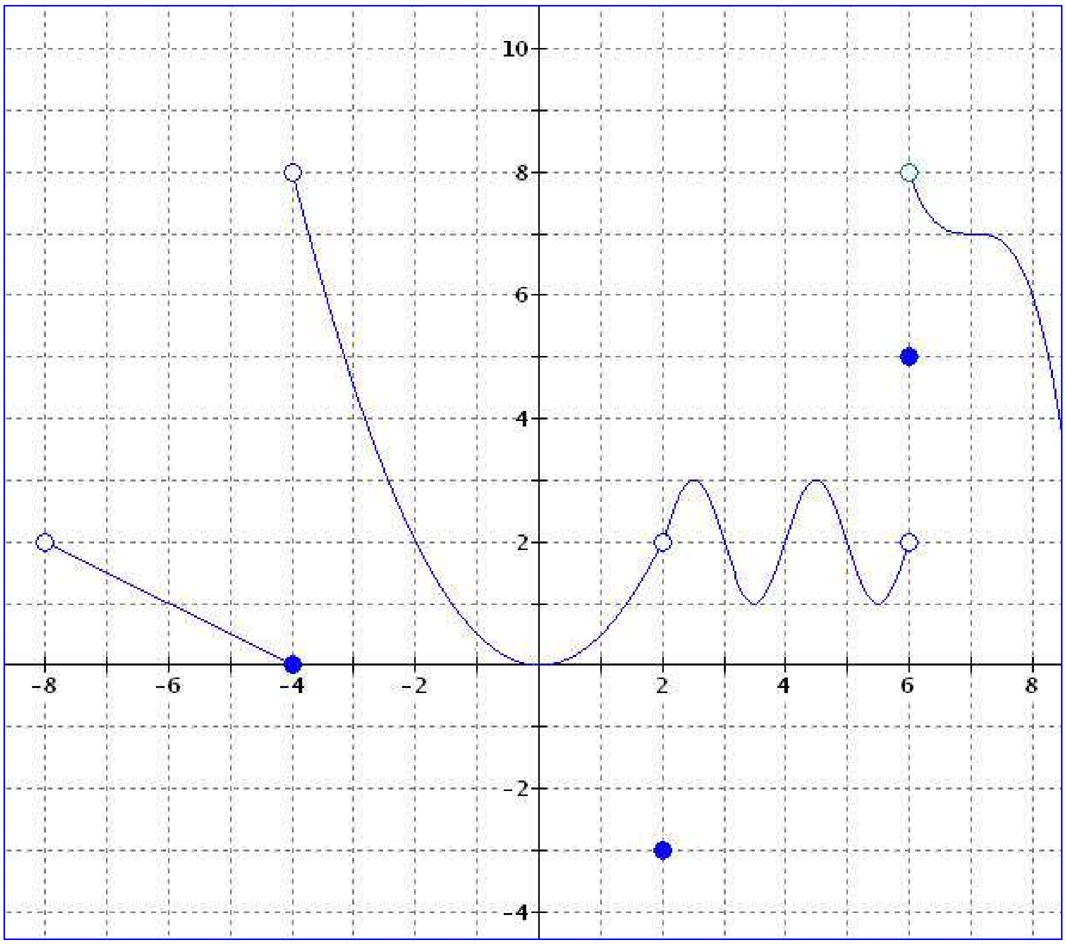
Finding limits of a piecewise defined function

Calculus I Tutorial, by Dave Collins

I. From the graph
II. From the algebraic representation of the function

Let's start with the graph. Suppose you have the graph of a piecewise defined function:



$f(x)$

First, make sure you recall the algebra – being able to evaluate the function. For example, can you determine the following?

$$f(-8) = \qquad f(2) =$$
$$f(-4) = \qquad f(4) =$$
$$f(-2) = \qquad f(6) =$$
$$f(0) = \qquad f(8) =$$

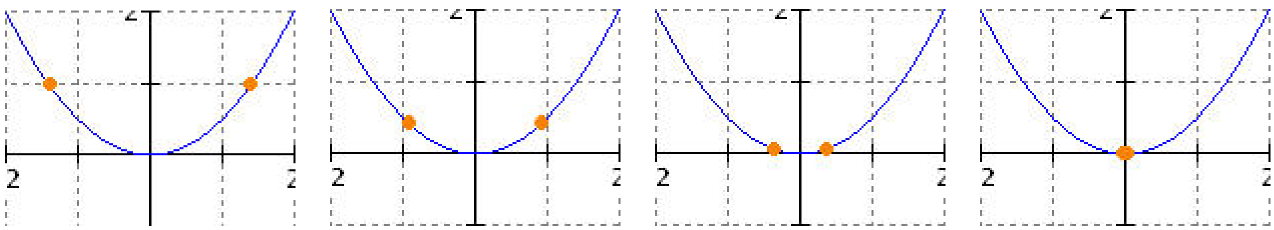
In this case, simply look at the graph and try to determine the exact value of y at the desired x coordinate. Did you get the following solutions?

$$f(-8) = undefined \qquad f(2) = -3$$
$$f(-4) = 0 \qquad f(4) = 2$$
$$f(-2) = 2 \qquad f(6) = 5$$
$$f(0) = 0 \qquad f(8) = 7$$

Now, in Calculus we're concerned with values of y as <u>x approaches</u> a given value. For example,

$$\lim_{x \to 0} f(x) = ?$$

is asking, "What does the value of y start approaching as x gets close to zero?". To determine this, just start tracing the graph (yes, you can use your fingers!) from **<u>both directions</u>** (the left and the right) as x gets closer and closer to zero. The next series of graphs should give you the feel of this tracing…



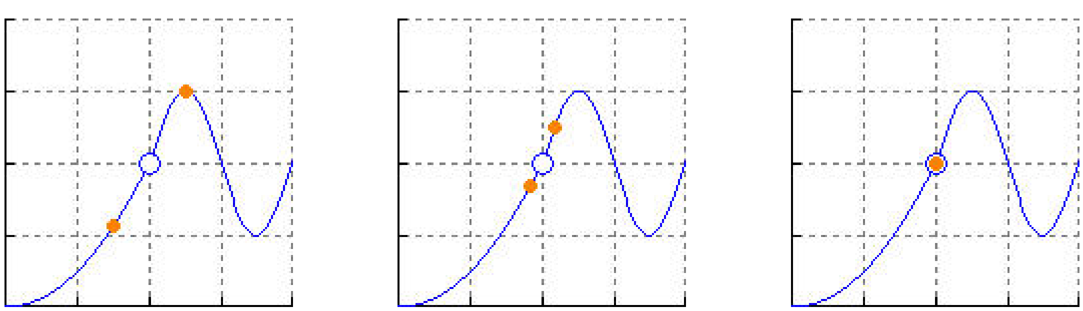
Since the **<u>y-value</u>** of the point seems to approach the same value <u>from both sides</u>, we say that the limit does exist, and the value of this limit is <u>y=0.</u> You would write,

$$\lim_{x \to 0} f(x) = 0$$

<u>It does not matter what happens at the point in order for the limit to exist</u>!!! Previously, it was shown that the value of the curve at x = 2 is y = -3. However, can you find the <u>limit</u> of this function as x approaches 2? In other words, what is:

$$\lim_{x \to 2} f(x) = ?$$

Again, start tracing the function from both sides to see if the value of y approaches the same number… The next series of graphs should depict what you should be visualizing:



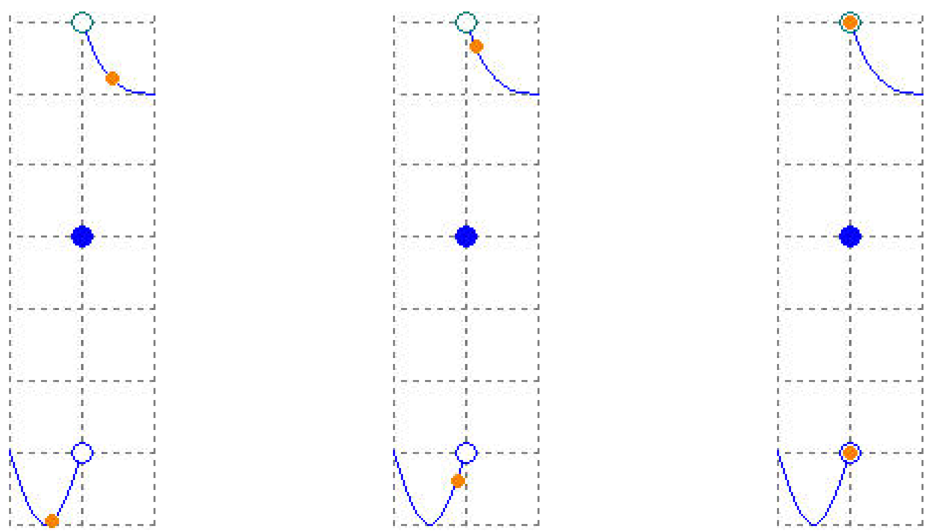
So, in this example, even though we found the function value to be -3, you should be able to visualize that it <u>appears as if</u> the value of y approaches 2, when x gets close to 2. Or,

$$\lim_{x \to 2} f(x) = 2$$

It must be emphasized that it **<u>does not matter the value of the function at 2</u>** to determine the limit. The next example shows that even though the value of the function exists, the limit may not.

$$\lim_{x \to 6} f(x) = ?$$

Again, tracing the curve as x approaches 6 <u>from both directions</u> should show you that the value of y does not appear to be the same:



So, the **<u>limit from the left</u>** appears to be y = 2, while the **<u>limit from the right</u>** appears to be y = 8. Therefore, since these values are not the same, we say the limit does not exist.

$$\lim_{x \to 6} f(x) = DNE \quad \text{(does not exist)}$$

This example does bring yet another important concept of limits – the existence of **<u>one sided limits</u>**. You can restrict your attention to only approach a given value of x **<u>from one side</u>** – either the left or the right. To denote this, to approach from the left use a superscript of a minus sign, and to approach from the right use a superscript of a plus sign. For example, to ask, "What is the limit of the function as x approaches 2 from the left?" you would see:

$$\lim_{x \to 6^-} f(x) = ?$$

And from the work earlier, we know this limit is 2. Or,

$$\lim_{x \to 6^-} f(x) = 2$$

And to ask, "What is the limit of the function as x approaches 2 from the right?" …

$$\lim_{x \to 6^+} f(x) = ?$$

And again, from the work earlier, we saw this limit to be 8. So you would write,

$$\lim_{x \to 6^+} f(x) = 8$$

Now that you should be clear on how to determine limits from the graph, let's move on to trying to determine the limit given the algebraic representation. Consider the function,

$$g(x) = \begin{cases} -x, & x < -2 \\ \frac{1}{2}x^2 - 1, & -2 \le x < 2 \\ -x + 3, & x > 2 \end{cases}$$

Again, let's start off by asking a couple of algebra questions. Determine:

$$\begin{array}{ll} g(-5) = & g(-2) = \\ g(0) = & g(2) = \\ g(3) = & \end{array}$$

The three on the left should be simple. Each x value falls directly in only one piece of this function. At x = -5, it is the first piece. At x = 0, it is the second piece. And at x = 3, it is the third piece. Do the ones on the right fall into more than one? Actually, they do not. Pay close attention to the inequality symbols. For x = -2, the function follows the second piece. But, it does not appear as if the value x = 2 falls into any of the domains specified. This is correct. When this happens, it just means that the function is not defined for this particular x-value. So, you should have:

$$\begin{array}{ll} g(-5) = 5 & g(-2) = 1 \\ g(0) = -1 & g(2) = undefined \\ g(3) = 0 & \end{array}$$

Now let's look at some limits. Algebraically, you will approach the x-value from both directions. However, it is relevant you see that it is possible that the function follows <u>different pieces</u> from both directions. (It did from our graphical examples!) But when should you look at different pieces? Simple – when the x-values where the function changes definition. For this function, you will only need to worry about one-sided limits at x = 2, and x = -2.

Try to find the following limits:

$$\lim_{x \to -3} g(x) = ?$$
$$\lim_{x \to -1} g(x) = ?$$
$$\lim_{x \to 4} g(x) = ?$$

In order to determine these, just identify which piece of the function x is on. For the first,

$$\begin{aligned}\lim_{x \to -3} g(x) &= \lim_{x \to -3} \left(-x\right) \\ &= -(-3) \\ &= 3\end{aligned}$$

And the second,

$$\begin{aligned}\lim_{x \to -1} g(x) &= \lim_{x \to -1} \left(\frac{1}{2}x^2 - 1\right) \\ &= \frac{1}{2}(-1)^2 - 1 \\ &= \frac{1}{2} - 1 \\ &= -\frac{1}{2}\end{aligned}$$

And the third,

$$\begin{aligned}\lim_{x \to 4} g(x) &= \lim_{x \to 4} \left(-x + 3\right) \\ &= -(4) + 3 \\ &= -1\end{aligned}$$

Now, what about the limits as x approaches 2 and -2? Do we have to look at x = 2 since we found out that the function value doesn't even exist? Recall, <u>**we do not care what happens at the point in order for the limit to exist**</u>. So, how do you determine if the limit exists at these points where the function changes definition? Answer: <u>Take one-sided limits</u>. For example, $\lim_{x \to 2} g(x) = 1$

$$\lim_{x \to -2} g(x) = ?$$

This limit should be broken down into two pieces: the limit from the left and the limit from the right:

$$\lim_{x \to -2^-} g(x) = ? \qquad \lim_{x \to -2^+} g(x) = ?$$

In both these cases, because the direction has been restricted, each value of x <u>should only fall in one piece</u>. Evaluate these limits in the same way:

$$\begin{aligned}\lim_{x \to -2^-} g(x) &= \lim_{x \to -2^-} \left(-x\right) \\ &= -(-2) \\ &= 2\end{aligned}$$

And the limit as x approaches -2 from the right:

$$\begin{aligned}\lim_{x \to -2^+} g(x) &= \lim_{x \to -2^+} \left(\frac{1}{2}x^2 - 1\right) \\ &= \frac{1}{2}(-2)^2 - 1 \\ &= 2 - 1 \\ &= 1\end{aligned}$$

Since the one-sided limits <u>are not the same</u>, the limit itself does not exist:

$$\lim_{x \to -2} g(x) = DNE \quad \text{(does not exist)}$$

What about the limit as x approaches 2? Again, since the function changes definition at 2, break this limit into two pieces – from the left and from the right. The limit from the left,

$$\begin{aligned}\lim_{x \to 2^-} g(x) &= \lim_{x \to 2^-} \left(\frac{1}{2}x^2 - 1\right) \\ &= \frac{1}{2}(2)^2 - 1 \\ &= 2 - 1 \\ &= 1\end{aligned}$$

And the limit from the right,

$$\begin{aligned}\lim_{x\to 2^+} g(x) &= \lim_{x\to 2^+}\left(-x+3\right)\\ &= -(2)+3\\ &= 1\end{aligned}$$

Since the limits are the same, the limit does exist (even though the function does not!) at x = 2. We say:

$$\lim_{x\to 2} g(x) = 1$$

And since you already know how to determine the limits from a graph, look at the graph and justify to yourself the work you just did finding limits algebraically.