FINITE-DIMENSIONAL ATTRACTORS AND EXPONENTIAL ATTRACTORS FOR DEGENERATE DOUBLY NONLINEAR EQUATIONS

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Abstract. We consider the following doubly nonlinear parabolic equation in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$f(x, \partial_t u) = \Delta_x u - g(x, u)$$

where the nonlinearity f can is allowed to have the degeneracy with respect to $\partial_t u$ of the form $\partial_t u|\partial_t u|^p$ at some points $x \in \Omega$.

Under some natural assumptions on the nonlinearities f and g we prove the existence and uniqueness of a solution of that problem and establish the finite-dimensionality of global and exponential attractors of the semigroup associated with this equation in the appropriate phase space.

Introduction

It is well-known that many problems of the modern material sciences can be reduced to the following abstract doubly nonlinear equations in the appropriate functional spaces

$$A\left(\frac{d}{dt}u\right) = B(u) + f \tag{0.1}$$

$$\frac{d}{dt}A(u) = B(u) + f \tag{0.2}$$

(where A and B are some (nonlinear, unbounded) operators and f are external forces) which have been intensively studied by many authors, see [3-4], [6], [11-12], [18-22] and references therein. The standard approach to equations (0.1) and (0.2) uses the assumption that the operators A and B are maximal monotone in the proper functional spaces and is based on the general theory of monotone operators, see [11], [25].

It is however worth to note that, being a very effective technical tool for establishing the existence of solutions for such equations, the monotone operators approach has essential drawbacks. Indeed, this method usually gives only *weak* energy solutions of (0.1) or (0.2) which are not regular enough to be unique and the existence of more regular solutions is much more delicate problem wich requires principally different methods. Thus, even after the proving the existence of weak energy solutions via the monotonicity methods, the analytic structure of the problem considered can remain completely unclear especially in the case where one or two of operators A and B are singular or degenerate.

The main aim of the present paper is to give a detailed study of the following relatively simple (but still non-trivial) model example of a doubly nonlinear equation of the form (0.2):

$$f(x, \partial_t u) = \Delta_x u - g(x, u), \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \tag{0.3}$$

in a smooth bounded domain $x \in \Omega \subset \mathbb{R}^3$ avoiding the usage of monotone operators theory. In particular, doubly nonlinear equations of the form

$$b(u, \partial_t u)\partial_t u = \Delta_x u - g(u) + h(x) \tag{0.4}$$

were introduced by M.Gurtin in order to generalize the classical Allen-Cahn model, see [12]. In the case where b is independent of u, (0.4) has the form of (0.3) with $f(\partial_t u) = b(\partial_t u)\partial_t u$, see also Remark 1.6 below concerning the general case $f = f(x, u, \partial_t u)$.

Moreover, we allow the function f to have polynomial degeneration with respect to $\partial_t u$. To be more precise, that f can be represented in the form

$$f(x, \partial_t u) = a(x)\partial_t u + \phi(x, \partial_t u) \tag{0.5}$$

where $a(x) \geq 0$ and ϕ satisfies

$$C(1 + |v|^p) \geq \phi'_v(x, v) \geq \alpha|v|^p, \quad C, \alpha > 0 \tag{0.6}$$

for some $p > 0$. Thus, the function $f(x, v)$ degenerates as $v|v|^p$ at all points x where $a(x) = 0$. Concerning the second nonlinearity g, we assume the standard dissipativity assumptions to be satisfied, see Section 1 for their precise formulation.

We also note that equation (0.3) is a fully nonlinear degenerate second order parabolic problem, so the highly developed classical theory of *quasilinear* parabolic equations (see e.g. [15]) is not formally applicable to it. Moreover, the structure of equation (0.3) does not fit the assumptions of the general fully nonlinear theory of Krylov and Safonov, see e.g. [14]. However, as we will see below, equation (0.3) possesses very good regularity properties and, in contrast to quasilinear case, even with *degenerate* nonlinearity f, it has classical solutions. The key idea of our method is to differentiate equation (0.3) by t and study the obtained formally quasilinear equation with respect to $v = \partial_t u$ using the classical methods, see Section 1.

On the other hand, it is worth to emphasize that the standard energy phase space

$$\Phi_{en} := \{u_0 \in W_0^{1,2}(\Omega), \quad G(x, u_0(x)) \in L^1(\Omega)\}, \quad G(x, w) := \int_0^w g(x, v)\, dv \tag{0.7}$$

surprisingly occurs to be unrelevant for problem (0.3) (even in the non-degenerate case) due to the existence of "pathological" singular weak energy solutions, e.g. of the form

$$u(t,x)=\frac{v(t,x)}{|x|^\beta},\quad v(t,0)\neq 0 \tag{0.8}$$

with regular v and positive β, see Example 1.1 below. Instead of Φ_{en} one should take slightly more regular phase space

$$\Phi:=W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \tag{0.9}$$

Then, as we will show, the "pathological" solutions cannot exist any more and we will have only the solutions with usual and reasonable properties. We believe that this phenomena has a general nature and somehow clarify the difficulties related with finding the stronger solutions of more general doubly nonlinear equations of the form (0.1) or (0.2) via the energy method.

Inspite of this, the uniqueness of such solutions in the degenerate case is more delicate problem. Indeed, as the simplest ODE example

$$y'(t)|y'(t)|^p=y(t),\quad y(0)=0,\ p>0 \tag{0.10}$$

shows, we cannot have the uniqueness under assumptions (0.6) on the nonlinearity f (if f is allowed to be degenerate), so, in this case, the additional restrictions are necessary. Moreover, this example gueses that, in order exclude the non-uniqueness example (0.10), the right-hand side $\Delta_x u-g(x,u)$ of equation (0.3) should be monotonic with respect to u at all points $x\in\Omega$ where the left hand side $f(x,\partial_t u)$ is degenerate. In order to avoid the technicalities, we prove the uniqueness under the additional assumption in slightly stronger form:

$$K_R[a(x)]^{1/2}+\inf_{|u|\le R} g'_u(x,u)\ge 0,\quad x\in\Omega \tag{0.11}$$

for every $R>0$ and appropriate $K_R>0$ depending on R, see Theorem 2.2 (and Remark 2.4 for weaker assumptions). Here, we only note that (0.11) is automatically satisfied in the non-degenerate case $a(x)\ge a_0>0$, so not any additional restriction are required if f is non-degenerate.

We also study the long-time behavior of solutions of problem (0.3) in terms of the associated global and exponential attractors (in the degenerate case where the uniqueness theorem holds, see also [19] for multi-valued semigroup appoach to long-time behavior of doubly nonlinear equations without uniqueness and [20] for global and exponential attractors in the non-degenerate case). In particular, we establish the finite-dimensionality of the global attractor of (0.3) under the uniqueness assumption (0.11) and assumptions (0.5) and (0.6) on the nonlinearity f which thus can be degenerate.

It is worth to recall here that the degeneracy of the equation considered can change drastically the long-time behavior and the structure of the associated global attractor. In particular, the global attractor of following degenerate analogue of Chafee-Infante equation

$$\partial_t u=\Delta_x(u^3)+u-u^3,\quad x\in\Omega\subset\subset\mathbb{R}^n,\quad u\big|_{\partial\Omega}=0 \tag{0.12}$$

is infinite-dimensional (see [9]), although, in the non-degenerate case, the dimension of the attractors associated with dissipative systems in bounded domains is usually finite, see [1], [23] and references therein.

Fortunately, in our case of equation (0.3), the infinite-dimensionality of the global attractor is automatically excluded by the uniqueness condition (0.11), so using the proper generalization of the so-called l-trajectory method, we verify the finite-dimensionality for the degenerate case as well, see Theorem 3.2 below.

The paper is organized as follows. A number of a priory estimates for the "sufficiently regular" solutions of (0.3) which are crucial for our study are given in Section 1. In particular, the dissipative estimate for the solutions of (0.3) in the phase space Φ and the regularity $\partial_t u(t) \in L^\infty(\Omega)$ for $t > 0$ are verified there and the examples of singular weak energy solutions which do not possess any smoothing properties are also given in this section, see Example 1.1.

Based on these estimate, we prove (in Section 2) the existence of a solution for (0.3) and its uniqueness under the additional assumption (0.11).

In Section 3, we formulate and prove the theorems on the existence of finite-dimensional global and exponential attractors for the semigroup (0.3) which can be considered as the main result of the paper.

The proof of one compact embedding theorem which is required for our exponential attractor construction is given in Appendix.

Finally, some additional properties of solutions of (0.3) which are not important for the proof of our main result, but (as we belive) clarify the nature of the equation considered are collected in a number of remarks throughout of the paper.

Acknowledgements. This work was partially supported by the *CRDF* grant RUM1-2654-MO-05.

§1 A priori estimates and dissipativity

In this section, we obtain a number of estimates for the solutions of the following problem in a bounded smooth domain $\Omega \subset\subset \mathbb{R}^3$:

$$\begin{cases} f(x,\partial_t u) = \Delta_x u - g(x,u), \\ u\big|_{\partial\Omega} = 0, \ u\big|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Here $u = u(t,x)$ is an unknown function, Δ_x is a Laplacian with respect to variable x and f and g are given nonlinear functions.

We assume that the nonlinearity f has the following structure:

$$f(x,v) = a(x)v + \phi(x,v) \tag{1.2}$$

where the function $a \in L^\infty(\Omega)$ is nonnegative:

$$a(x) \geq 0, \quad x \in \Omega \tag{1.3}$$

and the function $\phi \in L^\infty(\Omega, C^2(\mathbb{R}))$ satisfies

$$\begin{cases} 1) \ \ \phi'_v(x,v) \geq \alpha |v|^p, \\ 2) \ \ \phi'_v(x,v) \leq C(|v|^p + 1) \leq C, \alpha > 0, \end{cases} \tag{1.4}$$

for some $p \geq 0$. Thus, the degeneration of the form $f(x,\partial_t u) \sim \partial_t u |\partial_t u|^p$ is allowed in the subdomain $\Omega_0 \subset \Omega$ where $a(x) = 0$.

We also assume that the second nonlinearity $g \in L^\infty(\Omega, C^2(\mathbb{R}))$ and satisfies the the standard dissipativity assumption

$$\liminf_{|v|\to\infty} \inf_{x\in\Omega} g'_v(x,v) \ge \beta > 0. \tag{1.5}$$

We start with the standard energy estimate for sufficiently regular solutions $u(t)$ of problem (1.1) (roughly speaking, in this section "sufficiently regular" means that it can be approximated by smooth solutions of the regularized problem (1.1) which allows to justify all a priori estimates formally deduced below, see Section 2 for details).

Proposition 1.1. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1). *Then, the following equality hold:*

$$\frac{d}{dt}[\|\nabla_x u(t)\|^2_{L^2(\Omega)} + 2(G(\cdot, u(t)), 1)] = -2(f(\cdot, \partial_t u(t)), \partial_t u(t)) \tag{1.6}$$

where $G(x,u) := \int_0^u g(x,v)\,dv$ and $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$. In particular,

$$\begin{aligned} \|\nabla_x u(t)\|^2_{L^2(\Omega)} + \|G(\cdot, u(t))\|_{L^1(\Omega)} + \\ + \int_0^t (a(\cdot)\partial_t u(s), \partial_t u(s)) + \|\partial_t u(s)\|^{p+2}_{L^{p+2}(\Omega)}\,ds \le \\ \le C(1 + \|\nabla_x u(0)\|^2_{L^2(\Omega)} + \|G(\cdot, u(0))\|_{L^1(\Omega)}) \end{aligned} \tag{1.7}$$

where the constant C is independent of u and t (here and below $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$).

Proof. Indeed, multiplying equation (1.1) by $\partial_t u(t)$ and integrating over $x \in \Omega$, we deduce (1.6). In order to obtain estimate (1.7), it is sufficient to integrate (1.6) by t and use assumptions (1.4) and the obvious fact that $G(x,v) \ge -C$ (due to (1.5)). Proposition 1.1 is proven.

We see that the energy estimate (1.7) gives the estimate of L^{p+2}-norm of $\partial_t u$ and, consequently, due to assumptions (1.4), the $L^{\frac{p+2}{p+1}}$-norm of $f(x, \partial_t u)$. Our next task is to obtain stronger estimates for $\partial_t u$ and $f(x, \partial_t u)$. To this end, we differentiate equation (1.1) by t and denote $v = \partial_t u$. Then, we get

$$\partial_t f(x,v) = \Delta_x v - g'_u(x,u)v. \tag{1.8}$$

The next theorem gives the local L^∞-estimates for the quasilinear second order parabolic equation (1.8).

Theorem 1.1. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1). *Then for every $q \in [p+2, \infty]$, the following estimate hold:*

$$\|\partial_t u(t)\|_{L^q(\Omega)} \le C_T(1 + \|\partial_t u(0)\|_{L^q(\Omega)}), \quad t \in [0,T], \tag{1.9}$$

where C_T depends on T, but is independent of t and u Moreover, if, in addition, $q > 3p/2$, then the following smoothing property holds:

$$\|\partial_t u(t)\|_{L^\infty(\Omega)} \le Q_T(1/t + \|\partial_t u(0)\|_{L^q(\Omega)}), \quad t \in (0,T] \tag{1.10}$$

where the monotonic function Q_T depends on T, but is independent of u and t.

Proof. Indeed, for any $q>0$, multiplying equation (1.8) by $v|v|^q$, we have

$$\partial_t(F_{p+q+2}(\cdot,\partial_t u(t)),1)+\frac{4(q+1)}{(q+2)^2}\||v|^{1+q/2}\|^2_{W^{1,2}(\Omega)}\le K\|v\|^{q+2}_{L^{q+2}(\Omega)} \tag{1.11}$$

where $F_{p+q+2}(x,v):=\int_0^v f(x,s)s|s|^q\,ds$ and K is independent of u and t (here we have used that $g'(x,u)\ge -K$ due to assumption (1.5)). Moreover, according to (1.4), one has

$$\frac{C}{q+2}(|v|^{p+q+2}+1)\ge F_{p+q+2}(x,v)\ge\frac{a(x)}{q+2}|v|^{q+2}+\frac{\alpha}{p+q+2}|v|^{p+q+2}. \tag{1.12}$$

Integrating (1.11) over $t\in[\tau,s]\subset[0,T]$, using (1.12) and the embedding $W^{1,2}\subset L^6$, we deduce that

$$\|v(s)\|^{p+q+2}_{L^{p+q+2}(\Omega)}+\int_\tau^s\|v(t_1)\|^{q+2}_{L^{3(q+2)}(\Omega)}\,dt_1\le \\ \le C(\|v(\tau)\|^2_{L^{p+q+2}(\Omega)}+1)+C(q+2)\int_\tau^s\|v(t_1)\|^{q+2}_{L^{q+2}(\Omega)}\,dt_1 \tag{1.13}$$

where the constant C is independent of u, τ, s and q.

Estimate (1.9) with $q<\infty$ is now an immediate corollary of (1.13), estimate $|v|^{q+2}\le 1+|v|^{p+q+2}$ and Gronwall's inequality. So, we now only need to verify (1.10) and (1.9) with $q=\infty$. For simplicity, we verify (1.10) for $t=1$ only (for the general case it can be obtained analogously by the appropriate scaling). In order to do so, we are going to iterate (1.13) infinitely many times step by step increasing the exponent q. To be more precise, we set

$$q_{n+1}+2+p=3(q_n+2),\quad \text{i.e.}\quad q_n+2=(q_0+2-\frac{p}{2})3^n+\frac{p}{2}. \tag{1.14}$$

We see that, in order to have increasing sequence of q_n, we need $q_0+2-p/2>0$ or $q_0+p+2>3p/2$ which corresponds to the restriction $q>3p/2$ in the statement of (1.10). We also set

$$t_0=0,\quad t_{n+1}-t_n=\frac{\beta}{q_n},\quad \lim_{n\to\infty}t_n=1. \tag{1.15}$$

We now assume that the estimate for

$$I_n:=\|v\|^{p+q_n+2}_{L^\infty([t_n,2],L^{p+q_n+2}(\Omega))}+\|v\|^{q_n+2}_{L^{q_n+2}([t_n,2],L^{3(q_n+2)}(\Omega))}. \tag{1.16}$$

is already obtained and deduce the recurrent estimate for I_{n+1}. Indeed, there exists $t^*\in[t_n,t_n+1]$ such that

$$\|v(t^*)\|^{q_n+2}_{L^{p+q_{n+1}+2}(\Omega)}=\|v(t^*)\|^{q_n+2}_{L^{3(q_n+2)}(\Omega)}\le \\ \le\frac{1}{t_{n+1}-t_n}\int_{t_n}^{t_{n+1}}\|v(s)\|^{q_n+2}_{L^{3(q_n+2)}(\Omega)}\,ds\le \\ \le C\beta^{-1}q_n\|v\|^{q_n+2}_{L^{q_n+2}([t_n,2],L^{3(q_n+2)}(\Omega))}\le C\beta^{-1}q_nI_n$$

Setting now $\tau = t^*$, $q = q_{n+1}$ and $s \in [t_{n+1}, 2]$ in the basic estimate (1.13) and using the last estimate, we arrive at

$$I_{n+1} \le [C_1(q_n+2)]^{\frac{p+q_{n+1}+2}{q_n+2}} \left(1 + I_n^{\frac{p+q_{n+1}+2}{q_n+2}} + I_n^{\frac{q_{n+1}+2}{p+q_n+2}}\right) \tag{1.17}$$

with the constant C_1 independent of n. Setting now

$$J_n := \max\{1, I_n^{\frac{1}{p+q_n+2}}\}, \tag{1.18}$$

we transform (1.17) as follows

$$J_{n+1} \le [C_2(q_n+2)]^{\frac{1}{q_n+2}} J_n^{1+\frac{p}{q_n+2}}. \tag{1.19}$$

Iterating this estimate, we will have

$$J_n \le A_n J_0^{B_n} \tag{1.20}$$

with

$$B_n := \Pi_{i=0}^{n-1}(1 + \frac{p}{q_i+2}), \quad A_n := \Pi_{i=0}^{n-1}[C_2(q_i+2)]^{\frac{1}{q_i+2}B_{n-i}}. \tag{1.21}$$

Since $q_n \sim 3^n$, then, obviously,

$$A_n \le A_\infty := \lim_{n\to\infty} A_n < \infty, \quad B_n \le \lim_{n\to\infty} B_n < \infty$$

and, consequently,

$$J_\infty := \limsup_{n\to\infty} \le A_\infty J_0^{B_\infty}. \tag{1.22}$$

On the other hand, we have

$$\|v\|_{L^\infty([1,2]\times\Omega)} = \lim_{n\to\infty} \|v\|_{L^{p+q_n+2}([0,1]\times\Omega)} \le J_\infty.$$

Thus, we have proven that

$$\|v\|_{L^\infty([1,2]\times\Omega)} \le Q(\|v\|^{p+q_0+2}_{L^\infty([0,2],L^{p+q_0+2}(\Omega))} + \|v\|_{L^{q_0+2}([0,2],L^{3(q_0+2)}(\Omega))}).$$

Combining this estimate with estimate (1.9) and (1.13) with $q = p + q_0 + 2$, we finally get

$$\|v\|_{L^\infty([1,2]\times\Omega)} \le Q_1(\|v(0)\|_{L^{p+q_0+2}(\Omega)})$$

which finishes the proof of estimate (1.10). Estimate (1.9) with $q = \infty$ can be proven analogously, but even simplier since we can now set $t_n \equiv 0$. Theorem 1.1 is proven.

We now formulate one more simple, but usefull interior estimate for the derivative $\partial_t u$ of the solution of (1.1).

Proposition 1.2. *Let the above assumptions hold and let u be a sufficiently regular solution of* (1.1) *and let $q \in [0,\infty)$. Then, the following estimate holds:*

$$(1.22') \quad t\|\partial_t u(t)\|^{p+q+2}_{L^{p+q+2}(\Omega)} + \int_0^1 s\|\nabla_x|\partial_t u(s)|^{1+q/2}\|^2_{L^2(\Omega)}\,ds \le \le C\int_0^1 \|\partial_t u(s)\|^{p+q+2}_{L^{p+q+2}(\Omega)}\,ds + C, \quad t\in[0,1]$$

where the constant C is independent of u and t.

Indeed, in order to obtain this estimate, it is sufficient to multiply equality (1.12) by t and integrate over $t \in [0,1]$.

Remark 1.1. It is worth to emphasize that the L^∞-estimate for $\partial_t u$ is crucial for the theory of equation (1.1). Indeed, after obtaining this estimate, one can rewrite the term $f(x,\partial_t u)$ in the form of $l(t,x)\partial_t u$ and obtain a second order quasilinear parabolic equation with *bounded* coefficients. Applying the de Giorgi theory to that equation, one obtains e.g. C^α-estimates for u at least in the non-degenerate case $a(x) > a_0 > 0$. In turns, considering now equation (1.8) as a second order quasilinear parabolic equation with respect to v, one obtains the standard $C^{2+\alpha}$-estimates for $v = \partial_t u$, see [15] and Section 2 for more details. Repeating this procedure, one can obtain as much regularity of a solution as it is allowed by the smoothness of Ω, f and g.

Thus, at least in the non-degenerate case, the analytical properties of the fully nonlinear parabolic problem (1.1) will be the same as for the standard case of second order quasilinear equations, if the L^∞-estimate of $\partial_t u$ is available. Theorem 1.1 and Proposition 1.2 show that, it is sufficient to be able to estimate only the integral $\int_0^1 \|\partial_t u(t)\|^q_{L^q(\Omega)}\,ds$ with $q > 3p/2$.

In particular, in the *subcritical* case $p < 4$, we have $p + 2 > 3/2p$, so the boundedness of that integral follows immediately from the energy estimate of Proposition 1.1. Thus, in that case any properly defined weak energy solution of (1.1) possesses internal estimate of the form (1.10) and becomes essentially more regular for $t > 0$.

In contrast to that, in the supercritical case $p > 4$, we have a gap between the minimal regularity obtained from the energy estimate (1.7) and the regularity required for the crucial L^∞-estimate of $\partial_t u$. As we will show below this gap is related with the existence of "pathological" weak energy solutions which are singular and do not possesses any regularity increasing for an arbitrary long time, see Example 1.1 below.

Remark 1.2. It is worth to note that, in contrast to the energy estimate (1.7), the further regularity estimate (1.9) is, up to the moment, *divergent* in time. Nevertheless, analysing equality (1.11) in a little more accurate way, based on the Gronwall-type estimates for the differential inequality:

$$y(t)' + [y(t)]^\kappa \le h(t), \quad 0 < \kappa < 1$$

and energy estimate (1.7), one can easily obtain the analogue of (1.9) with the constant C_T *independent* of T. We however prefer to use Theorem 1.1 only in a finite time interval and will deduce the dissipative estimates with respect to $t \to \infty$ from the analogous L^∞-estimates of $u(t)$ and the comparison principle for the solutions of (1.1).

The next proposition gives the basic dissipative estimate for the L^∞-norm of the solution u of problem (1.1).

Proposition 1.3. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1). *Then, the following estimate holds:*

$$\|u(t)\|_{L^\infty(\Omega)} \le Q(\|u_0\|_{L^\infty(\Omega)})e^{-\gamma t} + C_* \tag{1.23}$$

where the positive constants C_* and γ and the monotonic function Q is independent of u and t.

Proof. We obtain estimate (1.23) by comparing the solution $u(t)$ with the appropriate sub and super solutions. To this end, we note that, due to condition (1.5) there exists a constant $C_+ > 0$ such that

$$g(x,u) \ge \beta(u - C_+)/2, \quad u \ge C_+. \tag{1.24}$$

Let now the function $Y_+(t)$ solves the equation

$$\frac{\alpha}{p+1} Y_+'(t)|Y_+'(t)|^p = -\beta(Y_+(t) - C_+)/2, \quad Y_+(0) = \max\{C_+, \|u_0\|_{L^\infty(\Omega)}\} \tag{1.25}$$

where $\alpha > 0$ is the same as in (1.4). On the one hand, solving (1.25), we get

$$Y_+(t) = C_+ + \left((\|u_0\|_{L^\infty(\Omega)} - C_+)_+^{p/(p+1)} - \delta t\right)_+^{1+1/p}, \ \delta := \frac{p+1}{p}\left(\frac{\beta(p+1)}{2\alpha}\right)^{1/(p+1)}$$

where $z_+ := \max\{z, 0\}$ and, consequently, $Y_+(t)$ satisfies the analogue of the dissipative estimate (1.23)

$$Y_+(t) \le Q(\|u_0\|_{L^\infty(\Omega)})e^{-\gamma t} + C_+ \tag{1.26}$$

for the appropriate positive constant γ and monotonic Q.

On the other hand, using assumptions (1.4) and (1.24) and the fact that $Y_+(t) \ge C_+ > 0$, $Y_+'(t) \le 0$, we see that

$$f(x, \partial_t Y_+(t)) - \Delta_x Y_+(t) + g(x, Y_+(t)) \ge 0$$

and, consequently, $Y_+(t)$ is a super-solution of (1.1) and, by comparison principle, we have

$$u(t,x) \le Y_+(t), \quad (t,x) \in [0,T] \times \Omega$$

which together with (1.26) gives the upper bound for $u(t,x)$ in the form of (1.23). The lower bounds can be obtained analogously by constructing the corresponding sub-solution $Y_-(t)$. Thus, estimate (1.23) is verified and Proposition 1.3 is proven.

Remark 1.3. Being a pedant, one needs to justify the comparison principle for the solutions of (1.1) especially in the degenerate case. It can be done in a standard way, e.g., by approximating the "sufficiently regular" solutions of (1.1) by the classical solutions of the regularized versions of equation (1.1)

$$f_\varepsilon(x, \partial_t u_\varepsilon) = \Delta_x u_\varepsilon - g_\varepsilon(x, u_\varepsilon).$$

Then, for the classical solutions of the regularized equation the comparison principle is obvious (see e.g. [15]), and passing to the limit $\varepsilon \to 0$, we obtain the comparison principle for the "sufficiently regular" solutions of (1.1). Below, we will prove (in the next section) the existence of the solution u exactly by such regularization procedure (and will not consider the solutions which cannot be obtained by this procedure). That is the reason why we have omitted the rigorous proof of the comparison principle in Proposition 1.3.

Remark 1.4. If the nonlinearity g satisfies more strong dissipativity condition

$$g(x,u)\operatorname{sgn} u \ge -C + \beta|u|^{p+1+\delta}, \quad \delta > 0, \tag{1.27}$$

we will have stronger equation for the super-solution $Y_+(t)$, namely

$$\frac{\alpha}{p+1} Y_+'(t)|Y_+'(t)|^p + \beta Y_+^{p+1+\delta}(t) \le C. \tag{1.28}$$

It is not difficult to see that this "superlinear" equation possesses the "dissipative" estimate in much stronger form:

$$Y_+(t) \le Q(1/t), \quad t \ge 0$$

where the monotonic function Q depends on C, but is independent of $Y_+(0)$. This, in turns, gives the following interior estimate for the L^∞-norm of u

$$\|u(t)\|_{L^\infty(\Omega)} \le Q(1/t) \tag{1.29}$$

which shows that, in that case, every sufficiently regular solution of (1.1) becomes bounded for $t > 0$. As we will see below, it is not the case if the strong dissipativity condition (1.27) is violated.

The next proposition gives the dissipative analogue of energy estimate (1.7).

Proposition 1.4. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1)*. Then, the following estimate holds:*

$$\|u(t)\|_{L^\infty(\Omega)} + \|\nabla_x u(t)\|_{L^2(\Omega)} + \int_t^{t+1} \|\partial_t u(s)\|^{p+2}_{L^{p+2}(\Omega)}\,ds \le \\ \le Q(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)})e^{-\gamma t} + C_* \tag{1.30}$$

where the positive constants C_* and γ and the monotonic function Q is independent of t and u.

Proof. We note that the dissipative estimate for the L^∞-norm of u is already obtained in the previous Proposition, therefore, due to estimate (1.7), we may assume without loss of generality, we may assume that

$$\|u(t)\|_{L^\infty(\Omega)} \le 2C_* \tag{1.31}$$

for all $t \ge 0$ where C_* is the same as in (1.23).

Multiplying now equation (1.1) by $2u(t)$ integrating over Ω and summing the obtained relation with (1.6), we have

$$\frac{d}{dt}[\|\nabla_x u(t)\|^2_{L^2(\Omega)} + 2(G(\cdot,u(t)),1)] + \\ +2(f(\cdot,\partial_t u(t)),\partial_t u(t)) + \|\nabla_x u(t)\|^2_{L^2(\Omega)} + 2(g(\cdot,u(t)),u(t)) = -2(f(\cdot,\partial_t u(t),u(t)). \tag{1.32}$$

Using now estimate (1.31) and the obvious fact that $|f(x,\partial_t u)| \le 1/2|f(x,\partial_t u)|\cdot |\partial_t u| + C$ (see (1.4)), we deduce that

$$\frac{d}{dt}[\|\nabla_x u(t)\|^2_{L^2(\Omega)} + 2(G(\cdot,u(t)),1)]+ \\ + [\|\nabla_x u(t)\|^2_{L^2(\Omega)} + 2(G(\cdot,u(t)),1)] + \gamma\|\partial_t u(t)\|^{p+2}_{L^{p+2}(\Omega)} \le C'_*$$

where the positive constants C'_* and γ are independent of t and u. Applying the Gronwall's inequality to that relation, we deduce (1.30) and finish the proof of Proposition 1.4.

The next theorem estimates the L^q-norm of $\partial_t u$ in terms of the L^∞-norm of u.

Theorem 1.2. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1). *Then, for every $q \ge p+2$, the following estimate holds:*

$$\|\partial_t u\|_{L^q([t,t+1]\times\Omega)} \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}),\quad t\ge T>0 \tag{1.33}$$

where the function Q_T depends on t, but is independent of t and u.

Proof. Analogously to Theorem 1.1, we will prove estimate (1.33) by the iteration procedure. We first note that for $q=p+2$ estimate (1.33) is obtained in Proposition 1.4. We now assume that this estimate is known for $q=p+2+\kappa$ for some $\kappa\ge 0$. Then, due to Proposition 1.2, we have

$$\|\partial_t u(t)\|^{p+2+\kappa}_{L^{p+2+\kappa}(\Omega)} + \int_t^{t+1}(|\nabla_x\partial_t u(s)|^2, |\partial_t u(s)|^\kappa)\,ds \le \\ \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}),\quad t\ge T>0. \tag{1.34}$$

Moreover, interpreting (1.1) as an elliptic equation at every fixed t:

$$\Delta_x u(t) = h_u(t) := f(x,\partial_t u) + g(x,u), \tag{1.35}$$

estimating $f(x,\partial_t u)$ by (1.34) and $g(x,u)$ by (1.23) and using the classical L^q-regularity theorem for the Laplacian (see e.g. [24]), we deduce

$$\|u(t)\|_{W^{2,\frac{p+2+\kappa}{p+1}}(\Omega)} \le C\|h_u(t)\|_{L^{\frac{p+2+\kappa}{p+1}}(\Omega)} \le \\ \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}),\quad t\ge T>0. \tag{1.36}$$

Interpolating between $W^{2,\frac{p+2+\kappa}{p+1}}$ and L^∞, we deduce from (1.36) and (1.23) that

$$\|\nabla_x u(t)\|_{L^{2\frac{p+2+\kappa}{p+1}}(\Omega)} \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}),\quad t\ge T>0. \tag{1.37}$$

Multiplying now equation (1.1) by $\partial_t u|\partial_t u|^{\kappa_1}$ where $\kappa_1>\kappa$ will be determined later and integrating over $[t,t+1]\times\Omega$, we obtain the following inequality

$$\|\partial_t u\|^{p+2+\kappa_1}_{L^{p+2+\kappa_1}([t,t+1]\times\Omega)} \le C\int_t^{t+1}(|\nabla_x u(s)|, |\nabla_x\partial_t u(s)|\cdot|\partial_t u(s)|^{\kappa_1})\,ds+ \\ + C\int_t^{t+1}(|g(\cdot,u(s))|, |\partial_t u(s)|^{\kappa_1+1})\,ds. \tag{1.38}$$

Since the L^∞-estimate for $u(t)$ is already known, the last term in the right hand side of (1.38) can be easily estimated by Hölder inequality

$$(1.39)\quad C\int_t^{t+1}(|g(\cdot,u(s)|,|\partial_t u(s)|^{\kappa_1+1})\,ds\le \\ \le 1/2\|\partial_t u\|^{p+2+\kappa_1}_{L^{p+2+\kappa_1}([t,t+1]\times\Omega)}+Q(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}).$$

So, we only need to estimate the first integral in the right-hand side of (1.38). To this end, we use the Hölder inequality with exponents $q_1=2\frac{p+2+\kappa}{p+1}$, $q_2=1/2$ and $q_3=2\frac{p+2+\kappa}{1+\kappa}$:

$$(1.40)\quad \int_t^{t+1}(|\nabla_x u(s)|,(|\nabla_x\partial_t u(s)|\cdot|\partial_t u(s)|^{\kappa/2})\cdot|\partial_t u(s)|^{\kappa_1-\kappa/2})\,ds\le \\ \le\|\nabla_x u\|_{L^{q_1}([t,t+1]\times\Omega)}\left(\int_t^{t+1}(|\nabla_x\partial_t u(s)|^2,|\partial_t u(s)|^\kappa)\,ds\right)^{1/2}\|\partial_t u\|^{\kappa_1-\kappa/2}_{L^{q_3(\kappa_1-\kappa/2)}(\Omega)}.$$

We see that the first two terms in the right-hand side of (1.40) can be estimated by (1.37) and (1.34) respectively. In order to estimate the third term, we assume that $\kappa_1=\kappa_1(\kappa)$ is chosen in such way that

$$(\kappa_1-\kappa/2)q_3=p+\kappa+2,$$

i.e., $\kappa_1=\kappa+1/2$. Then, the third term is also controlled by (1.37) which, together with (1.38) and (1.39) gives

$$(1.41)\qquad \|\partial_t u\|_{L^{p+2+\kappa+1/2}(\Omega)}\le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)}).$$

Thus, the above described procedure allows to increase the exponent q by in estimate (1.33) by $1/2$. Thus, starting from $q_0=p+2$ (for which this estimate is known) and iterating this procedure sufficiently many times, we obtain (1.33) for any finite $q>p+2$. Theorem 1.2 is proven.

The next corollary combines Theorems 1.1 and 1.2.

Corollary 1.1. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of* (1.1). *Then, for any $1<q<\infty$ and every $t\ge T>0$, the following estimate holds:*

$$(1.42)\qquad \|\partial_t u(t)\|_{L^\infty(\Omega)}+\|u(t)\|_{W^{2,q}(\Omega)}\le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)})e^{-\gamma t}+C_T$$

where the $\gamma>0$ and the constant C_T and the monotonic function Q_T depend on T, but are independent of u and t.

Proof. Indeed, due to dissipative estimate (1.30), it is sufficient to verify (1.42) for $t\le 1$ only. In order to obtain it, we first estimate $\|\partial_t u\|_{L^{3p/2+\delta}([t,t+1]\times\Omega)}$, for some $\delta>0$, via the initial data by Theorem 1.2, after estimate $\|\partial_t u\|_{L^\infty([t,t+1],L^{3p/2+\delta}(\Omega))}$ by Proposition 1.2 and finally estimate the L^∞-norm of $\partial_t u$ by Theorem 1.1. The estimate for the $W^{2,q}$-norm of $u(t)$ can be obtained after that from the elliptic equation (1.35). Corollary 1.1 is proven.

The following example shows that interior estimate (1.42) can be violated for weak energy solutions of (1.1) with the initial data $u_0 \notin L^\infty(\Omega)$.

Example 1.1. Let us consider the following simplest version of equation (1.1):

$$\partial_t u(t)|\partial_t u(t)|^p = \Delta_x u(t), \quad p > 1, \tag{1.43}$$

and seek for its radially symmetric singular solution in the form $u_0(t,x) = \frac{\phi(t)}{|x|^{2/p}}$. Inserting this into equation (1.43), we deduce the equation for $\phi(t)$, namely,

$$\phi'(t)|\phi'(t)|^p + \frac{4(p-1)}{p^2}\phi(t) = 0. \tag{1.44}$$

Thus, (1.43) possesses a family of special solutions of the form

$$u_0(t,x) = \frac{(C-\delta t)_+^{1+1/p}}{|x|^{2/p}} \tag{1.45}$$

where $\delta = \delta(p)$ is some fixed constant and $C \geq 0$ is arbitrary.

Having this automodel solution of the degenerate equation (1.43), one can easily construct analogous singular solutions for the non-degenerate equations of the form (1.1) as well. Namely, using the obvious fact that $\partial_t u_0 = -(1+1/p)(C-t)^{-1}u_0$, one verifies that (1.45) also solves

$$\partial_t u + 2\partial_t u|\partial_t u|^p = \Delta_x u - (1+1/p)\frac{u}{(C-t)} - (1+1/p)^{p+1}\frac{u|u|^p}{(C-t)^{p+1}}. \tag{1.46}$$

Without loss of generality, we may assume that $0 \in \Omega$. Then, equation (1.46) has the form of (1.1) with non-degenerate $f(x, \partial_t u)$, non-autonomous $g = g(t,x,u)$ satisfying the dissipativity assumption (1.5) and smooth non-homogeneous boundary conditions $u\big|_{\partial\Omega} = u_0(t,x)$ (we will consider this equation on the time interval $t \in [0, C/2]$ only). So, as it is not difficult to see that all the above verified estimates remain true (after the corresponding minor changings) for such non-autonomous equations as well.

On the other hand, this equation possesses a singular week solution (1.45) which has not any regularizing on the time interval $t \in [0, C/2]$. Thus, the analogue of interior estimate (1.42) clearly does not take place for such weak solutions.

We also mention that the solution (1.45) has a finite energy ($u_0(t) \in W^{1,2}(\Omega)$, $G(\cdot, u_0(t)) \in L^1(\Omega)$) if and only if $p > 4$.

Remark 1.5. We see that, in the supercritical case $p > 4$, the energy phase space is too large in order to be the adequate phase space for problem (1.1) (since the non-regularizing "pathological" solutions like (1.45) are allowed). In contrast to this, the phase space $\Phi := L^\infty(\Omega) \cap W^{1,2}(\Omega)$ does not contain such solutions and, according to Corollary 1.1, gives, in a sense minimal, reasonable phase space for that problem. That is the reason why we will consider below equation (1.1) in the phase space Φ.

We also recall that, due to Remark 1.4, the above singular solutions cannot exists if g satisfies the strong dissipativity assumption (1.27) with arbitrarily small positive δ. As equation (1.46) shows such singular solutions can exist if $\delta = 0$. Thus, the above regularity analysis seems to be sharp.

We conclude the section by obtaining a little more regularity of $\partial_t u$ which is however important for our the existence of a solution in the next section.

Proposition 1.5. *Let the above assumptions hold and let $u(t)$ be a sufficiently regular solution of equation* (1.1). *Then, the following estimate hold:*

$$(1.47)\quad \|\partial_t u(t)\|_{W^{1,2}(\Omega)} + \int_t^{t+1} (|\partial_t^2 u(s)|^2, |\partial_t u(s)|^p)\,ds + \|\partial_t u\|_{W^{\beta,2}([t,t+1]\times\Omega)} \le \\ \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)})e^{-\alpha t} + C_*,\ t \ge T > 0$$

where α and β and C_* are some positive constants and the monotonic function Q_T depends on T, but is independent of t and u.

Proof. Indeed, due to Corollary 1.1 and Proposition 1.2, it is sufficient to prove estimate (1.47) under the additional assumption that

$$(1.48)\qquad \|\partial_t u(t)\|_{L^\infty(\Omega)} + \int_t^{t+1} \|\nabla_x \partial_t u(s)\|^2_{L^2(\Omega)}\,ds \le C'_*, \quad t \in \mathbb{R}_+.$$

In order to do so, let us multiply equation (1.8) by $t\partial_t v(t)$ and integrate over $x \in \Omega$. Then, using assumption (1.4), we get

$$(1.49)\quad \alpha t(|\partial_t v(t)|^2, |v(t)|^p) + \partial_t[t\|\nabla_x v(t)\|^2_{L^2(\Omega)} + t(g'(\cdot, u(t))v(t), v(t))] \le \\ \le \|\nabla_x v(t)\|^2_{L^2(\Omega)} + (g'(\cdot, u(t))v(t), v(t)) + t(g''(\cdot, u(t))v^2(t), v(t)).$$

Integrating this inequality by t from $\max\{0, t-1\}$ till $t+1$ and using (1.48), we deduce estimate (1.47) for the first two terms into the left-hand side. In order to obtain the last term in that estimate, we note that, from the first two ones, we infer that $|\partial_t u|^{1+p/2} \in W^{1,2}([t,t+1]\times\Omega)$ and

$$(1.50)\quad \||\partial_t u|^{1+p/2}\|_{W^{1,2}([t,t+1]\times\Omega)} \le Q_T(\|u_0\|_{L^\infty(\Omega)\cap W^{1,2}(\Omega)})e^{-\alpha t} + C_*,\ t \ge T > 0.$$

which, in turns, implies the required estimate for $\|\partial_t u\|_{W^{\beta,2}([t,t+1]\times\Omega)}$ with $\beta < \frac{2}{p+2}$ and Proposition 1.5 is proven.

Remark 1.6. It is not difficult to see that the technique developed below is applicable for slightly more general than (1.1) equation of the form

$$(1.51)\qquad f(x, u, \partial_t u) = \Delta_x u - g(x, u)$$

where the nonlinearity f depends also on u. Then, in addition to (1.4), one should pose the following assumption

$$(1.52)\qquad |f''_{u,\partial_t u}(x, u, \partial_t u)| \le C(|u|)(1 + |\partial_t u|^{p-1}).$$

Roughly speaking this assumption means that the "leading part" of f with respect to $\partial_t u$ is independent of u:

$$f(x, u, \partial_t u) = f_0(x, \partial_t u) + f_1(x, u, \partial_t u)$$

where f_0 satisfies (1.4) and $|f_1(x, u, \partial_t u)| \le C(u)(1 + |\partial_t u|^p)$. We also note that, unfortunately, assumption (1.52) is not satisfied in the quasilinear case

$$(1.53)\qquad f(x, u, \partial_t u) = \partial_t \phi(x, u)$$

which requires a different technique, e.g. related with the variable change $w = \phi(x, u)$ and reducing the equation to the the quasilinear second order parabolic equation in a standard form, see [15].

§2 Existence and uniqueness of solutions.

Using a priori estimates of the previous section, we establish here the existence of the appropriate solution of problem (1.1) and, under the additional assumptions that the domains of degeneration and nonmonotonicity of equation (1.1) are well-separated, we verify also that this solution is unique. As we have already noted in Remark 1.5, we will consider equation (1.1) in the phase space $\Phi := L^\infty(\Omega) \cap W^{1,2}(\Omega)$. To be more precise, we define a solution of (1.1) as follows.

Definition 2.1. A function $u = u(t,x)$ is a (bounded) solution of (1.1) if

$$u \in L^\infty([0,T]\times\Omega) \cap L^\infty([0,T], W^{1,2}(\Omega)), \quad \partial_t u \in L^{p+2}([0,T]\times\Omega) \tag{2.1}$$

and satisfies equation (1.1) in the sense of distributions.

The next theorem gives the existence of such solutions.

Theorem 2.1. *Let the assumptions of Section 1 hold. Then, for any $u_0 \in \Phi$, equation* (1.1) *possesses at least one solution $u(t)$ in the sense of Definition 2.1.*

Proof. We first approximate the nonlinearities f and g in equation (1.1) by the smooth ones $f_\varepsilon(x,v)$ and $g_\varepsilon(x,v)$, $\varepsilon > 0$ in such way that (1.2–1.5) will be satisfied uniformly with respect to $\varepsilon \to 0$ and, for every $R > 0$,

$$\sup_{|v|\le R} \|f_\varepsilon(\cdot,v) - f(\cdot,v)\|_{L^1(\Omega)} \to 0 \ \text{ as } \varepsilon \to 0 \tag{2.2}$$

and analogously for g_ε and g. Moreover, we also assume that

$$a_\varepsilon(x) > a_\varepsilon > 0 \tag{2.3}$$

and, consequently, the associated approximations

$$f_\varepsilon(x, \partial_t u_\varepsilon) = \Delta_x u_\varepsilon - g_\varepsilon(x, u_\varepsilon), \quad u\big|_{t=0} = u_0^\varepsilon \tag{2.4}$$

will be smooth and non-degenerate. Thus, all of the estimates of the previous section hold for equation (2.4) *uniformly* with respect to ε.

Let us first verify the existence of a solution for the approximate equation (2.4). To this end, we rewrite it with respect to the variable $v_\varepsilon := \partial_t u_\varepsilon$:

$$f'_\varepsilon(x, v_\varepsilon)\partial_t v_\varepsilon = \Delta_x v_\varepsilon - g'_\varepsilon(x, u_\varepsilon) v_\varepsilon, \quad v_\varepsilon\big|_{t=0} = v_0^\varepsilon \tag{2.5}$$

where v_0^ε solves

$$f_\varepsilon(x, v_0^\varepsilon) = \Delta_x u_0^\varepsilon - g_\varepsilon(x, u_0^\varepsilon). \tag{2.6}$$

It is not difficult to verify using our assumptions on f that v_0^ε is uniquely defined by u_0^ε and smooth if u_0^ε is smooth.

Then, according to Theorem 1.1 and Proposition 1.3, we have

$$\|\partial_t u_\varepsilon\|_{L^\infty([0,T]\times\Omega)} + \|u\|_{L^\infty([0,T]\times\Omega)} \le C(u_0^\varepsilon). \tag{2.7}$$

Moreover, due to (2.3), equation (2.5) can be now interpreted as a linear non-degenerate second order parabolic equation with the coefficients belonging to L^∞. Applying the classical De Giorgi estimate to this equation, we deduce

$$\|\partial_t U_\varepsilon\|_{C^\beta([0,T]\times\Omega)} \le C(u_0^\varepsilon), \tag{2.8}$$

for some $\beta > 0$, see e.g. [15, Chapter III, §10]. Returning back to equation (2.4) and using the classical C^β-estimates for parabolic equations, see e.g. [15, Chapter IV, Th.5.2] and estimate (2.8), we have

$$\|u_\varepsilon\|_{C^{1+\beta/2,2+\beta}([0,T]\times\Omega)} \le C(u_0^\varepsilon). \tag{2.9}$$

Finally, returning again to equation (2.5) and using (2.9) and the C^β-estimates mentioned before, we finally deduce

$$\|\partial_t u_\varepsilon\|_{C^{1+\beta/2,2+\beta}([0,T]\times\Omega)} + \|u_\varepsilon\|_{C^{1+\beta/2,2+\beta}([0,T]\times\Omega)} \le C(u_0^\varepsilon). \tag{2.10}$$

In order to verify the existence of a solution, we introduce the integration operator

$$I(v_\varepsilon)(t) := u_0^\varepsilon + \int_0^t v_\varepsilon(s)\,ds$$

and rewrite (2.5) in the following equivalent form:

$$\partial_t v_\varepsilon = \frac{1}{f'_\varepsilon(x,v_\varepsilon)}\Delta_x v_\varepsilon - \frac{g'_\varepsilon(x,I(v_\varepsilon))}{f'_\varepsilon(x,v_\varepsilon)} v_\varepsilon. \tag{2.11}$$

The existence of a solution for such quasilinear equation can be obtained based on a priori estimate (2.10) and Leray-Schauder principle, exactly as in [15, Chapter VI, §4].

Thus, the existence of the solutions $u_\varepsilon(t)$ for the auxiliary problem (2.4) is verified for any smooth u_0^ε.

Let us approximate the initial data $u_0 \in \Phi$ by a sequence of smooth ones u_0^ε which converges to u_0 as $\varepsilon \to 0$, say, in $L^q(\Omega) \cap W^{1,2}(\Omega)$, $q \gg 1$ and weakly-$*$ in $L^\infty(\Omega)$, construct the associated solutions $u_\varepsilon(t)$ of problem (2.4) and pass to the limit $\varepsilon \to 0$. We claim that this procedure gives the desired solution $u(t)$ of the initial problem (1.1)

Indeed, according to Proposition 1.4, the following estimate

$$\|u(t)\|_\Phi + \int_t^{t+1} \|\partial_t u(s)\|^{p+2}_{L^{p+2}(\Omega)}\,ds \le Q(\|u_0\|_\Phi)e^{-\alpha t} + C_* \tag{2.12}$$

holds *uniformly* with respect to $\varepsilon \to 0$. Thus, the limit function $u(t)$ will also satisfy this estimate and, therefore, will belong to the class (2.1). So, we only need to verify that $u(t)$ satisfies (1.1) in the sense of distributions.

To this end, applying Corollary 1.1 and Proposition 1.5 to the auxiliary problem (2.4) we infer

$$\|\partial_t u\|_{L^\infty([t,T]\times\Omega)} + \|\partial_t u\|_{W^{\beta,2}([t,T]\times\Omega)} + \|u\|_{L^\infty([t,T]\times W^{2,q}(\Omega))} \le C_{t,T} \tag{2.13}$$

where $t > 0$ and the constant $C_{t,T}$ is independent of ε. Thus, without loss of generality, we may assume that, for every $t > 0$ the sequence u_ε converges $*$-weakly to u in the spaces involved to the right-hand side of (2.13). In particular, these weak convergence implies that

$$(2.14)\quad u_\varepsilon \to u \quad \text{strongly in} \quad C([t,T]\times\Omega),\ \ \partial_t u_\varepsilon \to \partial_t u \ \text{ strongly in } \quad L^q([t,T]\times\Omega)$$

for every $q \geq 1$. Having this convergences, it is not difficult to check that equation (2.4) converges to (1.1) as $\varepsilon \to$ in the sense of distributions. In a fact, the convergence of the linear term $\Delta_x u_\varepsilon$ is obvious and the convergence in the nonlinear terms f_ε and g_ε follows immediately from the following standard convergence

$$(2.15)\quad \|f_\varepsilon(\cdot,\partial_t u_\varepsilon) - f(\cdot,\partial_t u)\|_{L^1([t,T]\times\Omega)}+ \\ + \|g_\varepsilon(\cdot,u_\varepsilon) - g(\cdot,u)\|_{L^1([t,T]\times\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Indeed, due to assumption (1.4) and the fact that the L^∞-norm of $\partial_t u$ is bounded, we have

$$\|f_\varepsilon(\cdot,\partial_t u_\varepsilon) - f(\cdot,\partial_t u)\|_{L^1([t,T]\times\Omega)} \leq \|f_\varepsilon(\cdot,\partial_t u_\varepsilon) - f_\varepsilon(\cdot,\partial_t u)\|_{L^1([t,T]\times\Omega)}+ \\ + \|f_\varepsilon(\cdot,\partial_t u) - f(\cdot,\partial_t u)\|_{L^1([t,T]\times\Omega)} \leq \\ \leq C_T\|\partial_t u_\varepsilon - \partial_t u\|_{L^1([t,T]\times\Omega)} + \|f_\varepsilon(\cdot,\partial_t u) - f(\cdot,\partial_t u)\|_{L^1([t,T]\times\Omega)}.$$

The first term in the right-hand side of this inequality tends to zero due to (2.14) and the second one – due to (2.3) and the fact that the L^∞-norm of $\partial_t u$ is bounded. The convergence of $g_\varepsilon(x,u)$ to $g(x,u)$ can be verified analogously. Theorem 2.1 is proven.

Corollary 2.1. *The solution $u(t)$ of problem* (1.1) *constructed in Theorem 2.1 satisfies all of the estimated obtained in the previous section and, thus, gives a rigorous interpretation of the notion of a "sufficiently regular solution" used there.*

Remark 2.1. Let us assume, in addition, that the functions f and g are Hölder continuous with respect to x. Then, more delicate analysis of equation (1.8) based on the adaptation of the De Giorgi technique to degenerate parabolic equations (see [2], [5] and [13]), allows to establish not only the boundedness of $\partial_t u$, but also its Hölder continuity with respect to (t,x) with a sufficiently small positive Hölder exponent β depending on p. Applying after that the usual C^β-regularity theorem for elliptic equations to (1.1) (and interpreting $f(x,\partial_t u)$ as the external force), we then establish that $u(t) \in C^{2+\beta}(\Omega)$. Thus, finally, we obtain that, even in the degenerate case, the solution u belongs to

$$(2.16)\qquad u \in C^{1+\beta,2+\beta}([t,T]\times\Omega)$$

where we can take $t = 0$ if the initial data is smooth enough.

Therefore, even in the degenerate case, equation (1.1) possesses *classical* solutions for every sufficiently smooth initial data u_0. This shows the principal difference between the degenerate fully nonlinear problem (1.1) and standard quasilinear degenerate/singular equations where the classical solutions usually do not exist, see [5], [13].

Our next task is to verify the uniqueness of a solution for problem (1.1). To this end, we need the additional assumption on the structure of the nonlinearities f and g

$$K_R[a(x)]^{1/2} + \inf_{|u|\le R} g'_u(x,u) \ge 0 \tag{2.17}$$

for every $R>0$ and some $K_R>0$ depending on R. The last assumption means, in particular, that $g'_v(x,v)\ge 0$ for every x such that $a(x)=0$. Thus, equation (1.1) can be non-monotonic only in the subdomain of Ω where f is non-degenerate. As we have already mentioned, this assumption is crucial for the uniqueness. Indeed, the simplest ODE example:

$$y'(t)|y'(t)|^p = y(t) \tag{2.18}$$

shows that the solution $u(t)$ of (1.1) can be non-unique if (1.6) is violated. In contrast to this, the next theorem shows that the solution is indeed unique if (2.18) holds.

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold and let, in addition,* (2.17) *be satisfied. Then, for every two solutions $u_1(t)$ and $u_2(t)$ of* (1.1) *(in the sense of Definition 2.1), the following estimate holds:*

$$\|u_1(t)-u_2(t)\|_{W^{1,2}(\Omega)} \le Ce^{Kt}\|u_1(0)-u_2(0)\|_{W^{1,2}(\Omega)} \tag{2.19}$$

where the constants C and K depend on the Φ-norms of $u_1(0)$ and $u_2(0)$, but are independent of t. In particular, the solution $u(t)$ of problem (1.1) *is uniquely determined by its initial data $u_0\in\Phi$.*

Proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of problem (1.1) with the initial data belonging to Φ and let $v(t):=u_1(t)-u_2(t)$. Then, this function solves

$$a(x)v + [\phi(x,\partial_t u_1)-\phi(x,\partial_t u_2)] = \Delta_x v - [g(x,u_1)-g(x,u_2)]. \tag{2.20}$$

Moreover, according to Definition 2.1, we have

$$\|u_1\|_{L^\infty([0,T]\times\Omega)} + \|u_2\|_{L^\infty([0,T],\times\Omega)} \le R \tag{2.21}$$

for some finite R. Multiplying now equation (2.20) by $\partial_t v$ and using assumption (1.4), the obvious formula

$$\begin{aligned}\partial_t[G(x,u_1)-G(x,u_2)-g(u_2)v] = (g(x,u_1)-g(x,u_2),v)+\\ +\partial_t u_2[g(x,u_1)-g(x,u_2)-g'_u(x,u_2)v]\end{aligned} \tag{2.22}$$

with $G(x,u):=\int_0^u g(x,s)\,ds$, the fact that $g\in C^2$ with respect to u and estimate (2.21), we have

$$\begin{aligned}(a(\cdot)\partial_t v(t),\partial_t v(t)) + \partial_t[1/2\|\nabla_x v(t)\|^2_{L^2(\Omega)}+\\ +(G(\cdot,u_1(t))-G(\cdot,u_2(t))-g(\cdot,u_2(t))v(t),1)] \le C_R(|\partial_t u_2(t),|v(t)|^2).\end{aligned} \tag{2.23}$$

We also note that

$$(2.24)\quad (G(\cdot,u_1(t))-G(\cdot,u_2(t))-g(\cdot,u_2(t))v(t),1)=$$
$$=\left(\int_0^1\int_0^1 g'_u(x,u_2+s_1s_2v)\,ds_1\,ds_2,|v|^2\right)\geq\left(\inf_{|u|\leq 2R}g'_u(x,u),|v|^2\right).$$

Taking now a sum of (2.23) with the following simple inequality:

$$(2.25)\quad \partial_t(K_{2R}[a(\cdot)]^{1/2}v,v)=2K_{2R}([a(\cdot)]^{1/2}\partial_t v,v)\leq$$
$$\leq 1/2(a(\cdot)\partial_t v,\partial_t v)+4K_{2R}^2\|v\|^2_{L^2(\Omega)}$$

we get

$$(2.26)\quad \partial_t\mathcal{G}(v(t))+1/2(a(x)\partial_t v(t),\partial_t v(t))\leq$$
$$\leq\|\partial_t u_2(t)\|_{L^{3/2}(\Omega)}\|v(t)\|^2_{W^{1,2}(\Omega)}+4K_{2R}^2\|v(t)\|^2_{L^2(\Omega)}$$

where

$$(2.26')\quad \mathcal{G}(v):=1/2\|\nabla_x v\|^2_{L^2(\Omega)}+$$
$$+(G(\cdot,u_1(t))-G(\cdot,u_2(t))-g(\cdot,u_2(t))v(t),1)+K_{2R}([a(\cdot)]^{1/2}v(t),v(t)).$$

Moreover, due to assumption (2.17) and estimate (2.24), we have

$$(2.27)\qquad C_R\|\nabla_x v(t)\|^2_{L^2(\Omega)}\geq\mathcal{G}(v(t))\geq 1/2\|\nabla_x v(t)\|^2_{L^2(\Omega)}.$$

Thus, (2.26) implies that

$$(2.28)\qquad \partial_t\mathcal{G}(v(t))\leq C_R(1+\|\partial_t u_2(t)\|_{L^{3/2}(\Omega)})\mathcal{G}(v(t)).$$

Applying now the Gronwall's inequality to (2.28) and using that

$$\int_0^T\|\partial_t u(t)\|_{L^{3/2}(\Omega)}\,dt\leq C(1+\int_0^t\|\partial_t u(t)\|^{p+2}_{L^{p+2}(\Omega)}\,dt)\leq C(T+1)$$

(due to Proposition 1.4), we obtain estimate (2.19) and finish the proof of Theorem 2.2.

Remark 2.3. The uniqueness theorem proved above shows, in particular, that any solution $u(t)$ of equation (1.1) in the sense of Definition 2.1 is "sufficiently regular" in the sense that it satisfies all of the estimates of Section 1.

It is also worth to note that the uniqueness condition (2.17) is automatically satisfied if equation (1.1) is non-degenerate:

$$(2.29)\qquad a(x)\geq a_0>0.$$

Thus, for the non-degenerate case, we have the uniqueness under the assumptions of the existence Theorem 2.1.

Remark 2.4. Arguing in a little more accurate way and using formulae (A.2) and (A.3), one can prove the uniqueness theorem under slightly weaker assumption that

$$(2.30)\qquad \inf_u g'_u(x,u)+\lambda_1>\varepsilon,\quad x\in\Omega_0:=\{x\in\Omega,\,a(x)=0\}$$

where $\varepsilon>0$ is arbitrarily small fixed and λ_1 is the first eigenvalue of the Laplacian in Ω.

§3 Attractors and exponential attractors

In this section, we study the long-time behavior of solutions of problem (1.1) under the uniqueness assumption (2.17). In this case, equation (1.1) generates a dissipative semigroup $\{S(t),\, t\ge 0\}$ in the phase space Φ:

$$S(t)u_0 := u(t), \quad u(t) \text{ solves (1.1) with } u(0)=u_0 \tag{3.1}$$

We recall that a compact subset $\mathcal{A}$ in Φ is a global attractor of the semigroup $S(t):\Phi\to\Phi$ if the following conditions are satisfied

1) The set $\mathcal{A}$ is strictly invariant: $S(t)\mathcal{A}=\mathcal{A}$, $t\ge 0$.

2) $\mathcal{A}$ attracts the images of all bounded subsets of Φ as time tends to infinity, i.e., for every neighborhood $\mathcal{O}(\mathcal{A})$ and every bounded subset B there exists time $T=T(\mathcal{O},B)$ such that

$$S(t)B\subset \mathcal{O}(\mathcal{A}) \tag{3.2}$$

for all $t\ge T$.

We recall that the second condition can be rewritten in the following equivalent form

$$\lim_{t\to\infty} \operatorname{dist}(S(t)B,\mathcal{A})=0, \quad \text{for all bounded } B\subset\Phi \tag{3.3}$$

where $\operatorname{dist}(X,Y)$ is a standard non-symmetric Hausdorff distance between sets X and Y in Φ:

$$\operatorname{dist}(X,Y):=\sup_{x\in X}\inf_{y\in Y}\|x-y\|_\Phi. \tag{3.4}$$

The following theorem establishes the existence of a global attractor for the semigroup (3.1) associated with equation (1.1).

Theorem 3.1. *Let the assumptions of Theorem 2.2 hold. Then, the semigroup $S(t)$ associated with equation* (3.1) *possesses a compact global attractor $\mathcal{A}$ in Φ which is bounded in $W^{2,q}(\Omega)$ for every finite q which can be described as follows:*

$$\mathcal{A}=\mathcal{K}\big|_{t=0} \tag{3.5}$$

where $\mathcal{K}$ is a set of solutions of (1.1) *defined for every $t\in\mathbb{R}$ and bounded*

$$\mathcal{K}:=\{u\in L^\infty(\mathbb{R},\Phi), \quad u \textit{ solves } (1.1)\}. \tag{3.6}$$

Proof. According to the standard attractor existence theorem, see e.g. [1], we only need to verify that

1) The semigroup $S(t)$ has a closed graph in Φ.

2) This semigroup possesses a (pre)compact absorbing set $\mathcal{B}$ in the phase space Φ. The latter means that for any bounded subset $B\subset\Phi$ there exists time $T=T(B)$ such that

$$S(t)B\subset\mathcal{B}, \quad t\ge T. \tag{3.7}$$

Let us verify these conditions for the semigroup (3.1). Indeed, the first condition is immediate, since, due to Theorem 2.2, the semigroup $S(t)$ is Lipschitz continuous in a weaker topology of $W^{1,2}(\Omega)$. Moreover, from Corollary 1.1 and estimate (1.42), we conclude that the set

$$\mathcal{B} = \mathcal{B}_q := \{u_0 \in W^{2,q}(\Omega), \ \ \|u_0\|_{W^{2,q}(\Omega)} \le R\} \tag{3.8}$$

will be absorbing for the semigroup $S(t)$ if $R = R(q)$ is large enough. It remains to note that $\mathcal{B}_q$ is compactly embedded in Φ if q is large enough.

Thus, the existence of the attractor $\mathcal{A}$ is proven. It is bounded in $W^{2,q}(\Omega)$ since $\mathcal{A} \subset \mathcal{B}_q$ and, finally, description (3.6) is also the standard corollary of the abstract attractor's existence theorem. Theorem 3.1 is proven.

Our next task is to verify the finite-dimensionality of the global attractor and to construct the associated *exponential* attractor. We recall that a compact set $\mathcal{M}$ in Φ is an exponential attractor for the semigroup $S(t)$ if the following conditions are satisfied:

1) The set $\mathcal{M}$ is semi-invariant in the sense that $S(t)\mathcal{M} \subset \mathcal{M}$, $t \ge 0$;

2) The fractal dimension of $\mathcal{M}$ in Φ is finite:

$$\dim_f(\mathcal{M}, \Phi) \le C < \infty; \tag{3.9}$$

3) The set $\mathcal{M}$ attracts *exponentially* the images of all bounded sets in Φ, i.e., for every bounded $B \subset \Phi$, one has

$$\operatorname{dist}(S(t)B, \mathcal{M}) \le Q(\|B\|_\Phi)e^{-\beta t} \tag{3.10}$$

for some positive β and some monotonic function Q.

Remark 3.1. It is worth to recall that, in the global attractors theory, it is usually extremely difficult to estimate the rate of convergence in (3.3) or/and to express it in terms of the physical parameters of the system considered and this is one of the main drawbacks of the theory. Indeed, as the simplest examples show, the rate of convergence in (3.3) can be arbitrarily slow and non-uniform with respect to the parameters of the system considered. This, in turns, makes the global attractor sensitive to small perturbations and, in a sense, unobservable in experiments.

The concept of *exponential* attractor (suggested in [7]) overcomes this difficulty. Indeed, in contrast to the global attractors theory, the constant $\beta > 0$ and the function Q in (3.10) can be explicitly found in terms of the physical parameters and, moreover, the exponential attractor occurs robust (continuous and even Hölder continuous with respect to perturbations, see [7-10] for more details.

The main disadvantage of this theory is, of course, the non-uniqueness of the exponential attractor which makes its concrete choice artificial. This problem is partially solved in [9] by finding a rather simple explicit construction of the exponential attractor which gives a one-valued "branch" of exponential attractors depending in a Hölder continuous way on the dynamical system considered.

The next theorem establish the existence of the exponential attractor for problem (1.1) using some modification of the so-called method of l-trajectories.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. Then, semigroup $S(t)$ associated with equation* (1.1) *possesses an exponential attractor $\mathcal{M}$ in Φ in the sense of Definition 3.1.*

Proof. The proof of this theorem is based on the following abstract exponential attractor's existence theorem.

Proposition 3.1. *Let H, V, V_1 be Banach spaces such that the embedding $V_1 \hookrightarrow V$ is compact. Let B be a closed bounded subset of H, and let $\mathbb{S} : B \to B$ be a map. Assume also that there exists a uniformly Lipschitz continuous map $\mathbb{T} : B \to V_1$, i.e.*

$$\|\mathbb{T}b_1 - \mathbb{T}b_2\|_{V_1} \le L\|b_1 - b_2\|_H, \qquad \forall b_1, b_2 \in B, \tag{3.11}$$

for some $L \ge 0$, such that

$$\|\mathbb{S}b_1 - \mathbb{S}b_2\|_H \le \vartheta\|b_1 - b_2\|_H + K\|\mathbb{T}b_1 - \mathbb{T}b_2\|_V, \qquad \forall b_1, b_2 \in B, \tag{3.12}$$

for some $\vartheta < 1/2$ and $K \ge 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_d \subset B$ of the semigroup $\{S(n) := S^n, n \in \mathbb{Z}_+\}$ with discrete time in the phase space H (see Definition 3.1).

The proof of this proposition in the particular instance when $H = V_1$ and $\mathbb{T}$ is the identity map is given in [8]. The general proof repeats word by word this particular case and so thus omitted (see also [9] and [26]).

In order to apply this abstract result to our problem (1.1), we first note that it is sufficient to verify the existence of an exponential attractor not in the whole phase space Φ, but only for the restriction of $S(t)$ on some absorbing set in Φ. In order to construct the proper absorbing set, we recall that, due to Proposition 1.4, the ball

$$B(R) := \{u_0 \in \Phi, \|u_0\|_\Phi \le R\} \tag{3.13}$$

will be absorbing for semigroup $S(t)$ if R is large enough. Since we want the absorbing set to be semi-invariant with respect to the semigroup, we further set

$$B_1 = [\cup_{t \ge 0} S(t)B(R)]_\Phi \tag{3.14}$$

where $[\cdot]$ denotes the closure in the space Φ. And, finally, in order to have some compactness, we set

$$B := S(1)B_1. \tag{3.15}$$

Then, on the one hand, it is not difficult to verify that B is a compact semi-invariant subset of the phase space Φ, i.e.

$$S(t)B \subset B, \quad t \ge 0, \tag{3.16}$$

and, on the other hand, due to Corollary 1.1, we have

$$\|\partial_t u(t)\|_{L^\infty(\Omega)} + \|u(t)\|_{W^{2,q}(\Omega)} \le C \tag{3.17}$$

for every trajectory $u(t)$ of equation (1.1) starting from $u_0 \in B$ (and the constant $C = C_q$ depends on q, but is independent of the choice of $u_0 \in B$). In particular, we see that $B \subset W^{2,q}(\Omega)$ for every finite q.

Thus, we will construct below the exponential attractor $\mathcal{M}$ for the restriction of the semigroup $S(t)$ on the above absorbing set B. To this end, we need the following Lemma which gives the key estimate of the form (3.11).

Lemma 3.1. *Let the above assumptions hold. Then, for every two solutions $u_1(t)$ and $u_2(t)$ such that $u_i(0) \in B$, $i = 1, 2$, the following estimate holds:*

$$\begin{aligned}(3.18)\quad &\|u_1(t) - u_2(t)\|^2_{W^{1,2}(\Omega)} + \gamma\int_0^t (a(\cdot), |\partial_t u_1(s) - \partial_t u_2(s)|^2)\,ds \le \\ &\le Ce^{-\beta t}\|u_1(0) - u_2(0)\|^2_{W^{1,2}(\Omega)} + Ce^{kt}\int_0^t \|\chi_{\Omega\setminus\Omega_0}(\cdot)(u_1(s) - u_2(s))\|^2_{L^2(\Omega)}\,ds\end{aligned}$$

where $\Omega_0 := \{x \in \Omega,\ \ a(x) = 0\}$, $\chi_V(\cdot)$ means the characteristic function of the set V and the positive constants β, γ, C and K are independent of u_1 and u_2.

Proof. We first note that (2.25) can be improved as follows

$$(3.19)\qquad \partial_t(K_{2R}[a(\cdot)]^{1/2}v, v) \le 1/2(a(\cdot)\partial_t v, \partial_t v) + 4K_{2R}^2\|\chi_{\Omega\setminus\Omega_0}(\cdot)v\|^2_{L^2(\Omega)}$$

and, consequently, (2.26) reads

$$\begin{aligned}(3.20)\quad &\partial_t\mathcal{G}(v(t)) + 1/2(a(x)\partial_t v(t), \partial_t v(t)) + (l_{u_1,u_2}\partial_t v(t), \partial_t v(t)) \le \\ &\le \|\partial_t u_2(t)\|_{L^{3/2}(\Omega)}\|v(t)\|^2_{W^{1,2}(\Omega)} + 4K_{2R}\|\chi_{\Omega\setminus\Omega_0}v(t)\|^2_{L^2(\Omega)}\end{aligned}$$

where the function $\mathcal{G}$ is defined by (2.26′) and $l_{u_1,u_2} := \int_0^1 \phi'_{\partial_t u}(x, s\partial_t u_1 + (1-s)\partial_t u_2)\,ds$.

Multiplying now equation (2.20) by $v(t) := u_1(t) - u_2(t)$,

$$\begin{aligned}(3.21)\quad &\|\nabla_x v(t)\|^2_{L^2(\Omega)} + (g(\cdot, u_1(t)) - g(\cdot, u_2(t)), v(t)) + \\ &+ (l_{u_1,u_2}(t)\partial_t v(t), v(t)) + 1/2\partial_t(a(\cdot)v(t), v(t)) + \\ &+ (g(\cdot, u_1(t)) - g(\cdot, u_2(t)), v(t)) = 0.\end{aligned}$$

Due to conditions (1.4), we may assume, without loss of generality that $\phi'_{\partial_t u}(x, 0) = 0$. Then, using estimate (3.17) and the fact that $l_{u_1,u_2}(t) \ge 0$, we have

$$\begin{aligned}(3.22)\quad &|(l_{u_1,u_2}(t)\partial_t v(t), v(t))| \le 1/4(l_{u_1,u_2}\partial_t v(t), \partial_t v(t)) + \\ &+ (l_{u_1,u_2}v(t), v(t)) \le 1/4(l_{u_1,u_2}\partial_t v(t), \partial_t v(t)) + \\ &+ C(\|\partial_t u_1(t)\|_{L^{3/2}(\Omega)} + \|\partial_t u_2(t)\|_{L^{3/2}(\Omega)})\|\nabla_x v(t)\|^2_{W^{1,2}(\Omega)}.\end{aligned}$$

Moreover, due to the uniqueness assumption (2.17),

$$(3.23)\qquad (g(\cdot, u_1(t)) - g(\cdot, u_2(t)), v(t)) \ge -C\|\chi_{\Omega\setminus\Omega_0}v(t)\|^2_{L^2(\Omega)}.$$

Inserting estimates (3.22) and (3.23) into (3.21), we arrive at

$$\begin{aligned}(3.24)\quad &\|\nabla_x v(t)\|^2_{L^2(\Omega)} \le 1/4[(l_{u_1,u_2}\partial_t v(t), \partial_t v(t)) + (a(\cdot)\partial_t v(t), \partial_t v(t))] + \\ &+ C(\|\partial_t u_1(t)\|_{L^{3/2}(\Omega)} + \|\partial_t u_2(t)\|_{L^{3/2}(\Omega)})\|\nabla_x v(t)\|^2_{W^{1,2}(\Omega)} + C\|\chi_{\Omega\setminus\Omega_0}v(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Taking a sum of (3.20) and (3.24) and using estimates (2.27), we finally infer

$$\begin{aligned}(3.25)\quad &\partial_t\mathcal{G}(v(t)) + [\beta - C(\|\partial_t u_1(t)\|_{L^{3/2}(\Omega)} + \|\partial_t u_2(t)\|_{L^{3/2}(\Omega)})]\mathcal{G}(v(t)) + \\ &+ 1/4(a(\cdot)\partial_t v(t), \partial_t v(t)) \le C\|\chi_{\Omega\setminus\Omega_0}v(t)\|^2_{L^2(\Omega)}\end{aligned}$$

for some positive β, independent of u_1 and u_2.

In order to deduce estimate (3.18) from (3.26), we note that the existence of a global Lyapunov function (1.6) together with assumptions (1.4) implies that the following dissipation integrals are finite:

$$\int_0^\infty \|\partial_t u_1(s)\|^{p+2}_{L^{p+2}(\Omega)}\,ds + \int_0^\infty \|\partial_t u_2(s)\|^{p+2}_{L^{p+2}(\Omega)}\,ds \le C < \infty \tag{3.26}$$

where the constant C is independent of u_1 and u_2 (with the initial data belonging to the absorbing set B). These integrals imply that

$$\int_0^t (\|\partial_t u_1(s)\|_{L^{3/2}(\Omega)} + \|\partial_t u_2(s)\|_{L^{3/2}(\Omega)})\,ds \le \varepsilon t + C_\varepsilon, \tag{3.27}$$

where $\varepsilon > 0$ can be arbitrary and C_ε depends only on ε. Applying now the Gronwall's inequality to (3.25) and using (3.27) with sufficiently small ε, we deduce the required estimate (3.18) and finish the proof of Lemma 3.1.

It is now not difficult to finish the proof of the theorem. Indeed, let us fix $T > 0$ in such way that $Ce^{-\beta T} < 1/2$ where C and β are the same as in Lemma 3.1 and let $S = S(T)$.

We also set $H := W^{1,2}(\Omega)$,

$$V := L^2([0,T], L^2(\Omega\backslash\Omega_0)) \tag{3.28}$$

and

$$V_1 := \{u \in L^2([0,T], W^{1,2}(\Omega)),\ \int_0^T (a(\cdot)\partial_t u(s), \partial_t u(s))\,ds < \infty\}. \tag{3.29}$$

Finally, we define the operator $\mathbb{T} : B \to V_1$ as the solving operator of problem (3.1), i.e

$$\mathbb{T}u_0 := u \in V_1 \ \text{ where } u(t),\ t \in [0,T] \text{ solves (1.1) with } u(0) = u_0. \tag{3.30}$$

We claim that the operator $S : B \to B$, the spaces H, V and V_1 and the operator $\mathbb{T}$ thus defined satisfy all the assumptions of Proposition 3.1. Indeed, the compactness of the embedding $V_1 \subset V$ is verified in Appendix (see Lemma A.1), the global Lipschitz continuity of $\mathbb{T}$ is an immediate corollary of Theorem 3.2 and estimate (3.11) follows from Lemma 3.1. Thus, due to Proposition 3.1, the semigroup $S(n)$ generated by iterations of the operator $S : B \to B$ possesses an exponential attractor $\mathcal{M}_d$ in B endowed by the topology of $H = W^{1,2}(\Omega)$.

In order to constract the exponential attractor $\mathcal{M}$ for the semigroup $S(t)$ with continuous time, we note that, due to Theorem 3.2, this semigroup is Lipschitz continuous with respect to the initial data in the topology of H. Moreover, since the derivative $\partial_t u(t)$ is uniformly bounded for any trajectory $u(t)$ starting from B, this semigroup is also uniformly Lipschitz continuous in time in the $L^\infty(\Omega)$-metric. Since B is bounded in $W^{2,q}$, for any finite q, the last assertion together with the appropriate interpolation inequality gives the uniform Hölder continuity in time in the metric of H. Thus, we have verified that the map $(t, u_0) \to S(t)u_0$ is uniformly Hölder continuous on $[0,T] \times B$ where B is endowed by the H-metric. Therefore, the required exponential attractor $\mathcal{M}$ for the case of continuous time can be defined by the standard expression:

$$\mathcal{M} := \cup_{t\in[0,T]} S(t)\mathcal{M}_d \tag{3.31}$$

where $\mathcal{M}_d$ is the exponential attractor for the discrete semigroup associated with the map $S = S(T)$ constructed above.

So, in order to finish the proof of the theorem, we only need to verify that $\mathcal{M}$ defined by (3.31) will be the exponential attractor for $S(t)$ restricted to B not only in H-metric, but also in more strong metric of the phase space Φ. But this is an immediate corollary of the fact that B is bounded in $W^{2,q}(\Omega)$ and the interpolation inequality

$$\|w\|_{L^\infty(\Omega)} \le C\|w\|^{\kappa}_{W^{1,2}(\Omega)}\|w\|^{1-\kappa}_{W^{2,q}(\Omega)} \tag{3.32}$$

with the appropriate exponent $0 < \kappa < 1$. Theorem 3.2 is proven.

Remark 3.2. Since the global attractor $\mathcal{A}$ is always contained in the exponential one $\mathcal{M}$, the proved theorem immediately implies that the fractal dimension of the global attractor $\mathcal{A}$ is also finite.

Remark 3.3. We see that the proof of key Lemma 3.1 uses the dissipation integral (3.26) and the fact that equation (1.1) possesses a global Lyapunov function. This, can be rather essential restriction which does not allow, in particular, to consider the *non-autonomous* equations of the form (1.1) or the non-gradient systems where the dissipation integral does not take place. However, in the non-degenerate case, it is not necessary since $\Omega\backslash\Omega_0 = \Omega$ and the terms

$$(l_{u_1,u_2}v, v) \quad \text{and} \quad (|\partial_t u_2|, |v|^2)$$

can be directly estimated by $C\|v\|^2_{L^2(\Omega)}$ without the usage of the dissipation integral.

Moreover, even in the degenerate case, the dissipation integral can be overcome by adding the terms

$$\int_0^t (l_{u_1,u_2}\partial_t v(s), \partial_t v(s))\, ds \quad \text{and} \quad \int_0^t (l_{u_1,u_2}v(s), v(s))\, ds$$

into the left and right-hand side of inequality (3.18) respectively and by using the more delicate version of Proposition 3.1 where the spaces V and V_1 can depend on the trajectories u_1 and u_2, see [9].

Thus, the global Lyapunov function is not crucial for the above theory and has been used above only in order to avoid the additional technicalities.

Appendix. One compact embedding.

In this concluding section we verify the compactness of the embedding $V_1 \subset V$ which is crucial for our construction of the exponential attractor.

Lemma A.1. *Let the function $a \in L^\infty(\Omega)$ be non-negative, let $\Omega_0 := \{x \in \Omega, a(x) = 0\}$ and let the spaces V and V_1 be defined by* (3.28) *and* (3.29) *respectively. Then, the embedding $V_1 \subset V$ is compact.*

Proof. We set

$$\Omega^+_\delta := \{x \in \Omega, \quad a(x) > \delta\}, \quad \delta \ge 0. \tag{A.1}$$

Then, obviously, $\Omega_0^+ = \Omega\backslash\Omega_0$. Moreover, due to the continuity of the Lebesgue measure, we have

$$\lim_{\delta\to 0} \operatorname{mes}\{\Omega_0^+\backslash\Omega_\delta^+\} = 0. \tag{A.2}$$

On the other hand, due to Hölder inequality and embedding $W^{1,2} \subset L^6$, we have

$$\|\chi_X(\cdot)v\|_{L^2(\Omega)} \le \operatorname{mes}\{X\}^{3/4}\|v\|_{L^6(X)} \le C \operatorname{mes}\{X\}^{3/4}\|v\|_{W^{1,2}(\Omega)}$$

for any set $X \subset \Omega$ and, consequently,

$$\|\chi_X(\cdot)v\|_{L^2([0,T]\times\Omega)} \le C \operatorname{mes}\{X\}^{3/4}\|v\|_{V_1} \tag{A.3}$$

where the constant C is independent of $v \in V_1$. Thus, for verifying the compactness of the embedding $V_1 \subset V$, it is sufficient to verify the compactness of the embedding

$$V_1 \subset V^\delta, \quad V^\delta := L^2([0,T] \times \Omega_\delta^+) \tag{A.4}$$

for any *positive* δ.

Let now $\delta > 0$ be fixed. Then, according to the Arzela-Ascoli theorem, we need to verify that there exists a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, $\lim_{z\to 0+}\mu(z) = 0$ such that

$$\int_0^T \int_\Omega \chi_{\Omega_\delta^+}(x)|u(t+s,x) - u(t,x)|^2\,dx\,dt \le \mu(|s|), \ \ s \in \mathbb{R} \tag{A.5}$$

and

$$\int_0^T \int_\Omega |\chi_{\Omega_\delta^+}(x+h)u(t,x+h) - \chi_{\Omega_\delta^+}(x)u(t,x)|^2\,dx\,dt \le \mu(|h|), \quad h \in \mathbb{R}^3 \tag{A.6}$$

uniformly with respect to all u belonging to the unit ball in V_1 (in these estimates function v is assumed to be extended by zero for $(t,x) \notin (0,T) \times \Omega$).

Let us first verify (A.5). Let $s > 0$ (the case $s < 0$ can be considered analogously). Then, using the obvious formula

$$u(t+s,x) - u(t,x) = s\int_0^1 \partial_t u(t+\kappa s, x)\,d\kappa$$

together with the fact that $a(x) > \delta$ if $x \in \Omega_\delta^+$, we have

$$\begin{aligned} \int_0^{T-s} \int_\Omega \chi_{\Omega_\delta^+}(x)|u(t+s,x) - u(t,x)|^2\,dx\,dt \le \\ \le s\int_0^T \int_\Omega \chi_{\Omega_\delta^+}(x)|\partial_t u(t,x)|^2\,dx\,dt \le \\ \le \delta^{-1}s\int_0^T \int_\Omega a(x)|\partial_t u(t,x)|^2\,dx\,dt \le \delta^{-1}s\|v\|_{V_1}^2. \end{aligned} \tag{A.7}$$

On the other hand, using that

$$\|u(\cdot,x)\|_{L^\infty([0,T])} \le C(\|\partial_t u(\cdot,x)\|_{L^2([0,T])} + \|u(\cdot,x)\|_{L^2([0,T])}),$$

we obtain

$$
\text{(A.8)}\quad \int_{T-s}^{T}\int_\Omega \chi_{\Omega_\delta^+}(x)|u(t+s,x)-u(t,x)|^2\,dx\,dt\le
$$
$$
\le Cs\int_\Omega \chi_{\Omega_\delta^+}(x)(\|\partial_t u(\cdot,x)\|^2_{L^2([0,T])}+\|u(\cdot,x)\|^2_{L^2([0,T])})\,dx\,dt\le
$$
$$
\le C\delta^{-1}s\int_0^T\int_\Omega a(x)(|\partial_t u(t,x)|^2+|u(t,x)|^2)\,dx\,dt\le C\delta^{-1}\|u\|^2_{V_1}
$$

Estimates (A.7) and (A.8) show that (A.5) holds with $\mu(z):=2C\delta^{-1}z$.

Let us now verify now (A.6). Indeed, due to the estimate

$$
|\chi_{\Omega_\delta^+}(x+h)u(t,x+h)-\chi_{\Omega_\delta^+}(x)u(t,x)|\le
$$
$$
\le |\chi_{\Omega_\delta^+}(x+h)-\chi_{\Omega_\delta^+}(x)|\cdot|u(t,x)|+|u(t,x+h)-u(t,x)|
$$

and embedding $W^{1,2}\subset L^6$, we have

$$
\text{(A.9)}\quad \int_0^T\int_\Omega |\chi_{\Omega_\delta^+}(x+h)u(t,x+h)-\chi_{\Omega_\delta^+}(x)u(t,x)|^2\,dx\,dt\le
$$
$$
\le CT\|u\|^2_{V_1}\left(\int_\Omega|\chi_{\Omega_\delta^+}(x+h)-\chi_{\Omega_\delta^+}(x)|^3\,dx\right)^{2/3}+
$$
$$
+\int_0^T\int_\Omega|u(t,x+h)-u(t,x)|^2\,dx\,dt.
$$

The first term in the right-hand side of (A.9) tends to zero since $\chi_{\Omega_\delta^+}\in L^\infty(\Omega)\subset L^3(\Omega)$ and the second one tends to zero uniformly with respect to u analogously to (A.5). Thus, estimates (A.5) and (A.6) are verified and Lemma 1.1 is proven.

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