Fifth SI Edition

CHAPTER

3

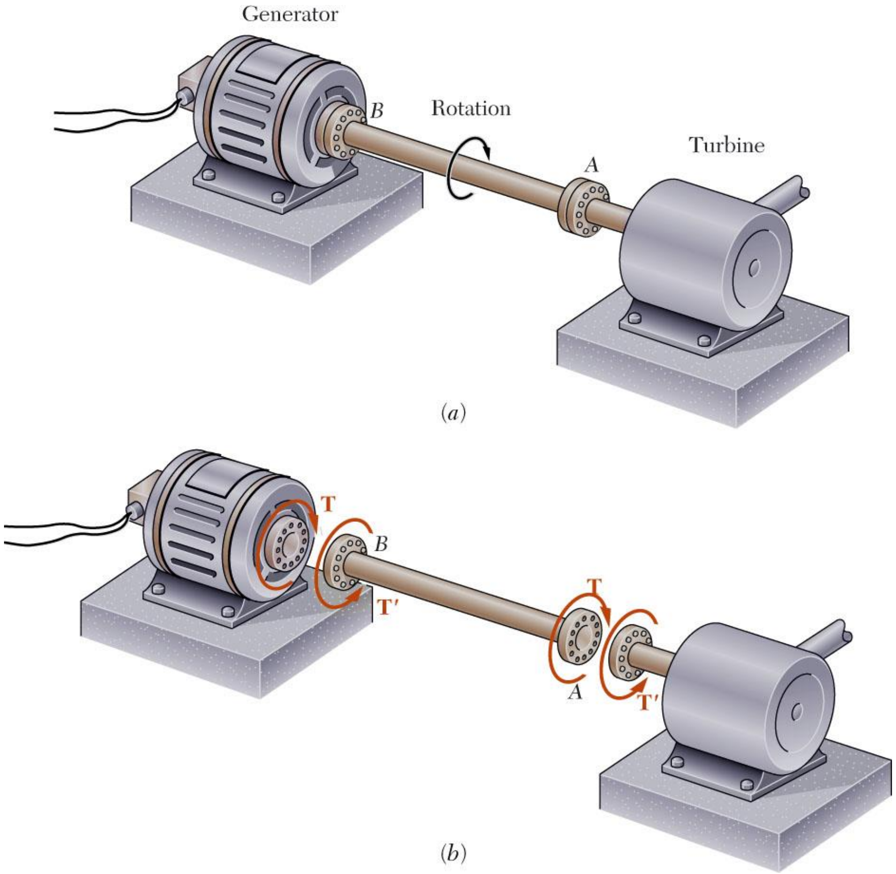
MECHANICS OF MATERIALS

Ferdinand P. Beer
E. Russell Johnston, Jr.
John T. DeWolf
David F. Mazurek

Lecture Notes:
J. Walt Oler
Texas Tech University

Torsion

Torsional Loads on Circular Shafts



- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator

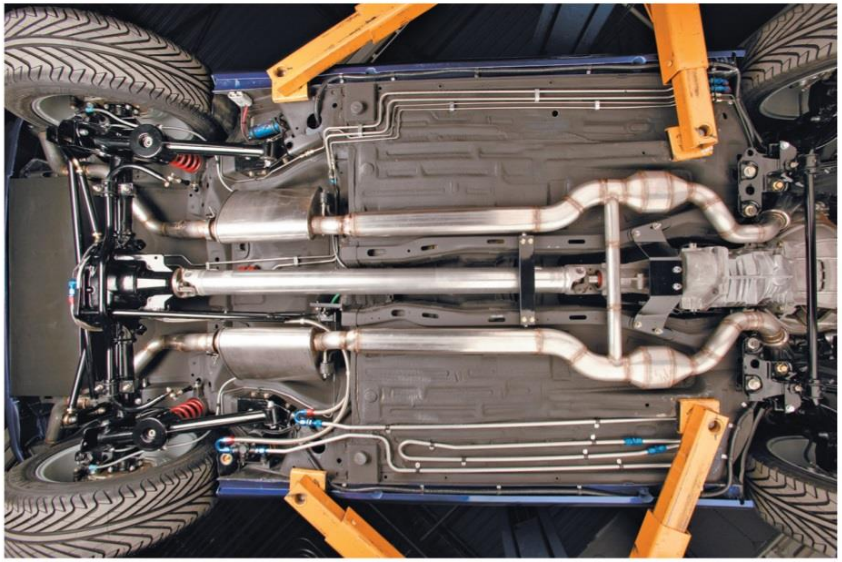
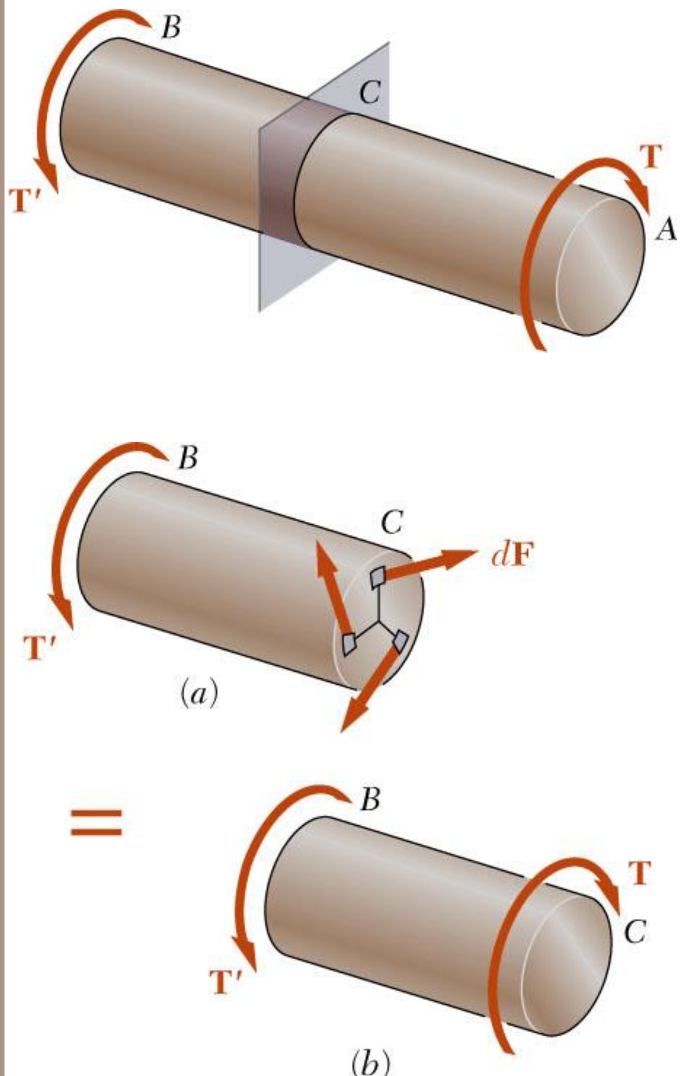


Photo 3.1 In this automotive power train, the shaft transmits power from the engine to the rear wheels.

3.1 Circular Shafts in Torsion (p150)



- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque, ρ is radius

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

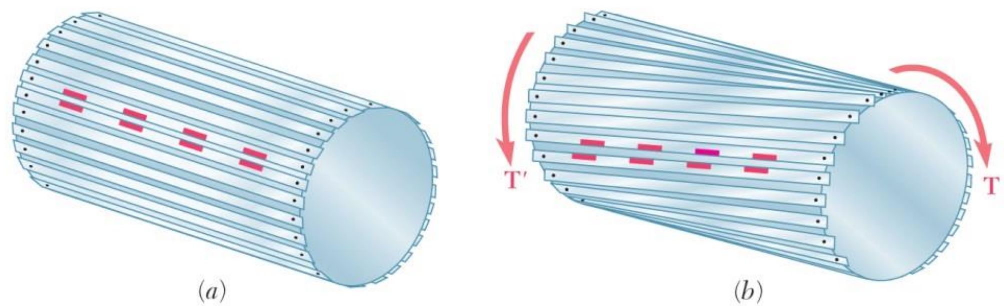
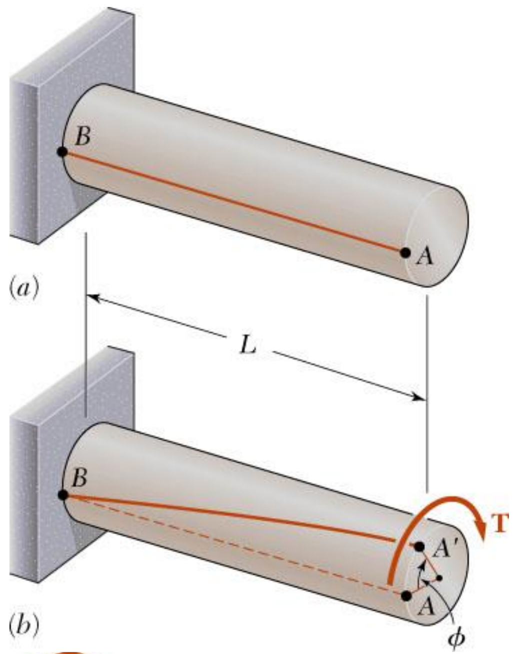
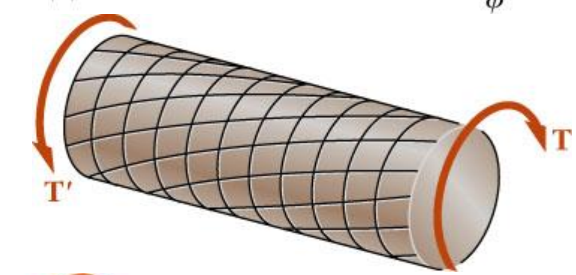


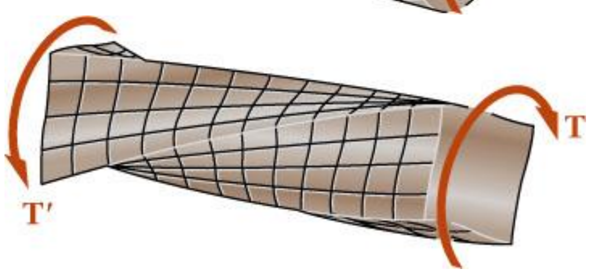
Fig. 3.6 Demonstration of shear in a shaft (*a*) undeformed; (*b*) loaded and deformed.

- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

Shaft Deformations (p151)







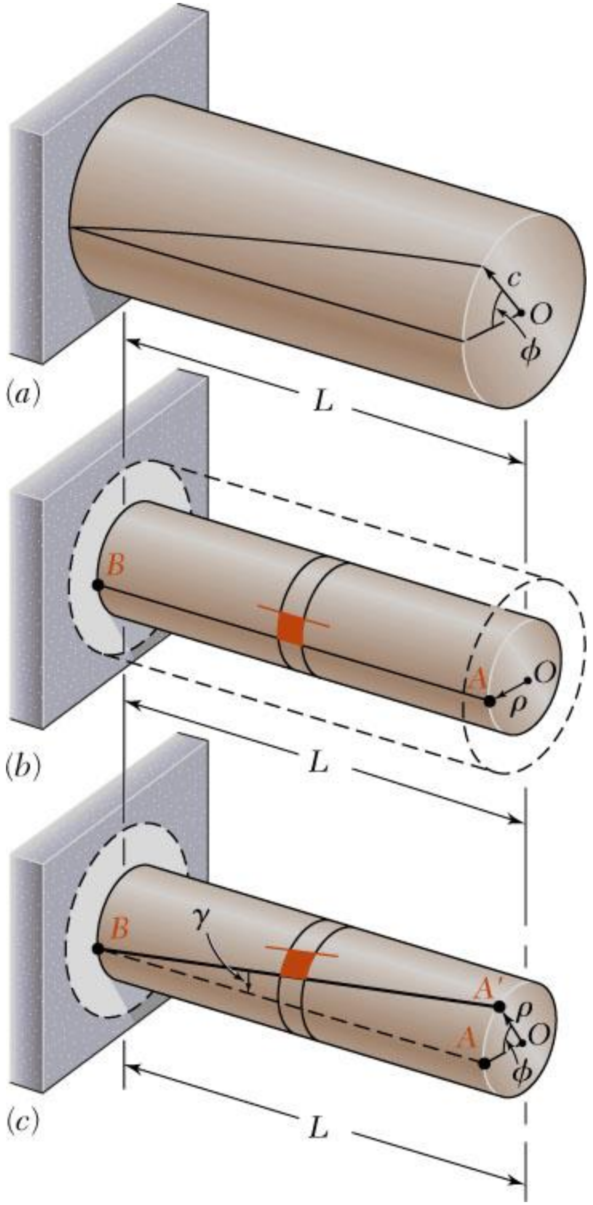
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain (p153)



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

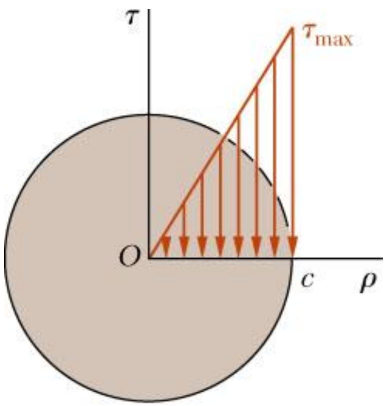
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

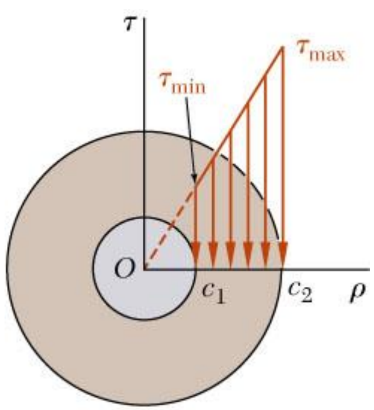
$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range (p153, 154)



(a)

$$J = \tfrac{1}{2}\pi c^4$$



(b)

$$J = \tfrac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

- Multiplying the previous equation by the shear modulus,

  $$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

  From Hooke's Law, $\tau = G\gamma$, so

  $$\tau = \frac{\rho}{c}\tau_{max}$$

  The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

  $$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

- The results are known as the *elastic torsion formulas*,

  $$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Concept Application 3.1

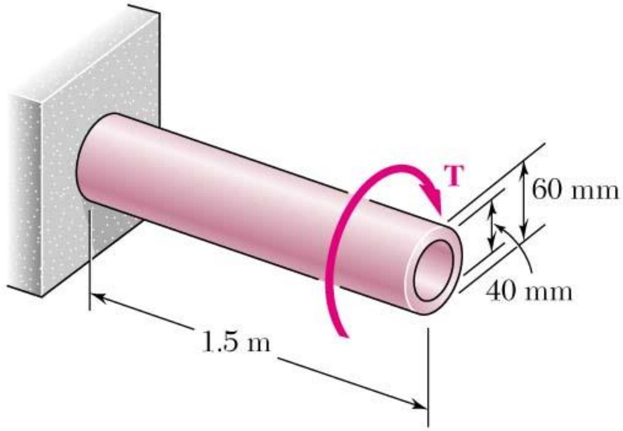
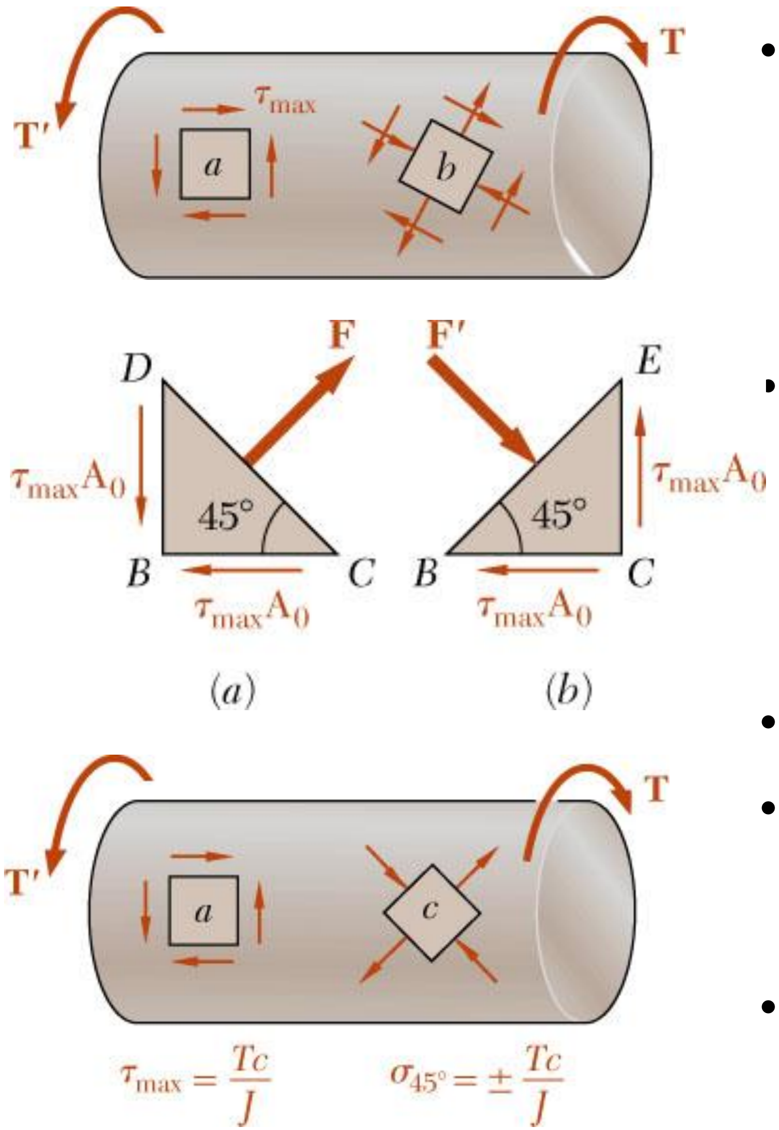


Fig. 3.16

Normal Stresses (p157)



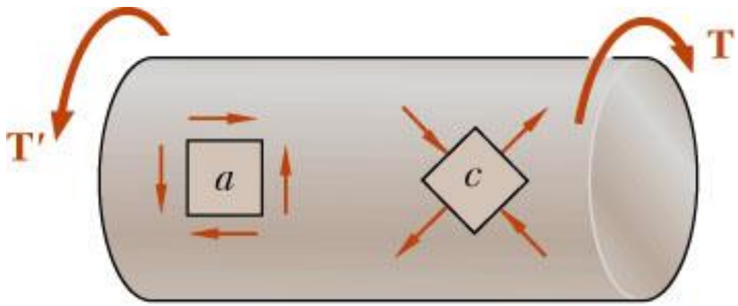
- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.
- Consider an element at 45° to the shaft axis,

$$F = 2(\tau_{max} A_0)\cos 45° = \tau_{max} A_0 \sqrt{2}$$

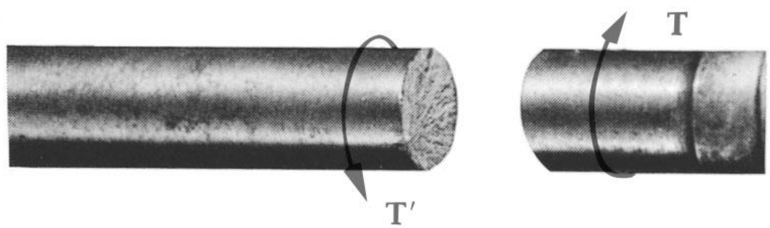
$$\sigma_{45°} = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{max}$$

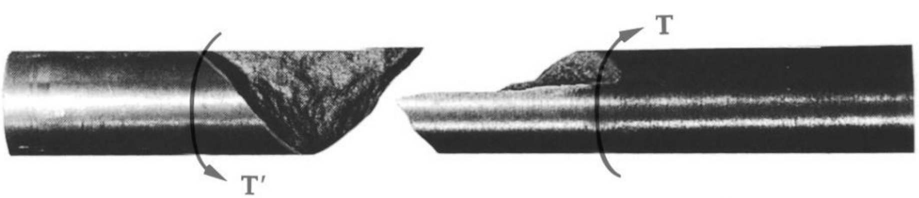
- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude

Torsional Failure Modes



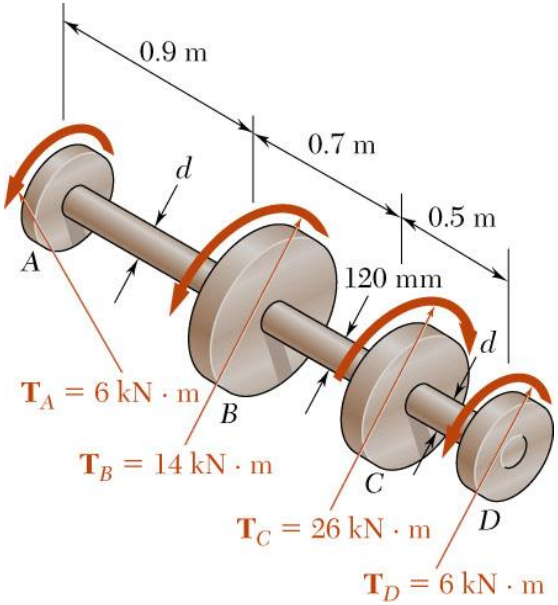
$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm \frac{Tc}{J}$$





- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.

Sample Problem 3.1 (p158)



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

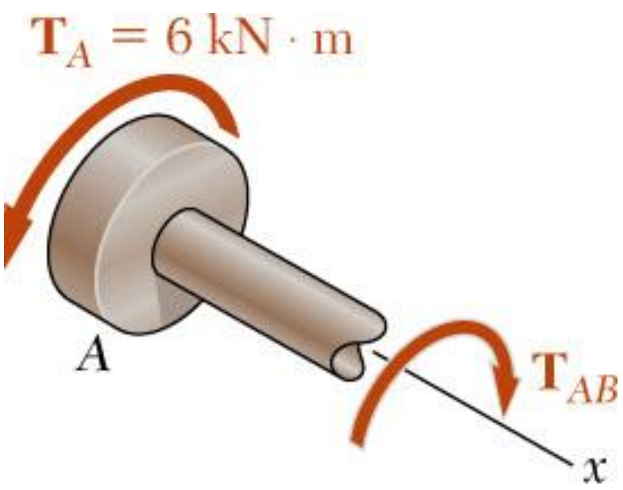
SOLUTION:

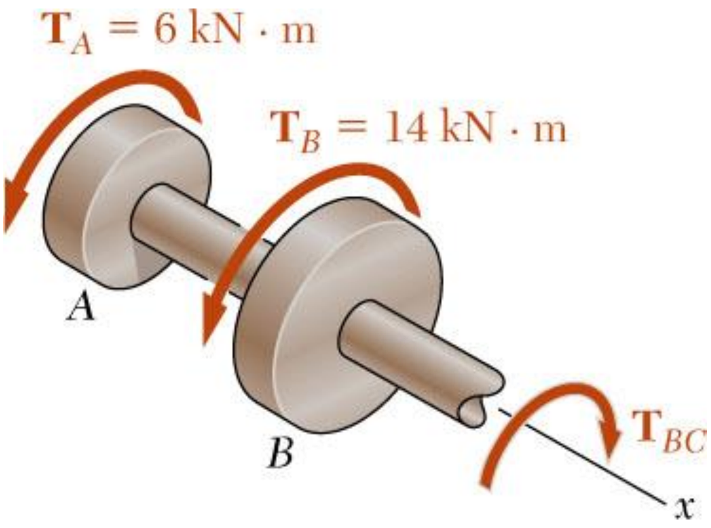
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

Sample Problem 3.1

SOLUTION:

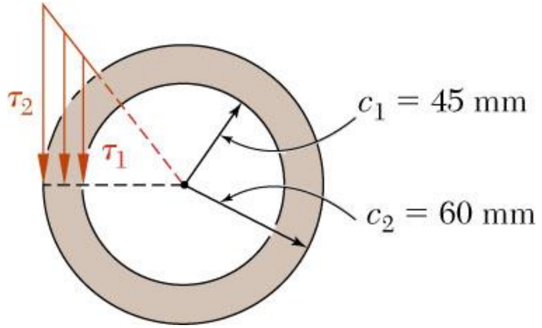
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.



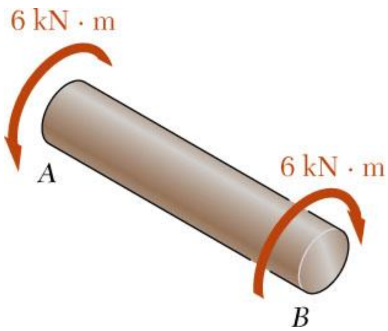


Sample Problem 3.1

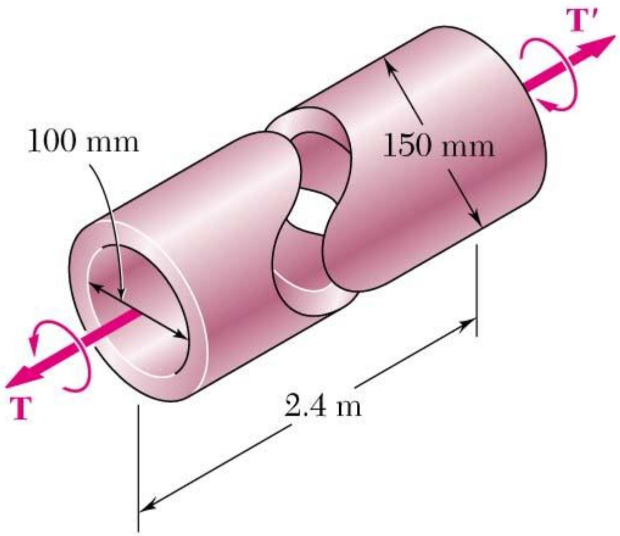
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.

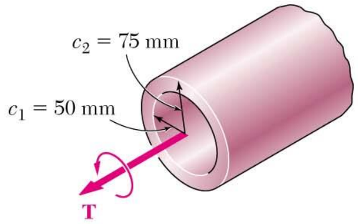


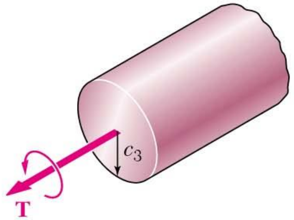
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

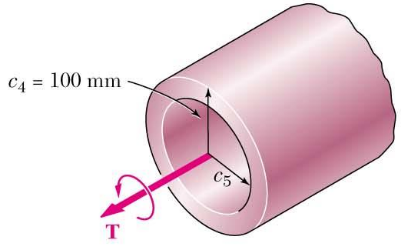


Sample Problem 3.2 (p159)





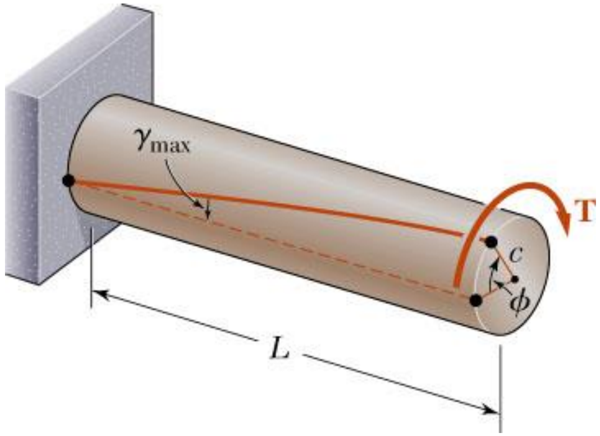


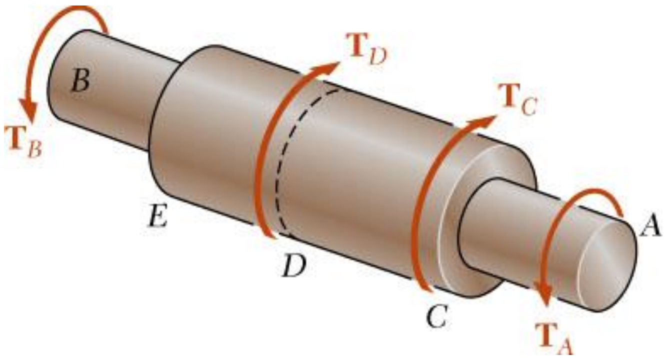


Problems

- What is moment of inertia?
- 3.11, 3.19 (p163, p164)

3.2 Angle of Twist in Elastic Range (p167)





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Concept Application 3.2

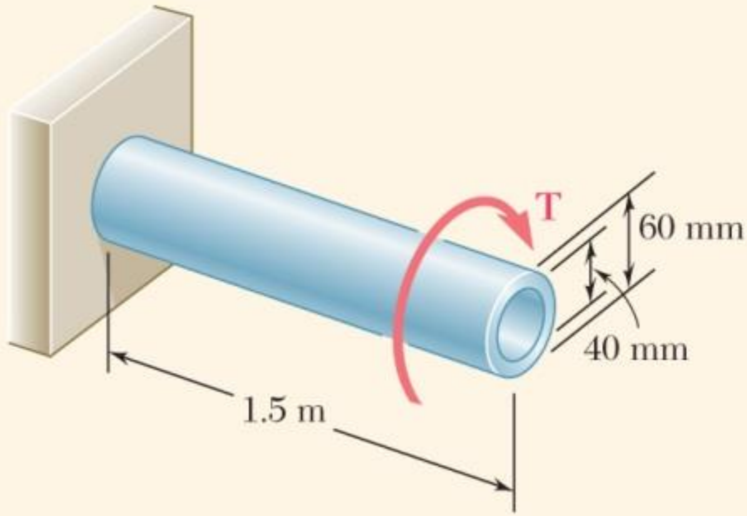


Fig. 3.15 Hollow, fixed-end shaft having torque *T* applied at end.

Concept Application 3.3

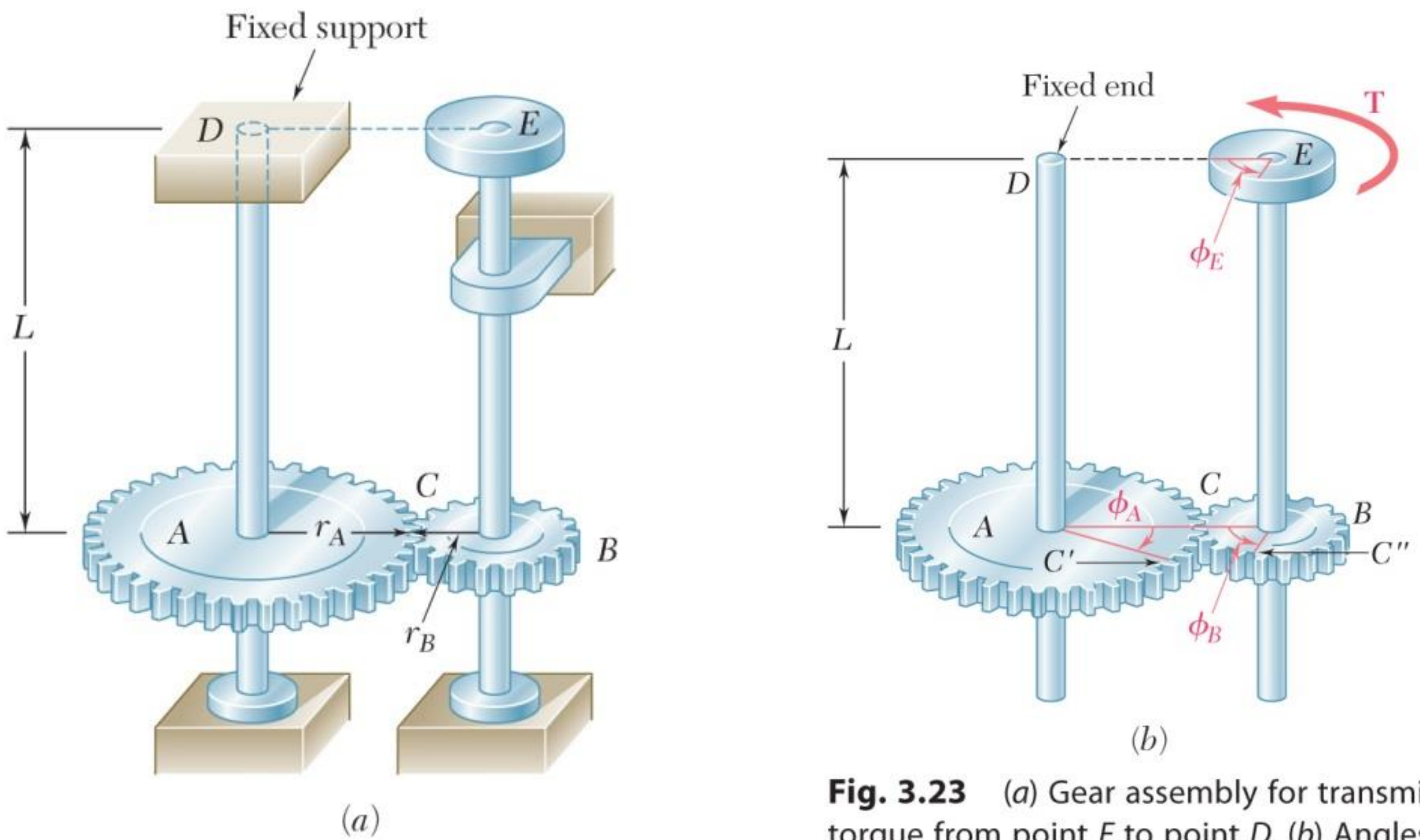


Fig. 3.23 (*a*) Gear assembly for transmitting torque from point *E* to point *D*. (*b*) Angles of twist at disk *E*, gear *B*, and gear *A*.

Concept Application 3.4

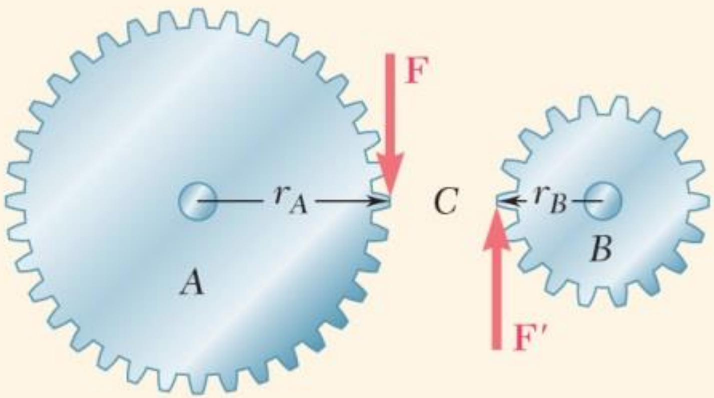
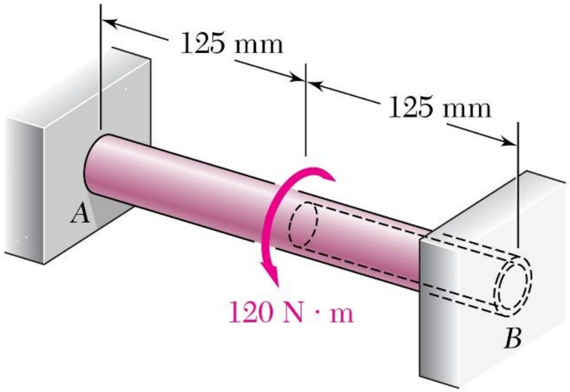
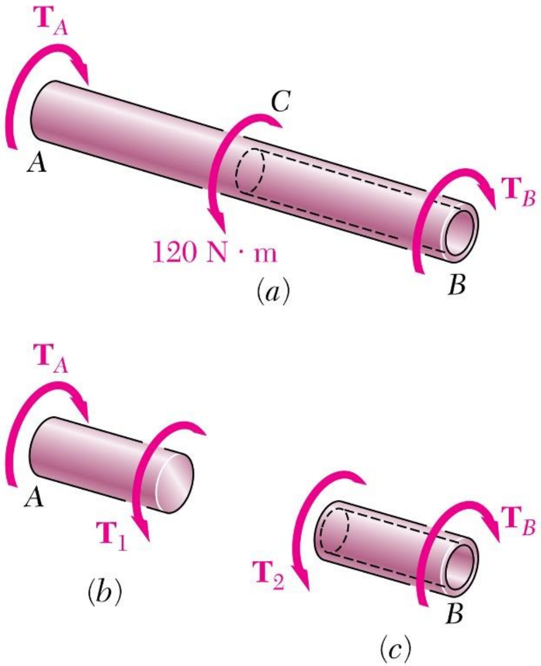


Fig. 3.24 Gear teeth forces for gears *A* and *B*.

Statically Indeterminate Shafts (p171)





- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

$$T_A + T_B = 120\,\mathrm{N\cdot m}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

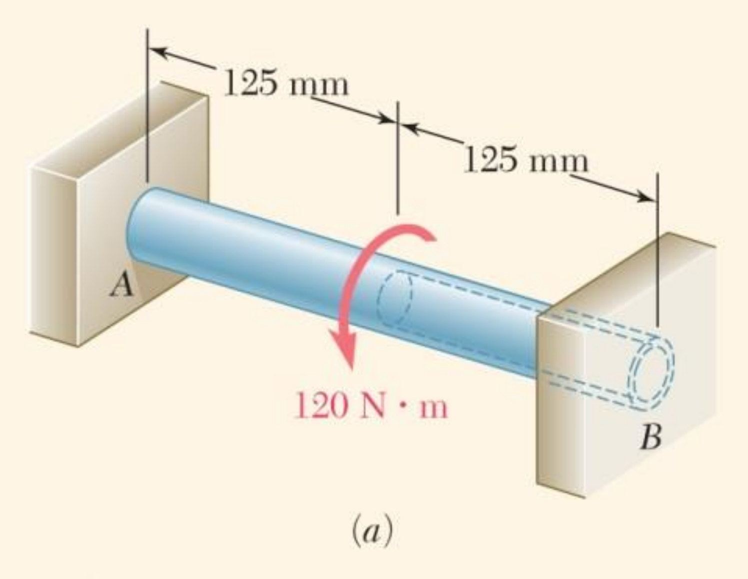
- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 120\,\mathrm{N\cdot m}$$

Concept Application 3.5



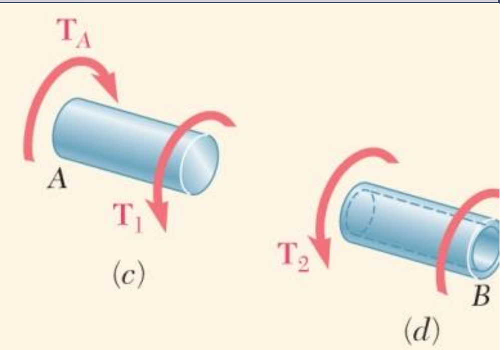
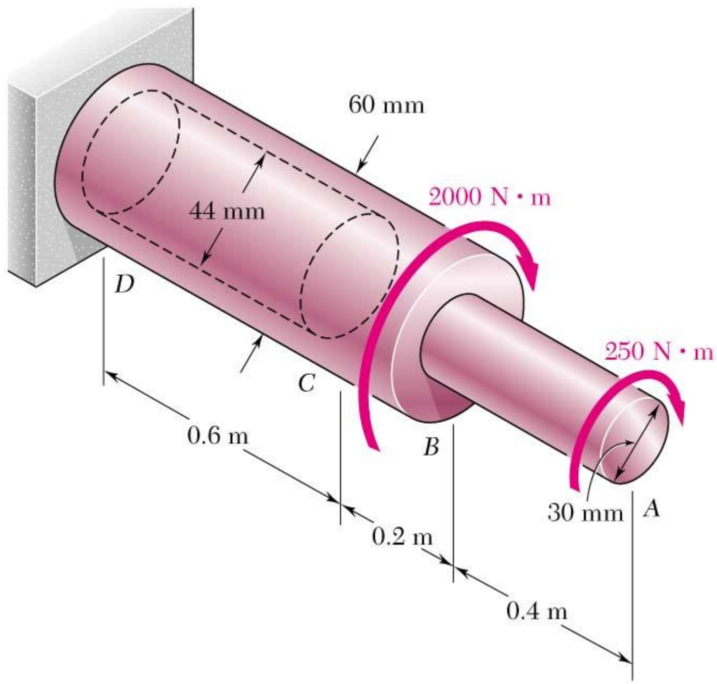
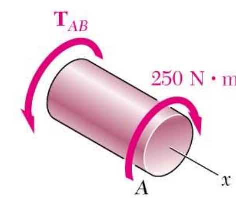
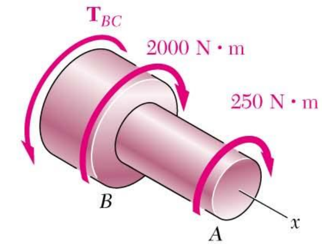


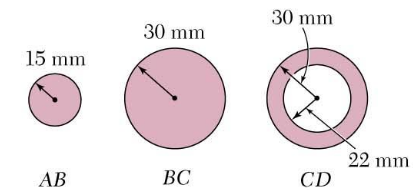
Fig. 3.25 (*a*) Shaft with central applied torque and fixed ends. (*b*) Free-body diagram of shaft *AB*. (*c*) Free-body diagrams for solid an hollow segments.

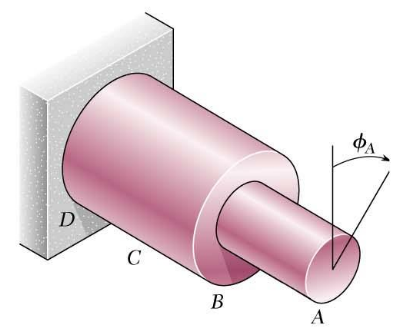
Sample Problem 3.3





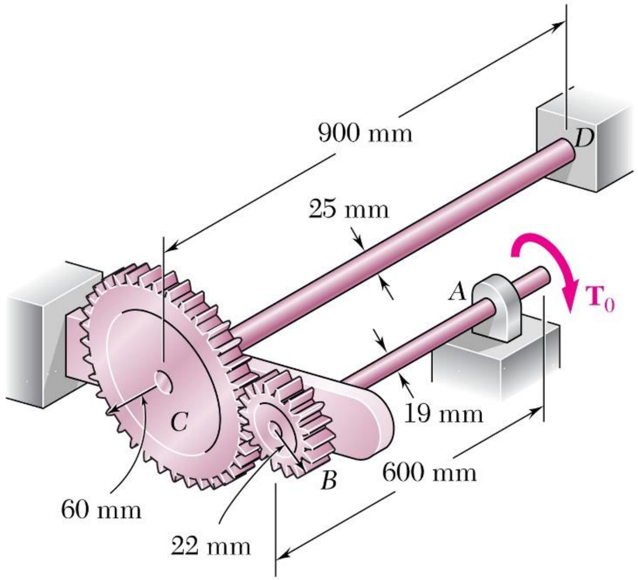






Sample Problem 3.3

Sample Problem 3.4 (p173)



Two solid steel shafts are connected by gears. Knowing that for each shaft G = 77 GPa and that the allowable shearing stress is 55 MPa, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

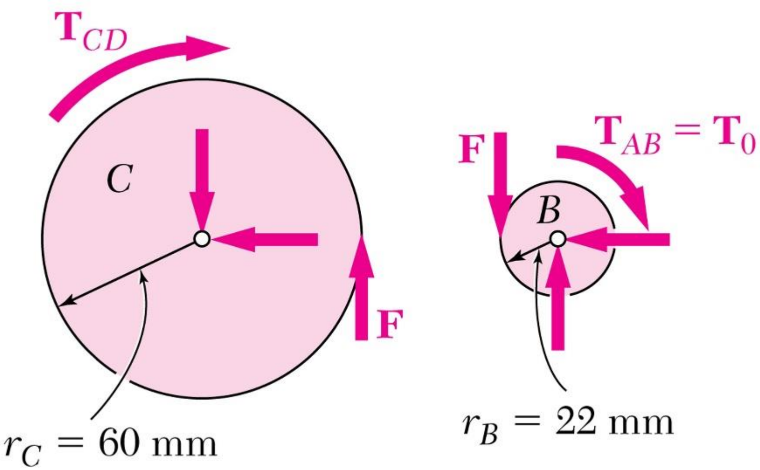
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0.
- Apply a kinematic analysis to relate the angular rotations of the gears.
- Find the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

Sample Problem 3.4

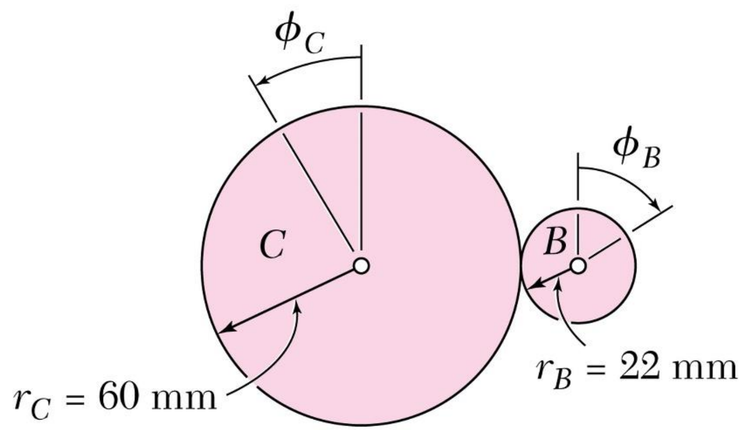
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .



$$\sum M_B = 0 = F(22\,\text{mm}) - T_0$$
$$\sum M_C = 0 = F(60\,\text{mm}) - T_{CD}$$
$$T_{CD} = 2.73 T_0$$

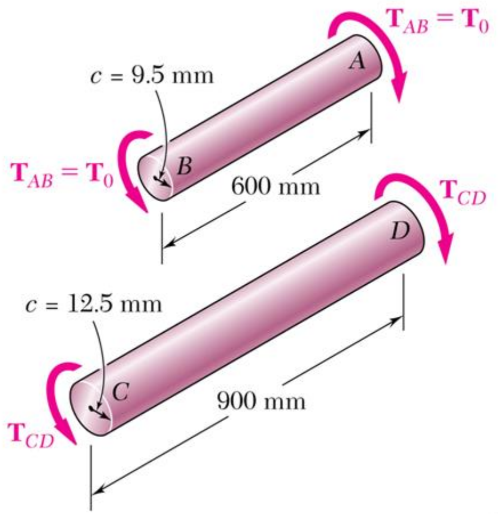
- Apply a kinematic analysis to relate the angular rotations of the gears.



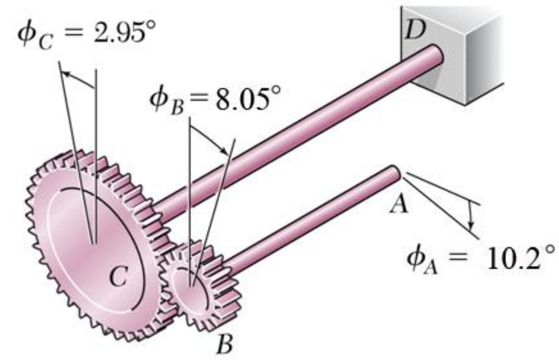
$$r_B \phi_B = r_C \phi_C$$
$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{60\,\text{mm}}{20\,\text{mm}}\phi_C$$
$$\phi_B = 2.73\phi_C$$

Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest.



- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.



Problems

- Sample problem 3.5
- 3.41 (p179)

3.4 Design of Transmission Shafts (p187)

$$P = T\omega$$

- ω is the angular velocity
- $\omega = 2\pi f$, f is frequency of rotation

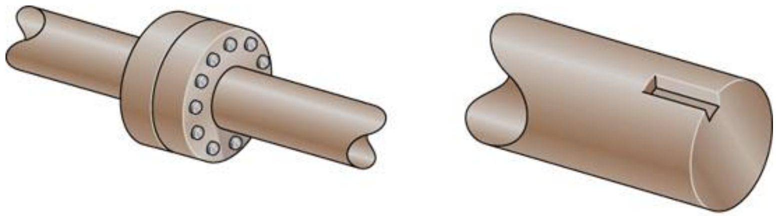
$$P = 2\pi f T \qquad T = \frac{P}{2\pi f}$$

$$\tau_{\max} = \frac{Tc}{J} \Rightarrow \frac{J}{c} = \frac{T}{\tau_{\max}}$$

Concept Application 3.6

Concept Application 3.7

3.5 Stress Concentrations in a Circular Shafts (p187)



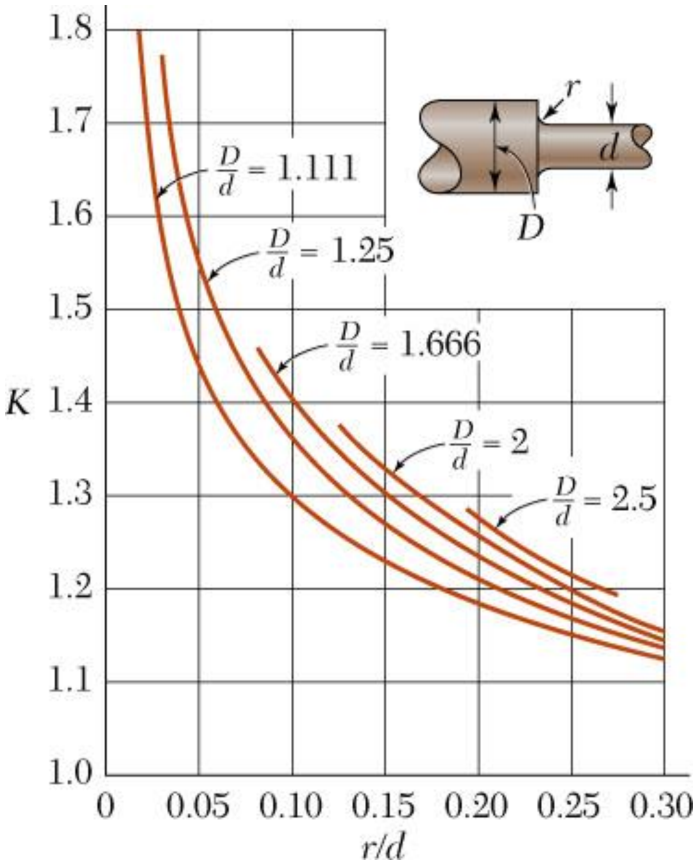


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

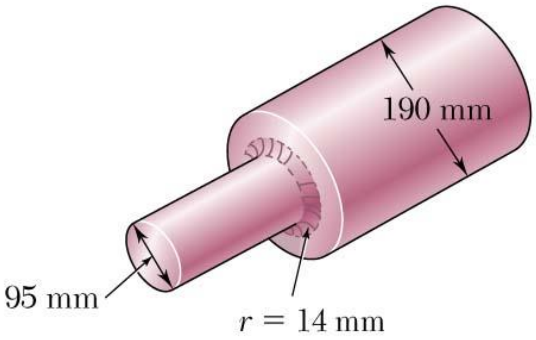
  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

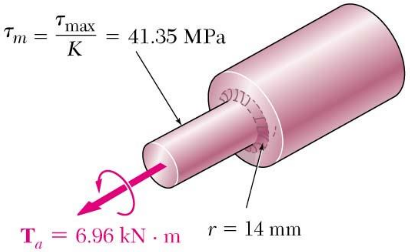
- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

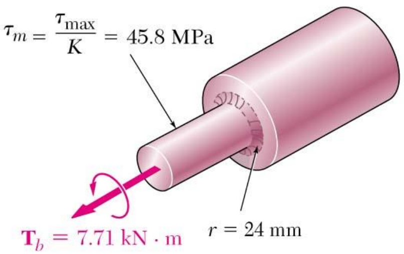
- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$

Sample Problem 3.6







Problems

- 3.85, 3.87 (p194)