y is also finite. Therefore $(X - U_y) \cup U_y = X$ is the union of two finite sets and is therefore finite. Viceversa, if X is finite, then x is open since $X - \{x\}$ is finite, and so is $\{y\}$. Now $\{x\} \cap \{y\} = \emptyset$, so X is Hausdorff.

We know that $\mathbb{R}$ with the usual topology is Hausdorff, but here is an example from using a different topology that we've seen before.

Exercise 8.3 (See [8], Exercise 8.2(b))**.** The line of Sorgenfrey is Hausdorff.

Solution. If $x, y \in \mathbb{R}$ are two distinct points with $x < y$, then we can always find two disjoint opens $x \in [x, \frac{x+y}{2})$ and $y \in [\frac{x+y}{2}, y+1)$ in the topology of Sorgenfrey on $\mathbb{R}$. In fact $[x, \frac{x+y}{2}) \cap [\frac{x+y}{2}, y+1) = \emptyset$.

The property of being Hausdorff is invariant under homeomorphisms.

Exercise 8.4 (See [8], Exercise 8.2(c))**.** Suppose that X and Y are homeomorphic topological spaces. X is Hausdorff if and only if Y is Hausdorff.

Solution. If X is Hausdorff, $x, y \in Y$ distinct points and $g : Y \to X$ be homeomorphism, then $g(x), g(y) \in X$ and $g(x) \neq g(y)$. This means there are U_x, U_y opens in X, such that $U_x \cap U_y = \emptyset$. On the other hand, $g^{-1}(U_x)$ and $g^{-1}(U_y)$ are open in Y, with $x \in g^{-1}(U_x)$, $y \in g^{-1}(U_y)$ such that $g^{-1}(U_x) \cap g^{-1}(U_y) = \emptyset$. Beginning from Y Hausdorff, the argument is analogous.

Actually all the axioms of separation are invariant under homeomorphisms.

Definition 8.5 (See [8], Definition 8.3)**.** Let k be one of the integers 0,1,2,3 or 4. A space X is said to be a *T_k-space*, or *a Tychonoff space of type k*, if it satisfies one of the following conditions:

T_0: For every pair of distinct points there is an open set containing one of them but not the other.

T_1: For every pair x, y of distinct points there are two open sets, one containing x but not y, and the other containing y but not x.

T_2: For every pair x, y of distinct points there are two disjoint open sets, one containing x and the other containing y.

T_3: X satisfies T_1 and for every closed subset F and every point x not in F there are two disjoint open sets, one containing F and the other containing x.

T_4: X satisfies T_1 and for every pair F_1, F_2 of disjoint closed subsets there are two disjoint open sets, one containing F_1 and the other containing F_2.

As mentioned before, all T_k are invariant under homeomorphisms.

Exercise 8.6 (See [8], Exercise 8.4(a))**.** Suppose that X and Y are homeomorphic spaces. Prove that X is a T_k-space if and only if Y is a T_k-space ($k = 0, 1, 2, 3, 4$).

Solution. The argument which we have seen in the case of T_2, that is, Hausdorff, may be used for all the cases T_k with $k \in \{0, 1, 3, 4\}$. Let's give details just for one case, omitting the details of the remaining cases which work by analogy. If X is T_0, $x, y \in Y$ distinct points and $g : Y \to X$ homeomorphism, then $g(x), g(y) \in X$ and $g(x) \neq g(y)$. This means there is U_x open in X, such that $g(x) \in U_x$ and $g(y) \notin U_x$. Then $x \in g^{-1}(U_x)$ is open and $y \notin g^{-1}(U_x)$, that is, X is T_0.

While T_k implies clearly T_{k-1} for all $k = 1, 2, 3, 4$ by definition, we offer examples that show that T_{k-1} doesn't imply a T_k in general.

Exercise 8.7 (See [8], Exercise 8.4(b))**.** Examples of topological spaces X_0, X_1, X_2 and X_3 with the property that X_k is a T_k but X_k is not a T_j for $j > k$ and $k \in \{0, 1, 2, 3\}$.

Solution. Consider $X_0 = \mathbb{R}$ and $\mathbb{Z}$ and the topology $\mathscr{U} = \{\mathbb{Z}, \emptyset, \mathbb{R}\}$ with three opens on $\mathbb{R}$. If $x, y \in \mathbb{R} - \mathbb{Z}$ and $x \neq y$, then an open U_x containing x is only $\mathbb{R}$ and an open U_y containing y is only $\mathbb{R}$. Then x, y are distinct and there is no open in this topology containing x and not containing y. This means that X_0 is neither T_0, nor T_1, nor T_2, nor T_3, nor T_4.

Consider $X_1 = \mathbb{R}$ with the topology of the left half lines. This space is clearly T_1 but is not Hausdorff, since any two distinct points x, y must belong to left half lines of the form $(-\infty, a)$ and $(-\infty, b)$, respectively, so their intersection cannot be empty.

Consider the *Moore plane* $X_2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ whose basis elements are the open disks induced in X_2 as well as elements we denote by

$$U_{a,r} = B((a, 0), r) \cap X_2 - \{(x, 0) \in \mathbb{R}^2 : x \in (a - r, a + r) - \{a\}\}.$$

Then X_2 with this topology is clearly T_2. Any two points not on the x-axis can be seperated by open balls. A point not on the x-axis can be seperated from a point on the x-axis by taking an open ball and open half-disk with a sufficiently small radius. The same goes for two points on the x-axis. However we claim that X_2 is not a T_3 space. Consider the following set:

$$F = \{(x, 0) \in \mathbb{R}^2 : -1 < x < 1\}.$$

This set is closed because $X_2 - F$ can be written as the union of $U_{-1,1}$ and $U_{1,1}$ as well as open balls that do not intersect F. Take the point $x = (1, 0)$. Then any open set containing x will contain a set of the form $U_{1,r}$ or $B((1, 0), r)$ where $r > 0$. But since $r \neq 0$, $U_{1,r}$ and $B((1, 0), r)$ will intersect any open set containg F and hence X_2 is not a T_3 space.

We now describe a fundamental property of T_3 spaces in terms of local bases.

Lemma 8.8. X is T_3 if and only if $\forall\ x \in X$ and $U_x \in \mathscr{U}(x)$, local basis for x, there is a closed neighbourhood of x contained in U_x (and containing x). In other words, a T_3 space X is characterized by having a local bases (for its points) of closed neighbourhoods.

We offer a proof of this fact, since the argument is classical.

Proof. We claim that: "$\forall\ x \in X$ and $x \in \mathring{A} = A$, $\exists$ closed neighbourhood of x contained in A". This is equivalent to the necessary condition.

Take $x \in X$ and $\mathring{A}$ with $x \in A$. We have $X - A$ is closed and $x \in X$, so by definition of T_3, they are seperable by two opens $X - A \subseteq U$ and $x \in V$ such that $U \cap V = \emptyset$. Of course, $X - U$ is closed and $X - U \subseteq A$. Moreover $V \subseteq X - U$. Then $X - U$ is the closed neighbourhood of x, contained in A, that we were looking for. Now we prove the sufficient condition. Let $x \in X$ and $F = \overline{F}$ with $x \in F$. Then $X - F$ is open, so there is a closed neighbourhood K with $x \in K$ and $K \subseteq X - F$. Then x and F are seperated by $\mathring{K}$ and $X - K$ and this is enough to conlclude that the space is T_3. □

We are able to show a classical result which is usually referred as the result of "normality of regular spaces with countable bases".

Theorem 8.9. *If X is T_3 and N_2, then X is T_4.*

Again we present a proof, in order to see classical arguments.

Proof. Consider $\overline{F_1} = F_1$ and $\overline{F_2} = F_2$ such that $\emptyset = F_1 \cap F_2$, $x \in F_1$. Note that $X - F_2$ is open, $F_1 \subseteq X - F_2$ and so $X - F_2$ is a neighbourhood of x. By Lemma 8.8, we may use the fact that a local basis for x is made by closed neighbourhoods of x. Then $\exists\ U_x \in \mathscr{U}(x)$ such that $\overline{U_x} \subseteq X - F_2$ and $\emptyset = \overline{U_x} \cap F_2$. By analogy, we may begin from $y \in F_2$ and consider $V_y \in \mathscr{U}(y)$ such that $\overline{V_y} \subseteq X - F_1$ and $\emptyset \subseteq \overline{V_y} \cap F_1$. This way we get two families: $\mathscr{F} = \{U_x \cap F_1\}_{x \in F_1}$ and $\mathscr{F} = \{V_y \cap F_2\}_{y \in F_2}$ which are done by opens in the induced topologies of F_1 and F_2, respectively. Moreover $\mathscr{F}_1$ and $\mathscr{F}_2$ have countable basis because we may refer to U_x and V_y as elements of the countable basis of X that we have by the assumptions. Therefore one may replace U_x by U_n and V_y by V_n, with $n \in \mathbb{N}$. This implies that $F_1 \subseteq \bigcup_{n \in \mathbb{N}} U_n$ and $F_2 \subseteq \bigcup_{n \in \mathbb{N}} V_n$ and $\overline{V_n} \cap F_1 = \emptyset$ and $\overline{U_n} \cap F_2 = \emptyset\ \forall n \in \mathbb{N}$. Set now

$$A_1 = U_1,\ B_1 = V_1 - \overline{A_1},\ A_2 = U_2 - \overline{B_1},\ B_2 = V_2 - (\overline{A_1 \cup A_2}),$$
$$\dots, A_n = U_n - (\overline{B_1 \cup \dots \cup B_{n-1}}),\ B_n = V_n - (\overline{A_1 \cup ... \cup A_n}), \dots$$

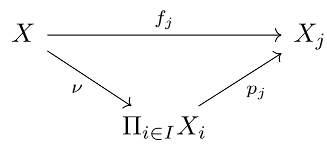
One can check that $A = \bigcup_{n \in \mathbb{N}} A_n$ and $B = \bigcup_{n \in \mathbb{N}} B_n$ are two opens of X and that they are exactly the opens that we need. □

Since metrizable spaces satisfy most of the axioms of separations, one can ask if the previous argument gives indeed a metrizable space. This theorem is classical and due to Urysohn (see [5]).

Theorem 8.10 (of Urysohn)**.** *If X is T_3 and N_2, then X is metrizable.*

Before to see the details of this proof, we mention a few facts.

Fact 1. Consider any topological space X_i with arbitrary index set I and $\Pi_{i \in I} X_i$ be the product space of the spaces X_i. Define $p_j : \Pi_{i \in I} X_i \to X_j$ as the projection from $\Pi_{i\in I}X_i$ onto its j-component, being $j \in I$. Of course, p_j is surjective and continuous. Consider now $f_j : X \to X_j$, where X is an arbitrary space and assume that f_j are all continuous for all $j \in I$. The function $\nu : X \to \Pi_{i\in I}X_i$ is such that the diagram



is commutative (i.e $p_j \circ \nu = f_j$), is an embedding (that is X is homeomorphic to $\nu(X)$, or equivalently, ν is continuous and injective) if and only if X has the weakest topology, induced by the f_j and in addition f_j seperates points $\forall\ j \in I$, that is $f_j(x) \neq f_j(y)$ for all $x \neq y$ in X.

Fact 2. (Lemma of Urysohn) Given C and C' disjoint closed subsets of a normal topological space X, there is a continuous function $f : X \to [0, 1]$ such that $f(C) = 0$ and $f(C') = 1$.

Sketch of the lemma of Urysohn. Consider the set

$$D = \left\{ \frac{m}{2^n} : m, n \in \mathbb{N}, 1 \leq m \leq 2^n - 1 \right\}$$

of the diadic numbers. Of course, $D \subseteq \mathbb{Q} \cap [0, 1]$ and one can see that D is dense in $[0, 1]$. In particular, $\inf D = 0$. Put

$$D_n = \left\{ r \in D : r = \frac{m}{2^n}, 1 \leq m \leq 2^n - 1 \right\}$$

and note that $D_1 = \{1/2\}$, $D_2 = \{1/2, 1/4, 3/4\}$, ... , so $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$. Define $D = \bigcup_{n\in\mathbb{N}} D_n$. $\forall\ r \in D\ \ \exists\ A_r = \mathring{A}_r \subseteq X$ such that a) $C \subseteq A_r$ and $\overline{A_r} \cap C' = \emptyset$ (for all r), b) $r < s \Rightarrow \overline{A_r} \subseteq A_s$ (We want to show that $\forall\ r \in D$ there are $A_r = \mathring{A}_r$ with these properties. Begin with $D_1 = \{1/2\}$. Since X is normal, that is, X is T_4, there are $\mathring{A}_{1/2} \subseteq A_{1/2}$ such that $C \subseteq A_{1/2}$ and $\overline{A}_{1/2} \cap C' = \emptyset$. Now pass to $D_2 = \{1/2, 1/4, 3/4\}$. Again for $r \in D_2$ we may find $A_r = \mathring{A}_r$ such that $C \subseteq A_r$ and $\emptyset \subseteq \overline{A_r} \cap C'$, by the normality of X. This is of course true in general when we deal with D_n for n arbitrary large. Then a) is satisfied because if $r \in D$ then there is always some $n \in \mathbb{N}$ such that $r \in D_n$ (note that $\bigcup_{n\in\mathbb{N}} D_n \subseteq D$). In addition, we can always choose r small enough in some D_n in such a way that the condition b) is satisfied. In fact, it is enough to note that $D_n \subseteq D_{n+1}$ and look for the corresponding $\overline{A_r}$ and A_s when $r \in D_n$ and $s \in D_{n+1} - D_n$. Therefore we may define $f : x \in X \to f(x)$

$$f : x \in X \to f(x) = \begin{cases} 1 & \text{if } x \in A_r; \\ \{\inf r : x \in A_r\} & \text{otherwise.} \end{cases}$$

We claim that this f is the function that we are looking for. In fact, if $x \in C$, then $x \in A_r, \forall\ r \in D$ by a) and so $f(C) = 0 = \{\inf r : r \in C \subseteq A_r\}$ $(\forall r \in D)$. If $x \in C'$, then $x \notin \overline{A_r}$ for all $r \in D$, so $f(x) = 1$. This means that $f(C') = 1$. Until here, we have shown that f separates the closed C and C'. It remains to see that f is continuous. In order to do this, one needs to note that in the present situation

1. $f(x) < r \Rightarrow x \in A_r$ In fact, if $f(x) < r$, then the density of D in $[0,1]$ shows that there is some $s < r$ such that $x \in A_s \subseteq \overline{A_s}$, and by b), we have $x \in A_r$.
2. $r < f(x) \Rightarrow x \notin \overline{A_r}$ In fact, if $s \in D$ and $r < s < f(x)$, then $x \in A_s$ and so $x \in \overline{A_r}$ (note b) again is used here).
3. $x \in A_r \Rightarrow f(x) \leq r$.
4. $x \in \overline{A_r} \Rightarrow f(x) \geq r$.

In fact assuming $f(x) < r$ and $x \notin \overline{A_r}$, we have already seen that $x \in A_r \subseteq \overline{A_r}$, which is impossible. Using 1, 2, 3 and 4 above, one can show the continuity of f, concluding the proof. We skip the details here.

□

Now we are ready to prove Theorem 8.10.

Proof of theorem of Metrizability of Urysohn. Since X is N_2, we have that $\mathscr{B}$ generates the topology $\mathscr{U}$ of X and $|\mathscr{B}| = |\mathbb{N}|$. Consider

$$\mathscr{P} = \{(B_1, B_2) \in \mathscr{B} \times \mathscr{B} : \overline{B_1} \subseteq B_2\}.$$

Of course $\mathscr{P}$ is countable. Applying Fact 2, $\forall\ (B_1, B_2) \in \mathscr{P}\ \exists\ f_{B_1B_2} : X \to [0,1]$ continuous such that $f_{B_1B_2}(\overline{B_1}) = 0$ and $f_{B_1B_2}(X - B_2) = 1$. The set

$$\mathscr{F} = \{f_{B_1B_2} : (B_1, B_2) \in \mathscr{P}\}$$

is again countable. Moreover, if $x \in X$ and F is a closed set not containing x, we may always find $B_2 \in \mathscr{B}$ containing x and $B_2 \subseteq X - F$, so Lemma 8.8 implies the existence of a closed neighbourhood I of x contained in B_2. Consider now $B_1 \in \mathscr{B}$ and $x \in B_1$ and $B_1 \subseteq I$. Then $\overline{B_1} \subseteq I \subseteq B_2$. Then we may consider the functions $f_{B_1B_2} \in \mathscr{F}$ such that $f_{B_1B_2}(x) = 0$ and $f_{B_1B_2}(F) = 1$, that is, we may apply again Fact 2 to this specific situation, separating closed sets and points. Then we may apply Fact 1, that is, we have

$$\begin{array}{ccc} X & \xrightarrow{\quad f_g \quad} & I_g \\ & {}_{\nu}\searrow \qquad \nearrow^{p_g} & \\ & \Pi_{f \in \mathscr{F}} I_f & \end{array}$$

where $I_f = [0,1]$ for all $f \in \mathscr{F}$, $f_g = f_{B_1,B_2}$, $I_g = [0,1]$ and p_g is the projection onto the factor I_g. Note that f_g seperates points by closed sets, so separates points and we may conclude that ν is an embedding of X in the Tychonoff cube. Since $\Pi_{f \in \mathscr{F}} I_f$ is metrizable and subspaces of metrizable spaces are still metrizable, $\nu(X)$ is metrizable and ν being a homeomorphism, this means that also X is metrizable.

□

There is an example of Hewitt that shows a regular T_1-space on which every continuous real-valued is constant. These type of examples are sophisticated and deals on this specific problem in the theory of compactifications. Details can be found in [15], [3], [4], [5], [6], [9], [10], [11]. What we can see more easily is that the compactness is enough, combined with T_2, to give T_4, so, in a certain sense, is a good property that may replace T_3 plus the axiom N_2.

Exercise 8.11 (See [8], Exercise 8.4(c))**.** Compact Hausdorff spaces are T_4.

Solution. Let X be a compact Hausdorff space, and E, F two disjoint closed sets in X. Then E, F are compact and, given $a \in E$ and $b \in F$, there are disjoint open sets V_a, U_b, where $a \in V_a$ and $b \in U_b$. The set $\{V_a \ : \ a \in E\}$ is an open cover of E so there exists a finite subcover $\{V_a(1), V_a(2), ..., V_a(n)\}$ which covers E. We can say the same for $\{U_b \ : \ b \in F\}$ from which it is possible to extract the subcover $\{U_b(1), U_b(2), ..., U_b(n)\}$ of F. Note that $U_b(i) \cap V_a(i) = \emptyset$ for all $i \in \{1, \ldots, n\}$. Now $U_B = U_b(1) \cup U_b(2) \cup \cdots U_b(n) \supseteq F$ is an open disjoint from each open of the cover $\{V_a(1), V_a(2), ..., V_a(n)\}$, so U_B is disjoint from $V_a(1) \cup ... \cup V_a(n) \supseteq E$. Therefore we have separated E and F, realizing the definition of T_4-space for X.

We can determine the topology on a Hausdorff space which appears as codomain of a continuous surjective map.

Exercise 8.12 (See [8], Exercise 8.14(a))**.** Let $f : X \to Y$ be a continuous surjective map of a compact space X onto a Hausdorff space Y. Prove that a subset U of Y is open if and only if $f^{-1}(U)$ is open in X. Deduce that Y has the quotient topology determined by f.

Solution. Note that the subset C of Y is closed if and only if $f^{-1}(C)$ is closed in X. In fact, if C is closed in Y, then $f^{-1}(C)$ is closed in X since f is continuous. Viceversa, if $f^{-1}(C)$ be closed in X, then $f^{-1}(C)$ is compact since X is compact, so $f(f^{-1}(C)) = C$ is compact in Y. Since Y is Hausdorff, C is closed, in fact in a Hausdorff spaces all the compact subspaces are characterized to be closed.

Once we have seen this, it is clear that a subset U of Y is open if and only if $f^{-1}(U)$ is open in X, just working on the complementary of U which will be closed by definition. Finally, $f(X) = Y$ and U open in Y implies $f^{-1}(U)$ open in X, so Y has the quotient topology.

Given a continuous map $f : x \in X \mapsto f(x) = y \in Y$ between topological spaces, the *nucleus* of f is defined by the set

$$N_f = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} \subseteq X \times X,$$

that is, by the elements of X having the same image via f in Y. One can note that $x_1 \sim x_2$ if and only if $x_1, x_2 \in N_f$ determines an equivalence relation in X and that the quotient set X/N_f formed by the equivalence classes modulo N_f (and equipped by the quotient topology, induced by the natural projection $\pi : x \in N \mapsto p(x) = [x]_{N_f} \in X/N_f$) is homeomorphic with $f(X)$ with the topology induced by Y. When X has the additional assumption of being Hausdorff, the following fact holds

Exercise 8.13 (See [8], Exercise 8.14(c))**.** Given $f : X \to Y$ continuous map and Y is Hausdorff, the set N_f is a closed in $X \times X$.

Solution. Of course, $(X \times X) - N_f = \{(x_1, x_2) \in X \times X : f(x_1) \neq f(x_2)\}$ and, since Y is Hausdorff, there are open sets $U_1, U_2 \subset Y$ with $f(x_1) \in U_1, f(x_2) \in U_2$, where $f(x_1) \neq f(x_2)$ and $U_1 \cap U_2 = \emptyset$. Since f is continuous, $f^{-1}(U_1) \times f^{-1}(U_2)$ is an open set in $X \times X$ and $(x_1, x_2) \in f^{-1}(U_1) \times f^{-1}(U_2) \subseteq (X \times X) - N_f$. So N_f is closed in $X \times X$.

In particular, if we consider the special case of the identic map, then we can argue in a similar way and conclude that Y is Hausdorif if and only if $\mathrm{diag}(Y)$ is a closed subset of $Y \times Y$. We can actually characterize a Hausdorff space by the closure of its nucleus.

Exercise 8.14 (See [8], Exercise 8.14(d))**.** Let $f : X \to Y$ be a map which is continuous, open and onto. Then Y is a Hausdorff if and only if N_f is closed.

Solution. If Y is Hausdorff, then N_f is a closed subset of $X \times X$ by what we have seen before. Viceversa, if N_f is a closed subset of $X \times X$ and $(x_1, x_2) \in (X \times X) - N_f$, then there are open sets $U_1, U_2 \subset X$ such that $(x_1, x_2) \in U_1 \times U_2 \subseteq (X \times X) - N_f$. So $x_1 \neq x_2$ and $x_1 \notin U_2$, which means $f(x_2) \in f(U_2) \subseteq Y - \{f(x_1)\}$, where $f(U_2)$ is open in Y. So $f(x_1)$ is closed in Y and since f is onto this is true for all elements in Y, thus Y is Hausdorff.

The behaviour of the property of being Hausdorff is illustrated in the next exercise for quotients.

Exercise 8.15 (See [8], Exercise 8.14(e))**.** Let X be a compact Hausdorff space and let Y be a quotient space determined by a map $f : X \to Y$. Then Y is Hausdorff if and only if f is a closed map. In particular, Y is Hausdorff if and only if N_f is closed.

Solution. If Y is Hausdorff and C closed of X, then C is compact and $f(C)$ is compact in Y. Note in fact that continuous images of compacts are compacts. This means $f(C)$ is closed since Y is Hausdorff. So f is closed. Viceversa, if f is closed, the space Y, as quotient of X, is completely determined by the surjective mapping $f : X \to Y$, so if X is compact Hausdorff and f is closed, so is Y. The rest is clear.

An application of the previous results, we show that the top half of S^n can be looked at as a disc by "flattening".

Exercise 8.16 (See [8], Exercise 8.14(h))**.** The projection

$$f : (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mapsto f(x_1, x_2, ..., x_{n+1}) = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$$

induces a homeomorphism from the half-sphere of dimension n

$$S^n_+ = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1, x_{n+1} \geq 0\} \subseteq S^n \subseteq \mathbb{R}^{n+1}$$

and the closed disk $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Solution. Let $f_S : S^n_+ \to D^n$ be induced by f and $x \in D^n$. Then

$$\sqrt{\|x\|} = x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1 \Rightarrow 0 \leq 1 - (x_1^2 + x_2^2 + \cdots + x_n^2).$$

If $1 - (x_1^2 + x_2^2 + \cdots + x_n^2) = a$, then $(x_1, x_2, ..., x_n, \sqrt{a}) \in\in S^n_+$ so f_S is surjective.

On the othe hand, if $(x_1, x_2, ..., x_n, x_{n+1}), (y_1, y_2, ..., y_n, y_{n+1}) \in D^n$, then

$$x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1 \Rightarrow x_{n+1} = \sqrt{1 - (x_1^2 + x_2^2 + \cdots + x_n^2)}$$

$$y_1^2 + y_2^2 + \cdots + y_n^2 + y_{n+1}^2 = 1 \Rightarrow y_{n+1} = \sqrt{1 - (y_1^2 + y_2^2 + \cdots + y_n^2)}$$

and $f_S((x_1, x_2, ..., x_n, x_{n+1})) = f_S((y_1, y_2, ..., y_n, y_{n+1}))$ implies $x_{n+1} = y_{n+1}$, so $(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$ and f_S is injective.

Of course f is continuous because it is a projection and its restriction f_S is also continuous with the corresponding induced topologies on domain and codomain. Note that Exercise 9.12 applies to f_S. In fact S^n is compact Hausdorff and S^n_+ is compact Hausdorff, because closed in a compact space. Moreover D_n is manifesfly Hausdorff and $S^n_+/N_{f_S} \sim D^n$, so D_n can be regarded as quotient space of S^n_+ via f_S. Then f_S is closed continuous bijective, so it is homeomorphism.

The methods above allow us to regard S^n in a different perspective.

Exercise 8.17 (See [8], Exercise 8.14(k))**.** There are homeomorphisms such that

$$S^n \cong (\mathbb{R}^n)^\infty \cong D^n/S^{n-1} \cong I^n/\partial I^n.$$

Solution. The homemomorphism between S^n and $(\mathbb{R}^n)^\infty$ has been described at the end of Chapter 5 when $n = 1$. We must indeed consider the stereographic projection φ of Example 5.7 and extending it, sending the northern pole $(0, 0, \ldots, 0, 1)$ into $\{\infty\}$. The fact that this realizes a homeomorphism does not need to be rechecked. The homeomorphism h between D^n and I^n has been seen in Exercise 7.14, so it does not need to be rechecked here. Of course, h restricts to a homeomorphism on the respective boundaries, so the quotient space D^n/S^{n-1}, realized by the identification of points of D^n modulo $\partial D^n = S^{n-1}$, is clearly homemomorphic to the points of I^n, identified modulo ∂I^n. Finally, we consider the function

$$k : [(x_1, \ldots, x_n)]_{S^{n-1}} \in D^n/S^{n-1} \mapsto k([x_1, \ldots x_n]_{S^{n-1}}) \in (\mathbb{R}^n)^\infty,$$

defined by

$$k([x_1, \ldots x_n]_{S^{n-1}}) = \begin{cases} (x_1, \ldots, x_n) & \text{if } (x_1, \ldots, x_n) \in D^n - S^{n-1} \\ \infty & \text{if } (x_1, \ldots, x_n) \in S^{n-1}. \end{cases}$$

Of course, two equivalent points in D^n/S^{n-1} are characterized to be either in $\mathring{D}^n$, in wich case they have the same components, or to belong to the boundary S^{n-1}, in which case they are mapped onto ∞. This shows that h is well defined and injective. The fact that h is surjective is clear by construction. Therefore h is a bijective function. In particular, we can endow $(\mathbb{R}^n)^\infty$ with the quotient topology induced by h, and, since D^n/S^{n-1} is compact and $(\mathbb{R}^n)^\infty$ Hausdorff, we may conclude that h is a closed map. This shows that h is homeomorphism by Exercise 9.12.

Due to the importance of Exercise 9.12, we may generalize it.

Exercise 8.18 (See [8], Exercise 8.14(l))**.** Let Y be the quotient space of X determined by the surjective mapping $f : X \to Y$. Suppose that X is a Hausdorff space, f is a closed mapping and $f^{-1}(y)$ is compact for all $y \in Y$. Then Y is a Hausdorff space.

Solution. Let $y_1, y_2 \in Y$ and $a, b \in X$ such that $a \in f^{-1}(y_1)$ and $b \in f^{-1}(y_2)$. Since X is Hausdorff, there are U_a, U_b opens in X such that $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \emptyset$. Now $U_a \subseteq f^{-1}(y_1)$ and $U_b \subseteq f^{-1}(y_2)$. Since f is closed, $W_1 = Y - f(X - U_a)$ and $W_2 = Y - f(X - U_b)$ are opens in Y. Noting that $f(X) - f(U) \subseteq f(X - U))$, we get

$$\begin{aligned} W_1 \cap W_2 &= (Y - f(X - U_a)) \cap (Y - f(X - U_b)) = Y - (f(X - U_a) \cup f(X - U_b)) \\ &\subseteq Y - \Big((f(X) - f(U_a)) \cup (f(X) - f(U_b))\Big) = Y - \Big(f(X) - (f(U_a) \cap f(U_b))\Big) \\ &= Y - \Big(Y - (f(U_a) \cap f(U_b))\Big) = f(U_a) \cap f(U_b) = \emptyset \end{aligned}$$

which implies that Y is Hausdorff.

9. Connected spaces

We devote this chapter to the notion of connected topological space. It is a fundamental notion in algebraic topology, as we will see later on.

Definition 9.1 (See [8], Definition 9.1)**.** A topological space X is connected if the only subsets of X which are both open and closed are $\emptyset$ and X. A subset of X is connected if it is connected as a space with the induced topology.

A first important property of connected spaces is that they are still connected under continuous images.

Theorem 9.2. *The image of a connected space under a continuous mapping is connected.*

We offer the argument of [8, Theorem 9.4].

Proof. Suppose that X is connected and $f : X \to Y$ is a continuous surjective map. If U is open and closed in Y then $f^{-1}(U)$ is open and closed in X which means that $f^{-1}(U) = \emptyset$ or X and $U = \emptyset$ or Y. Thus Y is connected. □

Another important property of connected spaces is that they are invariant under unions, provided that there is a common point.

Lemma 9.3. Suppose that $\{Y_j : j \in J\}$ is a collection of connected subsets of a space X. If $\bigcap_{j\in J} Y_j \neq \emptyset$ then $Y = \bigcup_{j\in J} Y_j$ is connected.

Again it is helpful to see a proof of this fact and we report [8, Theorem 9.6].

Proof. Suppose that U is a non-empty open and closed subset of Y. Then $U \cap Y_i \neq \emptyset$ for some $i \in J$ and $U \cap Y_i$ is both open and closed in Y_i. But Y_i is connected so $U \cap Y_i = Y_i$ and hence $Y_i \subseteq U$. The set Y_i intersects every other Y_j, $j \in J$ and so U also intersects every Y_j, $j \in J$. By repeating the argument we deduce that $Y_j \subseteq U$ for all $j \in J$ and hence $U = Y$. □

We show an argument for the invariance under countably many products.

Theorem 9.4. *Let $\{X_n : n \in \mathbb{N}\}$ be a collection of topological spaces. Then $\Pi_{n\in\mathbb{N}} X_n$ is connected if and only if X_n are connected for all $n \in \mathbb{N}$.*

Proof. If the product space is connected, then the factors are connected since we may use the projections which are continuous maps and Theorem 9.2.

Viceversa, we will use induction on n. Suppose that X and Y are connected. Since $X \cong X \times \{y\}$ and $Y \cong \{x\} \times Y$ for all $x \in X$, $y \in Y$ we see that $X \times \{y\}$ and $\{x\} \times Y$ are connected. Now $(X \times \{y\}) \cap (\{x\} \times Y) = \{(x, y)\} \neq \emptyset$ and so $(X \times \{y\}) \cup (\{x\} \times Y)$ is connected by Lemma 9.3. We may write

$$X \times Y = \bigcup_{x\in X} ((X \times \{y\}) \cup (\{x\} \times Y))$$

for some fixed $y \in Y$. Since $\bigcap_{x\in X} ((X \times \{y\}) \cup (\{x\} \times Y)) \neq \emptyset$ we deduce that $X \times Y$ is connected. Conversely, suppose that $X \times Y$ is connected. That X and Y are connected follows from Lemma 9.3 and the fact that $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous surjective maps.

Now X_1 and X_2 connected imply $X_1 \times X_2$ connected. Assume true the result for $n = k$ and consider $n \leq k+1$. We have $\Pi_{n \leq k+1} X_n \cong \Pi_{n \leq k} \times X_{k+1}$ which is connected by what we have seen before. Therefore the result follows.

□

Of course, one can note that the notion of being path connected is equivalent to the abstract notion of being connected in $\mathbb{R}^n$. This may help the intuition when we want to figure out connected and disconnected subsets of $\mathbb{R}^n$. We report some examples of connected and not connected subsets of spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance, $\{(x, y) \ : \ x^2 + y^2 \neq 1\} = \mathbb{R}^2 - S^1$ is disconnected because we cannot join with a path a point in $\mathring{D}^2$ with a point out from $\mathring{D}^2$ keeping the path in $\mathbb{R}^2 - S^1$. For the same reason, $\{(x, y, z) \ : \ x \neq 1\} = \mathbb{R}^3 - A$ is disconnected, where A is the plane of $\mathbb{R}^3$ of equation $x = 1$. Even here we may split $\mathbb{R}^3$ in two regions B and C in such a way that it is impossible to join with a path a point in B with a point in C, keeping the path in $\mathbb{R}^3 - A$. Similarly, one can note that $\mathbb{R} - \{x\}$ is disconnected, while $\mathbb{R}^2 - F$ is connected for any $F \subseteq \mathbb{R}^2$ of $|F| \leq |\mathbb{N}|$. In general, one can note that points disconnect $\mathbb{R}$, points do not disconnect $\mathbb{R}^2$ (even in a countable number), points and lines do not disconnect $\mathbb{R}^3$ (even in a countable number), but planes disconnect $\mathbb{R}^3$, and so on, if we regard $\mathbb{R}^n$ as vector space of dimension n and we put the usual topology on it, then it is possible to note that any vector subspace of dimension k of $\mathbb{R}^n$ (and actually a countable number of them) does not disconnect $\mathbb{R}^n$ when $k \in \{0, 1, 2, \ldots, n-2\}$, but for $k = n-1$ we get hyperplanes and they always disconnect $\mathbb{R}^n$.

We provide a direct proof for the connectedness of familiar spaces.

Exercise 9.5 (See [8], Exercise 9.8(h))**.** The punctured space $\mathbb{R}^{n+1} - \{0\}$ is connected, if $n \geq 1$. Moreover, S^n and $\mathbb{RP}^n$ are connected for $n \geq 1$.

Solution. Let $Y_{1,1} = (0, \infty) \times \mathbb{R} \times \cdots \times \mathbb{R}$, $Y_{1,2} = (-\infty, 0) \times \mathbb{R} \times \cdots \times \mathbb{R}$, $Y_{2,1} = \mathbb{R} \times (0, \infty) \times \cdots \times \mathbb{R}$, $Y_{2,2} = \mathbb{R} \times (-\infty, 0) \times \cdots \times \mathbb{R}$ and so on, until $Y_{n+1,1} = \mathbb{R} \times \cdots \times \mathbb{R} \times (0, \infty)$ and $Y_{n+1,2} = \mathbb{R} \times \cdots \times \mathbb{R} \times (-\infty, 0)$. All of them are connected subsets of $\mathbb{R}^{n+1}$, because they are direct products of connected spaces. Denoting $\mathbf{0} = (0, 0, ..., 0)$ and a generic element of these sequence as $Y_{j,k}$ with $j \in J = \{1, 2, ..., n+1\}$ and $k \in \{1, 2\}$, we note that $\mathbf{0} \in Y_{j,k}$ for all j and k, but, more important, we have

$$\bigcup_{\substack{k=1,2 \\ j=1,2,\ldots,n+1}} Y_{j,k} \ = \mathbb{R}^{n+1} - \{\mathbf{0}\}$$

and so $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ is realized as union of connected sets, which are two by two not disjoint. This fact allows us to conclude that $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ is connected. We consider the inverse of the stereographic projection, in the form of the function ψ in Exercise 5.7, but this time we restrict the domain and extend the codomain, that is, we consider ψ in the following form:

$$\psi : x \in \mathbb{R}^{n+1} - \{\mathbf{0}\} \longmapsto \psi(x) = \frac{x}{\|x\|} \in S^n.$$

Of course, ψ is surjective and continuous, so S^n is connected, being a continuous image of a connected space. On the other hand, $\mathbb{RP}^n$ may be realized as the quotient space under the continuous action of the multiplicative group $G = \{-1, 1\}$ with two elements on S^n via $(\pm 1, x) \in G \times S^n \mapsto \pm x \in S^n$. This action induces the natural projection $\pi : x \in S^n \mapsto \pi(x) = \mathrm{Orb}_G(x) \in S^n/G = \mathbb{RP}^n$ which is of course continuous and surjective and allows us to conclude that also $\mathbb{RP}^n$ is connected.

The following exercise gives an example of using algebra and topology simultaneously.

Exercise 9.6 (See [8], Exercise 9.8(k))**.** Let X be a topological space and $\{0,1\}$ the space with two elements with the discrete topology. Define

$$H(X) = \{f : X \to \{0,1\} \mid f \text{ is continuous}\}$$

and for all $f, g \in H(X)$, define $(f+g)(x) = f(x) + g(x) \mod 2$. The function $f+g$ is continuous and $H(X)$ is an abelian group with respect to this operation. Moreover X is connected if and only if $H(X)$ has two elements.

Solution. Of course, the sum of two continuous functions is continuous. The neutral element is the constant function to 0. Therefore one can define $-f(x)$ as that function that we need to get the constant function to 0 for each $f(x)$. It is hence easy to check the axioms of abelian group for $H(X)$. A characterization of connected space is that any continuous function from X to $\{0,1\}$ (as in our situation) must be constant. Therefore X is connected if and only if $H(X)$ reduces to the constant functions to 0 and to 1. This means that $H(X)$ is the group with two elements.

Let's introduce now an important notion in this chapter.

Definition 9.7 (See [8], Page 92)**.** A continuous mapping $f : [0,1] \to X$ is called a path in X. A space X is said to be path connected if given any two points x_0, x_1 in X there is a path in X from x_0 to x_1.

Any space with the trivial topology is path connected, in fact any function whose codomain has the trivial topology is continuous, so we can always take two points and join them with a path.

The next result is called the *gluing lemma* and is a key result for defining operations on functions that will be used in defining the composition of loops in the fundamental group.

Exercise 9.8 (See [8], Exercise 12.10(c))**.** Let W, X be topological spaces and suppose that $W = A \cup B$ with A, B both closed subsets of W. If $f : A \to X$ and $g : B \to X$ are continuous functions such that $f(w) = g(w)$ for all $w \in A \cap B$ then $h : W \to X$ defined by

$$h(w) = \begin{cases} f(w) & \text{if } w \in A; \\ g(w) & \text{if } w \in B. \end{cases}$$

is continuous.

Solution. Note that h is well defined. Suppose that C is closed of X. Then

$$h^{-1}(C) = h^{-1}(C) \cap (A \cup B) = (h^{-1}(C) \cap A) \cup (h^{-1}(C) \cap B) = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous, $f^{-1}(C)$ is closed in A and hence in W since A is closed in W. Similarly $g^{-1}(C)$ is closed in W. Hence $h^{-1}(C)$ is closed in W and h is continuous.

Now we illustrate a situation which makes non-intuitive the notion of being connected. There are in fact connected space which are not path connectedness.

Exercise 9.9 (See [8], Exercise 12.10(d))**.** Let $X = A \cup B$ be the subspace of $\mathbb{R}^2$ where

$$A = \{(x, y) : x = 0, -1 \leq y \leq 1\} \text{ and } B = \{(x, y) : 0 < x \leq 1, y = \cos(\pi/x)\}.$$

The space X is connected but not path connected.

Solution. First note that A is connected since it is an interval, and B is connected because it is the graph of a continuous function and is therefore homeomorphic to $(0, 1]$ (see Exercises 4.5 and 4.6) which is connected. In addition we can also note that both A and B are path connected.

To show that X is connected, assume that $X = U \cup V$, where U, V are open and closed subsets of X which are disjoint. Assume $a \in A$ is an element of U. Then U is non-empty, and since $U \cap A$ is open and closed, $U \cap A = A$ and we have that $A \subseteq U$. Since U is open in X, $\exists\, \varepsilon > 0$ such that $B_\varepsilon(a) \cap X \subseteq U$ and so $\exists\, 0 < b < \varepsilon$ such that $(b, \cos(x/b)) \in U$. This means that $U \cap B$ is non-empty and both open and closed, therefore $U \cap B = B$, meaning $B \subseteq U$. Hence $X = U$, which means X is connected.

Now we prove X is not path-connected. More specifically, we will show that there is no continuous function $f : [0, 1] \to X$ with $f(0) \in B$ and $f(1) \in A$. This will be a proof by contradiction. Thanks to the path connectedness of A we can extend our path to $f(1) = (0, 1)$. Choose $\varepsilon = \frac{1}{2} > 0$. By continuity, for some small $\delta > 0$ we have $f(t) \in B_\varepsilon((0, 1)) \cap X$ whenever $1 - \delta < t \leq 1$. Consider the image $f((1 - \delta, 1])$, which must be connected since f is continuous and $(1 - \delta, 1]$ is connected. Let $p \in (1 - \delta, 1]$ and $f(p) = (x_0, y_0)$. Consider the composite function $\pi \circ f|_{(1-\delta,1]}$, and $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $(x, y) \mapsto x$ is the natural projection. This function is continuous since π and $f|_{(1-\delta,1]}$ are continuous, so the image of this composite function is a connected subset of $\mathbb{R}$ which contains 0 (the x-coordinate of $f(1)$) and x_0 (the x-coordinate of $f(p)$). But since connected subsets of $\mathbb{R}$ must be intervals, it follows that the set of x-coordinates of points in $f((1 - \delta])$ includes the entire interval $[0, x_0]$. Thus for all $x_1 \in (0, x_0]$ there exists $t_1 \in (1 - \delta, 1]$ such that $f(t_1) = (x_1, \cos(1/x_1))$. In particular, if $x_1 = 1/(2n\pi - \pi)$ for large enough $n \in \mathbb{N}$ then $0 < x_1 < x_0$ yet $\cos(1/x_1) = \cos(-\pi) = -1$. Thus, the point $(1/(2n\pi - \pi), -1)$ has the form $f(t)$ for some $t_0 \in (1 - \delta, 1]$, and hence this point lies within a distance of $1/2$ from the point $(0, 1)$. But that's a contradiction, since the distance from $(1/(2n\pi - \pi), -1)$ to $(0, 1)$ is clearly at least 2 (as is the distance between any point on the line $y = 1$ and any other point on the line $y = -1$).

It is possible to give a more general notion of path connectivity which is local.

Definition 9.10. An open neighbourhood of a point $x \in X$ is an open set U such that $x \in U$. A space X is said to be *locally path connected* if for all $x \in X$ every open neighbourhood of x contains a path connected open neighbourhood of x.

The property of being locally path connected is stable for open subspaces. Moreover $\mathbb{R}^n$ is both path connected and locally path connected, but there are examples of path connected spaces which are not locally path connected spaces.

Exercise 9.11 (See [8], Exercise 12.10(j))**.** If X is locally path connected and $U \subseteq X$ is open in X, then U is locally path connected. Moreover $\mathbb{R}^n$ is locally path connected (and every open subset of $\mathbb{R}^n$ is locally path connected). On the other hand, if X is locally path connected and connected, then X is path connected, (but the converse is false in general).

Solution Let $V \subseteq X$ be an open neighbourhood of $x \in U \subseteq X$, then $U \cap V$ is open in X. So $\exists\ N_x$, path connected, open neighbourhood of x in $U \cap V$. So U with the induced topology would be locally path connected since $V \cap U$ is open in U and $U \cap N_x = N_x$. This shows the first claim.

Now we pass to the second claim. Let U be a neighbourhood of point $x \in \mathbb{R}^n$ then $\exists\ \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Since $B_\varepsilon(x)$ is homeomorphic to $\mathbb{R}^n$, and $B_\varepsilon(x)$ is open, then $B_\varepsilon(x)$ is a path connected, open neighbourhood of x. And hence, by the above, $\mathbb{R}^n$ is locally path connected.

We give an argument to show the third claim. Let $p \in X$ and let F be the subset of X that consists of those points in X that can be joined to p by a path in X. Clearly $p \in F$ and now we claim that F is open. To prove this we need only recognise that X is locally path connected, therefore there exists an open neighbourhood of p, U, such that U is path connected. Therefore $p \in U \subset F$. So F is open. We also claim that F is closed. To see this consider $G = X - F$, this means that G consists of the points in X that are not path connected to p. Let $q \in G$, then since X is locally path connected, there exists an open neighbourhood of q, V, that is path connected. So all points in V cannot have a path to p or else there would exist a path connected p to q. Therefore $q \in V \subseteq G$. So G is open, and therefore F is closed. Since F is non-empty, open and closed in a connected space X, $F = X$. And thus every point in X has a path connecting it to p, meaning every point in X has a path joining it to every other point, so X is path connected.

As an application of the notions of compact topological space and of connected topological space, one can look at varieties and, in particular, at manifolds.

Definition 9.12 (See [8], Definition 11.1)**.** Let n be a non-negative integer. An *n-dimensional manifold* is a Hausdorff space in which each point has an open neighbourhood homeomorphic to the open n-dimensional disc $\mathring{D}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$. Note that $\mathring{D}^n \cong \mathbb{R}^n$, so that we could equally require that each point has a neighbourhood homeomorphic to $\mathbb{R}^n$. For brevity we talk about an *n-manifold.* In particular, compact connected 2-dimensional manifolds are called *surfaces.*

One can see that manifolds are not only connected but indeed path connected.

Exercise 9.13 (See [8], Exercise 12.10(m))**.** Connected n-manifolds are path connected.

Solution. In a n-manifold every point has a neighbourhood that is homeomorphic to $\mathbb{R}^n$ and since $\mathbb{R}^n$ is path connected, an n-manifold is locally path connected and connectes, so path connected by Exercise 9.8.

This is an example that demonstrates that path connectedness does not imply local path connectedness.

Exercise 9.14 (See [8], Exercise 12.10(o))**.** The *Polish space* $Y = A \cup B \cup C$,

$$A = \{(x, y) : x^2 + y^2 = 1, y \geq 0\},\ B = \{(x, y) : -1 \leq x \leq 0, y = 0\},$$

$$C = \Big\{(x, y) : 0 < x \leq 1, y = \frac{1}{2}\sin(\pi/x)\Big\},$$

is path connected but not locally path connected.

Solution. A is half circle which is homeomorphic to $[a,b], a,b \in \mathbb{R}$ by "straightening" and "stretching" or "compressing". B is also clearly homeomorphic to $[-1,0]$. C on the other hand is the graph of a continuous function so it is homeomorphic to $(0,1]$. These sets are all homeomorphic to path connected spaces and are therefore path connected. Since $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, Y is path connected. On the other hand, let $0 < \varepsilon < 1$. Then $B_\varepsilon((0,0)) \cap Y$ is open in Y, and $B_\varepsilon((0,0)) \cap A = \emptyset$, $B_\varepsilon((0,0)) \cap B \neq \emptyset$, $B_\varepsilon((0,0)) \cap C \neq \emptyset$. But since $B \cap C = \emptyset$, $(0,0)$ cannot be connected to points in $B_\varepsilon((0,0)) \cap C$, confirming that Y is not locally path connected.

We want to end this chapter with another technical notion of connectivity, which will be clarified later on with the presence of the fundamental group and of its properties.

Definition 9.15. A topological space X is *semi-locally simply connected* if every point $x \in X$ has a neighborhood U_x with the property that every loop in U (i.e.: a loop is a path $f:[0,1] \to U$ with $f(0) = f(1)$) can be contracted to a single point within X.

The word "contracted" will be formalized with the notion of "homotopy" in the next chapter. For now, let's think at a contractbile space as a space which can be reduced with continuity to a single point (in itself), and let's abuse of terminology for a moment, thinking at a "simply connected space" as "a space which can be contracted to one of its points". The open balls of $\mathbb{R}^n$ are examples of contractible spaces, as suggested by the intution. Now one can see that in the previous definition, U_x need not be simply connected; though every loop in U_x must be contractible within X, the contraction is not required to take place inside of U_x. For this reason, a space can be semi-locally simply connected without being locally simply connected.

Most of the main theorems about covering spaces, including the existence of a universal cover, require a space to be path-connected, locally path-connected, and semi-locally simply connected. In particular, this condition is necessary for a space to have a simply connected covering space. We will formalize these notions.

The *Hawaiian earring* H is the topological space, defined by the union of circles

$$H = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \Big\{(x,y) \in \mathbb{R}^2 \mid \Big(x - \frac{1}{n}\Big)^2 + y^2 = \frac{1}{n^2}\Big\}.$$

The space H can be seen to be homeomorphic to the one-point compactification of the union of a countably infinite family of open intervals. Moreover one can also show that H is a compact and complete metric space. The main reason why we talk about H in this chapter is that H is path connected but neither semilocally simply connected, nor simply connected. Among its interesting properties, one can find that its fundamental group $\pi(H)$ contains the free group $F(X)$ with X set of generators of $|X| = |\mathbb{N}|$ as a proper subgroup. Another very interesting property of $\pi(X)$ is that it embeds into the inverse limit of the free groups F_n with n generators, where the bonding map from F_n to F_{n-1} send each generator in itself, except for the last one which gets to the identity element. Moreover $|\pi(X)| \geq \mathbb{R}|$ and $\pi(H)$ is not a free group. There are several more interesting properties of this group, and recent investigations are mentioned in [1].

10. Homotopy and fundamental group

We begin to formalize the notion of "deformation with continuity" which we have used until here in an intuitive way, or via equivalence relations, or via actions of groups, in different contexts.

Definition 10.1 (See [8], Definitions 13.1, 13.2, 13.5)**.** Two continuous maps $f_0, f_1 : X \to Y$ are said to be *homotopic* if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. When this happens, we briefly write $f_0 \simeq f_1$

Suppose that $A \subseteq X$. We say that f_0 is *homotopic relative* to A, if f_0 is homotopic to f_1 and A is fixed by F in its first variable, that is, $F(a,t) = f_0(a)$ for all $a \in A$ and $t \in [0,1]$. In this case, we write $f_0 \simeq_A f_1$.

In the special case of $[0,1]$ as domain of f_0, f_1 and X as codomain of f_0, f_1 and $\{0,1\} = A$, we say that $f_0 \simeq_{\{0,1\}} f_1$ (or briefly $f_0 \sim f_1$), if there is a continuous map $F : (t,s) \in [0,1] \times [0,1] \to X$ such that $F(t,0) = f_0(t)$, $F(t,1) = f_1(t)$, $F(0,s) = f_0(0)$, $F(1,s) = f_0(1)$ for all $s, t \in [0,1]$. This means that F is deforming the path f_0 with continuity to the path f_1 (with continuity within X), fixing the extremes $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$ along the deformation.

Now we have a formal meaning for the adjective "deformation": a space X is *homotopically equivalent* to a space Y, if there are two continuous functions $f_0 : X \to Y$ and $f_1 : Y \to X$ such that $f_0 \circ f_1 \simeq 1_Y$ and $f_1 \circ f_0 \simeq 1_X$. The adjective "contracted" has now a formal meaning: if X is homotopically equivalent to a point, we say that X *is contractible* (or that X can be contracted to a point). A cone and a cylinder are clear examples of contractible spaces.

Exercise 10.2 (See [8], Exercise 13.4 (a))**.** Let X be a space, $f : S^1 \to X$ a continuous map. Then f is *nullhomotopic* (i.e. homotopic to a constant map) if and only if there is a continuous map $g : D^2 \to X$ with $g|S^1 = f$.

Solution. Assume f is nullhomotopic and F the homotopy such that $F(x,0) = c$, where c is a constant map, and $F(x,1) = f(x)$ for all $x \in X$. We may consider the continuous function

$$\pi : s \in D^2 \mapsto \pi(s) = \left(\frac{s}{|s|}, |s|\right) \in S^1 \times [0,1]$$

and define $g : D^2 \to X$ by $g = F \circ \pi$. Now the restriction $g|S^1$ is continuous (because restriction of the composition of two continuous functions) and

$$g|S^1 : x \in S^1 \mapsto g|S^1(x) = F\Big(\frac{x}{|x|}, |x|\Big) = F(x,1) = f(x) \in X.$$

Viceversa, if there is a continuous map $g : D^2 \to X$ with $g|S^1 = f$, then $h : (x,t) \in S^1 \times [0,1] \to h(x,t) = xt \in D^2$ is continuous because multiplication of two continuous functions and $F = g \circ h$ has the desired properties.

It is possible to give more sophisticated notions of deformation, looking at the nature of the fixed points along the homotopy.

Definition 10.3. A subset A of a topological space X is a *retract* of X if there is a continuous map $r : X \to A$, called *retraction* of X onto A, such that $r \circ i = 1_A$, where $i : A \to X$ is the inclusion map.

In the same situation, we say that A is a *retract of deformation* of X, if $i \circ r \sim 1_X$, or, that A is a *strong retract of deformation* of X, if $i \circ r \simeq_A 1_X$, or, finally, that A is a *weak retract of deformation* of X, if $r \circ i \simeq 1_A$.

A classical example of strong retract of deformation is offered by

$$X = C_1 \cup C_2 = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}$$

with the topology induced from the usual topology on $\mathbb{R}^2$. Here $(0, 0)$ is strong retract of deformation of X.

On the other hand, a strong retract of deformation is clearly a retract of deformation, but the viceversa is false. The subset $S^1 \times \{(z_0, t_0)\}$ is a retract of $S^1 \times S^1$ via $r : ((x, y), (z, t)) \in S^1 \times S^1 \mapsto ((x, y), (z_0, t_0)) \in S^1 \times \{(z_0, t_0)\}$, which act as projection on the first factor, but $S^1 \times \{(z_0, t_0)\}$ is not a strong retract of deformation of $S^1 \times S^1$. The following exercise gives us another example of homotopically equivalent spaces via the notion of retract of deformation.

Exercise 10.4 (See [8], Exercise 13.10 (a))**.** The Mobius strip and the cylinder are homotopically equivalent.

Solution. First of all, we need to note that two homeomorphic spaces are homotopically equivalent (while the viceversa is not true in general). Let $M = ([0, 1] \times [0, 1])/\sim$, where $\sim$ the relation $(x, y) \sim (x', y')$ if and only if $(x, y) = (x', y')$, or $\{x, x'\} = \{0, 1\}$, and $y = 1 - y'$. This space M denotes the Mobius strip. The subset $A = \{[(x, y) \in M \ : \ y = 0.5\}$ of M is homeomorphic to $[0, 1]$ and it turns out to be homeomorphic to S^1 (by one of the exercises that we have seen previously). Now the functions $r : [(x, y)] \in M \mapsto [(x, 0.5)] \in A$ and $i : [(x, 0.5)] \in A \mapsto [(x, 0.5)] \in M$ satisfy the condition $r \circ i = 1_A$. On the other hand,

$$F : ([(x, y)], t) \in M \times [0, 1] \mapsto [(x, 0.5t + (1 - t)y)] \in M$$

is an homotopy between 1_A and 1_X since

$$F([(x, y)], 0) = [(x, 0.5 \cdot 0 + (1 - 0)y)] = [(x, (1)y)] = [(x, y)]$$

$$F([(x, y)], 1) = [(x, 0.5 \cdot 1 + (1 - 1)y)] = [(x, 0.5 + (0)y)] = [(x, 0.5)]$$

$$F([(x, 0.5)], t) = [(x, 0.5t + (1 - t) \cdot 0.5)] = [(x, 0.5t + 0.5 - 0.5t)] = [(x, 0.5)]$$

Actually we have shown that A is a deformation retract of M, but A is homemomorphic to S^1, so this means that S^1 is a deformation retract of M. Since S^1 is a deformation retract of the cylinder, we may conclude that S^1 is a deformation retract of both the cylinder and the Mobius strip. This means that the cylinder and the Moebius strip have the same homotopy type of the circle, and, by transitivity, the cylinder and the Moebius strip must have the same homotopy type.

One way is to check that a space X is contractible if and only if the identity map $1 : X \to X$ is homotopic to a constant map. The proof of this fact is elementary and involves just the way in which one has to write the homotopy.

We come back to the notion of connectivity that we have seen in previous chapters, in order to see that it is invariant under homotopies.

Exercise 10.5 (See [8], Exercise 13.10 (d))**.** If X is connected and homotopically equivalent to Y, then Y is also connected.

Solution. Since X and Y are homotopically equivalent, $\exists\ f : X \to Y$ and $g : Y \to X$. Let $Y = U \cup V$ as disjoint union of open and closed sets. Then $X = f^{-1}(U) \cup f^{-1}(V)$ is also a disjoint union of open and closed sets. As X is connected this implies that one of these is the empty set, say $f^{-1}(U)$. Then $f \circ g(Y) \subseteq V$ (since $f^{-1}(V) = X$), and since $f \circ g \simeq 1 : Y \to Y$ via a homotopy, say F to Y. That is $F(Y, 0) \subseteq V$, while $F(Y, 1) = Y$. We find that for any $y \in Y$, there is a path (we get by varying only the second component of the homotopy) to $f(g(y)) \in V$. So every point y in Y is path connected to V, so it follows then that y cannot be in U, that is $U = \emptyset$. We conclude that Y is connected.

An interesting example is the following.

Exercise 10.6 (See [8], Exercise 13.10 (e))**.** A retract of deformation is obviously also a weak retract deformation but the converse is not true in general and the following example shows this fact.

Solution. We consider an example, reported from [16]. Let

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, x = 0 \text{ or } x = 1/n, n \in \mathbb{N}\} \\ &\quad \cup \{(x,y) \in \mathbb{R}^2 : y = 0, 0 \le x \le 1\} \\ &= \Big(\bigcup_{n=1}^{\infty} \Big\{\frac{1}{n}\Big\} \times [0,1]\Big) \cup ([0,1] \times \{0\}) \end{aligned}$$

be a comb space, and let $X = [0, 1] \times [0, 1]$.

Then A is a weak retract of X because both A and X are contractible. We show that A is not a retract of X. Suppose that there were a retraction $r : X \to A$. Let $x_0 = (0, 1) \in A$. Then $r(x_0) = x_0$. Let $U = B_{1/2}(x_0) \cap A$ be an open neighborhood of x_0. There is an open neighborhood V of x_0 in I^2 such that $r(V) \subseteq U$. Let ε be a small positive number such that $B_\varepsilon(x_0) \cap I^2 \subseteq V$. Since $B_\varepsilon(x_0) \cap I^2$ is path-connected, the image $r(B_\varepsilon(x_0) \cap I^2) \subseteq U$ is path-connected in U. Let $m \neq n$ be positive integers such that $1/m$, $1/n < \varepsilon$. Then $(1/m, 1), (1/n, 1) \in r(B_\varepsilon(x_0) \cap I^2)$ because r is a retraction and so there is a path λ in $r(B_\varepsilon(x_0) \cap I^2) \subseteq U$ joining them. This contracts to $(1/m, 1)$ and $(1/n, 1)$ lie in different path-connected components of U.

We offere another example of retract in a product space.

Exercise 10.7 (See [8], Exercise 13.10 (h))**.** Let A be a subspace of X and let Y be a non-empty topological space. Then $A \times Y$ is a retract of deformation of $X \times Y$ if and only if A is a retract of deformation of X.

Solution. Assume $A \times Y$ is a retract of $X \times Y$. So we may consider $r : X \to Y \to A \times Y$ such that $r \circ i_{A\times Y} = 1_{A\times Y}$, with $i_{A\times Y} : A \times Y \to X \times Y$ inclusion map. Since X is homeomorphic to $X \times \{y_0\}$ via an homeomorphism g, we may consider the natural projection $\pi : X \times Y \to X$ on the first factor and i_A the inclusion map of A into X, getting $r_1 = \pi \circ r \circ g$ from X to A. This function is continuous and $(r_1 \circ i_A)(a) = \pi(r(g(i_A(a)))) = \pi(r(g(a))) = \pi(r(a, y_0)) = \pi(a, y_0) = a$, hence $r_1 \circ i_A = 1_A$ and A is a retract of deformation of X.

Viceversa, if A is a retract of X and $r : X \to A$ the retraction such that $r \circ i_A = 1_A$, where $i_A : A \to A$ is the inclusion map, then $A \times Y \subseteq X \times Y$. We get $r_1 : (x, y) \in X \times Y \mapsto (r(x), y) \in A \times Y$ as required retraction and, considering the inclusion map $i_1 : A \times Y \to X \times Y$, we have indeed $r_1 \circ i_1 = 1_{A\times Y}$. So $A \times Y$ is a retract of $X \times Y$.

This exercises is a proof of the transitivity of the relation "is a retract of" in the category of all topological spaces with corresponding homomorphisms. We have already used this fact at an intuitive level in the final part of Exercise 10.4.

Exercise 10.8 (See [8], Exercise 13.10 (i))**.** The relation "is a retract of" is an equivalence relation in the category of topological spaces.

Solution. Of course, the aforementioned relation is reflexive and symmetric. Let's show that it is transitive, that is, if A is a retract of B and B is a retract of C, then A is a retract of C. If A is a retract of B so $A \subseteq B$, there are a retraction $r_1 : B \to A$ and an inclusion $i_1 : A \to B$ such that $r_1 \circ i_1 = 1_A$. The same happens for B retract of C via $r_2 : C \to B$ and $i_2 : B \to C$ such that $r_2 \circ i_2 = 1_B$. Hence $A \subseteq B \subseteq C$ and $r_3 = r_1 \circ r_2 : C \to A$ and $i_3 = i_2 \circ i_1$ realize the definition of retracted of deformation A of C.

This illustrates how the circle is a deformation retract of the punctured plane.

Exercise 10.9 (See [8], Exercise 13.10 (k))**.** Given $x_0 \in \mathbb{R}^2$, S^1 is a strong deformation retract of $\mathbb{R}^2 - \{x_0\}$.

Solution. The circle $\|x - x_0\| = 1$ is translated from the unit circle $\partial B_1(0)$ with center at the origin and radius one, shifting the origin in x_0, and let $\partial B_1(x_0)$ be this translated circle. The functions

$$r : x \in \mathbb{R}^2 \mapsto \frac{x - x_0}{\|x - x_0\|} + x_0 \in \partial B_1(x_0) \text{ and } i : y \in \partial B_1(x_0) \mapsto y \in \mathbb{R}^2$$

are continuous and such that $(r \circ i)(y) = r(y)$ so that

$$F(x,t) = t\left(\frac{x - x_0}{\|x - x_0\|} + x_0\right) + (1-t)x$$

is the homotopy needed. In fact for all $a \in \partial B_1(x_0)$ and $x \in \mathbb{R}^2$

$$F(x,0) = 0\left(\frac{x - x_0}{\|x - x_0\|} + x_0\right) + (1-0)x = x,$$

$$F(x,1) = 1\left(\frac{x - x_0}{\|x - x_0\|} + x_0\right) + (1-1)x = \frac{x - x_0}{\|x - x_0\|} + x_0,$$

$$\begin{aligned} F(a,t) &= t\left(\frac{a - x_0}{\|a - x_0\|} + x_0\right) + (1-t)a = t\left(\frac{a - x_0}{1} + x_0\right) + (1-t)a \\ &= t(a - x_0 + x_0) + (1-t)a = ta + (1-t)a = a. \end{aligned}$$

Another useful example of a strong deformation retract is the following.

Exercise 10.10 (See [8], Exercise 13.10 (l))**.** Let T be a torus, and X the complement of a point in T. Then there is a subset Y of X, homeomorphic to

$$C_1 \cup C_2 = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\},$$

and Y is a strong deformation retract of X.

Solution. We consider the torus as $M = [-1,1] \times [-1,1]$ with the relation $\sim$ given by $(x,y) \sim (x',y')$ if and only iff $(x,y) = (x',y')$ or $\{x,x'\} = \{-1,1\}$ and $y = y'$ or $x = x'$ and $\{y,y'\} = \{-1,1\}$. In other words, $T = M/_{\sim}$. Note that ∂T is homeomorphic to $C_1 \cup C_2$. We will show that this subset is a strong deformation retract of $X = T - \{[(0,0)]\}$. In order to see this, it is enough to consider $F : T \times [0,1] \to T$, defined by

$$F([(x,y)],t) = [((1-t)x,(1-t)y)] + \frac{[(tx,ty)]}{\max\{|x|,|y|\}},$$

which is clearly well defined and continuous. Moreover, $F([(x,y)],0) = [(x,y)]$, $F([(x,y)],1) = \frac{[(x,y)]}{\max\{|x|,|y|\}}$ is on the boundary of T and $F([(x,y)],t) = [((1-t)x,(1-t)y)] + [(tx,ty)] = [(x,y)]$ for all $[(x,y)] \in \partial T$, so the image of the points of ∂T in $T - (0,0)$ form a strong deformation retract.

The following shows the role of being Hausdorff in the retracts.

Exercise 10.11 (See [8], Exercise 13.10 (n))**.** A retract of a Hausdorff space must be a closed subset.

Solution. Let $x \in X - A$, since r is a retraction onto A we have that $r(x) \in A$, and so $r(x) \neq x$. Since X is Hausdorff $\exists\ V, W$ disjoint open sets containing x and $r(x)$ respectively. Since r is continuous it follows that $r^{-1}(W \cap A)$ is open ($W \cap A$ is open in A) and since $r(x) \in W$. We have that $x \in r^{-1}(W \cap A)$, so $U = V \cap r^{-1}(W \cap A)$ is also an open neighbourhood of x. If $y \in U$ then $r(y) \in W$ and since V and W are disjoint we have that $r(y) \notin V$, but since $y \in V$ it follows that $y \neq r(y)$, and so $y \notin A$ since otherwise $r(y) = y$. Therefore U is an open neighbourhood of x which is disjoint from A.

Two continuous maps in $\mathbb{R}^n$ are homotopic, when they are homotopic within subspaces.

Exercise 10.12 (See [8], Exercise 13.10 (o))**.** Let Y be a subspace of $\mathbb{R}^n$ and $f, g : X \to Y$ be two continuous maps. If for each $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight-line segment in Y, then $f \simeq g$. In particular, any two maps $f, g : X \to \mathbb{R}^n$ must be homotopic.

Solution. The straight line segment that joins $f(x)$ to $g(x)$ is the set $\{(1-t)f(x)+tg(x) : t \in [0,1]\}$, and this holds for all $x \in X$. Therefore $F : X \times [0,1] \to Y$ defined by $(x,t) \mapsto F(x,t) = (1-t)f(x) + tg(x)$ is an homotopy. Since we can find a straight line segment joining any two points in $\mathbb{R}^n$, any two continuous functions $f, g : X \to \mathbb{R}^n$ must be homotopic.

We give a technical variation of the idea which is behind a well known theorem of fixed point, due to Brower.

Exercise 10.13 (See [8], Exercise 13.10 (p))**.** Let X be any space and let $f, g : X \to S^n$ be two continuous maps such that $f(x) \neq -g(x)$ for all $x \in X$. Prove that $f \simeq g$. In particular, if $f : X \to S^n$ is a continuous map that is not surjective, then f is homotopic to a constant map.

Solution. Consider the function $F : X \times [01] \to S^n$ be defined by

$$(x,t) \mapsto F(x,t) = \frac{(1-t)f(x) + t(g)}{\|(1-t)f(x) + tg(x)\|}.$$

This is continuous as long as

$$\|(1-t)f(x)+tg(x)\| \neq 0$$

which will be the case for all $(x,t) \in X \times [0,1]$ because

$$\|(1-t)f(x)+tg(x)\| = 0 \iff (1-t)f(x)+tg(x) = 0 \Rightarrow f(x) = \frac{t}{1-t}(-g(x))$$

and this can only happen when

$$\left|\frac{t}{1-t}\right| = 1 \Rightarrow \frac{t}{1-t} = 1 \Rightarrow t = 1-t \Rightarrow 2t = 1$$

that is when

$$f(x) = \frac{0.5}{1-0.5}(-g(x)) = -g(x)$$

but we assumed is never the case for all $x \in X$. In addition, if $f : X \to S^n$ is a continuous map that isn't surjective, then it is homotopic to a constant map $g : X \to S^n$ where $x \mapsto c$, and $f(x) \neq c$ for all $x \in X$.

Because of the gluing lemma, we may define the multiplication between paths.

Lemma 10.14 (See [8], Lemma 12.1)**.** If f and g are two paths in X for which the final point of f coincides with the initial point of g then the function $f*g : [0,1] \to X$ defined by

$$(f*g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a path in X.

We do not repeat the proof of the above lemma, but we stress the fact that the re-parametrazation that we need, in order to define a new path, by gluing two paths such that the final point of the first is equal to the starting point of the other, may be done in a different way. Below there is another possible re-parametrization, which can allow us to define the multiplication of two paths.

Exercise 10.15 (See [8], Exercise 13.4 (c))**.** Let $0 < s < 1$. Given twp paths p and q with $p(1) = q(0)$ in X,

$$h(t) = \begin{cases} p(t/s) & 0 \leq t \leq s; \\ \\ q((t-s)/(1-s)) & s \leq t \leq 1. \end{cases}$$

defines a multiplication path $h \sim p * q$.

Solution. First of all,

$$(p*q)(t) = \begin{cases} p(2t) & 0 \leq t \leq \frac{1}{2}; \\ \\ q(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then we consider

$$F(t,\beta) = \begin{cases} p(\frac{t}{s} * (1-2s\beta)) & 0 \leq t \leq s - \beta(s-\frac{1}{2}); \\ \\ q(\frac{(t-s)}{(1-s)}\left(1-\beta\left(\frac{2st-t+1-2s}{1-s}\right)\right)) & s-\beta(s-\frac{1}{2}) \leq t \leq 1. \end{cases}$$

which is continuous, since it is a composition of continuous functions, and realizes the required homotopy, relative to $\{0,1\}$, since $F(0,\beta) = p(0)$ and $F(1,\beta) = q(1)$ for $\beta \in [0,1]$.

For a path f let $\overline{f}$ be the path given by

$$\overline{f}(t) = f(1-t)$$

for all $t \in [0,1]$. It is easy to see that $f \simeq g$ if and only if $\overline{f} \simeq \overline{g}$. Moreover, if f is a loop, then $f * \overline{f}$ gives the constant path to the base point of the loop, which means that we may look at $\overline{f}$, up to equivalence classes modulo the homotopy relative to $\{0,1\}$, as the inverse elements of f with respect to the operation $*$, which we have just defined. The operation $*$ is in fact compatible with the homotopies of loops. The following exercise gives us an important result for compositions of functions.

Exercise 10.16 (See [8], Exercise 13.4 (e))**.** If f_0 and f_1 are paths from X to Y homotopic relative to A and $g : Y \to Z$ is a continuous map, then $g \circ f_0$ is homotpic relative to A with $g \circ f_1$.

Solution. Let F be a homotopy between f_0, f_1 and $t \in [0,1]$. Then $g \circ F : X \times I \to Z$ realizes an homotopy between $g \circ f_0$ and $g \circ f_1$. Moreover, one can see that this homotopy doesn't depend on t, so it is relative to A.

We expect that composition of functions that are homotopic to one another will preserve the homotopy relation.

Exercise 10.17 (See [8], Exercise 13.4 (f))**.** Suppose that $f_0 \simeq f_1$ from X to Y, and $g_0 \simeq g_1$ from Y to Z. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$, defined from X to Z.

Solution. The logic of solution is very similar to that of the previous exercise, so we omit the details.

What we have seen until now will allow us to consider on the topological space X the set

$$\pi(X, x_0) = \{[f] \mid f : [0,1] \to X \text{ is a loop with basis at } x_0 = f(0) = f(1)\}$$

of equivalence classes

$$[f] = \{g : [0,1] \to X \mid g \sim f \text{ and } g(0) = g(1) = x_0\}$$

of loops which are homotopic to f (relative to $\{0,1\}$), and, using the $*$ product as binary operation, one can endow the set $\pi(X, x_0)$ of the structure of group with neutral element $[\varepsilon_{x_0}]$, where ε_{x_0} denotes the constant loop at x_0, and inverse $\overline{f}$ for each loop f.

In this situation one can consider a continuous map $\varphi : X \to Y$ betwen X and another topological space Y and take $[f], [g] \in \pi(X, x_0)$. The compositions $\varphi \circ f$ and $\varphi \circ g$ turn out to be loops in Y and, if $f \sim g$, then also $\varphi \circ f \sim \varphi \circ g$. The details can be checked easily. This allows us to define the so called *induced homomorphism*, when we have a continuous map like φ, and it is given by

$$\varphi_* : [f] \in \pi(X, x_0) \mapsto \varphi_*([f]) = [\varphi \circ f] \in \pi(Y, \varphi(x_0))$$

which turns out to be a homomorphism of groups with respect to $*$. The next two exercises show that we cannot expect the induced homomorphism to behave like the continuous mapping: in order words we may have injective continuous functions which do not induce necessarily injective induced homomorphisms on the fundamental groups.

Exercise 10.18 (See [8], Exercise 15.11 (a))**.** We give an example of an injective continuous map $\varphi : X \to Y$ for which φ_* is not injective.

Solution. Assume to know that $\pi(S^1, x) \cong \mathbb{Z}$ and that $\pi(D^2, x) = 0$. Let $\varphi = i : S^1 \to D^2$ be the inclusion map. This is injective, continuous map. This induces an homomorphism φ_* from $\pi(S^1, x)$ to $\pi(D^2, x)$, but it cannot be clearly an injective homomorphism of groups, because it would imply $|\varphi_*(\pi(S^1, x))| = |\mathbb{Z}| \leq 1$, which is a contradiction.

In a similar way, we can provide examples of surjective continuous maps which do not induce surjective homomorphisms on the fundamental groups.

Exercise 10.19 (See [8], Exercise 15.11 (b))**.** We give an example of a surjective continuous map $\varphi : X \to Y$ for which φ_* is not surjective.

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the natural projection of the first component of $\mathbb{R}^2$, then $f|_{D^2}$ is continuous and $f|_{D^2}(D^2) = [-1, 1]$, which is homeomorphic to $[0, 1]$, via a homeomorphism say g. We have already seen that it is possible to define a quotient map p from $[0, 1]$ to S^1, identifying the boundary $\{0, 1\}$ of $[0, 1]$ via an appropriate equivalence relation $\sim$ that allow us to regard $[0, 1]/\sim$ as S^1. Then $\varphi = p \circ g \circ f|_{D^2}$ is a surjective continuous map from D^2 to S^1, but $\pi(D^2, x) = 0$ and $\pi(S^1, \varphi(x)) \cong \mathbb{Z}$, and therefore φ_* cannot be surjective.

The real situation when we cannot distinguish two maps that induce the same homomorphism on the fundamental groups is when they are homotopic.

Exercise 10.20 (See [8], Exercise 15.11 (d))**.** Two continuous mappings $\varphi, \psi : X \to Y$, with $\varphi(x_0) = \psi(x_0)$ for some point $x_0 \in X$, induce the same homomorphism from $\pi(X, x_0)$ to $\pi(Y, \varphi(x_0))$, if φ and ψ are homotopic relative to x_0.

Solution. Let f be a closed path in X, then φf and ψf are closed paths in Y. And since φ are ψ are homotopic relative to x_0 then so are $\varphi \circ f$ and $\psi \circ f$. This implies $\varphi_*[f] = [\varphi \circ f] = [\psi \circ f] = \psi_* f$, and since $\varphi(x_0) = \psi(x_0)$, φ, ψ induce the same homomorphism from $\pi(X, x_0)$ to $\pi(Y, \varphi(x_0))$.

Let's see what happens for retractions and inclusions. In this situation, then, yes, surjective (resp., injective) continuous maps induce surjective (resp. injective) homomorphisms on the fundamental groups.

Exercise 10.21 (See [8], Exercise 15.11 (e))**.** Suppose that A is a retract of X with retraction $r : X \to A$. Then $i_* : \pi(A, a) \to \pi(X, a)$ is a monomorphism and $r_* : \pi(X, a) \to \pi(A, a)$ an epimorphism for any point $a \in A$.

Solution. If $[f], [g] \in \pi(A, a)$ and $1_A : A \to A$ is the identity map. Then

$$[f] \neq [g] \Rightarrow [1 \circ f] \neq [1 \circ g] \Rightarrow [r \circ i \circ f] \neq [r \circ i \circ g]$$
$$\Rightarrow r_*([i \circ f]) \neq r_*([i \circ g]) \Rightarrow [i \circ f] \neq [i \circ g] \Rightarrow i_*([f]) \neq i_*([g]).$$

So i_* is a monomorphism. To see that r_* is an epimorphism consider $[f] \in \pi(A, a)$, obviously $i_*([f]) = [i \circ f] \in \pi(X, a)$, and

$$r_*([i \circ f]) = [r \circ i \circ f] = [(r \circ i) \circ f] = [f].$$

So r_* is an epimorphism.

The above exercise suggest the presence of an interesting condition of splitting when we have retracts of deformations

Exercise 10.22 (See [8], Exercise 15.11 (f))**.** With the above notation, we have

$$\pi(X,a) \simeq i_*(\pi(A,a)) \times \ker r_*.$$

Solution. We have the following diagram

$$1 \longrightarrow \pi(A,a) \xrightarrow{i_*} \pi(X,a) \xrightarrow{r_*} \pi(A,a) \longrightarrow 1,$$

where i_* is an injective homomorphism of groups and r_* a surjective homomorphism of groups. Moreover $i_*(\pi(A,a)) = \ker r^*$. Now one can see that there is the first theorem of homomorphism of groups that can be applied and this shows that $\pi(X,a) \simeq i_*(\pi(A,a)) \times \ker r_*$.

This exercise shows how being able to deform a space into another one preserves the structure of the loops.

Exercise 10.23 (See [8], Exercise 15.11 (g))**.** If A is a strong deformation retract of X, then $i_* : \pi(A,a) \to \pi(X,a)$ is an isomorphism for any point $a \in A$.

Solution. We have already seen that i_* is a monomorphism, all that is left is to prove that it is an epimorphism as well. First note that since $i \circ r \simeq_{relA} 1 : X \to X$ (where r is a retraction from X to A), then if f is a closed path in X, $i \circ r \circ f$ is homotopic to f relative to A. So for $[f] \in \pi(X,a)$, $r_*([f]) = [r \circ f] \in \pi(A,a)$ and so $i_*([r \circ f]) = [i \circ r \circ f] = [f]$, as claimed.

There is another example of mappings that can induce isomorphisms: any map which is homotopic to the identic map. The proof of this fact is elementary and involves the way of writing an homotopy between the given map and the identic map. We omit the details, since we have already seen the argument in similar situations.

Let's show that the sphere S^n is always contractible, provided $n \geq 2$.

Exercise 10.24 (See [8], Exercise 15.16 (c))**.** Suppose that $X = U \cup V$ with U, V open and simply connected and $U \cap V$ path connected. Then X is simply connected. In particular, S^n is simply connected for all $n \geq 2$.

Solution. First note that X is path connected. If $U = V$ then X is simply connected. Otherwise let $x \in U$ and $x \notin V$ then $\pi(X,x) = \pi(U,x) = \{\mathbf{1}\}$, and since X is path connected it follows that X is simply connected as well. Let $U = S^n - \{(1,0,...,0)\}$ and $V = S^n - \{(0,...,0,1)\}$, then U and V are homeomorphic to $\mathbb{R}^n, n \geq 2$, which is a contractible space. See Exercise 4.7 for details. Clearly $U \cap V$ is path connected for all $n \geq 2$, therefore S^n is simply connected.

We end this chapter with an important example of algebra and topology, in which the notion of fundamental group is involved.

Definition 10.25. A *topological group* G is a group that is also a topological space in which the multiplication $\mu : (g_1, g_2) \in G \times G \mapsto \mu(g_1, g_2) = g_1 g_2 \in G$ is a continuous map and the inverse $\nu : g \in G \mapsto g^{-1} \in G$ is continuous as well.

Topological groups play an important role in several branches of pure and applied mathematics: from algebraic topology, to group theory, to Lie theory, and finally to Dynamical System.

Exercise 10.26 (See [8], Exercise 15.18 (d))**.** Let f, h be closed paths on a topological group G, based at the identity element of G and define for all $t \in [0,1]$

$$(f \cdot h)(t) = \mu(f(t), h(t)).$$

Then $f * h \sim f \cdot h \sim h * f$ and $\pi(G,1)$ is abelian. Furthermore ν_* and μ_* are homomorphisms.

Solution. A homotopy F between $f * h$ and $f \cdot h$ can be defined as follows:

$$F(t,s) = \begin{cases} f(2t)(1-s) + s\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ h(2t-1)(1-s) + s\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases}$$

F is continuous since it is the gluing of a composition of continuous functions. And

$$\begin{aligned} F(t,0) &= \begin{cases} f(2t)(1-0) + 0\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ h(2t-1)(1-0) + 0\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases} \\ &= \begin{cases} f(2t) & 0 \le t \le \frac{1}{2}; \\ \\ h(2t-1) & \frac{1}{2} \le t \le 1. \end{cases} \\ &= (f * h)(t) \end{aligned}$$

while

$$\begin{aligned} F(t,1) &= \begin{cases} f(2t)(1-1) + 1\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ h(2t-1)(1-1) + 1\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases} \\ &= \begin{cases} \mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ \mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases} \\ &= \mu(f(t),h(t)) = (f \cdot h)(t) \end{aligned}$$

A homotopy G between $f \cdot h$ and $h \cdot f$ can be defined as follows:

$$G(t,s) = \begin{cases} h(2t)s + (1-s)\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ f(2t-1)s + (1-s)\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases}$$

which is also continuous. Moreover

$$\begin{aligned} G(t,0) &= \begin{cases} h(2t)0 + (1-0)\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ f(2t-1)0 + (1-0)\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases} \\ &= \begin{cases} \mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ \mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases} \\ \\ &= \mu(f(t),h(t)) = (f \cdot h)(t) \end{aligned}$$

while

$$G(t,1) = \begin{cases} h(2t)1 + (1-1)\mu(f(t),h(t)) & 0 \le t \le \frac{1}{2}; \\ \\ f(2t-1)1 + (1-1)\mu(f(t),h(t)) & \frac{1}{2} \le t \le 1. \end{cases}$$
$$= \begin{cases} h(2t) & 0 \le t \le \frac{1}{2}; \\ \\ f(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$
$$= (f * h)(t)$$

So let $[f], [h] \in \pi(G, 1)$, keeping in mind that $f \cdot h$ is also a loop based at 1 since $(f \cdot h)(0) = \mu(f(0), h(0)) = f(0)h(0) = 1$ and $(f \cdot h)(1) = \mu(f(1), h(1)) = f(1)h(1) = e$, we get

$$[f] * [h] = [f * h] = [f \cdot h] = [h * f] = [h] * [f]$$

so $\pi(G, 1)$ is abelian.

Finally, it is easy to check that ν_* and μ_* are homomorphisms of groups.

We will endow S^1 with the structure of topological group via the observations we have just made.

Exercise 10.27 (See [8], Exercise 15.18 (e))**.** For $S^1 \subseteq \mathbb{C}$ define $\mu : S^1 \times S^1 \to S^1$ by $\mu(z_1, z_2) = z_1 z_2$, and $\nu : S^1 \to S^1$ by $\nu(z) = z^{-1}$. Then S^1 can be regarded as a topological group and $\pi(S^1, 1)$ is an abelian group.

Solution. Let $e^{\pi i}, e^{\phi i}, e^{\rho i} \in S^1$.

(1) $e^{\pi i} e^{\phi i} = e^{\pi i + \phi i} = e^{(\pi + \phi)i}$

(2) $(e^{\pi i} e^{\phi i}) e^{\rho i} = e^{(\pi+\phi)i} e^{\rho i} = e^{(\pi+\phi)i + \rho i} = e^{((\pi+\phi)+\rho)i} = e^{(\pi+\phi+\rho)i} = e^{(\pi+(\phi+\rho))i}$
$= e^{\pi i + (\phi+\rho)i} = e^{\pi i} e^{(\phi+\rho)i} = e^{\pi i}(e^{\phi i} e^{\rho i})$

(3) $1 e^{\pi i} = e^{0i} e^{\pi i} = e^{(0+\pi)i} = e^{\pi i} = e^{(\pi+0)i} = e^{\pi i} e^{0i} = e^{\pi i} 1$

(4) $e^{-\pi i} e^{\pi i} = e^{(-\pi+\pi)i} = e^{0i} = 1 = e^{0i} = e^{(\pi-\pi)i} = e^{\pi i} e^{-\pi i}$

We already know that S^1 is a topological space, so it is topological group. What we have seen above allows us to conclude that $\pi(S^1, 1)$ is an abelian group.

11. Main notions on the covering spaces

Given two topological spaces $\widetilde{X}$ and X, a continuous surjective map $p\colon \widetilde{X} \to X$ is called a *covering map* if for all $x \in X$, there exists an open neighborhood U of x such that

$$p^{-1}(U) = \bigcup_{i \in I} U_i, \text{ where } U_i \cap U_j = \emptyset \ \forall i \neq j$$

and

$$p|U_i : U_i \to U \text{ is homeomorphism } \forall i \in I.$$

In this situation the space $\widetilde{X}$ is called *a covering space* for X. For any point x,

$$p^{-1}(x) = \{\widetilde{x} \in \widetilde{X} \mid p(\widetilde{x}) = x\}$$

is necessarily a discrete space, called *the fiber over x*. The special open neighborhoods U of x, involved in the definition of covering map, are said to be *evenly covered neighborhoods* and they form an open cover of the space X. Indeed, most of the times, these opens come from a basis which is given a priori on X. The homeomorphic copies in $\widetilde{X}$ of an evenly covered neighborhood U are called the *sheets*, or *leaves*, over U. A covering map has the same behaviour of projection in the sense that there is a homeomorphism, as indicated above. This circumstance is reported as *locally trivial property* of a covering maps. An example is shown below.

Exercise 11.1 (See [8], Exercise 17.9 (c))**.** The following is a covering mapping

$$p : z \in \mathbb{C}^* \mapsto p(z) = z^n \in \mathbb{C}^*$$

Solution. Of course, p is continuous and surjective. Then we need to check only the other property in the definition of covering map. Assume that $p(z) = z^n = r^n e^{(\theta n i)} \in \mathbb{C}^*$ for some $\theta \in [0, 2\pi]$, $r = |z|$ and $n \geq 1$. Then

$$U = \left\{ re^{\phi i} \in \mathbb{C}^* : \frac{1}{2}r^n < r < \frac{3}{2}r^n, \quad \theta n - \frac{\pi}{n} < \phi < \theta n + \frac{\pi}{n} \right\}$$

is an open containing z^n. By De Moivre's formula, if $k \in \{1, 2, ..., n\}$, then

$$z = re^{\theta + \frac{2\pi k}{n}}$$

for every $z \in p^{-1}(z^n)$. So we may consider the opens

$$V_k = \left\{ re^{\phi i} \in \mathbb{C} : \frac{1}{2n}r < r < \frac{3}{2n}r, \ \theta + \frac{2\pi k}{n} - \frac{\pi}{n^2} < \phi < \theta + \frac{2\pi k}{n} + \frac{\pi}{n^2} \right\}$$

such that $p|V_k$ is homeomorphism for all k and $p^{-1}(U) = \bigcup_{k=1}^{n} V_k$. These sets are disjoint because there can be no overlap of the angles for k and $k+1$.

$$\left(\theta + \frac{2\pi(k+1)}{n} - \frac{\pi}{n^2}\right) - \left(\theta + \frac{2\pi k}{n} + \frac{\pi}{n^2}\right) = \frac{2\pi(k+1)}{n} - \frac{\pi}{n^2} - \frac{2\pi k}{n} - \frac{\pi}{n^2}$$

$$= \frac{2\pi}{n} - \frac{2\pi}{n^2} = \frac{2\pi n - 2\pi}{n^2} = \frac{2\pi(n-1)}{n^2}$$

and

$$\left(\theta + \frac{2\pi(k+1)}{n} + \frac{\pi}{n^2}\right) - \left(\theta + \frac{2\pi k}{n} + \frac{\pi}{n^2}\right)$$

$$= \frac{2\pi(k+1)}{n} - \frac{2\pi k}{n} = \frac{2\pi}{n}$$

As one may expect, it is possible to get on the product space of two topological spaces the structure of covering space, if we have covering spaces on the factors a priori.

Exercise 11.2 (See [8], Exercise 17.9 (d))**.** Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering maps. Then $p \times q : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is a covering map, so that $\widetilde{X} \times \widetilde{Y}$ is a product covering space. In particular, if $X = Y$ and

$$\widetilde{W} = \{(\widetilde{x}, \widetilde{y}) \in \widetilde{X} \times \widetilde{X} : p(\widetilde{x}) = q(\widetilde{y})\},$$

then $f : (\widetilde{x}, \widetilde{y}) \in \widetilde{W} \mapsto p(\tilde{x}) \in X$ is a covering map.

Solution. The product function on the product space

$$(p \times q)\ (\widetilde{X} \times \widetilde{Y}) = p(\widetilde{X}) \times q(\widetilde{Y}) = X \times Y$$

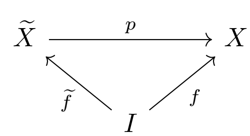
is clearly surjective and continuous, because so are p and q. Let $(x, y) \in X \times Y$, since p and q are covering maps there exists $U \subseteq X$ containing x and $V \subseteq Y$ containing y such that $p^{-1}(U) = \bigcup_{j \in J} U_j$ and $q^{-1}(V) = \bigcup_{k \in K} V_k$, where the U_j are mutually disjoint sets and $p|U_i$ is a homeomorpshim with U, similarly this happens for the sets V_k. Now

$$(p \times q)^{-1}\ (U \times V) = \bigcup_{(j,k) \in J \times K} U_j \times V_k$$

and if $(j, k) \neq (t, s)$, then either $j \neq t$ or $k \neq s$, hence either $U_j \cap U_t = \emptyset$ or $V_k \cap V_s = \emptyset$, in which case $(U_j \times V_k) \cap (U_t \times V_s) = \emptyset$. This means that the above union is made by mutually disjoint sets $U_j \times V_k$. Moreover $(p \times q)(U_j \times V_k) = p(U_j) \times q(V_k) = U \times V$, and this shows that $(p \times q)|(U_j \times V_k)$ is homeomorphism.

Since p is a covering map, $\exists\, U \subseteq X$ open neighbourhood of y such that $p^{-1}(U) = \bigcup_{j \in J} U_j$ open disjoint subsets of $\widetilde{X}$. So $f^{-1}(U) = \bigcup_{j \in J} U_j \times \widetilde{X}$ which is a union of disjoint sets. And $f(U_j \times \widetilde{X}) = p(U_j) = U$.

When $f : t \in I = [0, 1] \mapsto f(t) \in X$ is a path in X, we may consider the diagram



and define the *lifting* $\widetilde{f} : t \in [0, 1] \mapsto \widetilde{f}(t) \in \widetilde{X}$ of f as the continuous map, making commutative this diagram, that is, such that $p \circ \widetilde{f} = f$ and with $p(\widetilde{x}_0) = f(0) = x_0$ the starting point of the path f. Now one can consider relations between the fundamental group on $\widetilde{X}$ and that on X and if there are relations, involving coverings. The answer is positive and there are two fundamental theorems, which are always mentioned in this context.

Theorem 11.3. *(Existence of Liftings). Let $p : \widetilde{X} \to X$ be a covering.*

(i) *Given a path $f : I \to X$ and $a \in \widetilde{X}$ with $p(a) = f(0)$, there is a unique path $\tilde{f} : I \to \widetilde{X}$ such that $p\tilde{f} = f$ and $\tilde{f}(0) = a$.*

(ii) *Given a continuous map* $F : I \times I \to X$ *and* $a \in \widetilde{X}$ *with* $p(a) = F(0,0)$, *there is a unique continuous map* $\widetilde{F} : I \times I \to \widetilde{X}$ *such that* $p\widetilde{F} = F$ *and* $\widetilde{F}(0,0) = a$.

Proof. (i) For each $x \in X$, there is an open neighbourhood U_x of x that is evenly covered by p. By continuity, $f^{-1}(U_x)$ is an open subset of I, so it is a union of relatively open intervals in I; that is, intervals of the form $(a_i, b_i) \cap I$. This means that there is an open cover of I of the form $M = \{(a_j, b_j) \cap I : j \in J\}$, where for all $j \in J$, we have $f((a_j, b_j)) \subseteq U_x$ for some $x \in X$.

We use the compactness of I to obtain a minimal finite subcover M' of M, consisting of intervals:

$$[0, b_0), (a_1, b_1), (a_2, b_2), ..., (a_{n-1}, b_{n-1}), (a_n, 1].$$

By minimal, we mean that if any interval is removed from M', then it ceases to cover I. Therefore, there exists a permutation σ of the set $\{1, ..., n-1\}$ and points $r_1 \in [0, b_0) \cap (a_{\sigma(1)}, b_{\sigma(1)})$, $r_2 \in (a_{\sigma(1)}, b_{\sigma(1)}) \cap (a_{\sigma(2)}, b_{\sigma(2)})$, ..., $r_n \in (a_{\sigma(n-1)}, b_{\sigma(n-1)}) \cap (a_{\sigma(n)}, 1]$ such that $0 = r_0 < r_1 < \cdots < r_n < r_{n+1} = 1$. Once this is done, it is plain to see that each $[r_i, r_{i+1}]$ is contained in an interval in M', so $f([r_i, r_{i+1}]) \subseteq U_x$ for some $x \in X$.

Now we define the liftings $\tilde{f}_k : [0, r_k] \to \widetilde{X}$ inductively for $k = 0, 1, ..., n+1$. For $k = 0$, set $\tilde{f}_0(0) = a$. Suppose that $\tilde{f}_k : [0, r_k] \to \widetilde{X}$ is continuous and uniquely defined such that $p\tilde{f}_k = f|_{[0,r_k]}$. Since $f([r_k, r_{k+1}])$ is contained in some U_x evenly covered by p, we have that $p^{-1}(U_x)$ is a disjoint union of $\{W_j : j \in J\}$, with $p|_{W_j} : W_j \to U_x$ being a homeomorphism for each $j \in J$. Moreover, $\tilde{f}_k(r_k) \in W_\ell$ for some unique $\ell \in J$. Since $[r_k, r_{k+1}]$ is path-connected, it is necessary that $\tilde{f}_{k+1}([r_k, r_{k+1}]) \subseteq W_\ell$. Since $p|_{W_\ell}$ is a homeomorphism, there is a unique map $\theta : [r_k, r_{k+1}] \to W_\ell$ such that $p\theta = f|_{[r_k, r_{k+1}]}$ (In fact, $\theta = (p|_{W_\ell})^{-1} f|_{[r_k, r_{k+1}]}$). We now define $\tilde{f}_{k+1}$ by

$$\tilde{f}_{k+1}(s) = \begin{cases} \tilde{f}_k(s) & \text{if } 0 \leq s \leq r_k; \\ \theta(s) & \text{if } r_k \leq s \leq r_{k+1}. \end{cases}$$

This map is continuous by the gluing lemma, and unique by construction. Moreover, $\tilde{f}(0) = \tilde{f}_0(0) = a$ and $p\tilde{f} = f$. By induction, we obtain $\tilde{f}$.

(ii) Again, for each $x \in X$, there is an open neighbourhood U_x of x that is evenly covered by p. By continuity, $F^{-1}(U_x)$ is an open subset of I^2, so it is a union of relatively open squares $R_i = ((a_i, b_i) \times (a_i, b_i)) \cap I^2$ (All the sets of this form serve as a base for the product topology on $I^2 \subseteq \mathbb{R}^2$). So there is an open cover $CofI^2$ of the form $\{R_j : j \in J\}$ where $R_j = (a_i, b_i)$ and $F(R_j) \subseteq U_x$ for some $x \in X$. By compactness of I^2, there is a minimal finite subcover C' of C, consisting of open squares $R_1, R_2, ..., R_n$.

Now, renaming the R_k if necessary, there exist points $(0,0) = s_0 \in R_1$, $s_1 \in R_1 \cap R_2$, $s_2 \in (R_1 \cup R_2) \cap R_3$, ..., $s_{n-1} \in (\bigcup_{k=1}^{n-1} R_k) \cap R_n$, and $(1,0) = s_n \in R_n$.

We define the liftings $\widetilde{F}_k : \bigcup_{i=0}^{k} R_k \to \widetilde{X}$ inductively for $k = 0, 1, ..., n$, where we understand that $R_0 = \{(0,0)\}$. The main way in which this step differs from part

(i) is that, unlike the intervals $[r_k, r_{k+1}]$, the R_k may overlap. But this is not a problem.

For $k = 0$, set $\widetilde{X}_0(0,0) = a$. Now suppose that $\widetilde{F}_k : \bigcup_{i=0}^{k} R_k \to \widetilde{X}$ is continuous and uniquely defined, such that $p\widetilde{F}_k$ agrees with F on its domain. Recall that $F(R_{k+1})$ is contained in some U_x evenly covered by p, so $p^{-1}(U_x)$ is a disjoint union of $\{W_j : j \in J\}$, with $p|_{W_j} : W_j \to U_x$ being a homeomorphism for each $j \in J$. Moreover, $s_k \in (\bigcup_{i=1}^{k} R_i) \cap R_{k+1}$, so $\widetilde{F}_k(s_k) \in W_\ell$ for some unique $\ell \in J$. Since R_{k+1} is path-connected, it is necessary that $\widetilde{F}_{k+1}(R_{k+1}) \subseteq W_\ell$. Since $p|_{W_\ell}$ is a homeomorphism, there is a unique map $\theta : R_{k+1} \to W_\ell$ such that $p\theta = F|_{R_{k+1}}$. Finally, let

$$\tilde{F}_{k+1}(s) = \begin{cases} \tilde{F}_k(s) & \text{if } s \in \bigcup_{i=0}^{k} R_k; \\ \theta(s) & \text{if } s \in R_{k+1}. \end{cases}$$

This is continuous by the gluing lemma, and unique by construction. By induction, we obtain the lifting $\widetilde{F}$. □

Corollary 11.4. (Monodromy Theorem) Let $p : \widetilde{X} \to X$ be a covering, and suppose that $f, g : I \to X$ are homotopic paths relative to $\{0,1\}$. If $\tilde{f}(0) = \tilde{g}(0)$, then $\tilde{f}(1) = \tilde{g}(1)$.

Proof. Let $F : f \simeq g$ be a homotopy rel $\{0,1\}$. By the previous Theorem of Existence of Liftings, there exists a unique lifting $\widetilde{F} : I^2 \to \widetilde{X}$ with $\widetilde{F}(0,0) = \tilde{f}(0) = \tilde{g}(0)$. It can be shown that $\widetilde{F} : \tilde{f} \simeq \tilde{g}$ is a homotopy re; $\{0,1\}$. Observe that $F(t,0) = f(t)$, so by uniqueness of the lifting, $\widetilde{F}(t,0) = \tilde{f}(t)$ for all $t \in I$. Similarly, $F(t,1) = f(t)$, so by uniqueness of the lifting, $\widetilde{F}(1,s)$ is a path from $\tilde{f}(1)$ to $\tilde{g}(1)$, and $p\widetilde{F} = F$ implies that $\widetilde{F}(1,I) \subseteq p^{-1}(\{f(1)\})$. The point $f(1) = g(1) \in X$ has a neighbourhood U evenly covered by p, so $p^{-1}(U)$ is a disjoint union of open subsets $\{W_j : j \in J\}$ of $\widetilde{X}$, each of which is homeomorphic to U. The restrictions $p|_{W_j}$ are homeomorphisms for all $j \in J$, so whenever $a, b \in p^{-1}(\{f(1)\})$ and $a \neq b$, then there exist open subsets $W_\alpha = W_\beta$ of $\widetilde{X}$ such that $\{a\} = W_\alpha \cap p^{-1}(\{f(1)\})$ and $\{b\} = W_\beta \cap p^{-1}(\{f(1)\})$. In the subspace topology that $p^{-1}(\{f(1)\})$ inherits from $\widetilde{X}$, therefore, every singleton is open (in other words, $p^{-1}(\{f(1)\})$ is a discrete space). The only connected subsets of $p^{-1}(\{f(1)\})$, therefore are singletons. However, the interval I is connected, so $\widetilde{F}(1,I)$ must also be, and it followS that $\widetilde{F}(1,s)$ is constant. In conclusion, $\tilde{f}(1) = \widetilde{F}(1,0) = \widetilde{F}(1,1) = \tilde{g}(1)$. □

Now we can report some facts, which are useful to know once we gave the above notions. In a path connected space X with covering space $\widetilde{X}$ and covering map $p\colon \widetilde{X} \to X$ such that $p(\widetilde{x}_0) = \widetilde{x}_0$, the induced homomorphism

$$p_* : [\widetilde{f}] \in \pi(\widetilde{X}, \widetilde{x}_0) \to p_*([\widetilde{f}]) = [p \circ \widetilde{f}] \in \pi(X, x_0)$$

turns out to be (well defined and) injective. Therefore, the first theorem of homomorphism of groups shows that

$$\frac{\pi(\widetilde{X}, \widetilde{x}_0)}{\ker p_*} \simeq p_*(\pi(\widetilde{X}, \widetilde{x}_0))$$

$$= \{[\gamma] \ : \ \gamma_{\widetilde{X}} \ \text{ is a closed curve in } \ \widetilde{X} \ \text{ passing through } \ \widetilde{x}_0\}$$

and so a covering map p defines a conjugacy class of subgroups of $\pi(X, x_0)$ in such a way that equivalent covers of X define the same conjugacy class of subgroups of $\pi(X, x)$. In particular, one can see that if $\widetilde{X}$ and X are path-connected,

$$|p^{-1}(x_0)| = |\pi(X, x_0) : p_*(\pi(\widetilde{X}, \widetilde{x}_0))|.$$

A significant amount of covering maps may originate from properly discontinuous actions on path connectes spaces, where discrete groups are acting. We have seen several examples in the previous chapters. In particular, all we have said until now can be repeated when we replace the space $\widetilde{X}$ with the quotient space X/G of X under the action of G. Therefore, most of the times, looking for the fundamental group of a given topological space may reduced to looking for suitable quotients of spaces of orbits, provindg appropriate actions of discrete groups. We give and idea via the following example.

Exercise 11.5 (See [8], Exercise 17.9 (e))**.** Let $a : \mathbb{C} \to \mathbb{C}$ and $b : \mathbb{C} \to \mathbb{C}$ be the homeomorphisms of the complex plane $\mathbb{C}$ defined by $az = z + i$, $bz = \overline{z} + \frac{1}{2} + i$. Then $ba = a^{-1}b$ and

$$G = \{a^m b^{2n} b^{\varepsilon} \ : \ m, n \in \mathbb{Z}, \ \varepsilon \in \{0, 1\}\}$$

is a group of homeomorphisms of $\mathbb{C}$. Furthermore, the action of G is properly discontinuous on $\mathbb{C}$ and $\mathbb{C}/G$ is Hausdorff.

Solution. In order to check that G is not commutative, that is, $ba = a^{-1}b$ is true, we note that

$$baz = b(az) = b(z + i) = \overline{(z + i)} + \frac{1}{2} + i = \overline{z} - i + \frac{1}{2} + i = \overline{z} + \frac{1}{2}$$

while

$$a^{-1}bz = a^{-1}(bz) = a^{-1}(\overline{z} + \frac{1}{2} + i) = \overline{z} + \frac{1}{2} + i - i = \overline{z} + \frac{1}{2}.$$

Of course, G is given by homeomorphisms of $\mathbb{C}$ and is note hard to note that it is a discrete group and indeed G acts in a properly discontinuous way on $\mathbb{C}$. In particular, its actions preserve the property of being Hausdorff.

An important result of the covering space is reprhased below in an intuitive way:

Theorem 11.6. *There is a bijective map between equivalence classes of path-connected covers of X and the conjugacy classes of subgroups of the fundamental group $\pi(X, x)$.*

The main step in proving this result is establishing the existence of a *universal cover*, that is a cover corresponding to the trivial subgroup of $\pi(X, x)$. Once the existence of a universal cover $\widetilde{X}$ of X is established, if $H \leq \pi(X, x)$ is an arbitrary subgroup, the quotient space $\widetilde{X}/H$ may be regarded as the covering of X corresponding to H and so the bijective map may be constructed. In order to prove the theorem on the existence of a universal cover, one has to involve the notion of semi-locally simply connected space, since one can see that:

Theorem 11.7 (Theorem of existence of the universal cover)**.** *Assume X is a connected and locally path-connected space. Then X admits a universal cover if and only if X is semi-locally simply connected.*

The proof of the above result is quite technical and can be found in [8, Theorem 22.1]. We report some useful applications of the notions we have seen here.

12. Recent problems in geometric group theory

In the language of combinatorial group theory, if X is a topological space; U and V are open, path connected subspaces of X; $U \cap V$ is nonempty and path-connected; $w \in U \cap V$; then the natural inclusions $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ for the following commutive diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{i_1} & U \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ V & \xrightarrow{j_2} & X \end{array}$$

induce another commutative diagram on the corresponding fundamental groups, given by

$$\begin{array}{ccc} \pi(U \cap V, w) & \xrightarrow{(i_1)_*} & \pi(U, w) \\ {\scriptstyle (i_2)_*}\downarrow & & \downarrow{\scriptstyle (j_1)_*} \\ \pi(V, w) & \xrightarrow{(j_2)_*} & \pi(X, w) \end{array}$$

Here it is possible to interpret $\pi(X, w)$ as the *free product with amalgam* (see [14, Chapter 7] for more details on this notion) of $\pi(U, w)$ and $\pi(V, w)$ so that, given group presentations:

$$\pi(U, w) = \langle u_1, \cdots, u_k \mid \alpha_1, \cdots, \alpha_l \rangle;$$
$$\pi(V, w) = \langle v_1, \cdots, v_m \mid \beta_1, \cdots, \beta_n \rangle;$$
$$\pi(U \cap V, w) = \langle w_1, \cdots, w_p \mid \gamma_1, \cdots, \gamma_q \rangle;$$

one can describe $\pi(X, w)$ in terms of generators and relators as

$$\pi(X, w) = \langle u_1, \ldots, u_k, v_1, \cdots, v_m \mid$$
$$\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_n, (i_1)_*(w_1)(j_1)_*(w_1)^{-1}, \ldots, (i_2)_*(w_p)(j_2)_*(w_p)^{-1} \rangle.$$

The way in which one can get the presentation of this last group, beginning from the original data is indeed know as the Seifert-Van Kampen Theorem. Its proof is quite technical and can be found in [8, Chapters 23, 24, 25 and 26]: it is essentially based on combinatorial group theory and the way in which it is possible to pass from a given presentation to an equivalent one, without breaking the original conditions.

Among the various applications of this important result, one can classify surfaces of the usual space with three dimensions. We had no time to explore the connections with the differential geometry, so we briefly mention that a two dimensional compact and connected manifold is briefly called *surface*. Given two disjoint surfaces S_1 and S_2, their *connected sum* $S_1 \# S_2$ is formed by removing an open disk in S_1, an open disk in S_2, and gluing along the boundaries of the resulting holes. With this geometric operation in the hands, one has the so-called *classification theorem of surfaces*.

Theorem 12.1 (See [8], Theorem 11.3)**.** *Let S be a surface. Then S is homeomorphic to precisely one of the following surfaces:*

$$S^2 \# \underbrace{T \# T \# ... \# T}_{m} = S^2 \# mT \qquad (m \geq 0),$$

$$S^2 \# \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2 \# ... \# \mathbb{RP}^2}_{n} = S^2 \# n\mathbb{RP}^2 \qquad (n \geq 1).$$

Here S^2 is the sphere, $\mathbb{RP}^2$ the 2-dimensional real projective plane and T the torus.

The corresponding version for fundamental groups is the following.

Theorem 12.2 (See [8], Theorem 26.1)**.** *Let S be a surface. Then its fundamental group is*

$$\pi(S) = \langle c_1, d_1, \ldots, c_m, d_m, f_1, \ldots, f_n \mid [c_1, d_1] \ldots [c_m, d_m] f_1^2 \ldots f_n^2 = 1 \rangle,$$

where for all $1 \leq i \leq n$ and $1 \leq j \leq m$

$$\pi(\mathbb{RP}^2) = \langle f_i \mid f_i^2 = 1 \rangle \simeq \mathbb{Z}_2 \quad \textit{and} \quad \pi(T) = \langle c_j, d_j \mid [c_j, d_j] = 1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}.$$

While a group generated by a single element is abelian and isomorphic to the additive group $\mathbb{Z}$ of the integers, the situation is much more complicated for a group generated by only two elements withouth any relation. Intuitively, this is the idea of a 2-generated free group.

The importance of free groups in algebraic topology and their role in the actions of groups on manifolds is crucial and the reference [14] is helpful to understand some classical connections. There are more recent works, which investigate the role of finitely presented groups in terms of fundamental groups and the references [2, 13] may be consulted in this perspective.

Given a finitely presented group $\Gamma = \langle X \mid R \rangle$ and the free group $F(X)$ on X (where X is a nonemtpy set with a given number of elements that work as generators for F, and, consequently for Γ), if $w \in F(X)$ is a *freely reduced word* on Γ such that $w = 1$, then w can be written as

$$w = u_1 r_1 u_1^{-1} \cdots u_m r_m u_m^{-1} \in F(X),$$

where $m \geq 0$ and $r_i \in R \cup R^{-1}$ for all $i = 1, \ldots, m$. In this situation, we may define the *area* of w with respect to X and R as

$$\text{Area}(w) = \min\{m \geq 0 \mid w \text{ is a product in } F(X) \text{ of } m \text{ conjugates of elements of } R \cup R^{-1}\}.$$

Now it is possible to interpret Area(w) via certain diagrams which originate from the a combinatorial re-formulation of the theorem of Seifert and Van Kampen (these are the so called "van Kampen diagrams" over a given presentation).

We recall that an isoperimetric function (for a finite presentation) is a monotone non-decreasing function $f : \mathbb{N} \to [0, \infty[$ such that whenever $w \in F(X)$ is a freely reduced word satisfying $w = 1$ in Γ, then $\text{Area}(w) \leq f(|w|)$, where $|w|$ is the length of the word w. The *Dehn function* (of a finite presentation) is defined as

$$\text{Dehn}(n) = \max\{\text{Area}(w) \mid w = 1 \text{ in } \Gamma, |w| \leq n, w \text{ is freely reduced}\}.$$

Equivalently, one can see that Dehn(n) is the smallest isoperimetric function (for the given presentation).

The theorem of Seifert and van Kampen is historically at the origin of all, but one can define a Dehn function even in an abstract way, just beginning with a finitely presented group, that, in principle, is not given as a fundamental group of a surface. This algebraic approach has been considered in several contributions, as in [2, 12, 13]. Therefore one can make considerations of independent interest. The following problem originated recently:

Question 12.3 (Olshaskij and Sapir, see [12])**.** What it the structure of (finitely presented) groups with quadratic Dehn function?

There is a large interest in recent years in finding finitely presented groups with Dehn(n) such that

$$\lim_{n\to\infty} \frac{\mathrm{Dehn}(n)}{n^2 \log n} = 0.$$

Somehow this interest is related to certain corresponding problems of computability and of algorythms for words.

Most of the groups which we know from the geometry and topology have in fact quadratic Dehn function. For instance,

$$D_\infty = \langle x, y \mid y^2 = 1, y^{-1}xy = x^{-1}\rangle = \langle y\rangle \ltimes \langle x\rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}$$

is called infinite dihedral group and may be generalized getting to a well known class of groups, called *Baumsalg–Solitar groups*. The idea is the following: take the additive group of p-adic rationals $\mathbb{Q}_p$ for some odd prime p and consider the group

$$G = \langle x, a \ : \ a^{p^n} = x^{-1}ax\rangle = \langle x\rangle \ltimes \mathbb{Q}_p,$$

where

$$\langle a\rangle^G = \langle g^{-1}ag \mid \ \in G\rangle \simeq \mathbb{Q}_p$$

is the smallest normal subgroup of G containing $\langle a\rangle$ and where x induces in $\mathbb{Q}_p$ the automorphism $b \mapsto p^n \cdot b$. This group is a generalization of the dihedral group and also has quadratic Dehn function. Several other generalizations of these constructions turns out to have quadratic Dehn function, so the groups which haven't quadratic Dehn function turns out to have sophisticated constructions in which several notions of graph theory, algebraic topology and group theory are involved.

References

[1] A. Babaee, B. Mashayekhy, H. Mirebrahimi and H. Torabi, On a Van Kampen theorem for Hawaiian groups,*Topolo. Appl.* **241** (2018), 252–262.

[2] L. Funar and D.E. Otera, On the wgsc and qsf tameness conditions for finitely presented groups,*Groups Geom. Dyn.* **4** (2010), 549-596.

[3] E. Hewitt, A problem in set-theoretic topology, *Duke Math. J.* **10** (1943), 309-333.

[4] E. Hewitt, A remark on density characters, *Bull. Amer. Math. Soc.* **52** (1946), 641-643.

[5] E. Hewitt, On two problems of Urysohn, *Ann. of Math.* (2) **47** (1946), 503-509.

[6] E. Hewitt, Rings of real-valued continuous functions I, *Trans. Amer. Math. Soc.* **64** (1948), 45-99.

[7] E. Hewitt and K. Ross, *Abstract harmonic analysis I*, Springer, 1963, Berlin.

[8] C. Kosniowski, *Introduction to algebraic topology*, Cambridge University Press, 1980, Cambridge.

[9] L. Nachbin, Sur les Espaces Topologiques Ordonnes, *C. R. Acad. Sci. Paris* **226**, 381-382 (1948)

[10] L. Nachbin, Sur les espaces uniformes ordonnes, *C. R. Acad. Sci. Paris* **226** (1948), 774-775.

[11] L. Nachbin, Topological vector spaces of continuous functions, *Proc. Nat. Acad. Sci.* **40** (1954), 471-474.

[12] A.Yu. Olshanskij and M.V. Sapir, Groups with small Dehn functions and bipartite chord diagrams, *Geom. Funct. Anal.* **16** (2006), 1324–1376.

[13] D.E. Otera and F.G. Russo, On the wgsc property in some classes of groups, *Mediterr. J. Math.* **6** (2009), 501–508.

[14] J. Rotman, *An introduction to algebraic topology*, Springer, 1988, Berlin.

[15] S. Willard, *General topology*, Dover Publications, 2004, New York

[16] J. Wu, *Lecture notes on algebraic topology, department of mathematics*, University of Singapore, Singapore 119260, available at http://www.math.nus.edu.sg/ matwujie/ma5209