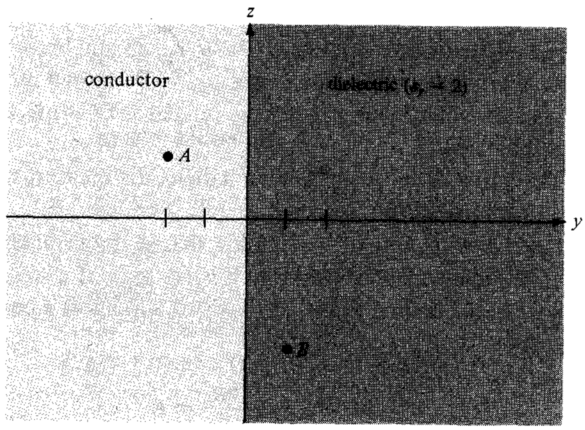


Figure 5.16 See Example 5.10.

Hence,

$$\mathbf{D} = 2\mathbf{a}_y \text{ nC/m}^2$$

and

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o \varepsilon_r} = 2 \times 10^{-9} \times \frac{36\pi}{2} \times 10^9 \, \mathbf{a}_y = 36\pi \mathbf{a}_y$$
$$= 113.1\mathbf{a}_y \text{ V/m}$$

PRACTICE EXERCISE 5.10

It is found that $\mathbf{E} = 60\mathbf{a}_x + 20\mathbf{a}_y - 30\mathbf{a}_z$ mV/m at a particular point on the interface between air and a conducting surface. Find $\mathbf{D}$ and ρ_S at that point.

Answer: $0.531\mathbf{a}_x + 0.177\mathbf{a}_y - 0.265\mathbf{a}_z$ pC/m^2, 0.619 pC/m^2.

SUMMARY

1. Materials can be classified roughly as conductors ($\sigma \gg 1, \varepsilon_r = 1$) and dielectrics ($\sigma \ll 1, \varepsilon_r \geq 1$) in terms of their electrical properties σ and ε_r, where σ is the conductivity and ε_r is the dielectric constant or relative permittivity.
2. Electric current is the flux of electric current density through a surface; that is,

$$I = \int \mathbf{J} \cdot d\mathbf{S}$$

3. The resistance of a conductor of uniform cross section is

$$R = \frac{\ell}{\sigma S}$$

4. The macroscopic effect of polarization on a given volume of a dielectric material is to "paint" its surface with a bound charge $Q_b = \oint_S \rho_{ps}\, dS$ and leave within it an accumulation of bound charge $Q_b = \int_v \rho_{pv}\, dv$ where $\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$ and $\rho_{pv} = -\nabla \cdot \mathbf{P}$.
5. In a dielectric medium, the **D** and **E** fields are related as $\mathbf{D} = \varepsilon\mathbf{E}$, where $\varepsilon = \varepsilon_o\varepsilon_r$ is the permittivity of the medium.
6. The electric susceptibility $\chi_e (= \varepsilon_r - 1)$ of a dielectric measures the sensitivity of the material to an electric field.
7. A dielectric material is linear if $\mathbf{D} = \varepsilon\mathbf{E}$ holds, that is, if ε is independent of **E**. It is homogeneous if ε is independent of position. It is isotropic if ε is a scalar.
8. The principle of charge conservation, the basis of Kirchhoff's current law, is stated in the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$$

9. The relaxation time, $T_r = \varepsilon/\sigma$, of a material is the time taken by a charge placed in its interior to decrease by a factor of $e^{-1} \simeq 37$ percent.
10. Boundary conditions must be satisfied by an electric field existing in two different media separated by an interface. For a dielectric–dielectric interface

$$E_{1t} = E_{2t}$$

$$D_{1n} - D_{2n} = \rho_S \qquad \text{or} \qquad D_{1n} = D_{2n} \qquad \text{if} \qquad \rho_S = 0$$

For a dielectric–conductor interface,

$$E_t = 0 \qquad D_n = \varepsilon E_n = \rho_S$$

because $\mathbf{E} = 0$ inside the conductor.

REVIEW QUESTIONS

5.1 Which is *not* an example of convection current?

(a) A moving charged belt
(b) Electronic movement in a vacuum tube
(c) An electron beam in a television tube
(d) Electric current flowing in a copper wire

5.2 When a steady potential difference is applied across the ends of a conducting wire,

(a) All electrons move with a constant velocity.
(b) All electrons move with a constant acceleration.
(c) The random electronic motion will, on the average, be equivalent to a constant velocity of each electron.
(d) The random electronic motion will, on the average, be equivalent to a nonzero constant acceleration of each electron.

5.3 The formula $R = \ell / (\sigma S)$ is for thin wires.

(a) True
(b) False
(c) Not necessarily

5.4 Sea water has $\varepsilon_r = 80$. Its permittivity is

(a) 81
(b) 79
(c) 5.162×10^{-10} F/m
(d) 7.074×10^{-10} F/m

5.5 Both ε_o and χ_e are dimensionless.

(a) True
(b) False

5.6 If $\nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{J} = \sigma \nabla \cdot \mathbf{E}$ in a given material, the material is said to be

(a) Linear
(b) Homogeneous
(c) Isotropic
(d) Linear and homogeneous
(e) Linear and isotropic
(f) Isotropic and homogeneous

5.7 The relaxation time of mica ($\sigma = 10^{-15}$ mhos/m, $\varepsilon_r = 6$) is

(a) 5×10^{-10} s
(b) 10^{-6} s
(c) 5 hours
(d) 10 hours
(e) 15 hours

5.8 The uniform fields shown in Figure 5.17 are near a dielectric–dielectric boundary but on opposite sides of it. Which configurations are correct? Assume that the boundary is charge free and that $\varepsilon_2 > \varepsilon_1$.

5.9 Which of the following statements are incorrect?

(a) The conductivities of conductors and insulators vary with temperature and frequency.
(b) A conductor is an equipotential body and $\mathbf{E}$ is always tangential to the conductor.
(c) Nonpolar molecules have no permanent dipoles.
(d) In a linear dielectric, P varies linearly with E.

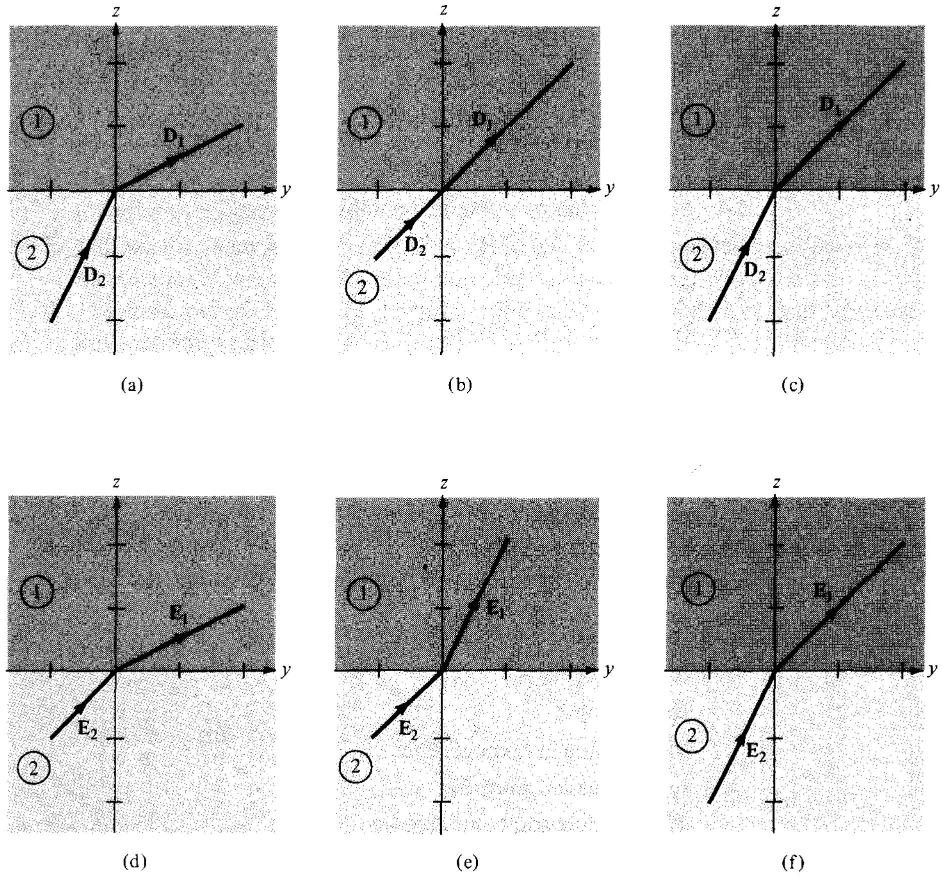


Figure 5.17 For Review Question 5.8.

5.10 The electric conditions (charge and potential) inside and outside an electric screening are completely independent of one another.

(a) True
(b) False

Answers: 5.1d, 5.2c, 5.3c, 5.4d, 5.5b, 5.6d, 5.7e, 5.8e, 5.9b, 5.10a.

PROBLEMS

5.1 In a certain region, $\mathbf{J} = 3r^2 \cos\theta\, \mathbf{a}_r - r^2 \sin\theta\, \mathbf{a}_\theta$ A/m, find the current crossing the surface defined by $\theta = 30°$, $0 < \phi < 2\pi$, $0 < r < 2$ m.

5.2 Determine the total current in a wire of radius 1.6 mm if $\mathbf{J} = \dfrac{500\mathbf{a}_z}{\rho}$ A/m^2.

5.3 The current density in a cylindrical conductor of radius a is

$$\mathbf{J} = 10e^{-(1-\rho/a)}\mathbf{a}_z \text{ A/m}^2$$

Find the current through the cross section of the conductor.

5.4 The charge $10^{-4}e^{-3t}$ C is removed from a sphere through a wire. Find the current in the wire at $t = 0$ and $t = 2.5$ s.

5.5 (a) Let $V = x^2y^2z$ in a region ($\varepsilon = 2\varepsilon_o$) defined by $-1 < x, y, z < 1$. Find the charge density ρ_v in the region.

(b) If the charge travels at $10^4 y\mathbf{a}_y$ m/s, determine the current crossing surface $0 < x, z < 0.5, y = 1$.

5.6 If the ends of a cylindrical bar of carbon ($\sigma = 3 \times 10^4$) of radius 5 mm and length 8 cm are maintained at a potential difference of 9 V, find: (a) the resistance of the bar, (b) the current through the bar, (c) the power dissipated in the bar.

5.7 The resistance of round long wire of diameter 3 mm is 4.04 Ω/km. If a current of 40 A flows through the wire, find

(a) The conductivity of the wire and identify the material of the wire

(b) The electric current density in the wire

5.8 A coil is made of 150 turns of copper wire wound on a cylindrical core. If the mean radius of the turns is 6.5 mm and the diameter of the wire is 0.4 mm, calculate the resistance of the coil.

5.9 A composite conductor 10 m long consists of an inner core of steel of radius 1.5 cm and an outer sheath of copper whose thickness is 0.5 cm.

(a) Determine the resistance of the conductor.

(b) If the total current in the conductor is 60 A, what current flows in each metal?

(c) Find the resistance of a solid copper conductor of the same length and cross-sectional areas as the sheath. Take the resistivities of copper and steel as 1.77×10^{-8} and 11.8×10^{-8} Ω · m, respectively.

5.10 A hollow cylinder of length 2 m has its cross section as shown in Figure 5.18. If the cylinder is made of carbon ($\sigma = 10^5$ mhos/m), determine the resistance between the ends of the cylinder. Take $a = 3$ cm, $b = 5$ cm.

5.11 At a particular temperature and pressure, a helium gas contains 5×10^{25} atoms/m^3. If a 10-kV/m field applied to the gas causes an average electron cloud shift of 10^{-18} m, find the dielectric constant of helium.

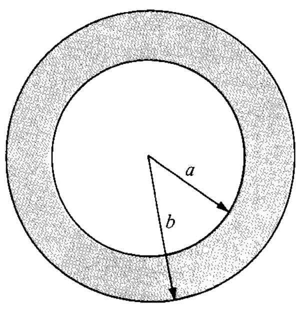


Figure 5.18 For Problems 5.10 and 5.15.

5.12 A dielectric material contains 2×10^{19} polar molecules/m^3, each of dipole moment 1.8×10^{-27} C/m. Assuming that all the dipoles are aligned in the direction of the electric field $\mathbf{E} = 10^5\ \mathbf{a}_x$ V/m, find $\mathbf{P}$ and ε_r.

5.13 In a slab of dielectric material for which $\varepsilon = 2.4\varepsilon_o$ and $V = 300z^2$ V, find: (a) $\mathbf{D}$ and ρ_v, (b) $\mathbf{P}$ and ρ_{pv}.

5.14 For $x < 0$, $\mathbf{P} = 5 \sin(\alpha y)\ \mathbf{a}_x$, where α is a constant. Find ρ_{ps} and ρ_{pv}.

5.15 Consider Figure 5.18 as a spherical dielectric shell so that $\varepsilon = \varepsilon_o\varepsilon_r$ for $a < r < b$ and $\varepsilon = \varepsilon_o$ for $0 < r < a$. If a charge Q is placed at the center of the shell, find

(a) $\mathbf{P}$ for $a < r < b$

(b) ρ_{pv} for $a < r < b$

(c) ρ_{ps} at $r = a$ and $r = b$

5.16 Two point charges when located in free space exert a force of 4.5 μN on each other. When the space between them is filled with a dielectric material, the force changes to 2 μN. Find the dielectric constant of the material and identify the material.

5.17 A conducting sphere of radius 10 cm is centered at the origin and embedded in a dielectric material with $\varepsilon = 2.5\varepsilon_o$. If the sphere carries a surface charge of 4 nC/m^2, find $\mathbf{E}$ at $(-3$ cm, 4 cm, 12 cm).

5.18 At the center of a hollow dielectric sphere ($\varepsilon = \varepsilon_o\varepsilon_r$) is placed a point charge Q. If the sphere has inner radius a and outer radius b, calculate $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$.

5.19 A sphere of radius a and dielectric constant ε_r has a uniform charge density of ρ_o.

(a) At the center of the sphere, show that

$$V = \frac{\rho_o a}{6\varepsilon_o\varepsilon_r}(2\varepsilon_r + 1)$$

(b) Find the potential at the surface of the sphere.

5.20 For static (time-independent) fields, which of the following current densities are possible?

(a) $\mathbf{J} = 2x^3y\mathbf{a}_x + 4x^2z^2\mathbf{a}_y - 6x^2yz\mathbf{a}_z$

(b) $\mathbf{J} = xy\mathbf{a}_x + y(z + 1)\mathbf{a}_y + 2y\mathbf{a}_z$

(c) $\mathbf{J} = \dfrac{z^2}{\rho}\mathbf{a}_\rho + z\cos\phi\ \mathbf{a}_z$

(d) $\mathbf{J} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r$

5.21 For an anisotropic medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Obtain $\mathbf{D}$ for: (a) $\mathbf{E} = 10\mathbf{a}_x + 10\mathbf{a}_y$ V/m, (b) $\mathbf{E} = 10\mathbf{a}_x + 20\mathbf{a}_y - 30\mathbf{a}_z$ V/m.

5.22 If $\mathbf{J} = \dfrac{100}{\rho^2} \mathbf{a}_\rho$ A/m^2, find: (a) the rate of increase in the volume charge density, (b) the total current passing through surface defined by $\rho = 2, 0 < z < 1, 0 < \phi < 2\pi$.

5.23 Given that $\mathbf{J} = \dfrac{5e^{-10^4 t}}{r} \mathbf{a}_r$ A/m^2, at $t = 0.1$ ms, find: (a) the amount of current passing surface $r = 2$ m, (b) the charge density ρ_v on that surface.

5.24 Determine the relaxation time for each of the following medium:

(a) Hard rubber ($\sigma = 10^{-15}$ S/m, $\varepsilon = 3.1\varepsilon_o$)
(b) Mica ($\sigma = 10^{-15}$ S/m, $\varepsilon = 6\varepsilon_o$)
(c) Distilled water ($\sigma = 10^{-4}$ S/m, $\varepsilon = 80\varepsilon_o$)

5.25 The excess charge in a certain medium decreases to one-third of its initial value in 20 μs. (a) If the conductivity of the medium is 10^{-4} S/m, what is the dielectric constant of the medium? (b) What is the relaxation time? (c) After 30 μs, what fraction of the charge will remain?

5.26 Lightning strikes a dielectric sphere of radius 20 mm for which $\varepsilon_r = 2.5$, $\sigma = 5 \times 10^{-6}$ mhos/m and deposits uniformly a charge of 10 μC. Determine the initial charge density and the charge density 2 μs later.

5.27 Region 1 ($z < 0$) contains a dielectric for which $\varepsilon_r = 2.5$, while region 2 ($z > 0$) is characterized by $\varepsilon_r = 4$. Let $\mathbf{E}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ V/m and find: (a) $\mathbf{D}_2$, (b) $\mathbf{P}_2$, (c) the angle between $\mathbf{E}_1$ and the normal to the surface.

5.28 Given that $\mathbf{E}_1 = 10\mathbf{a}_x - 6\mathbf{a}_y + 12\mathbf{a}_z$ V/m in Figure 5.19, find: (a) $\mathbf{P}_1$, (b) $\mathbf{E}_2$ and the angle $\mathbf{E}_2$ makes with the y-axis, (c) the energy density in each region.

5.29 Two homogeneous dielectric regions 1 ($\rho \leq 4$ cm) and 2 ($\rho \geq 4$ cm) have dielectric constants 3.5 and 1.5, respectively. If $\mathbf{D}_2 = 12\mathbf{a}_\rho - 6\mathbf{a}_\phi + 9\mathbf{a}_z$ nC/m^2, calculate: (a) $\mathbf{E}_1$ and $\mathbf{D}_1$, (b) $\mathbf{P}_2$ and ρ_{pv2}, (c) the energy density for each region.

5.30 A conducting sphere of radius a is half-embedded in a liquid dielectric medium of permittivity ε_1 as in Figure 5.20. The region above the liquid is a gas of permittivity ε_2. If the total free charge on the sphere is Q, determine the electric field intensity everywhere.

***5.31** Two parallel sheets of glass ($\varepsilon_r = 8.5$) mounted vertically are separated by a uniform air gap between their inner surface. The sheets, properly sealed, are immersed in oil ($\varepsilon_r = 3.0$) as shown in Figure 5.21. A uniform electric field of strength 2000 V/m in the horizontal direction exists in the oil. Calculate the magnitude and direction of the electric field in the glass and in the enclosed air gap when (a) the field is normal to the glass surfaces, and (b) the field in the oil makes an angle of 75° with a normal to the glass surfaces. Ignore edge effects.

5.32 (a) Given that $\mathbf{E} = 15\mathbf{a}_x - 8\mathbf{a}_z$ V/m at a point on a conductor surface, what is the surface charge density at that point? Assume $\varepsilon = \varepsilon_o$.

(b) Region $y \geq 2$ is occupied by a conductor. If the surface charge on the conductor is -20 nC/m^2, find $\mathbf{D}$ just outside the conductor.

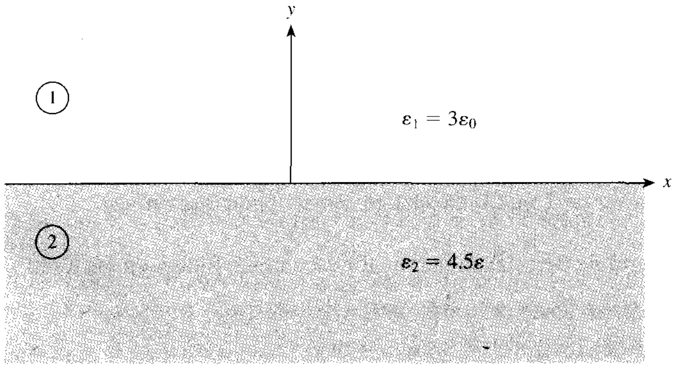


Figure 5.19 For Problem 5.28.

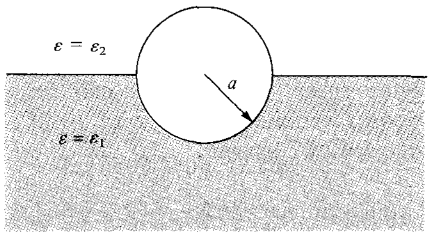


Figure 5.20 For Problem 5.30.

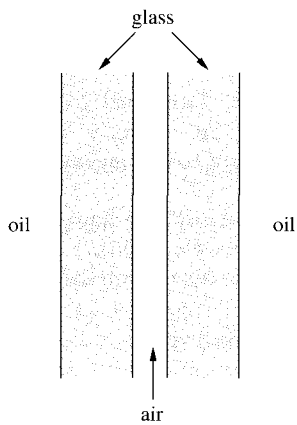


Figure 5.21 For Problem 5.31.

5.33 A silver-coated sphere of radius 5 cm carries a total charge of 12 nC uniformly distributed on its surface in free space. Calculate (a) $|\mathbf{D}|$ on the surface of the sphere, (b) $\mathbf{D}$ external to the sphere, and (c) the total energy stored in the field.

Chapter 6

ELECTROSTATIC BOUNDARY-VALUE PROBLEMS

Our schools had better get on with what is their overwhelmingly most important task: teaching their charges to express themselves clearly and with precision in both speech and writing; in other words, leading them toward mastery of their own language. Failing that, all their instruction in mathematics and science is a waste of time.

—JOSEPH WEIZENBAUM, M.I.T.

6.1 INTRODUCTION

The procedure for determining the electric field **E** in the preceding chapters has generally been using either Coulomb's law or Gauss's law when the charge distribution is known, or using $\mathbf{E} = -\nabla V$ when the potential V is known throughout the region. In most practical situations, however, neither the charge distribution nor the potential distribution is known.

In this chapter, we shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find **E** and V throughout the region. Such problems are usually tackled using Poisson's[1] or Laplace's[2] equation or the method of images, and they are usually referred to as *boundary-value* problems. The concepts of resistance and capacitance will be covered. We shall use Laplace's equation in deriving the resistance of an object and the capacitance of a capacitor. Example 6.5 should be given special attention because we will refer to it often in the remaining part of the text.

6.2 POISSON'S AND LAPLACE'S EQUATIONS

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium)

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \rho_v \tag{6.1}$$

[1]After Simeon Denis Poisson (1781–1840), a French mathematical physicist.

[2]After Pierre Simon de Laplace (1749–1829), a French astronomer and mathematician.

and

$$\mathbf{E} = -\nabla V \tag{6.2}$$

Substituting eq. (6.2) into eq. (6.1) gives

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v \tag{6.3}$$

for an inhomogeneous medium. For a homogeneous medium, eq. (6.3) becomes

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \tag{6.4}$$

This is known as *Poisson's equation.* A special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge-free region). Equation (6.4) then becomes

$$\boxed{\nabla^2 V = 0} \tag{6.5}$$

which is known as *Laplace's equation.* Note that in taking ε out of the left-hand side of eq. (6.3) to obtain eq. (6.4), we have assumed that ε is constant throughout the region in which V is defined; for an inhomogeneous region, ε is not constant and eq. (6.4) does not follow eq. (6.3). Equation (6.3) is Poisson's equation for an inhomogeneous medium; it becomes Laplace's equation for an inhomogeneous medium when $\rho_v = 0$.

Recall that the Laplacian operator ∇^2 was derived in Section 3.8. Thus Laplace's equation in Cartesian, cylindrical, or spherical coordinates respectively is given by

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \tag{6.6}$$

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0} \tag{6.7}$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0} \tag{6.8}$$

depending on whether the potential is $V(x, y, z)$, $V(\rho, \phi, z)$, or $V(r, \theta, \phi)$. Poisson's equation in those coordinate systems may be obtained by simply replacing zero on the right-hand side of eqs. (6.6), (6.7), and (6.8) with $-\rho_v/\varepsilon$.

Laplace's equation is of primary importance in solving electrostatic problems involving a set of conductors maintained at different potentials. Examples of such problems include capacitors and vacuum tube diodes. Laplace's and Poisson's equations are not only useful in solving electrostatic field problem; they are used in various other field problems.

For example, V would be interpreted as magnetic potential in magnetostatics, as temperature in heat conduction, as stress function in fluid flow, and as pressure head in seepage.

6.3 UNIQUENESS THEOREM

Since there are several methods (analytical, graphical, numerical, experimental, etc.) of solving a given problem, we may wonder whether solving Laplace's equation in different ways gives different solutions. Therefore, before we begin to solve Laplace's equation, we should answer this question: If a solution of Laplace's equation satisfies a given set of boundary conditions, is this the only possible solution? The answer is yes: there is only one solution. We say that the solution is unique. Thus any solution of Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used. This is known as the *uniqueness theorem.* The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

The theorem is proved by contradiction. We assume that there are two solutions V_1 and V_2 of Laplace's equation both of which satisfy the prescribed boundary conditions. Thus

$$\nabla^2 V_1 = 0, \qquad \nabla^2 V_2 = 0 \tag{6.9a}$$

$$V_1 = V_2 \qquad \text{on the boundary} \tag{6.9b}$$

We consider their difference

$$V_d = V_2 - V_1 \tag{6.10}$$

which obeys

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \tag{6.11a}$$

$$V_d = 0 \qquad \text{on the boundary} \tag{6.11b}$$

according to eq. (6.9). From the divergence theorem.

$$\int_v \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{6.12}$$

We let $\mathbf{A} = V_d \nabla V_d$ and use a vector identity

$$\nabla \cdot \mathbf{A} = \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d$$

But $\nabla^2 V_d = 0$ according to eq. (6.11), so

$$\nabla \cdot \mathbf{A} = \nabla V_d \cdot \nabla V_d \tag{6.13}$$

Substituting eq. (6.13) into eq. (6.12) gives

$$\int_v \nabla V_d \cdot \nabla V_d\, dv = \oint_S V_d \nabla V_d \cdot d\mathbf{S} \tag{6.14}$$

From eqs. (6.9) and (6.11), it is evident that the right-hand side of eq. (6.14) vanishes.

Hence:

$$\int_v |\nabla V_d|^2 \, dv = 0$$

Since the integration is always positive.

$$\nabla V_d = 0 \tag{6.15a}$$

or

$$V_d = V_2 - V_1 = \text{constant everywhere in } v \tag{6.15b}$$

But eq. (6.15) must be consistent with eq. (6.9b). Hence, $V_d = 0$ or $V_1 = V_2$ everywhere, showing that V_1 and V_2 cannot be different solutions of the same problem.

This is the **uniqueness theorem:** If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Similar steps can be taken to show that the theorem applies to Poisson's equation and to prove the theorem for the case where the electric field (potential gradient) is specified on the boundary.

Before we begin to solve boundary-value problems, we should bear in mind the three things that uniquely describe a problem:

1. The appropriate differential equation (Laplace's or Poisson's equation in this chapter)
2. The solution region
3. The prescribed boundary conditions

A problem does not have a unique solution and cannot be solved completely if any of the three items is missing.

6.4 GENERAL PROCEDURE FOR SOLVING POISSON'S OR LAPLACE'S EQUATION

The following general procedure may be taken in solving a given boundary-value problem involving Poisson's or Laplace's equation:

1. Solve Laplace's (if $\rho_v = 0$) or Poisson's (if $\rho_v \neq 0$) equation using either (a) direct integration when V is a function of one variable, or (b) separation of variables if V is a function of more than one variable. The solution at this point is not unique but expressed in terms of unknown integration constants to be determined.
2. Apply the boundary conditions to determine a unique solution for V. Imposing the given boundary conditions makes the solution unique.
3. Having obtained V, find $\mathbf{E}$ using $\mathbf{E} = -\nabla V$ and $\mathbf{D}$ from $\mathbf{D} = \varepsilon\mathbf{E}$.

4. If desired, find the charge Q induced on a conductor using $Q = \int \rho_S \, dS$ where $\rho_S = D_n$ and D_n is the component of $\mathbf{D}$ normal to the conductor. If necessary, the capacitance between two conductors can be found using $C = Q/V$.

Solving Laplace's (or Poisson's) equation, as in step 1, is not always as complicated as it may seem. In some cases, the solution may be obtained by mere inspection of the problem. Also a solution may be checked by going backward and finding out if it satisfies both Laplace's (or Poisson's) equation and the prescribed boundary conditions.

EXAMPLE 6.1

Current-carrying components in high-voltage power equipment must be cooled to carry away the heat caused by ohmic losses. A means of pumping is based on the force transmitted to the cooling fluid by charges in an electric field. The electrohydrodynamic (EHD) pumping is modeled in Figure 6.1. The region between the electrodes contains a uniform charge ρ_o, which is generated at the left electrode and collected at the right electrode. Calculate the pressure of the pump if $\rho_o = 25$ mC/m^3 and $V_o = 22$ kV.

Solution:

Since $\rho_v \neq 0$, we apply Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

The boundary conditions $V(z = 0) = V_o$ and $V(z = d) = 0$ show that V depends only on z (there is no ρ or ϕ dependence). Hence

$$\frac{d^2V}{dz^2} = \frac{-\rho_o}{\varepsilon}$$

Integrating once gives

$$\frac{dV}{dz} = \frac{-\rho_o z}{\varepsilon} + A$$

Integrating again yields

$$V = -\frac{\rho_o z^2}{2\varepsilon} + Az + B$$

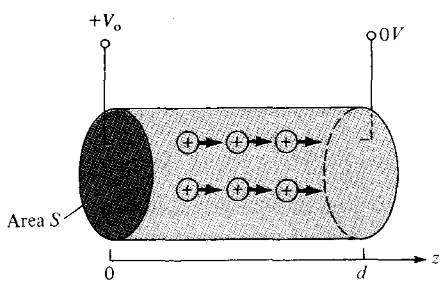


Figure 6.1 An electrohydrodynamic pump; for Example 6.1.

where A and B are integration constants to be determined by applying the boundary conditions. When $z = 0$, $V = V_o$,

$$V_o = -0 + 0 + B \rightarrow B = V_o$$

When $z = d$, $V = 0$,

$$0 = -\frac{\rho_o d^2}{2\varepsilon} + Ad + V_o$$

or

$$A = \frac{\rho_o d}{2\varepsilon} - \frac{V_o}{d}$$

The electric field is given by

$$\begin{aligned}\mathbf{E} &= -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = \left(\frac{\rho_o z}{\varepsilon} - A\right)\mathbf{a}_z \\ &= \left[\frac{V_o}{d} + \frac{\rho_o}{\varepsilon}\left(z - \frac{d}{2}\right)\right]\mathbf{a}_z\end{aligned}$$

The net force is

$$\begin{aligned}\mathbf{F} &= \int \rho_v \mathbf{E}\, dv = \rho_o \int dS \int_{z=0}^{d} \mathbf{E}\, dz \\ &= \rho_o S \left[\frac{V_o z}{d} + \frac{\rho_o}{2\varepsilon}(z^2 - dz)\right]\Bigg|_0^d \mathbf{a}_z \\ \mathbf{F} &= \rho_o S V_o \mathbf{a}_z\end{aligned}$$

The force per unit area or pressure is

$$p = \frac{F}{S} = \rho_o V_o = 25 \times 10^{-3} \times 22 \times 10^3 = 550 \text{ N/m}^2$$

PRACTICE EXERCISE 6.1

In a one-dimensional device, the charge density is given by $\rho_v = \rho_o x/a$. If $\mathbf{E} = 0$ at $x = 0$ and $V = 0$ at $x = a$, find V and $\mathbf{E}$.

Answer: $\frac{\rho_o}{6\varepsilon a}(a^3 - x^3)$, $\frac{\rho_o x^2}{2a\varepsilon}\mathbf{a}_x$

EXAMPLE 6.2

The xerographic copying machine is an important application of electrostatics. The surface of the photoconductor is initially charged uniformly as in Figure 6.2(a). When light from the document to be copied is focused on the photoconductor, the charges on the lower

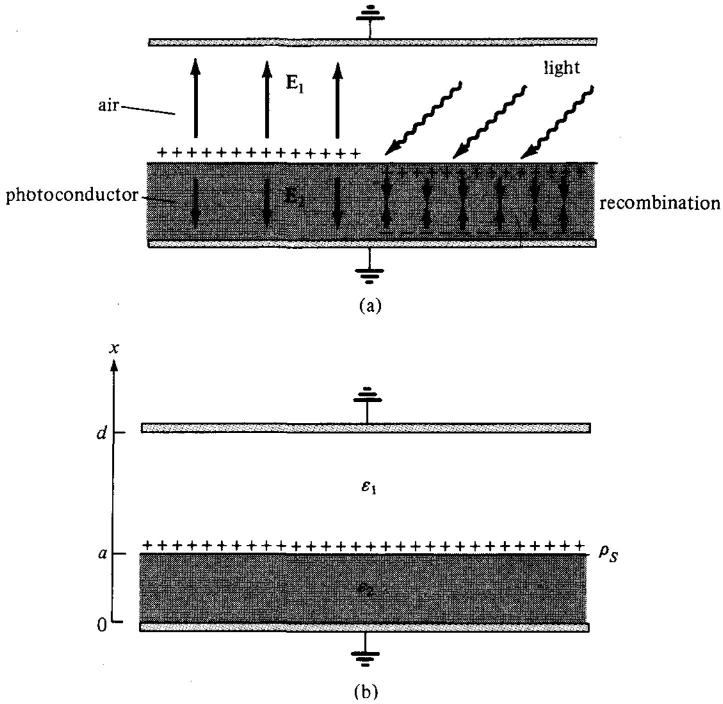


Figure 6.2 For Example 6.2.

surface combine with those on the upper surface to neutralize each other. The image is developed by pouring a charged black powder over the surface of the photoconductor. The electric field attracts the charged powder, which is later transferred to paper and melted to form a permanent image. We want to determine the electric field below and above the surface of the photoconductor.

Solution:

Consider the modeled version of Figure 6.2(a) as in Figure 6.2(b). Since $\rho_v = 0$ in this case, we apply Laplace's equation. Also the potential depends only on x. Thus

$$\nabla^2 V = \frac{d^2V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

Let the potentials above and below be V_1 and V_2, respectively.

$$V_1 = A_1x + B_1, \qquad x > a \tag{6.2.1a}$$

$$V_2 = A_2x + B_2, \qquad x < a \tag{6.2.1b}$$

The boundary conditions at the grounded electrodes are

$$V_1(x = d) = 0 \tag{6.2.2.a}$$

$$V_2(x = 0) = 0 \tag{6.2.2b}$$

At the surface of the photoconductor,

$$V_1(x = a) = V_2(x = a) \tag{6.2.3a}$$

$$D_{1n} - D_{2n} = \rho_S \Big|_{x=a} \tag{6.2.3b}$$

We use the four conditions in eqs. (6.2.2) and (6.2.3) to determine the four unknown constants A_1, A_2, B_1, and B_2. From eqs. (6.2.1) and 6.2.2),

$$0 = A_1 d + B_1 \to B_1 = -A_1 d \tag{6.2.4a}$$

$$0 = 0 + B_2 \to B_2 = 0 \tag{6.2.4b}$$

From eqs. (6.2.1) and (6.2.3a),

$$A_1 a + B_1 = A_2 a \tag{6.2.5}$$

To apply eq. (6.2.3b), recall that $\mathbf{D} = \varepsilon\mathbf{E} = -\varepsilon\nabla V$ so that

$$\rho_S = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = -\varepsilon_1 \frac{dV_1}{dx} + \varepsilon_2 \frac{dV_2}{dx}$$

or

$$\rho_S = -\varepsilon_1 A_1 + \varepsilon_2 A_2 \tag{6.2.6}$$

Solving for A_1 and A_2 in eqs. (6.2.4) to (6.2.6), we obtain

$$\mathbf{E}_1 = -A_1 \mathbf{a}_x = \frac{\rho_S \mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}$$

$$\mathbf{E}_2 = -A_2 \mathbf{a}_x = \frac{-\rho_S \left(\dfrac{d}{a} - 1\right)\mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}$$

PRACTICE EXERCISE 6.2

For the model of Figure 6.2(b), if $\rho_S = 0$ and the upper electrode is maintained at V_o while the lower electrode is grounded, show that

$$\mathbf{E}_1 = \frac{-V_o \mathbf{a}_x}{d - a + \dfrac{\varepsilon_1}{\varepsilon_2} a}, \qquad \mathbf{E}_2 = \frac{-V_o \mathbf{a}_x}{a + \dfrac{\varepsilon_2}{\varepsilon_1} d - \dfrac{\varepsilon_2}{\varepsilon_1} a}$$

EXAMPLE 6.3

Semiinfinite conducting planes $\phi = 0$ and $\phi = \pi/6$ are separated by an infinitesimal insulating gap as in Figure 6.3. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V, calculate V and $\mathbf{E}$ in the region between the planes.

Solution:

As V depends only on ϕ, Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

Since $\rho = 0$ is excluded due to the insulating gap, we can multiply by ρ^2 to obtain

$$\frac{d^2V}{d\phi^2} = 0$$

which is integrated twice to give

$$V = A\phi + B$$

We apply the boundary conditions to determine constants A and B. When $\phi = 0$, $V = 0$,

$$0 = 0 + B \rightarrow B = 0$$

When $\phi = \phi_o$, $V = V_o$,

$$V_o = A\phi_o \rightarrow A = \frac{V_o}{\phi_o}$$

Hence:

$$V = \frac{V_o}{\phi_o}\phi$$

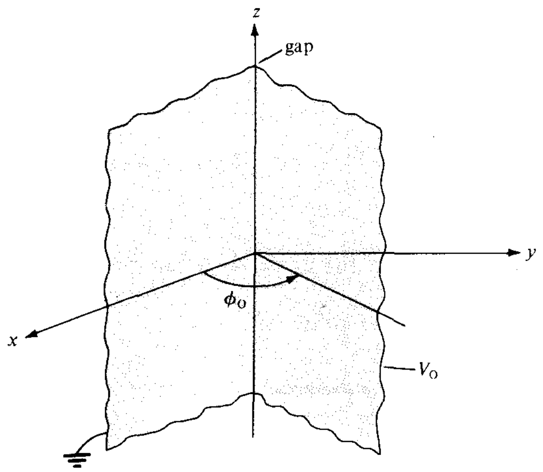


Figure 6.3 Potential $V(\phi)$ due to semi-infinite conducting planes.

and

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi$$

Substituting $V_o = 100$ and $\phi_o = \pi/6$ gives

$$V = \frac{600}{\pi}\phi \quad \text{and} \quad \mathbf{E} = \frac{600}{\pi\rho}\mathbf{a}_\phi$$

Check: $\nabla^2 V = 0$, $V(\phi = 0) = 0$, $V(\phi = \pi/6) = 100$.

PRACTICE EXERCISE 6.3

Two conducting plates of size 1×5 m are inclined at 45° to each other with a gap of width 4 mm separating them as shown in Figure 6.4. Determine an approximate value of the charge per plate if the plates are maintained at a potential difference of 50 V. Assume that the medium between them has $\varepsilon_r = 1.5$.

Answer: 22.2 nC.

EXAMPLE 6.4

Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V, find V and $\mathbf{E}$ between the cones.

Solution:

Consider the coaxial cone of Figure 6.5, where the gap serves as an insulator between the two conducting cones. V depends only on θ, so Laplace's equation in spherical coordinates becomes

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dV}{d\theta}\right] = 0$$

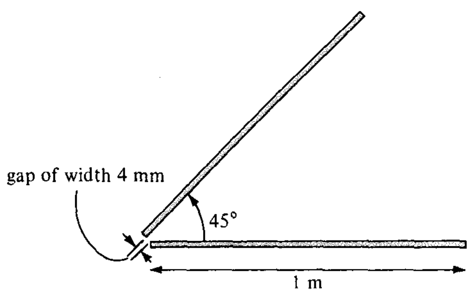


Figure 6.4 For Practice Exercise 6.3.

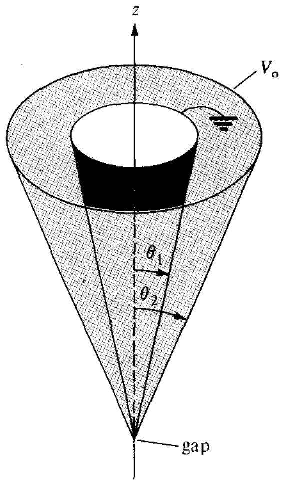


Figure 6.5 Potential $V(\phi)$ due to conducting cones.

Since $r = 0$ and $\theta = 0, \pi$ are excluded, we can multiply by $r^2 \sin\theta$ to get

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating once gives

$$\sin\theta \frac{dV}{d\theta} = A$$

or

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating this results in

$$\begin{aligned} V &= A \int \frac{d\theta}{\sin\theta} = A \int \frac{d\theta}{2\cos\theta/2 \sin\theta/2} \\ &= A \int \frac{1/2 \sec^2\theta/2 \, d\theta}{\tan\theta/2} \\ &= A \int \frac{d(\tan\theta/2)}{\tan\theta/2} \\ &= A \ln(\tan\theta/2) + B \end{aligned}$$

We now apply the boundary conditions to determine the integration constants A and B.

$$V(\theta = \theta_1) = 0 \rightarrow 0 = A \ln(\tan\theta_1/2) + B$$

or

$$B = -A \ln(\tan\theta_1/2)$$

Hence

$$V = A \ln\left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

Also

$$V(\theta = \theta_2) = V_o \rightarrow V_o = A \ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]$$

or

$$A = \frac{V_o}{\ln\left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}$$

Thus

$$V = \frac{V_o \ln\left[\dfrac{\tan \theta/2}{\tan \theta_1/2}\right]}{\ln\left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}$$

$$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{A}{r \sin\theta}\,\mathbf{a}_\theta$$

$$= -\frac{V_o}{r \sin\theta \ln\left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}\,\mathbf{a}_\theta$$

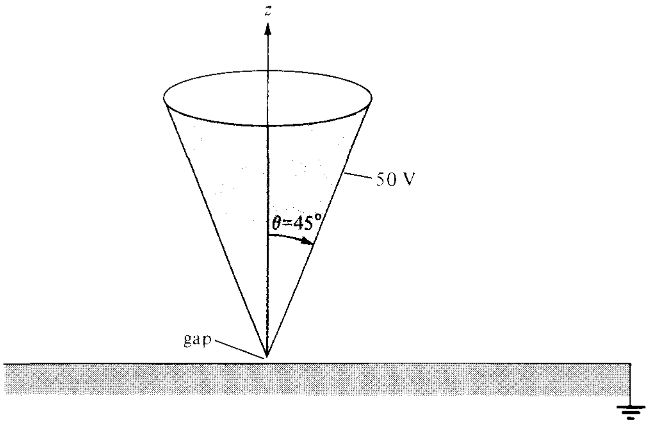
Taking $\theta_1 = \pi/10$, $\theta_2 = \pi/6$, and $V_o = 50$ gives

$$V = \frac{50 \ln\left[\dfrac{\tan \theta/2}{\tan \pi/20}\right]}{\ln\left[\dfrac{\tan \pi/12}{\tan \pi/20}\right]} = 95.1 \ln\left[\frac{\tan \theta/2}{0.1584}\right] V$$

and

$$\mathbf{E} = -\frac{95.1}{r \sin\theta}\,\mathbf{a}_\theta \text{ V/m}$$

Check: $\nabla^2 V = 0$, $V(\theta = \pi/10) = 0$, $V(\theta = \pi/6) = V_o$.



For Practice Exercise 6.4.

PRACTICE EXERCISE 6.4

A large conducting cone ($\theta = 45°$) is placed on a conducting plane with a tiny gap separating it from the plane as shown in Figure 6.6. If the cone is connected to a 50-V source, find V and $\mathbf{E}$ at $(-3, 4, 2)$.

Answer: 22.13 V, 11.36 $\mathbf{a}_\theta$ V/m.

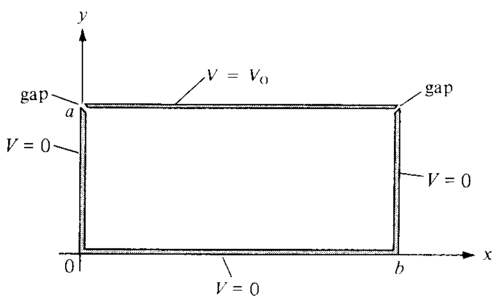
(a) Determine the potential function for the region inside the rectangular trough of infinite length whose cross section is shown in Figure 6.7.

(b) For $V_o = 100$ V and $b = 2a$, find the potential at $x = a/2$, $y = 3a/4$.

Solution:

(a) The potential V in this case depends on x and y. Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{6.5.1}$$



Potential $V(x, y)$ due to a conducting rectangular trough.

We have to solve this equation subject to the following boundary conditions:

$$V(x = 0, 0 \leq y \leq a) = 0 \tag{6.5.2a}$$

$$V(x = b, 0 \leq y \leq a) = 0 \tag{6.5.2b}$$

$$V(0 \leq x \leq b, y = 0) = 0 \tag{6.5.2c}$$

$$V(0 \leq x \leq b, y = a) = V_o \tag{6.5.2d}$$

We solve eq. (6.5.1) by the method of *separation of variables;* that is, we seek a product solution of V. Let

$$V(x, y) = X(x)\, Y(y) \tag{6.5.3}$$

when X is a function of x only and Y is a function of y only. Substituting eq. (6.5.3) into eq. (6.5.1) yields

$$X''Y + Y''X = 0$$

Dividing through by XY and separating X from Y gives

$$-\frac{X''}{X} = \frac{Y''}{Y} \tag{6.5.4a}$$

Since the left-hand side of this equation is a function of x only and the right-hand side is a function of y only, for the equality to hold, both sides must be equal to a constant λ; that is

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda \tag{6.5.4b}$$

The constant λ is known as the *separation constant.* From eq. (6.5.4b), we obtain

$$X'' + \lambda X = 0 \tag{6.5.5a}$$

and

$$Y'' - \lambda Y = 0 \tag{6.5.5b}$$

Thus the variables have been separated at this point and we refer to eq. (6.5.5) as *separated equations.* We can solve for *X(x)* and *Y(y)* separately and then substitute our solutions into eq. (6.5.3). To do this requires that the boundary conditions in eq. (6.5.2) be separated, if possible. We separate them as follows:

$$V(0, y) = X(0)Y(y) = 0 \rightarrow X(0) = 0 \tag{6.5.6a}$$

$$V(b, y) = X(b)Y(y) = 0 \rightarrow X(b) = 0 \tag{6.5.6b}$$

$$V(x, 0) = X(x)Y(0) = 0 \rightarrow Y(0) = 0 \tag{6.5.6c}$$

$$V(x, a) = X(0)Y(a) = V_o \text{ (inseparable)} \tag{6.5.6d}$$

To solve for $X(x)$ and $Y(y)$ in eq. (6.5.5), we impose the boundary conditions in eq. (6.5.6). We consider possible values of λ that will satisfy both the separated equations in eq. (6.5.5) and the conditions in eq. (6.5.6).

CASE A.

If $\lambda = 0$, then eq. (6.5.5a) becomes

$$X'' = 0 \qquad \text{or} \qquad \frac{d^2X}{dx^2} = 0$$

which, upon integrating twice, yields

$$X = Ax + B \tag{6.5.7}$$

The boundary conditions in eqs. (6.5.6a) and (6.5.6b) imply that

$$X(x = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = A \cdot b + 0 \qquad \text{or} \qquad A = 0$$

because $b \neq 0$. Hence our solution for X in eq. (6.5.7) becomes

$$X(x) = 0$$

which makes $V = 0$ in eq. (6.5.3). Thus we regard $X(x) = 0$ as a trivial solution and we conclude that $\lambda \neq 0$.

CASE B.

If $\lambda < 0$, say $\lambda = -\alpha^2$, then eq. (6.5.5a) becomes

$$X'' - \alpha^2 X = 0 \qquad \text{or} \qquad (D^2 - \alpha^2)X = 0$$

where $D = \dfrac{d}{dx}$

that is,

$$DX = \pm \alpha X \tag{6.5.8}$$

showing that we have two possible solutions corresponding to the plus and minus signs. For the plus sign, eq. (6.5.8) becomes

$$\frac{dX}{dx} = \alpha X \qquad \text{or} \qquad \frac{dX}{X} = \alpha\, dx$$

Hence

$$\int \frac{dX}{X} = \int \alpha\, dx \qquad \text{or} \qquad \ln X = \alpha x + \ln A_1$$

where $\ln A_1$ is a constant of integration. Thus

$$X = A_1 e^{\alpha x} \tag{6.5.9a}$$

Similarly, for the minus sign, solving eq. (6.5.8) gives

$$X = A_2 e^{-\alpha x} \tag{6.5.9b}$$

The total solution consists of what we have in eqs. (6.5.9a) and (6.5.9b); that is,

$$X(x) = A_1 e^{\alpha x} + A_2 e^{-\alpha x} \tag{6.5.10}$$

Since $\cosh \alpha x = (e^{\alpha x} + e^{-\alpha x})/2$ and $\sinh \alpha x = (e^{\alpha x} - e^{-\alpha x})/2$ or $e^{\alpha x} = \cosh \alpha x + \sinh \alpha x$ and $e^{-\alpha x} = \cosh \alpha x - \sinh \alpha x$, eq. (6.5.10) can be written as

$$X(x) = B_1 \cosh \alpha x + B_2 \sinh \alpha x \tag{6.5.11}$$

where $B_1 = A_1 + A_2$ and $B_2 = A_1 - A_2$. In view of the given boundary conditions, we prefer eq. (6.5.11) to eq. (6.5.10) as the solution. Again, eqs. (6.5.6a) and (6.5.6b) require that

$$X(x = 0) = 0 \rightarrow 0 = B_1 \cdot (1) + B_2 \cdot (0) \qquad \text{or} \qquad B_1 = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + B_2 \sinh \alpha b$$

Since $\alpha \neq 0$ and $b \neq 0$, $\sinh \alpha b$ cannot be zero. This is due to the fact that $\sinh x = 0$ if and only if $x = 0$ as shown in Figure 6.8. Hence $B_2 = 0$ and

$$X(x) = 0$$

This is also a trivial solution and we conclude that λ cannot be less than zero.

CASE C.

If $\lambda > 0$, say $\lambda = \beta^2$, then eq. (6.5.5a) becomes

$$X'' + \beta^2 X = 0$$

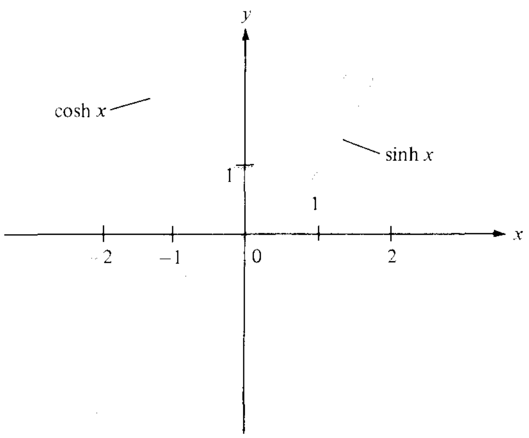


Figure 6.8 Sketch of $\cosh x$ and $\sinh x$ showing that $\sinh x = 0$ if and only if $x = 0$.

that is,

$$(D^2 + \beta^2)X = 0 \qquad \text{or} \qquad DX = \pm j\beta X \tag{6.5.12}$$

where $j = \sqrt{-1}$. From eqs. (6.5.8) and (6.5.12), we notice that the difference between Cases 2 and 3 is replacing α by $j\beta$. By taking the same procedure as in Case 2, we obtain the solution as

$$X(x) = C_o e^{j\beta x} + C_1 e^{-j\beta x} \tag{6.5.13a}$$

Since $e^{j\beta x} = \cos\beta x + j\sin\beta x$ and $e^{-j\beta x} = \cos\beta x - j\sin\beta x$, eq. (6.5.13a) can be written as

$$X(x) = g_o\cos\beta x + g_1\sin\beta x \tag{6.5.13b}$$

where $g_o = C_o + C_1$ and $g_1 = C_o - jC_1$.

In view of the given boundary conditions, we prefer to use eq. (6.5.13b). Imposing the conditions in eqs. (6.5.6a) and (6.5.6b) yields

$$X(x = 0) = 0 \rightarrow 0 = g_o \cdot (1) + 0 \qquad \text{or} \qquad g_o = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + g_1\sin\beta b$$

Suppose $g_1 \neq 0$ (otherwise we get a trivial solution), then

$$\sin\beta b = 0 = \sin n\pi$$

$$\beta = \frac{n\pi}{b}, \qquad n = 1, 2, 3, 4, \ldots \tag{6.5.14}$$

Note that, unlike $\sinh x$, which is zero only when $x = 0$, $\sin x$ is zero at an infinite number of points as shown in Figure 6.9. It should also be noted that $n \neq 0$ because $\beta \neq 0$; we have already considered the possibility $\beta = 0$ in Case 1 where we ended up with a trivial solution. Also we do not need to consider $n = -1, -2, -3, -4, \ldots$ because $\lambda = \beta^2$

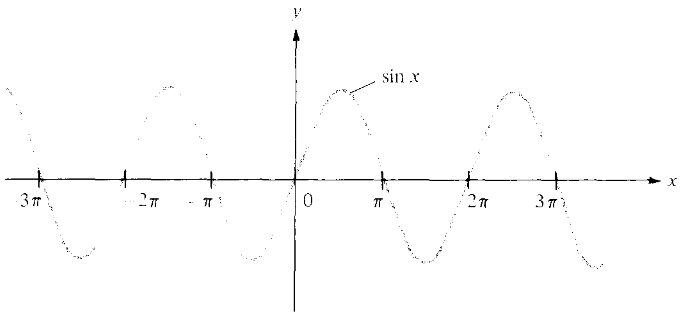


Figure 6.9 Sketch of $\sin x$ showing that $\sin x = 0$ at infinite number of points.

would remain the same for positive and negative values of n. Thus for a given n, eq. (6.5.13b) becomes

$$X_n(x) = g_n \sin \frac{n\pi x}{b} \tag{6.5.15}$$

Having found $X(x)$ and

$$\lambda = \beta^2 = \frac{n^2\pi^2}{b^2} \tag{6.5.16}$$

we solve eq. (6.5.5b) which is now

$$Y'' - \beta^2 Y = 0$$

The solution to this is similar to eq. (6.5.11) obtained in Case 2 that is,

$$Y(y) = h_o \cosh \beta y + h_1 \sinh \beta y$$

The boundary condition in eq. (6.5.6c) implies that

$$Y(y = 0) = 0 \rightarrow 0 = h_o \cdot (1) + 0 \qquad \text{or} \qquad h_o = 0$$

Hence our solution for $Y(y)$ becomes

$$Y_n(y) = h_n \sinh \frac{n\pi y}{b} \tag{6.5.17}$$

Substituting eqs. (6.5.15) and (6.5.17), which are the solutions to the separated equations in eq. (6.5.5), into the product solution in eq. (6.5.3) gives

$$V_n(x, y) = g_n h_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}$$

This shows that there are many possible solutions V_1, V_2, V_3, V_4, and so on, for $n =$ 1, 2, 3, 4, and so on.

By the *superposition theorem,* if $V_1, V_2, V_3, \ldots, V_n$ are solutions of Laplace's equation, the linear combination

$$V = c_1V_1 + c_2V_2 + c_3V_3 + \cdots + c_nV_n$$

(where $c_1, c_2, c_3 \ldots, c_n$ are constants) is also a solution of Laplace's equation. Thus the solution to eq. (6.5.1) is

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{6.5.18}$$

where $c_n = g_nh_n$ are the coefficients to be determined from the boundary condition in eq. (6.5.6d). Imposing this condition gives

$$V(x, y = a) = V_o = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b} \tag{6.5.19}$$

which is a Fourier series expansion of V_o. Multiplying both sides of eq. (6.5.19) by $\sin m\pi x/b$ and integrating over $0 < x < b$ gives

$$\int_0^b V_o \sin\frac{m\pi x}{b}\,dx = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi a}{b}\int_0^b \sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}\,dx \tag{6.5.20}$$

By the orthogonality property of the sine or cosine function (see Appendix A.9).

$$\int_0^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}$$

Incorporating this property in eq. (6.5.20) means that all terms on the right-hand side of eq. (6.5.20) will vanish except one term in which $m = n$. Thus eq. (6.5.20) reduces to

$$\int_0^b V_o \sin\frac{n\pi x}{b}\,dx = c_n \sinh\frac{n\pi a}{b}\int_0^b \sin^2\frac{n\pi x}{b}\,dx$$

$$-V_o\frac{b}{n\pi}\cos\frac{n\pi x}{b}\bigg|_0^b = c_n \sinh\frac{n\pi a}{b}\,\frac{1}{2}\int_0^b\left(1-\cos\frac{2n\pi x}{b}\right)dx$$

$$\frac{V_o b}{n\pi}(1-\cos n\pi) = c_n\sinh\frac{n\pi a}{b}\cdot\frac{b}{2}$$

or

$$c_n \sinh\frac{n\pi a}{b} = \frac{2V_o}{n\pi}(1-\cos n\pi) = \begin{cases}\dfrac{4V_o}{n\pi}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots\end{cases}$$

that is,

$$c_n = \begin{cases}\dfrac{4V_o}{n\pi\sinh\dfrac{n\pi a}{b}}, & n = \text{odd} \\ 0, & n = \text{even}\end{cases} \tag{6.5.21}$$

Substituting this into eq. (6.5.18) gives the complete solution as

$$V(x, y) = \frac{4V_o}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\sin\dfrac{n\pi x}{b}\sinh\dfrac{n\pi y}{b}}{n\sinh\dfrac{n\pi a}{b}} \tag{6.5.22}$$

Check: $\nabla^2 V = 0$, $V(x = 0, y) = 0 = V(x = b, y) = V(x, y, = 0)$, $V(x, y = a) = V_o$. The solution in eq. (6.5.22) should not be a surprise; it can be guessed by mere observation of the potential system in Figure 6.7. From this figure, we notice that along x, V varies from

0 (at $x = 0$) to 0 (at $x = b$) and only a sine function can satisfy this requirement. Similarly, along y, V varies from 0 (at $y = 0$) to V_o (at $y = a$) and only a hyperbolic sine function can satisfy this. Thus we should expect the solution as in eq. (6.5.22).

To determine the potential for each point (x, y) in the trough, we take the first few terms of the convergent infinite series in eq. (6.5.22). Taking four or five terms may be sufficient.

(b) For $x = a/2$ and $y = 3a/4$, where $b = 2a$, we have

$$\begin{aligned} V\left(\frac{a}{2}, \frac{3a}{4}\right) &= \frac{4V_o}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin n\pi/4 \sinh 3n\pi/8}{n \sinh n\pi/2} \\ &= \frac{4V_o}{\pi}\left[\frac{\sin \pi/4 \sinh 3\pi/8}{\sinh \pi/2} + \frac{\sin 3\pi/4 \sinh 9\pi/8}{3 \sinh 3\pi/2} \right. \\ &\quad \left. + \frac{\sin 5\pi/4 \sinh 15\pi/4}{5 \sinh 5\pi/4} + \cdots\right] \\ &= \frac{4V_o}{\pi}(0.4517 + 0.0725 - 0.01985 - 0.00645 + 0.00229 + \cdots) \\ &= 0.6374V_o \end{aligned}$$

It is instructive to consider a special case when $A = b = 1$ m and $V_o = 100$ V. The potentials at some specific points are calculated using eq. (6.5.22) and the result is displayed in Figure 6.10(a). The corresponding flux lines and equipotential lines are shown in Figure 6.10(b). A simple Matlab program based on eq. (6.5.22) is displayed in Figure 6.11. This self-explanatory program can be used to calculate $V(x, y)$ at any point within the trough. In Figure 6.11, $V(x = b/4, y = 3a/4)$ is typically calculated and found to be 43.2 volts.

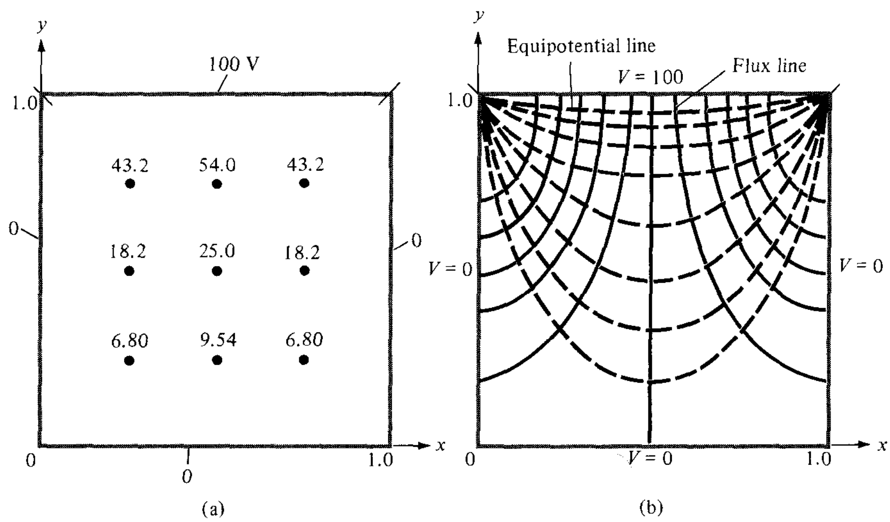


Figure 6.10 For Example 6.5: **(a)** $V(x, y)$ calculated at some points, **(b)** sketch of flux lines and equipotential lines.

```
% SOLUTION OF LAPLACE'S EQUATION
% ------------------------------
% THIS PROGRAM SOLVES THE TWO-DIMENSIONAL
% BOUNDARY-VALUE PROBLEM DESCRIBED IN FIG. 6.7
% a AND b ARE THE DIMENSIONS OF THE TROUGH
% x AND y ARE THE COORDINATES OF THE POINT
% OF INTEREST

P = [ ];
Vo = 100.0;
a = 1.0;
b = a;
x = b/4;
y= 3.*a/4.;
c = 4.*Vo/pi
sum = 0.0;
for k=1:10
   n = 2*k - 1
   a1 = sin(n*pi*x/b);
   a2 = sinh(n*pi*y/b);
   a3 = n*sinh(n*pi*a/b);
   sum = sum + c*a1*a2/a3;
   P = [n, sum]
end
diary test.out
P
diary off
```

Figure 6.11 Matlab program for Example 6.5.

PRACTICE EXERCISE 6.5

For the problem in Example 6.5, take $V_o = 100$ V, $b = 2a = 2$ m, find V and $\mathbf{E}$ at

(a) $(x, y) = (a, a/2)$

(b) $(x, y) = (3a/2, a/4)$

Answer: (a) 44.51 V, $-99.25\,\mathbf{a}_y$ V/m, (b) 16.5 V, $20.6\,\mathbf{a}_x - 70.34\,\mathbf{a}_y$ V/m.

EXAMPLE 6.6

In the last example, find the potential distribution if V_o is not constant but

(a) $V_o = 10 \sin 3\pi x/b$, $y = a$, $0 \leq x \leq b$

(b) $V_o = 2 \sin \dfrac{\pi x}{b} + \dfrac{1}{10} \sin \dfrac{5\pi x}{b}$, $y = a$, $0 \leq x \leq b$

Solution:

(a) In the last example, every step before eq. (6.5.19) remains the same; that is, the solution is of the form

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{6.6.1}$$

as per eq. (6.5.18). But instead of eq. (6.5.19), we now have

$$V(y = a) = V_o = 10 \sin \frac{3\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

By equating the coefficients of the sine terms on both sides, we obtain

$$c_n = 0, \qquad n \neq 3$$

For $n = 3$,

$$10 = c_3 \sinh \frac{3\pi a}{b}$$

or

$$c_3 = \frac{10}{\sinh \dfrac{3\pi a}{b}}$$

Thus the solution in eq. (6.6.1) becomes

$$V(x, y) = 10 \sin \frac{3\pi x}{b} \frac{\sinh \dfrac{3\pi y}{b}}{\sinh \dfrac{3\pi a}{b}}$$

(b) Similarly, instead of eq. (6.5.19), we have

$$V_o = V(y = a)$$

or

$$2 \sin \frac{\pi x}{b} + \frac{1}{10} \sinh \frac{5\pi x}{b} = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

Equating the coefficient of the sine terms:

$$c_n = 0, \qquad n \neq 1, 5$$

For $n = 1$,

$$2 = c_1 \sinh \frac{\pi a}{b} \quad \text{or} \quad c_1 = \frac{2}{\sinh \frac{\pi a}{b}}$$

For $n = 5$,

$$\frac{1}{10} = c_5 \sinh \frac{5\pi a}{b} \quad \text{or} \quad c_5 = \frac{1}{10 \sinh \frac{5\pi a}{b}}$$

Hence,

$$V(x, y) = \frac{2 \sin \frac{\pi x}{b} \sinh \frac{\pi y}{b}}{\sinh \frac{\pi a}{b}} + \frac{\sin \frac{5\pi x}{b} \sinh \frac{5\pi y}{b}}{10 \sinh \frac{5\pi a}{b}}$$

PRACTICE EXERCISE 6.6

In Example 6.5, suppose everything remains the same except that V_o is replaced by $V_o \sin \frac{7\pi x}{b}$, $0 \leq x \leq b$, $y = a$. Find $V(x, y)$.

Answer: $\dfrac{V_o \sin \frac{7\pi x}{b} \sinh \frac{7\pi y}{b}}{\sinh \frac{7\pi a}{b}}$

EXAMPLE 6.7

Obtain the separated differential equations for potential distribution $V(\rho, \phi, z)$ in a charge-free region.

Solution:

This example, like Example 6.5, further illustrates the method of separation of variables. Since the region is free of charge, we need to solve Laplace's equation in cylindrical coordinates; that is,

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6.7.1}$$

We let

$$V(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z) \tag{6.7.2}$$

where R, Φ, and Z are, respectively, functions of ρ, ϕ, and z. Substituting eq. (6.7.2) into eq. (6.7.1) gives

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2Z}{dz^2} = 0 \tag{6.7.3}$$

We divide through by $R\Phi Z$ to obtain

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \tag{6.7.4}$$

The right-hand side of this equation is solely a function of z whereas the left-hand side does not depend on z. For the two sides to be equal, they must be constant; that is,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2 \tag{6.7.5}$$

where $-\lambda^2$ is a separation constant. Equation (6.7.5) can be separated into two parts:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda^2 \tag{6.7.6}$$

or

$$Z'' - \lambda^2 Z = 0 \tag{6.7.7}$$

and

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \lambda^2\rho^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{6.7.8}$$

Equation (6.7.8) can be written as

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \lambda^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu^2 \tag{6.7.9}$$

where μ^2 is another separation constant. Equation (6.7.9) is separated as

$$\Phi'' = \mu^2\Phi = 0 \tag{6.7.10}$$

and

$$\rho^2R'' + \rho R' + (\rho^2\lambda^2 - \mu^2)R = 0 \tag{6.7.11}$$

Equations (6.7.7), (6.7.10), and (6.7.11) are the required separated differential equations. Equation (6.7.7) has a solution similar to the solution obtained in Case 2 of Example 6.5; that is,

$$Z(z) = c_1 \cosh \lambda z + c_2 \sinh \lambda z \tag{6.7.12}$$

The solution to eq. (6.7.10) is similar to the solution obtained in Case 3 of Example 6.5; that is,

$$\Phi(\phi) = c_3 \cos \mu\phi + c_4 \sin \mu\phi \tag{6.7.13}$$

Equation (6.7.11) is known as the *Bessel differential equation* and its solution is beyond the scope of this text.[3]

PRACTICE EXERCISE 6.7

Repeat Example 6.7 for $V(r, \theta, \phi)$.

Answer: If $V(r, \theta, \phi) = R(r)\, F(\theta)\, \Phi(\phi)$, $\Phi'' + \lambda^2 \Phi = 0$, $R'' + \frac{2}{r}R' - \frac{\mu^2}{r^2}R = 0$, $F'' + \cot\theta\, F' + (\mu^2 - \lambda^2 \operatorname{cosec}^2 \theta)\, F = 0$.

6.5 RESISTANCE AND CAPACITANCE

In Section 5.4 the concept of resistance was covered and we derived eq. (5.16) for finding the resistance of a conductor of uniform cross section. If the cross section of the conductor is not uniform, eq. (5.16) becomes invalid and the resistance is obtained from eq. (5.17):

$$\boxed{R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}}} \tag{6.16}$$

The problem of finding the resistance of a conductor of nonuniform cross section can be treated as a boundary-value problem. Using eq. (6.16), the resistance R (or conductance $G = 1/R$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume V_o as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine $\mathbf{E}$ from $\mathbf{E} = -\nabla V$ and I from $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$.
4. Finally, obtain R as V_o/I.

In essence, we assume V_o, find I, and determine $R = V_o/I$. Alternatively, it is possible to assume current I_o, find the corresponding potential difference V, and determine R from $R = V/I_o$. As will be discussed shortly, the capacitance of a capacitor is obtained using a similar technique.

[3]For a complete solution of Laplace's equation in cylindrical or spherical coordinates, see, for example, D. T. Paris and F. K. Hurd, *Basic Electromagnetic Theory.* New York: McGraw-Hill, 1969, pp. 150–159.

Generally speaking, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the *plates* of the capacitor. The plates may be separated by free space or a dielectric.

Consider the two-conductor capacitor of Figure 6.12. The conductors are maintained at a potential difference V given by

$$V = V_1 - V_2 = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} \tag{6.17}$$

where **E** is the electric field existing between the conductors and conductor 1 is assumed to carry a positive charge. (Note that the **E** field is always normal to the conducting surfaces.)

We define the *capacitance* C of the capacitor as the ratio of the magnitude of the charge on one of the plates to the potential difference between them; that is,

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}} \tag{6.18}$$

The negative sign before $V = -\int \mathbf{E} \cdot d\mathbf{l}$ has been dropped because we are interested in the absolute value of V. The capacitance C is a physical property of the capacitor and in measured in farads (F). Using eq. (6.18), C can be obtained for any given two-conductor capacitance by following either of these methods:

1. Assuming Q and determining V in terms of Q (involving Gauss's law)
2. Assuming V and determining Q in terms of V (involving solving Laplace's equation)

We shall use the former method here, and the latter method will be illustrated in Examples 6.10 and 6.11. The former method involves taking the following steps:

1. Choose a suitable coordinate system.
2. Let the two conducting plates carry charges $+Q$ and $-Q$.

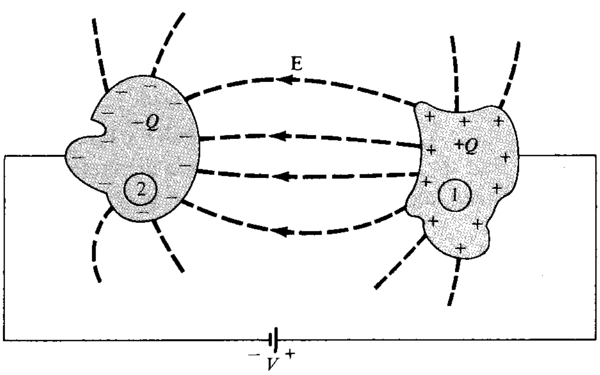


Figure 6.12 A two-conductor capacitor.

3. Determine **E** using Coulomb's or Gauss's law and find V from $V = -\int \mathbf{E} \cdot d\mathbf{l}$. The negative sign may be ignored in this case because we are interested in the absolute value of V.
4. Finally, obtain C from $C = Q/V$.

We will now apply this mathematically attractive procedure to determine the capacitance of some important two-conductor configurations.

A. Parallel-Plate Capacitor

Consider the parallel-plate capacitor of Figure 6.13(a). Suppose that each of the plates has an area S and they are separated by a distance d. We assume that plates 1 and 2, respectively, carry charges $+Q$ and $-Q$ uniformly distributed on them so that

$$\rho_S = \frac{Q}{S} \tag{6.19}$$

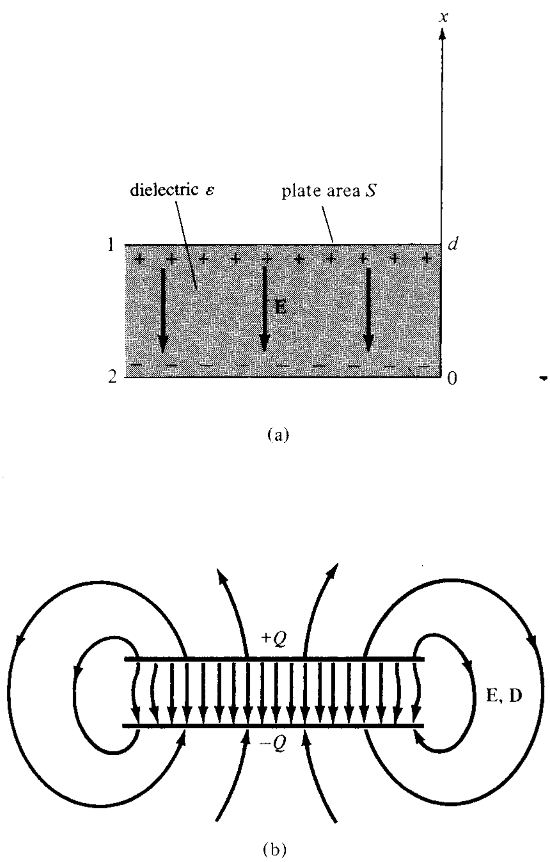


Figure 6.13 **(a)** Parallel-plate capacitor, **(b)** fringing effect due to a parallel-plate capacitor.

An ideal parallel-plate capacitor is one in which the plate separation d is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates, as illustrated in Figure 6.13(b), can be ignored so that the field between them is considered uniform. If the space between the plates is filled with a homogeneous dielectric with permittivity ε and we ignore flux fringing at the edges of the plates, from eq. (4.27), $\mathbf{D} = -\rho_S \mathbf{a}_x$ or

$$\begin{aligned} \mathbf{E} &= \frac{\rho_S}{\varepsilon}(-\mathbf{a}_x) \\ &= -\frac{Q}{\varepsilon S}\mathbf{a}_x \end{aligned} \tag{6.20}$$

Hence

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\varepsilon S}\mathbf{a}_x\right] \cdot dx\, \mathbf{a}_x = \frac{Qd}{\varepsilon S} \tag{6.21}$$

and thus for a parallel-plate capacitor

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon S}{d}} \tag{6.22}$$

This formula offers a means of measuring the dielectric constant ε_r of a given dielectric. By measuring the capacitance C of a parallel-plate capacitor with the space between the plates filled with the dielectric and the capacitance C_o with air between the plates, we find ε_r from

$$\varepsilon_r = \frac{C}{C_o} \tag{6.23}$$

Using eq. (4.96), it can be shown that the energy stored in a capacitor is given by

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}} \tag{6.24}$$

To verify this for a parallel-plate capacitor, we substitute eq. (6.20) into eq. (4.96) and obtain

$$\begin{aligned} W_E &= \frac{1}{2}\int_v \varepsilon \frac{Q^2}{\varepsilon^2 S^2}\, dv = \frac{\varepsilon Q^2 S d}{2\varepsilon^2 S^2} \\ &= \frac{Q^2}{2}\left(\frac{d}{\varepsilon S}\right) = \frac{Q^2}{2C} = \frac{1}{2}QV \end{aligned}$$

as expected.

B. Coaxial Capacitor

This is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length L of two coaxial conductors of inner radius a and outer radius b $(b > a)$ as shown in Figure 6.14. Let the space between the conductors be filled with a homogeneous dielectric with permittivity ε. We assume that conductors 1 and 2, respectively, carry $+Q$ and $-Q$ uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ $(a < \rho < b)$, we obtain

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L \tag{6.25}$$

Hence:

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho \tag{6.26}$$

Neglecting flux fringing at the cylinder ends,

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho\right] \cdot d\rho\,\mathbf{a}_\rho \tag{6.27a}$$

$$= \frac{Q}{2\pi\varepsilon L} \ln\frac{b}{a} \tag{6.27b}$$

Thus the capacitance of a coaxial cylinder is given by

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\dfrac{b}{a}}} \tag{6.28}$$

C. Spherical Capacitor

This is the case of two concentric spherical conductors. Consider the inner sphere of radius a and outer sphere of radius b $(b > a)$ separated by a dielectric medium with permittivity ε as shown in Figure 6.15. We assume charges $+Q$ and $-Q$ on the inner and outer spheres

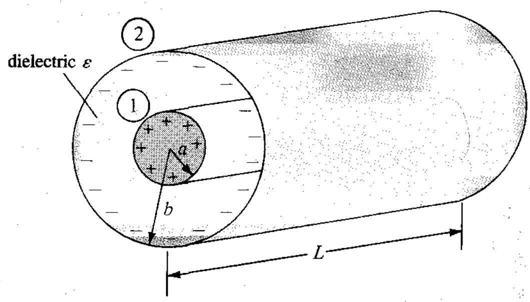


Figure 6.14 Coaxial capacitor.

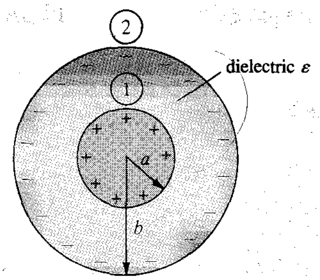


Figure 6.15 Spherical capacitor.

respectively. By applying Gauss's law to an arbitrary Gaussian spherical surface of radius $r\,(a < r < b)$,

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2 \tag{6.29}$$

that is,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r \tag{6.30}$$

The potential difference between the conductors is

$$\begin{aligned} V &= -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\right] \cdot dr\,\mathbf{a}_r \\ &= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \end{aligned} \tag{6.31}$$

Thus the capacitance of the spherical capacitor is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{6.32}$$

By letting $b \to \infty$, $C = 4\pi\varepsilon a$, which is the capacitance of a spherical capacitor whose outer plate is infinitely large. Such is the case of a spherical conductor at a large distance from other conducting bodies—the *isolated sphere*. Even an irregularly shaped object of about the same size as the sphere will have nearly the same capacitance. This fact is useful in estimating the stray capacitance of an isolated body or piece of equipment.

Recall from network theory that if two capacitors with capacitance C_1 and C_2 are in series (i.e., they have the same charge on them) as shown in Figure 6.16(a), the total capacitance is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{6.33}$$

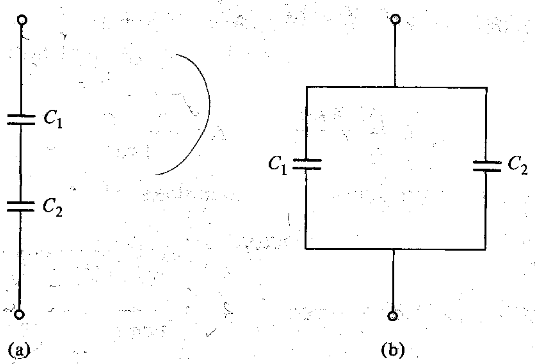


Figure 6.16 Capacitors in **(a)** series, and **(b)** parallel.

If the capacitors are in parallel (i.e., they have the same voltage across their plates) as shown in Figure 6.16(b), the total capacitance is

$$C = C_1 + C_2 \tag{6.34}$$

Let us reconsider the expressions for finding the resistance R and the capacitance C of an electrical system. The expressions were given in eqs. (6.16) and (6.18):

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{6.16}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}} \tag{6.18}$$

The product of these expressions yields

$$\boxed{RC = \frac{\varepsilon}{\sigma}} \tag{6.35}$$

which is the relaxation time T_r of the medium separating the conductors. It should be remarked that eq. (6.35) is valid only when the medium is homogeneous; this is easily inferred from eqs. (6.16) and (6.18). Assuming homogeneous media, the resistance of various capacitors mentioned earlier can be readily obtained using eq. (6.35). The following examples are provided to illustrate this idea.

For a parallel-plate capacitor,

$$C = \frac{\varepsilon S}{d}, \qquad R = \frac{d}{\sigma S} \tag{6.36}$$

For a cylindrical capacitor,

$$C = \frac{2\pi\varepsilon L}{\ln \frac{b}{a}}, \qquad R = \frac{\ln \frac{b}{a}}{2\pi\sigma L} \tag{6.37}$$

For a spherical capacitor,

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}, \qquad R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma} \tag{6.38}$$

And finally for an isolated spherical conductor,

$$C = 4\pi\varepsilon a, \qquad R = \frac{1}{4\pi\sigma a} \tag{6.39}$$

It should be noted that the resistance R in each of eqs. (6.35) to (6.39) is not the resistance of the capacitor plate but the leakage resistance between the plates; therefore, σ in those equations is the conductivity of the dielectric medium separating the plates.

EXAMPLE 6.8

A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 6.17. Show that (a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln \frac{b}{a}}{\sigma\pi t}$$

and (b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Solution:

(a) Between the vertical curved ends located at $\rho = a$ and $\rho = b$, the bar has a nonuniform cross section and hence eq. (5.16) does not apply. We have to use eq. (6.16). Let a potential difference V_o be maintained between the curved surfaces at $\rho = a$ and $\rho = b$ so that

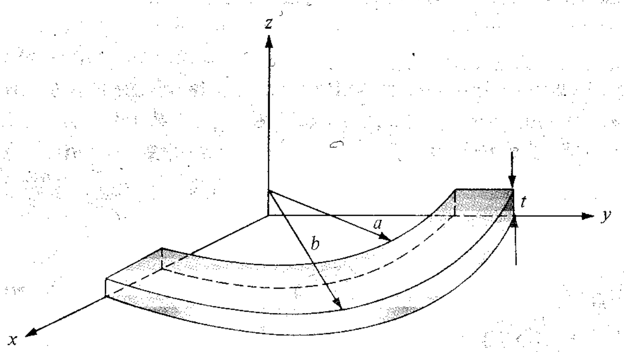


Figure 6.17 Metal bar of Example 6.8.

$V(\rho = a) = 0$ and $V(\rho = b) = V_o$. We solve for V in Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates. Since $V = V(\rho)$,

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

As $\rho = 0$ is excluded, upon multiplying by ρ and integrating once, this becomes

$$\rho \frac{dV}{d\rho} = A$$

or

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating once again yields

$$V = A \ln \rho + B$$

where A and B are constants of integration to be determined from the boundary conditions.

$$V(\rho = a) = 0 \rightarrow 0 = A \ln a + B \qquad \text{or} \qquad B = -A \ln a$$

$$V(\rho = b) = V_o \rightarrow V_o = A \ln b + B = A \ln b - A \ln a = A \ln \frac{b}{a} \qquad \text{or} \qquad A = \frac{V_o}{\ln \frac{b}{a}}$$

Hence,

$$V = A \ln \rho - A \ln a = A \ln \frac{\rho}{a} = \frac{V_o}{\ln \frac{b}{a}} \ln \frac{\rho}{a}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = -\frac{A}{\rho}\mathbf{a}_\rho = -\frac{V_o}{\rho \ln \frac{b}{a}}\mathbf{a}_\rho$$

$$\mathbf{J} = \sigma \mathbf{E}, \qquad d\mathbf{S} = -\rho\, d\phi\, dz\, \mathbf{a}_\rho$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/2} \int_{z=0}^{t} \frac{V_o \sigma}{\rho \ln \frac{b}{a}}\, dz\, \rho\, d\phi = \frac{\pi}{2} \frac{t V_o \sigma}{\ln \frac{b}{a}}$$

Thus

$$R = \frac{V_o}{I} = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

as required.

(b) Let V_o be the potential difference between the two horizontal surfaces so that $V(z = 0) = 0$ and $V(z = t) = V_o$. $V = V(z)$, so Laplace's equation $\nabla^2 V = 0$ becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating twice gives

$$V = Az + B$$

We apply the boundary conditions to determine A and B:

$$V(z = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

$$V(z = t) = V_o \rightarrow V_o = At \qquad \text{or} \qquad A = \frac{V_o}{t}$$

Hence,

$$V = \frac{V_o}{t} z$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{V_o}{t}\mathbf{a}_z$$

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma V_o}{t}\mathbf{a}_z, \qquad d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^{b} \int_{\phi=0}^{\pi/2} \frac{V_o\sigma}{t} \rho \, d\phi \, d\rho$$

$$= \frac{V_o\sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2}\bigg|_a^b = \frac{V_o \, \sigma \, \pi \, (b^2 - a^2)}{4t}$$

Thus

$$R' = \frac{V_o}{I} = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Alternatively, for this case, the cross section of the bar is uniform between the horizontal surfaces at $z = 0$ and $z = t$ and eq. (5.16) holds. Hence,

$$R' = \frac{\ell}{\sigma S} = \frac{t}{\sigma \frac{\pi}{4}(b^2 - a^2)}$$

$$= \frac{4t}{\sigma\pi(b^2 - a^2)}$$

as required.

PRACTICE EXERCISE 6.8

A disc of thickness t has radius b and a central hole of radius a. Taking the conductivity of the disc as σ, find the resistance between

(a) The hole and the rim of the disc

(b) The two flat sides of the disc

Answer: (a) $\dfrac{\ln b/a}{2\pi t\sigma}$, (b) $\dfrac{t}{\sigma\pi(b^2 - a^2)}$.

A coaxial cable contains an insulating material of conductivity σ. If the radius of the central wire is a and that of the sheath is b, show that the conductance of the cable per unit length is (see eq. (6.37))

$$G = \frac{2\pi\sigma}{\ln b/a}$$

Solution:

Consider length L of the coaxial cable as shown in Figure 6.14. Let V_o be the potential difference between the inner and outer conductors so that $V(\rho = a) = 0$ and $V(\rho = b) = V_o$. V and $\mathbf{E}$ can be found just as in part (a) of the last example. Hence:

$$\mathbf{J} = \sigma\mathbf{E} = \frac{-\sigma V_o}{\rho \ln b/a}\,\mathbf{a}_\rho, \qquad d\mathbf{S} = -\rho d\phi\, dz\, \mathbf{a}_\rho$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{L} \frac{V_o\sigma}{\rho \ln b/a}\,\rho\, dz\, d\phi$$
$$= \frac{2\pi L\sigma V_o}{\ln b/a}$$

The resistance per unit length is

$$R = \frac{V_o}{I}\cdot\frac{1}{L} = \frac{\ln b/a}{2\pi\sigma}$$

and the conductance per unit length is

$$G = \frac{1}{R} = \frac{2\pi\sigma}{\ln b/a}$$

as required.

PRACTICE EXERCISE 6.9

A coaxial cable contains an insulating material of conductivity σ_1 in its upper half and another material of conductivity σ_2 in its lower half (similar to the situation in Figure 6.19b). If the radius of the central wire is a and that of the sheath is b, show that the leakage resistance of length ℓ of the cable is

$$R = \frac{1}{\pi\ell(\sigma_1 + \sigma_2)} \ln \frac{b}{a}$$

Answer: Proof.

EXAMPLE 6.10

Conducting spherical shells with radii $a = 10$ cm and $b = 30$ cm are maintained at a potential difference of 100 V such that $V(r = b) = 0$ and $V(r = a) = 100$ V. Determine V and $\mathbf{E}$ in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.

Solution:

Consider the spherical shells shown in Figure 6.18. V depends only on r and hence Laplace's equation becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Since $r \neq 0$ in the region of interest, we multiply through by r^2 to obtain

$$\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Integrating once gives

$$r^2 \frac{dV}{dr} = A$$

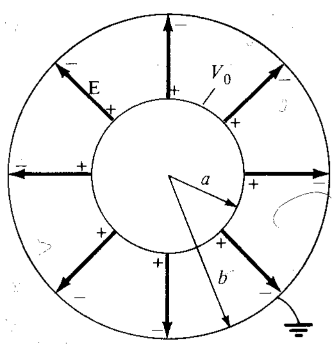


Figure 6.18 Potential $V(r)$ due to conducting spherical shells.

or

$$\frac{dV}{dr} = \frac{A}{r^2}$$

Integrating again gives

$$V = -\frac{A}{r} + B$$

As usual, constants A and B are determined from the boundary conditions.

When $r = b, V = 0 \rightarrow 0 = -\frac{A}{b} + B \quad \text{or} \quad B = \frac{A}{b}$

Hence

$$V = A\left[\frac{1}{b} - \frac{1}{r}\right]$$

Also when $r = a, V = V_o \rightarrow V_o = A\left[\frac{1}{b} - \frac{1}{a}\right]$

or

$$A = \frac{V_o}{\frac{1}{b} - \frac{1}{a}}$$

Thus

$$V = V_o \frac{\left[\frac{1}{r} - \frac{1}{b}\right]}{\frac{1}{a} - \frac{1}{b}}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\,\mathbf{a}_r = -\frac{A}{r^2}\,\mathbf{a}_r$$

$$= \frac{V_o}{r^2\left[\frac{1}{a} - \frac{1}{b}\right]}\,\mathbf{a}_r$$

$$Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\varepsilon_o \varepsilon_r V_o}{r^2\left[\frac{1}{a} - \frac{1}{b}\right]}\, r^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{4\pi\varepsilon_o\varepsilon_r V_o}{\frac{1}{a} - \frac{1}{\text{b}}}$$

The capacitance is easily determined as

$$C = \frac{Q}{V_o} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

which is the same as we obtained in eq. (6.32); there in Section 6.5, we assumed Q and found the corresponding V_o, but here we assumed V_o and found the corresponding Q to determine C. Substituting $a = 0.1$ m, $b = 0.3$ m, $V_o = 100$ V yields

$$V = 100 \frac{\left[\frac{1}{r} - \frac{10}{3}\right]}{10 - 10/3} = 15\left[\frac{1}{r} - \frac{10}{3}\right] V$$

Check: $\nabla^2 V = 0$, $V(r = 0.3 \text{ m}) = 0$, $V(r = 0.1 \text{ m}) = 100$.

$$\mathbf{E} = \frac{100}{r^2 [10 - 10/3]} \mathbf{a}_r = \frac{15}{r^2} \mathbf{a}_r \text{ V/m}$$

$$Q = \pm 4\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{(2.5) \cdot (100)}{10 - 10/3}$$
$$= \pm 4.167 \text{ nC}$$

The positive charge is induced on the inner shell; the negative charge is induced on the outer shell. Also

$$C = \frac{|Q|}{V_o} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \text{ pF}$$

PRACTICE EXERCISE 6.10

If Figure 6.19 represents the cross sections of two spherical capacitors, determine their capacitances. Let $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 0.53 pF, (b) 0.5 pF

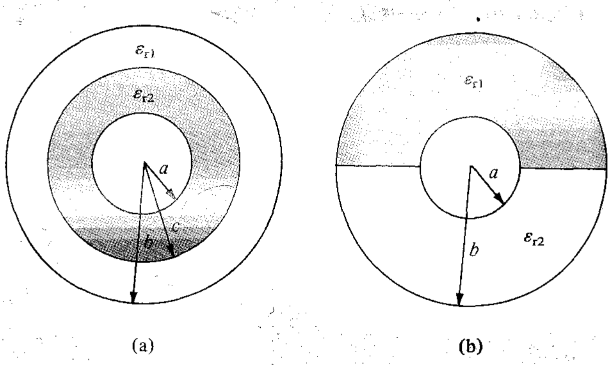


Figure 6.19 For Practice Exercises 6.9, 6.10, and 6.12.

EXAMPLE 6.11

In Section 6.5, it was mentioned that the capacitance $C = Q/V$ of a capacitor can be found by either assuming Q and finding V or by assuming V and finding Q. The former approach was used in Section 6.5 while we have used the latter method in the last example. Using the latter method, derive eq. (6.22).

Solution:

Assume that the parallel plates in Figure 6.13 are maintained at a potential difference V_o so that $V(x = 0)$ and $V(x = d) = V_o$. This necessitates solving a one-dimensional boundary-value problem; that is, we solve Laplace's equation

$$\nabla^2 V = \frac{d^2 V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

where A and B are integration constants to be determined from the boundary conditions. At $x = 0$, $V = 0 \rightarrow 0 = 0 + B$, or $B = 0$, and at $x = d$, $V = V_o \rightarrow V_o = Ad + 0$ or $A = V_o/d$.

Hence

$$V = \frac{V_o}{d} x$$

Notice that this solution satisfies Laplace's equation and the boundary conditions.

We have assumed the potential difference between the plates to be V_o. Our goal is to find the charge Q on either plate so that we can eventually find the capacitance $C = Q/V_o$. The charge on either plate is

$$Q = \int \rho_S \, dS$$

But $\rho_S = \mathbf{D} \cdot \mathbf{a}_n = \varepsilon \mathbf{E} \cdot \mathbf{a}_n$, where

$$\mathbf{E} = -\nabla V = -\frac{dV}{dx} \mathbf{a}_x = -A\mathbf{a}_x = -\frac{V_o}{d} \mathbf{a}_x$$

On the lower plates, $\mathbf{a}_n = \mathbf{a}_x$, so

$$\rho_S = -\frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = -\frac{\varepsilon V_o S}{d}$$

On the upper plates, $\mathbf{a}_n = -\mathbf{a}_x$, so

$$\rho_S = \frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = \frac{\varepsilon V_o S}{d}$$

As expected, Q is equal but opposite on each plate. Thus

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon S}{d}$$

which is in agreement with eq. (6.22).

PRACTICE EXERCISE 6.11

Derive the formula for the capacitance $C = Q/V_o$ of a cylindrical capacitor in eq. (6.28) by assuming V_o and finding Q.

EXAMPLE 6.12

Determine the capacitance of each of the capacitors in Figure 6.20. Take $\varepsilon_{r1} = 4$, $\varepsilon_{r2} = 6$, $d = 5$ mm, $S = 30$ cm^2.

Solution:

(a) Since **D** and **E** are normal to the dielectric interface, the capacitor in Figure 6.20(a) can be treated as consisting of two capacitors C_1 and C_2 in series as in Figure 6.16(a).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S}{d/2} = \frac{2\varepsilon_o \varepsilon_{r1} S}{d}, \qquad C_2 = \frac{2\varepsilon_o \varepsilon_{r2} S}{d}$$

The total capacitor C is given by

$$\begin{aligned} C &= \frac{C_1 C_2}{C_1 + C_2} = \frac{2\varepsilon_o S}{d} \frac{(\varepsilon_{r1}\varepsilon_{r2})}{\varepsilon_{r1} + \varepsilon_{r2}} \\ &= 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{5 \times 10^{-3}} \cdot \frac{4 \times 6}{10} \qquad (6.12.1) \\ C &= 25.46 \text{ pF} \end{aligned}$$

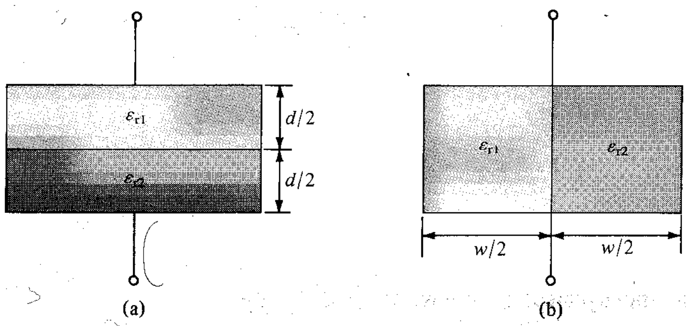


Figure 6.20 For Example 6.12.

(b) In this case, **D** and **E** are parallel to the dielectric interface. We may treat the capacitor as consisting of two capacitors C_1 and C_2 in parallel (the same voltage across C_1 and C_2) as in Figure 6.16(b).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S/2}{d} = \frac{\varepsilon_o \varepsilon_{r1} S}{2d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_{r2} S}{2d}$$

The total capacitance is

$$\begin{aligned} C &= C_1 + C_2 = \frac{\varepsilon_o S}{2d}(\varepsilon_{r1} + \varepsilon_{r2}) \\ &= \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{2 \cdot (5 \times 10^{-3})} \cdot 10 \qquad (6.12.2) \\ C &= 26.53 \text{ pF} \end{aligned}$$

Notice that when $\varepsilon_{r1} = \varepsilon_{r2} = \varepsilon_r$, eqs. (6.12.1) and (6.12.2) agree with eq. (6.22) as expected.

PRACTICE EXERCISE 6.12

Determine the capacitance of 10 m length of the cylindrical capacitors shown in Figure 6.19. Take $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 1.41 nF, (b) 1.52 nF.

A cylindrical capacitor has radii $a = 1$ cm and $b = 2.5$ cm. If the space between the plates is filled with an inhomogeneous dielectric with $\varepsilon_r = (10 + \rho)/\rho$, where ρ is in centimeters, find the capacitance per meter of the capacitor.

Solution:

The procedure is the same as that taken in Section 6.5 except that eq. (6.27a) now becomes

$$\begin{aligned} V &= -\int_b^a \frac{Q}{2\pi\varepsilon_o\varepsilon_r \rho L}\, d\rho = -\frac{Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{\rho\left(\dfrac{10+\rho}{\rho}\right)} \\ &= \frac{-Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{10+\rho} = \frac{-Q}{2\pi\varepsilon_o L}\ln(10+\rho)\bigg|_b^a \\ &= \frac{Q}{2\pi\varepsilon_o L}\ln\frac{10+b}{10+a} \end{aligned}$$

Thus the capacitance per meter is ($L = 1$ m)

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_o}{\ln \dfrac{10 + b}{10 + a}} = 2\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{\ln \dfrac{12.5}{11.0}}$$

$$C = 434.6 \text{ pF/m}$$

PRACTICE EXERCISE 6.13

A spherical capacitor with $a = 1.5$ cm, $b = 4$ cm has an inhomogeneous dielectric of $\varepsilon = 10\varepsilon_o/r$. Calculate the capacitance of the capacitor.

Answer: 1.13 nF.

6.6 METHOD OF IMAGES

The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine V, **E**, **D**, and ρ_S due to charges in the presence of conductors. By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

theory states that a given charge configuration above an infinite conducting plane may be replaced by the charge configuration [illegible] an equipotential surface in place of the conducting plane.

Typical examples of point, line, and volume charge configurations are portrayed in Figure 6.21(a), and their corresponding image configurations are in Figure 6.21(b).

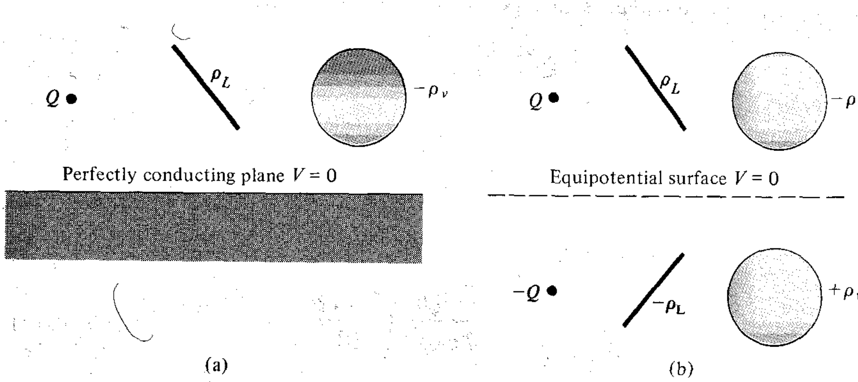


Figure 6.21 Image system: **(a)** charge configurations above a perfectly conducting plane; **(b)** image configuration with the conducting plane replaced by equipotential surface.

In applying the image method, two conditions must always be satisfied:

1. The image charge(s) must be located in the conducting region.
2. The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.

The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied. Let us now apply the image theory to two specific problems.

A. A Point Charge Above a Grounded Conducting Plane

Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent as in Figure 6.22(a). The image configuration is in Figure 6.22(b). The electric field at point $P(x, y, z)$ is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.40}$$

$$= \frac{Q\,\mathbf{r}_1}{4\pi\varepsilon_o r_1^3} + \frac{-Q\,\mathbf{r}_2}{4\pi\varepsilon_o r_2^3} \tag{6.41}$$

The distance vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h) \tag{6.42}$$

$$\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h) \tag{6.43}$$

so eq. (6.41) becomes

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + y\mathbf{a}_y + (z - h)\mathbf{a}_z}{[x^2 + y^2 + (z - h)^2]^{3/2}} - \frac{x\mathbf{a}_x + y\mathbf{a}_y + (z + h)\mathbf{a}_z}{[x^2 + y^2 + (z + h)^2]^{3/2}}\right] \tag{6.44}$$

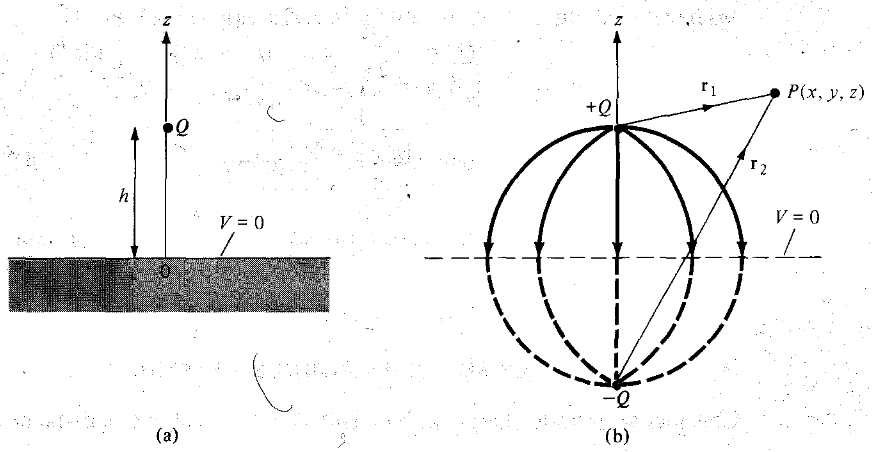


Figure 6.22 **(a)** Point charge and grounded conducting plane, **(b)** image configuration and field lines.

It should be noted that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is easily obtained from eq. (6.41) or (6.44) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= \frac{Q}{4\pi\varepsilon_o r_1} + \frac{-Q}{4\pi\varepsilon_o r_2} \\ V &= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z-h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z+h)^2]^{1/2}}\right\} \end{aligned} \tag{6.45}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge density of the induced charge can also be obtained from eq. (6.44) as

$$\begin{aligned} \rho_S = D_n = \varepsilon_o E_n \bigg|_{z=0} \\ = \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}} \end{aligned} \tag{6.46}$$

The total induced charge on the conducting plane is

$$Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-Qh \, dx \, dy}{2\pi[x^2 + y^2 + h^2]^{3/2}} \tag{6.47}$$

By changing variables, $\rho^2 = x^2 + y^2$, $dx\,dy = \rho\, d\rho\, d\phi$.

$$Q_i = -\frac{Qh}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}} \tag{6.48}$$

Integrating over ϕ gives 2π, and letting $\rho\, d\rho = \frac{1}{2} d\,(\rho^2)$, we obtain

$$\begin{aligned} Q_i &= -\frac{Qh}{2\pi} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2}\, \frac{1}{2}\, d(\rho^2) \\ &= \frac{Qh}{[\rho^2 + h^2]^{1/2}} \bigg|_0^{\infty} \\ &= -Q \end{aligned} \tag{6.49}$$

as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.

B. A Line Charge above a Grounded Conducting Plane

Consider an infinite charge with density ρ_L C/m located at a distance h from the grounded conducting plane $z = 0$. The same image system of Figure 6.22(b) applies to the line charge except that Q is replaced by ρ_L. The infinite line charge ρ_L may be assumed to be at

$x = 0, z = h$ and the image $-\rho_L$ at $x = 0, z = -h$ so that the two are parallel to the y-axis. The electric field at point P is given (from eq. 4.21) by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.50}$$

$$= \frac{\rho_L}{2\pi\varepsilon_o\rho_1}\mathbf{a}_{\rho 1} + \frac{-\rho_L}{2\pi\varepsilon_o\rho_2}\mathbf{a}_{\rho 2} \tag{6.51}$$

The distance vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are given by

$$\boldsymbol{\rho}_1 = (x, y, z) - (0, y, h) = (x, 0, z - h) \tag{6.52}$$

$$\boldsymbol{\rho}_2 = (x, y, z) - (0, y, -h) = (x, 0, z + h) \tag{6.53}$$

so eq. (6.51) becomes

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + (z - h)\mathbf{a}_z}{x^2 + (z - h)^2} - \frac{x\mathbf{a}_x + (z + h)\mathbf{a}_z}{x^2 + (z + h)^2}\right] \tag{6.54}$$

Again, notice that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is obtained from eq. (6.51) or (6.54) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_o}\ln\rho_2 \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{\rho_1}{\rho_2} \end{aligned} \tag{6.55}$$

Substituting $\rho_1 = |\boldsymbol{\rho}_1|$ and $\rho_2 = |\boldsymbol{\rho}_2|$ in eqs. (6.52) and (6.53) into eq. (6.55) gives

$$V = -\frac{\rho_L}{2\pi\varepsilon_o}\ln\left[\frac{x^2 + (z - h)^2}{x^2 + (z + h)^2}\right]^{1/2} \tag{6.56}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge induced on the conducting plane is given by

$$\rho_S = D_n = \varepsilon_o E_z\Big|_{z=0} = \frac{-\rho_L h}{\pi(x^2 + h^2)} \tag{6.57}$$

The induced charge per length on the conducting plane is

$$\rho_i = \int \rho_S\, dx = -\frac{\rho_L h}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2 + h^2} \tag{6.58}$$

By letting $x = h\tan\alpha$, eq. (6.58) becomes

$$\begin{aligned} \rho_i &= -\frac{\rho_L h}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d\alpha}{h} \\ &= -\rho_L \end{aligned} \tag{6.59}$$

as expected.

EXAMPLE 6.14

A point charge Q is located at point $(a, 0, b)$ between two semiinfinite conducting planes intersecting at right angles as in Figure 6.23. Determine the potential at point $P(x, y, z)$ and the force on Q.

Solution:

The image configuration is shown in Figure 6.24. Three image charges are necessary to satisfy the conditions in Section 6.6. From Figure 6.24(a), the potential at point $P(x, y, z)$ is the superposition of the potentials at P due to the four point charges; that is,

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

where

$$r_1 = [(x - a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_2 = [(x + a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_3 = [(x + a)^2 + y^2 + (z + b)^2]^{1/2}$$
$$r_4 = [(x - a)^2 + y^2 + (z + b)^2]^{1/2}$$

From Figure 6.24(b), the net force on Q

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= -\frac{Q^2}{4\pi\varepsilon_o(2b)^2}\mathbf{a}_z - \frac{Q^2}{4\pi\varepsilon_o(2a)^2}\mathbf{a}_x + \frac{Q^2(2a\mathbf{a}_x + 2b\mathbf{a}_z)}{4\pi\varepsilon_o[(2a)^2 + (2b)^2]^{3/2}}\\ &= \frac{Q^2}{16\pi\varepsilon_o}\left\{\left[\frac{a}{(a^2 + b^2)^{3/2}} - \frac{1}{a^2}\right]\mathbf{a}_x + \left[\frac{b}{(a^2 + b^2)^{3/2}} - \frac{1}{b^2}\right]\mathbf{a}_z\right\}\end{aligned}$$

The electric field due to this system can be determined similarly and the charge induced on the planes can also be found.

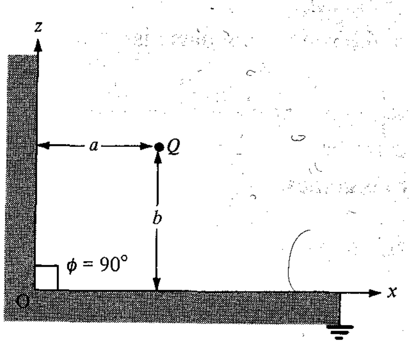


Figure 6.23 Point charge between two semiinfinite conducting planes.

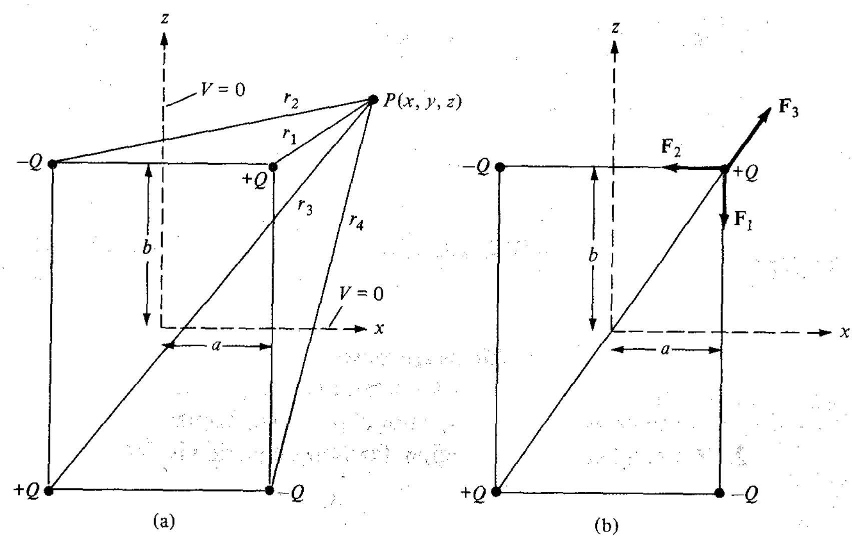


Figure 6.24 Determining **(a)** the potential at P, and **(b)** the force on charge Q.

In general, when the method of images is used for a system consisting of a point charge between two semiinfinite conducting planes inclined at an angle ϕ (in degrees), the number of images is given by

$$\boxed{N = \left(\frac{360°}{\phi} - 1\right)}$$

because the charge and its images all lie on a circle. For example, when $\phi = 180°$, $N = 1$ as in the case of Figure 6.22; for $\phi = 90°$, $N = 3$ as in the case of Figure 6.23; and for $\phi = 60°$, we expect $N = 5$ as shown in Figure 6.25.

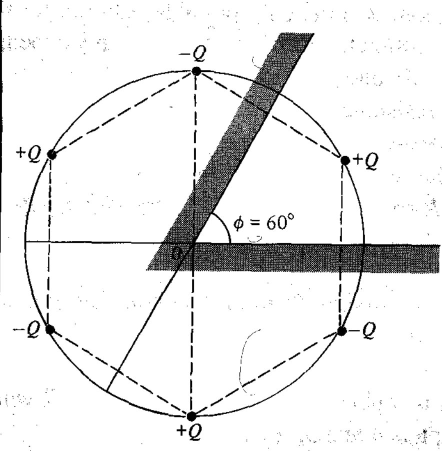


Figure 6.25 Point charge between two semiinfinite conducting walls inclined at $\phi = 60°$ to each.

PRACTICE EXERCISE 6.14

If the point charge $Q = 10$ nC in Figure 6.25 is 10 cm away from point O and along the line bisecting $\phi = 60°$, find the magnitude of the force on Q due to the charge induced on the conducting walls.

Answer: 60.53 μN.

SUMMARY

1. Boundary-value problems are those in which the potentials at the boundaries of a region are specified and we are to determine the potential field within the region. They are solved using Poisson's equation if $\rho_v \neq 0$ or Laplace's equation if $\rho_v = 0$.
2. In a nonhomogeneous region, Poisson's equation is

$$\nabla \cdot \varepsilon \nabla V = -\rho_v$$

For a homogeneous region, ε is independent of space variables. Poisson's equation becomes

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In a charge-free region ($\rho_v = 0$), Poisson's equation becomes Laplace's equation; that is,

$$\nabla^2 V = 0$$

3. We solve the differential equation resulting from Poisson's or Laplace's equation by integrating twice if V depends on one variable or by the method of separation of variables if V is a function of more than one variable. We then apply the prescribed boundary conditions to obtain a unique solution.
4. The uniqueness theorem states that if V satisfies Poisson's or Laplace's equation and the prescribed boundary condition, V is the only possible solution for that given problem. This enables us to find the solution to a given problem via any expedient means because we are assured of one, and only one, solution.
5. The problem of finding the resistance R of an object or the capacitance C of a capacitor may be treated as a boundary-value problem. To determine R, we assume a potential difference V_o between the ends of the object, solve Laplace's equation, find $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$, and obtain $R = V_o/I$. Similarly, to determine C, we assume a potential difference of V_o between the plates of the capacitor, solve Laplace's equation, find $Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S}$, and obtain $C = Q/V_o$.
6. A boundary-value problem involving an infinite conducting plane or wedge may be solved using the method of images. This basically entails replacing the charge configuration by itself, its image, and an equipotential surface in place of the conducting plane. Thus the original problem is replaced by "an image problem," which is solved using techniques covered in Chapters 4 and 5.

REVIEW QUESTIONS

6.1 Equation $\nabla \cdot (-\varepsilon \nabla V) = \rho_v$ may be regarded as Poisson's equation for an inhomogeneous medium.

(a) True
(b) False

6.2 In cylindrical coordinates, equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} + 10 = 0$$

is called

(a) Maxwell's equation
(b) Laplace's equation
(c) Poisson's equation
(d) Helmholtz's equation
(e) Lorentz's equation

6.3 Two potential functions V_1 and V_2 satisfy Laplace's equation within a closed region and assume the same values on its surface. V_1 must be equal to V_2.

(a) True
(b) False
(c) Not necessarily

6.4 Which of the following potentials does not satisfy Laplace's equation?

(a) $V = 2x + 5$
(b) $V = 10\,xy$
(c) $V = r\cos\phi$
(d) $V = \dfrac{10}{r}$
(e) $V = \rho\cos\phi + 10$

6.5 Which of the following is not true?

(a) $-5\cos 3x$ is a solution to $\phi''(x) + 9\phi(x) = 0$
(b) $10\sin 2x$ is a solution to $\phi''(x) - 4\phi(x) = 0$
(c) $-4\cosh 3y$ is a solution to $R''(y) - 9R(y) = 0$
(d) $\sinh 2y$ is a solution to $R''(y) - 4R(y) = 0$
(e) $\dfrac{g''(x)}{g(x)} = -\dfrac{h''(y)}{h(y)} = f(z) = -1$ where $g(x) = \sin x$ and $h(y) = \sinh y$

6.6 If $V_1 = X_1Y_1$ is a product solution of Laplace's equation, which of these are not solutions of Laplace's equation?

(a) $-10X_1Y_1$
(b) $X_1Y_1 + 2xy$
(c) $X_1Y_1 - x + y$
(d) $X_1 + Y_1$
(e) $(X_1 - 2)(Y_1 + 3)$

6.7 The capacitance of a capacitor filled by a linear dielectric is independent of the charge on the plates and the potential difference between the plates.

(a) True
(b) False

6.8 A parallel-plate capacitor connected to a battery stores twice as much charge with a given dielectric as it does with air as dielectric, the susceptibility of the dielectric is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

6.9 A potential difference V_o is applied to a mercury column in a cylindrical container. The mercury is now poured into another cylindrical container of half the radius and the same potential difference V_o applied across the ends. As a result of this change of space, the resistance will be increased

(a) 2 times
(b) 4 times
(c) 8 times
(d) 16 times

6.10 Two conducting plates are inclined at an angle 30° to each other with a point charge between them. The number of image charges is

(a) 12
(b) 11
(c) 6
(d) 5
(e) 3

Answers: 6.1a, 6.2c, 6.3a, 6.4c, 6.5b, 6.6d,e, 6.7a, 6.8b, 6.9d, 6.10b.

PROBLEMS

6.1 In free space, $V = 6xy^2z + 8$. At point $P(1, 2, -5)$, find $\mathbf{E}$ and ρ_v.

6.2 Two infinitely large conducting plates are located at $x = 1$ and $x = 4$. The space between them is free space with charge distribution $\frac{x}{6\pi}$ nC/m^3. Find V at $x = 2$ if $V(1) = -50$ V and $V(4) = 50$ V.

6.3 The region between $x = 0$ and $x = d$ is free space and has $\rho_v = \rho_o(x - d)/d$. If $V(x = 0) = 0$ and $V(x = d) = V_o$, find: (a) V and $\mathbf{E}$, (b) the surface charge densities at $x = 0$ and $x = d$.

6.4 Show that the exact solution of the equation

$$\frac{d^2V}{dx^2} = f(x) \qquad 0 < x < L$$

subject to

$$V(x = 0) = V_1 \qquad V(x = L) = V_2$$

is

$$V(x) = \left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda\right]\frac{x}{L} + V_1 + \int_0^x \int_0^\lambda f(\mu)\, d\mu\, d\lambda$$

6.5 A certain material occupies the space between two conducting slabs located at $y = \pm 2$ cm. When heated, the material emits electrons such that $\rho_v = 50(1 - y^2)\ \mu$C/m^3. If the slabs are both held at 30 kV, find the potential distribution within the slabs. Take $\varepsilon = 3\varepsilon_o$.

6.6 Determine which of the following potential field distributions satisfy Laplace's equation.

(a) $V_1 = x^2 + y^2 - 2z^2 + 10$

(b) $V_2 = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$

(c) $V_3 = \rho z \sin\phi + \rho^2$

(d) $V_4 = \dfrac{10 \sin\theta \sin\phi}{r^2}$

6.7 Show that the following potentials satisfy Laplace's equation.

(a) $V = e^{-5x} \cos 13y \sinh 12z$

(b) $V = \dfrac{z \cos\phi}{\rho}$

(c) $V = \dfrac{30 \cos\theta}{r^2}$

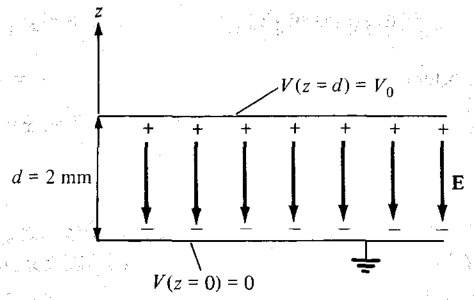


Figure 6.26 For Problem 6.11.

6.8 Show that $\mathbf{E} = (E_x, E_y, E_z)$ satisfies Laplace's equation.

6.9 Let $V = (A \cos nx + B \sin nx)(Ce^{ny} + De^{-ny})$, where A, B, C, and D are constants. Show that V satisfies Laplace's equation.

6.10 The potential field $V = 2x^2yz - y^3z$ exists in a dielectric medium having $\varepsilon = 2\varepsilon_o$. (a) Does V satisfy Laplace's equation? (b) Calculate the total charge within the unit cube $0 < x,y,z < 1$ m.

6.11 Consider the conducting plates shown in Figure 6.26. If $V(z = 0) = 0$ and $V(z = 2 \text{ mm}) = 50$ V, determine V, $\mathbf{E}$, and $\mathbf{D}$ in the dielectric region ($\varepsilon_r = 1.5$) between the plates and ρ_S on the plates.

6.12 The cylindrical capacitor whose cross section is in Figure 6.27 has inner and outer radii of 5 mm and 15 mm, respectively. If $V(\rho = 5 \text{ mm}) = 100$ V and $V(\rho = 15 \text{ mm}) = 0$ V, calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 10$ mm and ρ_S on each plate. Take $\varepsilon_r = 2.0$.

6.13 Concentric cylinders $\rho = 2$ cm and $\rho = 6$ cm are maintained at $V = 60$ V and $V = -20$ V, respectively. Calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 4$ cm.

6.14 The region between concentric spherical conducting shells $r = 0.5$ m and $r = 1$ m is charge free. If $V(r = 0.5) = -50$ V and $V(r = 1) = 50$ V, determine the potential distribution and the electric field strength in the region between the shells.

6.15 Find V and $\mathbf{E}$ at (3, 0, 4) due to the two conducting cones of infinite extent shown in Figure 6.28.

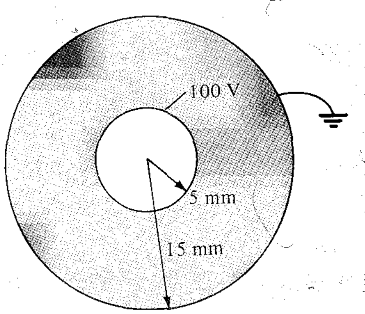


Figure 6.27 Cylindrical capacitor of Problem 6.12.

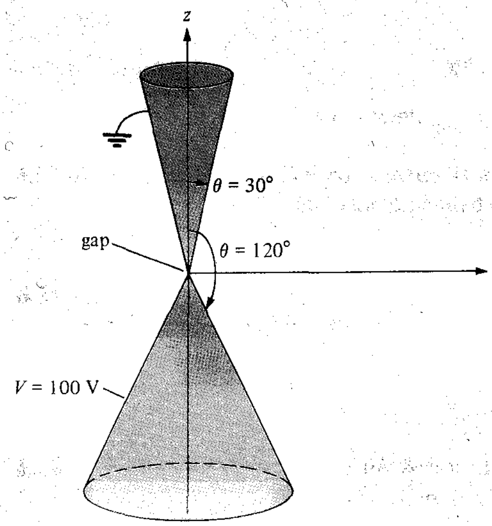


Figure 6.28 Conducting cones of Problem 6.15.

***6.16** The inner and outer electrodes of a diode are coaxial cylinders of radii $a = 0.6$ m and $b = 30$ mm, respectively. The inner electrode is maintained at 70 V while the outer electrode is grounded. (a) Assuming that the length of the electrodes $\ell \gg a, b$ and ignoring the effects of space charge, calculate the potential at $\rho = 15$ mm. (b) If an electron is injected radially through a small hole in the inner electrode with velocity 10^7 m/s, find its velocity at $\rho = 15$ mm.

6.17 Another method of finding the capacitance of a capacitor is using energy considerations, that is

$$C = \frac{2W_E}{V_o^2} = \frac{1}{V_o^2}\int \varepsilon|\mathbf{E}|^2\, dv$$

Using this approach, derive eqs. (6.22), (6.28), and (6.32).

6.18 An electrode with a hyperbolic shape ($xy = 4$) is placed above an earthed right-angle corner as in Figure 6.29. Calculate V and $\mathbf{E}$ at point (1, 2, 0) when the electrode is connected to a 20-V source.

***6.19** Solve Laplace's equation for the two-dimensional electrostatic systems of Figure 6.30 and find the potential $V(x, y)$.

***6.20** Find the potential $V(x, y)$ due to the two-dimensional systems of Figure 6.31.

6.21 By letting $V(\rho, \phi) = R(\rho)\Phi(\phi)$ be the solution of Laplace's equation in a region where $\rho \neq 0$, show that the separated differential equations for R and Φ are

$$R'' + \frac{R'}{\rho} - \frac{\lambda}{\rho^2}R = 0$$

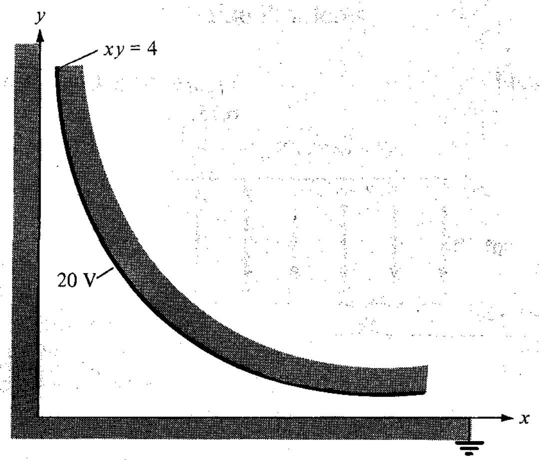


Figure 6.29 For Problem 6.18.

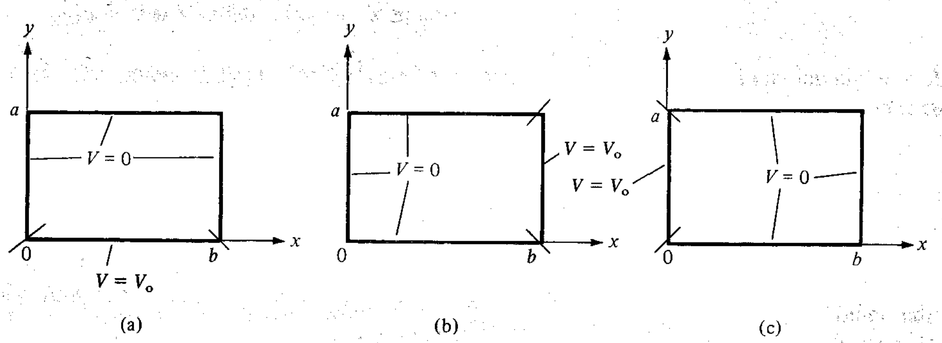


Figure 6.30 For Problem 6.19.

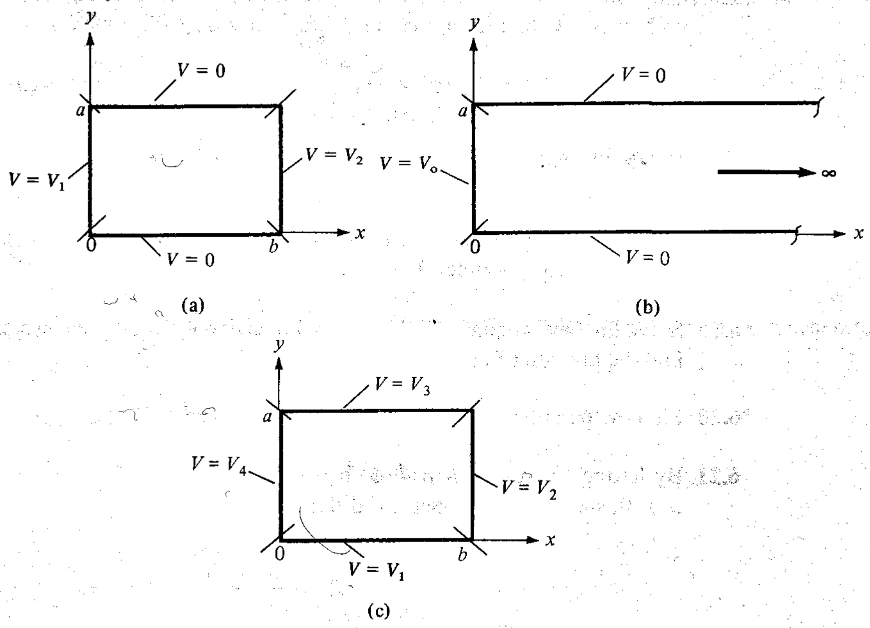


Figure 6.31 For Problem 6.20.

and

$$\Phi'' + \lambda\Phi = 0$$

where λ is the separation constant.

6.22 A potential in spherical coordinates is a function of r and θ but not ϕ. Assuming that $V(r, \theta) = R(r)F(\theta)$, obtain the separated differential equations for R and F in a region for which $\rho_v = 0$.

6.23 Show that the resistance of the bar of Figure 6.17 between the vertical ends located at $\phi = 0$ and $\phi = \pi/2$ is

$$R = \frac{\pi}{2\sigma t \ln b/a}$$

***6.24** Show that the resistance of the sector of a spherical shell of conductivity σ, with cross section shown in Figure 6.32 (where $0 \leq \phi < 2\pi$), between its base is

$$R = \frac{1}{2\pi\sigma(1 - \cos\alpha)}\left[\frac{1}{a} - \frac{1}{b}\right]$$

***6.25** A hollow conducting hemisphere of radius a is buried with its flat face lying flush with the earth surface thereby serving as an earthing electrode. If the conductivity of earth is σ, show that the leakage conductance between the electrode and earth is $2\pi a\sigma$.

6.26 The cross section of an electric fuse is shown in Figure 6.33. If the fuse is made of copper and of thickness 1.5 mm, calculate its resistance.

6.27 In an integrated circuit, a capacitor is formed by growing a silicon dioxide layer ($\varepsilon_r = 4$) of thickness 1 μm over the conducting silicon substrate and covering it with a metal electrode of area S. Determine S if a capacitance of 2 nF is desired.

6.28 The parallel-plate capacitor of Figure 6.34 is quarter-filled with mica ($\varepsilon_r = 6$). Find the capacitance of the capacitor.

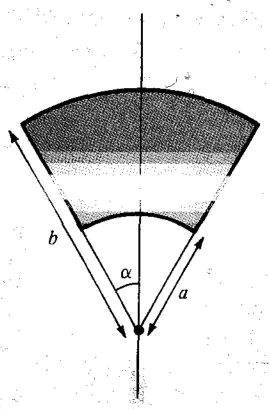


Figure 6.32 For Problem 6.24.

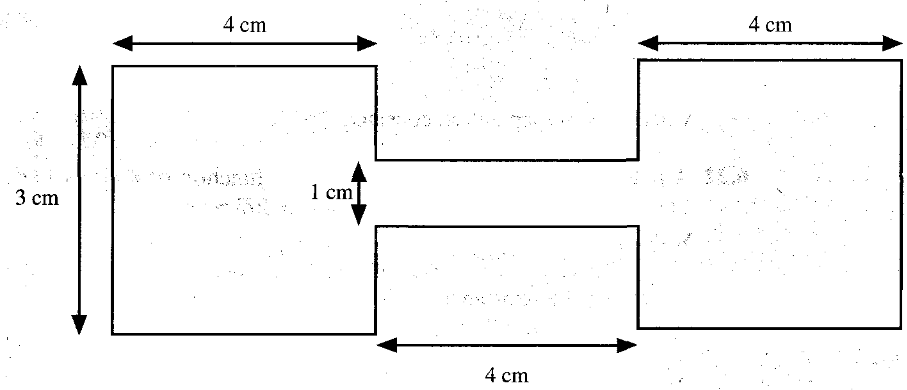


Figure 6.33 For Problem 6.26.

***6.29** An air-filled parallel plate capacitor of length L, width a, and plate separation d has its plates maintained at constant potential difference V_o. If a dielectric slab of dielectric constant ε_r is slid between the plates and is withdrawn until only a length x remains between the plates as in Figure 6.35, show that the force tending to restore the slab to its original position is

$$F = \frac{\varepsilon_o(\varepsilon_r - 1)\, a\, V_o^2}{2d}$$

6.30 A parallel-plate capacitor has plate area 200 cm^2 and plate separation 3 mm. The charge density is 1 $\mu\text{C/m}^2$ with air as dielectric. Find

(a) The capacitance of the capacitor

(b) The voltage between the plates

(c) The force with which the plates attract each other

6.31 Two conducting plates are placed at $z = -2$ cm and $z = 2$ cm and are, respectively, maintained at potentials 0 and 200 V. Assuming that the plates are separated by a polypropylene ($\varepsilon = 2.25\varepsilon_o$). Calculate: (a) the potential at the middle of the plates, (b) the surface charge densities at the plates.

6.32 Two conducting parallel plates are separated by a dielectric material with $\varepsilon = 5.6\varepsilon_o$ and thickness 0.64 mm. Assume that each plate has an area of 80 cm^2. If the potential field distribution between the plates is $V = 3x + 4y - 12z + 6$ kV, determine: (a) the capacitance of the capacitor, (b) the potential difference between the plates.

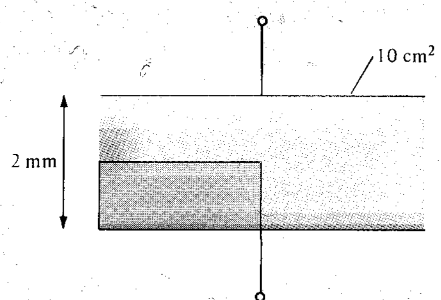


Figure 6.34 For Problem 6.28.

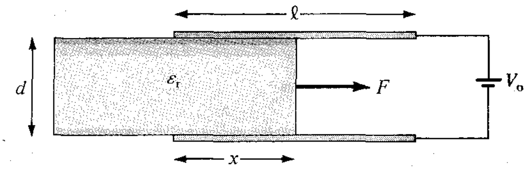


Figure 6.35 For Problem 6.29.

6.33 The space between spherical conducting shells $r = 5$ cm and $r = 10$ cm is filled with a dielectric material for which $\varepsilon = 2.25\varepsilon_o$. The two shells are maintained at a potential difference of 80 V. (a) Find the capacitance of the system. (b) Calculate the charge density on shell $r = 5$ cm.

6.34 Concentric shells $r = 20$ cm and $r = 30$ cm are held at $V = 0$ and $V = 50$, respectively. If the space between them is filled with dielectric material ($\varepsilon = 3.1\varepsilon_o$, $\sigma = 10^{-12}$ S/m), find: (a) V, **E**, and **D**, (b) the charge densities on the shells, (c) the leakage resistance.

6.35 A spherical capacitor has inner radius a and outer radius d. Concentric with the spherical conductors and lying between them is a spherical shell of outer radius c and inner radius b. If the regions $d < r < c$, $c < r < b$, and $b < r < a$ are filled with materials with permittivites ε_1, ε_2, and ε_3, respectively, determine the capacitance of the system.

6.36 Determine the capacitance of a conducting sphere of radius 5 cm deeply immersed in sea water ($\varepsilon_r = 80$).

6.37 A conducting sphere of radius 2 cm is surrounded by a concentric conducting sphere of radius 5 cm. If the space between the spheres is filled with sodium chloride ($\varepsilon_r = 5.9$), calculate the capacitance of the system.

***6.38** In an ink-jet printer the drops are charged by surrounding the jet of radius 20 μm with a concentric cylinder of radius 600 μm as in Figure 6.36. Calculate the minimum voltage required to generate a charge 50 fC on the drop if the length of the jet inside the cylinder is 100 μm. Take $\varepsilon = \varepsilon_o$.

6.39 A given length of a cable, the capacitance of which is 10 μF/km with a resistance of insulation of 100 MΩ/km, is charged to a voltage of 100 V. How long does it take the voltage to drop to 50 V?

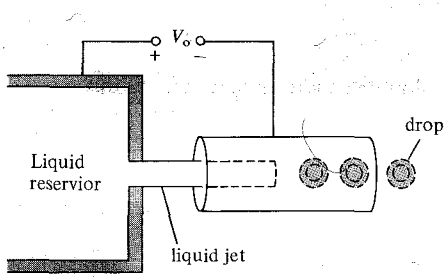


Figure 6.36 Simplified geometry of an ink-jet printer; for Problem 6.38.

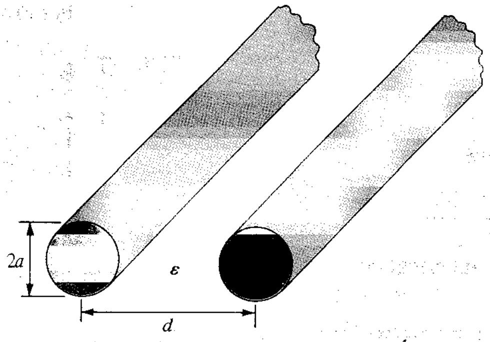


Figure 6.37 For Problem 6.40.

6.40 The capacitance per unit length of a two-wire transmission line shown in Figure 6.37 is given by

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\left[\dfrac{d}{2a}\right]}$$

Determine the conductance per unit length.

***6.41** A spherical capacitor has an inner conductor of radius a carrying charge Q and maintained at zero potential. If the outer conductor contracts from a radius b to c under internal forces, prove that the work performed by the electric field as a result of the contraction is

$$W = \frac{Q^2(b - c)}{8\pi\varepsilon bc}$$

***6.42** A parallel-plate capacitor has its plates at $x = 0, d$ and the space between the plates is filled with an inhomogeneous material with permittivity $\varepsilon = \varepsilon_o\left(1 + \frac{x}{d}\right)$. If the plate at $x = d$ is maintained at V_o while the plate at $x = 0$ is grounded, find:

(a) V and $\mathbf{E}$

(b) $\mathbf{P}$

(c) ρ_{ps} at $x = 0, d$

6.43 A spherical capacitor has inner radius a and outer radius b and filled with an inhomogeneous dielectric with $\varepsilon = \varepsilon_o k/r^2$. Show that the capacitance of the capacitor is

$$C = \frac{4\pi\varepsilon_o k}{b - a}$$

6.44 A cylindrical capacitor with inner radius a and outer radius b is filled with an inhomogeneous dielectric having $\varepsilon = \varepsilon_o k/\rho$, where k is a constant. Calculate the capacitance per unit length of the capacitor.

6.45 If the earth is regarded a spherical capacitor, what is its capacitance? Assume the radius of the earth to be approximately 6370 km.

6.46 A point charge of 10 nC is located at point $P(0, 0, 3)$ while the conducting plane $z = 0$ is grounded. Calculate

(a) V and **E** at $R(6, 3, 5)$

(b) The force on the charge due to induced charge on the plane.

6.47 Two point charges of 3 nC and -4 nC are placed, respectively, at (0, 0, 1 m) and (0, 0, 2 m) while an infinite conducting plane is at $z = 0$. Determine

(a) The total charge induced on the plane

(b) The magnitude of the force of attraction between the charges and the plane

6.48 Two point charges of 50 nC and -20 nC are located at $(-3, 2, 4)$ and $(1, 0, 5)$ above the conducting ground plane $z = 2$. Calculate (a) the surface charge density at $(7, -2, 2)$, (b) **D** at (3, 4, 8), and (c) **D** at (1, 1, 1).

***6.49** A point charge of 10 μC is located at (1, 1, 1), and the positive portions of the coordinate planes are occupied by three mutually perpendicular plane conductors maintained at zero potential. Find the force on the charge due to the conductors.

6.50 A point charge Q is placed between two earthed intersecting conducting planes that are inclined at 45° to each other. Determine the number of image charges and their locations.

6.51 Infinite line $x = 3, z = 4$ carries 16 nC/m and is located in free space above the conducting plane $z = 0$. (a) Find **E** at $(2, -2, 3)$. (b) Calculate the induced surface charge density on the conducting plane at $(5, -6, 0)$.

6.52 In free space, infinite planes $y = 4$ and $y = 8$ carry charges 20 nC/m^2 and 30 nC/m^2, respectively. If plane $y = 2$ is grounded, calculate **E** at $P(0, 0, 0)$ and $Q(-4, 6, 2)$.

PART 3

MAGNETOSTATICS

Chapter 7

MAGNETOSTATIC FIELDS

No honest man can be all things to all people.

—ABRAHAM LINCOLN

7.1 INTRODUCTION

In Chapters 4 to 6, we limited our discussions to static electric fields characterized by **E** or **D**. We now focus our attention on static magnetic fields, which are characterized by **H** or **B**. There are similarities and dissimilarities between electric and magnetic fields. As **E** and **D** are related according to $\mathbf{D} = \varepsilon\mathbf{E}$ for linear material space, **H** and **B** are related according to $\mathbf{B} = \mu\mathbf{H}$. Table 7.1 further shows the analogy between electric and magnetic field quantities. Some of the magnetic field quantities will be introduced later in this chapter, and others will be presented in the next. The analogy is presented here to show that most of the equations we have derived for the electric fields may be readily used to obtain corresponding equations for magnetic fields if the equivalent analogous quantities are substituted. This way it does not appear as if we are learning new concepts.

A definite link between electric and magnetic fields was established by Oersted[1] in 1820. As we have noticed, an electrostatic field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced. A magnetostatic field is produced by a constant current flow (or direct current). This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires. In this chapter, we consider magnetic fields in free space due to direct current. Magnetostatic fields in material space are covered in Chapter 8.

Our study of magnetostatics is not a dispensable luxury but an indispensable necessity. The development of the motors, transformers, microphones, compasses, telephone bell ringers, television focusing controls, advertising displays, magnetically levitated high-speed vehicles, memory stores, magnetic separators, and so on, involve magnetic phenomena and play an important role in our everyday life.[2]

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[1]Hans Christian Oersted (1777–1851), a Danish professor of physics, after 13 years of frustrating efforts discovered that electricity could produce magnetism.

[2]Various applications of magnetism can be found in J. K. Watson, *Applications of Magnetism.* New York: John Wiley & Sons, 1980.

TABLE 7.1 Analogy between Electric and Magnetic Fields*

Term	Electric	Magnetic
Basic laws	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^2}\mathbf{a}_r$	$d\mathbf{B} = \dfrac{\mu_o I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$
	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$
Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
Source element	dQ	$Q\mathbf{u} = I\, d\mathbf{l}$
Field intensity	$E = \dfrac{V}{\ell}$ (V/m)	$H = \dfrac{I}{\ell}$ (A/m)
Flux density	$\mathbf{D} = \dfrac{\Psi}{S}$ (C/m^2)	$\mathbf{B} = \dfrac{\Psi}{S}$ (Wb/m^2)
Relationship between fields	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
Potentials	$\mathbf{E} = -\nabla V$	$\mathbf{H} = -\nabla V_m\ (\mathbf{J} = 0)$
	$V = \displaystyle\int \frac{\rho_L dl}{4\pi\varepsilon r}$	$\mathbf{A} = \displaystyle\int \frac{\mu I\, d\mathbf{l}}{4\pi R}$
Flux	$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$	$\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\dfrac{dV}{dt}$	$V = L\dfrac{dI}{dt}$
Energy density	$w_E = \dfrac{1}{2}\mathbf{D} \cdot \mathbf{E}$	$w_m = \dfrac{1}{2}\mathbf{B} \cdot \mathbf{H}$
Poisson's equation	$\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$	$\nabla^2 \mathbf{A} = -\mu\mathbf{J}$

*A similar analogy can be found in R. S. Elliot, "Electromagnetic theory: a simplified representation," *IEEE Trans. Educ.*, vol. E-24, no. 4, Nov. 1981, pp. 294–296.

There are two major laws governing magnetostatic fields: (1) Biot–Savart's law,[3] and (2) Ampere's circuit law.[4] Like Coulomb's law, Biot–Savart's law is the general law of magnetostatics. Just as Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot–Savart's law and is easily applied in problems involving symmetrical current distribution. The two laws of magnetostatics are stated and applied first; their derivation is provided later in the chapter.

[3]The experiments and analyses of the effect of a current element were carried out by Ampere and by Jean-Baptiste and Felix Savart, around 1820.

[4]Andre Marie Ampere (1775–1836), a French physicist, developed Oersted's discovery and introduced the concept of current element and the force between current elements.

7.2 BIOT–SAVART'S LAW

Biot–Savart's law states that the magnetic field intensity dH produced at a point P, as shown in Figure 7.1, by the differential current element $I\,dl$ is proportional to the product $I\,dl$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

That is,

$$dH \propto \frac{I\,dl \sin \alpha}{R^2} \tag{7.1}$$

or

$$dH = \frac{kI\,dl \sin \alpha}{R^2} \tag{7.2}$$

where k is the constant of proportionality. In SI units, $k = 1/4\pi$, so eq. (7.2) becomes

$$dH = \frac{I\,dl \sin \alpha}{4\pi R^2} \tag{7.3}$$

From the definition of cross product in eq. (1.21), it is easy to notice that eq. (7.3) is better put in vector form as

$$\boxed{d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}} \tag{7.4}$$

where $R = |\mathbf{R}|$ and $\mathbf{a}_R = \mathbf{R}/R$. Thus the direction of $d\mathbf{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $d\mathbf{H}$ as shown in Figure 7.2(a). Alternatively, we can use the right-handed screw rule to determine the direction of $d\mathbf{H}$: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $d\mathbf{H}$ as in Figure 7.2(b).

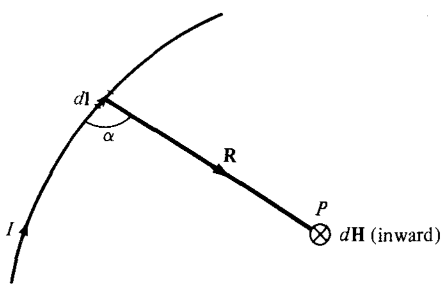


Figure 7.1 magnetic field $d\mathbf{H}$ at P due to current element $I\,d\mathbf{l}$.

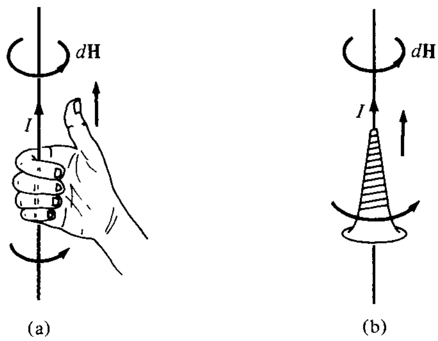


Figure 7.2 Determining the direction of $d\mathbf{H}$ using **(a)** the right-hand rule, or **(b)** the right-handed screw rule.

It is customary to represent the direction of the magnetic field intensity **H** (or current *I*) by a small circle with a dot or cross sign depending on whether **H** (or *I*) is out of, or into, the page as illustrated in Figure 7.3.

Just as we can have different charge configurations (see Figure 4.5), we can have different current distributions: line current, surface current, and volume current as shown in Figure 7.4. If we define **K** as the surface current density (in amperes/meter) and **J** as the volume current density (in amperes/meter square), the source elements are related as

$$I\,d\mathbf{l} \equiv \mathbf{K}\,dS \equiv \mathbf{J}\,dv \tag{7.5}$$

Thus in terms of the distributed current sources, the Biot–Savart law as in eq. (7.4) becomes

$$\mathbf{H} = \int_L \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(line current)} \tag{7.6}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\,dS \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(surface current)} \tag{7.7}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\,dv \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(volume current)} \tag{7.8}$$

As an example, let us apply eq. (7.6) to determine the field due to a *straight* current carrying filamentary conductor of finite length AB as in Figure 7.5. We assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles

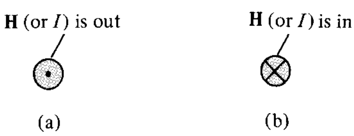


Figure 7.3 Conventional representation of **H** (or *I*) **(a)** out of the page and **(b)** into the page.

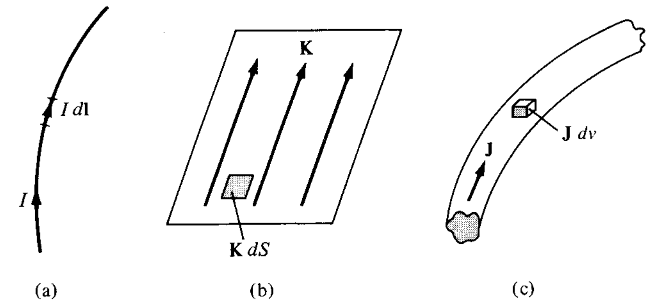


Figure 7.4 Current distributions: **(a)** line current, **(b)** surface current, **(c)** volume current.

α_2 and α_1 at P, the point at which $\mathbf{H}$ is to be determined. Particular note should be taken of this assumption as the formula to be derived will have to be applied accordingly. If we consider the contribution $d\mathbf{H}$ at P due to an element $d\mathbf{l}$ at $(0, 0, z)$,

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3} \tag{7.9}$$

But $d\mathbf{l} = dz\, \mathbf{a}_z$ and $\mathbf{R} = \rho \mathbf{a}_\rho - z\mathbf{a}_z$, so

$$d\mathbf{l} \times \mathbf{R} = \rho\, dz\, \mathbf{a}_\phi \tag{7.10}$$

Hence,

$$\mathbf{H} = \int \frac{I\rho\, dz}{4\pi[\rho^2 + z^2]^{3/2}}\, \mathbf{a}_\phi \tag{7.11}$$

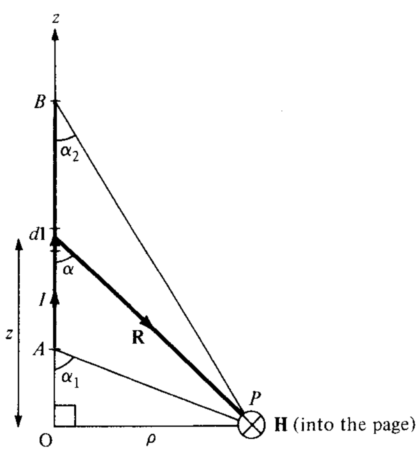


Figure 7.5 Field at point P due to a straight filamentary conductor.

Letting $z = \rho \cot \alpha$, $dz = -\rho \operatorname{cosec}^2 \alpha \, d\alpha$, and eq. (7.11) becomes

$$\mathbf{H} = -\frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2 \alpha \, d\alpha}{\rho^3 \operatorname{cosec}^3 \alpha} \mathbf{a}_\phi$$

$$= -\frac{I}{4\pi\rho} \mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin \alpha \, d\alpha$$

or

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_\phi} \tag{7.12}$$

This expression is generally applicable for any straight filamentary conductor of finite length. Notice from eq. (7.12) that **H** is always along the unit vector $\mathbf{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P. As a special case, when the conductor is *semiinfinite* (with respect to P) so that point A is now at $O(0, 0, 0)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 90°$, $\alpha_2 = 0°$, and eq. (7.12) becomes

$$\mathbf{H} = \frac{I}{4\pi\rho} \mathbf{a}_\phi \tag{7.13}$$

Another special case is when the conductor is *infinite* in length. For this case, point A is at $(0, 0, -\infty)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 180°$, $\alpha_2 = 0°$, so eq. (7.12) reduces to

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi} \tag{7.14}$$

To find unit vector $\mathbf{a}_\phi$ in eqs. (7.12) to (7.14) is not always easy. A simple approach is to determine $\mathbf{a}_\phi$ from

$$\boxed{\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho} \tag{7.15}$$

where $\mathbf{a}_\ell$ is a unit vector along the line current and $\mathbf{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point.

EXAMPLE 7.1

The conducting triangular loop in Figure 7.6(a) carries a current of 10 A. Find **H** at (0, 0, 5) due to side 1 of the loop.

Solution:

This example illustrates how eq. (7.12) is applied to any straight, thin, current-carrying conductor. The key point to keep in mind in applying eq. (7.12) is figuring out α_1, α_2, ρ, and $\mathbf{a}_\phi$. To find **H** at (0, 0, 5) due to side 1 of the loop in Figure 7.6(a), consider Figure

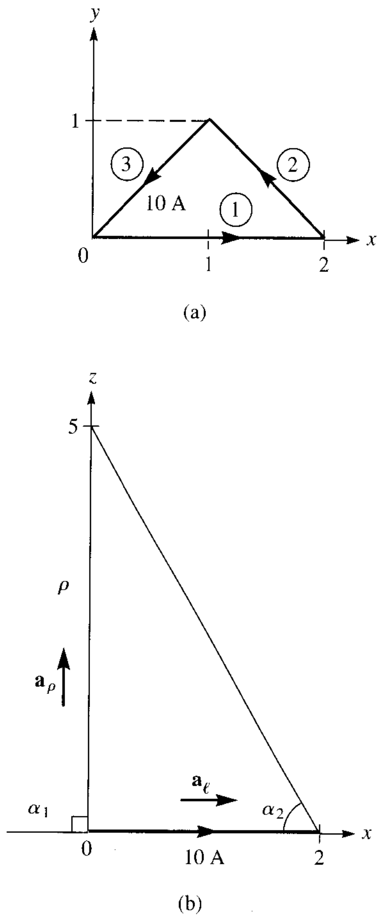


Figure 7.6 For Example 7.1: **(a)** conducting triangular loop, **(b)** side 1 of the loop.

7.6(b), where side 1 is treated as a straight conductor. Notice that we join the point of interest (0, 0, 5) to the beginning and end of the line current. Observe that α_1, α_2, and ρ are assigned in the same manner as in Figure 7.5 on which eq. (7.12) is based.

$$\cos \alpha_1 = \cos 90° = 0, \qquad \cos \alpha_2 = \frac{2}{\sqrt{29}}, \qquad \rho = 5$$

To determine $\mathbf{a}_\phi$ is often the hardest part of applying eq. (7.12). According to eq. (7.15), $\mathbf{a}_\ell = \mathbf{a}_x$ and $\mathbf{a}_\rho = \mathbf{a}_z$, so

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

Hence,

$$\begin{aligned}\mathbf{H}_1 &= \frac{I}{4\pi\rho}(\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-\mathbf{a}_y) \\ &= -59.1\mathbf{a}_y \text{ mA/m}\end{aligned}$$

PRACTICE EXERCISE 7.1

Find **H** at (0, 0, 5) due to side 3 of the triangular loop in Figure 7.6(a).

Answer: $-30.63\mathbf{a}_x + 30.63\mathbf{a}_y$ mA/m.

EXAMPLE 7.2

Find **H** at $(-3, 4, 0)$ due to the current filament shown in Figure 7.7(a).

Solution:

Let $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z$, where $\mathbf{H}_x$ and $\mathbf{H}_z$ are the contributions to the magnetic field intensity at $P(-3, 4, 0)$ due to the portions of the filament along x and z, respectively.

$$\mathbf{H}_z = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\mathbf{a}_\phi$$

At $P(-3, 4, 0)$, $\rho = (9 + 16)^{1/2} = 5$, $\alpha_1 = 90°$, $\alpha_2 = 0°$, and $\mathbf{a}_\phi$ is obtained as a unit vector along the circular path through P on plane $z = 0$ as in Figure 7.7(b). The direction of $\mathbf{a}_\phi$ is determined using the right-handed screw rule or the right-hand rule. From the geometry in Figure 7.7(b),

$$\mathbf{a}_\phi = \sin\theta\,\mathbf{a}_x + \cos\theta\,\mathbf{a}_y = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

Alternatively, we can determine $\mathbf{a}_\phi$ from eq. (7.15). At point P, $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ are as illustrated in Figure 7.7(a) for $\mathbf{H}_z$. Hence,

$$\mathbf{a}_\phi = -\mathbf{a}_z \times \left(-\frac{3}{5}\mathbf{a}_x + \frac{4}{5}\mathbf{a}_y\right) = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

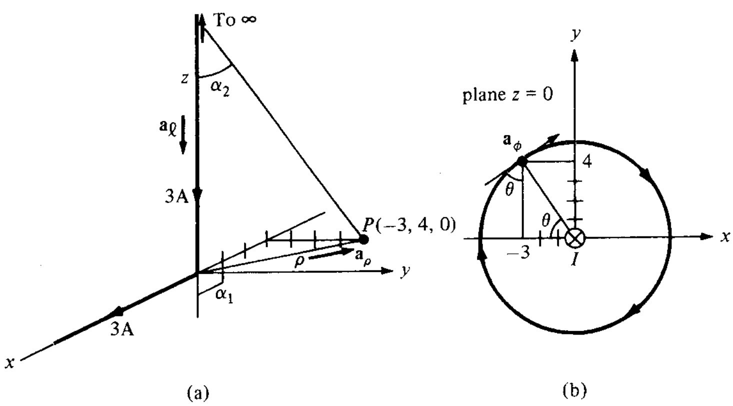


Figure 7.7 For Example 7.2: **(a)** current filament along semiinfinite x- and z-axes; $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ for $\mathbf{H}_z$ only; **(b)** determining $\mathbf{a}_\rho$ for $\mathbf{H}_z$.

as obtained before. Thus

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)\frac{(4\mathbf{a}_x + 3\mathbf{a}_y)}{5}$$
$$= 38.2\mathbf{a}_x + 28.65\mathbf{a}_y \text{ mA/m}$$

It should be noted that in this case $\mathbf{a}_\phi$ happens to be the negative of the regular $\mathbf{a}_\phi$ of cylindrical coordinates. $\mathbf{H}_z$ could have also been obtained in cylindrical coordinates as

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)(-\mathbf{a}_\phi)$$
$$= -47.75\mathbf{a}_\phi \text{ mA/m}$$

Similarly, for $\mathbf{H}_x$ at P, $\rho = 4$, $\alpha_2 = 0°$, $\cos\alpha_1 = 3/5$, and $\mathbf{a}_\phi = \mathbf{a}_z$ or $\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$. Hence,

$$\mathbf{H}_x = \frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)\mathbf{a}_z$$
$$= 23.88\ \mathbf{a}_z \text{ mA/m}$$

Thus

$$\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z = 38.2\mathbf{a}_x + 28.65\mathbf{a}_y + 23.88\mathbf{a}_z \text{ mA/m}$$

or

$$\mathbf{H} = -47.75\mathbf{a}_\phi + 23.88\mathbf{a}_z \text{ mA/m}$$

Notice that although the current filaments appear semiinfinite (they occupy the positive z- and x-axes), it is only the filament along the z-axis that is semiinfinite with respect to point P. Thus $\mathbf{H}_z$ could have been found by using eq. (7.13), but the equation could not have been used to find $\mathbf{H}_x$ because the filament along the x-axis is not semiinfinite with respect to P.

PRACTICE EXERCISE 7.2

The positive y-axis (semiinfinite line with respect to the origin) carries a filamentary current of 2 A in the $-\mathbf{a}_y$ direction. Assume it is part of a large circuit. Find **H** at

(a) $A(2, 3, 0)$

(b) $B(3, 12, -4)$

Answer: (a) $145.8\mathbf{a}_z$ mA/m, (b) $48.97\mathbf{a}_x + 36.73\mathbf{a}_z$ mA/m.

EXAMPLE 7.3

A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\,d\mathbf{l}$ is given by Biot–Savart's law:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$).

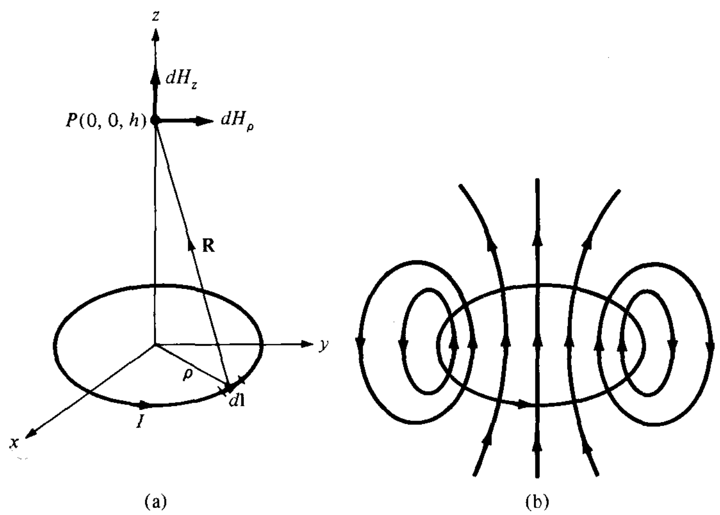


Figure 7.8 For Example 7.3: **(a)** circular current loop, **(b)** flux lines due to the current loop.

Integrating cos ϕ or sin ϕ over $0 \leq \phi \leq 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z \, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2 \, d\phi \, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\,\mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$

The flux lines due to the circular current loop are sketched in Figure 7.8(b).

PRACTICE EXERCISE 7.3

A thin ring of radius 5 cm is placed on plane $z = 1$ cm so that its center is at (0, 0, 1 cm). If the ring carries 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at

(a) (0, 0, −1 cm)

(b) (0, 0, 10 cm)

Answer: (a) $400\mathbf{a}_z$ mA/m, (b) $57.3\mathbf{a}_z$ mA/m.

EXAMPLE 7.4

A solenoid of length ℓ and radius a consists of N turns of wire carrying current I. Show that at point P along its axis,

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

where $n = N/\ell$, θ_1 and θ_2 are the angles subtended at P by the end turns as illustrated in Figure 7.9. Also show that if $\ell \gg a$, at the center of the solenoid,

$$\mathbf{H} = nI\mathbf{a}_z$$

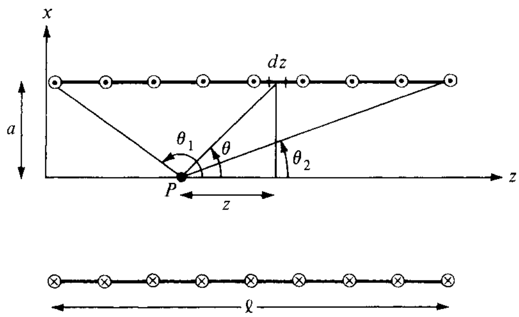


Figure 7.9 For Example 7.4; cross section of a solenoid.

Solution:

Consider the cross section of the solenoid as shown in Figure 7.9. Since the solenoid consists of circular loops, we apply the result of Example 7.3. The contribution to the magnetic field H at P by an element of the solenoid of length dz is

$$dH_z = \frac{I\,dl\,a^2}{2[a^2 + z^2]^{3/2}} = \frac{Ia^2 n\,dz}{2[a^2 + z^2]^{3/2}}$$

where $dl = n\,dz = (N/\ell)\,dz$. From Figure 7.9, $\tan\theta = a/z$; that is,

$$dz = -a\operatorname{cosec}^2\theta\,d\theta = -\frac{[z^2 + a^2]^{3/2}}{a^2}\sin\theta\,d\theta$$

Hence,

$$dH_z = -\frac{nI}{2}\sin\theta\,d\theta$$

or

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

Thus

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

as required. Substituting $n = N/\ell$ gives

$$\mathbf{H} = \frac{NI}{2\ell}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

At the center of the solenoid,

$$\cos\theta_2 = \frac{\ell/2}{[a^2 + \ell^2/4]^{1/2}} = -\cos\theta_1$$

and

$$\mathbf{H} = \frac{In\ell}{2[a^2 + \ell^2/4]^{1/2}}\,\mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$,

$$\mathbf{H} = nI\mathbf{a}_z = \frac{NI}{\ell}\,\mathbf{a}_z$$

PRACTICE EXERCISE 7.4

If the solenoid of Figure 7.9 has 2,000 turns, a length of 75 cm, a radius of 5 cm, and carries a current of 50 mA along $\mathbf{a}_\phi$, find **H** at

(a) (0, 0, 0)
(b) (0, 0, 75 cm)
(c) (0, 0, 50 cm)

Answer: (a) $66.52\mathbf{a}_z$ A/m, (b) $66.52\mathbf{a}_z$ A/m, (c) $131.7\mathbf{a}_z$ A/m.

7.3 AMPERE'S CIRCUIT LAW—MAXWELL'S EQUATION

Ampere's circuit law states that the line integral of the tangential component of **H** around a *closed* path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of **H** equals I_{enc}; that is,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} \tag{7.16}$$

Ampere's law is similar to Gauss's law and it is easily applied to determine **H** when the current distribution is symmetrical. It should be noted that eq. (7.16) always holds whether the current distribution is symmetrical or not but we can only use the equation to determine **H** when symmetrical current distribution exists. Ampere's law is a special case of Biot–Savart's law; the former may be derived from the latter.

By applying Stoke's theorem to the left-hand side of eq. (7.16), we obtain

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \tag{7.17}$$

But

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{7.18}$$

Comparing the surface integrals in eqs. (7.17) and (7.18) clearly reveals that

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{7.19}$$

This is the third Maxwell's equation to be derived; it is essentially Ampere's law in differential (or point) form whereas eq. (7.16) is the integral form. From eq. (7.19), we should observe that $\nabla \times \mathbf{H} = \mathbf{J} \neq 0$; that is, magnetostatic field is not conservative.

7.4 APPLICATIONS OF AMPERE'S LAW

We now apply Ampere's circuit law to determine **H** for some symmetrical current distributions as we did for Gauss's law. We will consider an infinite line current, an infinite current sheet, and an infinitely long coaxial transmission line.

A. Infinite Line Current

Consider an infinitely long filamentary current I along the z-axis as in Figure 7.10. To determine **H** at an observation point P, we allow a closed path pass through P. This path, on which Ampere's law is to be applied, is known as an *Amperian path* (analogous to the term Gaussian surface). We choose a concentric circle as the Amperian path in view of eq. (7.14), which shows that **H** is constant provided ρ is constant. Since this path encloses the whole current I, according to Ampere's law

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$

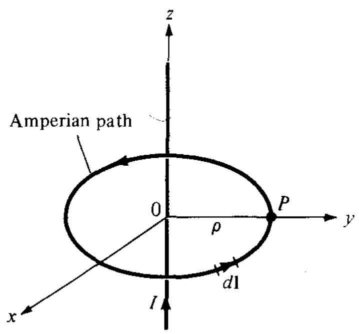


Figure 7.10 Ampere's law applied to an infinite filamentary line current.

or

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi} \tag{7.20}$$

as expected from eq. (7.14).

B. Infinite Sheet of Current

Consider an infinite current sheet in the $z = 0$ plane. If the sheet has a uniform current density $\mathbf{K} = K_y\mathbf{a}_y$ A/m as shown in Figure 7.11, applying Ampere's law to the rectangular closed path (Amperian path) gives

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b \tag{7.21a}$$

To evaluate the integral, we first need to have an idea of what $\mathbf{H}$ is like. To achieve this, we regard the infinite sheet as comprising of filaments; $d\mathbf{H}$ above or below the sheet due to a pair of filamentary currents can be found using eqs. (7.14) and (7.15). As evident in Figure 7.11(b), the resultant $d\mathbf{H}$ has only an x-component. Also, $\mathbf{H}$ on one side of the sheet is the negative of that on the other side. Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of $\mathbf{H}$ for a pair are the same for the infinite current sheets, that is,

$$\mathbf{H} = \begin{cases} H_o\mathbf{a}_x & z > 0 \\ -H_o\mathbf{a}_x & z < 0 \end{cases} \tag{7.21b}$$

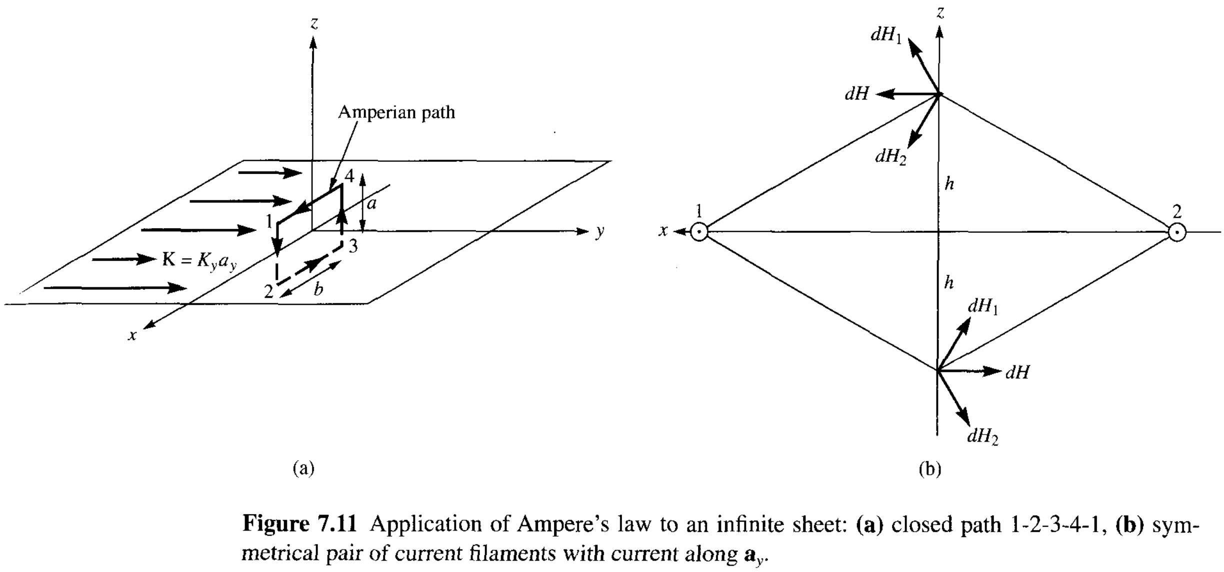


Figure 7.11 Application of Ampere's law to an infinite sheet: **(a)** closed path 1-2-3-4-1, **(b)** symmetrical pair of current filaments with current along $\mathbf{a}_y$.

where H_o is yet to be determined. Evaluating the line integral of **H** in eq. (7.21b) along the closed path in Figure 7.11(a) gives

$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right) \mathbf{H} \cdot d\mathbf{l}$$
$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) \qquad (7.21c)$$
$$= 2H_o b$$

From eqs. (7.21a) and (7.21c), we obtain $H_o = \frac{1}{2} K_y$. Substituting H_o in eq. (7.21b) gives

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases} \qquad (7.22)$$

In general, for an infinite sheet of current density **K** A/m,

$$\boxed{\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n} \qquad (7.23)$$

where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

C. Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The cross section of the line is shown in Figure 7.12, where the z-axis is out of the page. The inner conductor has radius a and carries current I while the outer conductor has inner radius b and thickness t and carries return current $-I$. We want to determine **H** everywhere assuming that current is uniformly distributed in both conductors. Since the current distribution is symmetrical, we apply Ampere's law along the Am-

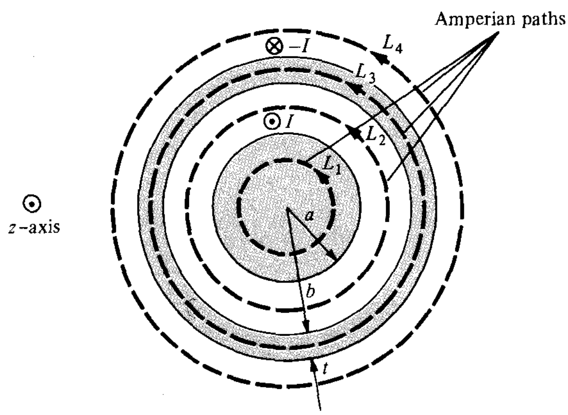


Figure 7.12 Cross section of the transmission line; the positive z-direction is out of the page.

perian path for each of the four possible regions: $0 \leq \rho \leq a$, $a \leq \rho \leq b$, $b \leq \rho \leq b + t$, and $\rho \geq b + t$.

For region $0 \leq \rho \leq a$, we apply Ampere's law to path L_1, giving

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{S} \tag{7.24}$$

Since the current is uniformly distributed over the cross section,

$$\mathbf{J} = \frac{I}{\pi a^2} \mathbf{a}_z, \qquad d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{S} = \frac{I}{\pi a^2} \iint \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2 = \frac{I\rho^2}{a^2}$$

Hence eq. (7.24) becomes

$$H_\phi \int dl = H_\phi \, 2\pi\rho = \frac{I\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \tag{7.25}$$

For region $a \leq \rho \leq b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = I$$

$$H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho} \tag{7.26}$$

since the whole current I is enclosed by L_2. Notice that eq. (7.26) is the same as eq. (7.14) and it is independent of a. For region $b \leq \rho \leq b + t$, we use path L_3, getting

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_\phi \cdot 2\pi\phi = I_{\text{enc}} \tag{7.27a}$$

where

$$I_{\text{enc}} = I + \int \mathbf{J} \cdot d\mathbf{S}$$

and **J** in this case is the current density (current per unit area) of the outer conductor and is along $-\mathbf{a}_z$, that is,

$$\mathbf{J} = -\frac{I}{\pi[(b+t)^2 - b^2]}\,\mathbf{a}_z$$

Thus

$$\begin{aligned} I_{enc} &= I - \frac{I}{\pi[(b+t)^2 - t^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho\, d\rho\, d\phi \\ &= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \end{aligned}$$

Substituting this in eq. (7.27a), we have

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \tag{7.27b}$$

For region $\rho \geq b + t$, we use path L_4, getting

$$\oint_{L_4} \mathbf{H} \cdot d\mathbf{l} = I - I = 0$$

or

$$H_\phi = 0 \tag{7.28}$$

Putting eqs. (7.25) to (7.28) together gives

$$\mathbf{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi, & 0 \leq \rho \leq a \\ \dfrac{I}{2\pi\rho}\,\mathbf{a}_\phi, & a \leq \rho \leq b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\mathbf{a}_\phi, & b \leq \rho \leq b + t \\ 0, & \rho \geq b + t \end{cases} \tag{7.29}$$

The magnitude of **H** is sketched in Figure 7.13.

Notice from these examples that the ability to take **H** from under the integral sign is the key to using Ampere's law to determine **H**. In other words, Ampere's law can only be used to find **H** due to symmetric current distributions for which it is possible to find a closed path over which **H** is constant in magnitude.

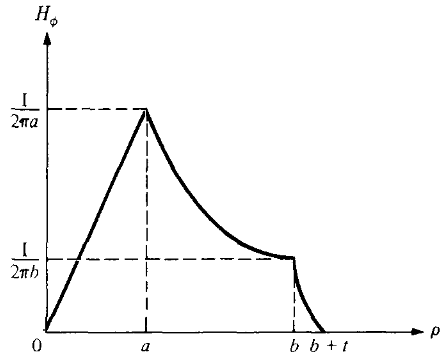


Figure 7.13 Plot of H_ϕ against ρ.

EXAMPLE 7.5

Planes $z = 0$ and $z = 4$ carry current $\mathbf{K} = -10\mathbf{a}_x$ A/m and $\mathbf{K} = 10\mathbf{a}_x$ A/m, respectively. Determine $\mathbf{H}$ at

(a) (1, 1, 1)

(b) (0, −3, 10)

Solution:

Let the parallel current sheets be as in Figure 7.14. Also let

$$\mathbf{H} = \mathbf{H}_o + \mathbf{H}_4$$

where $\mathbf{H}_o$ and $\mathbf{H}_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively. We make use of eq. (7.23).

(a) At (1, 1, 1), which is between the plates ($0 < z = 1 < 4$),

$$\mathbf{H}_o = 1/2\, \mathbf{K} \times \mathbf{a}_n = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\, \mathbf{K} \times \mathbf{a}_n = 1/2\,(10\mathbf{a}_x) \times (-\mathbf{a}_z) = 5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 10\mathbf{a}_y \text{ A/m}$$

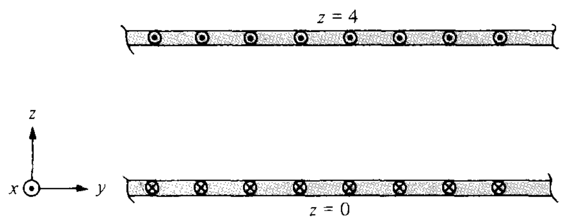


Figure 7.14 For Example 7.5; parallel infinite current sheets.

(b) At (0, −3, 10), which is above the two sheets ($z = 10 > 4 > 0$),

$$\mathbf{H}_o = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$
$$\mathbf{H}_4 = 1/2\,(10\mathbf{a}_x) \times \mathbf{a}_z = -5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 0 \text{ A/m}$$

PRACTICE EXERCISE 7.5

Plane $y = 1$ carries current $\mathbf{K} = 50\mathbf{a}_z$ mA/m. Find **H** at

(a) (0, 0, 0)

(b) (1, 5, −3)

Answer: (a) $25\mathbf{a}_x$ mA/m, (b) $-25\mathbf{a}_x$ mA/m.

EXAMPLE 7.6

A toroid whose dimensions are shown in Figure 7.15 has N turns and carries current I. De termine H inside and outside the toroid.

Solution:

We apply Ampere's circuit law to the Amperian path, which is a circle of radius ρ show dotted in Figure 7.15. Since N wires cut through this path each carrying current I, the ne current enclosed by the Amperian path is NI. Hence,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} \rightarrow H \cdot 2\pi\rho = NI$$

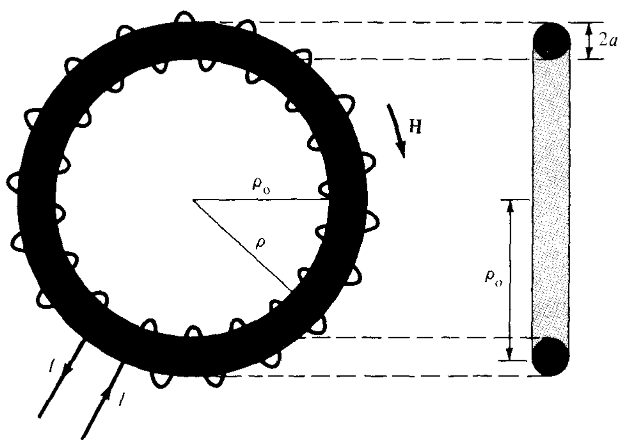


Figure 7.15 For Example 7.6; a toroid with a circular cross section.

or

$$H = \frac{NI}{2\pi\rho}, \qquad \text{for } \rho_o - a < \rho < \rho_o + a$$

where ρ_o is the mean radius of the toroid as shown in Figure 7.15. An approximate value of H is

$$H_{\text{approx}} = \frac{NI}{2\pi\rho_o} = \frac{NI}{\ell}$$

Notice that this is the same as the formula obtained for H for points well inside a very long solenoid ($\ell \gg a$). Thus a straight solenoid may be regarded as a special toroidal coil for which $\rho_o \to \infty$. Outside the toroid, the current enclosed by an Amperian path is $NI - NI = 0$ and hence $H = 0$.

PRACTICE EXERCISE 7.6

A toroid of circular cross section whose center is at the origin and axis the same as the z-axis has 1000 turns with $\rho_o = 10$ cm, $a = 1$ cm. If the toroid carries a 100-mA current, find $|H|$ at

(a) (3 cm, −4 cm, 0)

(b) (6 cm, 9 cm, 0)

Answer: (a) 0, (b) 147.1 A/m.

7.5 MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux density **B** is similar to the electric flux density **D**. As $\mathbf{D} = \varepsilon_o \mathbf{E}$ in free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to

$$\boxed{\mathbf{B} = \mu_o \mathbf{H}} \tag{7.30}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys/meter (H/m) and has the value of

$$\boxed{\mu_o = 4\pi \times 10^{-7} \text{ H/m}} \tag{7.31}$$

The precise definition of the magnetic field **B**, in terms of the magnetic force, will be given in the next chapter.

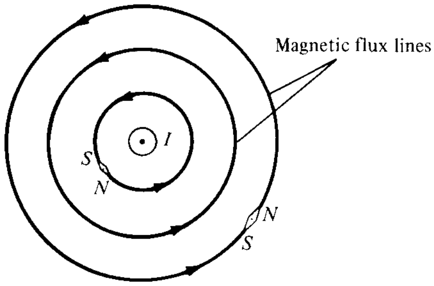


Figure 7.16 Magnetic flux lines due to a straight wire with current coming out of the page.

The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{7.32}$$

where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers/square meter (Wb/m^2) or teslas.

The magnetic flux line is the path to which $\mathbf{B}$ is tangential at every point in a magnetic field. It is the line along which the needle of a magnetic compass will orient itself if placed in the magnetic field. For example, the magnetic flux lines due to a straight long wire are shown in Figure 7.16. The flux lines are determined using the same principle followed in Section 4.10 for the electric flux lines. The direction of $\mathbf{B}$ is taken as that indicated as "north" by the needle of the magnetic compass. Notice that each flux line is closed and has no beginning or end. Though Figure 7.16 is for a straight, current-carrying conductor, it is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed; that is, $\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$. Thus it is possible to have an isolated electric charge as shown in Figure 7.17(a), which also reveals that electric flux lines are not necessarily closed. Unlike electric flux lines, magnetic flux lines always close upon themselves as in Figure 7.17(b). This is due to the fact that *it is not possible to have isolated magnetic*

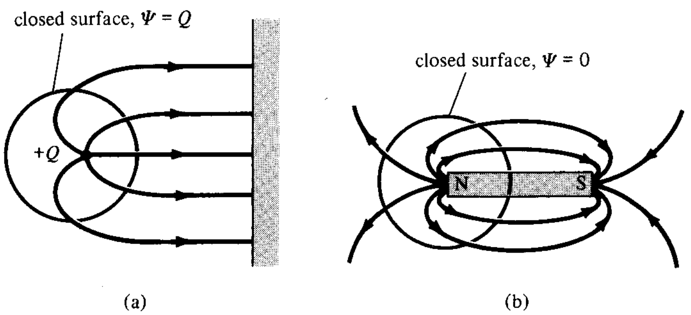


Figure 7.17 Flux leaving a closed surface due to: **(a)** isolated electric charge $\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$, **(b)** magnetic charge, $\Psi = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$.

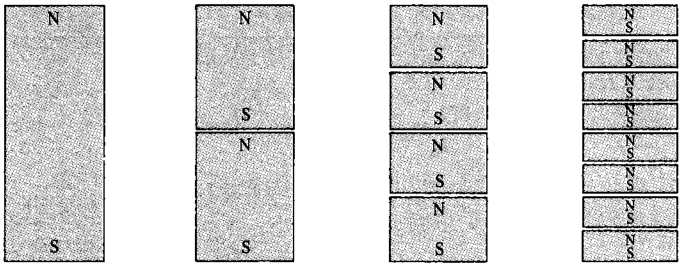


Figure 7.18 Successive division of a bar magnet results in pieces with north and south poles, showing that magnetic poles cannot be isolated.

poles (or magnetic charges). For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles as illustrated in Figure 7.18. We find it impossible to separate the north pole from the south pole.

An **isolated magnetic** charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is,

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{S} = 0} \tag{7.33}$$

This equation is referred to as the *law of conservation of magnetic flux* or *Gauss's law for magnetostatic fields* just as $\oint \mathbf{D} \cdot d\mathbf{S} = Q$ is Gauss's law for electrostatic fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq. (7.33), we obtain

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

or

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{7.34}$$

This equation is the fourth Maxwell's equation to be derived. Equation (7.33) or (7.34) shows that magnetostatic fields have no sources or sinks. Equation (7.34) suggests that magnetic field lines are always continuous.

7.6 MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

Having derived Maxwell's four equations for static electromagnetic fields, we may take a moment to put them together as in Table 7.2. From the table, we notice that the order in which the equations were derived has been changed for the sake of clarity.

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

The choice between differential and integral forms of the equations depends on a given problem. It is evident from Table 7.2 that a vector field is defined completely by specifying its curl and divergence. A field can only be electric or magnetic if it satisfies the corresponding Maxwell's equations (see Problems 7.26 and 7.27). It should be noted that Maxwell's equations as in Table 7.2 are only for static EM fields. As will be discussed in Chapter 9, the divergence equations will remain the same for time-varying EM fields but the curl equations will have to be modified.

7.7 MAGNETIC SCALAR AND VECTOR POTENTIALS

We recall that some electrostatic field problems were simplified by relating the electric potential V to the electric field intensity $\mathbf{E}$ ($\mathbf{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field $\mathbf{B}$. In fact, the magnetic potential could be scalar V_m or vector $\mathbf{A}$. To define V_m and $\mathbf{A}$ involves recalling two important identities (see Example 3.9 and Practice Exercise 3.9):

$$\nabla \times (\nabla V) = 0 \tag{7.35a}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{7.35b}$$

which must always hold for any scalar field V and vector field $\mathbf{A}$.

Just as $\mathbf{E} = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to $\mathbf{H}$ according to

$$\boxed{\mathbf{H} = -\nabla V_m} \qquad \text{if } \mathbf{J} = 0 \tag{7.36}$$

The condition attached to this equation is important and will be explained. Combining eq. (7.36) and eq. (7.19) gives

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0 \tag{7.37}$$

since V_m must satisfy the condition in eq. (7.35a). Thus the magnetic scalar potential V_m is only defined in a region where $\mathbf{J} = 0$ as in eq. (7.36). We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields; hence,

$$\nabla^2 V_m = 0, \qquad (\mathbf{J} = 0) \tag{7.38}$$

We know that for a magnetostatic field, $\nabla \cdot \mathbf{B} = 0$ as stated in eq. (7.34). In order to satisfy eqs. (7.34) and (7.35b) simultaneously, we can define the *vector magnetic potential* **A** (in Wb/m) such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{7.39}$$

Just as we defined

$$V = \int \frac{dQ}{4\pi\varepsilon_o r} \tag{7.40}$$

we can define

$$\boxed{\mathbf{A} = \int_L \frac{\mu_o I\, d\mathbf{l}}{4\pi R}} \quad \text{for line current} \tag{7.41}$$

$$\boxed{\mathbf{A} = \int_S \frac{\mu_o \mathbf{K}\, dS}{4\pi R}} \quad \text{for surface current} \tag{7.42}$$

$$\boxed{\mathbf{A} = \int_v \frac{\mu_o \mathbf{J}\, dv}{4\pi R}} \quad \text{for volume current} \tag{7.43}$$

Rather than obtaining eqs. (7.41) to (7.43) from eq. (7.40), an alternative approach would be to obtain eqs. (7.41) to (7.43) from eqs. (7.6) to (7.8). For example, we can derive eq. (7.41) from eq. (7.6) in conjunction with eq. (7.39). To do this, we write eq. (7.6) as

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I\, d\mathbf{l}' \times \mathbf{R}}{R^3} \tag{7.44}$$

where **R** is the distance vector from the line element $d\mathbf{l}'$ at the source point (x', y', z') to the field point (x, y, z) as shown in Figure 7.19 and $R = |\mathbf{R}|$, that is,

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \tag{7.45}$$

Hence,

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{R}}{R^3}$$

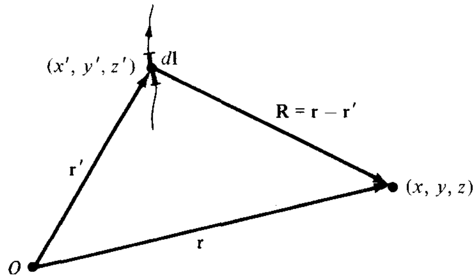


Figure 7.19 Illustration of the source point (x', y', z') and the field point (x, y, z).

or

$$\frac{\mathbf{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \quad \left(= \frac{\mathbf{a}_R}{R^2}\right) \tag{7.46}$$

where the differentiation is with respect to x, y, and z. Substituting this into eq. (7.44), we obtain

$$\mathbf{B} = -\frac{\mu_o}{4\pi}\int_L I\, d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) \tag{7.47}$$

We apply the vector identity

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F} \tag{7.48}$$

where f is a scalar field and $\mathbf{F}$ is a vector field. Taking $f = 1/R$ and $\mathbf{F} = d\mathbf{l}'$, we have

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{l}' - \nabla \times \left(\frac{d\mathbf{l}'}{R}\right)$$

Since ∇ operates with respect to (x, y, z) while $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = 0$. Hence,

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\mathbf{l}'}{R} \tag{7.49}$$

With this equation, eq. (7.47) reduces to

$$\mathbf{B} = \nabla \times \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R} \tag{7.50}$$

Comparing eq. (7.50) with eq. (7.39) shows that

$$\mathbf{A} = \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R}$$

verifying eq. (7.41).

By substituting eq. (7.39) into eq. (7.32) and applying Stokes's theorem, we obtain

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

or

$$\boxed{\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}} \tag{7.51}$$

Thus the magnetic flux through a given area can be found using either eq. (7.32) or (7.51). Also, the magnetic field can be determined using either V_m or $\mathbf{A}$; the choice is dictated by the nature of the given problem except that V_m can only be used in a source-free region. The use of the magnetic vector potential provides a powerful, elegant approach to solving EM problems, particularly those relating to antennas. As we shall notice in Chapter 13, it is more convenient to find $\mathbf{B}$ by first finding $\mathbf{A}$ in antenna problems.

EXAMPLE 7.7

Given the magnetic vector potential $\mathbf{A} = -\rho^2/4\ \mathbf{a}_z$ Wb/m, calculate the total magnetic flux crossing the surface $\phi = \pi/2$, $1 \leq \rho \leq 2$ m, $0 \leq z \leq 5$ m.

Solution:

We can solve this problem in two different ways: using eq. (7.32) or eq. (7.51).

Method 1:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \frac{\rho}{2}\mathbf{a}_\phi, \qquad d\mathbf{S} = d\rho\, dz\, \mathbf{a}_\phi$$

Hence,

$$\Psi = \int \mathrm{B} \cdot d\mathbf{S} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho\, d\rho\, dz = \frac{1}{4}\rho^2 \Big|_2^1 (5) = \frac{15}{4}$$

$$\Psi = 3.75 \text{ Wb}$$

Method 2:
We use

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

where L is the path bounding surface S; Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are, respectively, the evaluations of $\int \mathbf{A} \cdot d\mathbf{l}$ along the segments of L labeled 1 to 4 in Figure 7.20. Since $\mathbf{A}$ has only a z-component,

$$\Psi_1 = 0 = \Psi_3$$

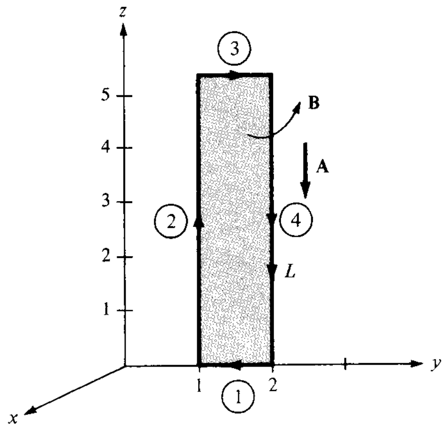


Figure 7.20 For Example 7.7.

That is,

$$\Psi = \Psi_2 + \Psi_4 = -\frac{1}{4}\left[(1)^2 \int_0^5 dz + (2)^2 \int_5^0 dz\right]$$

$$= -\frac{1}{4}(1 - 4)(5) = \frac{15}{4}$$

$$= 3.75 \text{ Wb}$$

as obtained previously. Note that the direction of the path L must agree with that of $d\mathbf{S}$.

PRACTICE EXERCISE 7.7

A current distribution gives rise to the vector magnetic potential $\mathbf{A} = x^2y\mathbf{a}_x + y^2x\mathbf{a}_y - 4xyz\mathbf{a}_z$ Wb/m. Calculate

(a) $\mathbf{B}$ at $(-1, 2, 5)$

(b) The flux through the surface defined by $z = 1, 0 \leq x \leq 1, -1 \leq y \leq 4$

Answer: (a) $20\mathbf{a}_x + 40\mathbf{a}_y + 3\mathbf{a}_z$ Wb/m^2, (b) 20 Wb.

EXAMPLE 7.8

If plane $z = 0$ carries uniform current $\mathbf{K} = K_y\mathbf{a}_y$,

$$\mathbf{H} = \begin{cases} 1/2\, K_y\mathbf{a}_x, & z > 0 \\ -1/2\, K_y\mathbf{a}_x, & z < 0 \end{cases}$$

This was obtained in Section 7.4 using Ampere's law. Obtain this by using the concept of vector magnetic potential.

Solution:

Consider the current sheet as in Figure 7.21. From eq. (7.42),

$$d\mathbf{A} = \frac{\mu_o \mathbf{K}\, dS}{4\pi R}$$

In this problem, $\mathbf{K} = K_y \mathbf{a}_y$, $dS = dx'\, dy'$, and for $z > 0$,

$$\begin{aligned} R = |\mathbf{R}| &= |(0, 0, z) - (x', y', 0)| \\ &= [(x')^2 + (y')^2 + z^2]^{1/2} \end{aligned} \tag{7.8.1}$$

where the primed coordinates are for the source point while the unprimed coordinates are for the field point. It is necessary (and customary) to distinguish between the two points to avoid confusion (see Figure 7.19). Hence

$$d\mathbf{A} = \frac{\mu_o K_y\, dx'\, dy'\, \mathbf{a}_y}{4\pi[(x')^2 + (y')^2 + z^2]^{1/2}}$$

$$\begin{aligned} d\mathbf{B} = \nabla \times d\mathbf{A} &= -\frac{\partial}{\partial z} dA_y\, \mathbf{a}_x \\ &= \frac{\mu_o K_y z\, dx'\, dy'\, \mathbf{a}_x}{4\pi[(x')^2 + (y')^2 + z^2]^{3/2}} \end{aligned}$$

$$\mathbf{B} = \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx'\, dy'}{[(x')^2 + (y')^2 + z^2]^{3/2}} \tag{7.8.2}$$

In the integrand, we may change coordinates from Cartesian to cylindrical for convenience so that

$$\begin{aligned} \mathbf{B} &= \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} \int_{\rho'=0}^{\infty} \int_{\phi'=0}^{2\pi} \frac{\rho'\, d\phi'\, d\rho'}{[(\rho')^2 + z^2]^{3/2}} \\ &= \frac{\mu_o K_y z \mathbf{a}_x}{4\pi} 2\pi \int_0^{\infty} [(\rho')^2 + z^2]^{-3/2}\, 1/2\, d[(\rho')^2] \\ &= \frac{\mu_o K_y z \mathbf{a}_x}{2} \left. \frac{-1}{[(\rho')^2 + z^2)^{1/2}} \right|_{\rho'=0}^{\infty} \\ &= \frac{\mu_o K_y \mathbf{a}_x}{2} \end{aligned}$$

Hence

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_o} = \frac{K_y}{2} \mathbf{a}_x, \qquad \text{for } z > 0$$

By simply replacing z by $-z$ in eq. (7.8.2) and following the same procedure, we obtain

$$\mathbf{H} = -\frac{K_y}{2} \mathbf{a}_x, \qquad \text{for } z < 0$$

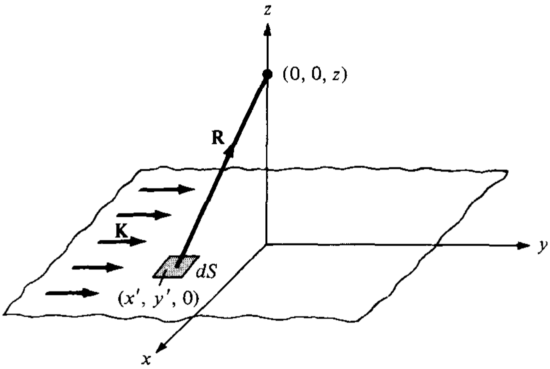


Figure 7.21 For Example 7.8; infinite current sheet.

PRACTICE EXERCISE 7.8

Repeat Example 7.8 by using Biot–Savart's law to determine **H** at points $(0, 0, h)$ and $(0, 0, -h)$.

†7.8 DERIVATION OF BIOT–SAVART'S LAW AND AMPERE'S LAW

Both Biot–Savart's law and Ampere's law may be derived using the concept of magnetic vector potential. The derivation will involve the use of the vector identities in eq. (7.48) and

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{7.52}$$

Since Biot–Savart's law as given in eq. (7.4) is basically on line current, we begin our derivation with eqs. (7.39) and (7.41); that is,

$$\mathbf{B} = \nabla \times \oint_L \frac{\mu_o I \, d\mathbf{l}'}{4\pi R} = \frac{\mu_o I}{4\pi} \oint_L \nabla \times \frac{1}{R} \, d\mathbf{l}', \tag{7.53}$$

where R is as defined in eq. (7.45). If the vector identity in eq. (7.48) is applied by letting $\mathbf{F} = d\mathbf{l}$ and $f = 1/R$, eq. (7.53) becomes

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint_L \left[\frac{1}{R}\nabla \times d\mathbf{l}' + \left(\nabla \frac{1}{R}\right) \times d\mathbf{l}'\right] \tag{7.54}$$

Since ∇ operates with respect to (x, y, z) and $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = 0$. Also

$$\frac{1}{R} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} \tag{7.55}$$

$$\nabla\left[\frac{1}{R}\right] = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{a}_R}{R^2} \tag{7.56}$$

where $\mathbf{a}_R$ is a unit vector from the source point to the field point. Thus eq. (7.54) (upon dropping the prime in $d\mathbf{l}'$) becomes

$$\mathbf{B} = \frac{\mu_o I}{4\pi} \oint_L \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2} \tag{7.57}$$

which is Biot–Savart's law.

Using the identity in eq. (7.52) with eq. (7.39), we obtain

$$\nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{7.58}$$

It can be shown that for a static magnetic field

$$\boxed{\nabla \cdot \mathbf{A} = 0} \tag{7.59}$$

so that upon replacing $\mathbf{B}$ with $\mu_o\mathbf{H}$ and using eq. (7.19), eq. (7.58) becomes

$$\nabla^2\mathbf{A} = -\mu_o \nabla \times \mathbf{H}$$

or

$$\boxed{\nabla^2\mathbf{A} = -\mu_o\mathbf{J}} \tag{7.60}$$

which is called the *vector Poisson's equation.* It is similar to Poisson's equation ($\nabla^2 V = -\rho_v/\varepsilon$) in electrostatics. In Cartesian coordinates, eq. (7.60) may be decomposed into three scalar equations:

$$\begin{aligned}\nabla^2 A_x &= -\mu_o J_x \\ \nabla^2 A_y &= -\mu_o J_y \\ \nabla^2 A_z &= -\mu_o J_z\end{aligned} \tag{7.61}$$

which may be regarded as the *scalar Poisson's equations.*

It can also be shown that Ampere's circuit law is consistent with our definition of the magnetic vector potential. From Stokes's theorem and eq. (7.39),

$$\begin{aligned}\oint_L \mathbf{H} \cdot d\mathbf{l} &= \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} \\ &= \frac{1}{\mu_o} \int_S \nabla \times (\nabla \times \mathbf{A}) \cdot d\mathbf{S}\end{aligned} \tag{7.62}$$

From eqs. (7.52), (7.59), and (7.60),

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} = \mu_o\mathbf{J}$$

Substituting this into eq. (7.62) yields

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = I$$

which is Ampere's circuit law.

SUMMARY

1. The basic laws (Biot–Savart's and Ampere's) that govern magnetostatic fields are discussed. Biot–Savart's law, which is similar to Coulomb's law, states that the magnetic field intensity $d\mathbf{H}$ at $\mathbf{r}$ due to current element $I\,d\mathbf{l}$ at $\mathbf{r}'$ is

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3} \qquad \text{(in A/m)}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$. For surface or volume current distribution, we replace $I\,d\mathbf{l}$ with $\mathbf{K}\,dS$ or $\mathbf{J}\,dv$ respectively; that is,

$$I\,d\mathbf{l} \equiv \mathbf{K}\,dS \equiv \mathbf{J}\,dv$$

2. Ampere's circuit law, which is similar to Gauss's law, states that the circulation of $\mathbf{H}$ around a closed path is equal to the current enclosed by the path; that is,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \text{(third Maxwell's equation to be derived).}$$

When current distribution is symmetric so that an Amperian path (on which $\mathbf{H} = H_\phi \mathbf{a}_\phi$ is constant) can be found, Ampere's law is useful in determining $\mathbf{H}$; that is,

$$H_\phi \oint dl = I_{enc} \qquad \text{or} \qquad H_\phi = \frac{I_{enc}}{\ell}$$

3. The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(in Wb)}$$

where $\mathbf{B}$ is the magnetic flux density in Wb/m^2. In free space,

$$\mathbf{B} = \mu_o \mathbf{H}$$

where $\mu_o = 4\pi \times 10^{-7}$ H/m = permeability of free space.

4. Since an isolated or free magnetic monopole does not exist, the net magnetic flux through a closed surface is zero;

$$\Psi = \oint \mathbf{B} \cdot d\mathbf{S} = 0$$

or

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(fourth Maxwell's equation to be derived).}$$

5. At this point, all four Maxwell's equations for static EM fields have been derived, namely:

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$

6. The magnetic scalar potential V_m is defined as

$$\mathbf{H} = -\nabla V_m \qquad \text{if } \mathbf{J} = 0$$

and the magnetic vector potential $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where $\nabla \cdot \mathbf{A} = 0$. With the definition of $\mathbf{A}$, the magnetic flux through a surface S can be found from

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

where L is the closed path defining surface S (see Figure 3.20). Rather than using Biot–Savart's law, the magnetic field due to a current distribution may be found using $\mathbf{A}$, a powerful approach that is particularly useful in antenna theory. For a current element $I\,d\mathbf{l}$ at $\mathbf{r}'$, the magnetic vector potential at $\mathbf{r}$ is

$$\mathbf{A} = \int \frac{\mu_o I\, d\mathbf{l}}{4\pi R}, \qquad R = |\mathbf{r} - \mathbf{r}'|$$

7. Elements of similarity between electric and magnetic fields exist. Some of these are listed in Table 7.1. Corresponding to Poisson's equation $\nabla^2 V = -\rho_v/\varepsilon$, for example, is

$$\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}$$

REVIEW QUESTIONS

7.1 One of the following is not a source of magnetostatic fields:

(a) A dc current in a wire
(b) A permanent magnet
(c) An accelerated charge
(d) An electric field linearly changing with time
(e) A charged disk rotating at uniform speed

7.2 Identify the configuration in Figure 7.22 that is not a correct representation of I and $\mathbf{H}$.

7.3 Consider points A, B, C, D, and E on a circle of radius 2 as shown in Figure 7.23. The items in the right list are the values of $\mathbf{a}_\phi$ at different points on the circle. Match these items with the points in the list on the left.

(a) A	(i) $\mathbf{a}_x$
(b) B	(ii) $-\mathbf{a}_x$
(c) C	(iii) $\mathbf{a}_y$
(d) D	(iv) $-\mathbf{a}_y$
(e) E	(v) $\dfrac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
	(vi) $\dfrac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$
	(vii) $\dfrac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
	(viii) $\dfrac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$

7.4 The z-axis carries filamentary current of 10π A along $\mathbf{a}_z$. Which of these is incorrect?

(a) $\mathbf{H} = -\mathbf{a}_x$ A/m at $(0, 5, 0)$
(b) $\mathbf{H} = \mathbf{a}_\phi$ A/m at $(5, \pi/4, 0)$
(c) $\mathbf{H} = -0.8\mathbf{a}_x - 0.6\mathbf{a}_y$ at $(-3, 4, 0)$
(d) $\mathbf{H} = -\mathbf{a}_\phi$ at $(5, 3\pi/2, 0)$

7.5 Plane $y = 0$ carries a uniform current of $30\mathbf{a}_z$ mA/m. At $(1, 10, -2)$, the magnetic field intensity is

(a) $-15\mathbf{a}_x$ mA/m
(b) $15\mathbf{a}_x$ mA/m

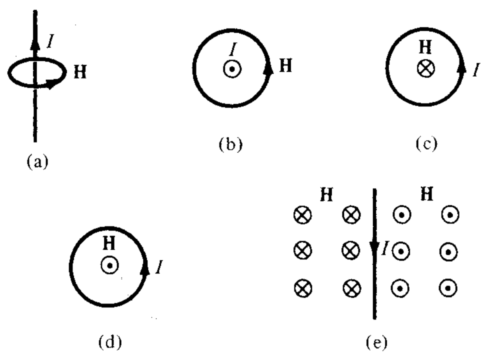


Figure 7.22 For Review Question 7.2.

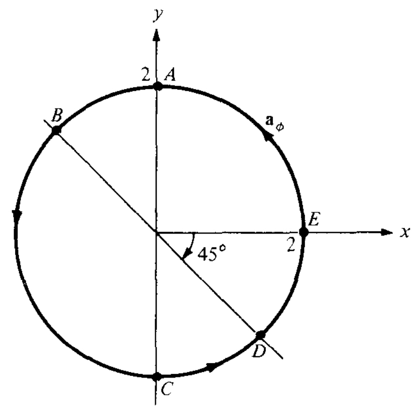


Figure 7.23 For Review Question 7.3.

(c) $477.5\mathbf{a}_y$ μA/m
(d) $18.85\mathbf{a}_y$ nA/m
(e) None of the above

7.6 For the currents and closed paths of Figure 7.24, calculate the value of $\oint_L \mathbf{H} \cdot d\mathbf{l}$.

7.7 Which of these statements is not characteristic of a static magnetic field?

(a) It is solenoidal.
(b) It is conservative.
(c) It has no sinks or sources.
(d) Magnetic flux lines are always closed.
(e) The total number of flux lines entering a given region is equal to the total number of flux lines leaving the region.

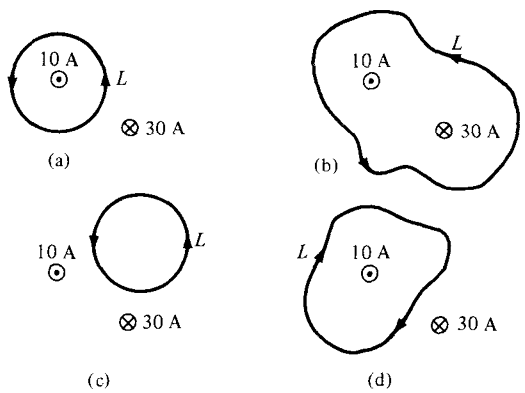


Figure 7.24 For Review Question 7.6.

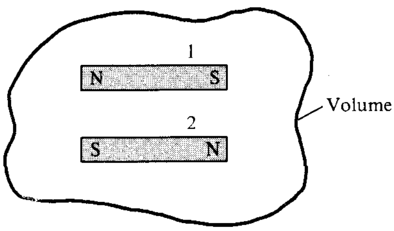


Figure 7.25 For Review Question 7.10.

7.8 Two identical coaxial circular coils carry the same current I but in opposite directions. The magnitude of the magnetic field $\mathbf{B}$ at a point on the axis midway between the coils is

(a) Zero
(b) The same as that produced by one coil
(c) Twice that produced by one coil
(d) Half that produced by one coil.

7.9 One of these equations is not Maxwell's equation for a static electromagnetic field in a linear homogeneous medium.

(a) $\nabla \cdot \mathbf{B} = 0$
(b) $\nabla \times \mathbf{D} = 0$
(c) $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_o I$
(d) $\oint \mathbf{D} \cdot d\mathbf{S} = Q$
(e) $\nabla^2 \mathbf{A} = \mu_o \mathbf{J}$

7.10 Two bar magnets with their north poles have strength $Q_{m1} = 20$ A · m and $Q_{m2} = 10$ A · m (magnetic charges) are placed inside a volume as shown in Figure 7.25. The magnetic flux leaving the volume is

(a) 200 Wb
(b) 30 Wb
(c) 10 Wb
(d) 0 Wb
(e) −10 Wb

Answers: 7.1c, 7.2c, 7.3 (a)-(ii), (b)-(vi), (c)-(i), (d)-(v), (e)-(iii), 7.4d, 7.5a, 7.6 (a) 10 A, (b) −20 A, (c) 0, (d) −10 A, 7.7b, 7.8a, 7.9e, 7.10d.

PROBLEMS

7.1 (a) State Biot–Savart's law

(b) The y- and z-axes, respectively, carry filamentary currents 10 A along $\mathbf{a}_y$ and 20 A along $-\mathbf{a}_z$. Find $\mathbf{H}$ at $(-3, 4, 5)$.

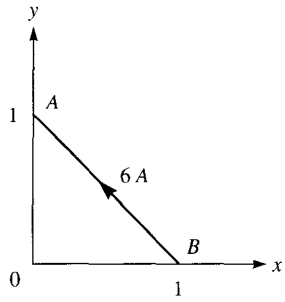


Figure 7.26 For Problem 7.3.

7.2 A conducting filament carries current I from point $A(0, 0, a)$ to point $B(0, 0, b)$. Show that at point $P(x, y, 0)$,

$$\mathbf{H} = \frac{I}{4\pi\sqrt{x^2 + y^2}^3}\left[\frac{b}{\sqrt{x^2 + y^2 + b^2}} - \frac{a}{\sqrt{x^2 + y^2 + a^2}}\right]\mathbf{a}_y$$

7.3 Consider AB in Figure 7.26 as part of an electric circuit. Find $\mathbf{H}$ at the origin due to AB.

7.4 Repeat Problem 7.3 for the conductor AB in Figure 7.27.

7.5 Line $x = 0$, $y = 0$, $0 \le z \le 10$ m carries current 2 A along $\mathbf{a}_z$. Calculate $\mathbf{H}$ at points

(a) (5, 0, 0)
(b) (5, 5, 0)
(c) (5, 15, 0)
(d) (5, −15, 0)

***7.6** (a) Find $\mathbf{H}$ at (0, 0, 5) due to side 2 of the triangular loop in Figure 7.6(a).
(b) Find $\mathbf{H}$ at (0, 0, 5) due to the entire loop.

7.7 An infinitely long conductor is bent into an L shape as shown in Figure 7.28. If a direct current of 5 A flows in the current, find the magnetic field intensity at (a) (2, 2, 0), (b) (0, −2, 0), and (c) (0, 0, 2).

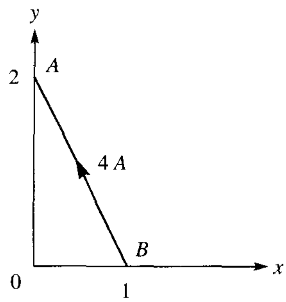


Figure 7.27 For Problem 7.4.

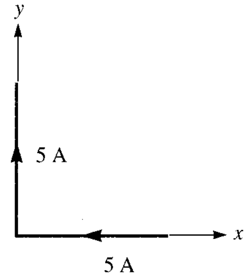


Figure 7.28 Current filament for Problem 7.7.

7.8 Find **H** at the center C of an equilateral triangular loop of side 4 m carrying 5 A of current as in Figure 7.29.

7.9 A rectangular loop carrying 10 A of current is placed on $z = 0$ plane as shown in Figure 7.30. Evaluate **H** at

(a) (2, 2, 0)
(b) (4, 2, 0)
(c) (4, 8, 0)
(d) (0, 0, 2)

7.10 A square conducting loop of side $2a$ lies in the $z = 0$ plane and carries a current I in the counterclockwise direction. Show that at the center of the loop

$$\mathbf{H} = \frac{\sqrt{2}I}{\pi a}\,\mathbf{a}_z$$

***7.11** (a) A filamentary loop carrying current I is bent to assume the shape of a regular polygon of n sides. Show that at the center of the polygon

$$H = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

where r is the radius of the circle circumscribed by the polygon.

(b) Apply this to cases when $n = 3$ and $n = 4$ and see if your results agree with those for the triangular loop of Problem 7.8 and the square loop of Problem 7.10, respectively.

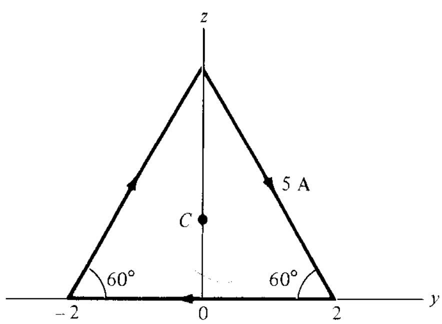


Figure 7.29 Equilateral triangular loop for Problem 7.8.

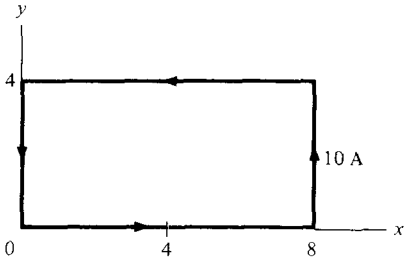


Figure 7.30 Rectangular loop of Problem 7.9.

(c) As n becomes large, show that the result of part (a) becomes that of the circular loop of Example 7.3.

7.12 For the filamentary loop shown in Figure 7.31, find the magnetic field strength at O.

7.13 Two identical current loops have their centers at (0, 0, 0) and (0, 0, 4) and their axes the same as the z-axis (so that the "Helmholtz coil" is formed). If each loop has radius 2 m and carries current 5 A in $\mathbf{a}_\phi$, calculate $\mathbf{H}$ at

(a) (0, 0, 0)

(b) (0, 0, 2)

7.14 A 3-cm-long solenoid carries a current of 400 mA. If the solenoid is to produce a magnetic flux density of 5 mWb/m^2, how many turns of wire are needed?

7.15 A solenoid of radius 4 mm and length 2 cm has 150 turns/m and carries current 500 mA. Find: (a) $|\mathbf{H}|$ at the center, (b) $|\mathbf{H}|$ at the ends of the solenoid.

7.16 Plane $x = 10$ carries current 100 mA/m along $\mathbf{a}_z$ while line $x = 1$, $y = -2$ carries filamentary current 20π mA along $\mathbf{a}_z$. Determine $\mathbf{H}$ at (4, 3, 2).

7.17 (a) State Ampere's circuit law.

(b) A hollow conducting cylinder has inner radius a and outer radius b and carries current I along the positive z-direction. Find $\mathbf{H}$ everywhere.

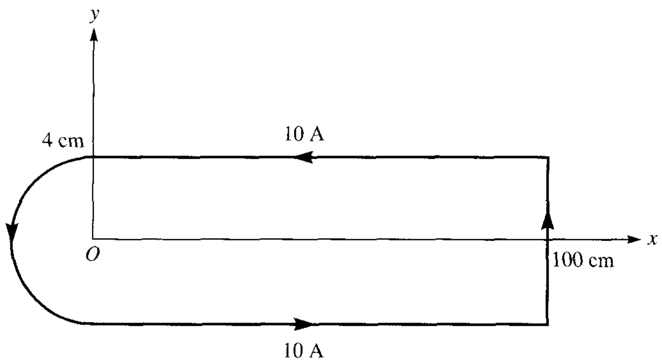


Figure 7.31 Filamentary loop of Problem 7.12; not drawn to scale.

7.18 (a) An infinitely long solid conductor of radius a is placed along the z-axis. If the conductor carries current I in the $+z$ direction, show that

$$\mathbf{H} = \frac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi$$

within the conductor. Find the corresponding current density.

(b) If $I = 3$ A and $a = 2$ cm in part (a), find $\mathbf{H}$ at (0, 1 cm, 0) and (0, 4 cm, 0).

7.19 If $\mathbf{H} = y\mathbf{a}_x - x\mathbf{a}_y$ A/m on plane $z = 0$, (a) determine the current density and (b) verify Ampere's law by taking the circulation of $\mathbf{H}$ around the edge of the rectangle $z = 0, 0 < x < 3, -1 < y < 4$.

7.20 In a certain conducting region,

$$\mathbf{H} = yz(x^2 + y^2)\mathbf{a}_x - y^2xz\mathbf{a}_y + 4x^2y^2\mathbf{a}_z \text{ A/m}$$

(a) Determine $\mathbf{J}$ at $(5, 2, -3)$

(b) Find the current passing through $x = -1, 0 < y, z < 2$

(c) Show that $\nabla \cdot \mathbf{B} = 0$

7.21 An infinitely long filamentary wire carries a current of 2 A in the $+z$-direction. Calculate

(a) $\mathbf{B}$ at $(-3, 4, 7)$

(b) The flux through the square loop described by $2 \le \rho \le 6, 0 \le z \le 4, \phi = 90°$

7.22 The electric motor shown in Figure 7.32 has field

$$\mathbf{H} = \frac{10^6}{\rho}\sin 2\phi\,\mathbf{a}_\rho \text{ A/m}$$

Calculate the flux per pole passing through the air gap if the axial length of the pole is 20 cm.

7.23 Consider the two-wire transmission line whose cross section is illustrated in Figure 7.33. Each wire is of radius 2 cm and the wires are separated 10 cm. The wire centered at (0, 0)

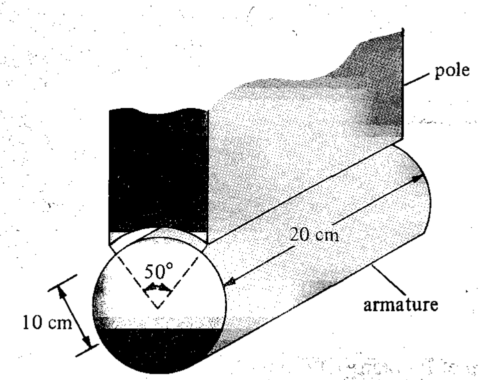


Figure 7.32 Electric motor pole of Problem 7.22.

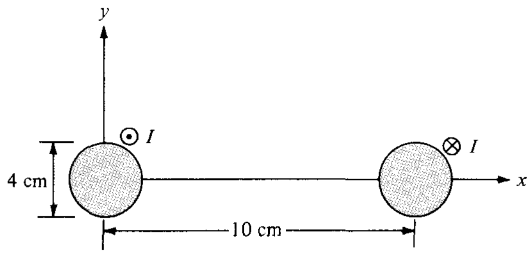


Figure 7.33 Two-wire line of Problem 7.23.

carries current 5 A while the other centered at (10 cm, 0) carries the return current. Find **H** at

(a) (5 cm, 0)

(b) (10 cm, 5 cm)

7.24 Determine the magnetic flux through a rectangular loop ($a \times b$) due to an infinitely long conductor carrying current I as shown in Figure 7.34. The loop and the straight conductors are separated by distance d.

***7.25** A brass ring with triangular cross section encircles a very long straight wire concentrically as in Figure 7.35. If the wire carries a current I, show that the total number of magnetic flux lines in the ring is

$$\Psi = \frac{\mu_o I h}{2\pi b}\left[b - a \ln \frac{a + b}{b}\right]$$

Calculate Ψ if a = 30 cm, b = 10 cm, h = 5 cm, and I = 10 A.

7.26 Consider the following arbitrary fields. Find out which of them can possibly represent electrostatic or magnetostatic field in free space.

(a) $\mathbf{A} = y \cos ax\mathbf{a}_x + (y + e^{-x})\mathbf{a}_z$

(b) $\mathbf{B} = \dfrac{20}{\rho}\mathbf{a}_\rho$

(c) $\mathbf{C} = r^2 \sin\theta\, \mathbf{a}_\phi$

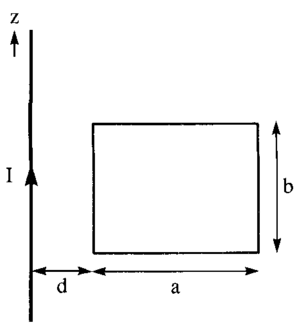


Figure 7.34 For Problem 7.24

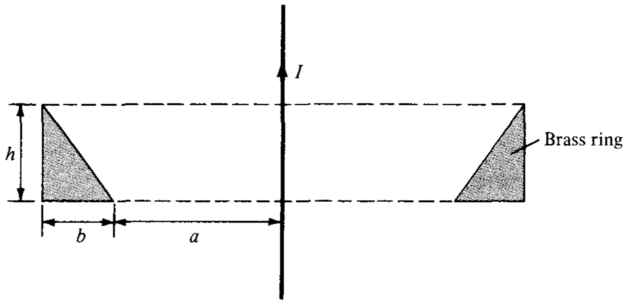


Figure 7.35 Cross section of a brass ring enclosing a long straight wire; for Problem 7.25.

7.27 Reconsider the previous problem for the following fields.

(a) $\mathbf{D} = y^2z\mathbf{a}_x + 2(x + 1)yz\mathbf{a}_y - (x + 1)z^2\mathbf{a}_z$

(b) $\mathbf{E} = \dfrac{(z + 1)}{\rho}\cos\phi\,\mathbf{a}_\rho + \dfrac{\sin\phi}{\rho}$

(c) $\mathbf{F} = \dfrac{1}{r^2}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$

7.28 For a current distribution in free space,

$$\mathbf{A} = (2x^2y + yz)\mathbf{a}_x + (xy^2 - xz^3)\mathbf{a}_y - (6xyz - 2x^2y^2)\mathbf{a}_z \text{ Wb/m}$$

(a) Calculate **B**.

(b) Find the magnetic flux through a loop described by $x = 1, 0 < y, z < 2$.

(c) Show that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

7.29 The magnetic vector potential of a current distribution in free space is given by

$$\mathbf{A} = 15e^{-\rho}\sin\phi\,\mathbf{a}_z \text{ Wb/m}$$

Find **H** at $(3, \pi/4, -10)$. Calculate the flux through $\rho = 5, 0 \leq \phi \leq \pi/2, 0 \leq z \leq 10$.

7.30 A conductor of radius a carries a uniform current with $\mathbf{J} = J_o\mathbf{a}_z$. Show that the magnetic vector potential for $\rho > a$ is

$$\mathbf{A} = -\frac{1}{4}\mu_o J_o \rho^2 \mathbf{a}_z$$

7.31 An infinitely long conductor of radius a is placed such that its axis is along the z-axis. The vector magnetic potential, due to a direct current I_o flowing along $\mathbf{a}_z$ in the conductor, is given by

$$\mathbf{A} = \frac{-I_o}{4\pi a^2}\mu_o(x^2 + y^2)\,\mathbf{a}_z \text{ Wb/m}$$

Find the corresponding **H**. Confirm your result using Ampere's law.

7.32 The magnetic vector potential of two parallel infinite straight current filaments in free space carrying equal current I in opposite direction is

$$\mathbf{A} = \frac{\mu I}{2\pi} \ln \frac{d - \rho}{\rho} \mathbf{a}_z$$

where d is the separation distance between the filaments (with one filament placed along the z-axis). Find the corresponding magnetic flux density **B**.

7.33 Find the current density **J** to

$$\mathbf{A} = \frac{10}{\rho^2} \mathbf{a}_z \text{ Wb/m}$$

in free space.

7.34 Prove that the magnetic scalar potential at $(0, 0, z)$ due to a circular loop of radius a shown in Figure 7.8(a) is

$$V_m = \frac{I}{2}\left[1 - \frac{z}{[z^2 + a^2]^{1/2}}\right]$$

***7.35** A coaxial transmission line is constructed such that the radius of the inner conductor is a and the outer conductor has radii $3a$ and $4a$. Find the vector magnetic potential within the outer conductor. Assume $A_z = 0$ for $\rho = 3a$.

7.36 The z-axis carries a filamentary current 12 A along $\mathbf{a}_z$. Calculate V_m at $(4, 30°, -2)$ if $V_m = 0$ at $(10, 60°, 7)$.

7.37 Plane $z = -2$ carries a current of $50\mathbf{a}_y$ A/m. If $V_m = 0$ at the origin, find V_m at

(a) $(-2, 0, 5)$

(b) $(10, 3, 1)$

7.38 Prove in cylindrical coordinates that

(a) $\nabla \times (\nabla V) = 0$

(b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

7.39 If $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$, show that

$$\nabla \frac{1}{\mathbf{R}} = -\nabla' \frac{1}{\mathbf{R}} = -\frac{\mathbf{R}}{R^3}$$

where ∇ and ∇' are del operators with respect to (x, y, z) and (x', y', z), respectively.

Chapter 8

MAGNETIC FORCES, MATERIALS, AND DEVICES

Do all the good you can,
By all the means you can,
In all the ways you can,
In all the places you can,
At all the times you can,
To all the people you can,
As long as ever you can.

—JOHN WESLEY

8.1 INTRODUCTION

Having considered the basic laws and techniques commonly used in calculating magnetic field **B** due to current-carrying elements, we are prepared to study the force a magnetic field exerts on charged particles, current elements, and loops. Such a study is important to problems on electrical devices such as ammeters, voltmeters, galvanometers, cyclotrons, plasmas, motors, and magnetohydrodynamic generators. The precise definition of the magnetic field, deliberately sidestepped in the previous chapter, will be given here. The concepts of magnetic moments and dipole will also be considered.

Furthermore, we will consider magnetic fields in material media, as opposed to the magnetic fields in vacuum or free space examined in the previous chapter. The results of the preceding chapter need only some modification to account for the presence of materials in a magnetic field. Further discussions will cover inductors, inductances, magnetic energy, and magnetic circuits.

8.2 FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be (a) due to a moving charged particle in a **B** field, (b) on a current element in an external **B** field, or (c) between two current elements.

A. Force on a Charged Particle

According to our discussion in Chapter 4, the electric force $\mathbf{F}_e$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity $\mathbf{E}$ as

$$\mathbf{F}_e = Q\mathbf{E} \tag{8.1}$$

This shows that if Q is positive, $\mathbf{F}_e$ and $\mathbf{E}$ have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

This clearly shows that $\mathbf{F}_m$ is perpendicular to both $\mathbf{u}$ and $\mathbf{B}$.

From eqs. (8.1) and (8.2), a comparison between the electric force $\mathbf{F}_e$ and the magnetic force $\mathbf{F}_m$ can be made. $\mathbf{F}_e$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy. Unlike $\mathbf{F}_e$, $\mathbf{F}_m$ depends on the charge velocity and is normal to it. $\mathbf{F}_m$ cannot perform work because it is at right angles to the direction of motion of the charge ($\mathbf{F}_m \cdot d\mathbf{l} = 0$); it does not cause an increase in kinetic energy of the charge. The magnitude of $\mathbf{F}_m$ is generally small compared to $\mathbf{F}_e$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

or

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})} \tag{8.3}$$

This is known as the *Lorentz force equation.*[1] It relates mechanical force to electrical force. If the mass of the charged particle moving in $\mathbf{E}$ and $\mathbf{B}$ fields is m, by Newton's second law of motion.

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{8.4}$$

The solution to this equation is important in determining the motion of charged particles in $\mathbf{E}$ and $\mathbf{B}$ fields. We should bear in mind that in such fields, energy transfer can be only by means of the electric field. A summary on the force exerted on a charged particle is given in Table 8.1.

Since eq. (8.2) is closely parallel to eq. (8.1), which defines the electric field, some authors and instructors prefer to begin their discussions on magnetostatics from eq. (8.2) just as discussions on electrostatics usually begin with Coulomb's force law.

[1]After Hendrik Lorentz (1853–1928), who first applied the equation to electric field motion.

TABLE 8.1 Force on a Charged Particle

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

B. Force on a Current Element

To determine the force on a current element $I\,d\mathbf{l}$ of a current-carrying conductor due to the magnetic field $\mathbf{B}$, we modify eq. (8.2) using the fact that for convection current [see eq. (5.7)]:

$$\mathbf{J} = \rho_v \mathbf{u} \tag{8.5}$$

From eq. (7.5), we recall the relationship between current elements:

$$I\,d\mathbf{l} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{8.6}$$

Combining eqs. (8.5) and (8.6) yields

$$I\,d\mathbf{l} = \rho_v \mathbf{u}\,dv = dQ\,\mathbf{u}$$

Alternatively, $I\,d\mathbf{l} = \dfrac{dQ}{dt}\,d\mathbf{l} = dQ\,\dfrac{d\mathbf{l}}{dt} = dQ\,\mathbf{u}$

Hence,

$$\boxed{I\,d\mathbf{l} = dQ\,\mathbf{u}} \tag{8.7}$$

This shows that an elemental charge dQ moving with velocity $\mathbf{u}$ (thereby producing convection current element $dQ\,\mathbf{u}$) is equivalent to a conduction current element $I\,d\mathbf{l}$. Thus the force on a current element $I\,d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is found from eq. (8.2) by merely replacing $Q\mathbf{u}$ by $I\,d\mathbf{l}$; that is,

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B} \tag{8.8}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\boxed{\mathbf{F} = \oint_L I\,d\mathbf{l} \times \mathbf{B}} \tag{8.9}$$

In using eq. (8.8) or (8.9), we should keep in mind that the magnetic field produced by the current element $I\,d\mathbf{l}$ does not exert force on the element itself just as a point charge does not exert force on itself. The $\mathbf{B}$ field that exerts force on $I\,d\mathbf{l}$ must be due to another element. In other words, the $\mathbf{B}$ field in eq. (8.8) or (8.9) is external to the current element $I\,d\mathbf{l}$. If instead of the line current element $I\,d\mathbf{l}$, we have surface current elements $\mathbf{K}\,dS$

or a volume current element $\mathbf{J}\,dv$, we simply make use of eq. (8.6) so that eq. (8.8) becomes

$$d\mathbf{F} = \mathbf{K}\,dS \times \mathbf{B} \quad \text{or} \quad d\mathbf{F} = \mathbf{J}\,dv \times \mathbf{B} \tag{8.8a}$$

while eq. (8.9) becomes

$$\mathbf{F} = \int_S \mathbf{K}\,dS \times \mathbf{B} \quad \text{or} \quad \mathbf{F} = \int_v \mathbf{J}\,dv \times \mathbf{B} \tag{8.9a}$$

From eq. (8.8)

The **magnetic field B** is defined as the force per unit current element.

Alternatively, **B** may be defined from eq. (8.2) as the vector which satisfies $\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$ just as we defined electric field **E** as the force per unit charge, $\mathbf{F}_e/q$. Both of these definitions of **B** show that **B** describes the force properties of a magnetic field.

C. Force between Two Current Elements

Let us now consider the force between two elements $I_1\,d\mathbf{l}_1$ and $I_2\,d\mathbf{l}_2$. According to Biot–Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\mathbf{F}_1)$ on element $I_1\,d\mathbf{l}_1$ due to the field $d\mathbf{B}_2$ produced by element $I_2\,d\mathbf{l}_2$ as shown in Figure 8.1. From eq. (8.8),

$$d(d\mathbf{F}_1) = I_1\,d\mathbf{l}_1 \times d\mathbf{B}_2 \tag{8.10}$$

But from Biot–Savart's law,

$$d\mathbf{B}_2 = \frac{\mu_o I_2\,d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2} \tag{8.11}$$

Hence,

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\,d\mathbf{l}_1 \times (I_2\,d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2} \tag{8.12}$$

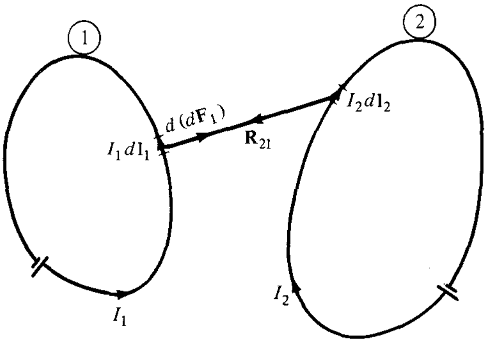


Figure 8.1 Force between two current loops.

This equation is essentially the law of force between two current elements and is analogous to Coulomb's law, which expresses the force between two stationary charges. From eq. (8.12), we obtain the total force $\mathbf{F}_1$ on current loop 1 due to current loop 2 shown in Figure 8.1 as

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \tag{8.13}$$

Although this equation appears complicated, we should remember that it is based on eq. (8.10). It is eq. (8.9) or (8.10) that is of fundamental importance.

The force $\mathbf{F}_2$ on loop 2 due to the magnetic field $\mathbf{B}_1$ from loop 1 is obtained from eq. (8.13) by interchanging subscripts 1 and 2. It can be shown that $\mathbf{F}_2 = -\mathbf{F}_1$; thus $\mathbf{F}_1$ and $\mathbf{F}_2$ obey Newton's third law that action and reaction are equal and opposite. It is worthwhile to mention that eq. (8.13) was experimentally established by Oersted and Ampere; Biot and Savart (Ampere's colleagues) actually based their law on it.

EXAMPLE 8.1

A charged particle of mass 2 kg and charge 3 C starts at point $(1, -2, 0)$ with velocity $4\mathbf{a}_x + 3\mathbf{a}_z$ m/s in an electric field $12\mathbf{a}_x + 10\mathbf{a}_y$ V/m. At time $t = 1$ s, determine

(a) The acceleration of the particle

(b) Its velocity

(c) Its kinetic energy

(d) Its position

Solution:

(a) This is an initial-value problem because initial values are given. According to Newton's second law of motion,

$$\mathbf{F} = m\mathbf{a} = Q\mathbf{E}$$

where $\mathbf{a}$ is the acceleration of the particle. Hence,

$$\mathbf{a} = \frac{Q\mathbf{E}}{m} = \frac{3}{2}(12\mathbf{a}_x + 10\mathbf{a}_y) = 18\mathbf{a}_x + 15\mathbf{a}_y \text{ m/s}^2$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(u_x, u_y, u_z) = 18\mathbf{a}_x + 15\mathbf{a}_y$$

(b) Equating components gives

$$\frac{du_x}{dt} = 18 \rightarrow u_x = 18t + A \tag{8.1.1}$$

$$\frac{du_y}{dt} = 15 \rightarrow u_y = 15t + B \tag{8.1.2}$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C \tag{8.1.3}$$

where A, B, and C are integration constants. But at $t = 0$, $\mathbf{u} = 4\mathbf{a}_x + 3\mathbf{a}_z$. Hence,

$$u_x(t = 0) = 4 \rightarrow 4 = 0 + A \quad \text{or} \quad A = 4$$

$$u_y(t = 0) = 0 \rightarrow 0 = 0 + B \quad \text{or} \quad B = 0$$

$$u_z(t = 0) = 3 \rightarrow 3 = C$$

Substituting the values of A, B, and C into eqs. (8.1.1) to (8.1.3) gives

$$\mathbf{u}(t) = (u_x, u_y, u_z) = (18t + 4, 15t, 3)$$

Hence

$$\mathbf{u}(t = 1 \text{ s}) = 22\mathbf{a}_x + 15\mathbf{a}_y + 3\mathbf{a}_z \text{ m/s}$$

(c) Kinetic energy (K.E.) $= \frac{1}{2}m\,|\mathbf{u}|^2 = \frac{1}{2}(2)(22^2 + 15^2 + 3^2)$
$= 718$ J

(d) $\mathbf{u} = \frac{d\mathbf{l}}{dt} = \frac{d}{dt}(x, y, z) = (18t + 4, 15t, 3)$

Equating components yields

$$\frac{dx}{dt} = u_x = 18t + 4 \rightarrow x = 9t^2 + 4t + A_1 \tag{8.1.4}$$

$$\frac{dy}{dt} = u_y = 15t \qquad \rightarrow y = 7.5t^2 + B_1 \tag{8.1.5}$$

$$\frac{dz}{dt} = u_z = 3 \qquad \rightarrow z = 3t + C_1 \tag{8.1.6}$$

At $t = 0$, $(x, y, z) = (1, -2, 0)$; hence,

$$x(t = 0) = 1 \quad \rightarrow 1 \quad = 0 + A_1 \quad \text{or} \quad A_1 = 1$$

$$y(t = 0) = -2 \rightarrow -2 = 0 + B_1 \quad \text{or} \quad B_1 = -2$$

$$z(t = 0) = 0 \quad \rightarrow 0 \quad = 0 + C_1 \quad \text{or} \quad C_1 = 0$$

Substituting the values of A_1, B_1, and C_1 into eqs. (8.1.4) to (8.1.6), we obtain

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t) \tag{8.1.7}$$

Hence, at $t = 1$, $(x, y, z) = (14, 5.5, 3)$.
By eliminating t in eq. (8.1.7), the motion of the particle may be described in terms of x, y, and z.

PRACTICE EXERCISE 8.1

A charged particle of mass 1 kg and charge 2 C starts at the origin with zero initial velocity in a region where $\mathbf{E} = 3\mathbf{a}_z$ V/m. Find

(a) The force on the particle

(b) The time it takes to reach point $P(0, 0, 12 \text{ m})$

(c) Its velocity and acceleration at P

(d) Its K.E. at P.

Answer: (a) $6\mathbf{a}_z$ N, (b) 2 s, (c) $12\mathbf{a}_z$ m/s, $6\mathbf{a}_z$ m/s^2, (d) 72 J.

EXAMPLE 8.2

A charged particle of mass 2 kg and 1 C starts at the origin with velocity $3\mathbf{a}_y$ m/s and travels in a region of uniform magnetic field $\mathbf{B} = 10\mathbf{a}_z$ Wb/m^2. At $t = 4$ s, calculate

(a) The velocity and acceleration of the particle

(b) The magnetic force on it

(c) Its K.E. and location

(d) Find the particle's trajectory by eliminating t.

(e) Show that its K.E. remains constant.

Solution:

(a) $\mathbf{F} = m\dfrac{d\mathbf{u}}{dt} = Q\mathbf{u} \times \mathbf{B}$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{Q}{m}\mathbf{u} \times \mathbf{B}$$

Hence

$$\frac{d}{dt}(u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z) = \frac{1}{2}\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ 0 & 0 & 10 \end{vmatrix} = 5(u_y\mathbf{a}_x - u_x\mathbf{a}_y)$$

By equating components, we get

$$\frac{du_x}{dt} = 5u_y \tag{8.2.1}$$

$$\frac{du_y}{dt} = -5u_x \tag{8.2.2}$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C_o \tag{8.2.3}$$

We can eliminate u_x or u_y in eqs. (8.2.1) and (8.2.2) by taking second derivatives of one equation and making use of the other. Thus

$$\frac{d^2u_x}{dt^2} = 5\frac{du_y}{dt} = -25u_x$$

or

$$\frac{d^2u_x}{dt^2} + 25u_x = 0$$

which is a linear differential equation with solution (see Case 3 of Example 6.5)

$$u_x = C_1 \cos 5t + C_2 \sin 5t \tag{8.2.4}$$

From eqs. (8.2.1) and (8.2.4),

$$5u_y = \frac{du_x}{dt} = -5C_1 \sin 5t + 5C_2 \cos 5t \tag{8.2.5}$$

or

$$u_y = -C_1 \sin 5t + C_2 \cos 5t$$

We now determine constants C_o, C_1, and C_2 using the initial conditions. At $t = 0$, $\mathbf{u} = 3\mathbf{a}_y$. Hence,

$$u_x = 0 \rightarrow 0 = C_1 \cdot 1 + C_2 \cdot 0 \rightarrow C_1 = 0$$
$$u_y = 3 \rightarrow 3 = -C_1 \cdot 0 + C_2 \cdot 1 \rightarrow C_2 = 3$$
$$u_z = 0 \rightarrow 0 = C_o$$

Substituting the values of C_o, C_1, and C_2 into eqs. (8.2.3) to (8.2.5) gives

$$\mathbf{u} = (u_x, u_y, u_z) = (3 \sin 5t, 3 \cos 5t, 0) \tag{8.2.6}$$

Hence,

$$\begin{aligned}\mathbf{u}(t = 4) &= (3 \sin 20, 3 \cos 20, 0)\\ &= 2.739\mathbf{a}_x + 1.224\mathbf{a}_y \text{ m/s}\end{aligned}$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = (15 \cos 5t, -15 \sin 5t, 0)$$

and

$$\mathbf{a}(t = 4) = 6.101\mathbf{a}_x - 13.703\mathbf{a}_y \text{ m/s}^2$$

(b) $$\mathbf{F} = m\mathbf{a} = 12.2\mathbf{a}_x - 27.4\mathbf{a}_y \text{ N}$$

or

$$\begin{aligned}\mathbf{F} &= Q\mathbf{u} \times \mathbf{B} = (1)(2.739\mathbf{a}_x + 1.224\mathbf{a}_y) \times 10\mathbf{a}_z\\ &= 12.2\mathbf{a}_x - 27.4\mathbf{a}_y \text{ N}\end{aligned}$$

(c) K.E. $= 1/2m\,|\mathbf{u}|^2 = 1/2(2)\,(2.739^2 + 1.224^2) = 9$ J

$$u_x = \frac{dx}{dt} = 3 \sin 5t \rightarrow x = -\frac{3}{5}\cos 5t + b_1 \tag{8.2.7}$$

$$u_y = \frac{dy}{dt} = 3 \cos 5t \rightarrow y = \frac{3}{5}\sin 5t + b_2 \tag{8.2.8}$$

$$u_z = \frac{dz}{dt} = 0 \rightarrow z = b_3 \tag{8.2.9}$$

where b_1, b_2, and b_3 are integration constants. At $t = 0$, $(x, y, z) = (0, 0, 0)$ and hence,

$$x(t = 0) = 0 \rightarrow 0 = -\frac{3}{5}\cdot 1 + b_1 \rightarrow b_1 = 0.6$$

$$y(t = 0) = 0 \rightarrow 0 = \frac{3}{5}\cdot 0 + b_2 \rightarrow b_2 = 0$$

$$z(t = 0) = 0 \rightarrow 0 = b_3$$

Substituting the values of b_1, b_2, and b_3 into eqs. (8.2.7) to (8.2.9), we obtain

$$(x, y, z) = (0.6 - 0.6 \cos 5t, 0.6 \sin 5t, 0) \tag{8.2.10}$$

At $t = 4$ s,

$$(x, y, z) = (0.3552, 0.5478, 0)$$

(d) From eq. (8.2.10), we eliminate t by noting that

$$(x - 0.6)^2 + y^2 = (0.6)^2\,(\cos^2 5t + \sin^2 5t), \qquad z = 0$$

or

$$(x - 0.6)^2 + y^2 = (0.6)^2, \qquad z = 0$$

which is a circle on plane $z = 0$, centered at (0.6, 0, 0) and of radius 0.6 m. Thus the particle gyrates in an orbit about a magnetic field line.

(e)
$$\text{K.E.} = \frac{1}{2} m\,|\mathbf{u}|^2 = \frac{1}{2}(2)\,(9 \cos^2 5t + 9 \sin^2 5t) = 9 \text{ J}$$

which is the same as the K.E. at $t = 0$ and $t = 4$ s. Thus the uniform magnetic field has no effect on the K.E. of the particle.

Note that the angular velocity $\omega = QB/m$ and the radius of the orbit $r = u_o/\omega$, where u_o is the initial speed. An interesting application of the idea in this example is found in a common method of focusing a beam of electrons. The method employs a uniform magnetic field directed parallel to the desired beam as shown in Figure 8.2. Each electron emerging from the electron gun follows a helical path and is back on the axis at the same focal point with other electrons. If the screen of a cathode ray tube were at this point, a single spot would appear on the screen.

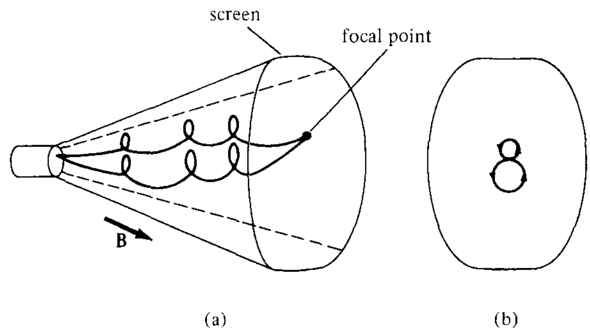


Figure 8.2 Magnetic focusing of a beam of electrons: **(a)** helical paths of electrons, **(b)** end view of paths.

PRACTICE EXERCISE 8.2

A proton of mass m is projected into a uniform field $\mathbf{B} = B_o\mathbf{a}_z$ with an initial velocity $\alpha\mathbf{a}_x + \beta\mathbf{a}_z$. (a) Find the differential equations that the position vector $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ must satisfy. (b) Show that a solution to these equations is

$$x = \frac{\alpha}{\omega}\sin\omega t, \qquad y = \frac{\alpha}{\omega}\cos\omega t, \qquad z = \beta t$$

where $\omega = eB_o/m$ and e is the charge on the proton. (c) Show that this solution describes a circular helix in space.

Answer: (a) $\dfrac{dx}{dt} = \alpha\cos\omega t$, $\dfrac{dy}{dt} = -\alpha\sin\omega t$, $\dfrac{dz}{dt} = \beta$, (b) and (c) Proof.

EXAMPLE 8.3

A charged particle moves with a uniform velocity $4\mathbf{a}_x$ m/s in a region where $\mathbf{E} = 20\,\mathbf{a}_y$ V/m and $\mathbf{B} = B_o\mathbf{a}_z$ Wb/m^2. Determine B_o such that the velocity of the particle remains constant.

Solution:

If the particle moves with a constant velocity, it implies that its acceleration is zero. In other words, the particle experiences no net force. Hence,

$$0 = \mathbf{F} = m\mathbf{a} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$0 = Q(20\mathbf{a}_y + 4\mathbf{a}_x \times B_o\mathbf{a}_z)$$

or

$$-20\mathbf{a}_y = -4B_o\mathbf{a}_y$$

Thus $B_o = 5$.

This example illustrates an important principle employed in a velocity filter shown in Figure 8.3. In this application, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{u}$ are mutually perpendicular so that $Q\mathbf{u} \times \mathbf{B}$ is

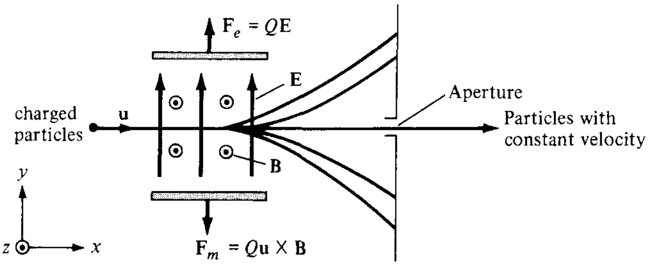


Figure 8.3 A velocity filter for charged particles.

directed opposite to $Q\mathbf{E}$, regardless of the sign of the charge. When the magnitudes of the two vectors are equal,

$$QuB = QE$$

or

$$u = \frac{E}{B}$$

This is the required (critical) speed to balance out the two parts of the Lorentz force. Particles with this speed are undeflected by the fields; they are "filtered" through the aperture. Particles with other speeds are deflected down or up, depending on whether their speeds are greater or less than this critical speed.

PRACTICE EXERCISE 8.3

Uniform **E** and **B** fields are oriented at right angles to each other. An electron moves with a speed of 8×10^6 m/s at right angles to both fields and passes undeflected through the field.

(a) If the magnitude of **B** is 0.5 mWb/m², find the value of **E**.

(b) Will this filter work for positive and negative charges and any value of mass?

Answer: (a) 4 kV/m, (b) Yes.

EXAMPLE 8.4

A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure 8.4(a). Show that the force experienced by the loop is given by

$$\mathbf{F} = -\frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right]\mathbf{a}_\rho \text{ N}$$

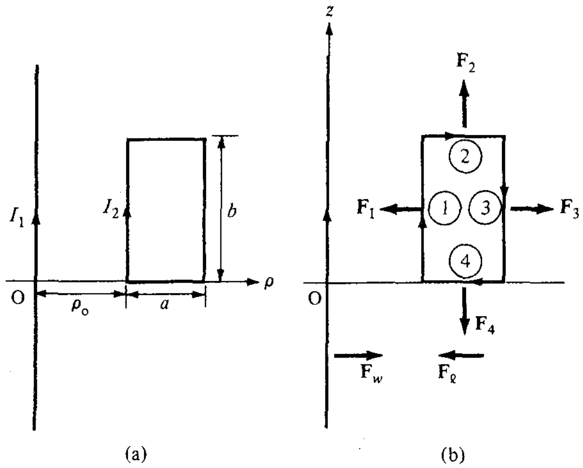


Figure 8.4 For Example 8.4: **(a)** rectangular loop inside the field produced by an infinitely long wire, **(b)** forces acting on the loop and wire.

Solution:

Let the force on the loop be

$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ are, respectively, the forces exerted on sides of the loop labeled 1, 2, 3, and 4 in Figure 8.4(b). Due to the infinitely long wire

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi$$

Hence,

$$\begin{aligned}\mathbf{F}_1 &= I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=0}^{b} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi \\ &= -\frac{\mu_o I_1 I_2 b}{2\pi\rho_o}\,\mathbf{a}_\rho \qquad \text{(attractive)}\end{aligned}$$

$\mathbf{F}_1$ is attractive because it is directed toward the long wire; that is, $\mathbf{F}_1$ is along $-\mathbf{a}_\rho$ due to the fact that loop side 1 and the long wire carry currents along the same direction. Similarly,

$$\begin{aligned}\mathbf{F}_3 &= I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=b}^{0} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi(\rho_o + a)}\,\mathbf{a}_\phi \\ &= \frac{\mu_o I_1 I_2 b}{2\pi(\rho_o + a)}\,\mathbf{a}_\rho \qquad \text{(repulsive)}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_2 &= I_2 \int_{\rho=\rho_o}^{\rho_o+a} d\rho\,\mathbf{a}_\rho \times \frac{\mu_o I_1\,\mathbf{a}_\phi}{2\pi\rho} \\ &= \frac{\mu_o I_1 I_2}{2\pi} \ln\frac{\rho_o + a}{\rho_o}\,\mathbf{a}_z \qquad \text{(parallel)}\end{aligned}$$

$$\mathbf{F}_4 = I_2 \int_{\rho=\rho_o+a}^{\rho_o} d\rho\, \mathbf{a}_\rho \times \frac{\mu_o I_1\, \mathbf{a}_\phi}{2\pi\rho}$$

$$= -\frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\, \mathbf{a}_z \qquad \text{(parallel)}$$

The total force $\mathbf{F}_\ell$ on the loop is the sum of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$; that is,

$$\mathbf{F}_\ell = \frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right](-\mathbf{a}_\rho)$$

which is an attractive force trying to draw the loop toward the wire. The force $\mathbf{F}_w$ on the wire, by Newton's third law, is $-\mathbf{F}_\ell$; see Figure 8.4(b).

PRACTICE EXERCISE 8.4

In Example 8.4, find the force experienced by the infinitely long wire if $I_1 = 10$ A, $I_2 = 5$ A, $\rho_o = 20$ cm, $a = 10$ cm, $b = 30$ cm.

Answer: $5\mathbf{a}_\rho$ μN.

8.3 MAGNETIC TORQUE AND MOMENT

Now that we have considered the force on a current loop in a magnetic field, we can determine the torque on it. The concept of a current loop experiencing a torque in a magnetic field is of paramount importance in understanding the behavior of orbiting charged particles, d.c. motors, and generators. If the loop is placed parallel to a magnetic field, it experiences a force that tends to rotate it.

The **torque T** (or mechanical moment of force) on the loop is the vector product of the force **F** and the moment arm **r**.

That is,

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{8.14}$$

and its units are Newton-meters (N · m).

Let us apply this to a rectangular loop of length ℓ and width w placed in a uniform magnetic field **B** as shown in Figure 8.5(a). From this figure, we notice that $d\mathbf{l}$ is parallel to **B** along sides 12 and 34 of the loop and no force is exerted on those sides. Thus

$$\mathbf{F} = I \int_2^3 d\mathbf{l} \times \mathbf{B} + I \int_4^1 d\mathbf{l} \times \mathbf{B}$$

$$= I \int_0^\ell dz\, \mathbf{a}_z \times \mathbf{B} + I \int_\ell^0 dz\, \mathbf{a}_z \times \mathbf{B}$$

XX1

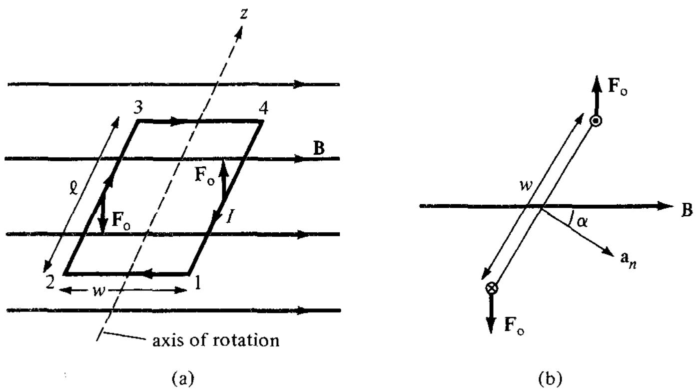


Figure 8.5 Rectangular planar loop in a uniform magnetic field.

or

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0 \tag{8.15}$$

where $|\mathbf{F}_o| = IB\ell$ because **B** is uniform. Thus, no force is exerted on the loop as a whole. However, $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points on the loop, thereby creating a couple. If the normal to the plane of the loop makes an angle α with **B**, as shown in the cross-sectional view of Figure 8.5(b), the torque on the loop is

$$|\mathbf{T}| = |\mathbf{F}_o|\, w \sin \alpha$$

or

$$T = BI\ell w \sin \alpha \tag{8.16}$$

But $\ell w = S$, the area of the loop. Hence,

$$T = BIS \sin \alpha \tag{8.17}$$

We define the quantity

$$\boxed{\mathbf{m} = IS\mathbf{a}_n} \tag{8.18}$$

as the *magnetic dipole moment* (in A/m^2) of the loop. In eq. (8.18), $\mathbf{a}_n$ is a unit normal vector to the plane of the loop and its direction is determined by the right-hand rule: fingers in the direction of current and thumb along $\mathbf{a}_n$.

The **magnetic dipole moment** is the product of current and area of the loop; its direction is normal to the loop.

Introducing eq. (8.18) in eq. (8.17), we obtain

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B}} \tag{8.19}$$

This expression is generally applicable in determining the torque on a planar loop of any arbitrary shape although it was obtained using a rectangular loop. The only limitation is that the magnetic field must be uniform. It should be noted that the torque is in the direction of the axis of rotation (the z-axis in the case of Figure 8.5a). It is directed such as to reduce α so that **m** and **B** are in the same direction. In an equilibrium position (when **m** and **B** are in the same direction), the loop is perpendicular to the magnetic field and the torque will be zero as well as the sum of the forces on the loop.

8.4 A MAGNETIC DIPOLE

A bar magnet or a small filamentary current loop is usually referred to as a *magnetic dipole.* The reason for this and what we mean by "small" will soon be evident. Let us determine the magnetic field **B** at an observation point $P(r, \theta, \phi)$ due to a circular loop carrying current I as in Figure 8.6. The magnetic vector potential at P is

$$\mathbf{A} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{l}}{r} \tag{8.20}$$

It can be shown that at far field ($r \gg a$, so that the loop appears small at the observation point), **A** has only ϕ-component and it is given by

$$\mathbf{A} = \frac{\mu_o I \pi a^2 \sin\theta \, \mathbf{a}_\phi}{4\pi r^2} \tag{8.21a}$$

or

$$\boxed{\mathbf{A} = \frac{\mu_o \, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2}} \tag{8.21b}$$

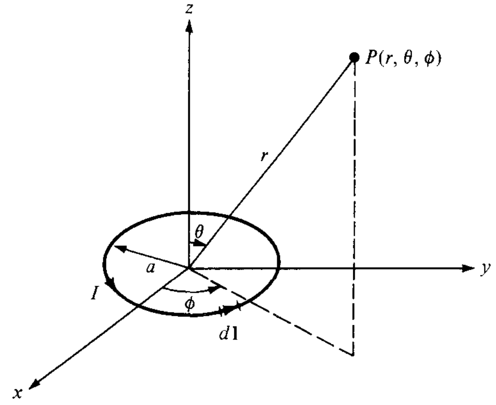


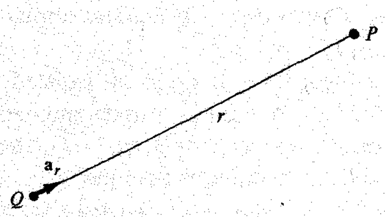
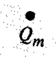
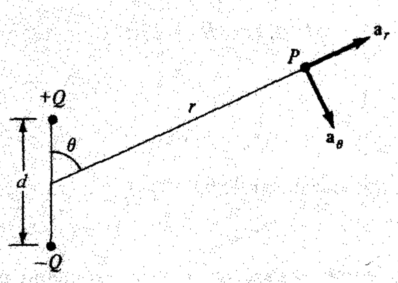
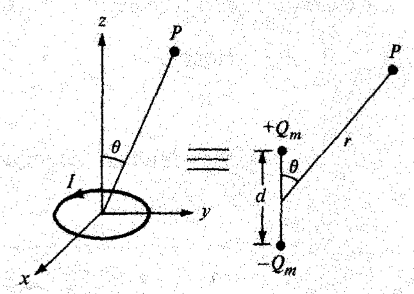
Figure 8.6 Magnetic field at P due to a current loop.

where $\mathbf{m} = I\pi a^2 \mathbf{a}_z$, the magnetic moment of the loop, and $\mathbf{a}_z \times \mathbf{a}_r = \sin\theta\, \mathbf{a}_\phi$. We determine the magnetic flux density $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$ as

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta) \tag{8.22}$$

It is interesting to compare eqs. (8.21) and (8.22) with similar expressions in eqs. (4.80) and (4.82) for electrical potential V and electric field intensity $\mathbf{E}$ due to an electric dipole. This comparison is done in Table 8.2, in which we notice the striking similari-

TABLE 8.2 Comparison between Electric and Magnetic Monopoles and Dipoles

Electric	Magnetic
$V = \dfrac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \dfrac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$  Monopole (point charge)	Does not exist  Monopole (point charge)
$V = \dfrac{Q\cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \dfrac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$  Dipole (two point charge)	$\mathbf{A} = \dfrac{\mu_o m \sin\theta\, \mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \dfrac{\mu_o m}{4\pi r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta)$  Dipole (small current loop or bar magnet)

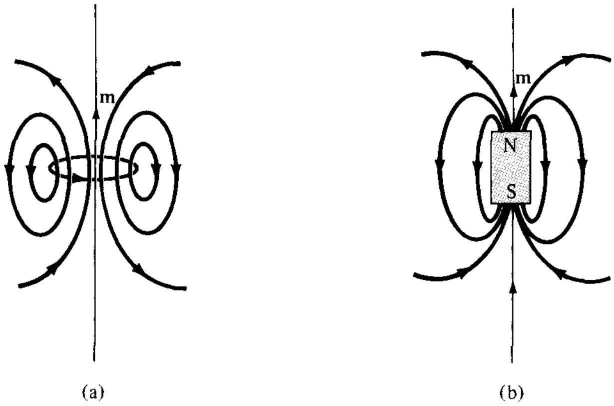


Figure 8.7 The **B** lines due to magnetic dipoles: **(a)** a small current loop with $\mathbf{m} = I\mathbf{S}$, **(b)** a bar magnet with $\mathbf{m} = Q_m\boldsymbol{\ell}$.

ties between **B** as far field due to a small current loop and **E** at far field due to an electric dipole. It is therefore reasonable to regard a small current loop as a magnetic dipole. The **B** lines due to a magnetic dipole are similar to the **E** lines due to an electric dipole. Figure 8.7(a) illustrates the **B** lines around the magnetic dipole $\mathbf{m} = I\mathbf{S}$.

A short permanent magnetic bar, shown in Figure 8.7(b), may also be regarded as a magnetic dipole. Observe that the **B** lines due to the bar are similar to those due to a small current loop in Figure 8.7(a).

Consider the bar magnet of Figure 8.8. If Q_m is an isolated magnetic charge (pole strength) and ℓ is the length of the bar, the bar has a dipole moment $Q_m\boldsymbol{\ell}$. (Notice that Q_m does exist; however, it does not exist without an associated $-Q_m$. See Table 8.2.) When the bar is in a uniform magnetic field **B**, it experiences a torque

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m\boldsymbol{\ell} \times \mathbf{B} \tag{8.23}$$

where $\boldsymbol{\ell}$ points in the direction south-to-north. The torque tends to align the bar with the external magnetic field. The force acting on the magnetic charge is given by

$$\mathbf{F} = Q_m\mathbf{B} \tag{8.24}$$

Since both a small current loop and a bar magnet produce magnetic dipoles, they are equivalent if they produce the same torque in a given **B** field; that is, when

$$T = Q_m\ell B = ISB \tag{8.25}$$

Hence,

$$Q_m\ell = IS \tag{8.26}$$

showing that they must have the same dipole moment.

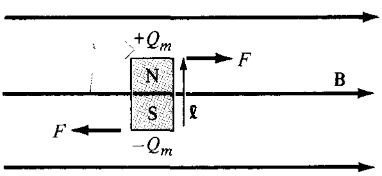


Figure 8.8 A bar magnet in an external magnetic field.

EXAMPLE 8.5

Determine the magnetic moment of an electric circuit formed by the triangular loop of Figure 8.9.

Solution:

From Problem 1.18(c), the equation of a plane is given by $Ax + By + Cz + D = 0$ where $D = -(A^2 + B^2 + C^2)$. Since points (2, 0, 0), (0, 2, 0), and (0, 0, 2) lie on the plane, these points must satisfy the equation of the plane, and the constants A, B, C, and D can be determined. Doing this gives $x + y + z = 2$ as the plane on which the loop lies. Thus we can use

$$\mathbf{m} = IS\mathbf{a}_n$$

where

$$S = \text{loop area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2\sqrt{2})(2\sqrt{2})\sin 60°$$
$$= 4 \sin 60°$$

If we define the plane surface by a function

$$f(x,y,z) = x + y + z - 2 = 0,$$

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$

We choose the plus sign in view of the direction of the current in the loop (using the right-hand rule, $\mathbf{m}$ is directed as in Figure 8.9). Hence

$$\mathbf{m} = 5\,(4 \sin 60°)\,\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$
$$= 10(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\ \text{A} \cdot \text{m}^2$$

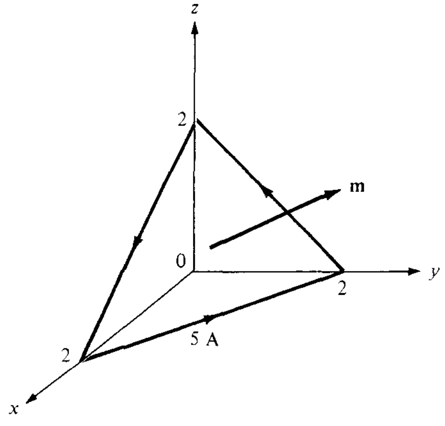


Figure 8.9 Triangular loop of Example 8.5.

PRACTICE EXERCISE 8.5

A rectangular coil of area 10 cm^2 carrying current of 50 A lies on plane $2x + 6y - 3z = 7$ such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

Answer: $(1.429\mathbf{a}_x + 4.286\mathbf{a}_y - 2.143\mathbf{a}_z) \times 10^{-2}\ \text{A} \cdot \text{m}^2$

EXAMPLE 8.6

A small current loop L_1 with magnetic moment $5\mathbf{a}_z$ A/m^2 is located at the origin while another small loop current L_2 with magnetic moment $3\mathbf{a}_y$ A · m^2 is located at $(4, -3, 10)$. Determine the torque on L_2.

Solution:

The torque $\mathbf{T}_2$ on the loop L_2 is due to the field $\mathbf{B}_1$ produced by loop L_1. Hence,

$$\mathbf{T}_2 = \mathbf{m}_2 \times \mathbf{B}_1$$

Since $\mathbf{m}_1$ for loop L_1 is along $\mathbf{a}_z$, we find $\mathbf{B}_1$ using eq. (8.22):

$$\mathbf{B}_1 = \frac{\mu_o m_1}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Using eq. (2.23), we transform $\mathbf{m}_2$ from Cartesian to spherical coordinates:

$$\mathbf{m}_2 = 3\mathbf{a}_y = 3\,(\sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi)$$

At $(4, -3, 10)$,

$$r = \sqrt{4^2 + (-3)^2 + 10^2} = 5\sqrt{5}$$

$$\tan\theta = \frac{\rho}{z} = \frac{5}{10} = \frac{1}{2} \rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \qquad \cos\theta = \frac{2}{\sqrt{5}}$$

$$\tan\phi = \frac{y}{x} = \frac{-3}{4} \rightarrow \sin\phi = \frac{-3}{5}, \qquad \cos\phi = \frac{4}{5}$$

Hence,

$$\mathbf{B}_1 = \frac{4\pi \times 10^{-7} \times 5}{4\pi\, 625\sqrt{5}}\left(\frac{4}{\sqrt{5}}\mathbf{a}_r + \frac{1}{\sqrt{5}}\mathbf{a}_\theta\right)$$
$$= \frac{10^{-7}}{625}(4\mathbf{a}_r + \mathbf{a}_\theta)$$

$$\mathbf{m}_2 = 3\left[-\frac{3\mathbf{a}_r}{5\sqrt{5}} - \frac{6\mathbf{a}_\theta}{5\sqrt{5}} + \frac{4\mathbf{a}_\phi}{5}\right]$$

and

$$\mathbf{T} = \frac{10^{-7}(3)}{625\,(5\sqrt{5})}(-3\mathbf{a}_r - 6\mathbf{a}_\theta + 4\sqrt{5}\mathbf{a}_\phi) \times (4\mathbf{a}_r + \mathbf{a}_\phi)$$
$$= 4.293 \times 10^{-11}(-6\mathbf{a}_r + 38.78\mathbf{a}_\theta + 24\mathbf{a}_\phi)$$
$$= -0.258\mathbf{a}_r + 1.665\mathbf{a}_\theta + 1.03\mathbf{a}_\phi \text{ nN} \cdot \text{m}$$

PRACTICE EXERCISE 8.6

If the coil of Practice Exercise 8.5 is surrounded by a uniform field $0.6\mathbf{a}_x + 0.4\mathbf{a}_y + 0.5\mathbf{a}_z$ Wb/m^2,

(a) Find the torque on the coil.

(b) Show that the torque on the coil is maximum if placed on plane $2x - 8y + 4z = \sqrt{84}$. Calculate the value of the maximum torque.

Answer: (a) $0.03\mathbf{a}_x - 0.02\mathbf{a}_y - 0.02\mathbf{a}_z$ N · m, (b) 0.04387 N · m.

8.5 MAGNETIZATION IN MATERIALS

Our discussion here will parallel that on polarization of materials in an electric field. We shall assume that our atomic model is that of an electron orbiting about a positive nucleus.

We know that a given material is composed of atoms. Each atom may be regarded as consisting of electrons orbiting about a central positive nucleus; the electrons also rotate (or spin) about their own axes. Thus an internal magnetic field is produced by electrons orbiting around the nucleus as in Figure 8.10(a) or electrons spinning as in Figure 8.10(b). Both of these electronic motions produce internal magnetic fields $\mathbf{B}_i$ that are similar to the magnetic field produced by a current loop of Figure 8.11. The equivalent current loop has a magnetic moment of $\mathbf{m} = I_b S \mathbf{a}_n$, where S is the area of the loop and I_b is the bound current (bound to the atom).

Without an external **B** field applied to the material, the sum of **m**'s is zero due to random orientation as in Figure 8.12(a). When an external **B** field is applied, the magnetic

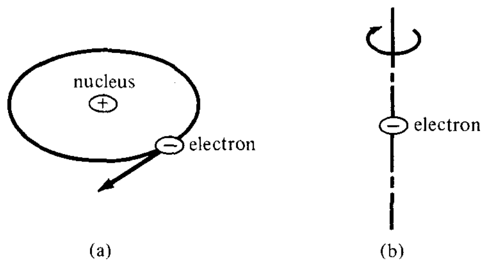


Figure 8.10 **(a)** Electron orbiting around the nucleus; **(b)** electron spin.

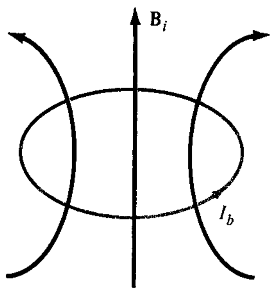


Figure 8.11 Circular current loop equivalent to electronic motion of Figure 8.10.

moments of the electrons more or less align themselves with **B** so that the net magnetic moment is not zero, as illustrated in Figure 8.12(b).

The **magnetization M** (in amperes/meter) is the magnetic dipole moment per unit volume.

If there are N atoms in a given volume Δv and the kth atom has a magnetic moment $\mathbf{m}_k$,

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v} \tag{8.27}$$

A medium for which **M** is not zero everywhere is said to be magnetized. For a differential volume dv', the magnetic moment is $d\mathbf{m} = \mathbf{M}\, dv'$. From eq. (8.21b), the vector magnetic potential due to $d\mathbf{m}$ is

$$d\mathbf{A} = \frac{\mu_o \mathbf{M} \times \mathbf{a}_R}{4\pi R^2}\, dv' = \frac{\mu_o \mathbf{M} \times \mathbf{R}}{4\pi R^3}\, dv'$$

According to eq. (7.46),

$$\frac{\mathbf{R}}{R^3} = \nabla' \frac{1}{R}$$

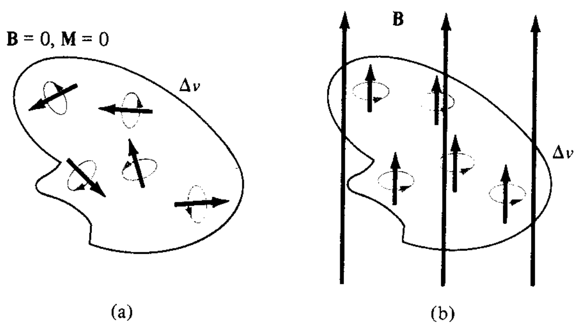


Figure 8.12 Magnetic dipole moment in a volume Δv: **(a)** before **B** is applied, **(b)** after **B** is applied.

Hence,

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int \mathbf{M} \times \nabla' \frac{1}{R} \, dv' \tag{8.28}$$

Using eq. (7.48) gives

$$\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R}$$

Substituting this into eq. (8.28) yields

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} \, dv' - \frac{\mu_o}{4\pi} \int_{v'} \nabla' \times \frac{\mathbf{M}}{R} \, dv'$$

Applying the vector identity

$$\int_{v'} \nabla' \times \mathbf{F} \, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{S}$$

to the second integral, we obtain

$$\begin{aligned} \mathbf{A} &= \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} \, dv' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}_n}{R} \, dS' \\ &= \frac{\mu_o}{4\pi} \int_{v'} \frac{\mathbf{J}_b \, dv'}{R} + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{K}_b \, dS'}{R} \end{aligned} \tag{8.29}$$

Comparing eq. (8.29) with eqs. (7.42) and (7.43) (upon dropping the primes) gives

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}} \tag{8.30}$$

and

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n} \tag{8.31}$$

where $\mathbf{J}_b$ is the *bound volume current density* or *magnetization volume current density* (in amperes per meter square), $\mathbf{K}_b$ is the *bound surface current density* (in amperes per meter), and $\mathbf{a}_n$ is a unit vector normal to the surface. Equation (8.29) shows that the potential of a magnetic body is due to a volume current density $\mathbf{J}_b$ throughout the body and a surface current $\mathbf{K}_b$ on the surface of the body. The vector $\mathbf{M}$ is analogous to the polarization $\mathbf{P}$ in dielectrics and is sometimes called the *magnetic polarization density* of the medium. In another sense, $\mathbf{M}$ is analogous to $\mathbf{H}$ and they both have the same units. In this respect, as $\mathbf{J} = \nabla \times \mathbf{H}$, so is $\mathbf{J}_b = \nabla \times \mathbf{M}$. Also, $\mathbf{J}_b$ and $\mathbf{K}_b$ for a magnetized body are similar to ρ_{pv} and ρ_{ps} for a polarized body. As is evident in eqs. (8.29) to (8.31), $\mathbf{J}_b$ and $\mathbf{K}_b$ can be derived from $\mathbf{M}$; therefore, $\mathbf{J}_b$ and $\mathbf{K}_b$ are not commonly used.

In free space, $\mathbf{M} = 0$ and we have

$$\nabla \times \mathbf{H} = \mathbf{J}_f \qquad \text{or} \qquad \nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \mathbf{J}_f \tag{8.32}$$

where $\mathbf{J}_f$ is the free current volume density. In a material medium $\mathbf{M} \neq 0$, and as a result, $\mathbf{B}$ changes so that

$$\begin{aligned}\nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) &= \mathbf{J}_f + \mathbf{J}_b = \mathbf{J} \\ &= \nabla \times \mathbf{H} + \nabla \times \mathbf{M}\end{aligned}$$

or

$$\boxed{\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})} \tag{8.33}$$

The relationship in eq. (8.33) holds for all materials whether they are linear or not. The concepts of linearity, isotropy, and homogeneity introduced in Section 5.7 for dielectric media equally apply here for magnetic media. For linear materials, $\mathbf{M}$ (in A/m) depends linearly on $\mathbf{H}$ such that

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}} \tag{8.34}$$

where χ_m is a dimensionless quantity (ratio of M to H) called *magnetic susceptibility* of the medium. It is more or less a measure of how susceptible (or sensitive) the material is to a magnetic field. Substituting eq. (8.34) into eq. (8.33) yields

$$\mathbf{B} = \mu_o(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{8.35}$$

or

$$\boxed{\mathbf{B} = \mu_o \mu_r \mathbf{H}} \tag{8.36}$$

where

$$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_o}} \tag{8.37}$$

The quantity $\mu = \mu_o \mu_r$ is called the *permeability* of the material and is measured in henrys/meter; the henry is the unit of inductance and will be defined a little later. The dimensionless quantity μ_r is the ratio of the permeability of a given material to that of free space and is known as the *relative permeability* of the material.

It should be borne in mind that the relationships in eqs. (8.34) to (8.37) hold only for linear and isotropic materials. If the materials are anisotropic (e.g., crystals), eq. (8.33) still holds but eqs. (8.34) to (8.37) do not apply. In this case, μ has nine terms (similar to ε in eq. 5.37) and, consequently, the fields $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ are no longer parallel.

8.6 CLASSIFICATION OF MAGNETIC MATERIALS

In general, we may use the magnetic susceptibility χ_m or the relative permeability μ_r to classify materials in terms of their magnetic property or behavior. A material is said to be *nonmagnetic* if $\chi_m = 0$ (or $\mu_r = 1$); it is magnetic otherwise. Free space, air, and materials with $\chi_m = 0$ (or $\mu_r \approx 1$) are regarded as nonmagnetic.

Roughly speaking, magnetic materials may be grouped into three major classes: diamagnetic, paramagnetic, and ferromagnetic. This rough classification is depicted in Figure 8.13. A material is said to be *diamagnetic* if it has $\mu_r \lesssim 1$ (i.e., very small negative χ_m). It is *paramagnetic* if $\mu_r \gtrsim 1$ (i.e., very small positive χ_m). If $\mu_r \gg 1$ (i.e., very large positive χ_m), the material is *ferromagnetic*. Table B.3 in Appendix B presents the values μ_r for some materials. From the table, it is apparent that for most practical purposes we may assume that $\mu_r \simeq 1$ for diamagnetic and paramagnetic materials. Thus, we may regard diamagnetic and paramagnetic materials as linear and nonmagnetic. Ferromagnetic materials are always nonlinear and magnetic except when their temperatures are above curie temperature (to be explained later). The reason for this will become evident as we more closely examine each of these three types of magnetic materials.

Diamagnetism occurs in materials where the magnetic fields due to electronic motions of orbiting and spinning completely cancel each other. Thus, the permanent (or intrinsic) magnetic moment of each atom is zero and the materials are weakly affected by a magnetic field. For most diamagnetic materials (e.g., bismuth, lead, copper, silicon, diamond, sodium chloride), χ_m is of the order of -10^{-5}. In certain types of materials called *superconductors* at temperatures near absolute zero, "perfect diamagnetism" occurs: $\chi_m = -1$ or $\mu_r = 0$ and $B = 0$. Thus superconductors cannot contain magnetic fields.[2] Except for superconductors, diamagnetic materials are seldom used in practice. Although the diamagnetic effect is overshadowed by other stronger effects in some materials, all materials exhibit diamagnetism.

Materials whose atoms have nonzero permanent magnetic moment may be paramagnetic or ferromagnetic. *Paramagnetism* occurs in materials where the magnetic fields pro-

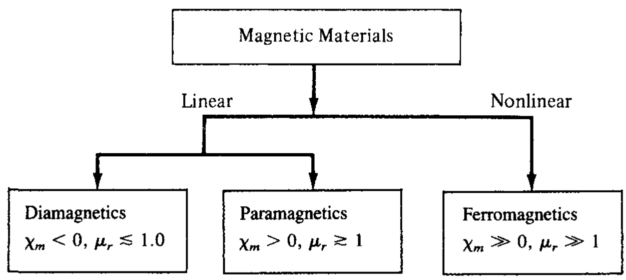


Figure 8.13 Classification of magnetic materials.

[2]An excellent treatment of superconductors is found in M. A. Plonus, *Applied Electromagnetics*. New York: McGraw-Hill, 1978, pp. 375–388. Also, the August 1989 issue of the *Proceedings of IEEE* is devoted to superconductivity.

duced by orbital and spinning electrons do not cancel completely. Unlike diamagnetism, paramagnetism is temperature dependent. For most paramagnetic materials (e.g., air, platinum, tungsten, potassium), χ_m is of the order $+10^{-5}$ to $+10^{-3}$ and is temperature dependent. Such materials find application in masers.

Ferromagnetism occurs in materials whose atoms have relatively large permanent magnetic moment. They are called ferromagnetic materials because the best known member is iron. Other members are cobalt, nickel, and their alloys. Ferromagnetic materials are very useful in practice. As distinct from diamagnetic and paramagnetic materials, ferromagnetic materials have the following properties:

1. They are capable of being magnetized very strongly by a magnetic field.
2. They retain a considerable amount of their magnetization when removed from the field.
3. They lose their ferromagnetic properties and become linear paramagnetic materials when the temperature is raised above a certain temperature known as the *curie temperature.* Thus if a permanent magnet is heated above its curie temperature (770°C for iron), it loses its magnetization completely.
4. They are nonlinear; that is, the constitutive relation $\mathbf{B} = \mu_o \mu_r \mathbf{H}$ does not hold for ferromagnetic materials because μ_r depends on $\mathbf{B}$ and cannot be represented by a single value.

Thus, the values of μ_r cited in Table B.3 for ferromagnetics are only typical. For example, for nickel $\mu_r = 50$ under some conditions and 600 under other conditions.

As mentioned in Section 5.9 for conductors, ferromagnetic materials, such as iron and steel, are used for screening (or shielding) to protect sensitive electrical devices from disturbances from strong magnetic fields. A typical example of an iron shield is shown in Figure 8.14(a) where the compass is protected. Without the iron shield, the compass gives an erroneous reading due to the effect of the external magnetic field as in Figure 8.14(b). For perfect screening, it is required that the shield have infinite permeability.

Even though $\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M})$ holds for all materials including ferromagnetics, the relationship between $\mathbf{B}$ and $\mathbf{H}$ depends on previous magnetization of a ferromagnetic

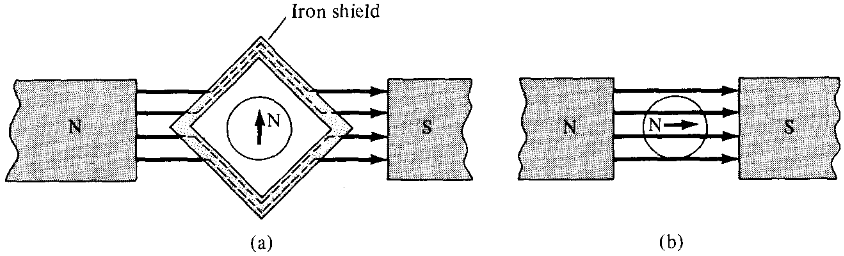


Figure 8.14 Magnetic screening: **(a)** iron shield protecting a small compass, **(b)** compass gives erroneous reading without the shield.

material—its "magnetic history." Instead of having a linear relationship between **B** and **H** (i.e., $\mathbf{B} = \mu\mathbf{H}$), it is only possible to represent the relationship by a *magnetization curve* or *B–H* curve.

A typical *B–H* curve is shown in Figure 8.15. First of all, note the nonlinear relationship between B and H. Second, at any point on the curve, μ is given by the ratio B/H and not by dB/dH, the slope of the curve.

If we assume that the ferromagnetic material whose *B–H* curve in Figure 8.15 is initially unmagnetized, as H increases (due to increase in current) from O to maximum applied field intensity H_{max}, curve OP is produced. This curve is referred to as the *virgin* or *initial magnetization curve.* After reaching saturation at P, if H is decreased, B does not follow the initial curve but lags behind H. This phenomenon of B lagging behind H is called *hysteresis* (which means "to lag" in Greek).

If H is reduced to zero, B is not reduced to zero but to B_r, which is referred to as the *permanent flux density.* The value of B_r depends on H_{max}, the maximum applied field intensity. The existence of B_r is the cause of having permanent magnets. If H increases negatively (by reversing the direction of current), B becomes zero when H becomes H_c, which is known as the *coercive field intensity.* Materials for which H_c is small are said to be magnetically hard. The value of H_c also depends on H_{max}.

Further increase in H in the negative direction to reach Q and a reverse in its direction to reach P gives a closed curve called a *hysteresis loop.* The shape of hysteresis loops varies from one material to another. Some ferrites, for example, have an almost rectangular hysteresis loop and are used in digital computers as magnetic information storage devices. The area of a hysteresis loop gives the energy loss (hysteresis loss) per unit volume during one cycle of the periodic magnetization of the ferromagnetic material. This energy loss is in the form of heat. It is therefore desirable that materials used in electric generators, motors, and transformers should have tall but narrow hysteresis loops so that hysteresis losses are minimal.

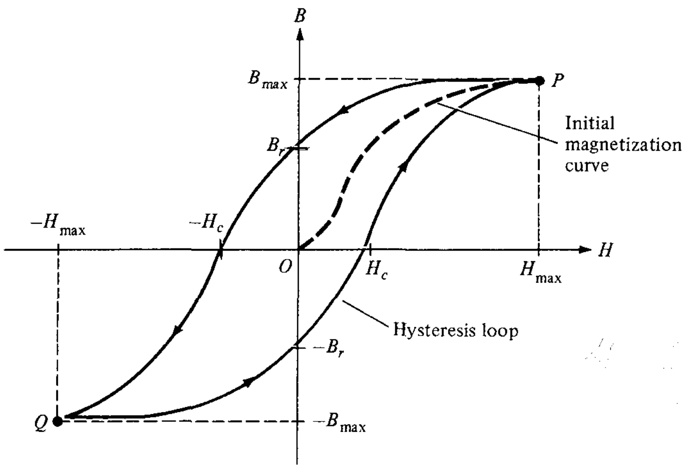


Figure 8.15 Typical magnetization (*B–H*) curve.

EXAMPLE 8.7

Region $0 \leq z \leq 2$ m is occupied by an infinite slab of permeable material ($\mu_r = 2.5$). If $\mathbf{B} = 10y\mathbf{a}_x - 5x\mathbf{a}_y$ mWb/m^2 within the slab, determine: (a) $\mathbf{J}$, (b) $\mathbf{J}_b$, (c) $\mathbf{M}$, (d) $\mathbf{K}_b$ on $z = 0$.

Solution:

(a) By definition,

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_o \mu_r} = \frac{1}{4\pi \times 10^{-7}(2.5)}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{a}_z$$
$$= \frac{10^6}{\pi}(-5 - 10)10^{-3}\mathbf{a}_z = -4.775\mathbf{a}_z \text{ kA/m}^2$$

(b) $\mathbf{J}_b = \chi_m \mathbf{J} = (\mu_r - 1)\mathbf{J} = 1.5(-4.775\mathbf{a}_z) \cdot 10^3$
$= -7.163\mathbf{a}_z$ kA/m^2

(c) $\mathbf{M} = \chi_m \mathbf{H} = \chi_m \dfrac{\mathbf{B}}{\mu_o \mu_r} = \dfrac{1.5(10y\mathbf{a}_x - 5x\mathbf{a}_y) \cdot 10^{-3}}{4\pi \times 10^{-7}(2.5)}$
$= 4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y$ kA/m

(d) $\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n$. Since $z = 0$ is the lower side of the slab occupying $0 \leq z \leq 2$, $\mathbf{a}_n = -\mathbf{a}_z$. Hence,

$$\mathbf{K}_b = (4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y) \times (-\mathbf{a}_z)$$
$$= 2.387x\mathbf{a}_x + 4.775y\mathbf{a}_y \text{ kA/m}$$

PRACTICE EXERCISE 8.7

In a certain region ($\mu = 4.6\mu_o$),

$$\mathbf{B} = 10e^{-y}\mathbf{a}_z \text{ mWb/m}^2$$

find: (a) χ_m, (b) $\mathbf{H}$, (c) $\mathbf{M}$.

Answer: (a) 3.6, (b) $1730e^{-y}\mathbf{a}_z$ A/m, (c) $6228e^{-y}\mathbf{a}_z$ A/m.

8.7 MAGNETIC BOUNDARY CONDITIONS

We define magnetic boundary conditions as the conditions that $\mathbf{H}$ (or $\mathbf{B}$) field must satisfy at the boundary between two different media. Our derivations here are similar to those in Section 5.9. We make use of Gauss's law for magnetic fields

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0 \tag{8.38}$$

and Ampere's circuit law

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \tag{8.39}$$

Consider the boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_2 as in Figure 8.16. Applying eq. (8.38) to the pillbox (Gaussian surface) of Figure 8.16(a) and allowing $\Delta h \to 0$, we obtain

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0 \tag{8.40}$$

Thus

$$\boxed{\mathbf{B}_{1n} = \mathbf{B}_{2n}} \qquad \text{or} \qquad \mu_1 \mathbf{H}_{1n} = \mu_2 \mathbf{H}_{2n} \tag{8.41}$$

since $\mathbf{B} = \mu\mathbf{H}$. Equation (8.41) shows that the normal component of **B** is continuous at the boundary. It also shows that the normal component of **H** is discontinuous at the boundary; **H** undergoes some change at the interface.

Similarly, we apply eq. (8.39) to the closed path *abcda* of Figure 8.16(b) where surface current K on the boundary is assumed normal to the path. We obtain

$$\begin{aligned} K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} \\ -H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2} \end{aligned} \tag{8.42}$$

As $\Delta h \to 0$, eq. (8.42) leads to

$$H_{1t} - H_{2t} = K \tag{8.43}$$

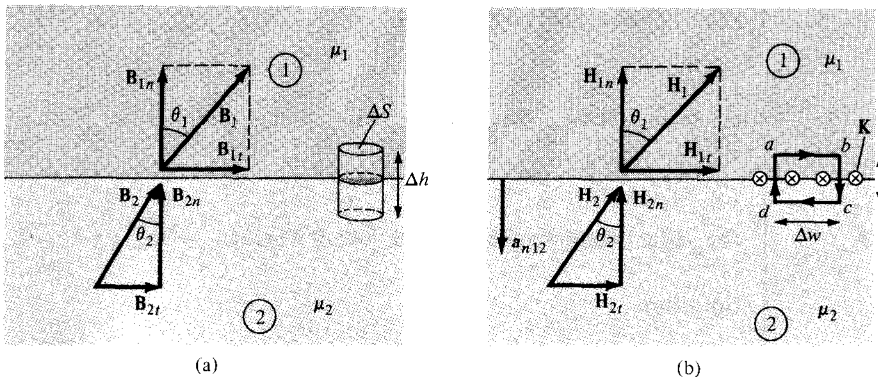


Figure 8.16 Boundary conditions between two magnetic media: **(a)** for **B**, **(b)** for **H**.

This shows that the tangential component of H is also discontinuous. Equation (8.43) may be written in terms of B as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \tag{8.44}$$

In the general case, eq. (8.43) becomes

$$\boxed{(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}} \tag{8.45}$$

where $\mathbf{a}_{n12}$ is a unit vector normal to the interface and is directed from medium 1 to medium 2. If the boundary is free of current or the media are not conductors (for K is free current density), $K = 0$ and eq. (8.43) becomes

$$\boxed{\mathbf{H}_{1t} = \mathbf{H}_{2t}} \quad \text{or} \quad \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2} \tag{8.46}$$

Thus the tangential component of **H** is continuous while that of **B** is discontinuous at the boundary.

If the fields make an angle θ with the normal to the interface, eq. (8.41) results in

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \tag{8.47}$$

while eq. (8.46) produces

$$\frac{B_1}{\mu_1} \sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin\theta_2 \tag{8.48}$$

Dividing eq. (8.48) by eq. (8.47) gives

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}} \tag{8.49}$$

which is [similar to eq. (5.65)] the law of refraction for magnetic flux lines at a boundary with no surface current.

EXAMPLE 8.8

Given that $\mathbf{H}_1 = -2\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$ A/m in region $y - x - 2 \leq 0$ where $\mu_1 = 5\mu_o$, calculate

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$

(b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in region $y - x - 2 \geq 0$ where $\mu_2 = 2\mu_o$

Solution:

Since $y - x - 2 = 0$ is a plane, $y - x \leq 2$ or $y \leq x + 2$ is region 1 in Figure 8.17. A point in this region may be used to confirm this. For example, the origin (0, 0) is in this

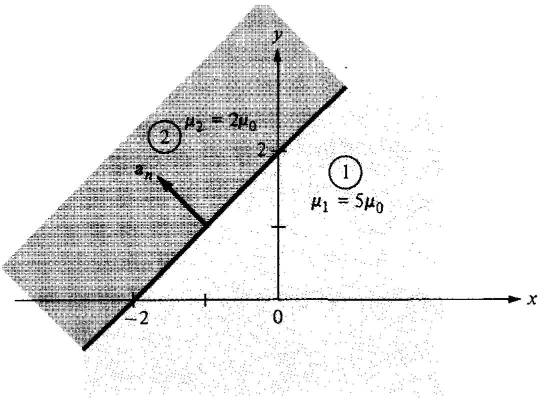


Figure 8.17 For Example 8.8.

region since $0 - 0 - 2 < 0$. If we let the surface of the plane be described by $f(x, y) = y - x - 2$, a unit vector normal to the plane is given by

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_y - \mathbf{a}_x}{\sqrt{2}}$$

(a)

$$\begin{aligned}\mathbf{M}_1 &= \chi_{m1}\mathbf{H}_1 = (\mu_{r1} - 1)\,\mathbf{H}_1 = (5 - 1)(-2, 6, 4) \\ &= -8\mathbf{a}_x + 24\mathbf{a}_y + 16\mathbf{a}_z \text{ A/m}\end{aligned}$$

$$\begin{aligned}\mathbf{B}_1 &= \mu_1\mathbf{H}_1 = \mu_o\mu_{r1}\mathbf{H}_1 = 4\pi \times 10^{-7}(5)(-2, 6, 4) \\ &= -12.57\mathbf{a}_x + 37.7\mathbf{a}_y + 25.13\mathbf{a}_z\ \mu\text{Wb/m}^2\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{H}_{1n} &= (\mathbf{H}_1 \cdot \mathbf{a}_n)\mathbf{a}_n = \left[(-2, 6, 4) \cdot \frac{(-1, 1, 0)}{\sqrt{2}}\right]\frac{(-1, 1, 0)}{\sqrt{2}} \\ &= -4\mathbf{a}_x + 4\mathbf{a}_y\end{aligned}$$

But

$$\mathbf{H}_1 = \mathbf{H}_{1n} + \mathbf{H}_{1t}$$

Hence,

$$\begin{aligned}\mathbf{H}_{1t} &= \mathbf{H}_1 - \mathbf{H}_{1n} = (-2, 6, 4) - (-4, 4, 0) \\ &= 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z\end{aligned}$$

Using the boundary conditions, we have

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \rightarrow \mu_2\mathbf{H}_{2n} = \mu_1\mathbf{H}_{1n}$$

or

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-4\mathbf{a}_x + 4\mathbf{a}_y) = -10\mathbf{a}_x + 10\mathbf{a}_y$$

Thus

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = -8\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z \text{ A/m}$$

and

$$\mathbf{B}_2 = \mu_2\mathbf{H}_2 = \mu_o\mu_{r2}\mathbf{H}_2 = (4\pi \times 10^{-7})(2)(-8, 12, 4)$$
$$= -20.11\mathbf{a}_x + 30.16\mathbf{a}_y + 10.05\mathbf{a}_z \ \mu\text{Wb/m}^2$$

PRACTICE EXERCISE 8.8

Region 1, described by $3x + 4y \geq 10$, is free space whereas region 2, described by $3x + 4y \leq 10$, is a magnetic material for which $\mu = 10\mu_o$. Assuming that the boundary between the material and free space is current free, find $\mathbf{B}_2$ if $\mathbf{B}_1 = 0.1\mathbf{a}_x + 0.4\mathbf{a}_y + 0.2\mathbf{a}_z$ Wb/m^2

Answer: $-1.052\mathbf{a}_x + 1.264\mathbf{a}_y + 2\mathbf{a}_z$ Wb/m^2

EXAMPLE 8.9

The xy-plane serves as the interface between two different media. Medium 1 ($z < 0$) is filled with a material whose $\mu_r = 6$, and medium 2 ($z > 0$) is filled with a material whose $\mu_r = 4$. If the interface carries current $(1/\mu_o)\ \mathbf{a}_y$ mA/m, and $\mathbf{B}_2 = 5\mathbf{a}_x + 8\mathbf{a}_z$ mWb/m^2, find $\mathbf{H}_1$ and $\mathbf{B}_1$.

Solution:

In the previous example $\mathbf{K} = 0$, so eq. (8.46) was appropriate. In this example, however, $\mathbf{K} \neq 0$, and we must resort to eq. (8.45) in addition to eq. (8.41). Consider the problem as illustrated in Figure 8.18. Let $\mathbf{B}_1 = (B_x, B_y, B_z)$ in mWb/m^2.

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} = 8\mathbf{a}_z \rightarrow B_z = 8 \tag{8.8.1}$$

But

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \text{ mA/m} \tag{8.8.2}$$

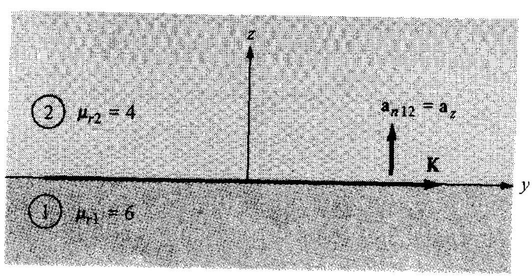


Figure 8.18 For Example 8.9.

and

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \text{ mA/m} \tag{8.8.3}$$

Having found the normal components, we can find the tangential components using

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

or

$$\mathbf{H}_1 \times \mathbf{a}_{n12} = \mathbf{H}_2 \times \mathbf{a}_{n12} + \mathbf{K} \tag{8.8.4}$$

Substituting eqs. (8.8.2) and (8.8.3) into eq. (8.8.4) gives

$$\frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \times \mathbf{a}_z = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \times \mathbf{a}_z + \frac{1}{\mu_o}\mathbf{a}_y$$

Equating components yields

$$B_y = 0, \qquad \frac{-B_x}{6} = \frac{-5}{4} + 1 \qquad \text{or} \qquad B_x = \frac{6}{4} = 1.5 \tag{8.8.5}$$

From eqs. (8.8.1) and (8.8.5),

$$\mathbf{B}_1 = 1.5\mathbf{a}_x + 8\mathbf{a}_z \text{ mWb/m}^2$$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_o}(0.25\mathbf{a}_x + 1.33\mathbf{a}_z) \text{ mA/m}$$

and

$$\mathbf{H}_2 = \frac{1}{\mu_o}(1.25\mathbf{a}_x + 2\mathbf{a}_z) \text{ mA/m}$$

Note that H_{1x} is $(1/\mu_o)$ mA/m less than H_{2x} due to the current sheet and also that $B_{1n} = B_{2n}$.

PRACTICE EXERCISE 8.9

A unit normal vector from region 2 ($\mu = 2\mu_o$) to region 1 ($\mu = \mu_o$) is $\mathbf{a}_{n21} = (6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z)/7$. If $\mathbf{H}_1 = 10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$

(b) The surface current density $\mathbf{K}$ on the interface

(c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface.

Answer: (a) 5.833, (b) $4.86\mathbf{a}_x - 8.64\mathbf{a}_y + 3.95\mathbf{a}_z$ A/m, (c)76.27°, 77.62°.

8.8 INDUCTORS AND INDUCTANCES

A circuit (or closed conducting path) carrying current I produces a magnetic field **B** which causes a flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$ to pass through each turn of the circuit as shown in Figure 8.19. If the circuit has N identical turns, we define the *flux linkage* λ as

$$\lambda = N\Psi \tag{8.50}$$

Also, if the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it; that is,

$$\lambda \propto I$$

$$\text{or} \qquad \lambda = LI \tag{8.51}$$

where L is a constant of proportionality called the *inductance* of the circuit. The inductance L is a property of the physical arrangement of the circuit. A circuit or part of a circuit that has inductance is called an *inductor.* From eqs. (8.50) and (8.51), we may define inductance L of an inductor as the ratio of the magnetic flux linkage λ to the current I through the inductor; that is,

$$\boxed{L = \frac{\lambda}{I} = \frac{N\Psi}{I}} \tag{8.52}$$

The unit of inductance is the henry (H) which is the same as webers/ampere. Since the henry is a fairly large unit, inductances are usually expressed in millihenrys (mH).

The inductance defined by eq. (8.52) is commonly referred to as *self-inductance* since the linkages are produced by the inductor itself. Like capacitances, we may regard inductance as a measure of how much magnetic energy is stored in an inductor. The magnetic energy (in joules) stored in an inductor is expressed in circuit theory as:

$$W_m = \frac{1}{2}LI^2 \tag{8.53}$$

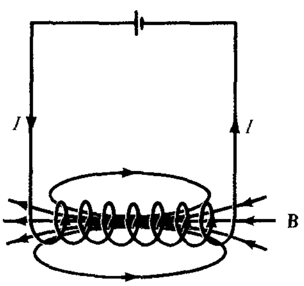


Figure 8.19 Magnetic field **B** produced by a circuit.

or

$$\boxed{L = \frac{2W_m}{I^2}} \tag{8.54}$$

Thus the self-inductance of a circuit may be defined or calculated from energy considerations.

If instead of having a single circuit we have two circuits carrying current I_1 and I_2 as shown in Figure 8.20, a magnetic interaction exists between the circuits. Four component fluxes Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} are produced. The flux Ψ_{12}, for example, is the flux passing through circuit 1 due to current I_2 in circuit 2. If $\mathbf{B}_2$ in the field due to I_2 and S_1 is the area of circuit 1, then

$$\Psi_{12} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S} \tag{8.55}$$

We define the *mutual inductance* M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1\Psi_{12}$ on circuit 1 to current I_2, that is,

$$\boxed{M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\Psi_{12}}{I_2}} \tag{8.56}$$

Similarly, the mutual inductance M_{21} is defined as the flux linkages of circuit 2 per unit current I_1; that is,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\Psi_{21}}{I_1} \tag{8.57a}$$

It can be shown by using energy concepts that if the medium surrounding the circuits is linear (i.e., in the absence of ferromagnetic material),

$$M_{12} = M_{21} \tag{8.57b}$$

The mutual inductance M_{12} or M_{21} is expressed in henrys and should not be confused with the magnetization vector $\mathbf{M}$ expressed in amperes/meter.

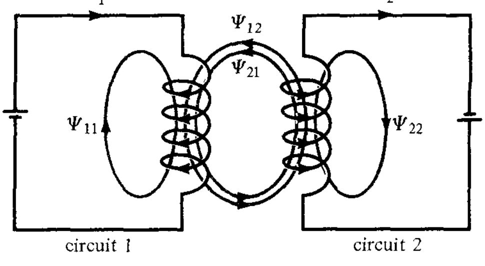


Figure 8.20 Magnetic interaction between two circuits.

We define the self-inductance of circuits 1 and 2, respectively, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Psi_1}{I_1} \tag{8.58}$$

and

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Psi_2}{I_2} \tag{8.59}$$

where $\Psi_1 = \Psi_{11} + \Psi_{12}$ and $\Psi_2 = \Psi_{21} + \Psi_{22}$. The total energy in the magnetic field is the sum of the energies due to L_1, L_2, and M_{12} (or M_{21}); that is,

$$\begin{aligned} W_m &= W_1 + W_2 + W_{12} \\ &= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2 \end{aligned} \tag{8.60}$$

The positive sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuits strengthen each other. If the currents flow such that their magnetic fields oppose each other, the negative sign is taken.

As mentioned earlier, an inductor is a conductor arranged in a shape appropriate to store magnetic energy. Typical examples of inductors are toroids, solenoids, coaxial transmission lines, and parallel-wire transmission lines. The inductance of each of these inductors can be determined by following a procedure similar to that taken in determining the capacitance of a capacitor. For a given inductor, we find the self-inductance L by taking these steps:

1. Choose a suitable coordinate system.
2. Let the inductor carry current I.
3. Determine $\mathbf{B}$ from Biot–Savart's law (or from Ampere's law if symmetry exists) and calculate Ψ from $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$.
4. Finally find L from $L = \dfrac{\lambda}{I} = \dfrac{N\Psi}{I}$.

The mutual inductance between two circuits may be calculated by taking a similar procedure.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the *internal inductance* L_{in} while that produced by the flux external to it is called *external inductance* L_{ext}. The total inductance L is

$$L = L_{\text{in}} + L_{\text{ext}} \tag{8.61}$$

Just as it was shown that for capacitors

$$RC = \frac{\varepsilon}{\sigma} \tag{6.35}$$

it can be shown that

$$\boxed{L_{\text{ext}}C = \mu\varepsilon} \tag{8.62}$$

Thus L_{ext} may be calculated using eq. (8.62) if C is known.

A collection of formulas for some fundamental circuit elements is presented in Table 8.3. All formulas can be derived by taking the steps outlined above.[3]

8.9 MAGNETIC ENERGY

Just as the potential energy in an electrostatic field was derived as

$$W_E = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\int \varepsilon E^2\, dv \tag{4.96}$$

we would like to derive a similar expression for the energy in a magnetostatic field. A simple approach is using the magnetic energy in the field of an inductor. From eq. (8.53),

$$W_m = \frac{1}{2} LI^2 \tag{8.53}$$

The energy is stored in the magnetic field $\mathbf{B}$ of the inductor. We would like to express eq. (8.53) in terms of $\mathbf{B}$ or $\mathbf{H}$.

Consider a differential volume in a magnetic field as shown in Figure 8.21. Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI.

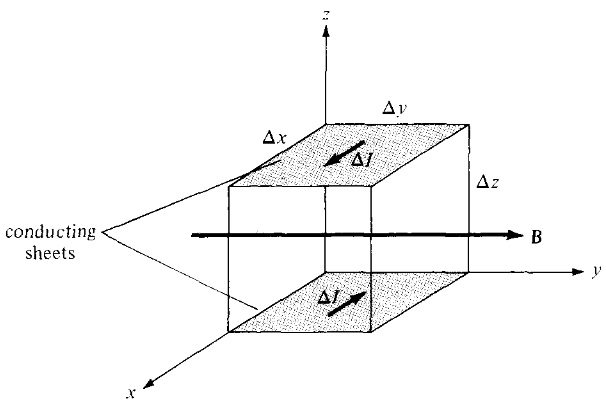


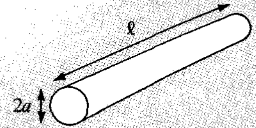
Figure 8.21 A differential volume in a magnetic field.

[3]Additional formulas can be found in standard electrical handbooks or in H. Knoepfel, *Pulsed High Magnetic Fields*. Amsterdam: North-Holland, 1970, pp. 312–324.

TABLE 8.3 A Collection of Formulas for Inductance of Common Elements

1. Wire

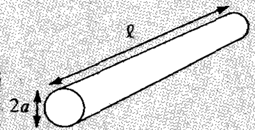
$$L = \frac{\mu_o \ell}{8\pi}$$



2. Hollow cylinder

$$L = \frac{\mu_o \ell}{2\pi}\left(\ln \frac{2\ell}{\mathbf{a}} - 1\right)$$

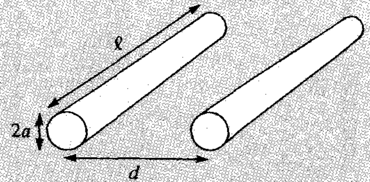
$$\ell \gg \mathbf{a}$$



3. Parallel wires

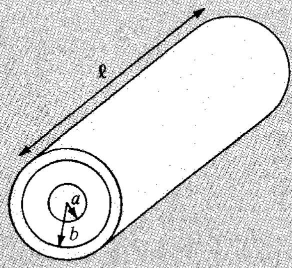
$$L = \frac{\mu_o \ell}{\pi} \ln \frac{d}{a}$$

$$\ell \gg d,\ d \gg a$$



4. Coaxial conductor

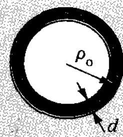
$$L = \frac{\mu_o \ell}{\pi} \ln \frac{b}{a}$$



5. Circular loop

$$L = \frac{\mu_o \ell}{2\pi}\left(\ln \frac{4\ell}{d} - 2.45\right)$$

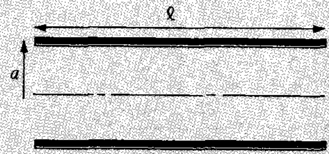
$$\ell = 2\pi\rho_o,\ \rho_o \gg d$$



6. Solenoid

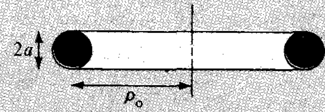
$$L = \frac{\mu_o N^2 S}{\ell}$$

$$\ell \gg a$$



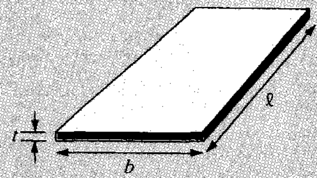
7. Torus (of circular cross section)

$$L = \mu_o N^2[\rho_o - \sqrt{\rho_o^2 - a^2}]$$



8. Sheet

$$L = \mu_o\, 2\ell\left(ln \frac{2\ell}{b + t} + 0.5\right)$$



We assume that the whole region is filled with such differential volumes. From eq. (8.52), each volume has an inductance

$$\Delta L = \frac{\Delta \Psi}{\Delta I} = \frac{\mu H \, \Delta x \, \Delta z}{\Delta I} \tag{8.63}$$

where $\Delta I = H \, \Delta y$. Substituting eq. (8.63) into eq. (8.53), we have

$$\Delta W_m = \frac{1}{2} \Delta L \, \Delta I^2 = \frac{1}{2} \mu H^2 \, \Delta x \, \Delta y \, \Delta z \tag{8.64}$$

or

$$\Delta W_m = \frac{1}{2} \mu H^2 \, \Delta v$$

The magnetostatic energy density w_m (in J/m^3) is defined as

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2$$

Hence,

$$w_m = \frac{1}{2} \mu H^2 = \frac{1}{2} B \cdot H = \frac{B^2}{2\mu} \tag{8.65}$$

Thus the energy in a magnetostatic field in a linear medium is

$$W_m = \int w_m \, dv$$

or

$$\boxed{W_m = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, dv = \frac{1}{2} \int \mu H^2 \, dv} \tag{8.66}$$

which is similar to eq. (4.96) for an electrostatic field.

EXAMPLE 8.10

Calculate the self-inductance per unit length of an infinitely long solenoid.

Solution:

We recall from Example 7.4 that for an infinitely long solenoid, the magnetic flux inside the solenoid per unit length is

$$B = \mu H = \mu I n$$

where $n = N/\ell$ = number of turns per unit length. If S is the cross-sectional area of the solenoid, the total flux through the cross section is

$$\Psi = BS = \mu InS$$

Since this flux is only for a unit length of the solenoid, the linkage per unit length is

$$\lambda' = \frac{\lambda}{\ell} = n\Psi = \mu n^2 IS$$

and thus the inductance per unit length is

$$L' = \frac{L}{\ell} = \frac{\lambda'}{I} = \mu n^2 S$$

$$\boxed{L' = \mu n^2 S} \qquad \text{H/m}$$

PRACTICE EXERCISE 8.10

A very long solenoid with 2 × 2 cm cross section has an iron core ($\mu_r = 1000$) and 4000 turns/meter. If it carries a current of 500 mA, find

(a) Its self-inductance per meter

(b) The energy per meter stored in its field

Answer: (a) 8.042 H/m, (b) 1.005 J/m.

EXAMPLE 8.11

Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

Solution:

The self-inductance of the inductor can be found in two different ways: by taking the four steps given in Section 8.8 or by using eqs. (8.54) and (8.66).

Method 1: Consider the cross section of the cable as shown in Figure 8.22. We recall from eq. (7.29) that by applying Ampere's circuit law, we obtained for region 1 ($0 \leq \rho \leq a$),

$$\mathbf{B}_1 = \frac{\mu I \rho}{2\pi a^2}\,\mathbf{a}_\phi$$

and for region 2 ($a \leq \rho \leq b$),

$$\mathbf{B}_2 = \frac{\mu I}{2\pi\rho}\,\mathbf{a}_\phi$$

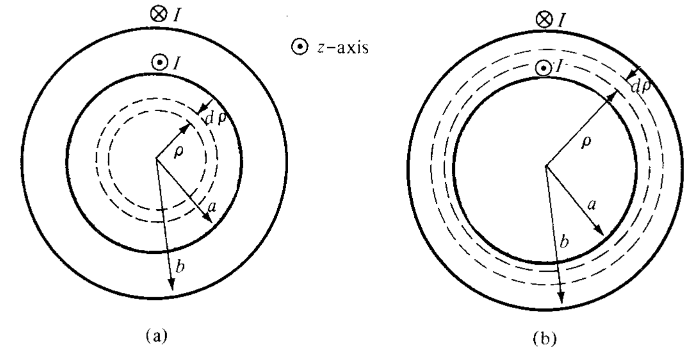


Figure 8.22 Cross section of the coaxial cable: **(a)** for region 1, $0 < \rho < a$, **(b)** for region 2, $a < \rho < b$; for Example 8.11.

We first find the internal inductance L_{in} by considering the flux linkages due to the inner conductor. From Figure 8.22(a), the flux leaving a differential shell of thickness $d\rho$ is

$$d\Psi_1 = B_1\, d\rho\, dz = \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz$$

The flux linkage is $d\Psi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area, that is,

$$d\lambda_1 = d\Psi_1 \cdot \frac{I_{enc}}{I} = d\Psi_1 \cdot \frac{\pi\rho^2}{\pi a^2}$$

because I is uniformly distributed over the cross section for d.c. excitation. Thus, the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu I \rho\, d\rho\, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

For length ℓ of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3\, d\rho\, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{in} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi} \tag{8.11.1}$$

The internal inductance per unit length, given by

$$\boxed{L'_{in} = \frac{L_{in}}{\ell} = \frac{\mu}{8\pi}} \quad \text{H/m} \tag{8.11.2}$$

is independent of the radius of the conductor or wire. Thus eqs. (8.11.1) and (8.11.2) are also applicable to finding the inductance of any infinitely long straight conductor of finite radius.

We now determine the external inductance L_{ext} by considering the flux linkages between the inner and the outer conductor as in Figure 8.22(b). For a differential shell of thickness $d\rho$,

$$d\Psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz$$

In this case, the total current I is enclosed within the path enclosing the flux. Hence,

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I\, d\rho\, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{\text{ext}} = \frac{\lambda_2}{I} = \frac{\mu \ell}{2\pi} \ln \frac{b}{a}$$

Thus

$$L = L_{\text{in}} + L_{\text{ext}} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

or the inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]} \qquad \text{H/m}$$

Method 2: It is easier to use eqs. (8.54) and (8.66) to determine L, that is,

$$W_m = \frac{1}{2} L I^2 \qquad \text{or} \qquad L = \frac{2W_m}{I^2}$$

where

$$W_m = \frac{1}{2}\int \mathbf{B} \cdot \mathbf{H}\, dv = \int \frac{B^2}{2\mu}\, dv$$

Hence

$$L_{\text{in}} = \frac{2}{I^2}\int \frac{B_1^2}{2\mu}\, dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}\, \rho\, d\rho\, d\phi\, dz$$
$$= \frac{\mu}{4\pi^2 a^4}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_0^{a} \rho^3\, d\rho = \frac{\mu\ell}{8\pi}$$

$$L_{\text{ext}} = \frac{2}{I^2}\int \frac{B_2^2}{2\mu}\, dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2}{4\pi^2 \rho^2}\, \rho\, d\rho\, d\phi\, dz$$
$$= \frac{\mu}{4\pi^2}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^{b} \frac{d\rho}{\rho} = \frac{\mu\ell}{2\pi}\ln\frac{b}{a}$$

and

$$L = L_{in} + L_{ext} = \frac{\mu \ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

as obtained previously.

PRACTICE EXERCISE 8.11

Calculate the self-inductance of the coaxial cable of Example 8.11 if the inner conductor is made of an inhomogeneous material having $\mu = 2\mu_o/(1 + \rho)$.

Answer: $\dfrac{\mu_o \ell}{8\pi} + \dfrac{\mu_o \ell}{\pi}\left[\ln\dfrac{b}{a} - \ln\dfrac{(1 + b)}{(1 + a)}\right]$

EXAMPLE 8.12

Determine the inductance per unit length of a two-wire transmission line with separation distance d. Each wire has radius a as shown in Figure 6.37.

Solution:

We use the two methods of the last example.

Method 1: We determine L_{in} just as we did in the last example. Thus for region $0 \leq \rho \leq a$, we obtain

$$\lambda_1 = \frac{\mu I \ell}{8\pi}$$

as in the last example. For region $a \leq \rho \leq d - a$, the flux linkages between the wires are

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{d-a} \int_{z=0}^{\ell} \frac{\mu I}{2\pi\rho}\, d\rho\, dz = \frac{\mu I \ell}{2\pi} \ln\frac{d - a}{a}$$

The flux linkages produced by wire 1 are

$$\lambda_1 + \lambda_2 = \frac{\mu I \ell}{8\pi} + \frac{\mu I \ell}{2\pi} \ln\frac{d - a}{a}$$

By symmetry, the same amount of flux produced by current $-I$ in wire 2. Hence the total linkages are

$$\lambda = 2(\lambda_1 + \lambda_2) = \frac{\mu I \ell}{\pi}\left[\frac{1}{4} + \ln\frac{d - a}{a}\right] = LI$$

If $d \gg a$, the self-inductance per unit length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{\pi}\left[\frac{1}{4} + \ln\frac{d}{a}\right]} \qquad \text{H/m}$$

Method 2: From the last example,

$$L_{\text{in}} = \frac{\mu\ell}{8\pi}$$

Now

$$\begin{aligned}
L_{\text{ext}} &= \frac{2}{I^2}\int \frac{B^2\,dv}{2\mu} = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2}{4\pi^2\rho^2}\,\rho\,d\rho\,d\phi\,dz \\
&= \frac{\mu}{4\pi^2}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^{d-a} \frac{d\rho}{\rho} \\
&= \frac{\mu\ell}{2\pi}\ln\frac{d-a}{a}
\end{aligned}$$

Since the two wires are symmetrical,

$$\begin{aligned}
L &= 2\,(L_{\text{in}} + L_{\text{ext}}) \\
&= \frac{\mu\ell}{\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right] \text{H}
\end{aligned}$$

as obtained previously.

PRACTICE EXERCISE 8.12

Two #10 copper wires (2.588 mm in diameter) are placed parallel in air with a separation distance d between them. If the inductance of each wire is 1.2 μH/m, calculate

(a) L_{in} and L_{ext} per meter for each wire

(b) The separation distance d

Answer: (a) 0.05, 1.15 μH/m, (b) 40.79 cm.

EXAMPLE 8.13

Two coaxial circular wires of radii a and b ($b > a$) are separated by distance h ($h \gg a, b$) as shown in Figure 8.23. Find the mutual inductance between the wires.

Solution:

Let current I_1 flow in wire 1. At an arbitrary point P on wire 2, the magnetic vector potential due to wire 1 is given by eq. (8.21a), namely

$$\mathbf{A}_1 = \frac{\mu I_1 a^2 \sin\theta}{4r^2}\,\mathbf{a}_\phi = \frac{\mu I_1 a^2 b\,\mathbf{a}_\phi}{4[h^2 + b^2]^{3/2}}$$

If $h \gg b$

$$\mathbf{A}_1 \simeq \frac{\mu I_1 a^2 b}{4h^3}\,\mathbf{a}_\phi$$

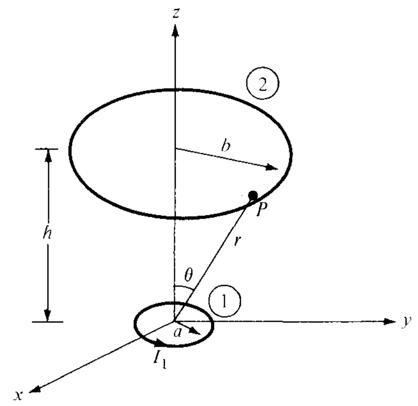


Figure 8.23 Two coaxial circular wires; for Example 8.13.

Hence,

$$\Psi_{12} = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2 = \frac{\mu I_1 a^2 b}{4h^3} 2\pi b = \frac{\mu \pi I_1 a^2 b^2}{2h^3}$$

and

$$M_{12} = \frac{\Psi_{12}}{I_1} = \frac{\mu \pi a^2 b^2}{2h^3}$$

PRACTICE EXERCISE 8.13

Find the mutual inductance of two coplanar concentric circular loops of radii 2 m and 3 m.

Answer: 2.632 μH.

8.10 MAGNETIC CIRCUITS

The concept of magnetic circuits is based on solving some magnetic field problems using circuit approach. Magnetic devices such as toroids, transformers, motors, generators, and relays may be considered as magnetic circuits. The analysis of such circuits is made simple if an analogy between magnetic circuits and electric circuits is exploited. Once this is done, we can directly apply concepts in electric circuits to solve their analogous magnetic circuits.

The analogy between magnetic and electric circuits is summarized in Table 8.4 and portrayed in Figure 8.24. The reader is advised to pause and study Table 8.4 and Figure 8.24. First, we notice from the table that two terms are new. We define the *magnetomotive force* (mmf) $\mathcal{F}$ (in ampere-turns) as

$$\boxed{\mathcal{F} = NI = \oint \mathbf{H} \cdot d\mathbf{l}} \quad (8.67)$$

TABLE 8.4 Analogy between Electric and Magnetic Circuits

Electric	Magnetic
Conductivity σ	Permeability μ
Field intensity E	Field intensity H
Current $I = \int \mathbf{J} \cdot d\mathbf{S}$	Magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
Current density $J = \dfrac{I}{S} = \sigma E$	Flux density $B = \dfrac{\Psi}{S} = \mu H$
Electromotive force (emf) V	Magnetomotive force (mmf) $\mathcal{F}$
Resistance R	Reluctance $\mathcal{R}$
Conductance $G = \dfrac{1}{R}$	Permeance $\mathcal{P} = \dfrac{1}{\mathcal{R}}$
Ohm's law $R = \dfrac{V}{I} = \dfrac{\ell}{\sigma S}$	Ohm's law $\mathcal{R} = \dfrac{\mathcal{F}}{\Psi} = \dfrac{\ell}{\mu S}$
or $V = E\ell = IR$	or $\mathcal{F} = H\ell = \Psi\mathcal{R} = NI$
Kirchoff's laws:	Kirchhoff's laws:
$\Sigma I = 0$	$\Sigma \Psi = 0$
$\Sigma V - \Sigma RI = 0$	$\Sigma \mathcal{F} - \Sigma \mathcal{R} \Psi = 0$

The source of mmf in magnetic circuits is usually a coil carrying current as in Figure 8.24. We also define *reluctance* $\mathcal{R}$ (in ampere-turns/weber) as

$$\boxed{\mathcal{R} = \frac{\ell}{\mu S}} \tag{8.68}$$

where ℓ and S are, respectively, the mean length and the cross-sectional area of the magnetic core. The reciprocal of reluctance is *permeance* $\mathcal{P}$. The basic relationship for circuit elements is Ohm's law ($V = IR$):

$$\boxed{\mathcal{F} = \Psi\mathcal{R}} \tag{8.69}$$

Based on this, Kirchhoff's current and voltage laws can be applied to nodes and loops of a given magnetic circuit just as in an electric circuit. The rules of adding voltages and for

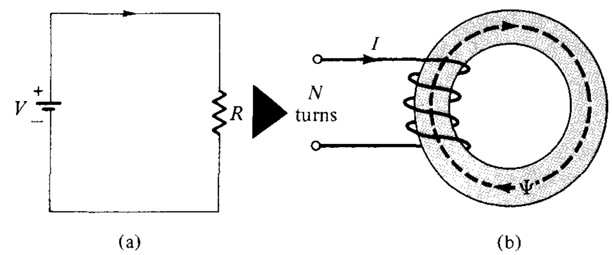


Figure 8.24 Analogy between **(a)** an electric circuit, and **(b)** a magnetic circuit.

combining series and parallel resistances also hold for mmfs and reluctances. Thus for n magnetic circuit elements in series

$$\Psi_1 = \Psi_2 = \Psi_3 = \cdots = \Psi_n \tag{8.70}$$

and

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n \tag{8.71}$$

For n magnetic circuit elements in parallel,

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \cdots + \Psi_n \tag{8.72}$$

and

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \cdots = \mathcal{F}_n \tag{8.73}$$

Some differences between electric and magnetic circuits should be pointed out. Unlike an electric circuit where current I flows, magnetic flux does not flow. Also, conductivity σ is independent of current density J in an electric circuit whereas permeability μ varies with flux density B in a magnetic circuit. This is because ferromagnetic (nonlinear) materials are normally used in most practical magnetic devices. These differences notwithstanding, the magnetic circuit concept serves as an approximate analysis of practical magnetic devices.

8.11 FORCE ON MAGNETIC MATERIALS

It is of practical interest to determine the force that a magnetic field exerts on a piece of magnetic material in the field. This is useful in electromechanical systems such as electromagnets, relays, rotating machines, and magnetic levitation. Consider, for example, an electromagnet made of iron of constant relative permeability as shown in Figure 8.25. The coil has N turns and carries a current I. If we ignore fringing, the magnetic field in the air gap is the same as that in iron ($B_{1n} = B_{2n}$). To find the force between the two pieces of iron, we calculate the change in the total energy that would result were the two pieces of the magnetic circuit separated by a differential displacement $d\mathbf{l}$. The work required to effect

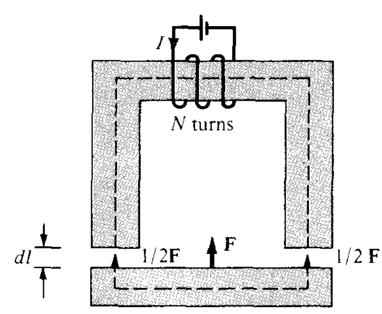


Figure 8.25 An electromagnet.

the displacement is equal to the change in stored energy in the air gap (assuming constant current), that is

$$-F\,dl = dW_m = 2\left[\frac{1}{2}\frac{B^2}{\mu_o}S\,dl\right] \tag{8.74}$$

where S is the cross-sectional area of the gap, the factor 2 accounts for the two air gaps, and the negative sign indicates that the force acts to reduce the air gap (or that the force is attractive). Thus

$$F = -2\left(\frac{B^2 S}{2\mu_o}\right) \tag{8.75}$$

Note that the force is exerted on the lower piece and not on the current-carrying upper piece giving rise to the field. The tractive force across a *single* gap can be obtained from eq. (8.75) as

$$\boxed{F = -\frac{B^2 S}{2\mu_o}} \tag{8.76}$$

Notice the similarity between eq. (8.76) and that derived in Example 5.8 for electrostatic case. Equation (8.76) can be used to calculate the forces in many types of devices including relays, rotating machines, and magnetic levitation. The tractive pressure (in N/m^2) in a magnetized surface is

$$p = \frac{F}{S} = \frac{B^2}{2\mu_o} = \frac{1}{2}BH \tag{8.77}$$

which is the same as the energy density w_m in the air gap.

EXAMPLE 8.14

The toroidal core of Figure 8.26(a) has $\rho_o = 10$ cm and a circular cross section with $a = 1$ cm. If the core is made of steel ($\mu = 1000\,\mu_o$) and has a coil with 200 turns, calculate the amount of current that will produce a flux of 0.5 mWb in the core.

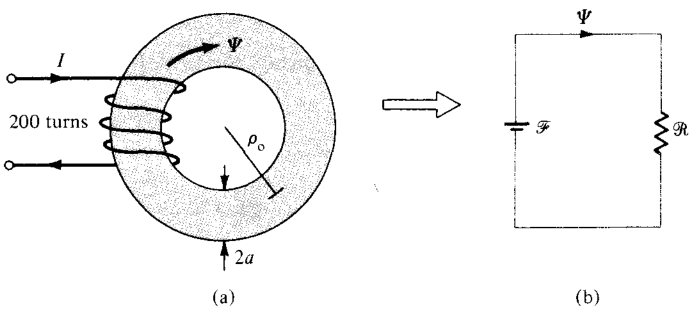


Figure 8.26 **(a)** Toroidal core of Example 8.14; **(b)** its equivalent electric circuit analog.

Solution:

This problem can be solved in two different ways: using the magnetic field approach (direct), or using the electric circuit analog (indirect).

Method 1: Since ρ_o is large compared with a, from Example 7.6,

$$B = \frac{\mu NI}{\ell} = \frac{\mu_o \mu_r NI}{2\pi\rho_o}$$

Hence,

$$\Psi = BS = \frac{\mu_o \mu_r NI\, \pi a^2}{2\pi\rho_o}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = \frac{2(10 \times 10^{-2})(0.5 \times 10^{-3})}{4\pi \times 10^{-7}(1000)(200)(1 \times 10^{-4})}$$
$$= \frac{100}{8\pi} = 3.979 \text{ A}$$

Method 2: The toroidal core in Figure 8.26(a) is analogous to the electric circuit of Figure 8.26(b). From the circuit and Table 8.4.

$$\mathcal{F} = NI = \Psi\mathcal{R} = \Psi \frac{\ell}{\mu S} = \Psi \frac{2\pi\rho_o}{\mu_o \mu_r \pi a^2}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = 3.979\ A$$

as obtained previously.

PRACTICE EXERCISE 8.14

A conductor of radius a is bent into a circular loop of mean radius ρ_o (see Figure 8.26a). If ρ_o = 10 cm and $2a$ = 1 cm, calculate the internal inductance of the loop.

Answer: 31.42 nH.

EXAMPLE 8.15 In the magnetic circuit of Figure 8.27, calculate the current in the coil that will produce a magnetic flux density of 1.5 Wb/m^2 in the air gap assuming that $\mu = 50\mu_o$ and that all branches have the same cross-sectional area of 10 cm^2.

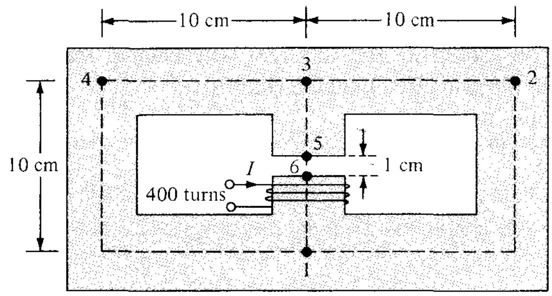


Figure 8.27 Magnetic circuit of Example 8.15.

Solution:

The magnetic circuit of Figure 8.27 is analogous to the electric circuit of Figure 8.28. In Figure 8.27, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_a$ are the reluctances in paths 143, 123, 35 and 16, and 56 (air gap), respectively. Thus

$$\mathcal{R}_1 = \mathcal{R}_2 = \frac{\ell}{\mu_o \mu_r S} = \frac{30 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})}$$
$$= \frac{3 \times 10^8}{20\pi}$$

$$\mathcal{R}_3 = \frac{9 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{0.9 \times 10^8}{20\pi}$$

$$\mathcal{R}_a = \frac{1 \times 10^{-2}}{(4\pi \times 10^{-7})(1)(10 \times 10^{-4})} = \frac{5 \times 10^8}{20\pi}$$

We combine $\mathcal{R}_1$ and $\mathcal{R}_2$ as resistors in parallel. Hence,

$$\mathcal{R}_1 \| \mathcal{R}_2 = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{\mathcal{R}_1}{2} = \frac{1.5 \times 10^8}{20\pi}$$

The total reluctance is

$$\mathcal{R}_T = \mathcal{R}_a + \mathcal{R}_3 + \mathcal{R}_1 \| \mathcal{R}_2 = \frac{7.4 \times 10^8}{20\pi}$$

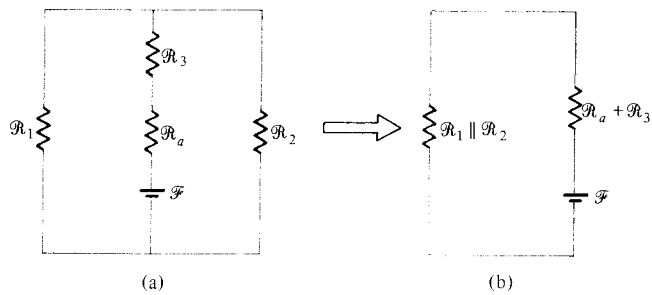


Figure 8.28 Electric circuit analog of the magnetic circuit in Figure 8.27.

The mmf is

$$\mathcal{F} = NI = \Psi_a R_T$$

But $\Psi_a = \Psi = B_a S$. Hence

$$I = \frac{B_a S \mathcal{R}_T}{N} = \frac{1.5 \times 10 \times 10^{-4} \times 7.4 \times 10^8}{400 \times 20\pi}$$
$$= 44.16 \text{ A}$$

PRACTICE EXERCISE 8.15

The toroid of Figure 8.26(a) has a coil of 1000 turns wound on its core. If $\rho_o = 10$ cm and $a = 1$ cm, what current is required to establish a magnetic flux of 0.5 mWb

(a) If the core is nonmagnetic
(b) If the core has $\mu_r = 500$

Answer: (a) 795.8 A, (b) 1.592 A.

EXAMPLE 8.16

A U-shaped electromagnet shown in Figure 8.29 is designed to lift a 400-kg mass (which includes the mass of the keeper). The iron yoke ($\mu_r = 3000$) has a cross section of 40 cm^2 and mean length of 50 cm, and the air gaps are each 0.1 mm long. Neglecting the reluctance of the keeper, calculate the number of turns in the coil when the excitation current is 1 A.

Solution:

The tractive force across the two air gaps must balance the weight. Hence

$$F = 2\frac{(B_a^2 S)}{2\mu_o} = mg$$

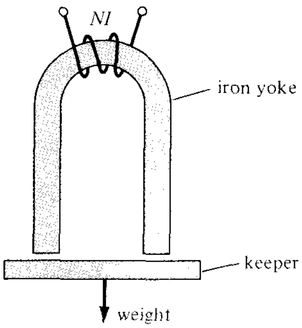


Figure 8.29 U-shaped electromagnet; for Example 8.16.

or

$$B_a^2 = \frac{mg\mu_o}{S} = \frac{400 \times 9.8 \times 4\pi \times 10^{-7}}{40 \times 10^{-4}}$$

$$B_a = 1.11 \text{ Wb/m}^2$$

But

$$\mathscr{F} = NI = \Psi(\mathscr{R}_a + \mathscr{R}_i)$$

$$\mathscr{R}_a = \frac{\ell_a}{\mu S} = \frac{2 \times 0.1 \times 10^{-3}}{4\pi \times 10^{-7} \times 40 \times 10^{-4}} = \frac{6 \times 10^6}{48\pi}$$

$$\mathscr{R}_i = \frac{\ell_i}{\mu_o \mu_r S} = \frac{50 \times 10^{-2}}{4\pi \times 10^{-7} \times 3000 \times 40 \times 10^{-4}} = \frac{5 \times 10^6}{48\pi}$$

$$\mathscr{F}_a = \frac{\mathscr{R}_a}{\mathscr{R}_a + \mathscr{R}_i}\mathscr{F} = \frac{6}{6+5}NI = \frac{6}{11}NI$$

Since

$$\mathscr{F}_a = H_a \ell_a = \frac{B_a \ell_a}{\mu_o}$$

$$N = \frac{11}{6}\frac{B_a \ell_a}{\mu_o I} = \frac{11 \times 1.11 \times 0.1 \times 10^{-3}}{6 \times 4\pi \times 10^{-7} \times 1}$$

$$N = 162$$

PRACTICE EXERCISE 8.16

Find the force across the air gap of the magnetic circuit of Example 8.15.

Answer: 895.2 N.

SUMMARY

1. The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = m\frac{d\mathbf{u}}{dt}$$

relates the force acting on a particle with charge Q in the presence of EM fields. It expresses the fundamental law relating EM to mechanics.

2. Based on the Lorentz force law, the force experienced by a current element $I d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$$

From this, the magnetic field $\mathbf{B}$ is defined as the force per unit current element.

3. The torque on a current loop with magnetic moment **m** in a uniform magnetic field **B** is

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = IS\mathbf{a}_n \times \mathbf{B}$$

4. A magnetic dipole is a bar magnet or a small filamental current loop; it is so called due to the fact that its **B** field lines are similar to the **E** field lines of an electric dipole.
5. When a material is subjected to a magnetic field, it becomes magnetized. The magnetization **M** is the magnetic dipole moment per unit volume of the material. For linear material,

$$\mathbf{M} = \chi_m \mathbf{H}$$

where χ_m is the magnetic susceptibility of the material.
6. In terms of their magnetic properties, materials are either linear (diamagnetic or paramagnetic) or nonlinear (ferromagnetic). For linear materials,

$$\mathbf{B} = \mu\mathbf{H} = \mu_o\mu_r\mathbf{H} = \mu_o(1 + \chi_m)\mathbf{H} = \mu_o(\mathbf{H} + \mathbf{M})$$

where μ = permeability and $\mu_r = \mu/\mu_o$ = relative permeability of the material. For nonlinear material, $B = \mu(H)\,H$, that is, μ does not have a fixed value; the relationship between B and H is usually represented by a magnetization curve.
7. The boundary conditions that **H** or **B** must satisfy at the interface between two different media are

$$\mathbf{B}_{1n} = \mathbf{B}_{2n}$$

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \quad \text{or} \quad \mathbf{H}_{1t} = \mathbf{H}_{2t} \quad \text{if } \mathbf{K} = 0$$

where $\mathbf{a}_{n12}$ is a unit vector directed from medium 1 to medium 2.
8. Energy in a magnetostatic field is given by

$$W_m = \frac{1}{2}\int \mathbf{B} \cdot \mathbf{H}\, dv$$

For an inductor carrying current I

$$W_m = \frac{1}{2} LI^2$$

Thus the inductance L can be found using

$$L = \frac{\int \mathbf{B} \cdot \mathbf{H}\, dv}{I^2}$$

9. The inductance L of an inductor can also be determined from its basic definition: the ratio of the magnetic flux linkage to the current through the inductor, that is,

$$L = \frac{\lambda}{I} = \frac{N\Psi}{I}$$

Thus by assuming current I, we determine $\mathbf{B}$ and $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$, and finally find $L = N\Psi/I$.

10. A magnetic circuit can be analyzed in the same way as an electric circuit. We simply keep in mind the similarity between

$$\mathcal{F} = NI = \oint \mathbf{H} \cdot d\mathbf{l} = \Psi\mathcal{R} \qquad \text{and} \qquad V = IR$$

that is,

$$\mathcal{F} \leftrightarrow V, \Psi \leftrightarrow I, \mathcal{R} \leftrightarrow R$$

Thus we can apply Ohms and Kirchhoff's laws to magnetic circuits just as we apply them to electric circuits.

11. The magnetic pressure (or force per unit surface area) on a piece of magnetic material is

$$P = \frac{F}{S} = \frac{1}{2}BH = \frac{B^2}{2\mu_o}$$

where B is the magnetic field at the surface of the material.

REVIEW QUESTIONS

8.1 Which of the following statements are not true about electric force $\mathbf{F}_e$ and magnetic force $\mathbf{F}_m$ on a charged particle?

(a) $\mathbf{E}$ and $\mathbf{F}_e$ are parallel to each other whereas $\mathbf{B}$ and $\mathbf{F}_m$ are perpendicular to each other.
(b) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ depend on the velocity of the charged particle.
(c) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ can perform work.
(d) Both $\mathbf{F}_e$ and $\mathbf{F}_m$ are produced when a charged particle moves at a constant velocity.
(e) $\mathbf{F}_m$ is generally small in magnitude compared to $\mathbf{F}_e$.
(f) $\mathbf{F}_e$ is an accelerating force whereas $\mathbf{F}_m$ is a purely deflecting force.

8.2 Two thin parallel wires carry currents along the same direction. The force experienced by one due to the other is

(a) Parallel to the lines
(b) Perpendicular to the lines and attractive
(c) Perpendicular to the lines and repulsive
(d) Zero

8.3 The force on differential length $d\mathbf{l}$ at point P in the conducting circular loop in Figure 8.30 is

(a) Outward along OP
(b) Inward along OP

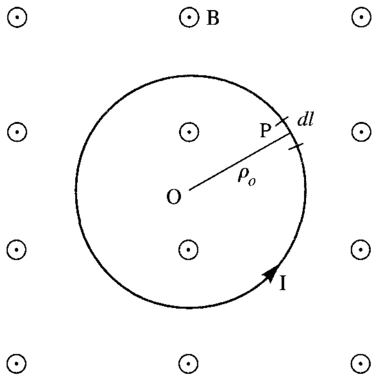


Figure 8.30 For Review Questions 8.3 and 8.4.

(c) In the direction of the magnetic field

(d) Tangential to the loop at P

8.4 The resultant force on the circular loop in Figure 8.30 has the magnitude of

(a) $2\pi\rho_o IB$

(b) $\pi\rho_o^2 IB$

(c) $2\rho_o IB$

(d) Zero

8.5 What is the unit of magnetic charge?

(a) Ampere-meter square

(b) Coulomb

(c) Ampere

(d) Ampere-meter

8.6 Which of these materials requires the least value of magnetic field strength to magnetize it?

(a) Nickel

(b) Silver

(c) Tungsten

(d) Sodium chloride

8.7 Identify the statement that is not true of ferromagnetic materials.

(a) They have a large χ_m.

(b) They have a fixed value of μ_r.

(c) Energy loss is proportional to the area of the hysteresis loop.

(d) They lose their nonlinearity property above the curie temperature.

8.8 Which of these formulas is wrong?

(a) $B_{1n} = B_{2n}$
(b) $B_2 = \sqrt{B_{2n}^2 + B_{2t}^2}$
(c) $H_1 = H_{1n} + H_{1t}$
(d) $\mathbf{a}_{n21} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$, where $\mathbf{a}_{n21}$ is a unit vector normal to the interface and directed from region 2 to region 1.

8.9 Each of the following pairs consists of an electric circuit term and the corresponding magnetic circuit term. Which pairs are not corresponding?

(a) V and $\mathcal{F}$
(b) G and $\mathcal{P}$
(c) ε and μ
(d) IR and $H\mathcal{R}$
(e) $\Sigma I = 0$ and $\Sigma \Psi = 0$

8.10 A multilayer coil of 2000 turns of fine wire is 20 mm long and has a thickness 5 mm of winding. If the coil carries a current of 5 mA, the mmf generated is

(a) 10 A-t
(b) 500 A-t
(c) 2000 A-t
(d) None of the above

Answers: 8.1 b,c, 8.2b, 8.3a, 8.4d, 8.5d, 8.6a, 8.7b, 8.8c, 8.9c,d, 8.10a.

PROBLEMS

8.1 An electron with velocity $\mathbf{u} = (3\mathbf{a}_x + 12\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s experiences no net force at a point in a magnetic field $\mathbf{B} = 10\mathbf{a}_x + 20\mathbf{a}_y + 30\mathbf{a}_z$ mWb/m^2. Find $\mathbf{E}$ at that point.

8.2 A charged particle of mass 1 kg and charge 2 C starts at the origin with velocity $10\mathbf{a}_z$ m/s in a magnetic field $\mathbf{B} = 1\mathbf{a}_x$ Wb/m^2. Find the location and the kinetic energy of the particle at $t = 2$ s.

***8.3** A particle with mass 1 kg and charge 2 C starts from rest at point (2, 3, −4) in a region where $\mathbf{E} = -4\mathbf{a}_y$ V/m and $\mathbf{B} = 5\mathbf{a}_x$ Wb/m^2. Calculate

(a) The location of the particle at $t = 1$ s
(b) Its velocity and K.E. at that location

8.4 A −2-mC charge starts at point (0, 1, 2) with a velocity of $5\mathbf{a}_x$ m/s in a magnetic field $\mathbf{B} = 6\mathbf{a}_y$ Wb/m^2. Determine the position and velocity of the particle after 10 s assuming that the mass of the charge is 1 gram. Describe the motion of the charge.

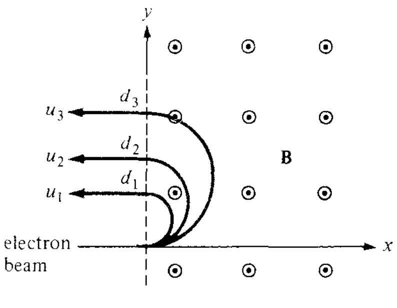


Figure 8.31 For Problem 8.5.

***8.5** By injecting an electron beam normally to the plane edge of a uniform field $B_o\mathbf{a}_z$, electrons can be dispersed according to their velocity as in Figure 8.31.

(a) Show that the electrons would be ejected out of the field in paths parallel to the input beam as shown.

(b) Derive an expression for the exit distance d above entry point.

8.6 Given that $\mathbf{B} = 6x\mathbf{a}_x - 9y\mathbf{a}_y + 3z\mathbf{a}_z$ Wb/m^2, find the total force experienced by the rectangular loop (on $z = 0$ plane) shown in Figure 8.32.

8.7 A current element of length 2 cm is located at the origin in free space and carries current 12 mA along $\mathbf{a}_x$. A filamentary current of $15\mathbf{a}_z$ A is located along $x = 3$, $y = 4$. Find the force on the current filament.

***8.8** Three infinite lines L_1, L_2, and L_3 defined by $x = 0, y = 0$; $x = 0, y = 4$; $x = 3, y = 4$, respectively, carry filamentary currents -100 A, 200 A, and 300 A along $\mathbf{a}_z$. Find the force per unit length on

(a) L_2 due to L_1

(b) L_1 due to L_2

(c) L_3 due to L_1

(d) L_3 due to L_1 and L_2. State whether each force is repulsive or attractive.

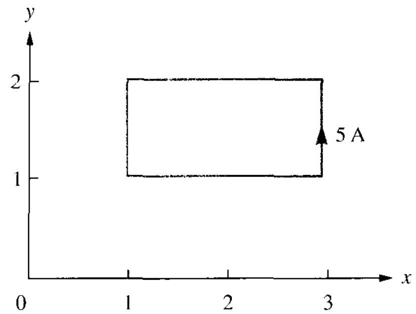


Figure 8.32 For Problem 8.6.

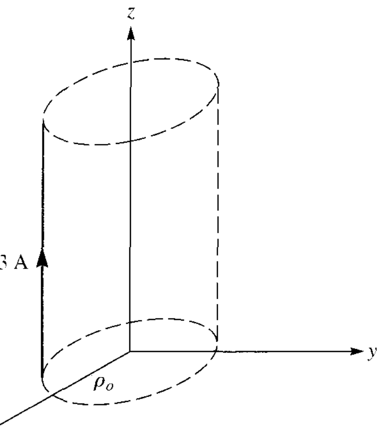


Figure 8.33 For Problem 8.9.

8.9 A conductor 2 m long carrying 3A is placed parallel to the z-axis at distance $\rho_o = 10$ cm as shown in Figure 8.33. If the field in the region is $\cos(\phi/3)\,\mathbf{a}_\rho$ Wb/m^2, how much work is required to rotate the conductor one revolution about the z-axis?

***8.10** A conducting triangular loop carrying a current of 2 A is located close to an infinitely long, straight conductor with a current of 5 A, as shown in Figure 8.34. Calculate (a) the force on side 1 of the triangular loop and (b) the total force on the loop.

***8.11** A three-phase transmission line consists of three conductors that are supported at points A, B, and C to form an equilateral triangle as shown in Figure 8.35. At one instant, conductors A and B both carry a current of 75 A while conductor C carries a return current of 150 A. Find the force per meter on conductor C at that instant.

***8.12** An infinitely long tube of inner radius a and outer radius b is made of a conducting magnetic material. The tube carries a total current I and is placed along the z-axis. If it is exposed to a constant magnetic field $B_o\mathbf{a}_\rho$, determine the force per unit length acting on the tube.

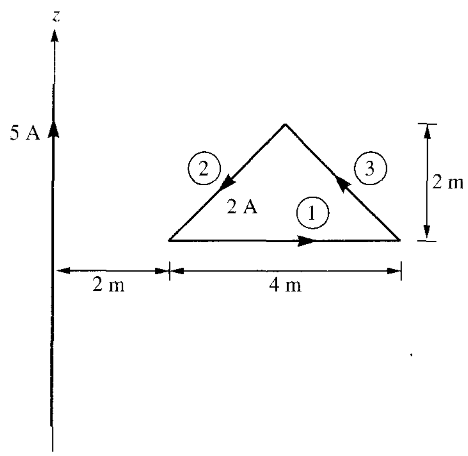


Figure 8.34 For Problem 8.10.

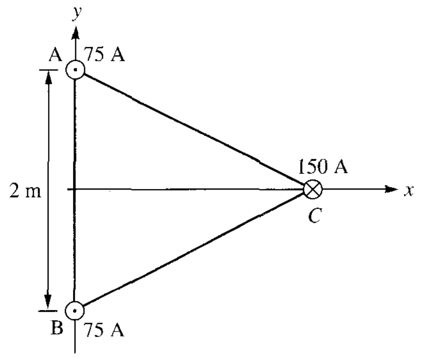


Figure 8.35 For Problem 8.11.

***8.13** An infinitely long conductor is buried but insulated from an iron mass ($\mu = 2000\mu_o$) as shown in Figure 8.36. Using image theory, estimate the magnetic flux density at point P.

8.14 A galvanometer has a rectangular coil of side 10 by 30 mm pivoted about the center of the shorter side. It is mounted in radial magnetic field so that a constant magnetic field of 0.4 Wb/m^2 always acts across the plane of the coil. If the coil has 1000 turns and carries current 2 mA, find the torque exerted on it.

8.15 A small magnet placed at the origin produces $\mathbf{B} = -0.5\mathbf{a}_z$ mWb/m^2 at (10, 0, 0). Find $\mathbf{B}$ at

(a) (0, 3, 0)
(b) (3, 4, 0)
(c) (1, 1, −1)

8.16 A block of iron ($\mu = 5000\mu_o$) is placed in a uniform magnetic field with 1.5 Wb/m^2. If iron consists of 8.5×10^{28} $atoms/m^3$, calculate: (a) the magnetization $\mathbf{M}$, (b) the average magnetic current.

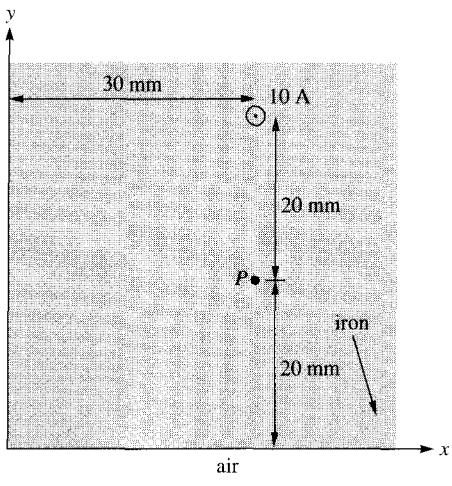


Figure 8.36 For Problem 8.13.

8.17 In a certain material for which $\mu = 6.5\mu_o$,

$$\mathbf{H} = 10\mathbf{a}_x + 25\mathbf{a}_y - 40\mathbf{a}_z \text{ A/m}$$

find

(a) The magnetic susceptibility χ_m of the material
(b) The magnetic flux density **B**
(c) The magnetization **M**,
(d) The magnetic energy density

8.18 In a ferromagnetic material ($\mu = 4.5\mu_o$),

$$\mathbf{B} = 4y\mathbf{a}_z \text{ mWb/m}^2$$

calculate: (a) χ_m, (b) **H**, (c) **M**, (d) $\mathbf{J}_b$.

8.19 The magnetic field intensity is $H = 1200$ A/m in a material when $B = 2$ Wb/m^2. When H is reduced to 400 A/m, $B = 1.4$ Wb/m^2. Calculate the change in the magnetization M.

8.20 An infinitely long cylindrical conductor of radius a and permeability $\mu_o\mu_r$ is placed along the z-axis. If the conductor carries a uniformly distributed current I along $\mathbf{a}_z$ find **M** and $\mathbf{J}_b$ for $0 < \rho < a$.

8.21 If $\mathbf{M} = \frac{k_o}{a}(-y\mathbf{a}_x + x\mathbf{a}_y)$ in a cube of size a, find $\mathbf{J}_b$. Assume k_o is a constant.

***8.22** (a) For the boundary between two magnetic media such as is shown in Figure 8.16, show that the boundary conditions on the magnetization vector are

$$\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K \qquad \text{and} \qquad \frac{\mu_1}{\chi_{m1}} m_{1n} = \frac{\mu_2}{\chi_{m2}} M_{2n}$$

(b) If the boundary is not current free, show that instead of eq. (8.49), we obtain

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left[1 + \frac{K\mu_2}{B_2\sin\theta_2}\right]$$

8.23 If $\mu_1 = 2\mu_o$ for region 1 ($0 < \phi < \pi$) and $\mu_2 = 5\mu_o$ for region 2 ($\pi < \phi < 2\pi$) and $\mathbf{B}_2 = 10\mathbf{a}_\rho + 15\mathbf{a}_\phi - 20\mathbf{a}_z$ mWb/m^2. Calculate: (a) $\mathbf{B}_1$, (b) the energy densities in the two media.

8.24 The interface $2x + y = 8$ between two media carries no current. If medium 1 ($2x + y \geq 8$) is nonmagnetic with $\mathbf{H}_1 = -4\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$ A/m. Find: (a) the magnetic energy density in medium 1, (b) $\mathbf{M}_2$ and $\mathbf{B}_2$ in medium 2 ($2x + y \leq 8$) with $\mu = 10\mu_o$, (c) the angles $\mathbf{H}_1$ and $\mathbf{H}_2$ make with the normal to the interface.

8.25 The interface $4x - 5z = 0$ between two magnetic media carries current $35\mathbf{a}_y$ A/m. If $\mathbf{H}_1 = 25\mathbf{a}_x - 30\mathbf{a}_y + 45\mathbf{a}_z$ A/m in region $4x - 5z \leq 0$ where $\mu_{r1} = 5$, calculate $\mathbf{H}_2$ in region $4x - 5z \geq 0$ where $\mu_{r2} = 10$.

8.26 The plane $z = 0$ separates air ($z \geq 0$, $\mu = \mu_o$) from iron ($z \leq 0$, $\mu = 200\mu_o$). Given that

$$\mathbf{H} = 10\mathbf{a}_x + 15\mathbf{a}_y - 3\mathbf{a}_z \text{ A/m}$$

in air, find **B** in iron and the angle it makes with the interface.

8.27 Region $0 \leq z \leq 2$ m is filled with an infinite slab of magnetic material ($\mu = 2.5\mu_o$). If the surfaces of the slab at $z = 0$ and $z = 2$, respectively, carry surface currents $30\mathbf{a}_x$ A/m and $-40\mathbf{a}_x$ A/m as in Figure 8.37, calculate **H** and **B** for

(a) $z < 0$
(b) $0 < z < 2$
(c) $z > 2$

8.28 In a certain region for which $\chi_m = 19$,

$$\mathbf{H} = 5x^2yz\mathbf{a}_x + 10xy^2z\mathbf{a}_y - 15xyz^2\mathbf{a}_z \text{ A/m}$$

How much energy is stored in $0 < x < 1, 0 < y < 2, -1 < z < 2$?

8.29 The magnetization curve for an iron alloy is approximately given by $B = \frac{1}{3}H + H^2\mu$ Wb/m^2. Find: (a) μ_r when $H = 210$ A/m, (b) the energy stored per unit volume in the alloy as H increases from 0 to 210 A/m.

***8.30** (a) If the cross section of the toroid of Figure 7.15 is a square of side a, show that the self-inductance of the toroid is

$$L = \frac{\mu_o N^2 a}{2\pi} \ln \left[\frac{2\rho_o + a}{2\rho_o - a}\right]$$

(b) If the toroid has a circular cross section as in Figure 7.15, show that

$$L = \frac{\mu_o N^2 a^2}{2\rho_o}$$

where $\rho_o \gg a$.

8.31 When two parallel identical wires are separated by 3 m, the inductance per unit length is 2.5 μH/m. Calculate the diameter of each wire.

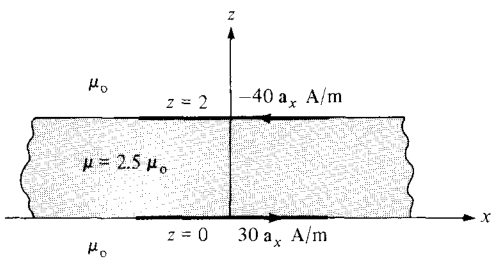


Figure 8.37 For Problem 8.27.

8.32 A solenoid with length 10 cm and radius 1 cm has 450 turns. Calculate its inductance.

8.33 The core of a toroid is 12 cm^2 and is made of material with $\mu_r = 200$. If the mean radius of the toroid is 50 cm, calculate the number of turns needed to obtain an inductance of 2.5 H.

8.34 Show that the mutual inductance between the rectangular loop and the infinite line current of Figure 8.4 is

$$M_{12} = \frac{\mu b}{2\pi} \ln \left[\frac{a + \rho_o}{\rho_o}\right]$$

Calculate M_{12} when $a = b = \rho_o = 1$ m.

***8.35** Prove that the mutual inductance between the closed wound coaxial solenoids of length ℓ_1 and ℓ_2 ($\ell_1 \gg \ell_2$), turns N_1 and N_2, and radii r_1 and r_2 with $r_1 \simeq r_2$ is

$$M_{12} = \frac{\mu N_1 N_2}{\ell_1} \pi r_1^2$$

8.36 A cobalt ring ($\mu_r = 600$) has a mean radius of 30 cm. If a coil wound on the ring carries 12 A, calculate the number of turns required to establish an average magnetic flux density of 1.5 Wb/m in the ring.

8.37 Refer to Figure 8.27. If the current in the coil is 0.5 A, find the mmf and the magnetic field intensity in the air gap. Assume that $\mu = 500\mu_o$ and that all branches have the same cross-sectional area of 10 cm^2.

8.38 The magnetic circuit of Figure 8.38 has current 10 A in the coil of 2000 turns. Assume that all branches have the same cross section of 2 cm^2 and that the material of the core is iron with $\mu_r = 1500$. Calculate R, $\mathcal{F}$, and Ψ for

(a) The core
(b) The air gap

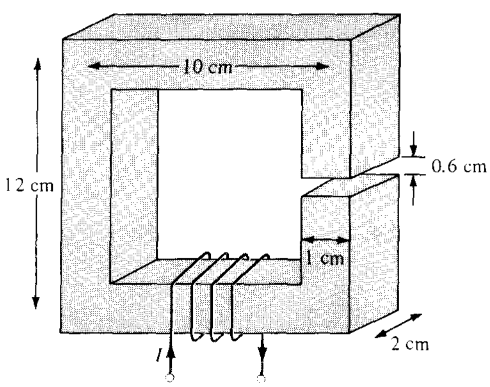


Figure 8.38 For Problem 8.38.

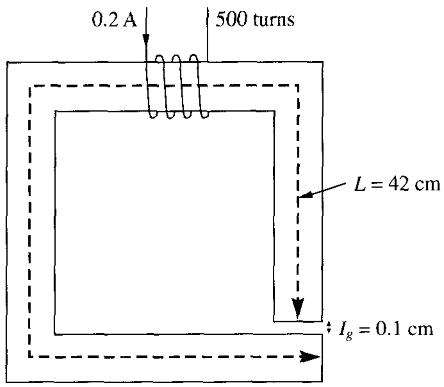


Figure 8.39 For Problem 8.39.

8.39 Consider the magnetic circuit in Figure 8.39. Assuming that the core ($\mu = 1000\mu_o$) has a uniform cross section of 4 cm^2, determine the flux density in the air gap.

8.40 An electromagnetic relay is modeled as shown in Figure 8.40. What force is on the armature (moving part) of the relay if the flux in the air gap is 2 mWb? The area of the gap is 0.3 cm^2, and its length 1.5 mm.

8.41 A toroid with air gap, shown in Figure 8.41, has a square cross section. A long conductor carrying current I_2 is inserted in the air gap. If $I_1 = 200$ mA, $N = 750$, $\rho_o = 10$ cm, $a = 5$ mm, and $\ell_a = 1$ mm, calculate

(a) The force across the gap when $I_2 = 0$ and the relative permeability of the toroid is 300

(b) The force on the conductor when $I_2 = 2$ mA and the permeability of the toroid is infinite. Neglect fringing in the gap in both cases.

8.42 A section of an electromagnet with a plate below it carrying a load is shown in Figure 8.42. The electromagnet has a contact area of 200 cm^2 per pole with the middle pole having a winding of 1000 turns with $I = 3$ A. Calculate the maximum mass that can be lifted. Assume that the reluctance of the electromagnet and the plate is negligible.

8.43 Figure 8.43 shows the cross section of an electromechanical system in which the plunger moves freely between two nonmagnetic sleeves. Assuming that all legs have the same cross-sectional area S, show that

$$\mathbf{F} = -\frac{2N^2I^2\mu_oS}{(a + 2x)^2}\mathbf{a}_x$$

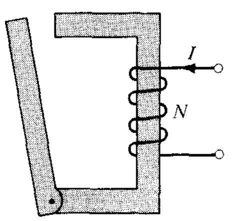


Figure 8.40 For Problem 8.40.

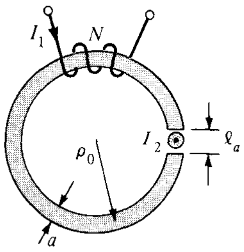


Figure 8.41 For Problem 8.41.

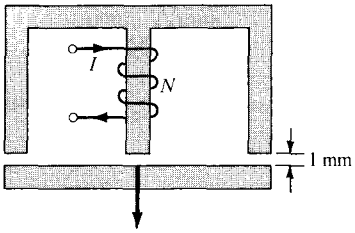


Figure 8.42 For Problem 8.42.

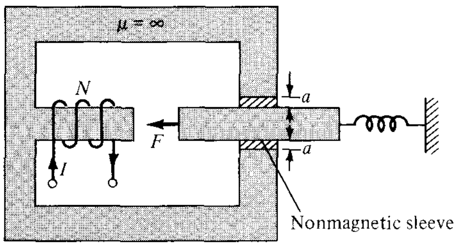


Figure 8.43 For Problem 8.43.

PART 4

WAVES AND APPLICATIONS

Chapter 9

MAXWELL'S EQUATIONS

Do you want to be a hero? Don't be the kind of person who watches others do great things or doesn't know what's happening. Go out and make things happen. The people who get things done have a burning desire to make things happen, get ahead, serve more people, become the best they can possibly be, and help improve the world around them.

—GLENN VAN EKEREN

9.1 INTRODUCTION

In Part II (Chapters 4 to 6) of this text, we mainly concentrated our efforts on electrostatic fields denoted by $\mathbf{E}(x, y, z)$; Part III (Chapters 7 and 8) was devoted to magnetostatic fields represented by $\mathbf{H}(x, y, z)$. We have therefore restricted our discussions to static, or time-invariant, EM fields. Henceforth, we shall examine situations where electric and magnetic fields are dynamic, or time varying. It should be mentioned first that in static EM fields, electric and magnetic fields are independent of each other whereas in dynamic EM fields, the two fields are interdependent. In other words, a time-varying electric field necessarily involves a corresponding time-varying magnetic field. Second, time-varying EM fields, represented by $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$, are of more practical value than static EM fields. However, familiarity with static fields provides a good background for understanding dynamic fields. Third, recall that electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying currents such as shown in Figure 9.1. Any pulsating current will produce radiation (time-varying fields). It is worth noting that pulsating current of the type shown in Figure 9.1(b) is the cause of radiated emission in digital logic boards. In summary:

stationary charges	$\rightarrow$ electrostatic fields
steady currents	$\rightarrow$ magnetostatic fields
time-varying currents	$\rightarrow$ electromagnetic fields (or waves)

Our aim in this chapter is to lay a firm foundation for our subsequent studies. This will involve introducing two major concepts: (1) electromotive force based on Faraday's experiments, and (2) displacement current, which resulted from Maxwell's hypothesis. As a result of these concepts, Maxwell's equations as presented in Section 7.6 and the boundary

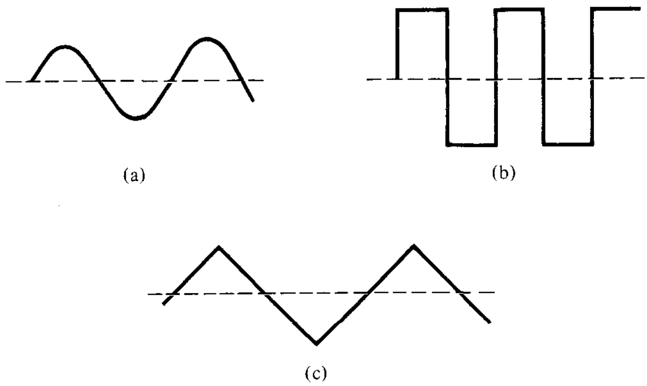


Figure 9.1 Various types of time-varying current: **(a)** sinusoidal, **(b)** rectangular, **(c)** triangular.

conditions for static EM fields will be modified to account for the time variation of the fields. It should be stressed that Maxwell's equations summarize the laws of electromagnetism and shall be the basis of our discussions in the remaining part of the text. For this reason, Section 9.5 should be regarded as the heart of this text.

9.2 FARADAY'S LAW

After Oersted's experimental discovery (upon which Biot–Savart and Ampere based their laws) that a steady current produces a magnetic field, it seemed logical to find out if magnetism would produce electricity. In 1831, about 11 years after Oersted's discovery, Michael Faraday in London and Joseph Henry in New York discovered that a time-varying magnetic field would produce an electric current.[1]

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (called *electromotive force* or simply emf) in a closed circuit, which causes a flow of current.

> Faraday discovered that the **induced emf,** V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called *Faraday's law,* and it can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \tag{9.1}$$

where N is the number of turns in the circuit and Ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux produc-

[1]For details on the experiments of Michael Faraday (1791–1867) and Joseph Henry (1797–1878), see W. F. Magie, *A Source Book in Physics.* Cambridge, MA: Harvard Univ. Press, 1963, pp. 472–519.

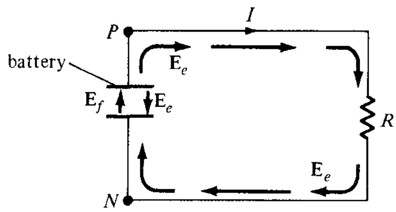


Figure 9.2 A circuit showing emf-producing field $\mathbf{E}_f$ and electrostatic field $\mathbf{E}_e$.

ing it. This is known as *Lenz's law,*[2] and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field.

Recall that we described an electric field as one in which electric charges experience force. The electric fields considered so far are caused by electric charges; in such fields, the flux lines begin and end on the charges. However, there are other kinds of electric fields not directly caused by electric charges. These are emf-produced fields. Sources of emf include electric generators, batteries, thermocouples, fuel cells, and photovoltaic cells, which all convert nonelectrical energy into electrical energy.

Consider the electric circuit of Figure 9.2, where the battery is a source of emf. The electrochemical action of the battery results in an emf-produced field $\mathbf{E}_f$. Due to the accumulation of charge at the battery terminals, an electrostatic field $\mathbf{E}_e$ ($= -\nabla V$) also exists. The total electric field at any point is

$$\mathbf{E} = \mathbf{E}_f + \mathbf{E}_e \tag{9.2}$$

Note that $\mathbf{E}_f$ is zero outside the battery, $\mathbf{E}_f$ and $\mathbf{E}_e$ have opposite directions in the battery, and the direction of $\mathbf{E}_e$ inside the battery is opposite to that outside it. If we integrate eq. (9.2) over the closed circuit,

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \oint_L \mathbf{E}_f \cdot d\mathbf{l} + 0 = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} \qquad \text{(through battery)} \tag{9.3a}$$

where $\oint \mathbf{E}_e \cdot d\mathbf{l} = 0$ because $\mathbf{E}_e$ is conservative. The emf of the battery is the line integral of the emf-produced field; that is,

$$V_{\text{emf}} = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} = -\int_N^P \mathbf{E}_e \cdot d\mathbf{l} = IR \tag{9.3b}$$

since $\mathbf{E}_f$ and $\mathbf{E}_e$ are equal but opposite within the battery (see Figure 9.2). It may also be regarded as the potential difference $(V_P - V_N)$ between the battery's open-circuit terminals. It is important to note that:

1. An electrostatic field $\mathbf{E}_e$ cannot maintain a steady current in a closed circuit since $\oint_L \mathbf{E}_e \cdot d\mathbf{l} = 0 = IR$.
2. An emf-produced field $\mathbf{E}_f$ is nonconservative.
3. Except in electrostatics, voltage and potential difference are usually not equivalent.

[2]After Heinrich Friedrich Emil Lenz (1804–1865), a Russian professor of physics.

9.3 TRANSFORMER AND MOTIONAL EMFs

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn ($N = 1$), eq. (9.1) becomes

$$\boxed{V_{\text{emf}} = -\frac{d\Psi}{dt}} \tag{9.4}$$

In terms of **E** and **B**, eq. (9.4) can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{9.5}$$

where Ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by the closed path L. It is clear from eq. (9.5) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that $d\mathbf{l}$ and $d\mathbf{S}$ in eq. (9.5) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in Figure 9.3. The variation of flux with time as in eq. (9.1) or eq. (9.5) may be caused in three ways:

1. By having a stationary loop in a time-varying **B** field
2. By having a time-varying loop area in a static **B** field
3. By having a time-varying loop area in a time-varying **B** field.

Each of these will be considered separately.

A. Stationary Loop in Time-Varying B Field (transformer emf)

This is the case portrayed in Figure 9.3 where a stationary conducting loop is in a time-varying magnetic **B** field. Equation (9.5) becomes

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}} \tag{9.6}$$

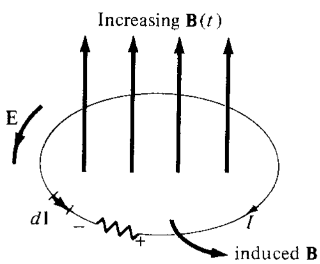


Figure 9.3 Induced emf due to a stationary loop in a time-varying **B** field.

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (9.6), we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{9.7}$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{9.8}$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time-varying **E** field is not conservative ($\nabla \times \mathbf{E} \neq 0$). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field. Observe that Figure 9.3 obeys Lenz's law; the induced current I flows such as to produce a magnetic field that opposes $\mathbf{B}(t)$.

B. Moving Loop in Static B Field (Motional emf)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. We recall from eq. (8.2) that the force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

We define the *motional electric field* $\mathbf{E}_m$ as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B} \tag{9.9}$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \tag{9.10}$$

This type of emf is called *motional emf* or *flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators. Figure 9.4 illustrates a two-pole dc machine with one armature coil and a two-bar commutator. Although the analysis of the d.c. machine is beyond the scope of this text, we can see that voltage is generated as the coil rotates within the magnetic field. Another example of motional emf is illustrated in Figure 9.5, where a rod is moving between a pair

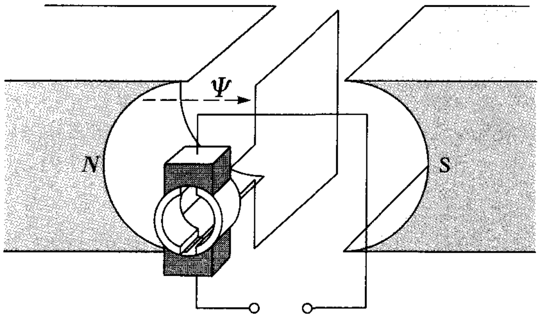


Figure 9.4 A direct-current machine.

of rails. In this example, **B** and **u** are perpendicular, so eq. (9.9) in conjunction with eq. (8.2) becomes

$$\mathbf{F}_m = I\boldsymbol{\ell} \times \mathbf{B} \tag{9.11}$$

or

$$F_m = I\ell B \tag{9.12}$$

and eq. (9.10) becomes

$$V_{\text{emf}} = uB\ell \tag{9.13}$$

By applying Stokes's theorem to eq. (9.10)

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

or

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})} \tag{9.14}$$

Notice that unlike eq. (9.6), there is no need for a negative sign in eq. (9.10) because Lenz's law is already accounted for.

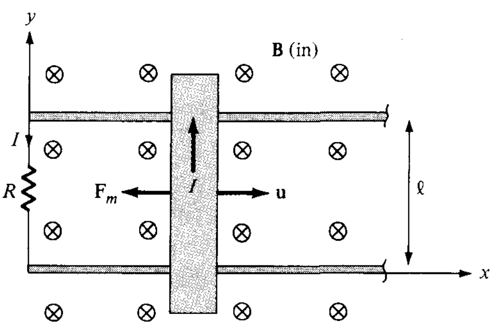


Figure 9.5 Induced emf due to a moving loop in a static **B** field.

To apply eq. (9.10) is not always easy; some care must be exercised. The following points should be noted:

1. The integral in eq. (9.10) is zero along the portion of the loop where $\mathbf{u} = 0$. Thus $d\mathbf{l}$ is taken along the portion of the loop that is cutting the field (along the rod in Figure 9.5), where $\mathbf{u}$ has nonzero value.
2. The direction of the induced current is the same as that of $\mathbf{E}_m$ or $\mathbf{u} \times \mathbf{B}$. The limits of the integral in eq. (9.10) are selected in the opposite direction to the induced current thereby satisfying Lenz's law. In eq. (9.13), for example, the integration over L is along $-\mathbf{a}_y$ whereas induced current flows in the rod along $\mathbf{a}_y$.

C. Moving Loop in Time-Varying Field

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining eqs. (9.6) and (9.10) gives the total emf as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \tag{9.15}$$

or from eqs. (9.8) and (9.14),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{9.16}$$

Note that eq. (9.15) is equivalent to eq. (9.4), so V_{emf} can be found using either eq. (9.15) or (9.4). In fact, eq. (9.4) can always be applied in place of eqs. (9.6), (9.10), and (9.15).

EXAMPLE 9.1

A conducting bar can slide freely over two conducting rails as shown in Figure 9.6. Calculate the induced voltage in the bar

(a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4 \cos 10^6 t \, \mathbf{a}_z$ mWb/m^2

(b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z$ mWb/m^2

(c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4 \cos (10^6 t - y) \, \mathbf{a}_z$ mWb/m^2

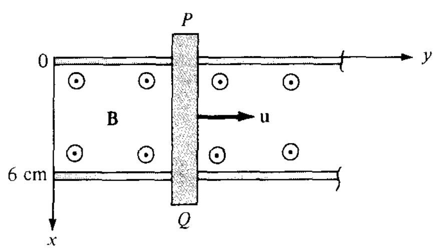


Figure 9.6 For Example 9.1.

Solution:

(a) In this case, we have transformer emf given by

$$
\begin{aligned}
V_{\text{emf}} &= -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^6) \sin 10^6 t \, dx \, dy \\
&= 4(10^3)(0.08)(0.06) \sin 10^6 t \\
&= 19.2 \sin 10^6 t \text{ V}
\end{aligned}
$$

The polarity of the induced voltage (according to Lenz's law) is such that point P on the bar is at lower potential than Q when $\mathbf{B}$ is increasing.

(b) This is the case of motional emf:

$$
\begin{aligned}
V_{\text{emf}} &= \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^{0} (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x \\
&= -uB\ell = -20(4.10^{-3})(0.06) \\
&= -4.8 \text{ mV}
\end{aligned}
$$

(c) Both transformer emf and motional emf are present in this case. This problem can be solved in two ways.

Method 1: Using eq. (9.15)

$$
V_{\text{emf}} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \tag{9.1.1}
$$

$$
\begin{aligned}
&= \int_{x=0}^{0.06} \int_0^{y} 4.10^{-3}(10^6) \sin(10^6 t - y')dy' \, dx \\
&\quad + \int_{0.06}^{0} [20\mathbf{a}_y \times 4.10^{-3} \cos(10^6 t - y)\mathbf{a}_z] \cdot dx \, \mathbf{a}_x \\
&= 240 \cos(10^6 t - y') \Big|_0^y - 80(10^{-3})(0.06) \cos(10^6 t - y) \\
&= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos(10^6 t - y) \\
&\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t
\end{aligned} \tag{9.1.2}
$$

because the motional emf is negligible compared with the transformer emf. Using trigonometric identity

$$
\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}
$$

$$
V_{\text{emf}} = 480 \sin\left(10^6 t - \frac{y}{2}\right) \sin \frac{y}{2} \text{ V} \tag{9.1.3}
$$

Method 2: Alternatively we can apply eq. (9.4), namely,

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t} \tag{9.1.4}$$

where

$$\begin{aligned}
\Psi &= \int \mathbf{B} \cdot d\mathbf{S} \\
&= \int_{y=0}^{y} \int_{x=0}^{0.06} 4 \cos(10^6 t - y)\, dx\, dy \\
&= -4(0.06) \sin(10^6 t - y) \Big|_{y=0}^{y} \\
&= -0.24 \sin(10^6 t - y) + 0.24 \sin 10^6 t \text{ mWb}
\end{aligned}$$

But

$$\frac{dy}{dt} = u \rightarrow y = ut = 20t$$

Hence,

$$\Psi = -0.24 \sin(10^6 t - 20t) + 0.24 \sin 10^6 t \text{ mWb}$$

$$\begin{aligned}
V_{\text{emf}} &= -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20) \cos(10^6 t - 20t) - 0.24(10^6) \cos 10^6 t \text{ mV} \\
&\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t \text{ V}
\end{aligned} \tag{9.1.5}$$

which is the same result in (9.1.2). Notice that in eq. (9.1.1), the dependence of y on time is taken care of in $\int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$, and we should not be bothered by it in $\partial \mathbf{B}/\partial t$. Why? Because the loop is assumed stationary when computing the transformer emf. This is a subtle point one must keep in mind in applying eq. (9.1.1). For the same reason, the second method is always easier.

PRACTICE EXERCISE 9.1

Consider the loop of Figure 9.5. If $\mathbf{B} = 0.5\mathbf{a}_z$ Wb/m^2, $R = 20\ \Omega$, $\ell = 10$ cm, and the rod is moving with a constant velocity of $8\mathbf{a}_x$ m/s, find

(a) The induced emf in the rod
(b) The current through the resistor
(c) The motional force on the rod
(d) The power dissipated by the resistor.

Answer: (a) 0.4 V, (b) 20 mA, (c) $-\mathbf{a}_x$ mN, (d) 8 mW.

EXAMPLE 9.2

The loop shown in Figure 9.7 is inside a uniform magnetic field $\mathbf{B} = 50\,\mathbf{a}_x$ mWb/m^2. If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz-plane at time $t = 0$, find

(a) The induced emf at $t = 1$ ms

(b) The induced current at $t = 3$ ms

Solution:

(a) Since the **B** field is time invariant, the induced emf is motional, that is,

$$V_{\text{emf}} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

where

$$d\mathbf{l} = d\mathbf{l}_{DC} = dz\,\mathbf{a}_z, \qquad \mathbf{u} = \frac{d\mathbf{l}'}{dt} = \frac{\rho\, d\phi}{dt}\mathbf{a}_\phi = \rho\omega\mathbf{a}_\phi$$

$$\rho = AD = 4 \text{ cm}, \qquad \omega = 2\pi f = 100\pi$$

As **u** and $d\mathbf{l}$ are in cylindrical coordinates, we transform **B** into cylindrical coordinates using eq. (2.9):

$$\mathbf{B} = B_o\mathbf{a}_x = B_o(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)$$

where $B_o = 0.05$. Hence,

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\omega & 0 \\ B_o\cos\phi & -B_o\sin\phi & 0 \end{vmatrix} = -\rho\omega B_o\cos\phi\,\mathbf{a}_z$$

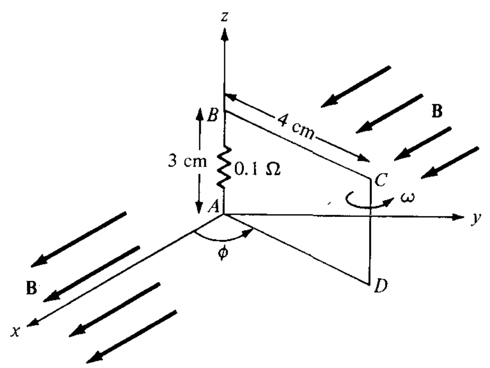


Figure 9.7 For Example 9.2; polarity is for increasing emf.

and

$$(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\rho\omega B_o \cos\phi\, dz = -0.04(100\pi)(0.05)\cos\phi\, dz$$
$$= -0.2\pi \cos\phi\, dz$$

$$V_{emf} = \int_{z=0}^{0.03} -0.2\pi\cos\phi\, dz = -6\pi\cos\phi \text{ mV}$$

To determine ϕ, recall that

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_o$$

where C_o is an integration constant. At $t = 0$, $\phi = \pi/2$ because the loop is in the yz-plane at that time, $C_o = \pi/2$. Hence,

$$\phi = \omega t + \frac{\pi}{2}$$

and

$$V_{emf} = -6\pi\cos\left(\omega t + \frac{\pi}{2}\right) = 6\pi\sin(100\pi t) \text{ mV}$$

$$\text{At } t = 1 \text{ ms},\ V_{emf} = 6\pi\sin(0.1\pi) = 5.825 \text{ mV}$$

(b) The current induced is

$$i = \frac{V_{emf}}{R} = 60\pi\sin(100\pi t) \text{ mA}$$

At $t = 3$ ms,

$$i = 60\pi\sin(0.3\pi) \text{ mA} = 0.1525 \text{ A}$$

PRACTICE EXERCISE 9.2

Rework Example 9.2 with everything the same except that the **B** field is changed to:

(a) $\mathbf{B} = 50\mathbf{a}_y$ mWb/m²—that is, the magnetic field is oriented along the y-direction

(b) $\mathbf{B} = 0.02t\,\mathbf{a}_x$ Wb/m²—that is, the magnetic field is time varying.

Answer: (a) −17.93 mV, −0.1108 A, (b) 20.5 μV, −41.92 mA.

EXAMPLE 9.3

The magnetic circuit of Figure 9.8 has a uniform cross section of 10^{-3} m². If the circuit is energized by a current $i_1(t) = 3\sin 100\pi t$ A in the coil of $N_1 = 200$ turns, find the emf induced in the coil of $N_2 = 100$ turns. Assume that $\mu = 500\,\mu_o$.

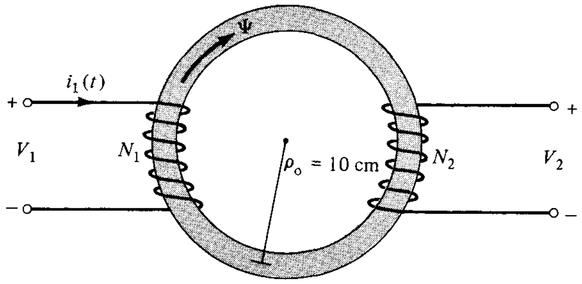


Figure 9.8 Magnetic circuit of Example 9.3.

Solution:
The flux in the circuit is

$$\Psi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{N_1 i_1}{\ell/\mu S} = \frac{N_1 i_1 \mu S}{2\pi\rho_o}$$

According to Faraday's law, the emf induced in the second coil is

$$\begin{aligned} V_2 &= -N_2 \frac{d\Psi}{dt} = -\frac{N_1 N_2 \mu S}{2\pi\rho_o} \frac{di_1}{dt} \\ &= -\frac{100 \cdot (200) \cdot (500) \cdot (4\pi \times 10^{-7}) \cdot (10^{-3}) \cdot 300\pi \cos 100\pi t}{2\pi \cdot (10 \times 10^{-2})} \\ &= -6\pi \cos 100\pi t \text{ V} \end{aligned}$$

PRACTICE EXERCISE 9.3

A magnetic core of uniform cross section 4 cm^2 is connected to a 120-V, 60-Hz generator as shown in Figure 9.9. Calculate the induced emf V_2 in the secondary coil.

Answer: 72 V

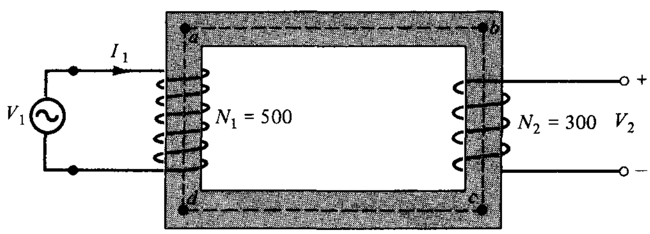


Figure 9.9 For Practice Exercise 9.3.

9.4 DISPLACEMENT CURRENT

In the previous section, we have essentially reconsidered Maxwell's curl equation for electrostatic fields and modified it for time-varying situations to satisfy Faraday's law. We shall now reconsider Maxwell's curl equation for magnetic fields (Ampere's circuit law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.17}$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \tag{9.18}$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \tag{9.19}$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \tag{9.20}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \tag{9.21}$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{9.22a}$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}} \tag{9.22b}$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{9.23}$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current

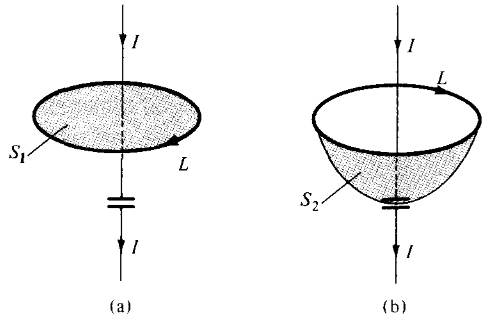


Figure 9.10 Two surfaces of integration showing the need for $\mathbf{J}_d$ in Ampere's circuit law.

density ($\mathbf{J} = \sigma\mathbf{E}$).[3] The insertion of $\mathbf{J}_d$ into eq. (9.17) was one of the major contributions of Maxwell. Without the term $\mathbf{J}_d$, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, $\mathbf{J}_d$ is usually neglected compared with $\mathbf{J}$. However, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. (9.23) could not be verified experimentally. It was years later that Hertz succeeded in generating and detecting radio waves thereby verifying eq. (9.23). This is one of the rare situations where mathematical argument paved the way for experimental investigation.

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \tag{9.24}$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates. This example, shown in Figure 9.10, serves to illustrate the need for the displacement current. Applying an unmodified form of Ampere's circuit law to a closed path L shown in Figure 9.10(a) gives

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{enc} = I \tag{9.25}$$

where I is the current through the conductor and S_1 is the flat surface bounded by L. If we use the balloon-shaped surface S_2 that passes between the capacitor plates, as in Figure 9.10(b),

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} = I_{enc} = 0 \tag{9.26}$$

because no conduction current ($\mathbf{J} = 0$) flows through S_2. This is contradictory in view of the fact that the same closed path L is used. To resolve the conflict, we need to include the

[3]*Recall that we also have* $\mathbf{J} = \rho_v \mathbf{u}$ *as the convection current density.*

displacement current in Ampere's circuit law. The total current density is $\mathbf{J} + \mathbf{J}_d$. In eq. (9.25), $\mathbf{J}_d = 0$ so that the equation remains valid. In eq. (9.26), $\mathbf{J} = 0$ so that

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J}_d \cdot d\mathbf{S} = \frac{d}{dt} \int_{S_2} \mathbf{D} \cdot d\mathbf{S} = \frac{dQ}{dt} = I \qquad (9.27)$$

So we obtain the same current for either surface though it is conduction current in S_1 and displacement current in S_2.

EXAMPLE 9.4

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage $50 \sin 10^3 t$ V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_o$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S \frac{d\rho_s}{dt} = S \frac{dD}{dt} = \varepsilon S \frac{dE}{dt} = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$
$$= 147.4 \cos 10^3 t \text{ nA}$$

PRACTICE EXERCISE 9.4

In free space, $\mathbf{E} = 20 \cos(\omega t - 50x)\, \mathbf{a}_y$ V/m. Calculate

(a) $\mathbf{J}_d$

(b) $\mathbf{H}$

(c) ω

Answer: (a) $-20\omega\varepsilon_o \sin(\omega t - 50x)\, \mathbf{a}_y$ A/m^2, (b) $0.4\, \omega\varepsilon_o \cos(\omega t - 50x)\, \mathbf{a}_z$ A/m, (c) 1.5×10^{10} rad/s.

9.5 MAXWELL'S EQUATIONS IN FINAL FORMS

James Clerk Maxwell (1831–1879) is regarded as the founder of electromagnetic theory in its present form. Maxwell's celebrated work led to the discovery of electromagnetic waves.[4] Through his theoretical efforts over about 5 years (when he was between 35 and 40), Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until they were later confirmed by Heinrich Rudolf Hertz (1857–1894), a German physics professor. Hertz was successful in generating and detecting radio waves.

The laws of electromagnetism that Maxwell put together in the form of four equations were presented in Table 7.2 in Section 7.6 for static conditions. The more generalized forms of these equations are those for time-varying conditions shown in Table 9.1. We notice from the table that the divergence equations remain the same while the curl equations have been modified. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to be "qualified" as an electromagnetic field, it must satisfy all four Maxwell's equations. The importance of Maxwell's equations cannot be overemphasized because they summarize all known laws of electromagnetism. We shall often refer to them in the remaining part of this text.

Since this section is meant to be a compendium of our discussion in this text, it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{9.28}$$

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

[4]The work of James Clerk Maxwell (1831–1879), a Scottish physicist, can be found in his book, *A Treatise on Electricity and Magnetism.* New York: Dover, vols. 1 and 2, 1954.

is associated with Maxwell's equations. Also the equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{9.29}$$

is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_o \mathbf{E} + \mathbf{P} \tag{9.30a}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_o (\mathbf{H} + \mathbf{M}) \tag{9.30b}$$

$$\mathbf{J} = \sigma \mathbf{E} + \rho_v \mathbf{u} \tag{9.30c}$$

hold for time-varying fields. Consequently, the boundary conditions

$$E_{1t} = E_{2t} \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_{n12} = 0 \tag{9.31a}$$

$$H_{1t} - H_{2t} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \tag{9.31b}$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = \rho_s \tag{9.31c}$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_{n12} = 0 \tag{9.31d}$$

remain valid for time-varying fields. However, for a perfect conductor ($\sigma \simeq \infty$) in a time-varying field,

$$\mathbf{E} = 0, \qquad \mathbf{H} = 0, \qquad \mathbf{J} = 0 \tag{9.32}$$

and hence,

$$\mathbf{B}_n = 0, \qquad \mathbf{E}_t = 0 \tag{9.33}$$

For a perfect dielectric ($\sigma \simeq 0$), eq. (9.31) holds except that $\mathbf{K} = 0$. Though eqs. (9.28) to (9.33) are not Maxwell's equations, they are associated with them.

To complete this summary section, we present a structure linking the various potentials and vector fields of the electric and magnetic fields in Figure 9.11. This electromagnetic flow diagram helps with the visualization of the basic relationships between field quantities. It also shows that it is usually possible to find alternative formulations, for a given problem, in a relatively simple manner. It should be noted that in Figures 9.10(b) and (c), we introduce ρ^m as the free magnetic density (similar to ρ_v), which is, of course, zero, $\mathbf{A}_e$ as the magnetic current density (analogous to $\mathbf{J}$). Using terms from stress analysis, the principal relationships are typified as:

(a) compatibility equations

$$\nabla \cdot \mathbf{B} = \rho^m = 0 \tag{9.34}$$

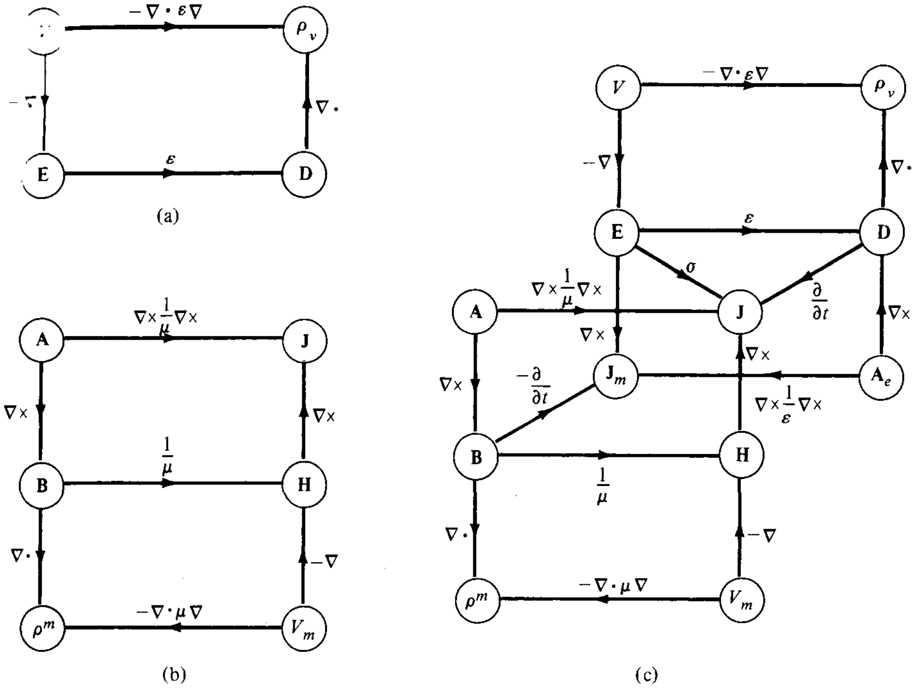


Figure 9.11 Electromagnetic flow diagram showing the relationship between the potentials and vector fields: **(a)** electrostatic system, **(b)** magnetostatic system, **(c)** electromagnetic system. [Adapted with permission from IEE Publishing Dept.]

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}_m \tag{9.35}$$

(b) constitutive equations

$$\mathbf{B} = \mu \mathbf{H} \tag{9.36}$$

and

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{9.37}$$

(c) equilibrium equations

$$\nabla \cdot \mathbf{D} = \rho_v \tag{9.38}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.39}$$

9.6 TIME-VARYING POTENTIALS

For static EM fields, we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v \, dv}{4\pi\varepsilon R} \tag{9.40}$$

and the magnetic vector potential as

$$\mathbf{A} = \int_v \frac{\mu J \, dv}{4\pi R} \tag{9.41}$$

We would like to examine what happens to these potentials when the fields are time varying. Recall that $\mathbf{A}$ was defined from the fact that $\nabla \cdot \mathbf{B} = 0$, which still holds for time-varying fields. Hence the relation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{9.42}$$

holds for time-varying situations. Combining Faraday's law in eq. (9.8) with eq. (9.42) gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \tag{9.43a}$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{9.43b}$$

Since the curl of the gradient of a scalar field is identically zero (see Practice Exercise 3.10), the solution to eq. (9.43b) is

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{9.44}$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{9.45}$$

From eqs. (9.42) and (9.45), we can determine the vector fields $\mathbf{B}$ and $\mathbf{E}$ provided that the potentials $\mathbf{A}$ and V are known. However, we still need to find some expressions for $\mathbf{A}$ and V similar to those in eqs. (9.40) and (9.41) that are suitable for time-varying fields.

From Table 9.1 or eq. (9.38) we know that $\nabla \cdot \mathbf{D} = \rho_v$ is valid for time-varying conditions. By taking the divergence of eq. (9.45) and making use of eqs. (9.37) and (9.38), we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$$

or

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\varepsilon} \tag{9.46}$$

Taking the curl of eq. (9.42) and incorporating eqs. (9.23) and (9.45) results in

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J} + \varepsilon\mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \mu \mathbf{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \end{aligned} \tag{9.47}$$

where $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ have been assumed. By applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{9.48}$$

to eq. (9.47),

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{9.49}$$

A vector field is uniquely defined when its curl and divergence are specified. The curl of **A** has been specified by eq. (9.42); for reasons that will be obvious shortly, we may choose the divergence of **A** as

$$\boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}} \tag{9.50}$$

This choice relates **A** and V and it is called the *Lorentz condition for potentials.* We had this in mind when we chose $\nabla \cdot \mathbf{A} = 0$ for magnetostatic fields in eq. (7.59). By imposing the Lorentz condition of eq. (9.50), eqs. (9.46) and (9.49), respectively, become

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}} \tag{9.51}$$

and

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}} \tag{9.52}$$

which are *wave equations* to be discussed in the next chapter. The reason for choosing the Lorentz condition becomes obvious as we examine eqs. (9.51) and (9.52). It uncouples eqs. (9.46) and (9.49) and also produces a symmetry between eqs. (9.51) and (9.52). It can be shown that the Lorentz condition can be obtained from the continuity equation; therefore, our choice of eq. (9.50) is not arbitrary. Notice that eqs. (6.4) and (7.60) are special static cases of eqs. (9.51) and (9.52), respectively. In other words, potentials V and **A** satisfy Poisson's equations for time-varying conditions. Just as eqs. (9.40) and (9.41) are the solutions, or the integral forms of eqs. (6.4) and (7.60), it can be shown that the solutions[5] to eqs. (9.51) and (9.52) are

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi\varepsilon R} \tag{9.53}$$

and

$$\mathbf{A} = \int_v \frac{\mu[\mathbf{J}]\, dv}{4\pi R} \tag{9.54}$$

The term $[\rho_v]$ (or $[\mathbf{J}]$) means that the time t in $\rho_v(x, y, z, t)$ [or $\mathbf{J}(x, y, z, t)$] is replaced by the *retarded time* t' given by

$$t' = t - \frac{R}{\mu} \tag{9.55}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$ and

$$u = \frac{1}{\sqrt{\mu\varepsilon}} \tag{9.56}$$

is the velocity of wave propagation. In free space, $u = c \simeq 3 \times 10^8$ m/s is the speed of light in a vacuum. Potentials V and $\mathbf{A}$ in eqs. (9.53) and (9.54) are, respectively, called the *retarded electric scalar potential* and the *retarded magnetic vector potential.* Given ρ_v and $\mathbf{J}$, V and $\mathbf{A}$ can be determined using eqs. (9.53) and (9.54); from V and $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ can be determined using eqs. (9.45) and (9.42), respectively.

9.7 TIME-HARMONIC FIELDS

So far, our time dependence of EM fields has been arbitrary. To be specific, we shall assume that the fields are *time harmonic.*

A **time-harmonic field** is one that varies periodically or sinusoidally with time.

Not only is sinusoidal analysis of practical value, it can be extended to most waveforms by Fourier transform techniques. Sinusoids are easily expressed in phasors, which are more convenient to work with. Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor.

A *phasor* z is a complex number that can be written as

$$z = x + jy = r \underline{/\phi} \tag{9.57}$$

[5] For example, see D. K. Cheng, *Field and Wave Electromagnetics*. Reading, MA: Addison-Wesley, 1983, pp. 291–292.

or

$$z = r\,e^{j\phi} = r\,(\cos\phi + j\sin\phi) \tag{9.58}$$

where $j = \sqrt{-1}$, x is the real part of z, y is the imaginary part of z, r is the magnitude of z, given by

$$r = |z| = \sqrt{x^2 + y^2} \tag{9.59}$$

and ϕ is the phase of z, given by

$$\phi = \tan^{-1}\frac{y}{x} \tag{9.60}$$

Here x, y, z, r, and ϕ should not be mistaken as the coordinate variables although they look similar (different letters could have been used but it is hard to find better ones). The phasor z can be represented in *rectangular form* as $z = x + jy$ or in *polar form* as $z = r\,\underline{/\phi} = r\,e^{j\phi}$. The two forms of representing z are related in eqs. (9.57) to (9.60) and illustrated in Figure 9.12. Addition and subtraction of phasors are better performed in rectangular form; multiplication and division are better done in polar form.

Given complex numbers

$$z = x + jy = r\,\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\,\underline{/\phi_1}, \qquad \text{and} \qquad z_2 = x_2 + jy_2 = r_2\,\underline{/\phi_2}$$

the following basic properties should be noted.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.61a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.61b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2\,\underline{/\phi_1 + \phi_2} \tag{9.61c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\underline{/\phi_1 - \phi_2} \tag{9.61d}$$

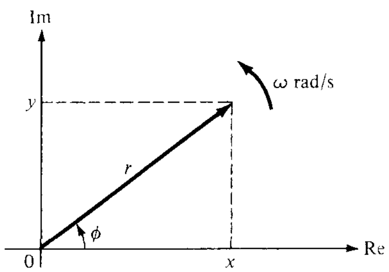


Figure 9.12 Representation of a phasor $z = x + jy = r\,\underline{/\phi}$.

Square Root:

$$\sqrt{z} = \sqrt{r} \,\underline{/\phi/2} \tag{9.61e}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{9.61f}$$

Other properties of complex numbers can be found in Appendix A.2.

To introduce the time element, we let

$$\phi = \omega t + \theta \tag{9.62}$$

where θ may be a function of time or space coordinates or a constant. The real (Re) and imaginary (Im) parts of

$$re^{j\phi} = re^{j\theta}e^{j\omega t} \tag{9.63}$$

are, respectively, given by

$$\text{Re}\,(re^{j\phi}) = r\cos\,(\omega t + \theta) \tag{9.64a}$$

and

$$\text{Im}\,(re^{j\phi}) = r\sin\,(\omega t + \theta) \tag{9.64b}$$

Thus a sinusoidal current $I(t) = I_o\cos(\omega t + \theta)$, for example, equals the real part of $I_o e^{j\theta}e^{j\omega t}$. The current $I'(t) = I_o\sin(\omega t + \theta)$, which is the imaginary part of $I_o e^{j\theta}e^{j\omega t}$, can also be represented as the real part of $I_o e^{j\theta}e^{j\omega t}e^{-j90°}$ because $\sin\alpha = \cos(\alpha - 90°)$. However, in performing our mathematical operations, we must be consistent in our use of either the real part or the imaginary part of a quantity but not both at the same time.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the *phasor* current, denoted by I_s; that is,

$$I_s = I_o e^{j\theta} = I_0 \underline{/\theta} \tag{9.65}$$

where the subscript s denotes the phasor form of $I(t)$. Thus $I(t) = I_o\cos(\omega t + \theta)$, the *instantaneous form*, can be expressed as

$$I(t) = \text{Re}\,(I_s e^{j\omega t}) \tag{9.66}$$

In general, a phasor could be scalar or vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the *phasor form* of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as

$$\boxed{\mathbf{A} = \text{Re}\,(\mathbf{A}_s e^{j\omega t})} \tag{9.67}$$

For example, if $\mathbf{A} = A_o\cos\,(\omega t - \beta x)\,\mathbf{a}_y$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \text{Re}\,(A_o e^{-j\beta x}\,\mathbf{a}_y e^{j\omega t}) \tag{9.68}$$

Comparing this with eq. (9.67) indicates that the phasor form of $\mathbf{A}$ is

$$\mathbf{A}_s = A_o e^{-j\beta x}\,\mathbf{a}_y \tag{9.69}$$

Notice from eq. (9.67) that

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial}{\partial t}\,\text{Re}\,(\mathbf{A}_s e^{j\omega t}) \\ &= \text{Re}\,(j\omega \mathbf{A}_s e^{j\omega t})\end{aligned} \tag{9.70}$$

showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$. That is,

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s \tag{9.71}$$

Similarly,

$$\int \mathbf{A}\,\partial t \rightarrow \frac{\mathbf{A}_s}{j\omega} \tag{9.72}$$

Note that the real part is chosen in eq. (9.67) as in circuit analysis; the imaginary part could equally have been chosen. Also notice the basic difference between the instantaneous form $\mathbf{A}(x, y, z, t)$ and its phasor form $\mathbf{A}_s(x, y, z)$; the former is time dependent and real whereas the latter is time invariant and generally complex. It is easier to work with $\mathbf{A}_s$ and obtain $\mathbf{A}$ from $\mathbf{A}_s$ whenever necessary using eq. (9.67).

We shall now apply the phasor concept to time-varying EM fields. The fields quantities $\mathbf{E}(x, y, z, t)$, $\mathbf{D}(x, y, z, t)$, $\mathbf{H}(x, y, z, t)$, $\mathbf{B}(x, y, z, t)$, $\mathbf{J}(x, y, z, t)$, and $\rho_v(x, y, z, t)$ and their derivatives can be expressed in phasor form using eqs. (9.67) and (9.71). In phasor form, Maxwell's equations for time-harmonic EM fields in a linear, isotropic, and homogeneous medium are presented in Table 9.2. From Table 9.2, note that the time factor $e^{j\omega t}$ disappears because it is associated with every term and therefore factors out, resulting in time-independent equations. Herein lies the justification for using phasors; the time factor can be suppressed in our analysis of time-harmonic fields and inserted when necessary. Also note that in Table 9.2, the time factor $e^{j\omega t}$ has been assumed. It is equally possible to have assumed the time factor $e^{-j\omega t}$, in which case we would need to replace every j in Table 9.2 with $-j$.

TABLE 9.2 Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\,dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

EXAMPLE 9.5

Evaluate the complex numbers

(a) $z_1 = \dfrac{j(3 - j4)^*}{(-1 + j6)(2 + j)^2}$

(b) $z_2 = \left[\dfrac{1 + j}{4 - j8}\right]^{1/2}$

Solution:

(a) This can be solved in two ways: working with z in rectangular form or polar form.

Method 1: (working in rectangular form):
Let

$$z_1 = \frac{z_3 z_4}{z_5 z_6}$$

where

$$z_3 = j$$

$$z_4 = (3 - j4)^* = \text{the complex conjugate of } (3 - j4) = 3 + j4$$

(To find the complex conjugate of a complex number, simply replace every j with $-j$.)

$$z_5 = -1 + j6$$

and

$$z_6 = (2 + j)^2 = 4 - 1 + j4 = 3 + j4$$

Hence,

$$z_3 z_4 = j(3 + j4) = -4 + j3$$

$$z_5 z_6 = (-1 + j6)(3 + j4) = -3 - j4 + j18 - 24 = -27 + j14$$

and

$$z_1 = \frac{-4 + j3}{-27 + j14}$$

Multiplying and dividing z_1 by $-27 - j14$ (rationalization), we have

$$z_1 = \frac{(-4 + j3)(-27 - j14)}{(-27 + j14)(-27 - j14)} = \frac{150 - j25}{27^2 + 14^2} = 0.1622 - j0.027 = 0.1644 \underline{/-9.46^\circ}$$

Method 2: (working in polar form):

$$z_3 = j = 1 \underline{/90^\circ}$$

$$z_4 = (3 - j4)^* = 5 \underline{/-53.13^\circ})^* = 5 \underline{/53.13^\circ}$$

$$z_5 = (-1 + j6) = \sqrt{37}\ \underline{/99.46^\circ}$$

$$z_6 = (2 + j)^2 = (\sqrt{5}\ \underline{/26.56^\circ})^2 = 5\ \underline{/53.13^\circ}$$

Hence,

$$\begin{aligned} z_1 &= \frac{(1\ \underline{/90^\circ})(5\ \underline{/53.13^\circ})}{(\sqrt{37}\ \underline{/99.46^\circ})(5\ \underline{/53.13^\circ})} \\ &= \frac{1}{\sqrt{37}}\ \underline{/90^\circ - 99.46^\circ} = 0.1644\ \underline{/-9.46^\circ} \\ &= 0.1622 - j0.027 \end{aligned}$$

as obtained before.

(b) Let

$$z_2 = \left[\frac{z_7}{z_8}\right]^{1/2}$$

where

$$z_7 = 1 + j = \sqrt{2}\ \underline{/45^\circ}$$

and

$$z_8 = 4 - j8 = 4\sqrt{5}\ \underline{/-63.4^\circ}$$

Hence

$$\begin{aligned} \frac{z_7}{z_8} &= \frac{\sqrt{2}\ \underline{/45^\circ}}{4\sqrt{5}\ \underline{/-63.4^\circ}} = \frac{\sqrt{2}}{4\sqrt{5}}\ \underline{/45^\circ - -63.4^\circ} \\ &= 0.1581\ \underline{/108.4^\circ} \end{aligned}$$

and

$$\begin{aligned} z_2 &= \sqrt{0.1581}\ \underline{/108.4^\circ/2} \\ &= 0.3976\ \underline{/54.2^\circ} \end{aligned}$$

PRACTICE EXERCISE 9.5

Evaluate these complex numbers:

(a) $j^3\left[\frac{1 + j}{2 - j}\right]^2$

(b) $6\ \underline{/30^\circ} + j5 - 3 + e^{j45^\circ}$

Answer: (a) $0.24 + j0.32$, (b) $2.903 + j8.707$.

EXAMPLE 9.6

Given that $\mathbf{A} = 10\cos(10^8 t - 10x + 60°)\,\mathbf{a}_z$ and $\mathbf{B}_s = (20/j)\,\mathbf{a}_x + 10\,e^{j2\pi x/3}\,\mathbf{a}_y$, express $\mathbf{A}$ in phasor form and $\mathbf{B}_s$ in instantaneous form.

Solution:

$$\mathbf{A} = \text{Re}\,[10e^{j(\omega t - 10x + 60°)}\,\mathbf{a}_z]$$

where $\omega = 10^8$. Hence

$$\mathbf{A} = \text{Re}\,[10e^{j(60° - 10x)}\,\mathbf{a}_z\,e^{j\omega t}] = \text{Re}\,(\mathbf{A}_s e^{j\omega t})$$

or

$$\mathbf{A}_s = 10\,e^{j(60° - 10x)}\mathbf{a}_z$$

If

$$\begin{aligned}\mathbf{B}_s &= \frac{20}{j}\,\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y = -j20\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y \\ &= 20e^{-j\pi/2}\mathbf{a}_x + 10e^{j2\pi x/3}\mathbf{a}_y\end{aligned}$$

$$\begin{aligned}\mathbf{B} &= \text{Re}\,(\mathbf{B}_s e^{j\omega t}) \\ &= \text{Re}\,[20e^{j(\omega t - \pi/2)}\mathbf{a}_x + 10e^{j(\omega t + 2\pi x/3)}\mathbf{a}_y] \\ &= 20\cos(\omega t - \pi/2)\mathbf{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y \\ &= 20\sin\omega t\,\mathbf{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\mathbf{a}_y\end{aligned}$$

PRACTICE EXERCISE 9.6

If $\mathbf{P} = 2\sin(10t + x - \pi/4)\,\mathbf{a}_y$ and $\mathbf{Q}_s = e^{jx}(\mathbf{a}_x - \mathbf{a}_z)\sin\pi y$, determine the phasor form of $\mathbf{P}$ and the instantaneous form of $\mathbf{Q}_s$.

Answer: $2e^{j(x - 3\pi/4)}\mathbf{a}_y$, $\sin\pi y\cos(\omega t + x)(\mathbf{a}_x - \mathbf{a}_z)$.

EXAMPLE 9.7

The electric field and magnetic field in free space are given by

$$\mathbf{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\,\mathbf{a}_\phi \text{ V/m}$$

$$\mathbf{H} = \frac{H_o}{\rho}\cos(10^6 t + \beta z)\,\mathbf{a}_\rho \text{ A/m}$$

Express these in phasor form and determine the constants H_o and β such that the fields satisfy Maxwell's equations.

Solution:

The instantaneous forms of **E** and **H** are written as

$$\mathbf{E} = \text{Re}\,(\mathbf{E}_s e^{j\omega t}), \qquad \mathbf{H} = \text{Re}\,(\mathbf{H}_s e^{j\omega t}) \tag{9.7.1}$$

where $\omega = 10^6$ and phasors $\mathbf{E}_s$ and $\mathbf{H}_s$ are given by

$$\mathbf{E}_s = \frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\phi, \qquad \mathbf{H}_s = \frac{H_o}{\rho}\, e^{j\beta z}\mathbf{a}_\rho \tag{9.7.2}$$

For free space, $\rho_v = 0$, $\sigma = 0$, $\varepsilon = \varepsilon_o$, and $\mu = \mu_o$ so Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \varepsilon_o \nabla \cdot \mathbf{E} = 0 \rightarrow \nabla \cdot \mathbf{E}_s = 0 \tag{9.7.3}$$

$$\nabla \cdot \mathbf{B} = \mu_o \nabla \cdot \mathbf{H} = 0 \rightarrow \nabla \cdot \mathbf{H}_s = 0 \tag{9.7.4}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \rightarrow \nabla \times \mathbf{H}_s = j\omega\varepsilon_o \mathbf{E}_s \tag{9.7.5}$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \rightarrow \nabla \times \mathbf{E}_s = -j\omega\mu_o \mathbf{H}_s \tag{9.7.6}$$

Substituting eq. (9.7.2) into eqs. (9.7.3) and (9.7.4), it is readily verified that two Maxwell's equations are satisfied; that is,

$$\nabla \cdot \mathbf{E}_s = \frac{1}{\rho}\frac{\partial}{\partial \phi}(E_{\phi s}) = 0$$

$$\nabla \cdot \mathbf{H}_s = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_{\rho s}) = 0$$

Now

$$\nabla \times \mathbf{H}_s = \nabla \times \left(\frac{H_o}{\rho}\, e^{j\beta z}\, \mathbf{a}_\rho\right) = \frac{jH_o\beta}{\rho}\, e^{j\beta z}\, \mathbf{a}_\phi \tag{9.7.7}$$

Substituting eqs. (9.7.2) and (9.7.7) into eq. (9.7.5), we have

$$\frac{jH_o\beta}{\rho}\, e^{j\beta z}\mathbf{a}_\phi = j\omega\varepsilon_o \frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\phi$$

or

$$H_o\beta = 50\,\omega\varepsilon_o \tag{9.7.8}$$

Similarly, substituting eq. (9.7.2) into (9.7.6) gives

$$-j\beta\frac{50}{\rho}\, e^{j\beta z}\mathbf{a}_\rho = -j\omega\mu_o \frac{H_o}{\rho}\, e^{j\beta z}\mathbf{a}_\rho$$

or

$$\frac{H_o}{\beta} = \frac{50}{\omega\mu_o} \tag{9.7.9}$$

Multiplying eq. (9.7.8) with eq. (9.7.9) yields

$$H_o^2 = (50)^2 \frac{\varepsilon_o}{\mu_o}$$

or

$$H_o = \pm 50\sqrt{\varepsilon_o/\mu_o} = \pm \frac{50}{120\pi} = \pm 0.1326$$

Dividing eq. (9.7.8) by eq. (9.7.9), we get

$$\beta^2 = \omega^2 \mu_o \varepsilon_o$$

or

$$\beta = \pm \omega\sqrt{\mu_o \varepsilon_o} = \pm \frac{\omega}{c} = \pm \frac{10^6}{3 \times 10^8}$$
$$= \pm 3.33 \times 10^{-3}$$

In view of eq. (9.7.8), $H_o = 0.1326$, $\beta = 3.33 \times 10^{-3}$ or $H_o = -0.1326$, $\beta = -3.33 \times 10^{-3}$; only these will satisfy Maxwell's four equations.

PRACTICE EXERCISE 9.7

In air, $\mathbf{E} = \dfrac{\sin \theta}{r} \cos (6 \times 10^7 t - \beta r)\, \mathbf{a}_\phi$ V/m.

Find β and $\mathbf{H}$.

Answer: 0.2 rad/m, $-\dfrac{1}{12\pi r^2} \cos \theta \sin (6 \times 10^7 t - 0.2r)\, \mathbf{a}_r - \dfrac{1}{120\pi r} \sin \theta \times \cos (6 \times 10^7 t - 0.2r)\, \mathbf{a}_\theta$ A/m.

EXAMPLE 9.8

In a medium characterized by $\sigma = 0$, $\mu = \mu_o$, ε_o, and

$$\mathbf{E} = 20 \sin (10^8 t - \beta z)\, \mathbf{a}_y \text{ V/m}$$

calculate β and $\mathbf{H}$.

Solution:

This problem can be solved directly in time domain or using phasors. As in the previous example, we find β and $\mathbf{H}$ by making $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's four equations.

Method 1 (time domain): Let us solve this problem the harder way—in time domain. It is evident that Gauss's law for electric fields is satisfied; that is,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_y}{\partial y} = 0$$

From Faraday's law,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int (\nabla \times \mathbf{E})\, dt$$

But

$$\nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z} \mathbf{a}_x + \frac{\partial E_y}{\partial x} \mathbf{a}_z$$
$$= 20\beta \cos (10^8 t - \beta z)\, \mathbf{a}_x + 0$$

Hence,

$$\mathbf{H} = -\frac{20\beta}{\mu} \int \cos (10^8 t - \beta z)\, dt\, \mathbf{a}_x$$
$$= -\frac{20\beta}{\mu 10^8} \sin (10^8 t - \beta z)\, \mathbf{a}_x \tag{9.8.1}$$

It is readily verified that

$$\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} = 0$$

showing that Gauss's law for magnetic fields is satisfied. Lastly, from Ampere's law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{E} = \frac{1}{\varepsilon} \int (\nabla \times \mathbf{H})\, dt \tag{9.8.2}$$

because $\sigma = 0$.
But

$$\nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial z} \mathbf{a}_y - \frac{\partial H_x}{\partial y} \mathbf{a}_z$$
$$= \frac{20\beta^2}{\mu 10^8} \cos (10^8 t - \beta z)\, \mathbf{a}_y + 0$$

where **H** in eq. (9.8.1) has been substituted. Thus eq. (9.8.2) becomes

$$\mathbf{E} = \frac{20\beta^2}{\mu\varepsilon 10^8} \int \cos (10^8 t - \beta z)\, dt\, \mathbf{a}_y$$
$$= \frac{20\beta^2}{\mu\varepsilon 10^{16}} \sin (10^8 t - \beta z)\, \mathbf{a}_y$$

Comparing this with the given **E**, we have

$$\frac{20\beta^2}{\mu\varepsilon 10^{16}} = 20$$

or

$$\beta = \pm 10^8\sqrt{\mu\varepsilon} = \pm 10^8\sqrt{\mu_o \cdot 4\varepsilon_o} = \pm\frac{10^8(2)}{c} = \pm\frac{10^8(2)}{3 \times 10^8}$$

$$= \pm\frac{2}{3}$$

From eq. (9.8.1),

$$\mathbf{H} = \pm\frac{20\ (2/3)}{4\pi \cdot 10^{-7}(10^8)} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x$$

or

$$\mathbf{H} = \pm\frac{1}{3\pi} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x \text{ A/m}$$

Method 2 (using phasors):

$$\mathbf{E} = \text{Im}\,(E_s e^{j\omega t}) \quad \rightarrow \quad \mathbf{E}_s = 20e^{-j\beta z}\,\mathbf{a}_y \tag{9.8.3}$$

where $\omega = 10^8$.
Again

$$\nabla \cdot \mathbf{E}_s = \frac{\partial E_{ys}}{\partial y} = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad \rightarrow \quad \mathbf{H}_s = \frac{\nabla \times \mathbf{E}_s}{-j\omega\mu}$$

or

$$\mathbf{H}_s = \frac{1}{-j\omega\mu}\left[-\frac{\partial E_{ys}}{\partial z}\mathbf{a}_x\right] = -\frac{20\beta}{\omega\mu}e^{-j\beta z}\,\mathbf{a}_x \tag{9.8.4}$$

Notice that $\nabla \cdot \mathbf{H}_s = 0$ is satisfied.

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s \quad \rightarrow \quad \mathbf{E}_s = \frac{\nabla \times \mathbf{H}_s}{j\omega\varepsilon} \tag{9.8.5}$$

Substituting $\mathbf{H}_s$ in eq. (9.8.4) into eq. (9.8.5) gives

$$\mathbf{E}_s = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z}\mathbf{a}_y = \frac{20\beta^2 e^{-j\beta z}}{\omega^2\mu\varepsilon}\mathbf{a}_y$$

Comparing this with the given $\mathbf{E}_s$ in eq. (9.8.3), we have

$$20 = \frac{20\beta^2}{\omega^2\mu\varepsilon}$$

or

$$\beta = \pm\omega\sqrt{\mu\varepsilon} = \pm\frac{2}{3}$$

as obtained before. From eq. (9.8.4),

$$\mathbf{H}_s = \pm\frac{20(2/3)\, e^{\pm j\beta z}}{10^8(4\pi \times 10^{-7})}\,\mathbf{a}_x = \pm\frac{1}{3\pi}\,e^{\pm j\beta z}\,\mathbf{a}_x$$

$$\mathbf{H} = \text{Im}\,(\mathbf{H}_s e^{j\omega t})$$
$$= \pm\frac{1}{3\pi}\sin\,(10^8 t \pm \beta z)\,\mathbf{a}_x \text{ A/m}$$

as obtained before. It should be noticed that working with phasors provides a considerable simplification compared with working directly in time domain. Also, notice that we have used

$$\mathbf{A} = \text{Im}\,(\mathbf{A}_s e^{j\omega t})$$

because the given **E** is in sine form and not cosine. We could have used

$$\mathbf{A} = \text{Re}\,(\mathbf{A}_s e^{j\omega t})$$

in which case sine is expressed in terms of cosine and eq. (9.8.3) would be

$$\mathbf{E} = 20\cos\,(10^8 t - \beta z - 90°)\,\mathbf{a}_y = \text{Re}\,(\mathbf{E}_s e^{j\omega t})$$

or

$$\mathbf{E}_s = 20e^{-j\beta z - j90°}\,\mathbf{a}_y = -j20e^{-j\beta z}\,\mathbf{a}_y$$

and we follow the same procedure.

PRACTICE EXERCISE 9.8

A medium is characterized by $\sigma = 0$, $\mu = 2\mu_o$ and $\varepsilon = 5\varepsilon_o$. If $\mathbf{H} = 2\cos\,(\omega t - 3y)\,\mathbf{a}_z$ A/m, calculate ω and **E**.

Answer: 2.846×10^8 rad/s, $-476.8\cos\,(2.846 \times 10^8 t - 3y)\,\mathbf{a}_x$ V/m.

SUMMARY

1. In this chapter, we have introduced two fundamental concepts: electromotive force (emf), based on Faraday's experiments, and displacement current, which resulted from Maxwell's hypothesis. These concepts call for modifications in Maxwell's curl equations obtained for static EM fields to accommodate the time dependence of the fields.
2. Faraday's law states that the induced emf is given by ($N = 1$)

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t}$$