A Course Material on

Finite Element Analysis

By

Mr. L.VINOTH ME

ASSISTANT PROFESSOR

DEPARTMENT OF MECHANICAL ENGINEERING

SASURIE COLLEGE OF ENGINEERING

VIJAYAMANGALAM – 638 056

QUALITY CERTIFICATE

This is to certify that the e-course material

Subject Code : **ME2353**

Subject : Finite Element Analysis

Class : III Year Mechanical

being prepared by me and it meets the knowledge requirement of the university curriculum.

Signature of the Author

Name: L.VINOTH M.E.

Designation: ASSISTANT PROFESSOR

This is to certify that the course material being prepared by Mr.L.Vinoth is of adequate quality. He has referred more than five books among them minimum one is from aboard author.

Signature of HD

Name: E.R SIVAKUMAR M.E.,(PhD)

SEAL

CONTENTS

UNIT – 3 TWO DIMENSIONAL FINITE ELEMENT ANALYSIS

UNIT – 4 DYNAMIC ANALYSIS USING ELEMENT METHOD

UNIT -5 APPLICATIONS IN HEAT TRANSFER &FLUID MECHANICS

TEXT BOOKS:

1. P.Seshu, "Text Book of Finite Element Analysis", Prentice-Hall of India Pvt. Ltd. New Delhi, 2007.
2. J.N.Reddy, "An Introduction to the Finite Element Method", McGraw-Hill International Editions(Engineering Mechanics Series), 1993.
3. Cook,Robert.D., Plesha,Michael.E & Witt,Robert.J. "Concepts and Applications of Finite Element Analysis",Wiley Student Edition, 2004.
4. Chandrupatla & Belagundu, "Introduction to Finite Elements in Engineering", 3rd Edition, Prentice-Hall of India, Eastern Economy Editions.

ME2353 FINITE ELEMENT ANALYSIS L T P C
3 1 0 4

INTRODUCTION (Not for examination) **5**

Solution to engineering problems – mathematical modeling – discrete and continuum modeling – need for numerical methods of solution – relevance and scope of finite element methods – engineering applications of FEA

UNIT I FINITE ELEMENT FORMULATION OF BOUNDARY VALUE PROBLEMS 5+3

Weighted residual methods –general weighted residual statement – weak formulation of the weighted residual statement –comparisons – piecewise continuous trial functions- example of a bar finite element – functional and differential forms – principle of stationary total potential – Rayleigh Ritz method – piecewise continuous trial functions – finite element method – application to bar element

UNIT II ONE DIMENSIONAL FINITE ELEMENT ANALYSIS 8+4

General form of total potential for 1-D applications – generic form of finite element equations – linear bar element – quadratic element –nodal approximation – development of shape functions – element matrices and vectors – example problems – extension to plane truss– development of element equations – assembly – element connectivity – global equations – solution methods –beam element – nodal approximation – shape functions – element matrices and vectors – assembly – solution – example problems

UNIT III TWO DIMENSIONAL FINITE ELEMENT ANALYSIS 10+4

Introduction – approximation of geometry and field variable – 3 noded triangular elements – four noded rectangular elements – higher order elements – generalized coordinates approach to nodal approximations – difficulties – natural coordinates and coordinate transformations – triangular and quadrilateral elements – iso-parametric elements – structural mechanics applications in 2-dimensions – elasticity equations – stress strain relations – plane problems of elasticity – element equations – assembly – need for quadrature formule – transformations to natural coordinates – Gaussian quadrature – example problems in plane stress, plane strain and axisymmetric applications

UNIT IV DYNAMIC ANALYSIS USING FINITE ELEMENT METHOD 8+4

Introduction – vibrational problems – equations of motion based on weak form – longitudinal vibration of bars – transverse vibration of beams – consistent mass matrices – element equations –solution of eigenvalue problems – vector iteration methods – normal modes – transient vibrations – modeling of damping – mode superposition technique – direct integration methods

UNIT V APPLICATIONS IN HEAT TRANSFER & FLUID MECHANICS 6+3

One dimensional heat transfer element – application to one-dimensional heat transfer problems- scalar variable problems in 2-Dimensions – Applications to heat transfer in 2- Dimension – Application to problems in fluid mechanics in 2-D

TEXT BOOK:

1. P.Seshu, "Text Book of Finite Element Analysis", Prentice-Hall of India Pvt. Ltd. New Delhi, 2007. ISBN-978-203-2315-5

REFERENCE BOOKS:

1. J.N.Reddy, "An Introduction to the Finite Element Method", McGraw-Hill International Editions(Engineering Mechanics Series), 1993. ISBN-0-07-051355-4
2. Chandrupatla & Belagundu, "Introduction to Finite Elements in Engineering", 3rd Edition, Prentice-Hall of India, Eastern Economy Editions. ISBN-978-81-203-2106-9
3. David V.Hutton,"Fundamentals of Finite Element Analysis", Tata McGraw-Hill Edition 2005. ISBN-0-07-239536-2
4. Cook,Robert.D., Plesha,Michael.E & Witt,Robert.J. "Concepts and Applications of Finite Element Analysis",Wiley Student Edition, 2004. ISBN-10 81-265-1336-5

Note: L- no. of lectures/week, T- no. of tutorials per week

UNIT I

FINITE ELEMENT FORMULATION OF BOUNDARY VALUE PROBLEMS

1.1 INTRODUCTION

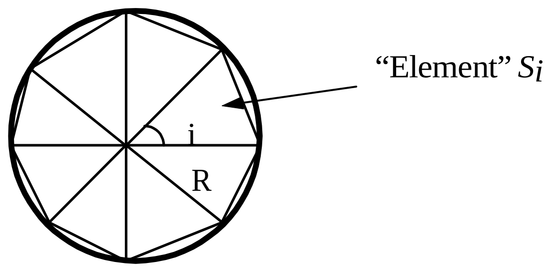
The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science

The finite element method (FEM), or finite element analysis (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in everyday life as well as in engineering.

Examples:

Lego (kids' play) Buildings

Approximation of the area of a circle:



Why Finite Element Method?

- Design analysis: hand calculations, experiments, and computer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications

1.1.1 A Brief History of the FEM

- 1943 --- Courant (variational method)
- 1956 --- Turner, clough, martin and top(stiffness)
- 1960 --- Clough (finite element plan problems)
- 1970 --- Applications on mainframe computer
- 1980 --- Microcomputers, pre and post processors
- 1990 --- Analysis of large structural systems

1.1.2 General Methods of the Finite Element Analysis

1. Force Method – Internal forces are considered as the unknowns of the problem.
2. Displacement or stiffness method – Displacements of the nodes are considered as the unknowns of the problem.

1.1.3 General Steps of the Finite Element Analysis

- Discretization of structure
- Numbering of Nodes and Elements
- Selection of Displacement function or interpolation function
- Define the material behavior by using Strain – Displacement and Stress – Strain relationships

- Derivation of element stiffness matrix and equations
- Assemble the element equations to obtain the global or total equations
- Applying boundary conditions
- Solution for the unknown displacements computation of the element strains and stresses from the nodal displacements
- Interpret the results (post processing).

1.1.4 Objectives of This FEM

- Understand the fundamental ideas of the FEM
- Know the behavior and usage of each type of elements covered in this course
- Be able to prepare a suitable FE model for given problems
- Can interpret and evaluate the quality of the results (know the physics of the problems)
- Be aware of the limitations of the FEM (don't misuse the
- FEM - a numerical tool)

1.1.5 Applications of FEM in Engineering

- Mechanical/Aerospace/Civil/Automobile Engineering Structure analysis (static/dynamic, linear/nonlinear) Thermal/fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics

1.2 WEIGHTED RESIDUAL METHOD

It is a powerful approximate procedure applicable to several problems. For non – structural problems, the method of weighted residuals becomes very useful. It has many types. The popular four methods are,

1. Point collocation method,

Residuals are set to zero at n different locations X_i, and the weighting function w_i is denoted as $\delta(x - x_i)$.

$$\int \delta(x - xi)\, R\,(x; a_1, a_2, a_3 \ldots an)\, dx = 0$$

2. Subdomain collocation method

3. Least square method,

$$\int [R\,(x; a_1, a_2, a_3 \ldots an)]^2\, dx = \text{minimum.}$$

4. Galerkin's method. $w_i = N_i\,(x)$

$$\int N_i\,(x)\,[R\,(x; a_1, a_2, a_3 \ldots an)]^2\, dx = 0, \qquad i = 1, 2, 3, \ldots n.$$

Problem I

Find the solution for the following differential equation.

$EI\frac{d4u}{dx} - q_o = 0$

The boundary conditions are $u(0)=0$, $\frac{du}{dx}(0)=0$,
$\frac{d2u}{dx2}(L)=0$, $\frac{d3u}{dx3}(L)=0$,

Given: The governing differential equation

$EI\frac{d4u}{dx} - q_o = 0$

Solution: assume a trial function

Let $u(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$…..

Apply 1st boundary condition

$x=0$, $u(x)=0$

$0=a_0+0$

$a_0=0$

Apply 2nd boundary condition

$x=0$, $\frac{du}{dx}=0$

$a_1=0$

Apply 3rd boundary condition

$x=L$, $\frac{d2u}{dx2}=0$

$a_2=-[3a_3L+6a_4L^2]$

Apply 4th boundary condition

$x=L$, $\frac{d3u}{dx3}=0$

$a_3=-4a_4L$

Substitute a_0, a1, a2 and $a_3$values in trial function

$u(x)= 0+0-[3a_3L+6a_4L^2] -4a_4L$

$u(x)= a_4[6 L^2x^2-4 Lx^3+ x^4]$

$\frac{du}{dx}= a_4[6 L^2 (2x)-12 Lx^2+ 4x^3]$

$\frac{d4u}{dx4}=24 a_4$

$R= EI\frac{d4u}{dx} - q_o = 0$

$a_4=\frac{qo}{24EI}$

Substitute $a_4$values in u(x)

$u(x) = \frac{qo}{24EI}[x_4-4Lx_3+6L^2x_2]$

Result:

Final solution $u(x) = \frac{qo}{24EI}[x_4-4Lx_3+6L^2x_2]$

Problem 2

The differential equation of a physical phenomenon is given by

$$\frac{d^2y}{dx^2} + y = 4, \quad 0 \le x \le 1$$

The boundary conditions are: y(0)=0

y(1)=1

Obtain one term approximate solution by using galerkin method

Solution:

Here the boundary conditions are not homogeneous so we assume a trial function as,

$y=a_1x(x-1)+x$

first we have to verify whether the trial function satisfies the boundary condition or not

$y=a_1x(x-1)+x$

when x=0, y=0

x=1, y=1

Resuldual R:

$Y=a_1x(x-1)+x=a_1(x^2-x)+x$

$\frac{dy}{dx}=a_1(2x-1)+1$

$\frac{d^2y}{dx^2} = 2a_1$

Substitute $\frac{d^2y}{dx^2}$value in given differential equation.

$2a_1+y=4x$

Substitute y vlue

$R=2a_1+a_1x(x-1)+x-4x$

In galerkin's method

$\int_0^1 wi\ R\ dx$

Substitute w_i and R value in equation

$a_1=0.83$

So one of the approximate solution is, $y= 0.83x(x-1)+x$

$= 0.83x^2-0.83x+x$

$y=0.83\ x^2+0.17x$

Problem 3

Find the deflection at the center of a simply supported beam of span length l subjected to uniform distributed load throughout its length as shown using (a) point collection method (b) Sub-domain method (c)least squared and (d) galerkin'n method.

Solution:

$EI\frac{d4y}{dx4}-\omega = 0,\ 0\le x \le l$

The boundary condition are y=0, x= 0and y= l

$EI\frac{d4y}{dx4}$=0 at x=0 and x= l

Where, $EI\frac{d4y}{dx4} = M$

Let us select the trail function for deflection as,

y= a sin $\pi x/l$

1.3 THE GENERAL WEIGHTED RESIDUAL STATEMENT

After understanding the basic techniques and successfully solved a few problem general weighted residual statement can be written as

R dx=0 for i= 1,2,…..n

Where $w_i=N_i$

The better result will be obtained by considering more terms in polynomial and trigonometric series.

1.4 WEAK FORMULATION OF THE WEIGHTED RESIDUAL STATEMENT.

The analysis in Section as applied to the model problem provides an attractive perspective to the solution of certain partial differential equations: the solution is identified with a "point", which minimizes an appropriately constructed functional over an admis- sible function space. Weak (variational) forms can be made fully equivalent to respective strong forms, as evidenced in the discussion of the weighted residual methods, under certain smoothness assumptions. However, the equivalence between weak (variational) forms and variational principles is not guaranteed: indeed, there exists no general method of construct-

ing functionals I [u], whose extremization recovers a desired weak (variational) form. In this

sense, only certain partial differential equations are amenable to analysis and solution by variational methods.

Vainberg's theorem provides the necessary and sufficient condition for the equivalence of a weak (variational) form to a functional extremization problem. If such equivalence holds, the functional is referred to as a potential.

Theorem (Vainberg)
Consider a weak (variational) form

G(u, δu) := B(u, δu) + (f, δu) + (q¯ ,δu)Γq = 0 ,

where u U , δu U0 , and f and q¯ are independent of u. Assume that G pos- sesses a Gˆateaux derivative in a neighborhood N of u, and the Gˆateaux differen- tial Dδu1 B(u, δu2) is continuous in u at every point of N .

Then, the necessary and sufficient condition for the above weak form to be derivable from a potential in N is that

Dδu1 G(u, δu2) = Dδu2 G(u, δu1) ,

Namely that Dδu1 G(u, δu2) be symmetric for all δu1, δu2 = U0 and all u = N .

Preliminary to proving the above theorem, introduce the following two lemmas:

Lemma 1 Show that D_V **I**[u] = lim

In the above derivation, note that operations and $|\omega=0$ are not interchangeable (as they both refer to the same variable ω), while $\lim_{\Delta\omega\to 0}$ and $|\omega=0$ are interchangeable, conditional upon sufficient smoothness of I [u].

Lemma 2 (Lagrange's formula)

Let I [u] be a functional with Gateaux derivatives everywhere, and u, u + δu be any points of U. Then,

$$I[u + \delta u] - I[u] = D_{\delta u} I[u + \varrho\, \delta u] \qquad 0 < \varrho < 1.$$

To prove Lemma 2, fix u and u + δu in U, and define function f on R as

$$f(\omega) := I[u + \omega\, \delta u].$$

It follows that

$$F = \frac{df}{d\omega} = \lim_{\Delta\omega\to 0} \frac{f(\omega + \Delta\omega) - f(\omega)}{\Delta\omega}$$

$$= \lim_{\Delta\omega\to 0} \frac{I[u + \omega\, \delta u + \Delta\omega\, \delta u] - I[u + \omega\, \delta u]}{\Delta\omega} = D_{\delta u} I[u + \omega\, \delta u],$$

Where Lemma 1 was invoked. Then, using the standard mean-value theorem of calculus,

1.5 PIECE WISE CONTINUOUS TRIAL FUNCTION

In weighted residual method the polynomial and trigonometric series are used as trial function. This trial function is a single composite function and it is valid over the entire solution domain this assumed trial function solution should match closely to the exact solution of the differential equation and the boundary conditions, it is nothing but a process of curve fitting. This curve fitting is carried out by piecewise method i.e., the more numbers of piece leads better curve fit. Piecewise method can be explained by the following simple problem.

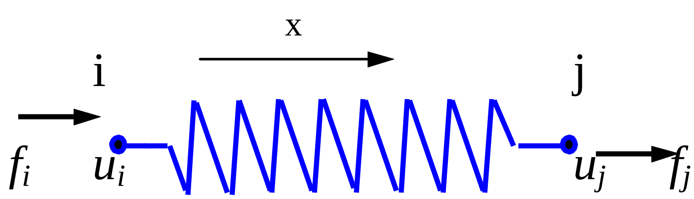
We know that the straight line can be drawn through any two points.

Let, $f(x)=\sin$— is the approximated function for straight line segments.

One straight line segment

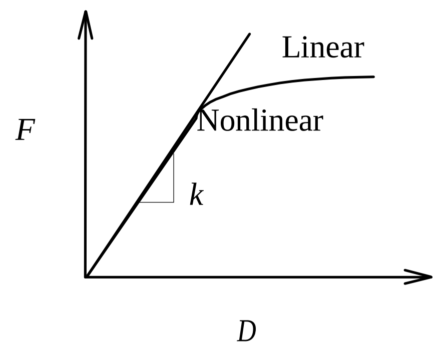
Two straight line segment

One Spring Element



Two nodes: i, j
Nodal displacements: u_i, u_j (in, m, mm)
Nodal forces: f_i, f_j (lb, Newton) Spring constant (stiffness): k (lb/in, N/m, N/mm)

Spring force-displacement relationship:



$k \quad F /$ (> 0) is the force needed to produce a unit stretch.

We only consider linear problems in this introductory course. Consider the equilibrium of forces for the spring.

At node 1 we have

$$f_i \quad F \quad k(u_j \quad u_i) \quad ku_i \quad ku_j$$

and at node j,

$$f_j \quad F \quad k(u_j \quad u_i) \quad ku_i \quad ku_j$$

In matrix form,

$$\begin{matrix} k & k & u_i & & f_i \\ k & k & u_j & & f_j \end{matrix}$$

or, where

(element) stiffness matrix

u = (element nodal) displacement vector

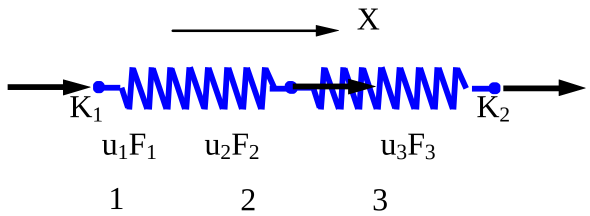
f = (element nodal) force vector

Note:

That k is symmetric. Is k singular or non singular? That is, can we solve the equation? If not, why?

Problem 4

To find the deformation of the shape



For element 1,

$$k_1 \qquad k_1 \quad u_2 \qquad f_1^{\,2}$$
$$k_1 \qquad k_1 \quad u_3 \qquad f_2^{\,2}$$

element 2,

$$k_2 \qquad k_2 \quad u_2 \qquad\qquad f^{\,2}$$
$$k_2 \quad k_2 \quad u_3 \qquad\qquad f^{\,2}$$

where f_I at node 2 F_2

M is the (internal) force acting on local node *i* of element Consider the quilibrium of forces at node

Checking the Results

Deformed shape of the structure
Balance of the external forces
Order of magnitudes of the numbers

Notes about the Spring Elements

Suitable for stiffness analysis
Not suitable for stress analysis of the spring itself
Can have spring elements with stiffness in the lateral direction,
Spring elements for torsion, etc.

1.6 EXAMPLES OF A BAR FINITE ELEMENT

The finite element method can be used to solve a variety of problem types in engineering, mathematics and science. The three main areas are mechanics of materials, heat transfer and fluid mechanics. The one-dimensional spring element belongs to the area of mechanics of materials, since it deals with the displacements, deformations and stresses involved in a solid body subjected to external loading.

Element dimensionality:

An element can be one-dimensional, two-dimensional or three-dimensional. A spring element is classified as one-dimensional.

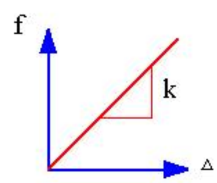
Geometric shape of the element

The geometric shape of element can be represented as a line, area, or volume. The one-dimensional spring element is defined geometrically as:

Spring law

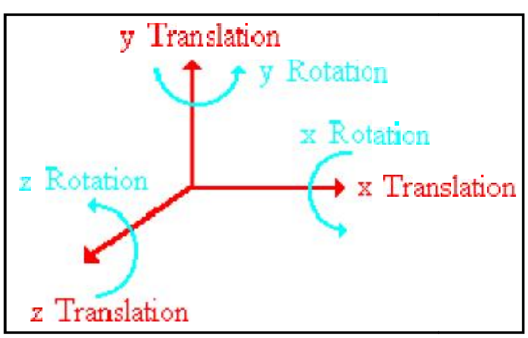
The spring is assumed to be linear. Force (f) is directly proportional to deformation (Δ) via the spring constant k, i.e.

$$f = k\Delta u$$



Types of degrees of freedom per node

Degrees of freedom are displacements and/or rotations that are associated with a node. A one-dimensional spring element has two translational degrees of freedom, which include, an axial (horizontal) displacement (u) at each node.



Element formulation

There are various ways to mathematically formulate an element. The simplest and limited approach is the direct method. More mathematically complex and general approaches are energy (variation) and weighted residual methods.

The direct method uses the fundamentals of equilibrium, compatibility and spring law from a sophomore level mechanics of material course. We will use the direct method to formulate the one-dimensional spring element because it is simple and based on a physical approach.

The direct method is an excellent setting for becoming familiar with such basis concepts of linear algebra, stiffness, degrees of freedom, etc., before using the mathematical formulation approaches as energy or weighted residuals.

Assumptions

Spring deformation

The spring law is a linear force-deformation as follows:

$f = k\ \Delta$

f - Spring Force (units: force)

k - Spring Constant (units: force/length)

Δ - Spring Deformation (units: length)

Spring Behaviour:

A spring behaves the same in tension and compression.

Spring Stiffness:

Spring stiffness k is always positive, i.e., k>0, for a physical linear system.

Nodal Force Direction:

Loading is uniaxial, i.e., the resultant force is along the element. Spring has no resistance to lateral force.

Weightless Member:

Element has no mass (weightless).

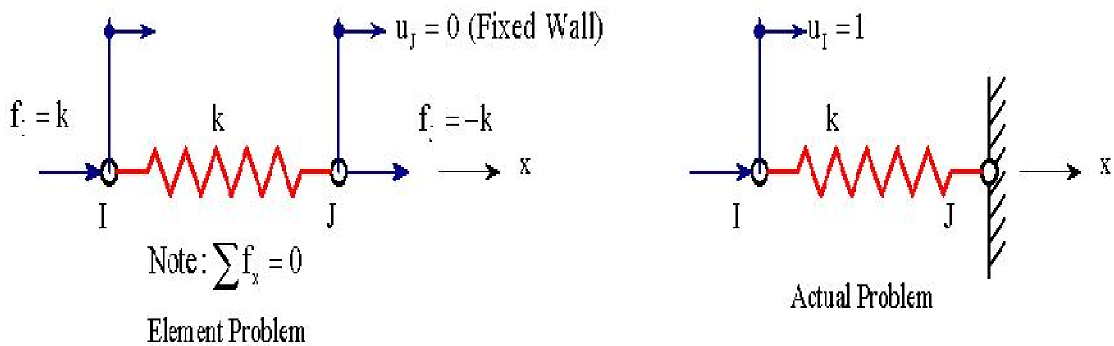
Node Location:

The geometric location of nodes I and J cannot coincide, i.e., $x_i \neq x_j$. The length of the element is only used to visually see the spring.

A column of K_E is a vector of nodal loads that must be applied to an element to sustain a deformed state in which responding nodal DOF has unit value and all other nodal DOF are zero. In other words, a column of K_E represents an equilibrium problem.

Example, $u_I = 1$, $u_J = 0$.

$$\begin{Bmatrix} f_I \\ f_J \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_I = 1 \\ u_J = 0 \end{Bmatrix} = \begin{Bmatrix} k \\ -k \end{Bmatrix}$$



Spring element has <u>one</u> rigid body mode.

Inter-Element Axial Displacement

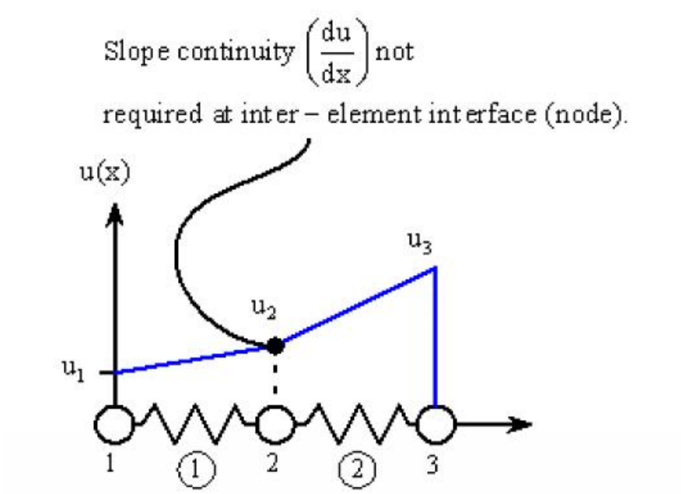
The axial displacement (u) is continuous through the assembled mesh and is described by a linear polynomial within each element. Each element in the mesh may be described by a different linear polynomial, depending on the spring rate (k), external loading, and constraints on the element.

Inter-Element Deformation

The deformation (Δ) is piecewise constant through the assembled mesh and is described by a constant within each element. Each element in the mesh may be described by a different constant, depending on the spring constant (k), external loading, and constraints on the element.

Inter-Element Internal Axial Force

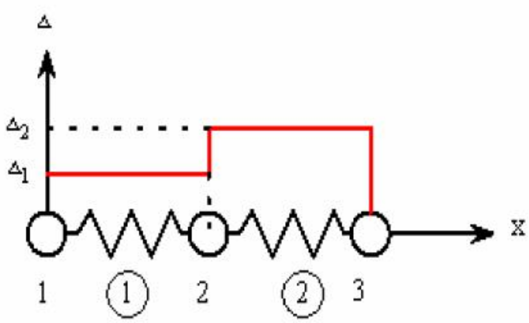
The internal axial force (f) is piecewise continuous through the assembled mesh and is described by a constant within each element. Each element in the mesh may be described by a different constant, depending on the spring constant, external loading, and constraints on the element.



u_1 – Axial displacement of node 1.

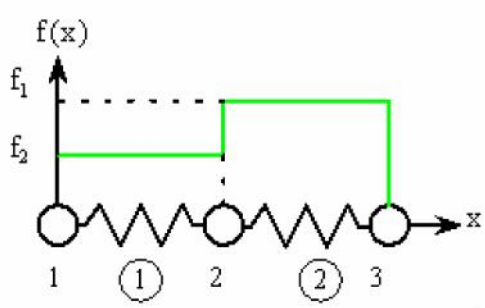
u_2 – Axial displacement of node 2.

u_3 – Axial displacement of node 3.



Δ_1 – Deformation of element (1).

Δ_2 – Deformation of element (2).



f_1 – Internal axial force of element (1).

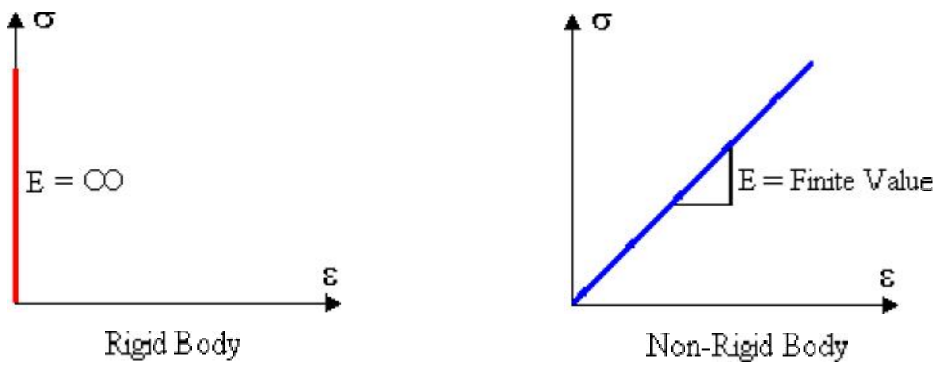
f_2 – Internal axial force of element (2).

1.6.1 Rigid Body

A body is considered rigid if it does not deform when a force is applied. Consider rigid and non-rigid bars subjected to a gradually applied axial force of increasing magnitude as shown.

The reader should note the following characteristics of rigid and non-rigid (flexible) bodies:

- Force Magnitude - Even if forces are large, a rigid body does not deform. A non-rigid body will deform even if a force is small. In reality, all bodies deform.
- Failure - A rigid body does not fail under any load; while a non-rigid body will result either in ductile or brittle failure when the applied load causes the normal stress to exceed the breaking (fracture) stress σ_b of the material. Brittle failure occurs when the applied load on the non-rigid bar shown above causes the breaking strength of the bar to be exceeded.
- Material - The material is not considered in a rigid body. Since a rigid body does not deform (ε = 0) this is equivalent to an infinite modulus of elasticity. In contrast the modulus of elasticity for a non-rigid material is finite, e.g., for steel, E_{steel} = 30 x 10^6 psi. (200 GPa). For rigid and non-rigid bars the material laws are:

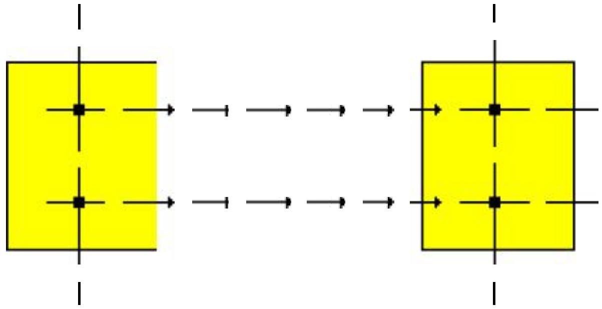


Rigid Body Motion

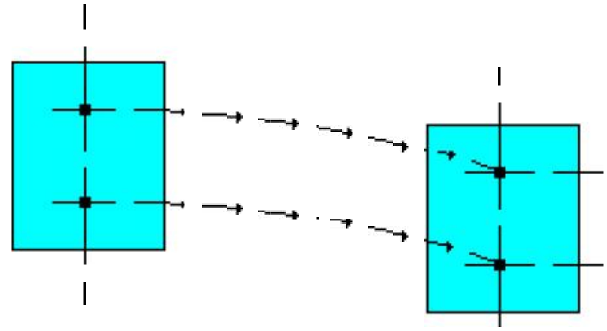
Rigid body motion occurs when forces and/or moments are applied to an unrestrained mesh (body), resulting in motion that occurs without any deformations in the entire mesh (body). Since no strains (deformations) occur during rigid body motion, there can be no stresses developed in the mesh.

A rigid body in general can be subjected to three types of motion, which are translation, rotation about a fixed axis, and general motion which consists of a combination of both translation and rotation. These three motion types are as follows:

Translation - If any line segment on the body remains parallel to its original direction during the motion, it is said to be in translation. When the path of motion is along a straight line, the motion is called rectilinear translation, while a curved path is considered as a curvilinear translation. The curvilinear motion shown below is a combination of two translational motions, one horizontal motion and one vertical motion.

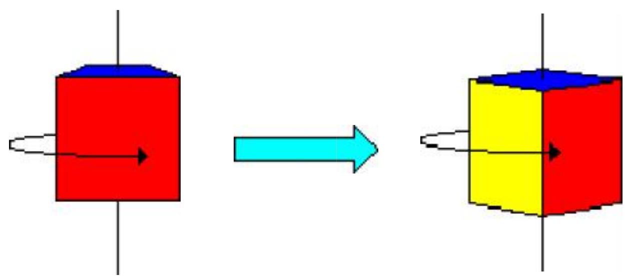


Rectilinear Translation



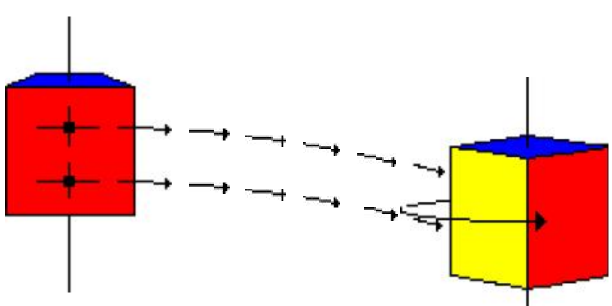
Curvilinear Translation

Rotation About a Fixed Axis - If all the particles of a rigid body move along circular paths, except the ones which lie on the axis of rotation, it is said to be in rotation about a fixed axis.



Rotation about a Fixed Axis

General Motion - Any motion of a rigid body that consists of the combination of both translations



General Motion

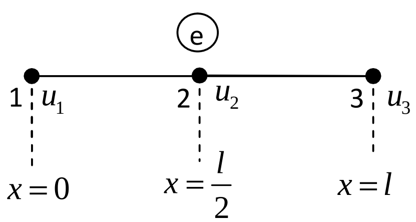
There are six rigid body modes in general three-dimensional situation; three translational along the x, y, and z axes and three rotational about x, y, and z axes. Illustrations of these rigid body modes are presented as follows:

Translational Rigid Body Modes	Rotational Rigid Body Modes
x-direction	about x-axis
y-direction	about y-axis
z-direction	about z-axis

1-D 3-NODED QUADRATIC BAR ELEMENT

Problem 6

A single 1-D 3-noded quadratic bar element has 3 nodes with local coordinates as shown in Figure



Note that node 2 is at the midpoint of the element.

The chosen approximation function for the field variable u is $u = a + bx + cx^2$

Let the field variable u have values u_1, u_2 and u_3 at nodes 1, 2 and 3, respectively.

To find the unknowns *a*, *b* and *c*, we apply the boundary conditions

at $x = 0, \quad u = u_1 \quad \Rightarrow \quad u_1 = a \quad \Rightarrow \quad a = u_1$

$$\left.\begin{aligned} &\text{at } x = \frac{l}{2}, \quad u = u_2 \quad \Rightarrow \quad u_2 = a + b\frac{l}{2} + c\frac{l^2}{4} \\ &\text{at } x = l, \quad u = u_3 \quad \Rightarrow \quad u_1 = a + bl + cl^2 \end{aligned}\right\} \text{ solving } \quad \begin{aligned} b &= \frac{-u_3 + 4u_2 - 3u_1}{l} \\ c &= \left(\frac{2}{l^2}\right)(u_3 - 2u_2 + u_1) \end{aligned}$$

Substituting the values of *a*, *b* and *c* in equation (1) and collecting the coefficients of u_1, u_2 and u_3

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$N_1 = 1 - 3\frac{x}{l} + 2\frac{x^2}{l^2} \qquad = \left(1-\frac{x}{l}\right)\left(1-\frac{2x}{l}\right) \qquad \frac{\partial N_1}{\partial x} = -\frac{3}{l} + 4\frac{x}{l^2}$$

Where

$$N_2 = 4\frac{x}{l} - 4\frac{x^2}{l^2} \qquad = 4\frac{x}{l}\left(1-\frac{x}{l}\right) \qquad \frac{\partial N_2}{\partial x} = \frac{4}{l} - 8\frac{x}{l^2}$$

$$N_3 = -\frac{x}{l} + 2\frac{x^2}{l^2} \qquad = -\frac{x}{l}\left(1-\frac{2x}{l}\right) \qquad \frac{\partial N_3}{\partial x} = -\frac{1}{l} + 4\frac{x}{l^2}$$

Derivation of stiffness matrix for 1-D 3-noded quadratic bar element:

$$[B]^T = \begin{Bmatrix} -\frac{3}{l} + 4\frac{x}{l^2} \\ \frac{4}{l} - 8\frac{x}{l^2} \\ -\frac{1}{l} + 4\frac{x}{l^2} \end{Bmatrix}$$

$[D] = E$ for a bar element (1-D case - only axial stress $(\pm_x)$ and strain $(£_x)$ exist $\Rightarrow \pm_x = E£_x$)

$\int_{volume} dV = \int_0^l A\,dx = A\int_0^l dx$ since the cross-sectional area A is constant for the total length of the bar.

$$[k] = A\int_0^l \begin{Bmatrix} -\frac{3}{l} + 4\frac{x}{l^2} \\ \frac{4}{l} - 8\frac{x}{l^2} \\ -\frac{1}{l} + 4\frac{x}{l^2} \end{Bmatrix} E \left\langle \left(-\frac{3}{l} + 4\frac{x}{l^2}\right) \quad \left(\frac{4}{l} - 8\frac{x}{l^2}\right) \quad \left(-\frac{1}{l} + 4\frac{x}{l^2}\right) \right\rangle dx$$

$$[k] = AE\int_0^l \begin{bmatrix} \left(-\frac{3}{l}+4\frac{x}{l^2}\right)\left(-\frac{3}{l}+4\frac{x}{l^2}\right) & \left(-\frac{3}{l}+4\frac{x}{l^2}\right)\left(\frac{4}{l}-8\frac{x}{l^2}\right) & \left(-\frac{3}{l}+4\frac{x}{l^2}\right)\left(-\frac{1}{l}+4\frac{x}{l^2}\right) \\ \left(\frac{4}{l}-8\frac{x}{l^2}\right)\left(-\frac{3}{l}+4\frac{x}{l^2}\right) & \left(\frac{4}{l}-8\frac{x}{l^2}\right)\left(\frac{4}{l}-8\frac{x}{l^2}\right) & \left(\frac{4}{l}-8\frac{x}{l^2}\right)\left(-\frac{1}{l}+4\frac{x}{l^2}\right) \\ \left(-\frac{1}{l}+4\frac{x}{l^2}\right)\left(-\frac{3}{l}+4\frac{x}{l^2}\right) & \left(-\frac{1}{l}+4\frac{x}{l^2}\right)\left(\frac{4}{l}-8\frac{x}{l^2}\right) & \left(-\frac{1}{l}+4\frac{x}{l^2}\right)\left(-\frac{1}{l}+4\frac{x}{l^2}\right) \end{bmatrix} dx$$

$$[k] = \int_{Volume} [B]^T [D][B]\,dV$$

$$[B] = \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \right\rangle = \left\langle \left(-\frac{3}{l} + 4\frac{x}{l^2}\right) \quad \left(\frac{4}{l} - 8\frac{x}{l^2}\right) \quad \left(-\frac{1}{l} + 4\frac{x}{l^2}\right) \right\rangle$$

To determine $\mathbf{K_{11}}$:

$$K_{11} = AE\int_0^l \left(\frac{-3}{l}+\frac{4x}{l^2}\right)\left(\frac{-3}{l}+\frac{4x}{l^2}\right)dx = AE\int_0^l \left(\frac{9}{l^2}-\frac{12x}{l^3}-\frac{12x}{l^3}+\frac{16x^2}{l^4}\right)dx$$

Integrating and applying limit we get,

$$K_{11} = AE\int_0^l \left(\frac{9}{l^2}-\frac{24x}{l^3}+\frac{16x^2}{l^4}\right)dx = AE\left[\frac{9x}{l^2}-\frac{24x^2}{2l^3}+\frac{16x^3}{3l^4}\right]_0^l = AE\left[\frac{9l}{l^2}-\frac{24l^2}{2l^3}+\frac{16l^3}{3l^4}\right]$$

$$K_{11} = AE\left[\frac{9}{l}-\frac{12}{l}+\frac{16}{3l}\right] = AE\left[\frac{27-36+16}{3l}\right] = \frac{AE}{3l}[7]$$

$$K_{11} = \frac{7AE}{3l}$$

To determine $\mathbf{K_{12}}$ (and $\mathbf{K_{21}}$):

$$K_{12} = AE\int_0^l \left(-\frac{3}{l}+\frac{4x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)dx = AE\int_0^l \left(-\frac{12}{l^2}+\frac{24x}{l^3}+\frac{16x}{l^3}-\frac{32x^2}{l^4}\right)dx$$

$$K_{12} = AE\left[-\frac{12x}{l^2}+\frac{40x^2}{2l^3}-\frac{32x^3}{3l^4}\right]_0^l$$

$$K_{12} = AE\left[-\frac{12x}{l^2}+\frac{40x^2}{2l^3}-\frac{32x^3}{3l^4}\right]_0^l = AE\left[-\frac{12l}{l^2}+\frac{40l^2}{2l^3}-\frac{32l^3}{3l^4}\right] = AE\left[-\frac{12}{l}+\frac{20}{l}-\frac{32}{3l}\right] = AE\left[\frac{-36+60-32}{3l}\right]$$

$$K_{12} = \frac{AE}{3l}[-8] = K_{21}$$

To determine $\mathbf{K_{13}}$ (and $\mathbf{K_{31}}$):

$$K_{13} = AE\int_0^l \left(-\frac{3}{l}+\frac{4x}{l^2}\right)\left(-\frac{1}{l}+\frac{4x}{l^2}\right)dx = AE\int_0^l \left(\frac{3}{l^2}-\frac{12x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right)dx$$

$$K_{13} = AE\int_0^l \left(\frac{3}{l^2}-\frac{16x}{l^3}+\frac{16x^2}{l^4}\right)dx = AE\left[\frac{3x}{l^2}-\frac{16x^2}{2l^3}+\frac{16x^3}{3l^4}\right]_0^l = AE\left[\frac{3l}{l^2}-\frac{16l^2}{2l^3}+\frac{16l^3}{3l^4}\right] = AE\left[\frac{3}{l}-\frac{8}{l}+\frac{16}{3l}\right]$$

$$K_{13} = AE\left[\frac{9-24+16}{3l}\right] = \frac{AE}{3l}[1] = K_{31}$$

To determine $\mathbf{K_{22}}$

$$K_{22} = AE\int_0^l \left(\frac{4}{l}-\frac{8x}{l^2}\right)\left(\frac{4}{l}-\frac{8x}{l^2}\right)dx = AE\int_0^l \left(+\frac{16}{l^2}-\frac{32x}{l^3}-\frac{32x}{l^3}+\frac{64x^2}{l^4}\right)dx$$

$$K_{22} = AE\int_0^l \left(\frac{16}{l^2}-\frac{64x}{l^3}+\frac{64x^2}{l^4}\right)dx = AE\left[\frac{16x}{l^2}-\frac{64x^2}{2l^3}+\frac{64x^3}{3l^4}\right]_0^l$$

$$K_{22} = AE\left[\frac{16l}{l^2}-\frac{64l^2}{2l^3}+\frac{64l^3}{3l^4}\right] = AE\left[\frac{16}{l}-\frac{32}{l}+\frac{64}{3l}\right] = AE\left[\frac{48-96+64}{3l}\right] = \frac{AE}{3l}[16]$$

$$K_{22} = \frac{16AE}{3l}$$

To determine $\mathbf{K_{23}}$ $(\text{and } \mathbf{K_{32}})$

$$K_{23} = AE\int_0^l \left(-\frac{4}{l^2}+\frac{24x}{l^3}-\frac{32x^2}{l^4}\right)dx = AE\left[-\frac{4x}{l^2}+\frac{24x^2}{2l^3}-\frac{32x^3}{3l^4}\right]_0^l$$

$$K_{23} = AE\left[-\frac{4l}{l^2}+\frac{24l^2}{2l^3}-\frac{32l^3}{3l^4}\right] = AE\left[\frac{-4}{l}+\frac{12}{l}-\frac{32}{3l}\right] = AE\left[\frac{-12+36-32}{3l}\right] = \frac{AE}{3l}[-8]$$

$$K_{23} = -\frac{8AE}{3l} = K_{32}$$

To determine $\mathbf{K_{33}}$

$$K_{33} = AE\int_0^l \left(-\frac{1}{l}+\frac{4x}{l^2}\right)\left(-\frac{1}{l}+\frac{4x}{l^2}\right)dx = AE\int_0^l \left(\frac{1}{l^2}-\frac{4x}{l^3}-\frac{4x}{l^3}+\frac{16x^2}{l^4}\right)dx$$

$$K_{23} = AE\int_0^l \left(\frac{4}{l}-\frac{8x}{l^2}\right)\left(-\frac{1}{l}+\frac{4x}{l^2}\right)dx = AE\int_0^l \left(-\frac{4}{l^2}+\frac{16x}{l^3}+\frac{8x}{l^3}-\frac{32x^2}{l^4}\right)dx$$

$$K_{33} = AE\int_0^l \left(\frac{1}{l^2}-\frac{8x}{l^3}+\frac{16x^2}{l^4}\right)dx = AE\left[\frac{x}{l^2}-\frac{8x^2}{2l^3}+\frac{16x^3}{3l^4}\right]_0^l$$

$$K_{33} = AE\left[\frac{l}{l^2}-\frac{8l^2}{2l^3}+\frac{16l^3}{3l^4}\right] = AE\left[\frac{1}{l}-\frac{4}{l}+\frac{16}{3l}\right] = AE\left[\frac{3-12+16}{3l}\right] = \frac{AE}{3l}[7]$$

$$K_{33} = \frac{7AE}{3l}$$

Assembling, we get $[k] = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$

1.7 PRINCIPLE OF STATIONERY TOTAL POTENTIAL (PSTP)

1.7.1 Potential energy in elastic bodies

Potential energy is the capacity to do the work by the force acting on deformable bodies; the forces acting on a body may be classified as external forces and internal forces. External forces are the applied loads while internal force is the stresses developed in the body. Hence the total potential energy is the sum of internal and external potential energy.

Consider a spring mass system let its stiffness be k and length L, due to a force P let it extend by u

The load P moves down by distance u. hence it loses its capacity to do work by P u. the external potential energy in this case is given by.

$$H = -P\,u$$

Average force = $\frac{K\,u}{2}$

The energy stored in the spring due to strain = Average force x Deflection

$$= \frac{K\,u}{2} \text{ x } u$$

$$= \frac{1}{2} K\,u^2$$

Total potential energy in the spring $\pi = \frac{1}{2} K\,u^2 - P\,u$

1.7.2 Principle of Minimum Potential Energy

From the expression for total potential energy,

$$\pi = U + H$$

$$\delta\pi = \delta U + \delta H$$

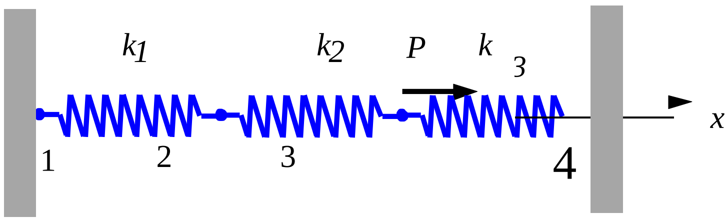
In principle of virtual work $\delta U = \delta H$

$$\delta\pi = 0$$

Hence we can conclude that a deformable body is in equilibrium when the potential energy is having stationary value.

Hence the principle of minimum potential energy states among all the displacement equations that internal compatibility and the boundary condition those that also satisfy the equation of equilibrium make the potential energy a minimum is a stable system

Problem 7



Given: For the spring system shown above,

k_1 100 N / mm,

k_2 200 N / mm,

k_3 100 N / mm

P 500 N,

u_1 0

u_4 0

Find: (a) The global stiffness matrix
(b) Displacements of nodes 2 and 3
(c) The reaction forces at nodes 1 and 4
(d) the force in the spring 2

Solution:

(a) The element stiffness matrices are

$$k_1 \quad \begin{matrix} 100 & 100 \\ 100 & 100 \end{matrix} \quad \text{(N/mm)} \qquad (1)$$

$$k_2 \quad \begin{matrix} 200 & 200 \\ 200 & 200 \end{matrix} \quad \text{(N/mm)} \qquad (2)$$

$$k_3 \quad \begin{matrix} 100 & 100 \\ 100 & 100 \end{matrix} \quad \text{(N/mm)} \qquad (3)$$

$$
\mathbf{K} \quad
\begin{matrix}
u_1 & u_2 & u_3 & u_4 & \\
100 & 100 & 0 & 0 & \\
100 & 100 \quad 200 & 200 & 0 & \\
0 & 200 & 200 \quad 100 & 100 & \\
 & 200 & 300 & 100 & \\
 & 0 & 100 & 100 &
\end{matrix}
$$

which is *symmetric* and ***banded***.

Equilibrium (FE) equation for the whole system is

$$
\begin{matrix}
100 & 100 & 0 & 0 \\
100 & 300 & 200 & 0 \\
0 & 200 & 300 & 100 \\
0 & 0 & 100 & 100
\end{matrix}
\quad
\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}
\quad
\begin{matrix} F_1 \\ 0 \\ P \\ F_4 \end{matrix}
\qquad (4)
$$

(b) Applying the BC

$$
\begin{matrix}
300 & 200 \\
200 & 300
\end{matrix}
\quad
\begin{matrix} u_2 \\ u_3 \end{matrix}
\quad
\begin{matrix} 0 \\ P \end{matrix}
\qquad (5)
$$

Solving Eq.(5), we obtain

$$
\begin{matrix} u_2 \\ u_3 \end{matrix}
\quad
\begin{matrix} P/250 \\ 3P/500 \end{matrix}
\quad
\begin{matrix} 2 \\ 3 \end{matrix}
\quad (\mathrm{mm})
\qquad (6)
$$

(c) From the 1st and 4th equations in (4), we get the reaction forces

$$F_1 \quad 100u_2 \quad 200 \ \ (\mathrm{N})$$

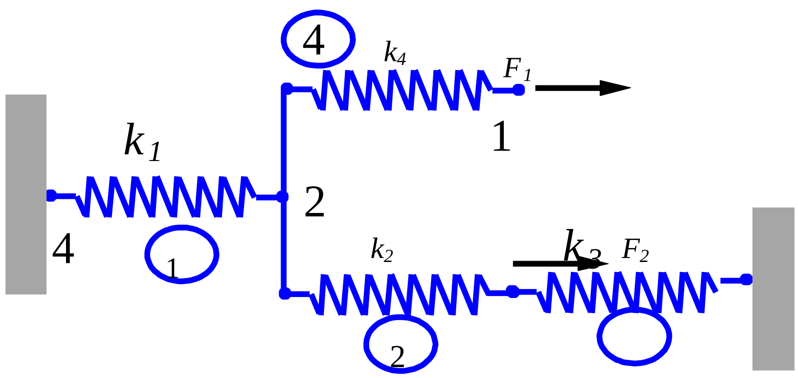
$$F_4 \quad 100u_3 \quad 300 \ \ (\mathrm{N})$$

(d) The FE equation for spring (element) 2 is

$$\begin{matrix} 200 & 200 \\ 200 & 200 \end{matrix} \quad \begin{matrix} u_i \\ u_j \end{matrix} \qquad \begin{matrix} f_i \\ f_j \end{matrix}$$

Here i = 2, j = 3 for element 2. Thus we can calculate the spring force as

Problem 8



For the spring system with arbitrarily numbered nodes and elements, as shown above, find the global stiffness matrix.

Solution:

First we construct the following

Element Connectivity Table

Element	***Node i (1)***	***Node j (2)***
1	4	2
2	2	3
3	3	5
4	2	1

Which specifies the global node numbers corresponding to the local node numbers for each element? Then we can write the element stiffness matrices as follows

$$\mathbf{k} \quad \begin{array}{cc} u_4 & u_2 \\ k_1 & k_1 \\ k_1 & k_1 \end{array} \qquad \mathbf{k}_2 \quad \begin{array}{cc} u_2 & u_3 \\ k_2 & k_2 \\ k_2 & k_2 \end{array}$$

$$\mathbf{k}_3 \quad \begin{array}{cc} u_3 & u_5 \\ k_3 & k_3 \\ k_3 & k_3 \end{array} \qquad \mathbf{k}_4 \quad \begin{array}{cc} u_2 & u_1 \\ k_4 & k_4 \\ k_4 & k_4 \end{array}$$

Finally, applying the superposition method, we obtain the global stiffness matrix as follows

We may note that N1 and N2 obey the definition of shape function that is the shape function will have a value equal to unity at the node to which it belong and zero value at other nodes.

K

	u_1	u_2	u_3	u_4	u_5
	k_4	k_4	0	0	0
	k_4	k_1 k_2 k_4	k_2	k_1	0
	0	k_2	k_2 k_3	0	k_3
	0	k_1	0	k_1	0
	0	0	k_3	0	k_3

1.8 RAYLEIGH – RITZ METHOD (VARIATIONAL APPROACH)

It is useful for solving complex structural problems. This method is possible only if a suitable functional is available. Otherwise, Galerkin's method of weighted residual is used.

Problems (I set)

1. A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using Rayleigh – Ritz method and compare with exact solutions.

2. A bar of uniform cross section is clamed at one end and left free at another end and it is subjected to a uniform axial load P. Calculate the displacement and stress in a bar by using two terms polynomial and three terms polynomial. Compare with exact solutions.

1.9 ADVANTAGES OF FINITE ELEMENT METHOD

1. FEM can handle irregular geometry in a convenient manner.
2. Handles general load conditions without difficulty
3. Non – homogeneous materials can be handled easily.
4. Higher order elements may be implemented.

1.10 DISADVANTAGES OF FINITE ELEMENT METHOD

1. It requires a digital computer and fairly extensive
2. It requires longer execution time compared with FEM.
3. Output result will vary considerably.

UNIT II

ONE DIMENSIONAL FINITE ELEMENT ANALYSIS

2.1 ONE DIMENSIONAL ELEMENTS

Bar and beam elements are considered as One Dimensional elements. These elements are often used to model trusses and frame structures.

- **Bar, Beam and Truss**

Bar is a member which resists only axial loads. A beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.

- **Stress, Strain and Displacement**

Stress is denoted in the form of vector by the variable x as σ_x, Strain is denoted in the form of vector by the variable x as e_x, Displacement is denoted in the form of vector by the variable x as u_x.

- **Types of Loading**

(1) Body force (f)

It is a distributed force acting on every elemental volume of the body. Unit is Force / Unit volume. Ex: Self weight due to gravity.

(2) Traction (T)

It is a distributed force acting on the surface of the body. Unit is Force / Unit area. But for one dimensional problem, unit is Force / Unit length. Ex: Frictional resistance, viscous drag and Surface shear.

(3) Point load (P)

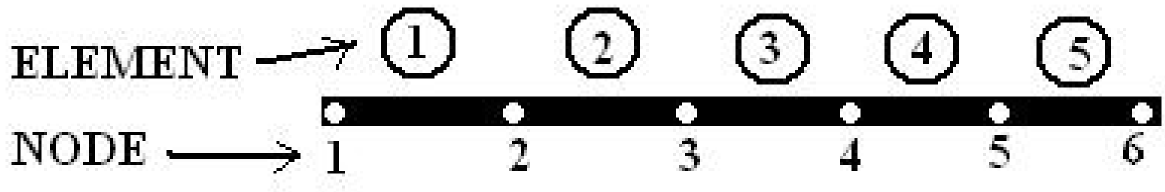
It is a force acting at a particular point which causes displacement.

- **Finite Element Modeling**

It has two processes.

(1) Discretization of structure

(2) Numbering of nodes.



- **CO – ORDINATES**

  (A) Global co – ordinates,

  (B) Local co – ordinates and

  (C) Natural co –ordinates.

- **Natural Co – Ordinate (ε)**

  Integration of polynomial terms in natural co – ordinates for two dimensional elements can be performed by using the formula,

  α β γ

- **Shape function**

  $N_1 N_2 N_3$ are usually denoted as shape function. In one dimensional problem, the displacement

  $$u = \Sigma N_i u_i = N_1 u_1$$

  For two noded bar element, the displacement at any point within the element is given by,

  $$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2$$

  For three noded triangular element, the displacement at any point within the element is given by,

  $$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

  $$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

  Shape function need to satisfy the following

  (a) First derivatives should be finite within an element; (b) Displacement should be continuous across the element boundary

- **Polynomial Shape function**

  Polynomials are used as shape function due to the following reasons, (1) Differentiation and integration of polynomials are quite easy.

  (2) It is easy to formulate and computerize the finite element equations.

  (3) The accuracy of the results can be improved by increasing the order of

- **Properties of Stiffness Matrix**

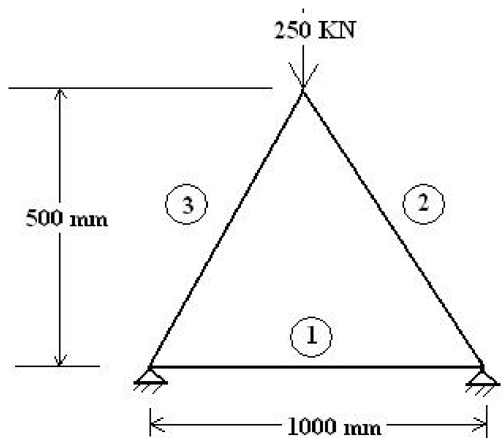
1. It is a symmetric matrix,
2. The sum of elements in any column must be equal to zero,
3. It is an unstable element. So the determinant is equal to zero.

- **Problem (I set)**

1. A two noded truss element is shown in figure. The nodal displacements are u_1 = 5 mm and u_2 = 8 mm. Calculate the displacement at x = ¼, 1/3 and ½.

- **Problem (II set)**

1. Consider a three bar truss as shown in figure. It is given that E = 2×10^5 N/mm^2. Calculate
(a) Nodal displacement,
(b) Stress in each member and
(c) Reactions at the support. Take Area of element 1 = 2000 mm^2, Area of element 2 = 2500 mm^2, Area of element 3 = 2500 mm^2.



- **Types of beam**

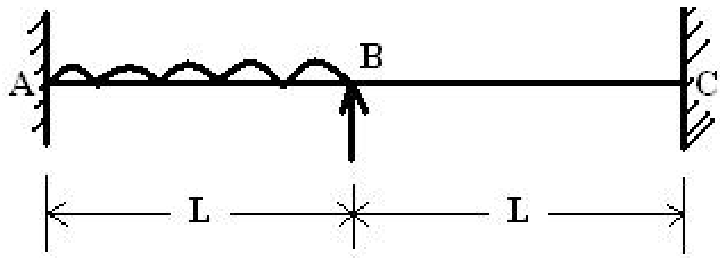
1. Cantilever beam,
2. Simply Supported beam,
3. Over hanging beam,
4. Fixed beam and
5. Continuous beam.

- **Types of Transverse Load**

1. Point or Concentrated Load,
2. Uniformly Distributed Load and
3. Uniformly

- **Problem (III set)**

  1. A fixed beam of length 2L m carries a uniformly distributed load of w (N/m) which runs over a length of L m from the fixed end. Calculate the rotation at Point B.



2.2 LINEAR STATIC ANALYSIS(BAR ELEMENT)

Most structural analysis problems can be treated as linear static problems, based on the following assumptions

1. Small deformations (loading pattern is not changed due to the deformed shape)
2. Elastic materials (no plasticity or failures)
3. Static loads (the load is applied to the structure in a slow or steady fashion)

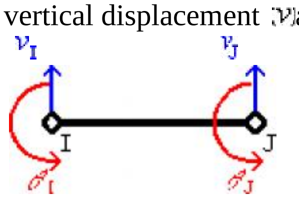
Linear analysis can provide most of the information about the behavior of a structure, and can be a good approximation for many analyses. It is also the bases of nonlinear analysis in most of the cases.

2.3 BEAM ELEMENT

A beam element is defined as a long, slender member (one dimension is much larger than the other two) that is subjected to vertical loads and moments, which produce vertical displacements and rotations. The degrees of freedom for a beam element are a vertical displacement and a rotation at each node, as opposed to only an horizontal displacement at each node for a truss element.

Degrees of Freedom

Degrees of freedom are defined as the number of independent coordinates necessary to specify the configuration of a system. The degrees of freedom for a general situation consists of three translations in the *x*, *y*, and *z* directions and three rotations about the *x*, *y*, and *z* axes. A one-dimensional beam element has four degrees of freedom, which include, a vertical displacement (v) and a rotation (θ) at each node.



Assumptions

Nodal Forces and Moments

Forces and moments can only be applied at the nodes of the beam element, not between the nodes. The nodal forces and moments, $\underline{f}_E$, are related to the nodal displacements and rotations, $\underline{v}_E$through the element stiffness matrix, $\underline{K}_E$.

Constant Load

The loads that are applied to the beam element are assumed to be static and not to vary over the time period being considered, this assumption is only valid if the rate of change of the force is much less than the applied force (*F* >> *dF/dt*). If the loads vary significantly, (if the variation in load is not much less than the applied force) then the problem must be considered as dynamic.

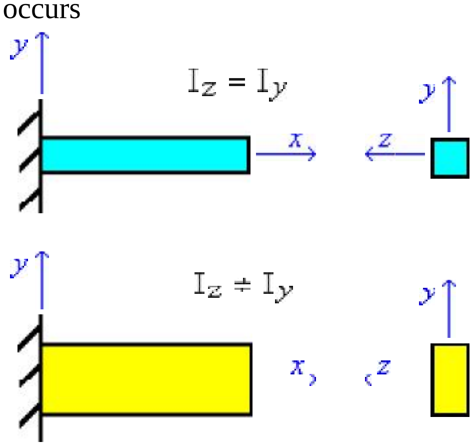
Weightless Member

The weight (W) of the beam is neglected, if it is much less than the total resultant forces (F) acting on the beam. If the weight of the beam is not neglected, then its effects must be represented as vertical forces acting at the nodes, by dividing up the weight and lumping it at the nodes, proportionally according to it's placement along the beam.

Prismatic Member

The beam element is assumed to have a constant cross-section, which means that the cross-sectional area and the moment of inertia will both be constant (i.e., the beam element is a prismatic member). If a beam is stepped, then it must be divided up into sections of constant cross-section, in order to obtain an exact solution. If a beam is tapered, then the beam can be approximated by using many small beam elements, each having the same cross-section as the middle of the tapered length it is approximating. The more sections that are used to approximate a tapered beam, the more accurate the solution will be.

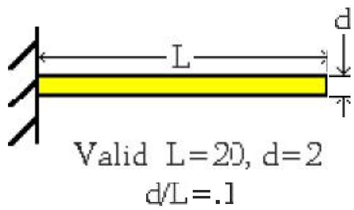
The moment of inertia is a geometric property of a beam element, which describes the beams resistance to bending and is assumed to be constant through the length of the element. The moment of inertia can be different along different axes if the beam element is not symmetric, we use the moment of inertia (I) of the axis about which the bending of the beam occurs

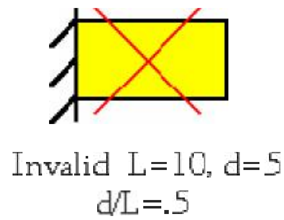


Where (I_z) refers to the moment of inertia, resisting bending about the "*z*" axis and (I_y) about the "*y*" axis.

The Beam Element is a Slender Member

A beam is assumed to be a slender member, when it's length (L) is more than 5 times as long as either of it's cross-sectional dimensions (d) resulting in (d/L<.2). A beam must be slender, in order for the beam equations to apply, that were used to derive our FEM equations.





The Beam Bends without Twisting.

It is assumed that the cross-section of the beam is symmetric about the plane of bending (*x-y* plane in this case) and will undergo symmetric bending (where no twisting of the beam occurs during the bending process). If the beam is not symmetric about this plane, then the beam will twist during bending and the situation will no longer be one-dimensional and must be approached as an unsymmetric bending problem (where the beam twists while bending) in order to obtain a correct solution.

Cross Section Remains Plane

When a beam element bends, it is assumed that it will deflect uniformly, thus the cross section will move uniformly and remain plane to the beam centerline. In other words, plane sections remain plane and normal to the *x* axis before and after bending.

Axially Rigid

The one-dimensional beam element is assumed to be axially rigid, meaning that there will be no axial displacement (*u*) along the beams centriodal axis. This implies that forces will only be applied perpendicular to the beams centriodal axis. The one-dimensional beam element can be used only when the degrees of freedom are limited to vertical displacements (perpendicular to the beams centriodal axis) and rotations in one plane. If axial displacements are present then a one-dimensional bar element must be superimposed with the one-dimensional beam element in order to obtain a valid solution.

Homogenous Material

A beam element has the same material composition throughout and therefore the same mechanical properties at every position in the material. Therefore, the modulus of elasticity E is constant throughout the beam element. A member in which the material properties varies from one point to the next in the member is called inhomogenous (non-homogenous). If a beam is composed of different types of materials, then it must be divide up into elements that are each of a single homogeneous material, otherwise the solution will not be exact.

Isotropic Material

A beam element has the same mechanical and physical properties in all directions, i.e., they are independent of direction. For instance, cutting out three tensile test specimens, one in the *x*-direction, one in the *y*-direction and the other oriented 45 degrees in the *x*-*y* plane, a tension test on each specimen, will result in the same value for the modulus of elasticity (E), yield strength (σ_Y) and ultimate strength (σ_u). Most metals are considered isotropic. In contrast fibrous materials, such as wood, typically have properties that are directionaly dependant and are generally considered anisotropic (not isotropic).

The Proportional Limit is not Exceeded

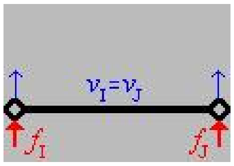
It is assumed that the beam element is initially straight and unstressed. It is also assumed that the material does not yield, therefore the beam will be straight after the load is released. These assumptions mean that the beam must be made of an elastic material, one which will return to it's original size and shape when all loads are removed, if not stressed past the materials elastic or proportional limit. It is also assumed that the beam is not stressed past the proportional limit, at which point the beam will take a permanent set and will not fully return to it's original size and shape, when all loads are removed. Below the proportional limit an elastic material is in the linear elastic range, where the strain (ε) varies linearly with the applied load and the stress (σ) varies linearly according to: $\sigma = E\varepsilon$, where E is the modulus of elasticity.

Rigid Body Modes for the One-Dimensional Beam Element

Rigid body motion occurs when forces and/or moments are applied to an unrestrained mesh (body), resulting in motion that occurs without any deformations in the entire mesh (body). Since no strains (deformations) occur during rigid body motion, there can be no stresses developed in the mesh. In order to obtain a unique FEM solution, rigid body motion must be constrained. If rigid body motion is not constrained, then a singular system of equations will result, since the determinate of the mesh stiffness matrix is equal to zero (i.e., $|\underline{K}| = 0$).

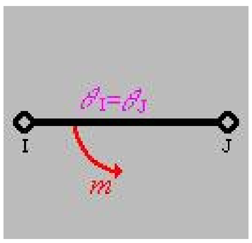
There are two rigid body modes for the one-dimensional beam element, a translation (displacement) only and a rotation only. These two rigid body modes can occur at the same time resulting in a displacement and a rotation simultaneously. In order to eliminate rigid body motion in a 1-D beam element (body), one must prescribe at least two nodal degrees of freedom (DOF), either two displacements or a displacement and a rotation. A DOF can be equal to zero or a non-zero known value, as long as the element is restrained from rigid body motion (deformation can take place when forces and moments are applied) .

For simplicity we will introduce the rigid body modes using a mesh composed of a single element. If only *translational rigid body motion* occurs, then the displacement at local node **I** will be equal to the displacement at local node **J**. Since the displacements are equal there is no strain developed in the element and the applied nodal forces cause the element to move in a rigid (non-deflected) vertical motion (which can be either up as shown below or it can be in the downward direction depending on the direction of the applied forces).



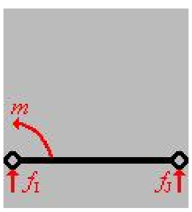
This rigid body mode can be suppressed by prescribing a vertical nodal displacement.

If rotational rigid body motion occurs, then the rotation at local node **I** will be equal to the rotation at local node **J** (i.e., in magnitude and direction). In this situation the nodal forces and/or moments applied to the element, cause the element to rotate as a rigid body (either clockwise as shown below or counterclockwise depending on the direction of the applied forces and/or moments).

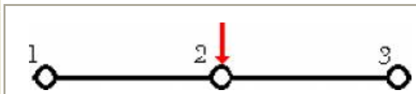
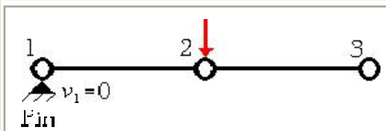


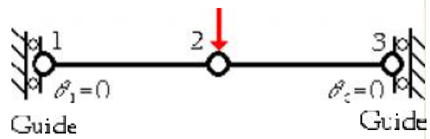
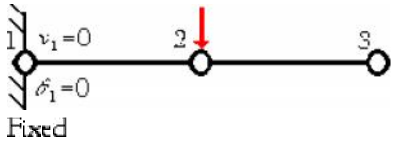
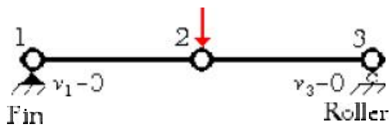
This rigid body mode can be suppressed by prescribing a nodal translation or rotation.

If translational *and* rotational rigid body motion occurs simultaneously then:



Simple Examples of Beam Problems with and without Rigid Body Motion

Case	**Stable/Unstable Structure**	**Rigid Body Mode(s) Present**	**Determinant of Mesh Stiffness Matrix**	**Equations**
	**Unstable**	θ **and** v	$\lvert K \rvert = 0$	**Dependent Equations**
	**Unstable**	θ	$\lvert K \rvert = 0$	**Dependent Equations**

	**Unstable**	v	$\lvert K \rvert = 0$	**Dependent Equations**
	**Stable**	**None**	$\lvert K \rvert \neq 0$	**Independen t Equations**
	**Stable**	**None**	$\lvert K \rvert \neq 0$	**Independen t Equations**

2.4 1-D 2-NODED CUBIC BEAM ELEMENT MATRICES

A single 1-d 2-noded cubic beam element has two nodes, with two degrees of freedom at each node (one vertical displacement and one rotation or slope). There is a total of 4 dof and the displacement polynomial function assumed should have 4 terms, so we choose a cubic polynomial for the vertical deflection. Slope is a derivative of the vertical deflections.

The vertical displacement $v = a + bx + cx^2 + dx^3$ …………………..(1)

The slope $\theta = \frac{dv}{dx} = b + 2cx + 3dx^2$ …………………..(2)

Apply the boundary conditions

$$\text{at } x = 0, \quad v = v_1 \quad \Rightarrow \quad v_1 = a \quad \Rightarrow \quad a = v_1$$
$$\text{at } x = 0, \quad \theta = \theta_1 \quad \Rightarrow \quad \theta_1 = b \quad \Rightarrow \quad b = \theta_1$$

$$\left.\begin{array}{lll} \text{at } x = l, & v = v_2 & \Rightarrow \quad v_2 = a + bl + cl^2 + dl^3 \\ \text{at } x = l, & \theta = \theta_2 & \Rightarrow \quad \theta_2 = b + 2cl + 3dl^2 \end{array}\right\} \text{ solving } \begin{array}{l} c = \frac{3}{l^2}(v_2 - v_1) - \frac{1}{l}(2\theta_1 + \theta_2) \\ d = \frac{2}{l^3}(v_1 - v_2) + \frac{1}{l^2}(\theta_1 + \theta_2) \end{array}$$

Substituting the values of *a*, *b*, *c* and *d* in equation (1), and collecting the coefficients of $v_1, \theta_1, v_2, \theta_2$ we obtain

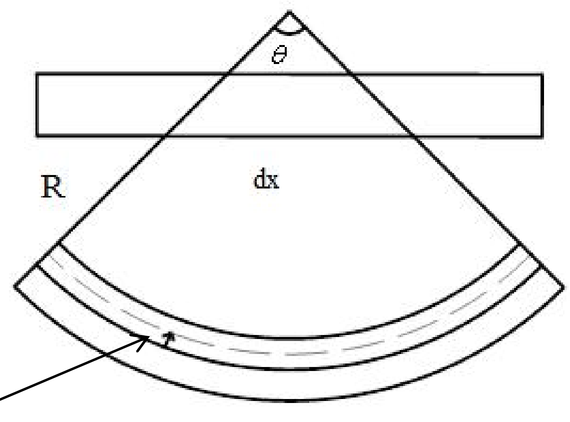
$$v = N_1 v_1 + N_2 \theta_1 + N_3 v_3 + N_4 \theta_2$$

where

$$N_1 = 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}, \quad N_2 = x - 2\frac{x^2}{l} + \frac{x^3}{l^2},$$

$$N_3 = \quad 3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}, \quad N_4 = \quad -\frac{x^2}{l} + \frac{x^3}{l^2}$$

2.5 DEVELOPMENT OF ELEMENT EQUATION



$$dx = R\bar{\ }$$

$$\in_x = \frac{(R-y)\bar{\ } - R\bar{\ }}{R\bar{\ }} = \frac{-y}{R} = -y\frac{d^2y}{dx^2}$$

$$v = N_1v_1 + N_2\bar{\ }_1 + N_3v_2 + N_4\bar{\ }_2$$

$$\in_x = -y\frac{d^2v}{dx^2} = -y\frac{d^2}{dx^2}\left(N_1v_1 + N_2\bar{\ }_1 + N_3v_2 + N_4\bar{\ }_2\right)$$

$$\in_x = -y\underbrace{\begin{pmatrix} \frac{d^2N_1}{dx^2} & \frac{d^2N_2}{dx^2} & \frac{d^2N_3}{dx^2} & \frac{d^2N_4}{dx^2} \end{pmatrix}}_{[B]}\underbrace{\begin{pmatrix} v_1 \\ \bar{\ }_1 \\ v_2 \\ \bar{\ }_2 \end{pmatrix}}_{\{a\}}$$

$$\in_x = [B]\{a\}$$

We Know that,

$$[K]=\int_v [B]^T D[B]\,dv$$

$$[K]=\int_{volume}(-y)\begin{pmatrix}\frac{d^2N_1}{dx^2}\\ \frac{d^2N_2}{dx^2}\\ \frac{d^2N_3}{dx^2}\\ \frac{d^2N_4}{dx^2}\end{pmatrix}E(-y)\begin{pmatrix}\frac{d^2N_1}{dx^2} & \frac{d^2N_2}{dx^2} & \frac{d^2N_3}{dx^2} & \frac{d^2N_4}{dx^2}\end{pmatrix}dv$$

$$[K]=E\int_v\int_0^l y^2\begin{pmatrix}\left(\frac{d^2N_1}{dx^2}\right)^2 & \left(\frac{d^2N_1}{dx^2}\frac{d^2N_2}{dx^2}\right) & \left(\frac{d^2N_1}{dx^2}\frac{d^2N_3}{dx^2}\right) & \left(\frac{d^2N_1}{dx^2}\frac{d^2N_4}{dx^2}\right)\\ \left(\frac{d^2N_2}{dx^2}\frac{d^2N_1}{dx^2}\right) & \left(\frac{d^2N_2}{dx^2}\right)^2 & \left(\frac{d^2N_2}{dx^2}\frac{d^2N_3}{dx^2}\right) & \left(\frac{d^2N_2}{dx^2}\frac{d^2N_4}{dx^2}\right)\\ \left(\frac{d^2N_3}{dx^2}\frac{d^2N_1}{dx^2}\right) & \left(\frac{d^2N_3}{dx^2}\frac{d^2N_2}{dx^2}\right) & \left(\frac{d^2N_3}{dx^2}\right)^2 & \left(\frac{d^2N_3}{dx^2}\frac{d^2N_4}{dx^2}\right)\\ \left(\frac{d^2N_4}{dx^2}\frac{d^2N_1}{dx^2}\right) & \left(\frac{d^2N_4}{dx^2}\frac{d^2N_2}{dx^2}\right) & \left(\frac{d^2N_4}{dx^2}\frac{d^2N_3}{dx^2}\right) & \left(\frac{d^2N_4}{dx^2}\right)^2\end{pmatrix}dA\,dx$$

Where, $\int_0^l y^2 dA = I$

$$[K]=EI\int_0^v\begin{pmatrix}\frac{d^2N_1}{dx^2}\\ \frac{d^2N_2}{dx^2}\\ \frac{d^2N_3}{dx^2}\\ \frac{d^2N_4}{dx^2}\end{pmatrix}\begin{pmatrix}\frac{d^2N_1}{dx^2} & \frac{d^2N_2}{dx^2} & \frac{d^2N_3}{dx^2} & \frac{d^2N_4}{dx^2}\end{pmatrix}dx$$

Where,

$$N_1=1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3}\Rightarrow\frac{dN_1}{dx}=\frac{-6x}{l^2}+\frac{6x^2}{l^3}\Rightarrow\frac{d^2N_1}{dx^2}=\frac{-6}{l^2}+\frac{12x}{l^3}$$

$$N_2=x-\frac{2x^2}{l}+\frac{x^3}{l^2}\Rightarrow\frac{dN_2}{dx}=1-\frac{4x}{l}+\frac{3x^2}{l^2}\Rightarrow\frac{d^2N_2}{dx^2}=\frac{-4}{l}+\frac{6x}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{3x^3}{l^3} \Rightarrow \frac{dN_3}{dx} = \frac{6x}{l^2} - \frac{6x^2}{l^3} \Rightarrow \frac{d^2N_3}{dx^2} = \frac{6}{l^2} - \frac{12x}{l^3}$$

$$N_4 = \frac{x^3}{l^2} - \frac{x^2}{l} \Rightarrow \frac{dN_4}{dx} = \frac{3x}{l^2} - \frac{2x}{l} \Rightarrow \frac{d^2N_4}{dx^2} = \frac{6x}{l^2} - \frac{2}{l}$$

$$[K] = EI\int_0^l \begin{pmatrix} \left(\frac{-6}{l^2} + \frac{12x}{l^3}\right) \\ \left(\frac{-4}{l} + \frac{6x}{l^2}\right) \\ \left(\frac{6}{l^2} - \frac{12x}{l^3}\right) \\ \left(\frac{6x}{l^2} - \frac{2}{l}\right) \end{pmatrix} \left(\left(\frac{-6}{l^2} + \frac{12x}{l^3}\right) \quad \left(\frac{-4}{l} + \frac{6x}{l^2}\right) \quad \left(\frac{6}{l^2} - \frac{12x}{l^3}\right) \quad \left(\frac{6x}{l^2} - \frac{2}{l}\right) \right) dx$$

$$K_{11} = EI\int_0^l \left(\frac{-6}{l^2} + \frac{12x}{l^3}\right)\left(\frac{-6}{l^2} + \frac{12x}{l^3}\right) dx$$

$$K_{11} = EI\int_0^l \left(\frac{-6l + 12x}{l^3}\right)\left(\frac{-6l + 12x}{l^3}\right) dx$$

$$K_{11} = EI\int_0^l \left(\frac{36l^2 - 72xl - 72xl + 144x^2}{l^6}\right) dx$$

$$K_{11} = EI\int_0^l \left(\frac{36l^2}{l^6} - \frac{144xl}{l^6} + \frac{144x^2}{l^6}\right) dx$$

$$K_{11} = EI\left[\frac{36xl^2}{l^6} - \frac{144x^2l}{2l^6} + \frac{144x^3}{3l^6}\right]_o^l$$

$$K_{11} = EI\left[\frac{-36}{l^3} - \frac{72}{l^3} + \frac{48}{l^3}\right]$$

$$\boxed{K_{11} = \frac{12EI}{l^3}}$$

$$K_{12} = EI\int_0^l \left(\frac{-6}{l^2} + \frac{12x}{l^3}\right)\left(\frac{-4}{l} + \frac{6x}{l^2}\right) dx$$

$$K_{12} = EI\int_0^l \left(\frac{-6l+12x}{l^3}\right)\left(\frac{-4l+6x}{l^2}\right)dx$$

$$K_{12} = EI\int_0^l \left(\frac{24l^2-48xl-36xl+72x^2}{l^5}\right)dx$$

$$K_{12} = EI\int_0^l \left(\frac{24l^2}{l^5}-\frac{84xl}{l^5}+\frac{72x^2}{l^5}\right)dx$$

$$K_{12} = EI\int_0^l \left[\frac{24xl^2}{l^5}-\frac{84x^2l}{2l^5}+\frac{72x^3}{3l^5}\right]_0^l$$

$$K_{12} = EI\left[\frac{24}{l^2}-\frac{42}{l^2}+\frac{24}{l^2}\right]$$

$$\boxed{K_{12} = \frac{6EI}{l^2}}$$

$$K_{13} = EI\int_0^l \left(\frac{-6}{l^2}+\frac{12x}{l^3}\right)\left(\frac{6}{l^2}+\frac{-12x}{l^3}\right)dx$$

$$K_{13} = EI\int_0^l \left(\frac{-6l+12x}{l^3}\right)\left(\frac{6l-12x}{l^3}\right)dx$$

$$K_{13} = EI\int_0^l \left(\frac{-36l^2+72xl+72xl-144x^2}{l^6}\right)dx$$

$$K_{13} = EI\int_0^l \left[\frac{-36xl^2}{l^6}+\frac{144x^2l}{2l^6}-\frac{144x^3}{3l^6}\right]_0^l$$

$$K_{13} = EI\left[\frac{-36+72-48}{l^3}\right]$$

$$\boxed{K_{13} = \frac{-12EI}{l^3}}$$

$$K_{14} = EI\int_0^l \left(\frac{-6}{l^2}+\frac{12x}{l^3}\right)\left(\frac{6x}{l^2}-\frac{2}{l}\right)dx$$

$$K_{14} = EI\int_0^l \left(\frac{-6l+12x}{l^3}\right)\left(\frac{6x-2l}{l^2}\right)dx$$

$$K_{14} = EI\int_0^l \left(\frac{12l^2 - 36xl - 24xl + 72x^2}{l^5} \right) dx$$

$$K_{14} = EI\int_0^l \left(\frac{12l^2}{l^5} - \frac{60xl}{l^5} + \frac{72x^2}{l^5} \right) dx$$

$$K_{14} = EI\int_0^l \left[\frac{12xl^2}{l^5} - \frac{60x^2l}{2l^5} + \frac{72x^3}{3l^5} \right]_0^l$$

$$K_{14} = EI\left[\frac{-30+12+24}{l^2} \right]$$

$$\boxed{K_{14} = \frac{6EI}{l^2}}$$

$$K_{21} = EI\int_0^l \left(\frac{-4}{l} + \frac{6x}{l^2} \right)\left(\frac{-6}{l^2} + \frac{12x}{l^3} \right) dx$$

$$\boxed{K_{21} = \frac{6EI}{l^2}}$$

$$K_{22} = EI\int_0^l \left(\frac{-4}{l} + \frac{6x}{l^2} \right)\left(\frac{-4}{l} + \frac{6x}{l^2} \right) dx$$

$$K_{22} = EI\int_0^l \left(\frac{-4l + 6x}{l^2} \right)\left(\frac{-4l + 6x}{l^2} \right) dx$$

$$K_{22} = EI\int_0^l \left(\frac{16l^2 - 24xl - 24xl + 36x^2}{l^4} \right) dx$$

$$K_{22} = EI\int_0^l \left(\frac{16l^2}{l^4} - \frac{48xl}{l^4} + \frac{36x^2}{l^4} \right) dx$$

$$K_{22} = EI\int_0^l \left[\frac{16xl^2}{l^4} - \frac{48x^2l}{2l^4} + \frac{36x^3}{3l^4} \right]_0^l$$

$$K_{22} = EI\left[\frac{16-24+12}{l} \right]$$

$$\boxed{K_{22} = \frac{4EI}{l}}$$

$$K_{23} = EI\int_0^l \left(\frac{-4}{l} + \frac{6x}{l^2}\right)\left(\frac{6}{l^2} - \frac{12x}{l^3}\right)dx$$

$$K_{23} = EI\int_0^l \left(\frac{-4l + 6x}{l^2}\right)\left(\frac{6l - 12x}{l^3}\right)dx$$

$$K_{23} = EI\int_0^l \left(\frac{-24l^2 + 36xl + 48xl - 72x^2}{l^5}\right)dx$$

$$K_{23} = EI\int_0^l \left(\frac{-24l^2 + 84xl - 72x^2}{l^5}\right)dx$$

$$K_{23} = EI\int_0^l \left[\frac{-24xl^2}{l^5} + \frac{84x^2 l}{2l^5} - \frac{72x^3}{3l^5}\right]_0^l$$

$$K_{23} = EI\left[\frac{-24 + 42 - 24}{l^2}\right]$$

$$\boxed{K_{23} = \frac{-6EI}{l^2}}$$

$$K_{24} = EI\int_0^l \left(\frac{-4}{l} + \frac{6x}{l^2}\right)\left(\frac{6x}{l^2} - \frac{2}{l}\right)dx$$

$$K_{24} = EI\int_0^l \left(\frac{-4l + 6x}{l^2}\right)\left(\frac{6x - 2l}{l^2}\right)dx$$

$$K_{24} = EI\int_0^l \left(\frac{8l^2 - 24xl - 12xl + 36x^2}{l^4}\right)dx$$

$$K_{24} = EI\int_0^l \left(\frac{8l^2 - 36xl + 36x^2}{l^4}\right)dx$$

$$K_{24} = EI\int_0^l \left[\frac{8xl^2}{l^4} - \frac{36x^2 l}{2l^4} + \frac{36x^3}{3l^4}\right]_0^l$$

$$K_{24} = EI\left[\frac{-18 + 12 + 8}{l}\right]$$

$$\boxed{K_{24} = \frac{2EI}{l}}$$

$$K_{31} = EI\int_0^l \left(\frac{6}{l^2} + \frac{-12x}{l^3}\right)\left(\frac{-6}{l^2} + \frac{12x}{l^3}\right)dx$$

$$\boxed{K_{31} = \frac{-12EI}{l^3}}$$

$$K_{32} = EI\int_0^l \left(\frac{6}{l^2} - \frac{12x}{l^3}\right)\left(\frac{-4}{l} + \frac{6x}{l^2}\right)dx$$

$$\boxed{K_{32} = \frac{-6EI}{l^2}}$$

$$K_{33} = EI\int_0^l \left(\frac{6}{l^2} - \frac{12x}{l^3}\right)\left(\frac{6}{l^2} - \frac{12x}{l^3}\right)dx$$

$$K_{33} = EI\int_0^l \left(\frac{6l-12x}{l^3}\right)\left(\frac{6l-12x}{l^3}\right)dx$$

$$K_{33} = EI\int_0^l \left(\frac{36l^2 - 72xl - 72xl + 144x^2}{l^6}\right)dx$$

$$K_{33} = EI\int_0^l \left(\frac{36l^2 - 144xl + 144x^2}{l^6}\right)dx$$

$$K_{33} = EI\int_0^l \left[\frac{36xl^2}{l^6} - \frac{144x^2l}{2l^6} + \frac{144x^3}{3l^6}\right]_0^l$$

$$K_{33} = EI\left[\frac{36-72+48}{l^3}\right]$$

$$\boxed{K_{33} = \frac{12EI}{l^3}}$$

$$K_{34} = EI\int_0^l \left(\frac{6}{l^2} - \frac{12x}{l^3}\right)\left(\frac{6x}{l^2} - \frac{2}{l}\right)dx$$

$$K_{34} = EI\int_0^l \left(\frac{6l-12x}{l^3}\right)\left(\frac{6x-2l}{l^2}\right)dx$$

$$K_{34} = EI\int_0^l \left(\frac{-12l^2 + 24xl + 36xl - 72x^2}{l^5}\right)dx$$

$$K_{34} = EI\int_0^l \left(\frac{-12l^2 + 60xl - 72x^2}{l^5} \right) dx$$

$$K_{34} = EI\int_0^l \left[\frac{-12xl^2}{l^5} + \frac{60x^2 l}{2l^5} - \frac{72x^3}{3l^5} \right]_0^l$$

$$K_{34} = EI\left[\frac{-12 + 30 - 24}{l^2} \right]$$

$$\boxed{K_{34} = \frac{-6EI}{l^2}}$$

$$K_{41} = EI\int_0^l \left(\frac{6x}{l^2} - \frac{2}{l} \right)\left(\frac{-6}{l^2} + \frac{12x}{l^3} \right) dx$$

$$\boxed{K_{41} = \frac{6EI}{l^2}}$$

$$K_{42} = EI\int_0^l \left(\frac{6x}{l^2} - \frac{2}{l} \right)\left(\frac{-4}{l} + \frac{6x}{l^2} \right) dx$$

$$\boxed{K_{42} = \frac{2EI}{l}}$$

$$K_{43} = EI\int_0^l \left(\frac{6x}{l^2} - \frac{2}{l} \right)\left(\frac{6}{l^2} - \frac{12x}{l^3} \right) dx$$

$$\boxed{K_{43} = \frac{-6EI}{l^2}}$$

$$K_{44} = EI\int_0^l \left(\frac{6x}{l^2} - \frac{2}{l} \right)\left(\frac{6x}{l^2} - \frac{2}{l} \right) dx$$

$$K_{44} = EI\int_0^l \left(\frac{6x - 2l}{l^2} \right)\left(\frac{6x - 2l}{l^2} \right) dx$$

$$K_{44} = EI\int_0^l \left(\frac{4l^2 - 12xl - 12xl + 36x^2}{l^4} \right) dx$$

$$K_{44} = EI\int_0^l \left(\frac{4l^2 - 24xl + 36x^2}{l^4} \right) dx$$

$$K_{44} = EI\int_0^l \left[\frac{4xl^2}{l^4} - \frac{24x^2l}{2l^4} + \frac{36x^3}{3l^4} \right]_0^l$$

$$K_{44} = EI\left[\frac{12+4-12}{l} \right]$$

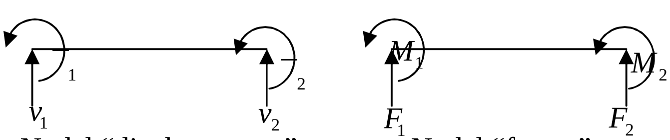
$$\boxed{K_{44} = \frac{4EI}{l}}$$

Therefore $[K]$ is

$$[K] = EI\begin{pmatrix} \frac{12}{l^3} & \frac{6}{l^2} & \frac{-12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^3} & \frac{-6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{pmatrix}$$

2.6 BEAM ELEMENT

A beam is a long, slender structural member generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects. An elemental length of a long beam subjected to arbitrary loading is considered for analysis. For this elemental beam length L, we assign two points of interest, i.e., the ends of the beam, which become the nodes of the beam element. The bending deformation is measured as a transverse (vertical) displacement and a rotation (slope). Hence, for each node, we have a vertical displacement and a rotation (slope) – two degrees of freedom at each node. For a single 2-noded beam element, we have a total of 4 degrees of freedom. The associated "forces" are shear force and bending moment at each node.



Nodal "displacements" Nodal "forces"

1st degree of freedom	vertical displacement at node i	1	v_i or v_1	corres-pond-ing to	shear force at node i	F_i or F_1	1
2nd degree of freedom	slope or rotation at node i	2	$–_i$ or $–_1$		bending moment at node i	M_i or M_1	2
3rd	vertical	3	v_j or v_2		shear force at node	F_j or F_2	3

degree of freedom	displacement at node j				i		
4^{th} degree of freedom	slope or rotation at node j	4	$-_j$ or $-_2$		bending moment at node j	M_j or M_2	4

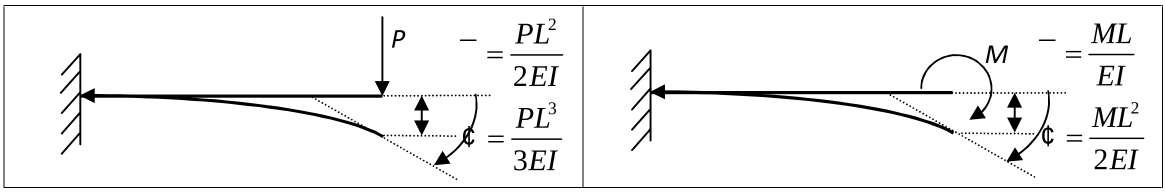
The stiffness term k_{ij} indicates the force (or moment) required at i to produce a unit deflection (or rotation) at j, while all other degrees of freedom are kept zero.

Sign conventions followed

Upward forces are positive and upward displacements are positive.

Counter-clockwise moments are positive and counter-clockwise rotations are positive.

Formulae required – cantilever beam subjected to concentrated load and moment.



2.6.1 ELEMENT MATRICES AND VECTORS

Derivation of first column of stiffness matrix: $v_1 = 1, \; -_1 = v_2 = -_2 = 0$, i.e., allow the first degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_2 = -_2 = 0$, we can fix node j. To produce an upward deflection at node i (i.e., allowing first degree of freedom to occur), apply an upward force k_{11} (first suffix indicates the force or moment DoF and the second suffix indicates the displacement or rotational DoF). $v_1 = \frac{k_{11}L^3}{3EI}$ upwards. Refer table for displacement DoF number and force DoF number. Now the beam configuration is given by Figure 1. We can observe from the figure that the slope at node i is not zero. To make the slope at i equal to zero, we need to apply a counter-clockwise moment k_{21}. Refer Figure 2. But this moment k_{21} will produce a downward deflection $\frac{k_{21}L^2}{2EI}$ at node i. Refer Figure

3. In order to have a resultant unit upward displacement at node i, upward displacement produced by force k_{11} must be greater than the downward displacement produced by the moment k_{21}. i.e., $\frac{k_{11}L^3}{3EI} - \frac{k_{21}L^2}{2EI} = 1$ …..(1). At the same time, the negative slope produced at node i by the force k_{11} must be cancelled by the positive slope produced by the moment k_{21}. i.e., $\frac{k_{11}L^2}{2EI} = \frac{k_{21}L}{EI}$ ….(2). Solving these two equations, k_{11} and k_{21} are found. The fixed end

reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{31} ……. $\left(\sum F_y = 0 \quad \Rightarrow \quad k_{11} + k_{31} = 0\right)$ and moment equilibrium equation about node i to find fixed end reaction moment k_{41} .... $\left(\sum M_i = 0 \quad \Rightarrow \quad k_{21} + k_{31} L + k_{41} = 0\right)$.

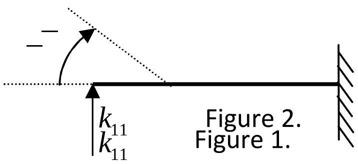


Figure 2.
Figure 1.

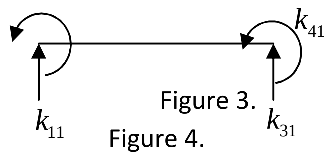
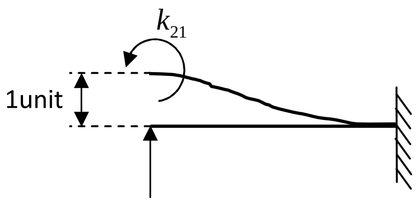
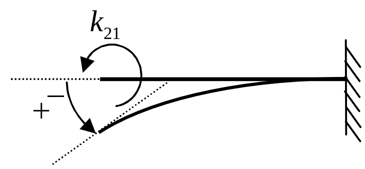


Figure 3.
Figure 4.





$$\begin{bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \end{bmatrix} = \begin{bmatrix} \dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \\ -\dfrac{12EI}{L^3} \\ \dfrac{6EI}{L^2} \end{bmatrix}$$

Derivation of second column of stiffness matrix: $v_1 = 0,\ \theta_1 = 1, v_2 = \theta_2 = 0$, i.e., allow the second degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_2 = \theta_2 = 0$, we can fix node j. To produce a counterclockwise (positive) rotation or slope at node i (i.e., allowing second degree of freedom to occur), apply a counterclockwise moment $k_{22}.\ \theta_1 = \frac{k_{22}L}{EI}$. Refer Figure 1. This moment k_{22} will produce a downward deflection $\frac{k_{22}L^2}{2EI}$. This downward deflection should be canceled by applying an upward force k_{12} at node i. The upward deflection produced by k_{12} is $\frac{k_{12}L^3}{3EI}$. Refer Figure 2. Equating these two deflections $\frac{k_{22}L^2}{2EI} = \frac{k_{12}L^3}{3EI}$ …(1) But this upward force k_{12} will also produce a negative slope at node i which is $\frac{k_{12}L^2}{2EI}$. Refer Figure 3. Hence the rotation produced by k_{22} should be greater than that produced by k_{12} so that the resultant rotation is 1 radians. $\frac{k_{22}L}{EI} - \frac{k_{12}L^2}{2EI} = 1$ ….(2). Solving these two equations, k_{12} and k_{22} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{32} … $\left(\sum F_y = 0 \quad \Rightarrow \quad k_{12} + k_{32} = 0\right)$ and moment equilibrium equation about node i to find fixed end reaction moment k_{42} …. $\left(\sum M_i = 0 \quad \Rightarrow \quad k_{22} + k_{32}L + k_{42} = 0\right)$.

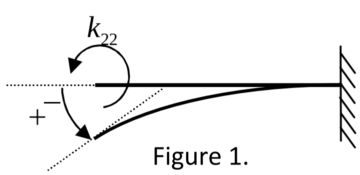


Figure 1.

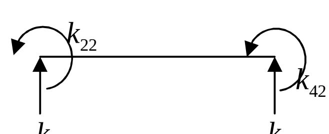


Figure 4.

$$\begin{bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{4EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{2EI}{L} \end{bmatrix}$$

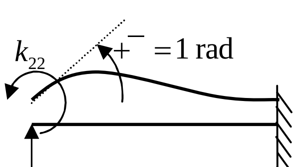


Figure 2.

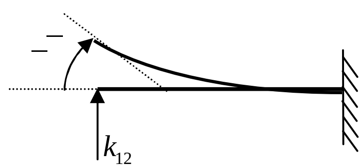


Figure 3.

Derivation of third column of stiffness matrix: $v_1 = 0,\ \theta_1 = 0, v_2 = 1, \theta_2 = 0$, i.e., allow the third degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_1 = {}^{-}{}_1 = 0$, we can fix node i. To produce an upward deflection at node j (i.e., allowing third degree of freedom to occur), apply an upward force k_{33}.

$v_2 = \frac{k_{33}L^3}{3EI}$ upwards. Now the beam configuration is given by Figure 1. We can observe from the figure that the slope at node j is not zero. To make the slope at j equal to zero, we need to apply a clockwise moment k_{43}. Refer Figure 2. But this moment k_{43} will produce a downward deflection $\frac{k_{43}L^2}{2EI}$ at node j. Refer Figure 3. In order to have a resultant unit upward displacement at node j, upward displacement produced by force k_{33} must be greater than the downward displacement produced by the moment k_{43}. i.e., $\frac{k_{33}L^3}{3EI} - \frac{k_{43}L^2}{2EI} = 1$ …..(1). At the same time, the positive slope produced at node j by the force k_{33} must be cancelled by the negative slope produced by the moment k_{43}. i.e., $\frac{k_{33}L^2}{2EI} = \frac{k_{43}L}{EI}$ ….(2). Solving these two equations, k_{33} and k_{43} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{13} … $\left(\sum F_y = 0 \quad \Rightarrow \quad k_{13} + k_{33} = 0\right)$ and moment equilibrium equation about node i to find fixed end reaction moment k_{23} …. $\left(\sum M_i = 0 \quad \Rightarrow \quad k_{23} + k_{33}L + k_{43} = 0\right)$.

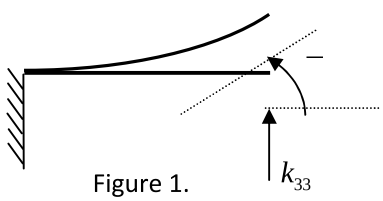


Figure 1.

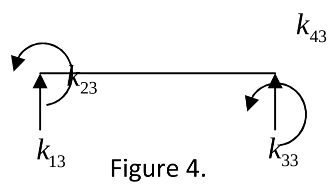


Figure 4.

$$\begin{bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \end{bmatrix} = \begin{bmatrix} -\frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \\ \frac{12EI}{L^3} \\ -\frac{6EI}{L^2} \end{bmatrix}$$

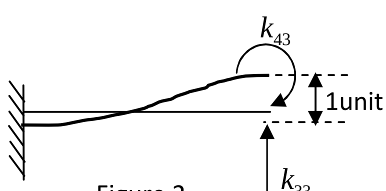


Figure 2.

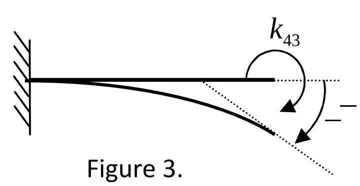


Figure 3.

Derivation of fourth column of stiffness matrix: $v_1 = {}^{-}{}_1 = 0, v_2 = 0, {}^{-}{}_2 = 1$, i.e., allow the fourth degree of freedom to occur and arrest all other DoF. (The deformed configuration is shown in Figure 2).

Initially you have a horizontal beam element. Since $v_1 = {}^{-}{}_1 = 0$, we can fix node i. To produce a counterclockwise (positive) rotation or slope at node j (i.e., allowing fourth degree of freedom to occur), apply a counterclockwise moment k_{44}. ${}^{-}{}_2 = \frac{k_{44}L}{EI}$. Refer Figure 1. This moment k_{44} will produce a upward deflection $\frac{k_{44}L^2}{2EI}$. This upward deflection should be canceled by applying a downward force k_{34} at node j. The downward deflection produced by k_{34} is $\frac{k_{34}L^3}{3EI}$. Refer Figure 2. Equating these two deflections $\frac{k_{44}L^2}{2EI} = \frac{k_{34}L^3}{3EI} \ldots(1)$ But this downward force k_{34} will also produce a negative slope at node j which is $\frac{k_{34}L^2}{2EI}$. Hence the rotation produced by k_{44} should be greater than that produced by k_{34} so that the resultant rotation is 1 radians. $\frac{k_{44}L}{EI} - \frac{k_{34}L^2}{2EI} = 1 \ldots.(2)$ Refer Figure 3. Solving these two equations, k_{34} and k_{44} are found. The fixed end reaction force and the reaction moment are assumed to be acting upwards and counterclockwise, respectively. Now use force equilibrium equation to find fixed end reaction force k_{14} … $\left(\sum F_y = 0 \quad \Rightarrow \quad k_{14} + k_{34} = 0\right)$ and moment equilibrium equation about node i to find fixed end reaction moment $k_{24} \ldots. \left(\sum M_i = 0 \quad \Rightarrow \quad k_{24} + k_{34}L + k_{44} = 0\right)$.

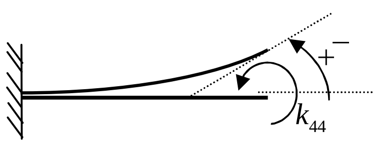


Figure 1.

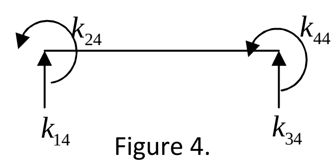


Figure 4.

$$\begin{bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{2EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{4EI}{L} \end{bmatrix}$$

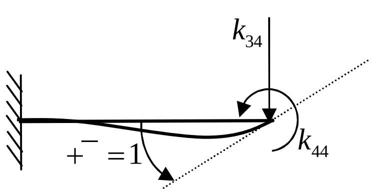


Figure 2.

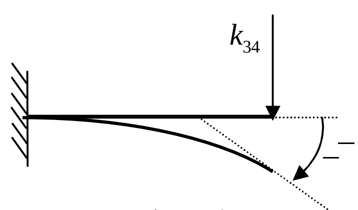
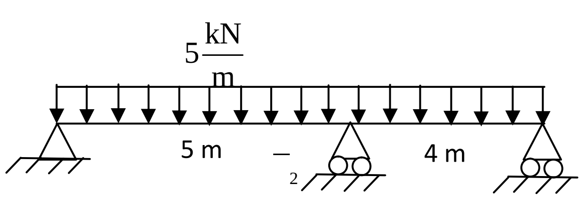


Figure 3.

Problem

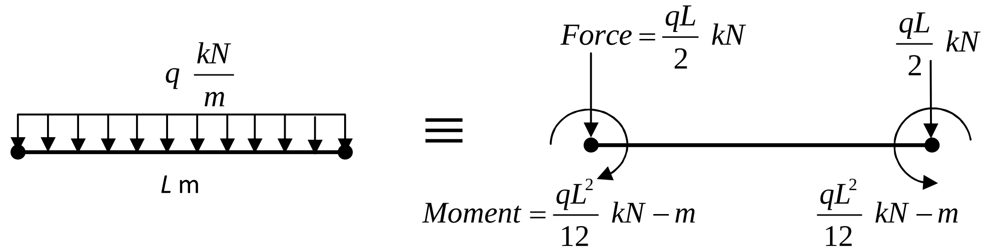
Find the slopes at the supports and support reaction forces and support reaction moments for the beam shown in Figure. Take E=210 GPa, I = 2×10^{-4} m^4. Daryl Logan P4-24 page 208.



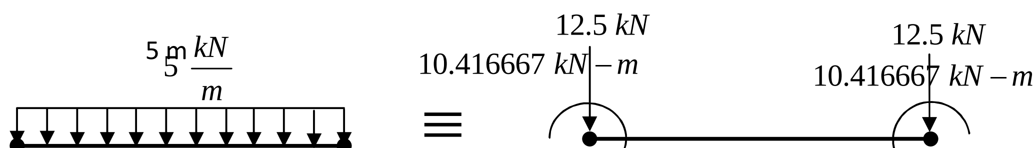
Finite element representation of the problem

v_2

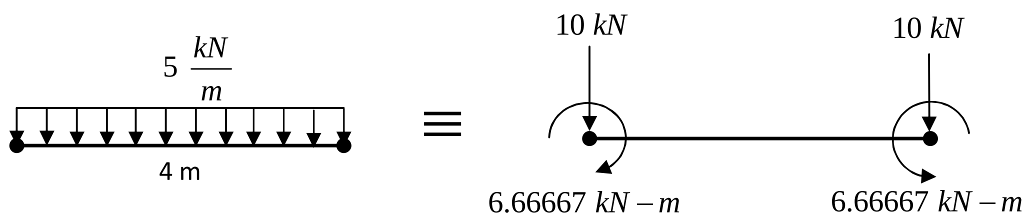
Conversion of UDL into nodal forces and nodal moments



for element 1,



for element 2,



$$EI = 210 \text{ GPa} \times 2\times10^{-4} \text{ m}^4 = 210\times10^6 \frac{kN}{m^2} \times 2\times10^{-4} \text{ m}^4 = 42000 \text{ kN-m}^2$$

Stiffness matrix for element 1

$$[K]^{(1)} = \begin{bmatrix} 4,032 & 10,080 & -4032 & 10,080 \\ 10,080 & 33,600 & -10,080 & 16,800 \\ -4032 & -10,080 & 4,032 & -10,080 \\ 10,080 & 16,800 & -10,080 & 33,600 \end{bmatrix}$$

Stiffness matrix for element 2

$$[K]^{(2)} = \begin{bmatrix} 7,875 & 15,750 & -7,875 & 15,750 \\ 15,750 & 42,000 & -15,750 & 21,000 \\ -7,875 & -15,750 & 7,875 & -15,750 \\ 15,750 & 21,000 & -15,750 & 42,000 \end{bmatrix}$$

Assembly of finite element equations

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_2 \\ M_3 \end{Bmatrix} + \begin{Bmatrix} -12.5 \\ -10.416667 \\ -12.5-10 \\ 10.416667-6.66667 \\ -10 \\ 6.66667 \end{Bmatrix} = \begin{Bmatrix} F_1-12.5 \\ 0-10.416667 \\ F_2-22.5 \\ 0+3.75 \\ F_2-10 \\ 0+6.66667 \end{Bmatrix}$$

support reactions　　　applied forces

> Support reaction moments at all simply supported ends are zero. $M_1 = M_2 = M_3 = 0$
> All support reaction forces are unknowns.

$v_1 = v_2 = v_3 = 0 \qquad \theta_1 = ? \quad \theta_2 = ? \quad \theta_3 = ?$

$$\begin{Bmatrix} F_1-12.5 \\ 0-10.416667 \\ F_2-22.5 \\ 0+3.75 \\ F_3-10 \\ 0+6.66667 \end{Bmatrix} = \begin{bmatrix} 4,032 & 10,080 & -4,032 & 10,080 & 0 & 0 \\ 10,080 & 33,600 & -10,080 & 16,800 & 0 & 0 \\ -4,032 & -10,080 & 4,032+7,875 & -10,080+15,750 & -7,875 & 15,750 \\ 10,080 & 16,800 & -10,080+15,750 & 33,600+42,000 & -15,750 & 21,000 \\ 0 & 0 & -7,875 & -15,750 & 7,875 & -15,750 \\ 0 & 0 & 15,750 & 21,000 & -15,750 & 42,000 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_1 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix}$$

Eliminating the first, third and fifth rows and columns of the stiffness matrix, the reduced matrix becomes

$$\begin{Bmatrix} -10.416667 \\ 3.75 \\ 6.66667 \end{Bmatrix} = \begin{bmatrix} 33,600 & 16,800 & 0 \\ 16,800 & 75,600 & 21,000 \\ 0 & 21,000 & 42,000 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Solving these equations

$\theta_1 = -3.59623\times10^{-4}$ rad $\quad \theta_2 = 9.9206349\times10^{-5}$ rad $\quad \theta_3 = 1.0912698\times10^{-4}$ rad

Substituting these values in the assembled matrix to find the support reactions, we find

$$F_1 - 12.5 = \left\{10,080\times\left(-3.59623\times10^{-4}\right)\right\} + \left\{10,080\times9.92\times10^{-5}\right\}$$
$$F_2 - 22.5 = \left\{-10,080\times\left(-3.59623\times10^{-4}\right)\right\} + \left\{5,670\times9.92\times10^{-5}\right\} + \left\{15,750\times1.0913\times10^{-4}\right\}$$
$$F_3 - 10 = \left\{15,750\times9.92\times10^{-5}\right\} - \left\{15,750\times1.0913\times10^{-4}\right\}$$

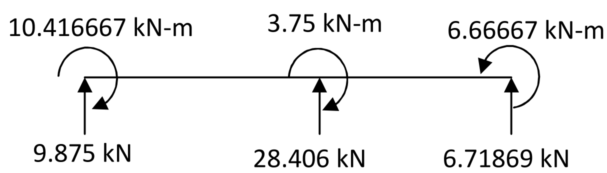
$F_1 - 12.5 = -2.625$ kN $\qquad F_1 = 9.875$ kN

$F_2 - 22.5 = 5.9062$ kN which means $F_2 = 28.406$ kN

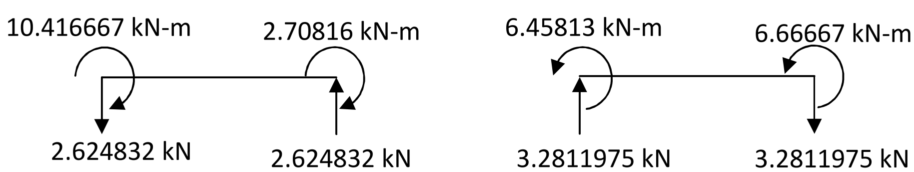
$F_3 - 12.5 = -3.2812$ kN $\qquad F_3 = 6.71869$ kN

It is verified that the total applied load $\left(5\frac{kN}{m}\times 5m = 45kN\right)$ is equal to the sum of the support reaction forces (9.875+28.406+6.71869 = 45 kN).

Total force and moment diagram



Individual force and moment diagrams



Individual force and moment calculations

Element 1

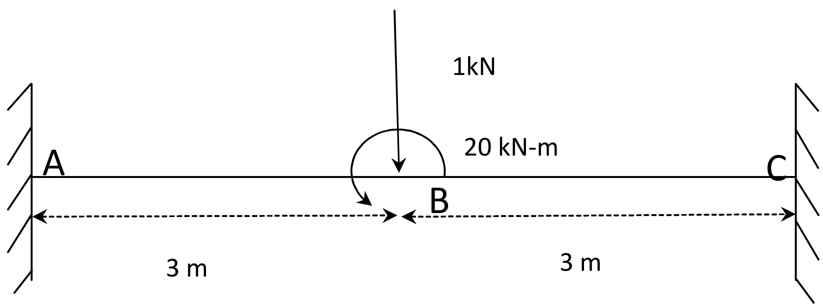
$$F_1^{(1)} = \left\{10,080\times\left(-3.59623\times10^{-4}\right)\right\} + \left\{10,080\times 9.92\times10^{-5}\right\} = -2.624832 \text{ kN}$$
$$M_1^{(1)} = \left\{33,600\times\left(-3.59623\times10^{-4}\right)\right\} + \left\{16,800\times 9.92\times10^{-5}\right\} = -10.416 \text{ kN-m}$$
$$F_2^{(1)} = \left\{-10,080\times\left(-3.59623\times10^{-4}\right)\right\} - \left\{10,080\times 9.92\times10^{-5}\right\} = 2.624832 \text{ kN}$$
$$M_2^{(1)} = \left\{16,800\times\left(-3.59623\times10^{-4}\right)\right\} + \left\{33,600\times 9.92\times10^{-5}\right\} = -2.70816 \text{ kN-m}$$

Element 2

$$F_1^{(1)} = \left\{15,750\times 9.92\times10^{-5}\right\} + \left\{15,750\times 1.0913\times10^{-4}\right\} = 3.2811975 \text{ kN}$$
$$M_1^{(1)} = \left\{42,000\times 9.92\times10^{-5}\right\} + \left\{21,000\times 1.0913\times10^{-4}\right\} = 6.45813 \text{ kN-m}$$
$$F_2^{(1)} = \left\{-15,750\times 9.92\times10^{-5}\right\} - \left\{15,750\times 1.0913\times10^{-4}\right\} = -3.2811975 \text{ kN}$$
$$M_2^{(1)} = \left\{21,000\times 9.92\times10^{-5}\right\} + \left\{42,000\times 1.0913\times10^{-4}\right\} = 6.66667 \text{ kN-m}$$

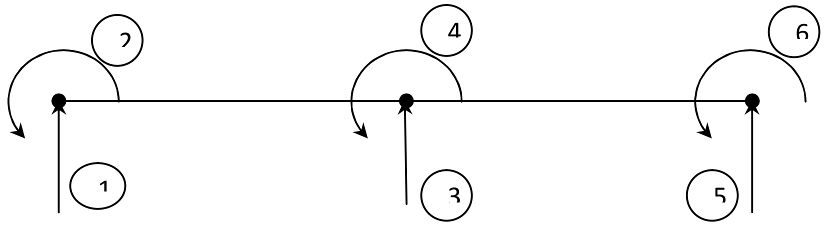
PROBLEM

Given that E=210 GPa and I=4×10^{-4} m^4, cross section of the beam is constant. Determine the deflection and slope at point C. calculate the reaction forces and moments. DARYL LOGAN P 171-172

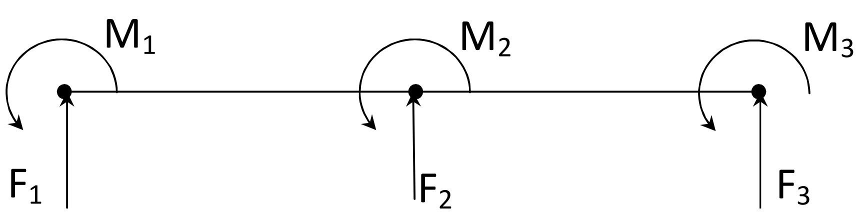


Solution:-

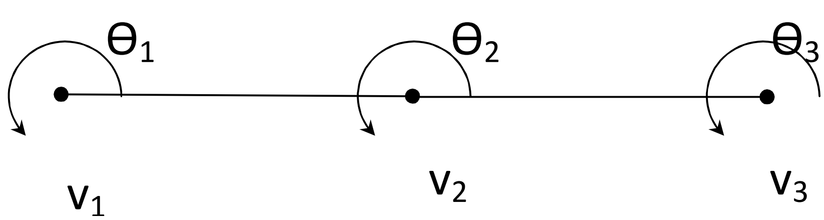
Degree of freedom in numbers:-



Degree of freedom of forces and moments:-



Degree of freedom of displacement and rotation:-



Stiffness matrix for element 1 and 2:-

$$[K]^1 = [K]^2 = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

$$[K]^1 = [K]^2 = 3.1\times10^6 \begin{bmatrix} 12 & 18 & -12 & 18 \\ 18 & 36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix}$$

Assembling:-

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = 3.1\times10^6 \begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & 0 & 0 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 12 & -18 \\ 0 & 0 & 18 & 18 & -18 & 36 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Boundary condition:-

F_2=-10 kN; M_2=20 kN-m $\quad$ $v_1 = v_3 = \Theta_1 = \Theta_3 = 0$

Therefore first, second, fifth, sixth columns are ineffective

and hence the reduced matrix is given by

$$\begin{Bmatrix} F_2 \\ m_2 \end{Bmatrix} = 3.1\times10^6 \begin{bmatrix} 24 & 0 \\ 0 & 72 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

Deflection and slope at point c:-

$V_2 = -1.34\times10^{-4}$ m = -0.134 mm

$\Theta_2 = 8.96\times10^{-5}$ rad

Reaction forces and moments:-

$$\begin{Bmatrix} F_2 \\ m_2 \\ F_3 \\ m_3 \end{Bmatrix} = 3.1\times10^6 \begin{bmatrix} -12 & 18 \\ -18 & 18 \\ -12 & -18 \\ 18 & 18 \end{bmatrix}$$

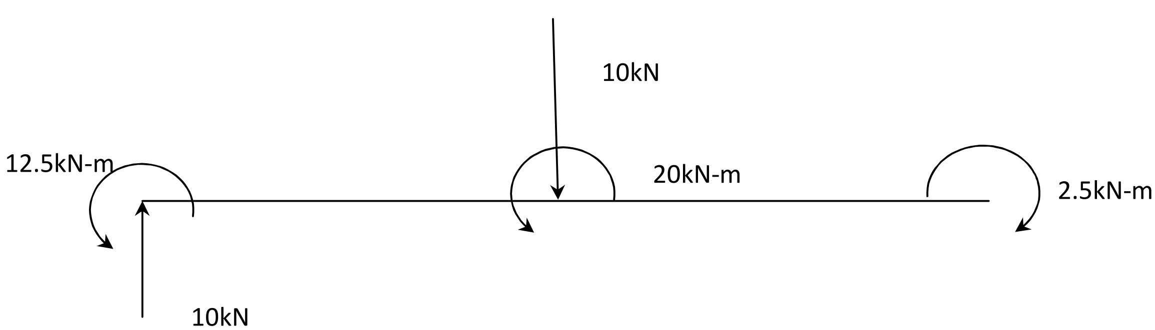
F_1=10000N

M_1=12500N-m

F_3=0

M_3= -2500N-m

individual element forces and moments are

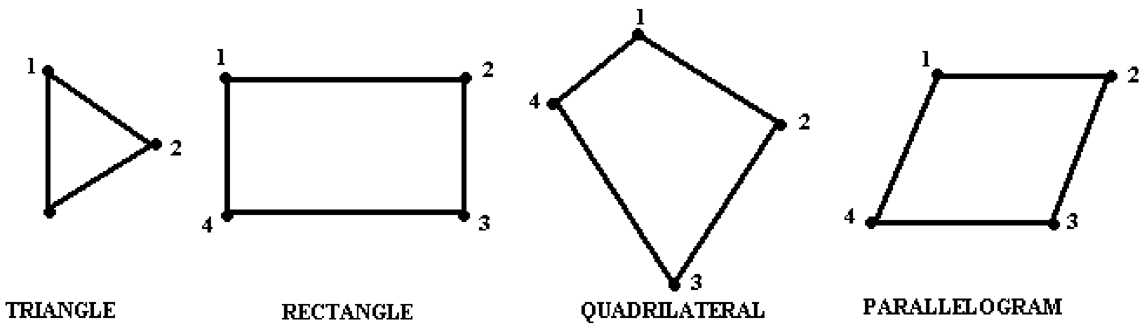


UNIT III

TWO DIMENSIONAL FINITE ELEMENT ANALYSIS

3.1 INTRODUCTION

Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element.



- **Plane Stress and Plane Strain**

The 2d element is extremely important for the Plane Stress analysis and Plane Strain analysis.

Plane Stress Analysis:

It is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

Plane Strain Analysis:

It is defined to be a state of strain in which the normal to the xy plane and the shear strain are assumed to be zero.

3.2 THREE NODED LINEAR TRIANGULAR ELEMENT

The physical domain considered is geometrically a 2-Dimensional domain, i.e., an area with uniform thickness and the single variable can be one of pressure, temperature, etc. (a scalar quantity, not a vector quantity). An example is the temperature distribution in a plate. At each point there can be only one temperature. We consider such an area meshed with triangular elements. Each triangular element has three nodes, (i.e., one node at each corner). Let us consider one such element with coordinates $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). The single variable (for example, temperature) at these nodes 1, 2 and 3 are u_1, u_2 and u_3, respectively. If so, then the unknown single variable u (temperature) at any non-nodal point (x, y) in the 2-D domain can be expressed in terms of the known nodal variables (temperatures) u_1, u_2 and u_3.

Let us assume that the single variable can be expressed as

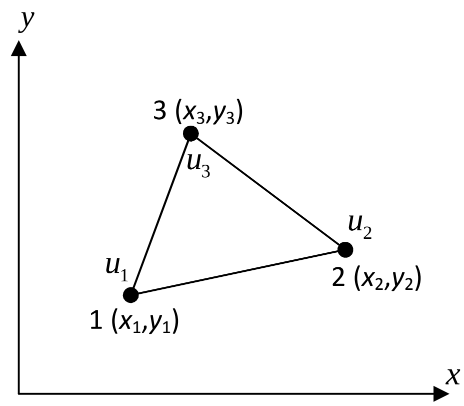
$$u = c_1 + c_2 x + c_3 y$$

In order to find the three unknowns c_1, c_2 and c_3, we apply the boundary conditions

at (x_1, y_1), $u = u_1 \quad \Rightarrow \quad u = c_1 + c_2 x_1 + c_3 y_1$

at (x_2, y_2), $u = u_2 \quad \Rightarrow \quad u = c_1 + c_2 x_2 + c_3 y_2$

at (x_3, y_3), $u = u_3 \quad \Rightarrow \quad u = c_1 + c_2 x_3 + c_3 y_3$



Writing the above three equations in matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

We need to find c_1, c_2 and c_3

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} = \frac{1}{2A} \begin{bmatrix} \sim_1 & \sim_2 & \sim_3 \\ {}_1 & {}_2 & {}_3 \\ \text{¥}_1 & \text{¥}_2 & \text{¥}_3 \end{bmatrix} \quad \text{where} \quad 2A = \sim_1 + \sim_2 + \sim_3 \quad \text{and}$$

$$\sim_i = x_j y_k - x_k y_j \qquad {}_i = y_j - y_k \qquad \text{¥}_i = -(x_j - x_k)$$

$$\sim_1 = x_2 y_3 - x_3 y_2 \qquad {}_1 = y_2 - y_3 \qquad \text{¥}_1 = -(x_2 - x_3)$$

$$\sim_2 = x_3 y_1 - x_1 y_3 \qquad {}_2 = y_3 - y_1 \qquad \text{¥}_2 = -(x_3 - x_1)$$

$$\sim_3 = x_1 y_2 - x_2 y_1 \qquad {}_3 = y_1 - y_2 \qquad \text{¥}_3 = -(x_1 - x_2)$$

A is the area of the triangle.

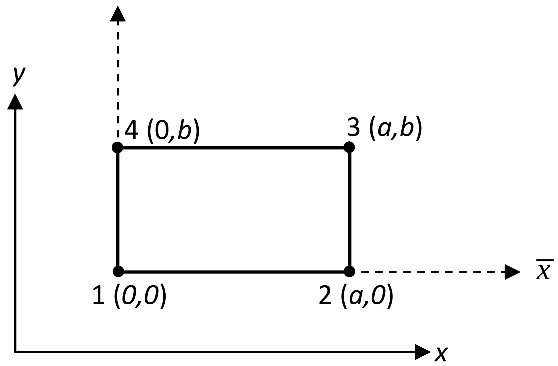
$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \sim_1 & \sim_2 & \sim_3 \\ {}_1 & {}_2 & {}_3 \\ \text{¥}_1 & \text{¥}_2 & \text{¥}_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Substituting the values of c_1, c_2 and c_3 in $u = c_1 + c_2 x + c_3 y$, we get

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$\text{where } N_i = \frac{1}{2A}\left(\sim_i + \ {}_i x + \text{¥}_i y\right), \qquad i = 1, 2, 3$$

3.3 FOUR NODED LINEAR RECTANGULAR ELEMENT



Let us assume that the single variable can be expressed as

$$u(\bar{x},\bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\,\bar{y} \qquad \dots\dots\dots\dots\dots\dots(1)$$

This polynomial contains four linearly independent terms and is linear in *x* and *y*, with a bilinear term in *x* and *y*. The polynomial requires an element with four nodes. There are two possible geometric shapes: a triangle with the fourth node at the centroid of the triangle or a rectangle with nodes at the vertices.

A triangle with a fourth node at the center does not provide a single-valued variation of *u* at inter-element boundaries, resulting in incompatible variation of *u* at inter-element boundaries and is therefore not admissible.

The linear rectangular element is a compatible element because on any side, the single variable u varies only linearly and there are two nodes to uniquely define it.

Here we consider an approximation of the form given in eqauation (1) and use a rectangular element with sides *a* and *b*. For the sake of convenience we choose a local coordinate system $(\bar{x},\bar{y})$ to derive the interpolation functions.

In order to find the three unknowns c_1, c_2 and c_3, we apply the boundary conditions

at $(0,0)$, $u = u_1 \quad \Rightarrow\; u = c_1$

at $(a,0)$, $u = u_2 \quad \Rightarrow\; u = c_1 + c_2 a$

at (a,b), $u = u_3 \quad \Rightarrow\; u = c_1 + c_2 a + c_3 ab$

at $(0,b)$, $u = u_3 \quad \Rightarrow\; u = c_1 + c_4 b$

Solving for c_1, c_2, c_3 and c_4

$\bar{y}$

$$c_1 = u_1, \qquad c_1 = \frac{u_2 - u_1}{a}, \qquad c_3 = \frac{u_4 - u_1}{b}, \qquad c_4 = \frac{u_1 - u_2 + u_3 - u_4}{ab},$$

3.4 TWO-VARIABLE 3-NODED LINEAR TRIANGULAR ELEMENT

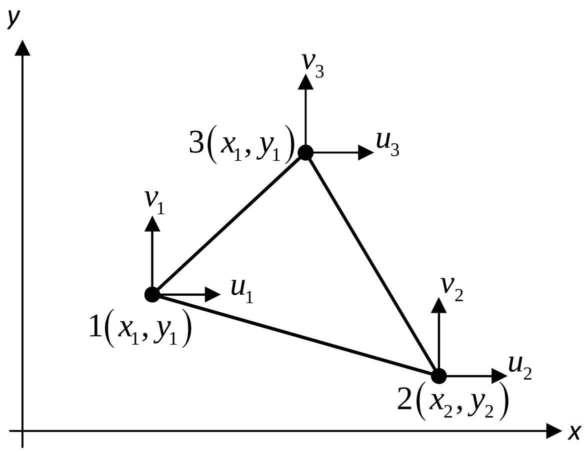


Figure shows a 2-D two-variable linear triangular element with three nodes and the two dof at each node. The nodes are placed at the corners of the triangle. The two variables (dof) are displacement in *x*-direction (*u*) and displacement in *y*-direction (*v*). Since each node has two dof, a single element has 6 dof. The nodal displacement vector is given by

$$\{U\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

We select a linear displacement function for each dof as

$$u(x, y) = c_1 + c_2 x + c_3 y$$
$$v(x, y) = c_4 + c_5 x + c_6 y$$

where $u(x, y)$ and $v(x, y)$ describe displacements at any interior point (x, y) of the element.

The above two algebraic equations can also be written as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}$$

Using steps we had developed for the 2-D single-variable linear triangular element, we can write

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \sim_1 & \sim_2 & \sim_3 \\ {}_1 & {}_2 & {}_3 \\ ¥_1 & ¥_2 & ¥_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \sim_1 & \sim_2 & \sim_3 \\ {}_1 & {}_2 & {}_3 \\ ¥_1 & ¥_2 & ¥_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

and using the interpolation functions we had developed for the 2-D single-variable linear triangular element, we can write

$$u(x, y) = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v(x, y) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

where

$$N_i = \frac{1}{2A}\left(\sim_i + {}_i x + ¥_i y\right), \qquad i = 1, 2, 3$$

Writing the above equations in matrix form $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$ The

$$\{U\} = [N]\{a\}$$

strains associated with the two-dimensional element are given by

$$\{£\} = \begin{Bmatrix} £_x \\ £_y \\ ¥_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

and note that

$$\frac{\partial N_i}{\partial x} = \quad {}_i \qquad \frac{\partial N_i}{\partial y} = ¥_i$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(N_1u_1 + N_2u_2 + N_3u_3\right) = \left\langle \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \left\langle \frac{\partial N_1}{\partial x} \quad 0 \quad \frac{\partial N_2}{\partial x} \quad 0 \quad \frac{\partial N_3}{\partial x} \quad 0 \right\rangle \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(N_1v_1 + N_2v_2 + N_3v_3\right) = \left\langle \frac{\partial N_1}{\partial y} \quad \frac{\partial N_2}{\partial y} \quad \frac{\partial N_3}{\partial y} \right\rangle \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \left\langle 0 \quad \frac{\partial N_1}{\partial y} \quad 0 \quad \frac{\partial N_2}{\partial y} \quad 0 \quad \frac{\partial N_3}{\partial y} \right\rangle \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(N_1u_1 + N_2u_2 + N_3u_3\right) + \frac{\partial}{\partial x}\left(N_1v_1 + N_2v_2 + N_3v_3\right) = \left\langle \frac{\partial N_1}{\partial y} \quad \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial y} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial y} \quad \frac{\partial N_3}{\partial x} \right\rangle \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{£\} = \begin{Bmatrix} £_x \\ £_y \\ ¥_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} {}_1 & 0 & {}_2 & 0 & {}_3 & 0 \\ 0 & ¥_1 & 0 & ¥_2 & 0 & ¥_3 \\ ¥_1 & {}_1 & ¥_2 & {}_2 & ¥_3 & {}_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{£\}_{3\times1} = [B]_{3\times6}\{a\}_{6\times1}$$

$$\{\pm\} = [D]\{\pounds\}$$

$$\begin{Bmatrix} \pm_x \\ \pm_y \\ 2_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \pounds_x \\ \pounds_y \\ \yen_{xy} \end{Bmatrix}$$

$$\{\pm\} = [D][B]\{a\}$$

The stiffness matrix is given by $[K]_{6\times6} = \int\limits_{Volume} [B]^T_{6\times3} [D]_{3\times3} [B]_{3\times6}\, dV = t \iint\limits_{Area} [B]^T [D][B]\, dxdy$.

where t is the thickness of the plate. The integrand $[B]^T [D][B]$ is not a function of x and y and hence can be taken outside the integral to yield

$$[K] = tA[B]^T [D][B]$$

D matrix is the material constitutive matrix, either for the plane-stress case or for the plane-strain case depending on the problem in hand.

and substituting them back in u, we get

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

where $N_1 = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right)$

$$N_2 = \frac{\bar{x}}{a}\left(1 - \frac{\bar{y}}{b}\right)$$

$$N_3 = \frac{\bar{x}}{a}\frac{\bar{y}}{b}$$

$$N_4 = \left(1 - \frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$

3.5 STRAIN – STRESS RELATION

$$\pounds_x = \frac{\pm_x}{E} - \neg\frac{\pm_y}{E} - \neg\frac{\pm_z}{E}$$

$$\pounds_y = -\neg\frac{\pm_x}{E} + \frac{\pm_y}{E} - \neg\frac{\pm_z}{E}$$

$$\pounds_z = -\neg\frac{\pm_x}{E} - \neg\frac{\pm_y}{E} + \frac{\pm_z}{E}$$

3.5.1 Plane stress conditions

σ_x, σ_y and τ_{xy} are present.

$\sigma_z = \tau_{xz} = \tau_{yz} = 0.$

since $\sigma_z = 0$, from equations 1 and 2

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

solving the above two equations

for σ_x and σ_y, we get

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) \text{ and}$$

$$\sigma_y = \frac{E}{(1-\nu^2)}(\nu\varepsilon_x + \varepsilon_y)$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$= \frac{(1-\nu^2)}{(1-\nu^2)}\frac{E}{2(1+\nu)}\gamma_{xy}$$

$$= \frac{(1+\nu)(1-\nu)}{(1-\nu^2)}\frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1-\nu^2)}\frac{(1-\nu)}{2}\gamma_{xy}$$

writing σ_x, σ_y and τ_{xy} in a matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

3.5.2 Plane strain conditions

σ_x, σ_y and τ_{xy} are present.

$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0.$

σ_z is not zero.

since $\varepsilon_z = 0$, we get from equation 3

$$\sigma_z = \nu(\sigma_x + \sigma_x)$$

substituting $\pm_z$ in equations 1 and 2

$$\pounds_x = \frac{\pm_x}{E} - \neg\frac{\pm_y}{E} - \neg^2\frac{(\pm_x+\pm_y)}{E}$$

$$\pounds_y = -\neg\frac{\pm_x}{E} + \frac{\pm_y}{E} - \neg^2\frac{(\pm_x+\pm_y)}{E}$$

rearranging the terms we get

$$\pounds_x = \frac{\pm_x}{E}(1-\neg^2) - \frac{\pm_y}{E}\neg(1+\neg)$$

$$\pounds_y = -\frac{\pm_x}{E}\neg(1+\neg) + \frac{\pm_y}{E}(1-\neg^2)$$

mutiplying by X by $\neg$ and Y by $(1-\neg)$

$$\neg\pounds_x = \frac{\pm_x}{E}\neg(1-\neg^2) - \frac{\pm_y}{E}\neg^2(1+\neg)$$

$$(1-\neg)\pounds_y = -\frac{\pm_x}{E}\neg(1-\neg)(1+\neg) + \frac{\pm_y}{E}(1-\neg)(1-\neg^2)$$

$$= -\frac{\pm_x}{E}\neg(1-\neg^2) + \frac{\pm_y}{E}(1-\neg)(1-\neg^2)$$

adding the above two equations to eliminate $\pm_x$

$$\neg\pounds_x + (1-\neg)\pounds_y = \frac{\pm_y}{E}\left[-\neg^2(1+\neg) + (1-\neg)(1-\neg^2)\right]$$

$$\neg\pounds_x + (1-\neg)\pounds_y = \frac{\pm_y}{E}\left[-\neg^2(1+\neg) + (1-\neg)(1+\neg)(1-\neg)\right]$$

$$\neg\pounds_x + (1-\neg)\pounds_y = \frac{\pm_y}{E}(1+\neg)\left[-\neg^2 + (1-\neg)(1-\neg)\right]$$

$$\frac{\neg\pounds_x + (1-\neg)\pounds_y}{(1+\neg)} = \frac{\pm_y}{E}\left[-\neg^2 + 1 + \neg^2 - 2\neg\right]$$

$$\pm_y = E\frac{\neg\pounds_x + (1-\neg)\pounds_y}{(1+\neg)(1-2\neg)}$$

similarly

$$\pm_x = E\frac{(1-\neg)\pounds_x + \neg\pounds_y}{(1+\neg)(1-2\neg)}$$

andasbefore

$${}^{2}{}_{xy} = \frac{E}{2(1+\neg)}\yen_{xy}$$

writing $\pm_x, \pm_y$ and ${}^{2}{}_{xy}$ in matrix form

$$\begin{Bmatrix} \pm_x \\ \pm_y \\ 2_{xy} \end{Bmatrix} = \frac{E}{(1+¬)(1-2¬)} \begin{bmatrix} (1-¬) & ¬ & 0 \\ ¬ & (1-¬) & 0 \\ 0 & 0 & \frac{(1-2¬)}{2} \end{bmatrix} \begin{Bmatrix} £_x \\ £_x \\ ¥_{xy} \end{Bmatrix}$$

It is difficult to represent the curved boundaries by straight edges element a large number of element may be used to obtain reasonable resembalance between original body and the assemblage

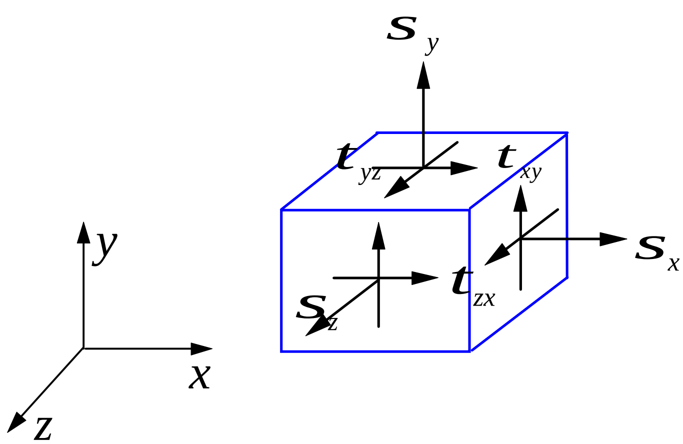
Two-Dimensional Problems

Review of the Basic Theory

In general, the stresses and strains in a structure consist of six components:

$s_x , s_y , s_z , t_{xy} , t_{yz} , t_{zx}$ for stresses,

and $e_x , e_y , e_z , g_{xy}, g_{yz} , g_{zx},$ for strains.



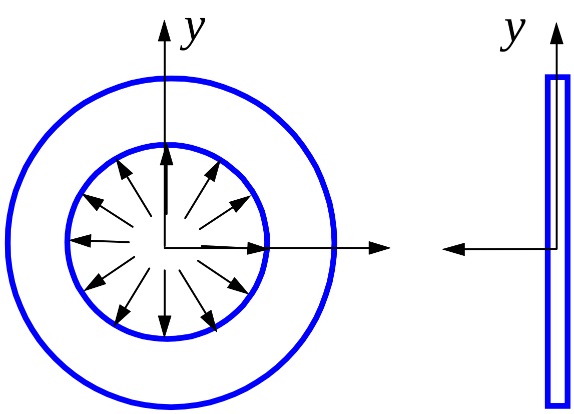
Under contain conditions, the state of stresses and strains can be simplified. A general 3-D structure analysis can, therefore, be reduced to a 2-D analysis.

t_{zx} 0 (e_z 0) (1)

Plane (2-D) Problems

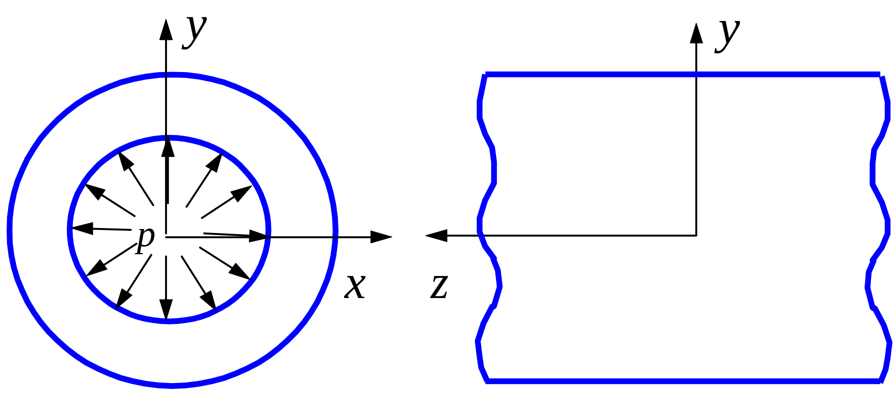
Plane stress:

s_z t_{yz}



Plane strain:

A long structure with a uniform cross section and transverse loading along its length (*z*-direction).



Stress-Strain-Temperature (Constitutive) Relations

For elastic and isotropic materials, we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & \nu/E & 0 \\ \nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_0$$

where ε_0 is the initial strain, E the Young's modulus, ν the Poisson's ratio and G the shear modulus. Note that, $G = \frac{E}{2(1-\nu)}$

which means that there are only two independent materials constants for homogeneous and isotropic materials.

We can also express stresses in terms of strains by solving the above equation,

The above relations are valid for plane stress case. For plane strain case, we need to replace the material constants in the above equations in the following fashion,

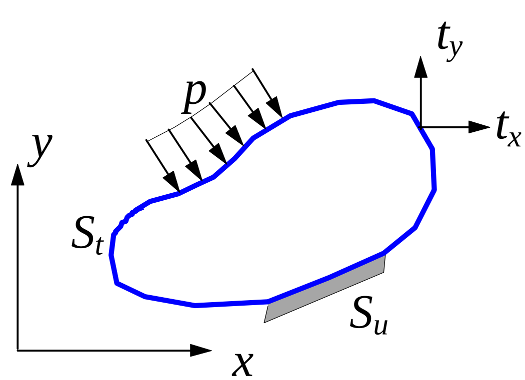
$$n \quad \frac{n}{1 - n}$$

For example, the stress is related to strain by

Initial strains due to *temperature change* (thermal loading) is given by,

where a is the coefficient of thermal expansion, T the change of temperature. Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

3.6 GENERALIZED COORDINATES APPROACH TO NODEL APPROXIMATIONS



The boundary S of the body can be divided into two parts, S_u and S_t. The boundary conditions (BC's) are described as, in which t_x and t_y are traction forces (stresses on the boundary) and the barred quantities are those with known values.

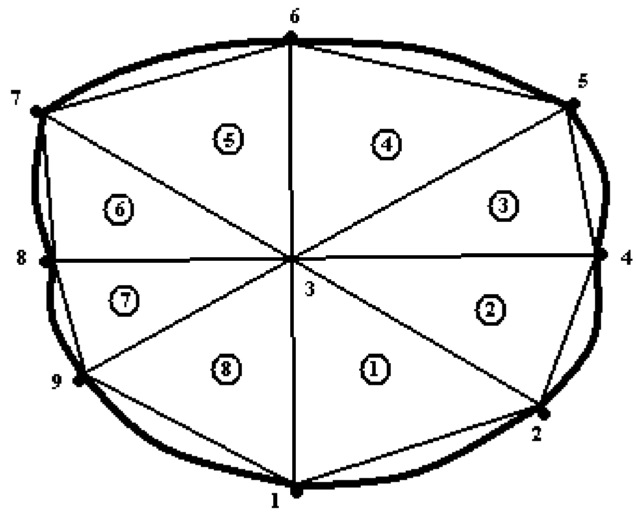
In FEM, all types of loads (distributed surface loads, body forces, concentrated forces and moments, etc.) are converted to point forces acting at the nodes.

Exact Elasticity Solution

The exact solution (displacements, strains and stresses) of a given problem must satisfy the equilibrium equations, the given boundary conditions and compatibility conditions (structures should deform in a continuous manner, no cracks and overlaps in the obtained displacement field)

3.7 ISOPARAMETRIC ELEMENTS

In one dimensional problem, each node is allowed to move only in ± x direction. But in two dimensional problem, each node is permitted to move in the two directions i.e., x and y.

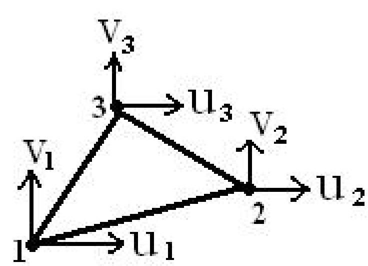


The element connectivity table for the above domain is explained as table.

Element (e)	Nodes
(1)	123
(2)	234
(3)	435
(4)	536
(5)	637
(6)	738
(7)	839
(8)	931

- **Constant Strain Triangular (CST) Element**

A three noded triangular element is known as constant strain triangular (CST) element. It has six unknown displacement degrees of freedom (u_1v_1, u_2v_2, u_3v_3).



- **Shape function for the CST element** Shape function $N_1 = (p_1 + q_1x + r_1y) / 2A$

Shape function $N_2 = (p_2 + q_2x + r_2y) / 2A$

Shape function $N_3 = (p_3 + q_3x + r_3y) / 2A$

- **Displacement function for the CST element**

Displacement function u = $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 \end{bmatrix} X \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \end{Bmatrix}$

- **Strain – Displacement matrix [B] for CST element**

Strain – Displacement matrix [B] = $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

Where, $q_1 = y_2 - y_3$ $\quad r_1 = x_3 - x_2$

$q_2 = y_3 - y_1$ $\quad r_2 = x_1 - x_3$

$q_3 = y_1 - y_2$ $\quad r_3 = x_2 - x_1$

- **Stress – Strain relationship matrix (or) Constitutive matrix [D] for two dimensional element**

$$E \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \end{bmatrix}$$

- **Stress – Strain relationship matrix for two dimensional plane stress problems**

  The normal stress σ_z and shear stresses τ_{xz}, τ_{yz} are zero.

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

- **Stress – Strain relationship matrix for two dimensional plane strain problems**

  Normal strain e_z and shear strains e_{xz}, e_{yz} are zero.

- **Stiffness matrix equation for two dimensional element (CST element)**

  Stiffness matrix $[k] = [B]^T [D] [B] A t$

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

For plane stress problems,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

For plane strain problems,

- **Temperature Effects**

Distribution of the change in temperature (ΔT) is known as strain. Due to the change in temperature can be considered as an initial strain e_0.

$$\sigma = D\,(Bu - e_0)$$

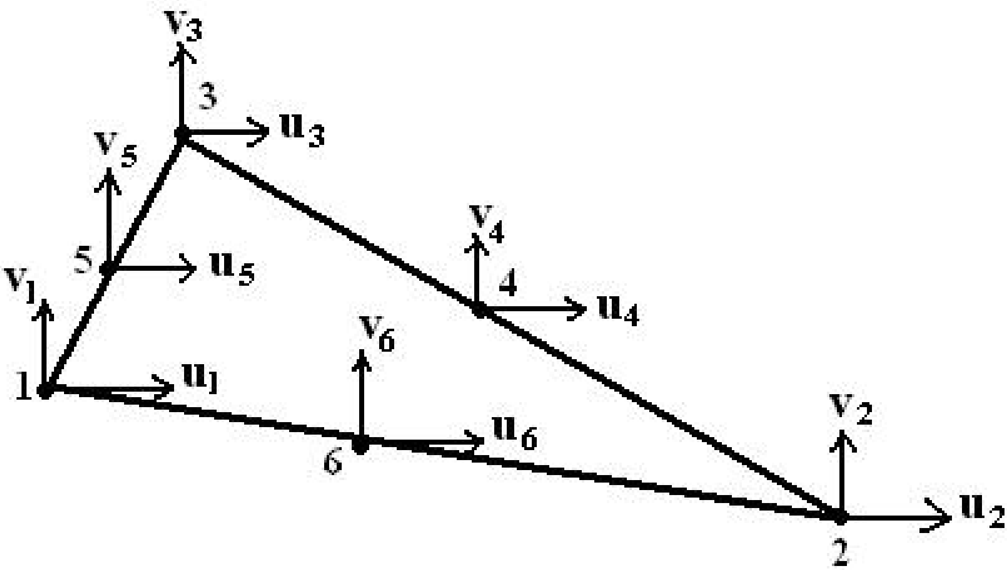
- **Galerkin Approach**

  Stiffness matrix $[K]_e = [B]^T [D][B] A t$.

  Force Vector $\{F\}_e = [K]_e \{u\}$

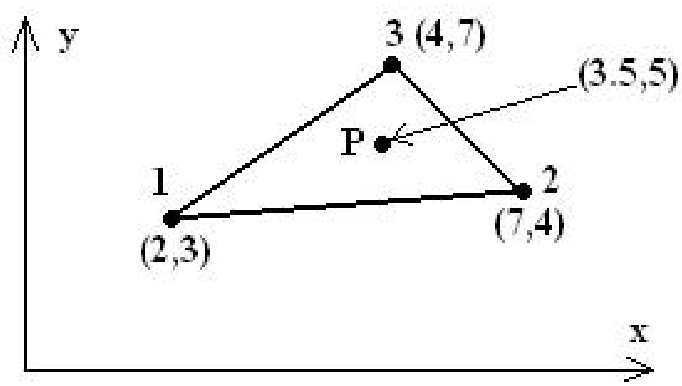
- **Linear Strain Triangular (LST) element**

A six noded triangular element is known as Linear Strain Triangular (LST) element. It has twelve unknown displacement degrees of freedom. The displacement functions of the element are quadratic instead of linear as in the CST.



- **Problem (I set)**

1. Determine the shape functions N_1, N_2 and N_3 at the interior point P for the triangular element for the given figure.

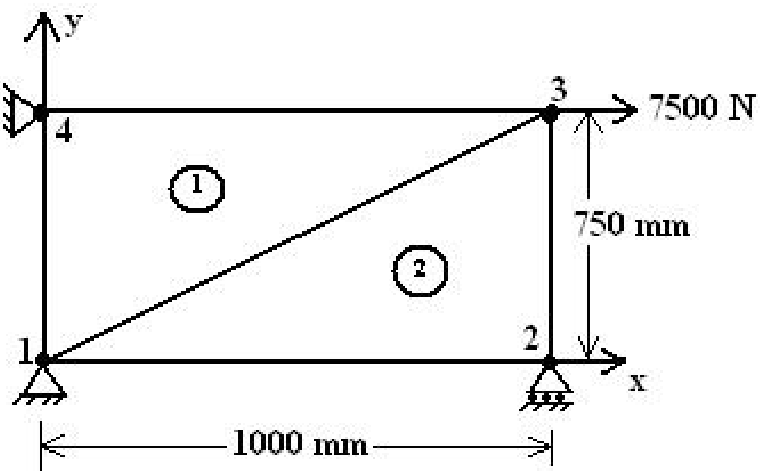


The two dimensional propped beam shown in figure. It is divided into two CST elements. Determine the nodal displacement and element stresses using plane stress conditions. Body force is neglected in comparison with the external forces.

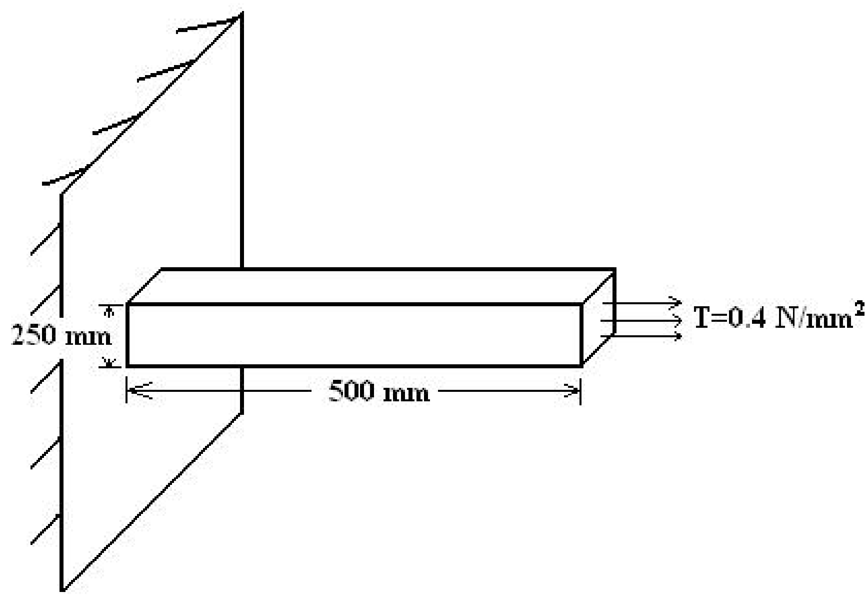
Take, Thickness (t) = 10mm,

Young's modulus (E) = 2×10^5 N/mm^2,

Poisson's ratio (v) = 0.25.



3. A thin plate is subjected to surface traction as in figure. Calculate the global stiffness matrix.



- **Scalar variable problems**

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stresses and strains are calculated for each element. In structural problems, the unknowns (displacements) are represented by the components of vector field. For example, in a two dimensional plate, the unknown quantity is the vector field u(x, y), where u is a (2x1) displacement vector.

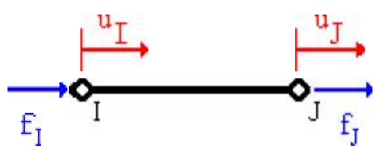
3.8 STRUCTURAL MECHANICS APPLICATIONS IN 2 DIMENSIONS

Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. Thest are four basic sets of elasticity equations they are

- Strain displacement relationship equations
- Stress strain relationship equations
- Equilibrium equations
- Compatibility equations

TRUSS ELEMENT

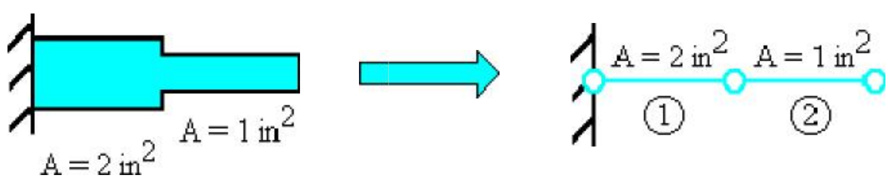
A truss element is defined as a deformable, two-force member that is subjected to loads in the axial direction. The loads can be tensile or compressive. The only degree of freedom for a one-dimensional truss (bar) element is axial (horizontal) displacement at each node.



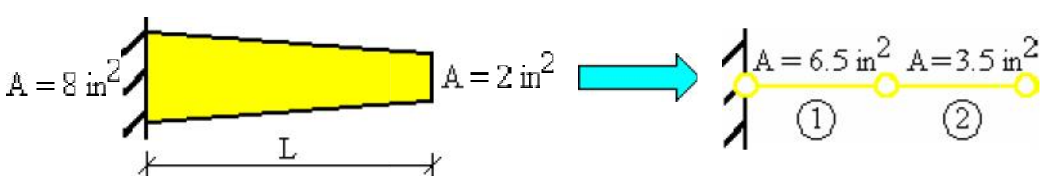
Assumptions for the One-Dimensional Truss Element

Prismatic Member

The truss element is assumed to have a constant cross-section, i.e., it is a prismatic member. If a truss structure is stepped, then it must be divided up into sections of constant cross-section in order to obtain an exact solution as shown below.



If a truss structure is tapered, then it can be approximated by using many small truss elements, each having the same cross-section as the middle of the tapered length it is approximating. The more sections that are used to approximate a tapered truss, the more accurate the solution will be.



Weightless Member

The weight (W) of the truss is neglected since it is assumed to be much less than the total resultant forces (F) acting on the truss. If the weight of the truss is not neglected, then its effects must be represented as vertical forces acting at the nodes. But since truss element is defined as two-force member it cannot have any vertical (shear) force, thus the member weight has to be neglected. If shear forces exist, then a beam element must be used to model the structure.

Nodal Forces

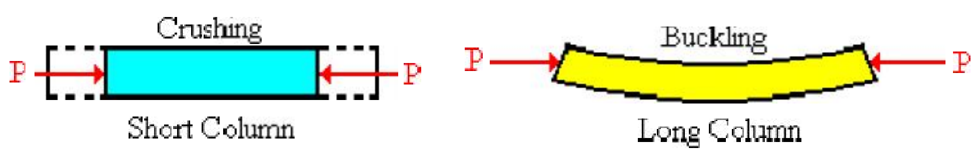
For one-dimensional truss element, forces (loads) can only be applied at the nodes of the element, but not between the nodes. This is consistent with the FEM equations which relate nodal forces to nodal displacements through the stiffness matrix.

Axially Loaded

For one-dimensional truss element, forces (loads) can only be applied at the centroid of the element cross-sectional area.

Buckling Effect not Considered

A bar element can be subjected to either tensile or compressive forces. Tensile forces can be applied to a bar of any cross-sectional area or member lengt*h*, and failure is associated with sudden fracture or general yielding. When compressive forces are applied to a member, it can either fail due to crushing or buckling. Buckling is present when the member bends and laterally deflects as shown on the right figure below.



Buckling is not accounted for in the formulation of the truss element. Members that do not buckle are classified as short columns and members that buckles are classified as long columns. The structural response of a short column can be predicted with a truss element.

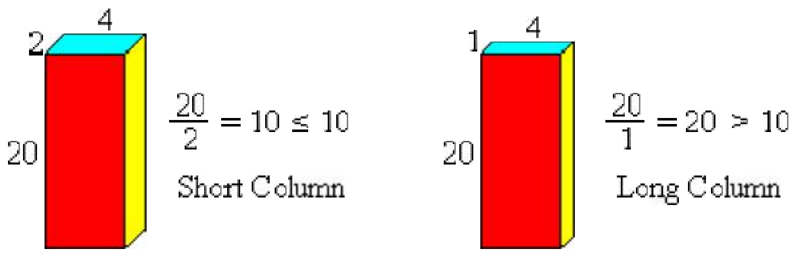
To determine if buckling will occur the reader should refer to a mechanics of material textbook. We will now introduce a simple geometric guideline to determine if buckling might occur. If the ratio between the member length and the least dimension of the cross-section is equal or less than 10, the member is considered a short column and buckling will not occur, i.e.,

$$\frac{L}{d} \leq 10$$

d - Least Cross-Sectional Dimension

L - Member Length

Two examples include



In the second case if a bar element is subjected to a compressive force, the element will not predict the buckling response. One should note that the above geometric rule is a simple guideline, however, in reality buckling depends not only on the member length and cross-sectional area, but material properties and support conditions.

Isotropic Material

A truss element has the same mechanical and physical properties in all directions, i.e., they are independent of direction. For instance, cutting out three tensile test specimens, one in the *x*-direction, one in the *y*-direction and the other oriented 45 degrees in the *x*-*y* plane, a tension test on each specimen, will result in the same mechanical values for the modulus of elasticity (E), yield strength τ_y and ultimate strength σ_u. Most metals are considered isotropic. In contrast fibrous materials, such as wood, typically have properties that are directionally dependant and are generally considered anisotropic (not isotropic).

Constant (Static) Load

The loads that are applied to the truss element are assumed to be static and not to vary over the time period being considered. This assumption is only valid if the rate of change of the force is much less than the applied force ($F >> dF/dt$), i.e., the loads are applied slowly. If the loads vary significantly, (if the variation in load is not much less than the applied force) then the problem must be considered as dynamic.

Poisson's Effect not Considered

Poisson's ratio is a material parameter. Poisson's effect is when a uniform cross-section bar is subject to a tensile load, and the axial stretching is accompanied by a contraction in the lateral dimension. For one-dimensional truss element., this effect is neglected for simplicity, i.e., $v = 0$.

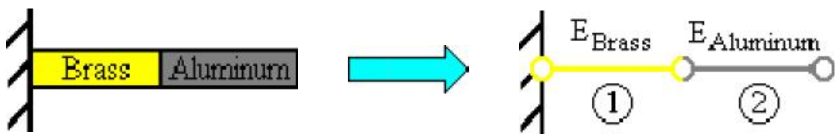
Cross Section Remains Plane

For one-dimensional element, although the force(s) are acting on only the centroid of the truss (bar) element, it is assumed that it has a uniform effect to the plane. Thus the cross section will move uniformly and remain plane and normal to the axial axis before and after loading.

Homogenous Material

A truss element has the same material composition throughout and therefore the same mechanical properties at every position in the material. Therefore, the modulus of elasticity E is constant throughout the truss element. A member in which the material properties varies from one point to the next in the member is called inhomogenous (non-homogenous). If a truss is composed of different types of materials, then it must be divide up into elements that are each of a single homogeneous material, otherwise the solution will not be exact.

The left figure shows a composite bar composed of brass and aluminum. This structure can be divided into two elements as shown on the right, one element for the brass with $E_1 = 15 \times 10^6$ psi and one for the aluminum with $E_2 = 10 \times 10^6$ psi.



TRUSS ELEMENT (OR SPAR ELEMENT OR LINK ELEMENT)

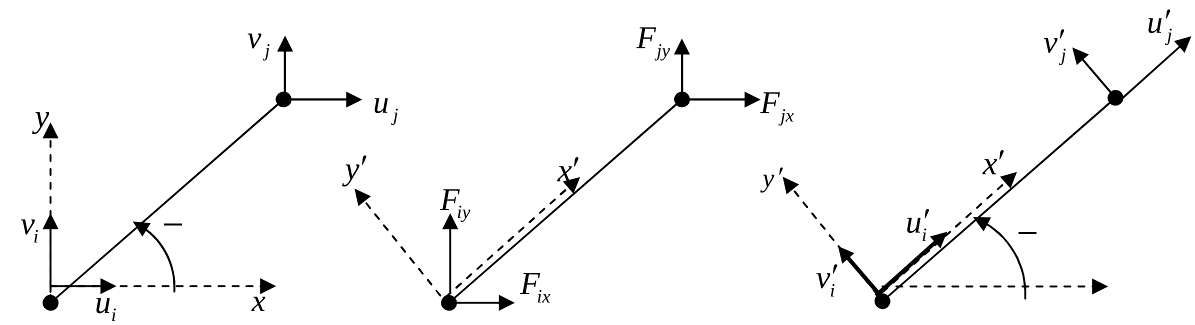
Differentiate between a truss and a frame.

Truss	Frame
Only concentrated loads act.	Concentrated loads, uniformly distributed loads, moments, all can act.
Loads act only at the joints.	Loads can be applied at the joints and/or in-between the joints
Truss members undergo only axial deformation (along the length of the member).	Frame members can undergo axial and bending deformations (translations as well as rotations).

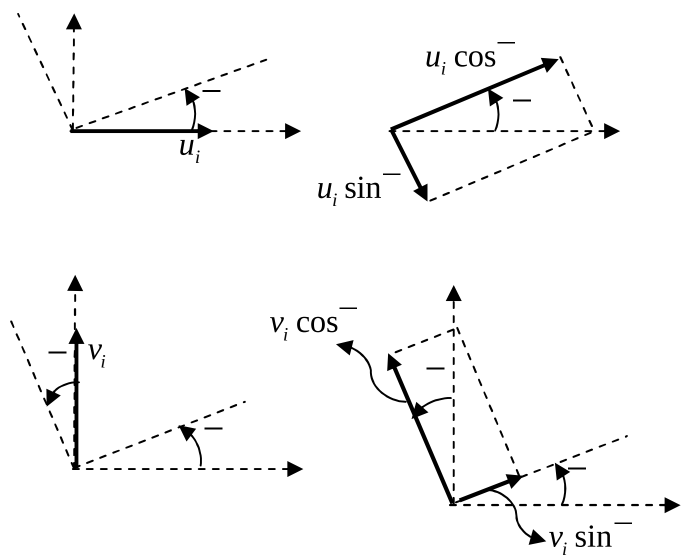
A **grid** is a structure on which loads applied perpendicular to the plane of the structure, as opposed to a plane frame, where loads are applied in the plane of the structure.

6.7.1 Derivation of stiffness matrix and finite element equation for a truss element.

There are two joints for an arbitrarily inclined single truss element (at an angle¯ , positive counter-clockwise from +ve *x*-axis). For each joint *i*, there are two degrees of freedom, i.e., a joint can have horizontal displacement (u_i) and vertical displacement (v_i). Hence, for a single truss element, there are 4 degrees of freedom. The nodal displacement degrees of freedom and the nodal force degrees of freedom are shown in the following figure.



Note that the deformations occurring in the truss members are so small that they are only axial. The axial displacement of the truss can be resolved along horizontal *x*-axis and vertical *y*-axis. But in our derivation, let us resolve the horizontal and vertical displacements (in *xy*-axes) of a joint along and perpendicular to the truss member (in $x'y'$-axes). Refer to the Figure in the next page. Note $u_i \sin\theta$ component acting towards negative y'-direction and all other components acting towards in +ve x'- and y'-directions.



$$u'_i = u_i \cos\theta + v_i \sin\theta$$
$$v'_i = -u_i \sin\theta + v_i \cos\theta$$

$$u'_j = u_j \cos\theta + v_j \sin\theta$$
$$v'_j = -u_j \sin\theta + v_j \cos\theta$$

The above equations can be written in the matrix form as follows

$$\begin{Bmatrix} u' \\ v'_i \\ u'_j \\ v'_j \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

$\{u'\} = [T]\{u\}$ where $[T]$ is the transformation matrix

It is important to note that the displacements v'_i and v'_j are both zero since there can be no displacements perpendicular to the length of the member. Also $[T]^{-1} = [T]^T$

Similarly, we resolve forces along the length of the member (positive x' direction) and perpendicular to the length of the member (positive y' direction)

$$\begin{Bmatrix} F'_{ix} \\ F'_{iy} \\ F'_{jx} \\ F'_{jy} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix}$$

$\{F'\} = [T]\{F\}$ where $[T]$ is the transformation matrix

The arbitrarily inclined truss member can be thought of as a simple bar element oriented at the same angle θ. Hence, we can write the finite element equation for this inclined bar element (in $x'y'$ coordinate system) as

$$\begin{Bmatrix} F'_{ix} \\ F'_{iy} \\ F'_{jx} \\ F'_{jy} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{Bmatrix}$$

$$\{F'\} = [k']\{u'\}$$

Substituting $\{F'\}$ and $\{u'\}$ from the previous equations, we can write

$$[T]\{F\} = [k'][T]\{u\}$$

Pre-multiplying the above equation by $[T]^{-1}$,

$$[T]^{-1}[T]\{F\} = [T]^{-1}[k'][T]\{u\}$$

But $[T]^{-1}[T] = 1$ and the above equation can be written as

$$\{F\} = [k]\{u\} \quad \text{where } [k] = [T]^{-1}[k'][T]$$

Carrying out the matrix multiplication for $[k]$, we obtain

$$\begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

where $c = \cos^{2}\theta$ and $s = \sin^{2}\theta$.

Computation of strain and stress in the truss element

The change in length of the truss member is equal to the change in axial displacement of the truss member in the $x'y'$ co-ordinate system

$$¢ = u'_j - u'_i$$

$$¢ = \left(u_j \cos¯ + v_j \sin¯\right) - \left(u_i \cos¯ + v_i \sin¯\right)$$

$$¢ = \langle -\cos¯ \quad -\sin¯ \quad \cos¯ \quad \sin¯ \rangle \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

Strain in the truss element is given by $£^e = \frac{¢}{L}$, i.e.,

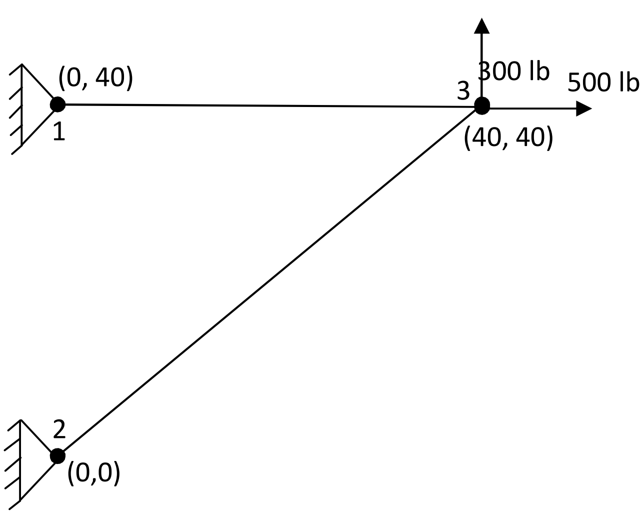
$$£^e = \frac{\langle -\cos¯ \quad -\sin¯ \quad \cos¯ \quad \sin¯ \rangle}{L} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

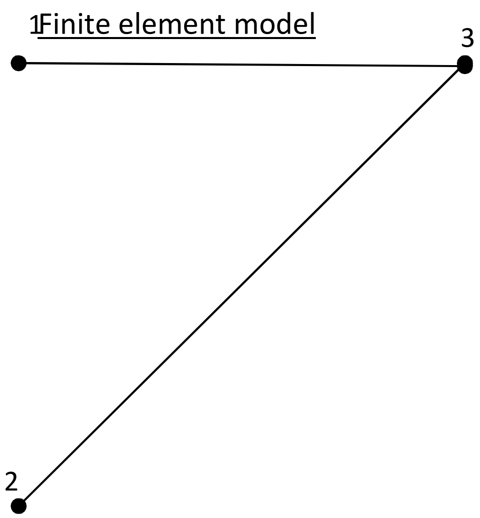
Stress in the truss element is given by $±^e = E£^e$, i.e.,

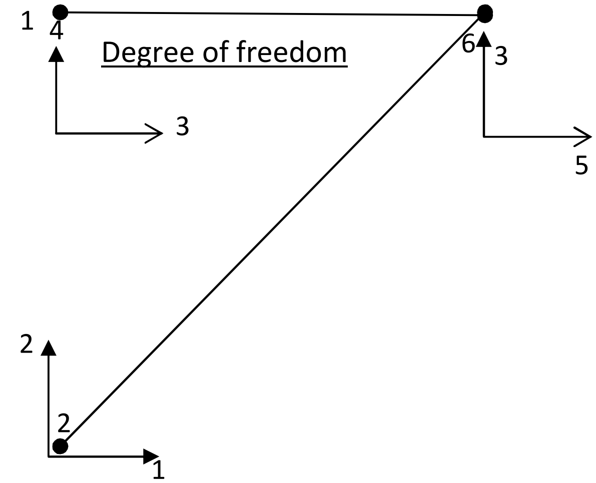
$$±^e = E \frac{\langle -\cos¯ \quad -\sin¯ \quad \cos¯ \quad \sin¯ \rangle}{L} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

Problem

The two-element truss is subjected to external loading as shown in figure. Using the same node and element numbering as shown in figure, determine the displacement components at node 3, the reaction components at nodes 1 and 2, and the element displacement, stresses and forces. The elements have modulus of elasticity $E_1 = E_2 = 10\times10^6$ $\frac{lb}{in^2}$ and cross-sectional areas $A_1 = A_2 = 1.5\ in^2$

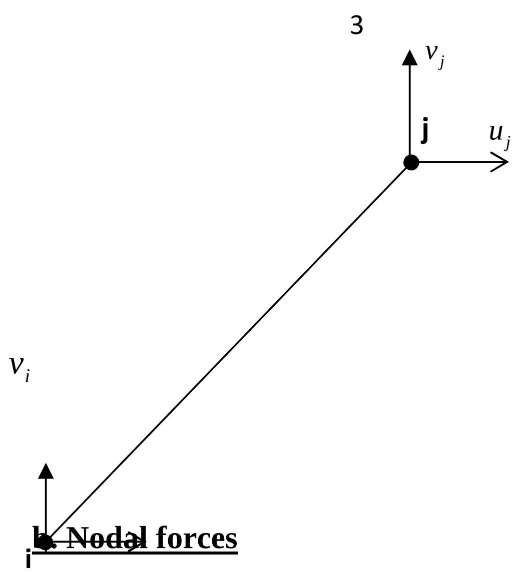






For element 1

For element 2



b. Nodal forces

element 1

element 2

FINITE ELEMENT EQUATION F_{ix}

$$\begin{pmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{pmatrix} = \frac{AE}{L} \begin{pmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & s^2 & cs & s^2 \end{pmatrix} \begin{pmatrix} u_i \\ v_i \\ u_j \\ v_j \end{pmatrix}$$

<u>For element 1</u>

$$- = 45, \qquad \frac{AE}{L} = \frac{1.5 \times 10 \times 10^6}{56.5685} = 2.65165 \times 10^5 \frac{lb}{in^2}$$

$$[K]^{(1)} = \frac{AE}{L} \begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \end{pmatrix} = 10^5 \begin{array}{c|cccc} & 1 & 2 & 5 & 6 \\ \hline 1 & 1.325826 & 1.325826 & -1.325826 & -1.325826 \\ 2 & 1.325826 & 1.325826 & -1.325826 & -1.325826 \\ 5 & -1.325826 & -1.325826 & 1.325826 & 1.325826 \\ 6 & -1.325826 & -1.325826 & 1.325826 & 1.325826 \end{array}$$

$$- = 0, \qquad \frac{AE}{L} = \frac{1.5 \times 10 \times 10^6}{40} = 3.75 \times 10^5$$

<u>For element 2</u>

$$[K]^{(2)} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 10^5 \begin{array}{c|cccc} & 3 & 4 & 5 & 6 \\ \hline 3 & 3.75 & 0 & -3.75 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 5 & -3.75 & 0 & 3.75 & 0 \\ 6 & 0 & 0 & 0 & 0 \end{array}$$

Assembly of finite element e

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{pmatrix} = 10^5 \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 \\ 2 & 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 \\ 3 & 0 & 0 & 3.75 & 0 & -3.75 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -1.325 & -1.325 & -3.75 & 0 & 5.0751 & 1.325 \\ 6 & -1.325 & -1.325 & 0 & 0 & 1.325 & 1.325 \end{array} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$ **quation**

Applying boundary conditions

$$F_{3x} = 500 \text{ lb}$$
$$F_{3y} = 300 \text{ lb}$$
$$u_1 = 0$$
$$u_2 = 0$$
$$v_1 = 0$$
$$v_2 = 0$$

knowns	unknowns
$F_{3x} = 500$ lb	F_{1x}
$F_{3y} = 300$ lb	F_{1y}
$u_1 = 0$	F_{2x}
$v_1 = 0$	F_{2y}
$u_2 = 0$	u_3
$v_2 = 0$	v_3

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ 500 \\ 300 \end{pmatrix} = 10^5 \begin{pmatrix} 1.325826 & 1.325826 & 0 & 0 & -1.325826 & -1.325826 \\ 1.325826 & 1.325826 & 0 & 0 & -1.325826 & -1.325826 \\ 0 & 0 & 3.75 & 0 & -3.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.325826 & -1.325826 & -3.75 & 0 & 5.0751826 & 1.325826 \\ -1.325826 & -1.325826 & 0 & 0 & 1.325826 & 1.325826 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{pmatrix}$$

Eliminating the 1st, 2nd, 3rd and 4th column to solve u_3 and v_3

$$500 = 10^5 (5.075u_3 + 1.325v_3) \longrightarrow$$

$$300 = 10^5 (1.325u_3 + 1.325v_3) \longrightarrow$$

Solve the equation and

$$\boxed{u_3 = 5.33 \times 10^{-4} \ in}$$

$$\boxed{v_3 = 1.731 \times 10^{-3} in}$$

$$F_{1x} = -300 \ lb$$
$$F_{1y} = -300 \ lb$$
$$F_{2x} = -200 \ lb$$
$$F_{2y} = 0$$

Finding element stresses

For element 1

$$\pm_1 = \frac{E}{L}[u_5 \cos 0^\circ - u_3 \sin 0^\circ] = \frac{10\times10^6}{40}\left[(0.5333\times1)-(0\times0)\right] = 133.325\frac{lb}{in^2}$$

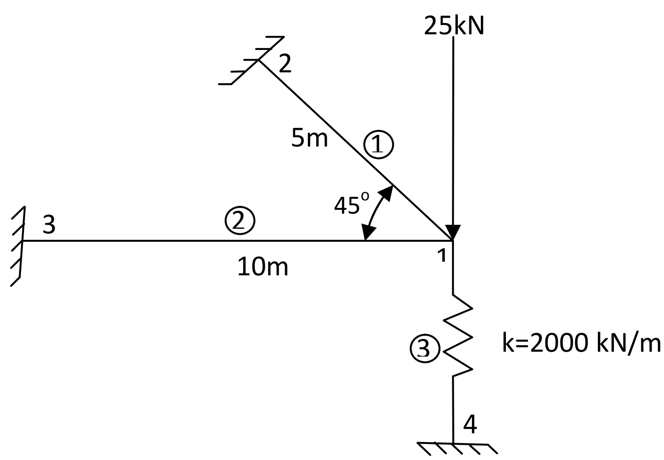
$$\pm_2 = \frac{E}{L}\left[(u_5 \cos 45^\circ - u_6 \sin 45^\circ)-(u_1 \cos 45^\circ + u_2 \sin 45^\circ)\right]$$

$$= \frac{10\times10^6}{56.57}\left[(0.5333\times10^{-3}\cos 45^\circ)+(1.731\times10^{-3}\sin 45^\circ)-0\right]$$

$$= \frac{10\times10^6}{56.57}\left[0.0003771+0.001224\right] = 283.03\frac{lb}{in^2}$$

PROBLEM

To illustrate how we can combine spring and bar element in one structure, we can solve the two-bar truss supported by a spring as shown below. Both bars have E = 210 GPa and A = 5.0 x10^{-4} m^2. Bar one has a length of 5 m and bar two a length of 10 m. the spring stiffness is k = 2000 kN/m.



Solution :

Given : E = 210 GPa

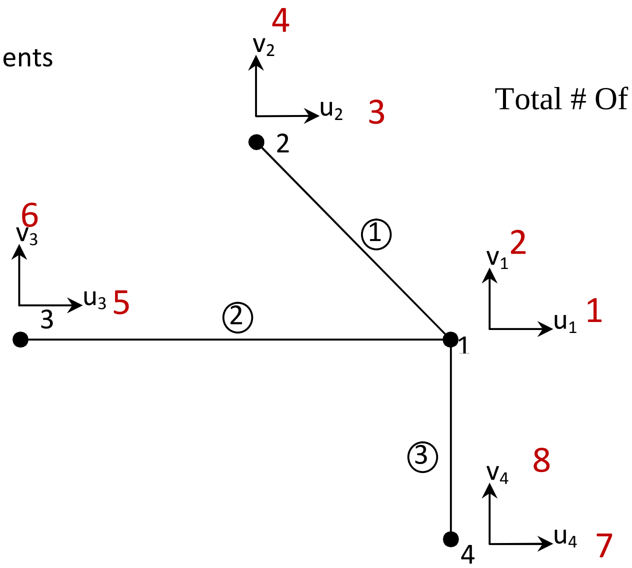
A = 5.0 x10^{-4} m^2 , L_1 = 5 m, L_2 = 10 m, K = 2 x $10^6 \frac{N}{m^2}$

NOTE: A spring is considered as a bar element whose stiffness is 2 x 10^6 $\frac{N}{m^2}$

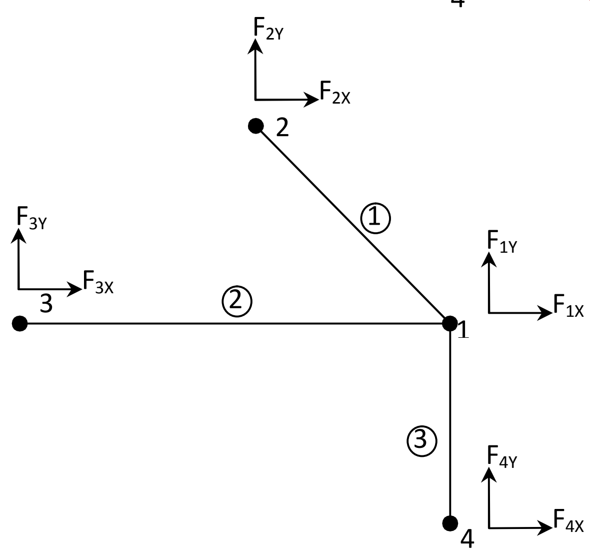
STEP 1 : Finite Element Representation Of Forces And Displacements

Displacements

Total # Of Degrees Of Freedom: 8



Forces



Step 2: Finite Element Equations

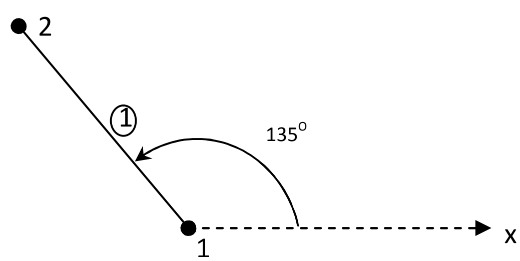
Element 1:

$\theta=135^0$

$l^2=\cos^2\theta=0.5$

$m^2=\sin^2\theta=0.5$

$lm=\cos\theta\,\sin\theta=-0.5$



$$[K]^{(1)}=\frac{\left(5\times10^{-4}\,\text{m}^2\right)\left(210\times10^{6}\,\text{kN/m}^2\right)}{5\text{ m}}\begin{array}{c} \\ 1\\2\\3\\4\end{array}\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}0.5 & -0.5 & -0.5 & 0.5\\ -0.5 & 0.5 & 0.5 & -0.5\\ -0.5 & 0.5 & 0.5 & -0.5\\ 0.5 & -0.5 & -0.5 & 0.5\end{bmatrix}\end{array}$$

$$[K]^{(1)}=105\times10^{5}\begin{array}{c} \\ 1\\2\\3\\4\end{array}\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\end{array}$$

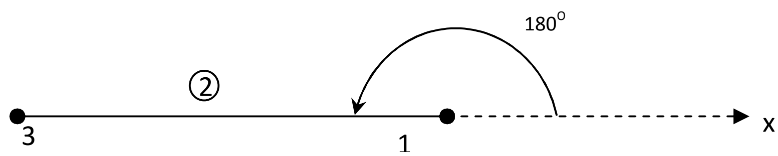
Element 2:

$\theta=180^0$

$l^2=\cos^2\theta=1$

$m^2=\sin^2\theta=0$

$lm=\cos\theta\sin\theta=0$



$$[K]^{(2)}=\frac{\left(5\times10^{-4}\,\text{m}^2\right)\left(210\times10^{6}\,\text{kN/m}^2\right)}{10\text{ m}}\begin{array}{c} \\ 1\\2\\3\\4\end{array}\begin{array}{c}\begin{array}{cccc}1 & 2 & 5 & 6\end{array}\\ \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

$$[K]^{(2)}=105\times10^{5}\begin{array}{c} \\ 1\\2\\5\\6\end{array}\begin{array}{c}\begin{array}{cccc}1 & 2 & 5 & 6\end{array}\\ \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

Element 3:

$\theta=270^0$

$l^2=\cos^2\theta=0$

$m^2=\sin^2\theta=1$

$lm=\cos\theta\sin\theta=0$

$$[K]^{(3)} = 2\times10^{6} \begin{matrix} & 1 & 2 & 7 & 8 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 \\ 7 & 0 & 0 & 0 & 0 \\ 8 & 0 & -1 & 0 & 1 \end{matrix}$$

STEP 3: Combination Of Finite Element Equations

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \\ F_{4X} \\ F_{4Y} \end{Bmatrix} = 10^5 \times \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 210 & -105 & -105 & 105 & -105 & 0 & 0 & 0 \\ 2 & -105 & 125 & 105 & -105 & 0 & 0 & 0 & -20 \\ 3 & -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ 4 & 105 & -108 & -105 & 105 & 0 & 0 & 0 & 0 \\ 5 & -105 & 0 & 0 & 0 & 105 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & -20 & 0 & 0 & 0 & 0 & 0 & 20 \end{matrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

STEP 4: Applying Boundary Conditions:

Since nodes 1, 2, and 3 are fixed, we have

$u_2 = v_2 = 0$; $u_3 = v_3 = 0$; $u_4 = v_4 = 0$;

$F_{1x} = 0$ and $F_{1y} = -25$ kN

$$\begin{Bmatrix} 0 \\ -25 \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \\ F_{4X} \\ F_{4Y} \end{Bmatrix} = 10^5 \times \begin{bmatrix} 210 & -105 & -105 & 105 & -105 & 0 & 0 & 0 \\ -105 & 125 & 105 & -105 & 0 & 0 & 0 & -20 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ 105 & -108 & -105 & 105 & 0 & 0 & 0 & 0 \\ -105 & 0 & 0 & 0 & 105 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Check whether there are as many unknowns as knowns.

STEP 5: SOLVING THE EQUATIONS:

Reduced matrix:

$$\begin{Bmatrix} 0 \\ -25 \end{Bmatrix} = 10^5 \times \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

On solving,

u_1=-1.724 x 10^{-3} m v_1=-3.4482 x 10^{-3} m

Find the reactions at supports by substituting the known nodal values

F_{2x} = -18.104 kN F_{2y} = 18.1041 kN

F_{3x} = 18.102 kN F_{3y} = 0

F_{4x} = 0 F_{4y} = 6.89 kN

STEP 6: Post Processing

Stress in element 1:

$$\sigma^{(1)} = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix} 10^{-3} \begin{Bmatrix} -1.724 \\ -3.4482 \\ 0 \\ 0 \end{Bmatrix}$$

$$\pm^{(1)} = 51.2 \text{ MPa (Tensile)}$$

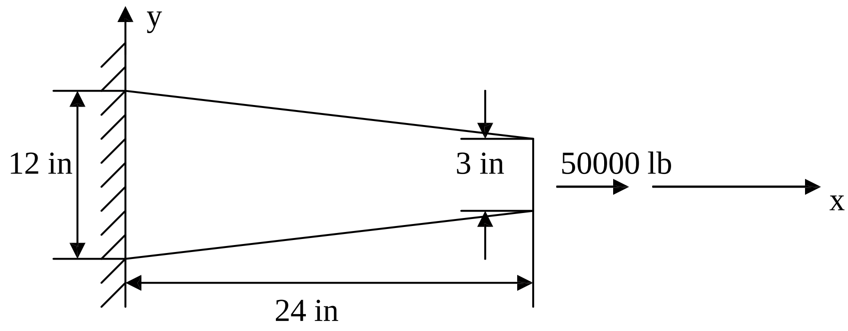
Stress in element 2:

$$\sigma^{(1)} = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix} 10^{-3} \begin{Bmatrix} -1.724 \\ -3.4482 \\ 0 \\ 0 \end{Bmatrix}$$

$$\pm^{(2)} = -36.2 \text{ MPa (Compressive)}$$

PROBLEM

A circular concrete beam structure is loaded as shown. Find the deflection of points at 8”,16”, and the end of the beam. E = 4 x 106 psi



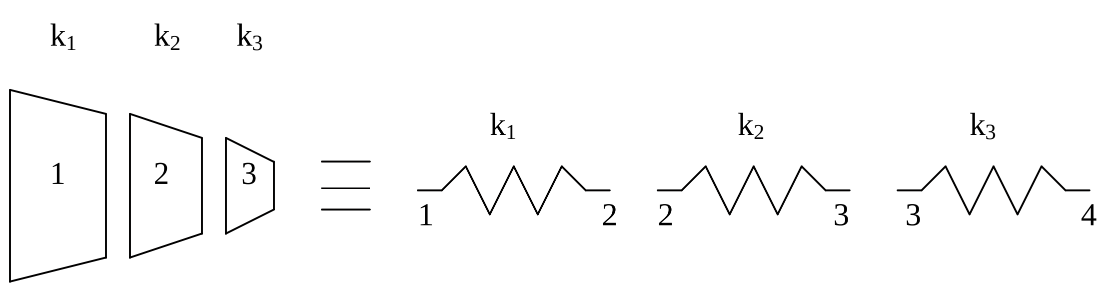
Solution

The beam structure looks very different from a spring. However, its behavior is very similar. Deflection occurs along the x-axis only. The only significant difference between the beam and a spring is that the beam has a variable cross-sectional area. An exact solution can be found if the beam is divided into an infinite number of elements, then, each element can be considered as a constant cross-section spring element, obeying the relation F = ku, where k is the stiffness constant of a beam element and is given by k = AE/L.

In order to keep size of the matrices small (for hand- calculations), let us divide the beam into only three elements. For engineering accuracy, the answer obtained will be in an acceptable range. If needed, accuracy can be improved by increasing the number of elements.

As mentioned earlier in this chapter, spring, truss, and beam elements are line-elements and the shape of the cross section of an element is irrelevant. Only the cross-sectional area is needed (also, moment of inertia for a beam element undergoing a bending load need to be defined). The beam elements and their computer models are shown

Here, the question of which cross-sectional area to be used for each beam section arises. A good approximation would be to take the diameter of the mid-section and use that to approximate the area of the element.



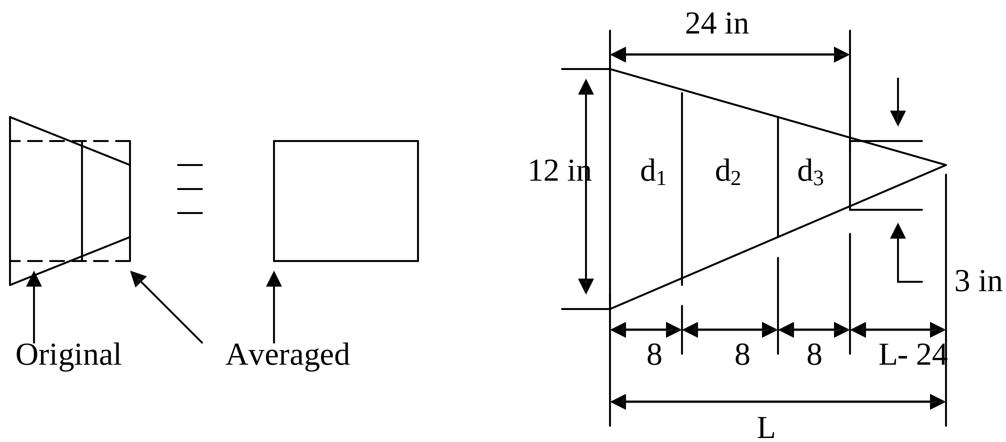
Cross-sectional area

The average diameters are: d_1 = 10.5 in., d_2 = 7.5 in., d_3 = 4.5. (diameters are taken at the mid sections and the values are found from the height and length ratio of the triangles shown in figure 2.10), which is given as

$12/L = 3/(L-24)$, $L = 32$

Average areas are:

A_1 = 86.59 in2 A_2 = 56.25 in2 A_3 = 15.9 in2



Stiffness

$k_1 = A_1 E/L_1 = (86.59)(4 \times 10^6/8) = 4.3295 \times 10^7$ lb./in., similarly,

$k_2 = A_2 E/L_2 = 2.8125 \times 10^7$ lb./in.

$k_3 = A_3 E/L_3 = 7.95 \times 10^6$ lb./in.

Element Stiffness Equations

$$[K^{(1)}] = 43.295 \times 10^7 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Similarly,

$$[K^{(2)}] = 28.125 \times 10^6 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[K^{(3)}] = 7.9500 \times 10^6 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Global stiffness matrix is

$$[K_g] = \begin{pmatrix} 43.295 & -43.295 & 0 & 0 \\ -43.295 & 43.295+28.125 & -28.125 & 0 \\ 0 & -28.125 & 28.125+7.95 & -7.95 \\ 0 & 0 & -7.95 & 7.95 \end{pmatrix} 10^6$$

Now the global structural equations can be written as,

$$10^6 \begin{pmatrix} 43.295 & -43.295 & 0 & 0 \\ -43.295 & 71.42 & -28.125 & 0 \\ 0 & -28.125 & 36.075 & -7.95 \\ 0 & 0 & -7.95 & 7.95 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \end{Bmatrix}$$

Applying the boundary conditions: $u_1 = 0$, and $F_1 = F_2 = F_3 = 0$, $F_4 = 5000$ lb., results in the reduced matrix,

$$10^6 \begin{pmatrix} 71.42 & -28.125 & 0 \\ -28.125 & 36.075 & -7.95 \\ 0 & -7.95 & 7.95 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5000 \end{Bmatrix}$$

Solving we get,

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.0012 \\ 0.0029 \\ 0.0092 \end{Bmatrix} \text{ in.}$$

The deflections u_2, u_3, and u_4 are only the approximate values, which can be improved by dividing the beam into more elements. As the number of elements increases, the accuracy will improve.

UNIT IV

DYNAMIC ANALYSIS USING ELEMENT METHOD

4.1 INTRODUCTION

It provides the basic equations necessary for structural dynamical analysis and developed both the lumped and the consistent mass matrix involved in the analysis of bar beam and spring elements.

4.1.1 Fundamentals of Vibration

Any motion which repeats itself after an interval of time is called vibration or oscillation or periodic motion

All bodies possessing mass and elasticity are capable of producing vibration.

4.1.2 Causes of Vibrations

- Unbalanced forces in the machine. These force are produced from within the machine itself
- Elastic nature of the system.
- Self excitations produced by the dry friction between the two mating surfaces.
- External excitations applied on the system.
- Wind may causes vibrations
- Earthquakes may causes vibrations

4.1.3 Types of Vibrations

1.According to the actuating force

Free or natural vibrations

Forced vibrations

Damped vibrations

Undamped vibrations

2.According to motion of system with respect to axis

Longitudinal vibrations

Transverse vibrations

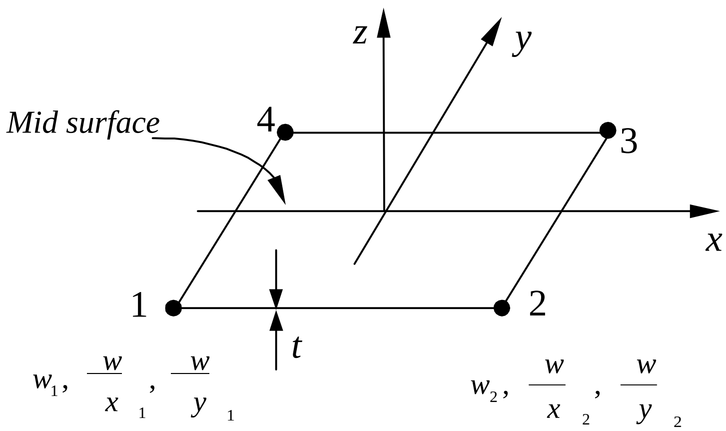
Torsional vibrations

4.2 EQUATION OF MOTION

There is two types of equation of motion

Longitudinal vibration of beam or axial vibration of a rod

Transverse vibration of a beam



DOF at each node: $w,\ v\ \frac{\partial w}{\partial y},\ \frac{\partial w}{\partial y}.$

On each element, the deflection *w(x,y)* is represented by

$$w(x, y) = \sum_{i=1} \left[N_i w_i + N_{xi} \left(\frac{\partial w}{\partial x}\right)_i + N_{yi} \left(\frac{\partial w}{\partial y}\right)_i \right],$$

where N_i, N_{xi} and N_{yi} are shape functions. This is an incompatible element! The stiffness matrix is still of the form

$$\mathbf{k} = \int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV,$$

where **B** is the strain-displacement matrix, and **E** the stress- strain matrix.

Minding Plate Elements:

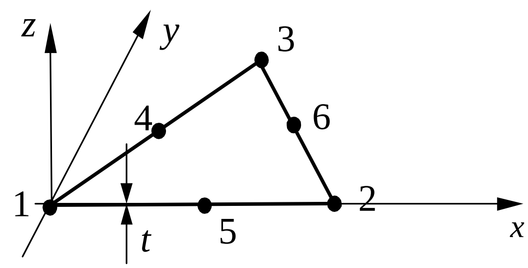
4-Node Quadrilateral 8-Node Quadrilateral

Three independent fields.

Deflection *w(x,y)* is linear for Q4, and quadratic for Q8.

Discrete Kirchhoff Element:

Triangular plate element (not available in *ANSYS*). Start with a 6-node riangular element,

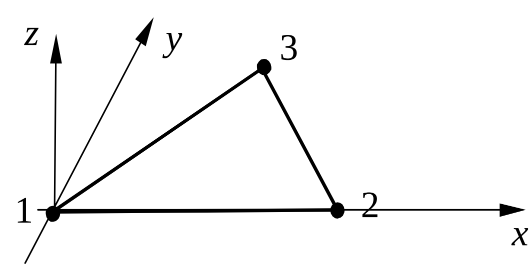


DOF at corner nodes: *w*, $\frac{w}{x}, \frac{w}{y}, x ; y$

DOF at mid side nodes: Total DOF x, y.

= 21.

Then, impose conditions xz yz 0, etc., at selected nodes to reduce the DOF (using relations in (15)). Obtain:

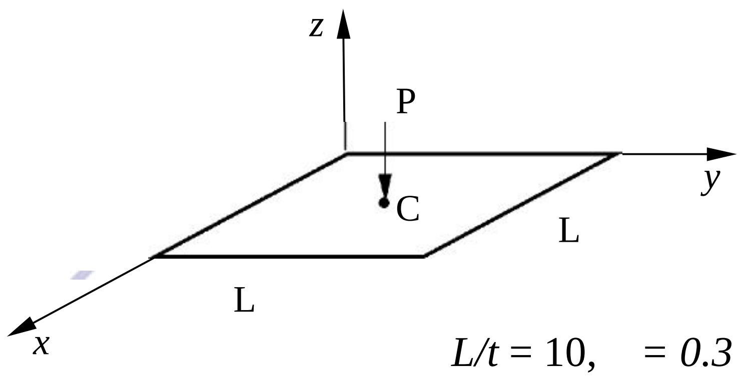


At each node: *w*, x $\frac{w}{x}$, y $\frac{w}{y}$.

Total DOF = 9 (DKT Element).

Incompatible *w(x,y)*; convergence is faster (*w* is cubic along each edge) and it is efficient.

Test Problem:



ANSYS 4-node quadrilateral plate element.

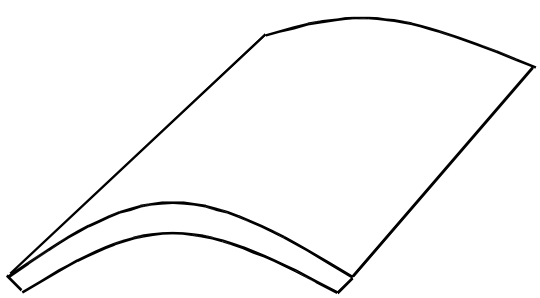
ANSYS Result for w_c

Mesh	w_c (PL^2/D)
2 2	0.00593
4 4	0.00598
8 8	0.00574
16 16	0.00565
:	:
Exact Solution	0.00560

Question: Converges from “above”? Contradiction to what we learnt about the nature of the FEA solution?
Reason: This is an incompatible element (See comments on p. 177).

Shells and Shell Elements

Shells – Thin structures witch span over curved surfaces.



Example:

Sea shell, egg shell (the wonder of the nature); Containers, pipes, tanks;

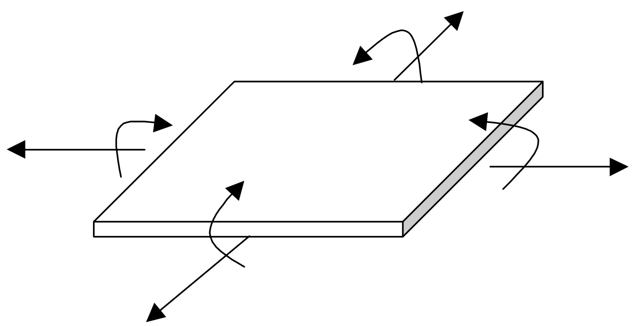
Car bodies;

Roofs, buildings (the Superdome), etc.

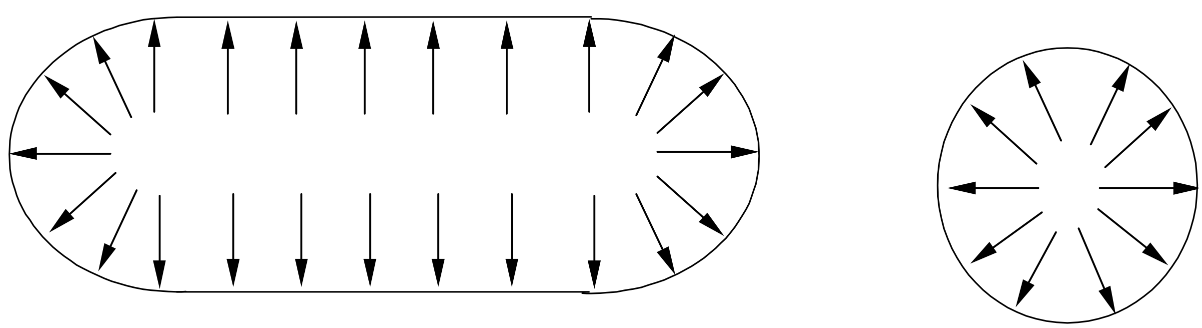
Forces in shells:

Membrane forces + Bending Moments

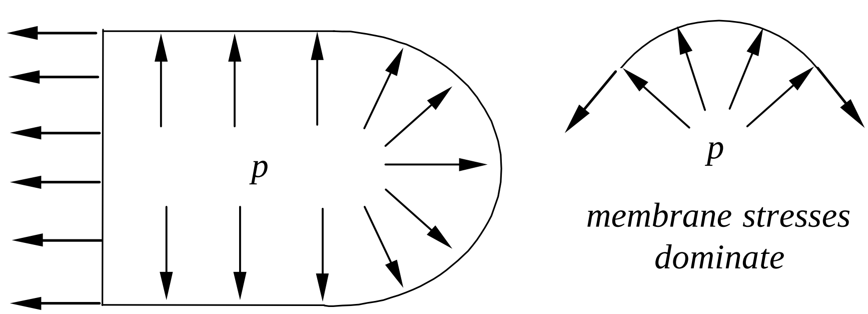
(cf. plates: bending only)



Example: A Cylindrical Container.



internal forces:



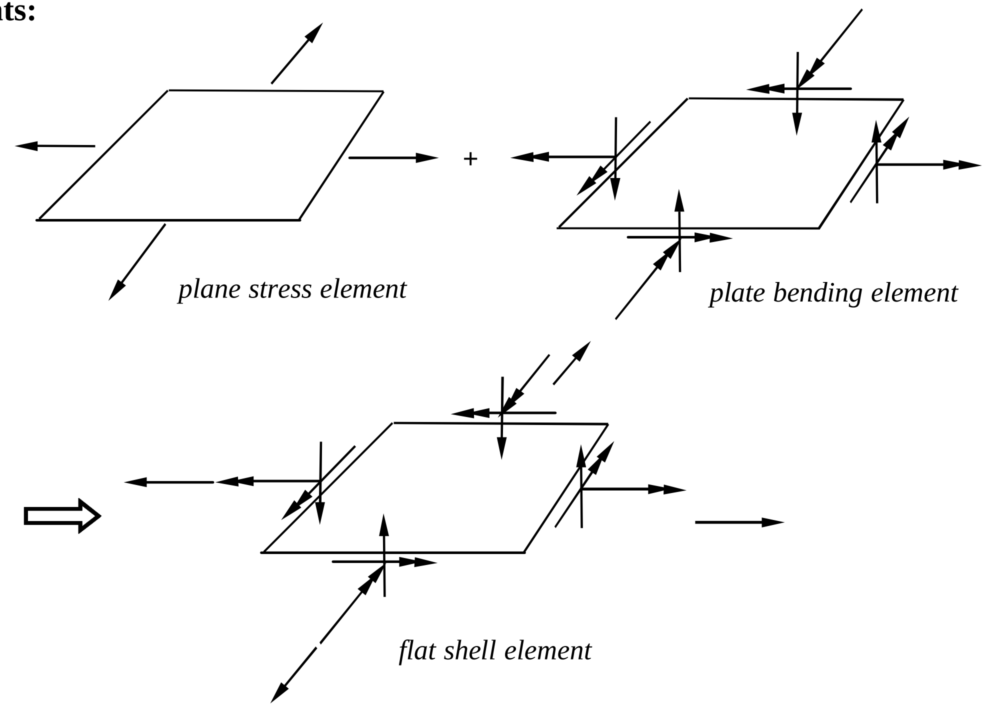
Shell Theory:

Thin shell theory

Shell theories are the most complicated ones to formulate and analyze in mechanics (Russian's contributions).

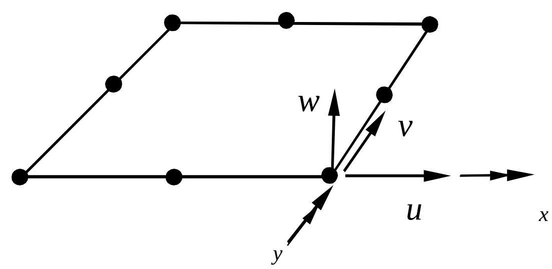
Engineering Craftsmanship Demand strong analytical skill

Shell Elements:



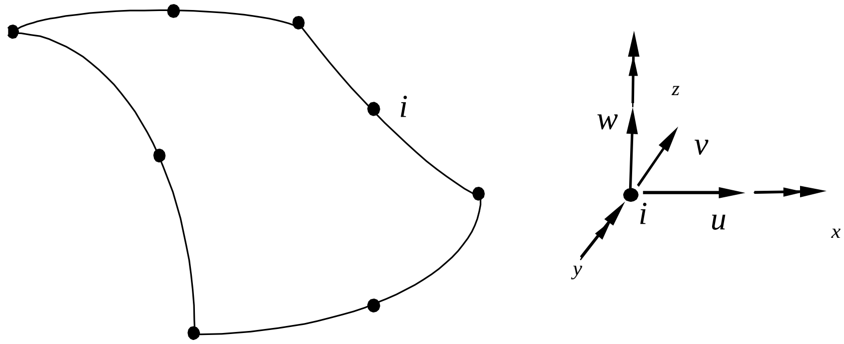
cf.: bar + simple beam element => general beam element.

DOF at each node:



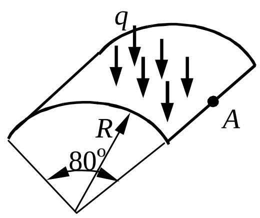
Q4 or Q8 shell element.

Curved shell elements:

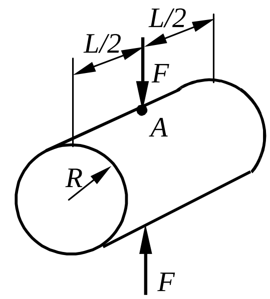


Based on shell theories;
Most general shell elements (flat shell and plate elements are subsets);
Complicated in formulation.

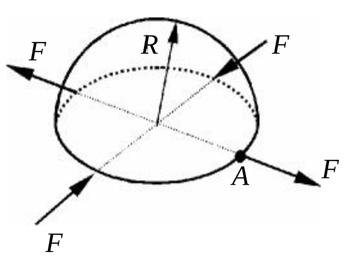
Test Cases:

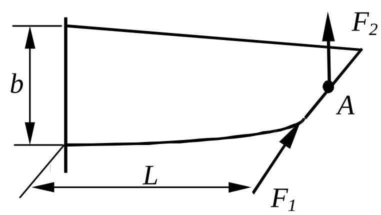


Roof



Pinched Cylinder





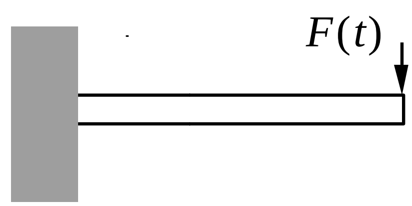
Pinched Hemisphere

4.3CONSISTENT MASS MATRICES

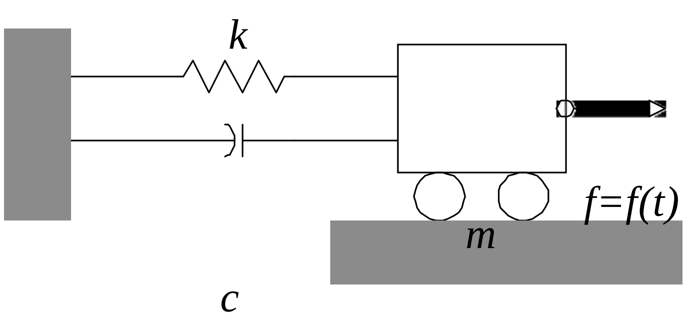
Natural frequencies and modes

Frequency response ($F(t)=F_O \sin wt$) Transient

response ($F(t)$ arbitrary)



4.3.1 Single DOF System



m - mass
k - stiffness
c - damping

Free Vibration:

$f(t) = 0$ and no damping ($c = 0$)

Eq. (1) becomes

$$m\ddot{u} + ku = 0$$

(meaning: inertia force + stiffness force = 0)

Assume:

$$u(t) = U\sin(\omega t),$$

where ω is the frequency of oscillation, U the amplitude.

Eq. (2) yields

$$-U\omega^2 m \sin(\omega t) + kU\sin(\omega t) = 0$$

i.e., $$\left[-\omega^2 m + k\right]U = 0.$$

For nontrivial solutions for U, we must have

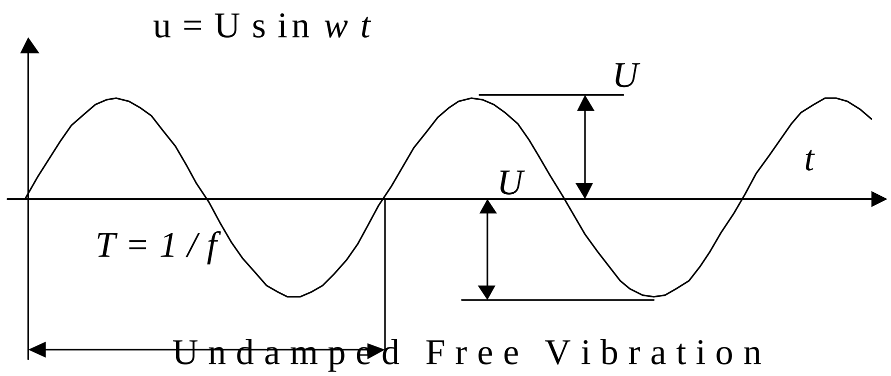
$$\left[-\omega^2 m + k\right] = 0,$$

which yields

$$\omega = \sqrt{\frac{k}{m}}.$$

This is the circular *natural frequency* of the single DOF system (rad/s). The cyclic frequency (1/*s* = Hz) is

$$f = \frac{\omega}{2\pi},$$



Undamped Free Vibration

With non-zero damping c, where

$$0 < c < c_c = 2mw = 2\sqrt{km} \qquad (c_c = \text{critical damping})$$

we have the damped natural frequency:

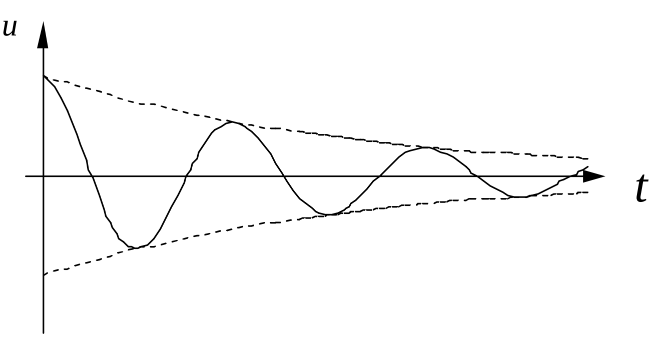
$$w_d = w\sqrt{1 - x^2},$$

where $x = \dfrac{c}{c_c}$ (damping ratio).

For structural damping: $0 \le x < 0.15$ (usually 1~5%)

$$w_d \approx w.$$

Thus, we can ignore damping in normal mode analysis.



Damped Free Vibration

4.3.2.Multiple DOF System

Equation of Motion

Equation of motion for the whole structure is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t), \qquad (8)$$

in which: **u** nodal displacement vector,

M mass matrix,

C damping matrix,

K stiffness matrix,

f forcing vector.

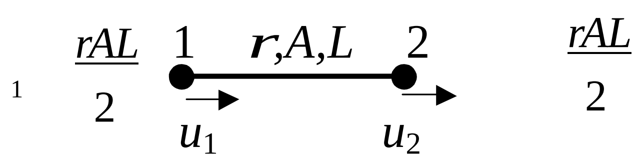
Physical meaning of Eq. (8):

Inertia forces + Damping forces + Elastic forces

= Applied forces

Mass Matrices

Lumped mass matrix (1-D bar element):

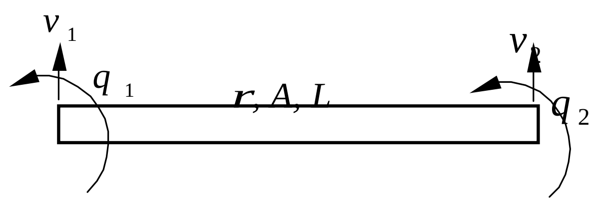


Element mass matrix is found to be

$$\mathbf{m} = \begin{bmatrix} \frac{rAL}{2} & 0 \\ 0 & \frac{rAL}{2} \end{bmatrix}$$

diagonal atrix

Simple Beam Element:



$$\mathbf{m} \quad {}_{V} \rho \mathbf{N}^T \mathbf{N} dV$$

$$\frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & 13L \\ 22L & 4L^2 & 13L & 3L^2 \\ 54 & 13L & 156 & 22L \\ 13L & 3L^2 & 22L & 4L^2 \end{bmatrix} . \begin{matrix} V_1 \\ \Theta_1 \\ v_2 \\ \Theta_2 \end{matrix}$$

Units in dynamic analysis (make sure they are consistent):

	Choice I	***Choice II***
t (time)	s	s
L (length)	m	mm
m (mass)	kg	Mg
a (accel.)	m/s^2	mm/s^2
f (force)	N	N
ρ (density)	kg/m^3	Mg/mm^3

4.4 VECTOR ITERATION METHODS

Study of the dynamic characteristics of a structure:

natural frequencies normal modes shapes)

Let $\mathbf{f}(t) = \mathbf{0}$ and $\mathbf{C} = \mathbf{0}$ (ignore damping) in the dynamic equation (8) and obtain

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

Assume that displacements vary harmonically with time, that is,

$$\mathbf{u}(t) = \bar{\mathbf{u}}\sin(\omega t),$$
$$\dot{\mathbf{u}}(t) = \omega\bar{\mathbf{u}}\cos(\omega t),$$
$$\ddot{\mathbf{u}}(t) = -\omega^2\bar{\mathbf{u}}\sin(\omega t),$$

where $\bar{\mathbf{u}}$ is the vector of nodal displacement amplitudes.

Eq. (12) yields,

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\bar{\mathbf{u}} = \mathbf{0}$$

This is a generalized eigenvalue problem (EVP).

Solutions?

This is an n-th order polynomial of from which we can find n solutions (roots) or eigenvalues

ω_i (i = 1, 2, …, n) are the natural frequencies (or characteristic frequencies) of the structure (the smallest one) is called the fundamental frequency. For each gives one solution (or eigen) vector

$$\left[\mathbf{K} - \omega_i^2\mathbf{M}\right]\bar{\mathbf{u}}_i = \mathbf{0}.$$

$\bar{\mathbf{u}}_i$ (i=1,2,…,n) are the *normal modes* (or *natural modes*, *mode shapes*, etc.).

Properties of Normal Modes

$$\bar{\mathbf{u}}_i^T \mathbf{K}\,\bar{\mathbf{u}}_j \quad 0,$$

$$\bar{\mathbf{u}}_i^T \mathbf{M}\,\bar{\mathbf{u}}_j \quad 0, \qquad \text{for } i \neq j,$$

if $w_i \; w_j$. That is, modes are orthogonal (or independent) to each other with respect to **K** and **M** matrices.

Note:

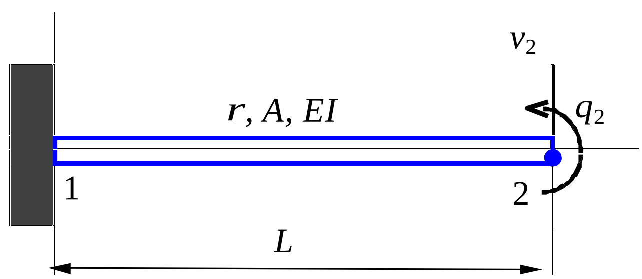
Magnitudes of displacements (modes) or stresses in normal mode analysis have no physical meaning.

For normal mode analysis, no support of the structure is necessary.

$_i = 0$ there are rigid body motions of the whole or a part of the structure. apply this to check the FEA model (check for mechanism or free elements in the models).

Lower modes are more accurate than higher modes in the FE calculations (less spatial variations in the lower modes fewer elements/wave length are needed).

Example:



$$\mathbf{K} \quad w^2 \mathbf{M}$$

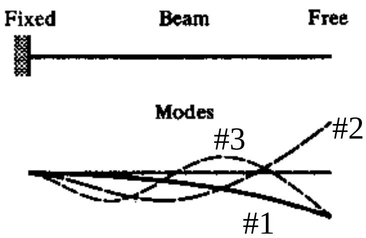
$$\mathbf{K} \quad \frac{EI}{L^3} \quad 12$$

EVP:

$$\begin{vmatrix} 12-156\lambda & 6L-22L\lambda \\ 6L-22L\lambda & 4L^2-4L^2\lambda \end{vmatrix} = 0,$$

in which $\lambda = \omega^2 \rho A L^4 / 420EI$.

Solving the EVP, we obtain,



$$\omega_1 = 3.533\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.38/L \end{Bmatrix},$$

$$\omega_2 = 34.81\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 7.62/L \end{Bmatrix}$$

Exact solutions:

$$\omega_1 = 3.516\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \omega_2 = 22.03\left(\frac{EI}{\rho AL^4}\right)^{1/2}.$$

4.5 MODELLING OF DAMPING

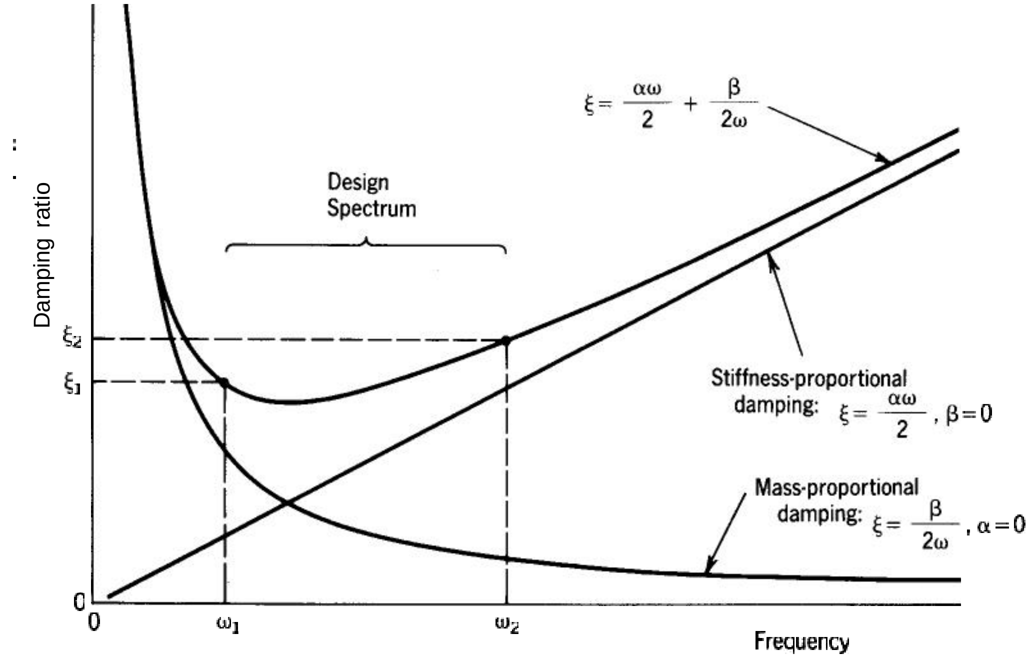
Two commonly used models for viscous damping.

4.5.1 Proportional Damping (Rayleigh Damping)

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \qquad (17)$$

where the constants α & β are found from

with ω_1, ω_2, ξ_1 & ξ_2 (damping ratio) being selected.



Modal Damping

Incorporate the viscous damping in modal equations.

Modal Equations

Use the normal modes (modal matrix) to transform the coupled system of dynamic equations to uncoupled system of equations.

We have

$$\left(\boldsymbol{K} - \omega_i^2 \boldsymbol{M}\right)\bar{\boldsymbol{u}}_i = \boldsymbol{0}, \qquad i = 1, 2, \ldots, n \tag{18}$$

where the normal mode $\bar{\mathbf{u}}_i$ satisfies:

$$\bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_j = 0, \qquad \bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_j = 0, \qquad \text{for } i \neq j,$$

and

$$\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_i = 1, \qquad \bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_i = \omega_i^2, \qquad \text{for } i = 1, 2, \ldots, n.$$

Form the *modal matrix*:

$$\text{Ö}_{(n \quad n)} \qquad \overline{\mathbf{u}_1 \quad \mathbf{u}_2 \quad \text{L} \quad \mathbf{u}_n}$$

are called *principal coordinates*.

Substitute (21) into the dynamic equation:

$$\mathbf{M} \quad \ddot{\mathbf{z}} \quad \mathbf{C} \quad \dot{\mathbf{z}} \quad \mathbf{K} \quad \mathbf{z} \qquad \mathbf{f}(t).$$

Pre-multiply by $^{\mathrm{T}}$, and apply (20):

$$\ddot{\mathbf{z}} \quad \mathbf{C} \quad \dot{\mathbf{z}} \quad \mathbf{z} \qquad \mathbf{p}(t),$$

where $\mathbf{C} \qquad \mathbf{I}$ (proportional damping),

$$\mathbf{p} \qquad ^{T}\mathbf{f}(t) \ .$$

Using *Modal Damping*

Can verify that

Transformation for the displacement vector,

$$z_1(t)\ z_2(t)\ \vdots \qquad z_n(t)$$

Equation (22) becomes,

$$\ddot{z}_i \quad 2 \quad _i \quad _i\ \dot{z}_i \qquad {}^{2}_{i}\, z_i \qquad p_i(t), \qquad i = 1,2,\ldots,\text{n}. \ (24)$$

Equations in (22) or (24) are called *modal equations*. These are uncoupled, second-order differential equations, which are much easier to solve than the original dynamic equation (coupled system).

To recover **u** from **z,** apply transformation (21) again, once **z** is obtained from (24).

Notes:

Only the first few modes may be needed in constructing the modal matrix Φ (i.e., Φ could be an n $\times$ m rectangular matrix with m<n). Thus, significant reduction in the size of the system can be achieved.

Modal equations are best suited for problems in which higher modes are not important (i.e., structural vibrations, but not shock loading).

4.5.2 Frequency Response Analysis

(Harmonic Response Analysis)

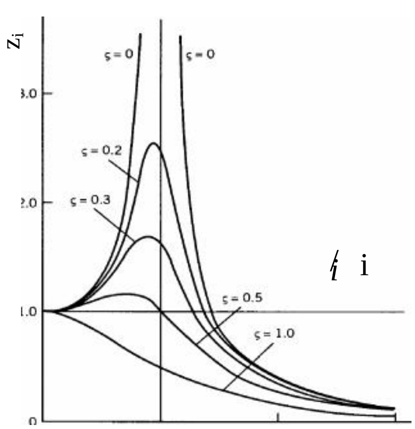
$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{Ku} = \mathbf{F}\sin\omega t \qquad \text{Harmonic loading} \tag{25}$$

Modal method: Apply the modal equations,

$$\ddot{z}_i + 2\xi_i\omega_i\dot{z}_i + \omega_i^2 z_i = p_i \sin\omega t, \qquad i=1,2,\dots,m. \tag{26}$$

These are 1-D equations. Solutions are

$$z_i(t) = \frac{p_i/\omega_i^2}{\sqrt{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2}}\sin(\omega t - \theta_i)$$



where

$$\xi_i = \frac{c_i}{c_c} = \frac{c_i}{2m\omega_i}, \text{ damping ratio}$$

Recover **u** from (21).

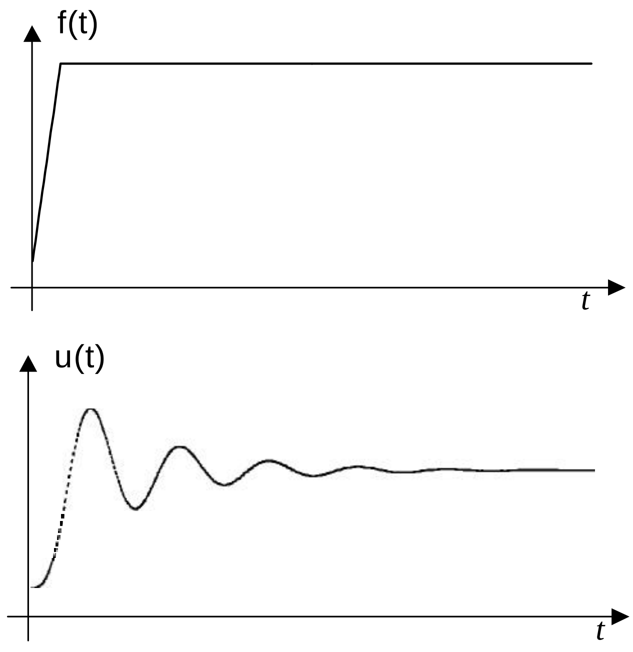
Direct Method: Solve Eq. (25) directly, that is, calculate the inverse. With $\mathbf{u} = \bar{\mathbf{u}}e^{i\omega t}$ (complex notation), Eq. (25) becomes

This equation is expensive to solve and matrix is ill- conditioned if ω is close

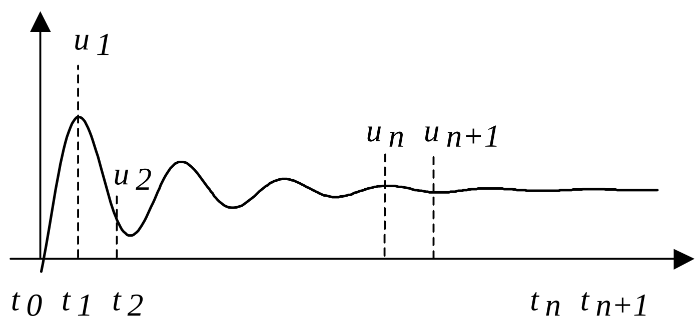
4.6TRANSIENT RESPONSE ANALYSIS

(Dynamic Response/Time-History Analysis)

Structure response to *arbitrary, time-dependent loading*.



Compute responses by integrating through time:



B. Modal Method

First, do the transformation of the dynamic equations using the modal matrix before the time marching:

Then, solve the uncoupled equations using an integration method. Can use, e.g., 10%, of the total modes (m= n/10).

Uncoupled system, Fewer equations,

No inverse of matrices,
More efficient for large problems.

4.6.1 Cautions in Dynamic Analysis

Symmetry: It should not be used in the dynamic analysis (normal modes, etc.) because symmetric structures can have antisymmetric modes.

Mechanism, rigid body motion means   = 0. Can use this to check FEA models to see if they are properly connected and/or supported.

Input for FEA: loading *F(t)* or *F( )* can be very complicated in real applications and often needs to be filtered first before used as input for FEA.

Examples

Impact, drop test, etc.

PROBLEM

In the spring structure shown k_1 = 10 lb./in., k_2 = 15 lb./in., k_3 = 20 lb./in., P= 5 lb. Determine the deflection at nodes 2 and 3.

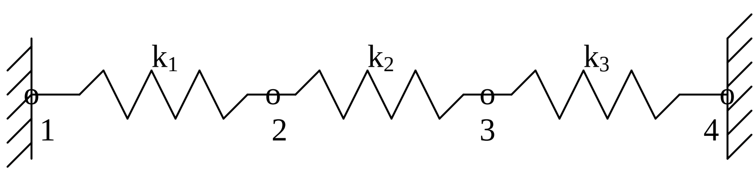


Figure 2.4

Solution:

Again apply the three steps outlined previously.

Step 1: Find the Element Stiffness Equations

Element 1:

$$[K^{(1)}] = \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}$$

Element 2:

$$[K^{(2)}] = \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \end{array} \begin{matrix} 2 \\ 3 \end{matrix}$$

Element 3:

$$[K^{(3)}] = \begin{array}{c} \begin{matrix} 3 & 4 \end{matrix} \\ \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \end{matrix}$$

Step 2: Find the Global stiffness matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$

$$\begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 10+15 & -15 & 0 \\ 0 & -15 & 15+20 & -20 \\ 0 & 0 & -20 & 20 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 25 & -15 & 0 \\ 0 & -15 & 35 & -20 \\ 0 & 0 & -20 & 20 \end{bmatrix}$$

Now the global structural equation can be written as,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 25 & -15 & 0 \\ 0 & -15 & 35 & -20 \\ 0 & 0 & -20 & 20 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Step 3: Solve for Deflections

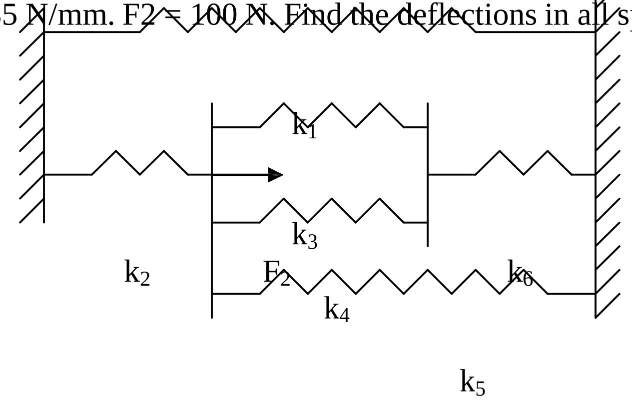
The known boundary conditions are: $u_1 = u_4 = 0$, $F_3 = P = 3$lb. Thus, rows and columns 1 and 4 will drop out, resulting in the following matrix equation,

$$\begin{Bmatrix} 0 \\ 3 \end{Bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 35 \end{bmatrix} \begin{Bmatrix} u2 \\ u3 \end{Bmatrix}$$

Solving, we get $u_2 = 0.0692$ & $u_3 = 0.1154$

PROBLEM

In the spring structure shown, $k_1 = 10$ N/mm, $k_2 = 15$ N/mm, $k_3 = 20$ N/mm, $k_4 = 25$ N/mm, $k_5 = 30$ N/mm, $k_6 = 35$ N/mm. F2 = 100 N. Find the deflections in all springs.



Solution:

Here again, we follow the three-step approach described earlier, without specifically mentioning at each step.

Element 1:

$$[K^{(1)}] = \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{4} \end{matrix} \\ \begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix} \end{array} \begin{matrix} \mathit{1} \\ \mathit{4} \end{matrix}$$

Element 2:

$$[K^{(2)}] = \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} \end{matrix} \\ \begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix} \end{array} \begin{matrix} \mathit{1} \\ \mathit{2} \end{matrix}$$

Element 3:

$$[K^{(3)}] = \begin{array}{c} \begin{matrix} \mathit{2} & \mathit{3} \end{matrix} \\ \begin{pmatrix} 20 & -20 \\ -20 & 20 \end{pmatrix} \end{array} \begin{matrix} \mathit{2} \\ \mathit{3} \end{matrix}$$

Element 4:

$$[K^{(4)}] = \begin{array}{c} \begin{matrix} \mathit{2} & \mathit{3} \end{matrix} \\ \begin{pmatrix} 25 & -25 \\ -25 & 25 \end{pmatrix} \end{array} \begin{matrix} \mathit{2} \\ \mathit{3} \end{matrix}$$

Element 5:

$$[K^{(5)}] = \begin{array}{c} \begin{matrix} \mathit{2} & \mathit{4} \end{matrix} \\ \begin{pmatrix} 30 & -30 \\ -30 & 30 \end{pmatrix} \end{array} \begin{matrix} \mathit{2} \\ \mathit{4} \end{matrix}$$

Element 6:

$$[K^{(6)}] = \begin{array}{c} \begin{matrix} \mathit{3} & \mathit{4} \end{matrix} \\ \begin{pmatrix} 35 & -35 \\ -35 & 35 \end{pmatrix} \end{array} \begin{matrix} \mathit{3} \\ \mathit{4} \end{matrix}$$

The global stiffness matrix is,

$$[K_g] = \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \end{matrix} \\ \begin{pmatrix} 10+15 & -15 & 0 & -10 \\ -15 & 15+20+25+30 & -20-25 & -30 \\ 0 & -20-25 & 20+25+35 & -35 \\ -10 & -30 & -35 & 10+30+35 \end{pmatrix} \end{array} \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{matrix}$$

And simplifying, we get

$$[K_g] = \begin{pmatrix} 25 & -15 & 0 & -10 \\ -15 & 90 & -45 & -30 \\ 0 & -45 & 80 & -35 \\ -10 & -30 & -35 & 75 \end{pmatrix}$$

And the structural equation is,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{pmatrix} 25 & -15 & 0 & -10 \\ -15 & 90 & -45 & -30 \\ 0 & -45 & 80 & -35 \\ -10 & -30 & -35 & 75 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Now, apply the boundary conditions, $u_1 = u_4 = 0$, $F_2 = 100$ N. This is carried out by deleting the rows 1 and 4, columns 1 and 4, and replacing F_2 by 100N. The final matrix equation is,

Which $\begin{Bmatrix} 100 \\ 0 \end{Bmatrix} = \begin{pmatrix} 90 & -45 \\ -45 & 80 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$ gives

Deflections:

Spring 1: $u_4 – u_1 = 0$

Spring 2: $u_2 – u_1 = 1.54590$

Spring 3: $u_3 – u_2 = -0.6763$

Spring 4: $u_3 – u_2 = -0.6763$

Spring 5: $u_4 – u_2 = -1.5459$

Spring 6: $u_4 – u_3 = -0.8696$

UNIT V

APPLICATIONS IN HEAT TRANSFER &FLUID MECHANICS

5.1 ONE DIMENSIONAL HEAT TRANSFER ELEMENT

In structural problem displacement at each nodel point is obtained. By using these displacement solutions, stresses and strains are calculated for each element. In structural problems, the unknowns are represented by the components of vector field. For example, in a two dimensional plate, the unknown quantity is the vector field u(x,y),where u is a (2x1)displacement vector.

Heat transfer can be defined as the transmission of energy from one region another region due to temperature difference. A knowledge of the temperature distribution within a body is important in many engineering problems. There are three modes of heat transfer.

They are: (i) Conduction

(ii) Convection

(iii) Radiation

5.1.1Strong Form for Heat Conduction in One Dimension with Arbitrary Boundary Conditions

Following the same procedure as in Section, the portion of the boundary where the temperature is prescribed, i.e. the essential boundary is denoted by T and the boundary where the flux is prescribed is recommended for Science and Engineering Track. Denoted by q ; these are the boundaries with natural boundary conditions. These boundaries are complementary, so

q = ¼ ; q ∖ T ¼ 0:

With the unit normal used in , we can express the natural boundary condition as qn ¼ q. For example, positive flux q causes heat inflow (negative q) on the left boundary point where qn ¼ q ¼ q and heat outflow (positive q) on the right boundary point where qn ¼ q ¼ q.

Strong form for 1D heat conduction problems

5.1.2Weak Form for Heat Conduction in One Dimension with Arbitrary Boundary Conditions

We again multiply the first two equations in the strong form by the weight function and integrate over the domains over which they hold, the domain for the differential equation and the domain q for the flux boundary condition, which yields ws dx with w ¼

Recalling that w ¼ 0 on T and combining with gives

Weak form for 1D heat conduction problems

Find T ðxÞ 2 U such that

Notice the similarity between

5.2 APPLICATION TO HEAT TRANSFER TWO-DIMENTIONAL

5.2.1Strong Form for Two-Point Boundary Value Problems

The equations developed in this chapter for heat conduction, diffusion and elasticity problems are all of the following form:

Such one-dimensional problems are called two-point boundary value problems. gives the particular meanings of the above variables and parameters for several applications. The natural boundary conditions can also be generalized as (based on Becker et al. (1981))

Þ ¼ 0 on : —

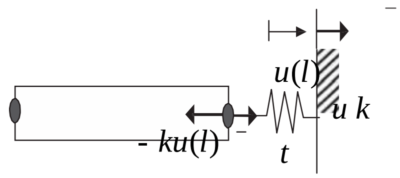
Equation is a natural boundary condition because the derivative of the solution appears in it. reduces to the standard natural boundary conditions considered in the previous sections when bðxÞ ¼ 0. Notice that the essential boundary condition can be recovered as a limiting case of when bðxÞ is a penalty parameter, i.e. a large number In this case, and Equation is called a generalized boundary condition.

An example of the above generalized boundary condition is an elastic bar with a spring attached as shown in In this case, bðlÞ ¼ k and reduces to

E(n-l) — (k-uð) uÞ ¼ 0 at x ¼ l;

where ¼ k is the spring constant. If the spring stiffness is set to a very large value, the above boundary condition enforces ¼ u; if we let k ¼ 0, the above boundary condition corresponds to a prescribed traction boundary. In practice, such generalized boundary conditions are often used to model the influence of the surroundings. For example, if the bar is a simplified model of a building and its foundation, the spring can represent the stiffness of the soil.

5.2.2 Two-Point Boundary Value Problem With Generalized Boundary Conditions



An example of the generalized boundary for elasticity problem.

Another example of the application of this boundary condition is convective heat transfer, where energy is transferred between the surface of the wall and the surrounding medium. Suppose convective heat transfer occurs at x ¼ l. Let T ðlÞ be the wall temperature at x ¼ l and T be the temperature in the medium. Then the flux at the boundary x ¼ l is given by qðlÞ ¼ hðT ðlÞ T Þ, so bðlÞ ¼ h and the boundary condition is

where h is convection coefficient, which has dimensions of W m ^{2} o C ^{1}. Note that when the convection coefficient is very large, the temperature T is immediately felt at x ¼ l and thus the essential boundary condition is again enforced as a limiting case of the natural boundary condition.

There are two approaches to deal with the boundary condition . We will call them the penalty and partition methods. In the penalty method, the essential boundary condition is enforced as a limiting case of the natural boundary condition by equating bðxÞ to a penalty parameter. The resulting strong form for the penalty method is given in.

General strong form for 1D problems-penalty method

Þ f ¼ 0 on ;

In the partition approach, the total boundary is partitioned into the natural boundary, and the complementary essential boundary, The natural boundary condition has the generalized form defined by The resulting strong form for the partition method is summarized in.

5.2.3 Weak Form for Two-Point Boundary Value Problems

In this section, we will derive the general weak form for two-point boundary value problems. Both the penalty and partition methods described in will be considered. To obtain the general weak form for the penalty method, we multiply the two equations in the strong by the weight function and integrate over the domains over which they hold: the domain for the differential equation and the domain for the generalized boundary condition.

5.3 SCALE VARIABLE PROBLEM IN 2 DIMENSIONS

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_2 \\ x_1 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

$N_1=1/4(1-\varepsilon)(1-\eta)$; $N_2=1/4(1+\varepsilon)(1-\eta)$; $N_3=1/4(1+\varepsilon)(1+\eta)$; $N_4=1/4(1-\varepsilon)(1+\eta)$.

- **Equation of Stiffness Matrix for 4 noded isoparametric quadrilateral element**

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix};$$

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1-v & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$, for plane stress conditions;

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$, for plane strain conditions.

- **Equation of element force vector**

$$\{F\}_e=[N]^T\begin{Bmatrix} F_x \\ F_y \end{Bmatrix};$$

N – Shape function, F_x – load or force along x direction,

F_y – load or force along y direction.

- **Numerical Integration (Gaussian Quadrature)**

The Gauss quadrature is one of the numerical integration methods to calculate the definite integrals. In FEA, this Gauss quadrature method is mostly preferred. In this method the numerical integration is achieved by the following expression,

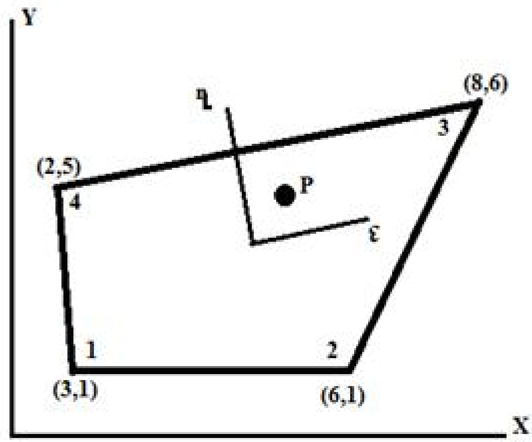
$$\int_{-1}^{1} f(x)dx=\sum_{i=1}^{n} w_i f(x_i)$$

Table gives gauss points for integration from -1 to 1.

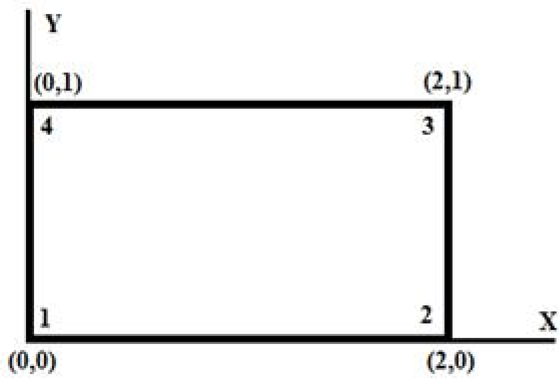
Number of Points n	Location x_i	Corresponding Weights w_i
1	$\mathbf{x_1 = 0.000}$	**2.000**
2	$\mathbf{x_1, x_2} = \pm\sqrt{\frac{1}{3}} = \pm 0.577350269189$	**1.000**
3	$\mathbf{x_1, x_3} = \pm\sqrt{\frac{3}{5}} = \pm 0.774596669241$ $\mathbf{x_2 = 0.000}$	$\frac{5}{9} = 0.555555$ $\frac{8}{9} = 0.888888$
4	$\mathbf{x_1, x_4} = \pm 0.8611363116$ $\mathbf{x_2, x_3} = \pm 0.3399810436$	**0.3478548451** **0.6521451549**

- **Problem (I set)**

1. Evaluate $I = \int_{-1}^{1} \cos\frac{\pi x}{2} dx$, by applying 3 point Gaussian quadrature and compare with exact solution.
2. Evaluate $I = \int_{-1}^{1}\left[3e^{x} + x^{2} + \frac{1}{x+2}\right]dx$, using one point and two point Gaussian quadrature. Compare with exact solution.
3. For the isoparametric quadrilateral element shown in figure, determine the local co –ordinates of the point P which has Cartesian co-ordinates (7, 4).



4. A four noded rectangular element is in figure. Determine (i) Jacobian matrix, (ii) Strain – Displacement matrix and (iii) Element Stresses. Take $E=2\times10^5 N/mm^2$, $\upsilon=0.25$, $u=[0,0,0.003,0.004,0.006, 0.004,0,0]^T$, $\varepsilon=0$, $\eta=0$. Assume plane stress condition.



5.4 2 DIMENTIONAL FLUID MECHANICS

The problem of linear elastostatics described in detail in can be extended to include the effects of inertia. The resulting equations of motion take the form

$$\begin{aligned}
\nabla\cdot\sigma + f &= \rho\ddot{u} && \text{in } \Omega\times\mathbf{I},\\
\sigma n &= \bar{t} && \text{on } \Gamma_q\times\mathbf{I},\\
u &= \bar{u} && \text{on } \Gamma_u\times\mathbf{I},\\
u(x_1,x_2,x_3,0) &= u_0(x_1,x_2,x_3) && \text{in } \Omega,\\
v(x_1,x_2,x_3,0) &= v_0(x_1,x_2,x_3) && \text{in } \Omega,
\end{aligned}$$

where $u = u(x_1, x_2, x_3, t)$ is the unknown displacement field, ρ is the mass density, and $\mathbf{I} = (0, T)$ with T being a given time. Also, u_0 and v_0 are the prescribed initial displacement and velocity fields. Clearly, two sets of boundary conditions are set on Γ_u and Γ_q, respectively, and are assumed to hold throughout the time interval $\mathbf{I}$. Likewise, two sets of initial conditions are set for the whole domain Ω at time $t = 0$. The strong form of the resulting initial/boundary- value problem is stated as follows: given functions f, $\bar{t}$, $\bar{u}$, u_0 and v_0, as well as a constitutive equation for σ, find u in $\Omega \times \mathbf{I}$, such that the equations are satisfied.

A Galerkin-based weak form of the linear elastostatics problem has been derived in Section In the elastodynamics case, the only substantial difference involves the inclusion of the term $\int_\Omega w \cdot \rho \ddot{u} \, d\Omega$, as long as one adopts the semi-discrete approach. As a result, the weak form at a fixed time can be expressed as

$$\int_\Omega w \cdot \rho \ddot{u} \, d\Omega + \int_\Omega \nabla_s w : \sigma \, d\Omega = \int_\Omega w \cdot f \, d\Omega + \int_\Gamma w \cdot \bar{\mathbf{t}} \, d\Gamma .$$

Following the development of Section 7.3, the discrete counterpart of can be written as

$$\int w_h \cdot \rho \ddot{u}_h \, d\Omega + \varrho(w_h) \cdot D \varrho(u_h) \, d\Omega = \int w_h \cdot f \, d\Omega + \int w_h \cdot \bar{\mathbf{t}} \, d\Gamma .$$

Following a standard procedure, the contribution of the forcing vector $F^{int,e}$ due to interele- ment tractions is neglected upon assembly of the global equations. As a result, the equations is give rise to their assembled counterparts in the form

$$Mu + K\hat{u} = F ,$$

where $\hat{u}$ is the global unknown displacement vector[1]. The preceding equations are, of course, subject to initial conditions that can be written in vectorial form as $\hat{u}(0) = \hat{u}_0$ and $\hat{v}(0) = \hat{v}_0$.

The most commonly employed method for the numerical solution of the system of coupled linear second-order ordinary differential equations is the Newmark method. This method is based on a time series expansion of $\hat{u}$ and $\hat{\dot{u}} := \hat{v}$. Concentrating on the time interval $(t_n, t_{n+1}]$, the Newmark method is defined by the equations

$$\hat{u}_{n+1} = \hat{u}_n + \hat{v}_n \Delta t_n + \frac{1}{2}[(1 - 2\beta)\hat{a}_n + 2\beta \hat{a}_{n+1}]\Delta t_n^2 ,$$

$$\hat{v}_{n+1} = \hat{v}_n + [(1 - \gamma)\hat{a}_n + \gamma \hat{a}_{n+1}]\Delta t_n ,$$

It is clear that the Newmark equations define a whole family of time integrators. It is important to distinguish this family into two categories, namely implicit and explicit integrators, corresponding to $\beta > 0$ and $\beta = 0$, respectively.

The overhead "hat" symbol is used to distinguish between the vector field $\mathbf{u}$ and the solution vector $\hat{u}$ emanating from the finite element approximation of the vector field $\mathbf{u}$.

The general implicit Newmark integration method may be implemented as follows: first, solve $(9.18)_1$ for $\hat{a}_{n+1}$, namely write

$$\hat{a}_{n+1} = \frac{}{\beta \Delta t_n^2} (\hat{u}_{n+1} - \hat{u}_n - \hat{v}_n \Delta t_n) \hat{a}_n$$

Then, substitute (9.19) into the semi-discrete form (9.17) evaluated at t_{n+1} to find that

$$M \underset{n}{+} K \quad \hat{u}_{n+1} = F_{n+1} \quad .$$

After solving for $\hat{u}_{n+1}$, one may compute the acceleration $\hat{a}_{n+1}$ from and the velocity $\hat{v}_{n+1}$ from.

Finally, the general explicit Newmark integration method may be implemented as follows: starting from the semi-discrete equations evaluated at t_{n+1}, one may substitute $\hat{u}_{n+1}$from to find that

$$M\hat{a}_{n+1} = -K(\hat{u}_n + \hat{v}_n \Delta t_n + \hat{a}_n \Delta t_n) + F_{n+1} \ .$$

If M is rendered diagonal (see discussion in Chapter 8), then $\hat{a}_{n+1}$ can be determined without solving any coupled linear algebraic equations. Then, the velocities $\hat{v}_{n+1}$ are immediately computed from $(9.18)_2$. Also, the displacements $\hat{u}_{n+1}$ are computed from indepen-dently of the accelerations $\hat{a}_{n+1}$.

QUESTION BANK

PART A QUESTIONS WITH ANSWERS

UNIT 1

1. What is meant by finite element?

   A small units having definite shape of geometry and nodes is called finite element.

2. What is meant by node or joint?

   Each kind of finite element has a specific structural shape and is inter- connected with the adjacent element by nodal point or nodes. At the nodes, degrees of freedom are located. The forces will act only at nodes at any others place in the element.

3. What is the basic of finite element method?

   Discretization is the basis of finite element method. The art of subdividing a structure in to convenient number of smaller components is known as discretization.

4. What are the types of boundary conditions?

   Primary boundary conditions

   Secondary boundary conditions

5. State the methods of engineering analysis?

   Experimental methods

   Analytical methods

   Numerical methods or approximate methods

6. What are the types of element?

7. 1D element

   2D element

   3D element

8. State the three phases of finite element method.

   Preprocessing

   Analysis

   Post Processing

9. What is structural problem?

   Displacement at each nodal point is obtained. By these displacements solution stress and strain in each element can be calculated.

10. What is non structural problem?

Temperature or fluid pressure at each nodal point is obtained. By using these values properties such as heat flow fluid flow for each element can be calculated.

10. What are the methods are generally associated with the finite element analysis?

Force method

Displacement or stiffness method.

11. Explain stiffness method.

Displacement or stiffness method, displacement of the nodes is considered as the unknown of the problem. Among them two approaches, displacement method is desirable.

12. What is meant by post processing?

Analysis and evaluation of the solution result is referred to as post processing. Postprocessor computer program help the user to interpret the result by displaying them in graphical form.

13. Name the variation methods.

Ritz method.

Ray-Leigh Ritz method.

14. What is meant by degrees of freedom?

When the force or reaction act at nodal point node is subjected to deformation. The deformation includes displacement rotation, and or strains. These are collectively known as degrees of freedom

15. What is meant by discretization and assemblage?

The art of subdividing a structure in to convenient number of smaller components is known as discretization. These smaller components are then put together. The process of uniting the various elements together is called assemblage.

16. What is Rayleigh-Ritz method?

It is integral approach method which is useful for solving complex structural problem, encountered in finite element analysis. This method is possible only if a suitable function is available.

17. What is Aspect ratio?

It is defined as the ratio of the largest dimension of the element to the smallest dimension. In many cases, as the aspect ratio increases the in accuracy of the solution increases. The conclusion of many researches is that the aspect ratio

18. What is truss element?

The truss elements are the part of a truss structure linked together by point joint which transmits only axial force to the element.

19. What are the h and p versions of finite element method?

It is used to improve the accuracy of the finite element method. In h version, the order of polynomial approximation for all elements is kept constant and the numbers of elements are increased. In p version, the numbers of elements are maintained constant and the order of polynomial approximation of element is increased.

20. Name the weighted residual method

Point collocation method

Sub domain collocation method

Least squares method

Galerkins method.

UNIT 2

21. List the two advantages of post processing.

Required result can be obtained in graphical form. Contour diagrams can be used to understand the solution easily and quickly.

22. During discretization, mention the places where it is necessary to place a node?

Concentrated load acting point

Cross-section changing point

Different material interjections

Sudden change in point load

23. What is the difference between static and dynamic analysis?

Static analysis: The solution of the problem does not vary with time is known as static analysis

Example: stress analysis on a beam

Dynamic analysis: The solution of the problem varies with time is known as dynamic analysis

24. Name any four FEA softwares.

ANSYS

NASTRAN

COSMOS

25. Differentiate between global and local axes.

Local axes are established in an element. Since it is in the element level, they change with the change in orientation of the element. The direction differs from element to element.

Global axes are defined for the entire system. They are same in direction for all the elements even though the elements are differently oriented.

26. Distinguish between potential energy function and potential energy functional

If a system has finite number of degree of freedom (q_1,q_2,and q_3), then the potential energy expressed as,

$\pi = f\,(q_1, q_2, \text{and } q_3)$

It is known as function. If a system has infinite degrees of freedom then the potential energy is expressed as

$$\int f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\right) dx$$

27. What are the types of loading acting on the structure?

Body force (f)

Traction force (T)

Point load (P)

28. Define the body force

A body force is distributed force acting on every elemental volume of the body Unit: Force per unit volume.

Example: Self weight due to gravity

29. Define traction force

Traction force is defined as distributed force acting on the surface of the body. Unit: Force per unit area.

Example: Frictional resistance, viscous drag, surface shear

30. What is point load?

Point load is force acting at a particular point which causes displacement.

31. What are the basic steps involved in the finite element modeling.

Discretization of structure.

Numbering of nodes.

32. Write down the general finite element equation.

$$\{F\}=[K]\{u\}$$

33. What is discretization?

The art of subdividing a structure in to a convenient number of smaller components is known as discretization.

34. What are the classifications of coordinates?

Global coordinates

Local coordinates

Natural coordinates

35. What is Global coordinates?

The points in the entire structure are defined using coordinates system is known as global coordinate system.

36. What is natural coordinates?

A natural coordinate system is used to define any point inside the element by a set of dimensionless number whose magnitude never exceeds unity. This system is very useful in assembling of stiffness matrices.

37. Define shape function.

Approximate relation $\varphi (x,y) = N_1 (x,y) \varphi_1 + N_2 (x,y) \varphi_2 + N_3 (x,y) \varphi_3$

Where $\varphi 1$, φ_2, and φ_3 are the values of the field variable at the nodes N_1, N_2, and N_3 are the interpolation functions.

N_1, N_2, and N_3 are also called shape functions because they are used to express the geometry or shape of the element.

38. What are the characteristic of shape function?

It has unit value at one nodal point and zero value at other nodal points. The sum of shape function is equal to one.

39. Why polynomials are generally used as shape function?

Differentiation and integration of polynomial are quit easy.

The accuracy of the result can be improved by increasing the order of the polynomial. It is easy to formulate and computerize the finite element equations

40. How do you calculate the size of the global stiffness matrix?

Global stiffness matrix size = Number of nodes X Degrees of freedom per node

UNIT 3

41. Write down the expression of stiffness matrix for one dimensional bar element.

$$[K] = \frac{AE}{l}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

42. State the properties of stiffness matrix

It is a symmetric matrix

The sum of elements in any column must be equal to zero
It is an unstable element. So the determinant is equal to zero.

43. Write down the expression of stiffness matrix for a truss element.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

44. Write down the expression of shape function N and displacement u for one dimensional bar element.

$U = N_1u_1 + N_2u_2$

$N_1 = 1 - X / l$

$N_2 = X / l$

45. Define total potential energy.

Total potential energy, π = Strain energy (U) + potential energy of the external forces (W)

46. State the principle of minimum potential energy.

Among all the displacement equations that satisfied internal compatibility and the boundary condition those that also satisfy the equation of equilibrium make the potential energy a minimum is a stable system.

47. Write down the finite element equation for one dimensional two noded bar element.

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

48. What is truss?

A truss is defined as a structure made up of several bars, riveted or welded together.

49. States the assumption are made while finding the forces in a truss.

All the members are pin jointed.

The truss is loaded only at the joint

The self weight of the members is neglected unless stated.

50. State the principles of virtual energy?

A body is in equilibrium if the internal virtual work equals the external virtual work for the every kinematically admissible displacement field

51. What is essential boundary condition?

Primary boundary condition or EBC Boundary condition which in terms of field variable is known as Primary boundary condition.

52. Natural boundary conditions?

Secondary boundary natural boundary conditions which are in the differential form of field variable is known as secondary boundary condition

53. How do you define two dimensional elements?

Two dimensional elements are define by three or more nodes in a two dimensional plane. The basic element useful for two dimensional analysis is the triangular element.

54. What is CST element?

Three noded triangular elements are known as CST. It has six unknown displacement degrees of freedom (u1, v1, u2, v2, u3, v3). The element is called CST because it has a constant strain throughout it.

55. What is LST element?

Six nodded triangular elements are known as LST. It has twelve unknown displacement degrees of freedom. The displacement function for the elements are quadratic instead of linear as in the CST.

56. What is QST element?

Ten nodded triangular elements are known as Quadratic strain triangle. It is also called as cubic displacement triangle.

58. What meant by plane stress analysis?

Plane stress is defined to be a state of stress in which the normal stress and shear stress directed perpendicular to the plane are assumed to be zero.

60. Define plane strain analysis.

Plane strain is defined to be state of strain normal to the xy plane and the shear strains are assumed to be zero.

UNIT 4

61. Write down the stiffness matrix equation for two dimensional CST elements.

Stiffness matrix $[K]=[B]^T[D][B]At$

$[B]^T$-Strain displacement $[D]$-Stress strain matrix $[B]$-Strain displacement matrix

62. Write down the stress strain relationship matrix for plane stress conditions.

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$$

63. What is axisymmetric element?

Many three dimensional problem in engineering exhibit symmetry about an axis of rotation such type of problem are solved by special two dimensional element called the axisymmetric element

64. What are the conditions for a problem to be axisymmetric?

The problem domain must be symmetric about the axis of revolution All boundary condition must be symmetric about the axis of revolution All loading condition must be symmetric about the axis of revolution

65. Give the stiffness matrix equation for an axisymmetric triangular element. Stiffness matrix $[K]=[B]^T[D][B]2\pi rA$

66. What is the purpose of Isoparametric element?

It is difficult to represent the curved boundaries by straight edges finite elements. A large number of finite elements may be used to obtain reasonable resemblance between original body and the assemblage.

67. Write down the shape functions for 4 noded rectangular elements using natural coordinate system.

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

68. Write down Jacobian matrix for 4 noded quadrilateral elements.

$$[J] = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

69. Write down stiffnes matrix equation for 4 noded isoparametric quadrilateral elements.

Stiffness matrix $[K] = t\int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] |J| \partial\varepsilon\partial\eta$

70. Define super parametric element.

If the number of nodes used for defining the geometry is more than of nodes used for defining the displacement is known as super parametric element

71. Define sub parametric element.

If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacement is known as sub parametric element.

72. What is meant by Isoparametric element?

If the number of nodes used for defining the geometry is same as number of nodes used for defining the displacement is known as Isoparametric element.

73. Is beam element an Isoparametric element?

Beam element is not an Isoparametric element since the geometry and displacement are defined by different order interpretation functions.

74. What is the difference between natural coordinate and simple natural coordinate?

L1 = 1-x/l

L2 = x/l

75. What is Area coordinates?

L1 = A1/A   L2 = A2/A   L3 = A3/A

76. What is simple natural coordinate?

A simple natural coordinate is one whose value between -1 and 1.

77. Give example for essential boundary conditions.

The geometry boundary condition are displacement, slope.

78. Give example for non essential boundary conditions.

The natural boundary conditions are bending moment, shear force

79. What is meant by degrees of freedom?

When the force or reaction act at nodal point node is subjected to deformation. The deformation includes displacement rotation, and or strains. These are collectively known as degrees of freedom.

80. What is QST element?

Ten noded triangular elements are known as Quadratic strain triangle. It is also called as cubic displacement triangle.

UNIT 5

81. What meant by plane stress analysis?

Plane stress is defined to be a state of stress in which the normal stress and shear stress directed perpendicular to the plane are assumed to be zero.

82. Define plane strain analysis.

Plane strain is defined to be state of strain normal to the x,y plane and the shear strains are assumed to be zero.

83. What is truss element?

The truss elements are the part of a truss structure linked together by point joint which transmits only axial force to the element.

84. List the two advantages of post processing.

Required result can be obtained in graphical form.

Contour diagrams can be used to understand the solution easily and quickly.

85. What are the h and p versions of finite element method?

It is used to improve the accuracy of the finite element method. In h version, the order of polynomial approximation for all elements is kept constant and the numbers of elements are increased. In p version, the numbers of elements are maintained constant and the order of polynomial approximation of element is increased.

86. During discretization, mention the places where it is necessary to place a node?

Concentrated load acting point Cross-
section changing point Different
material inter junction point Sudden
change in point load

87. What is the difference between static and dynamic analysis?

Static analysis: The solution of the problem does not vary with time is known as static analysis

Example: stress analysis on a beam

Dynamic analysis: The solution of the problem varies with time is known as dynamic analysis

Example: vibration analysis problem.

88. What is meant by discretization and assemblage?

The art of subdividing a structure in to convenient number of smaller components is known as discretization. These smaller components are then put together. The process of uniting the various elements together is called assemblage.

89. What is Rayleigh-Ritz method?

It is integral approach method which is useful for solving complex structural problem, encountered in finite element analysis. This method is possible only if a suitable function is available.

90. What is Aspect ratio?

It is defined as the ratio of the largest dimension of the element to the smallest dimension. In many cases, as the aspect ratio increases the in accuracy of the solution increases. The conclusion of many researches is that the aspect ratio should be close to unity as possible.

91. What is essential boundary condition?

Primary boundary condition or EBC, Boundary condition which in terms of field variable is known as Primary boundary condition

92. Natural boundary conditions.

Secondary boundary natural boundary conditions which are in the differential form of field variable is known as secondary boundary condition.

93. How do you define two dimensional elements?

Two dimensional elements are define by three or more nodes in a two dimensional plane. The basic element useful for two dimensional analysis is the triangular element.

94. State the principles of virtual energy?

A body is in equilibrium if the internal virtual work equals the external virtual work for the every kinematically admissible displacement field.

96. What is non-homogeneous form?

When the specified values of dependent variables are non-zero, the boundary conditi said to be non-homogeneous.

97. What is homogeneous form?

When the specified values of dependent variables is zero, the boundary condition are said to be homogeneous.

98. Define initial value problem.

An initial value problem is one in which the dependent variable and possibly is derivatives are specified initially.

99. Define boundary value problem.

A differential equation is said to describe a boundary value problem if the dependent variable and its derivatives are required to take specified values on the boundary.

Reg. No. : | | | | | | | | | | | | |

Question Paper Code : 51652

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2014.

Sixth Semester

Mechanical Engineering

ME 2353/ME 63/10122 ME 605 — FINITE ELEMENT ANALYSIS

(Common to Automobile Engineering, Mechanical and Automation Engineering, Industrial Engineering and Management)

(Regulation 2008/2010)

Time : Three hours Maximum : 100 marks

(Any missing data may be suitably assumed)

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. What is meant by node or joint?

2. What is Rayleigh-Ritz method?

3. Define shape function.

4. What is a truss?

5. How do you define two dimensional elements?

6. What is QST (Quadratic strain Triangle) element?

7. What is meant by transverse vibrations?

8. Define dynamic analysis.

9. Write down the governing equation for two-dimensional steady state heat conduction.

10. Define streamline.

PART B — (5 × 16 = 80 marks)

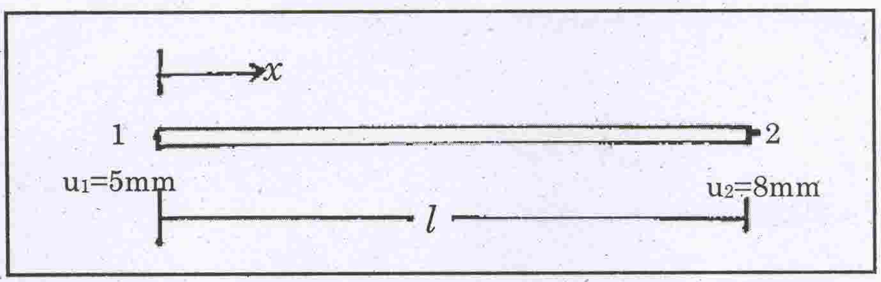
11. (a) List and briefly describe the general steps of the finite element method. (16)

Or

(b) The differential equation of a physical phenomenon is given by $\frac{d^2y}{dx^2} + y = 4x$, $0 \le x \le 1$

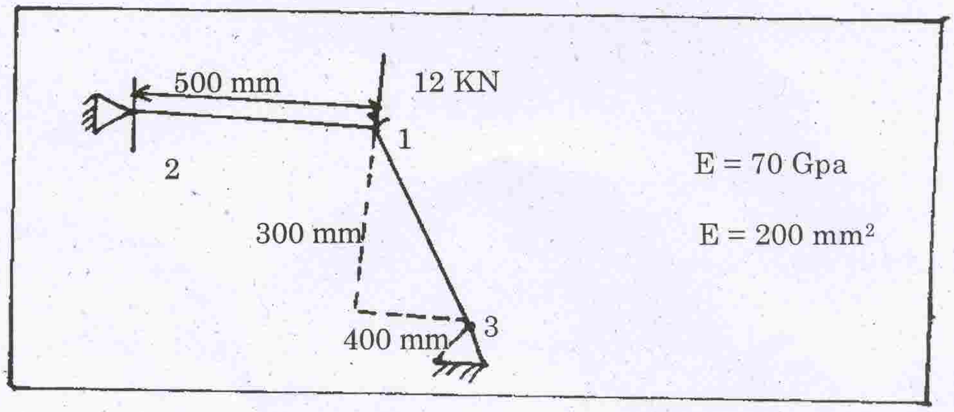
The boundary conditions are: $\begin{matrix} y(0) = 0 \\ y(1) = 1 \end{matrix}$. Obtain one term approximate solution by using Galerkin's method of weighted residuals. (16)

12. (a) A two noded truss element is shown in figure. The nodal displacements are $u_1 = 5\ mm$ and $u_2 = 8\ mm$. Calculate the displacement at $x = \frac{l}{4}, \frac{l}{3}$ and $\frac{l}{2}$. (16)

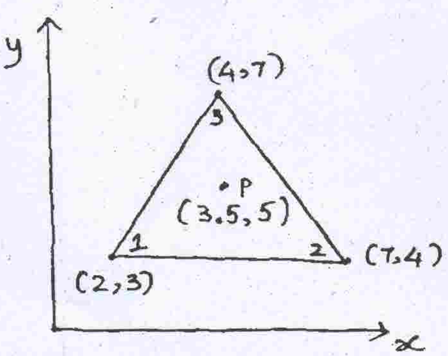


Or

(b) For the two bar truss shown in the figure, determine the displacements of node 1 and the stress in element 1-3.



13. (a) Determine the shape functions N_1, N_2 and N_3 at the interior point p for the triangular element shown in the figure. (16)



Or

(b) Determine the shape functions for a constant strain triangular (CST) element in terms of natural co-ordinate system. (16)

14. (a) Derive the equation of motion based on weak form for transverse vibration of a beam. (16)

Or

(b) Determine the eigen values and natural frequencies of a system whose stiffness and mass matrices are given below. (16)

$$[K] = \frac{2AE}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad M = \frac{\rho AL}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}.$$

15. (a) Derive a finite element equation for one dimensional heat conduction with free end convection. (16)

Or

(b) Derive the stiffness matrix and load vectors for fluid mechanics in two dimensional finite element. (16)

Question Paper Code : 31043

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2013.

Sixth Semester

Mechanical Engineering

080120032 — FINITE ELEMENT ANALYSIS

(Common to Automobile Engineering)

(Regulation 2008)

Time : Three hours | Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. State the advantages of Gaussian elimination technique.

2. What is Ritz method?

3. State the significance of shape function.

4. What is post processing? Give an example.

5. What is meant by primary and secondary node?

6. Distinguish between CST and LST elements.

7. Write the finite element equation used to analyse a two dimensional heat transfer problem.

8. State the applications of axisymmetric elements.

9. When are isoparameteric elements used?

10. What are force vectors? Give an example.

PART B — (5 × 16 = 80 marks)

11. (a) (i) Discuss the importance of FEA in assisting design process. (6)

(ii) Solve the ordinary differential equation

$$\left(\frac{d^2y}{dx^2}\right) + 10x^2 = 0 \text{ for } 0 \leq x \leq 1$$

Subject to the boundry conditions $y(0) = y(1) = 0$ using the Galerkin method with the trial functions $N_0(x) = 0$; $N_1(x) = x(1 - x^2)$. (10)

Or

(b) (i) Discuss the factors to be considered in descretisation of a domain. (10)

(ii) Solve the following equations using the gauss elimination method.

$2x_1 + 3x_2 + x_3 = 9$

$x_1 + 2x_2 + 3x_3 = 6$ (6)

$3x_1 + x_2 + 2x_3 = 0.$

12. (a) Fig.1 shows the pin-jointed configuration. Determine the nodal displacements and stresses in each element. (16)

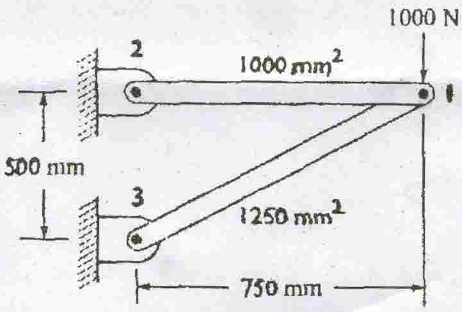


Fig.1

Or

(b) For the beam shown in Fig.2, determine

(i) The slopes at node 2 and 3 and

(ii) Vertical deflection at the mid-point of the distributed load. All the lemenets have E = 200 GPa and I = 5 × 10^6 mm^4. (16)

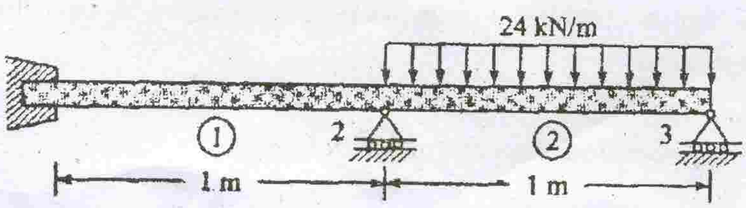
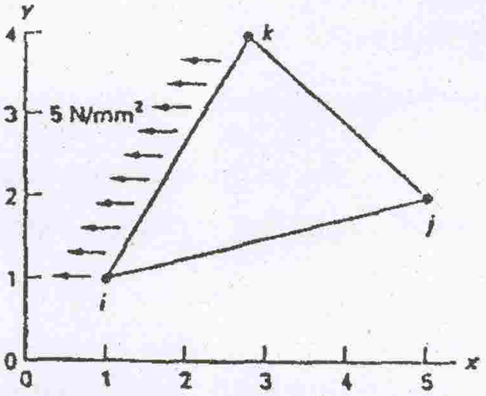


Fig. 2

13. (a) Compute the finite element equation for the LST element shown in Fig.3. (16)



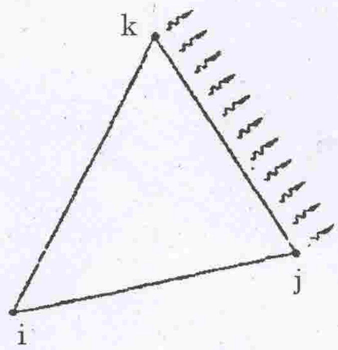
Thickness = 2 mm
E = 200 GPa
$\upsilon = 0.3$

Plane stress

Fig. 3

Or

(b) Determine the element matrices and vectors for the LST element shown in Fig.4.The nodal coordinates are i (1, 1), j (5, 2) and k (3, 5). Convection takes place along the edge jk.



K = 7.5 W/mm°C
h = 0.15 W/mm²°C
T_∞ = 30°C

Fig. 4

14. (a) Triangular elements are used for the stress analysis of plate subjected to inplane loads. The (x, y) coordinates of nodes i, j and k of an element are given by (2, 3), (4, 1), and (4, 5) mm respectively. The nodal displacements are given as :

u_1 = 2.0 mm, u_2 = 0.5 mm, u_3 = 3.0 mm

v_1 = 1.0 mm, v_2 = 0.0 mm, v_3 = 0.5 mm

Determine element stresses. Let E = 160 GPa, Poisson's ratio = 0.25 and thickness of the element t = 10 mm. (16)

Or

(b) (i) What are the non-zero strain and stress components of axisymmetric element? Explain. (4)

(ii) Derive the stiffness matrix of an axisymmetric element using potential approach. (12)

15. (a) (i) Consider the isoparametric quadrilateral element with nodes 1 – 4 at (5, 5), (11, 7), (12, 15), and (4, 10) respectively. Compute the Jacobian matrix and its determinant at the element centroid. (10)

(ii) Use Gaussian quadrature with two points to evaluate the integral

$$\int_{-1}^{1} \left(\cos x / \left(1 - x^2\right)\right) dx$$

The Gaussian points are ± 0.5774 and weights at the two points are equal to unity. (6)

Or

(b) The nodal displacements of a rectangular element having nodal coordinates (0, 0), (4, 0), (4,2) and (0,2) are : u_1 = 0 mm, v_1 = 0 mm, u_2 = 0.1 mm, v_2 = 0.05 mm, u_3 = 0.05 mm, v_3 = – 0.05, u_4 = 0 and v_4 = 0 mm respectively. Determine the stress matrix at r = 0 and s = 0 using the isoparametric formulation. Take E = 210 GPa and Poisson's ratio = 0.25.

Reg. No. : | | | | | | | | | | | | |

Question Paper Code : 21042

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2012.

Sixth Semester

Mechanical Engineering

080120032 –FINITE ELEMENT ANALYSIS

(Common to Automobile Engineering)

(Regulation 2008)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. Compare the Rayleigh Ritz method with Nodal Approximation method.
2. On what basis, collocation points are selected?
3. Define p-refinement.
4. What are the factors which govern the selection of nodes?
5. Define continuity.
6. When triangular element is preferred over quadrilateral elements?
7. Why variational formulation is called as weak formulation?
8. Differentiate between upper bound and lower bound solutions with an example.
9. What are the characteristics of shape functions?
10. What is meant by natural coordinate system?

PART B — ($5 \times 16 = 80$ marks)

11. (a) An alloy bar 1 m long and 200 mm^2 in cross section is fixed at one end is subjected to a compressive load of 20 kN. If the modulus of elasticity for the alloy is 100 GPa, find the decrease in the length of the bar. Also determine the stress developed and the decrease in length at 0.25 m, 0.5 m and 0.75m. Solve by collocation method. (16)

Or

(b) An alloy bar 1m long and 200 mm^2 in cross section is fixed at one end is subjected to a compressive load of 20 kN. If the modulus of elasticity for the alloy is 100 GPa, find the decrease in the length of the bar. Also determine the stress developed and the decrease in length at 0.25 m, 0.5 m and 0.75m. Solve by Ritz method. (16)

12. (a) A tapered bar of aluminum is having a length of 500 cm. The area of cross section at the fixed end is 80 cm^2 and the free end is 20 cm^2 with the variation of the sectional area as linear. The bar is subjected to an axial load of 10 kN at 240 mm from the fixed end. Calculate the maximum displacement and stress developed in the bar. (16)

Or

(b) A fixed beam AB of 5 m span carries a point load of 20 kN at a distance of 2m from A. Determine the slope and deflection under the load. (16)

Assume $EI = 10 \times 10^3 kN - m^2$.

13. (a) Find the temperature at a point P (2, 1.5) inside the triangular elements with nodal temperatures given as T_i=40°C, T_j=54°C and T_k=46°C. The nodal coordinates are 1(0, 0), J (4, 0.5) and K (3, 6). (16)

Or

(b) A circular aluminum rod is having a length of 700 cm. The area of cross section is 60cm^2. The bar is subjected to an axial compressive load of 50 kN at the fixed end. Calculate the maximum displacement and stress developed in the bar. Solve using two dimensional coordinates. (16)

14. (a) What are lagrangian interpolation functions? Using lagrangian polynomials derive the shape functions for ID quadratic element/cubic element. (16)

Or

(b) Derive constitutive matrix for axisymmetric analysis. (16)

15. (a) (i) Explain with an example of each of the following

(1) sub parametric element,

(2) iso parametric element,

(3) Super parametric element. (12)

(ii) Define bandwidth in finite element analysis and its significance in the solution of global system matrices? (4)

Or

(b) (i) Derive the shape function for the one dimensional quadratic element in Natural Coordinates? (8)

(ii) Derive the stiffness matrix for heat transfer using shape functions for a four noded quadrilateral element. (8)

Reg. No. : | | | | | | | | | | | | |

Question Paper Code : 13055

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2012.

Sixth Semester

Automobile Engineering

080120032 — FINITE ELEMENT ALALYSIS

(Common to Mechanical Engineering)

(Regulation 2008)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Give two sketches of structures that have both discrete elements and continuum.

2. Write about the Galerkin's residual method.

3. State the principle of minimum potential energy theorem.

4. What is the need for developing the overall stiffness matrix of the entire structure in terms of its global coordinate system? Give an example.

5. List out the limitations of CST element.

6. State Fourier's law of heat conduction used in FEA.

7. List the importance of two dimensional plane stress and plane strain analysis.

8. Give four examples of practical application of axisymmetric elements.

9. What is the salient feature of an isoparametric element?

10. Give the Lagrange equation of motion and obtain the equation of motion of a two degree of freedom system.

PART B — (5 × 16 = 80 marks)

11. (a) The differential equation for a phenomenon is given by $\frac{d^2y}{dx^2} + 500x^2 = 0; 0 \le x \le 5$. The boundary condition are $y(0) = 0$, $y(5) = 0$. Find the approximate solution using any classical technique. Start with minimal possible approximate solution.

Or

(b) (i) List and briefly describe the general steps of finite element method. (6)

(ii) Derive an equation to find the displacement at node 2 of fixed- fixed beam subjected to axial load P at node 2 using Rayleigh -Ritz method. (10)

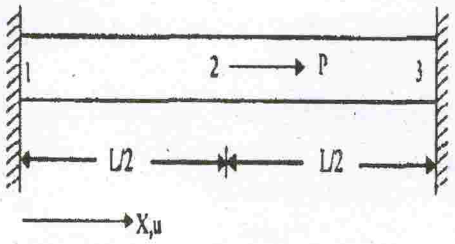


Figure 11(b)

12. (a) For the plane trusses supported by the spring at node 1 in figure 12 (a), determine the nodal displacement and stresses in each element. Let E = 210GPa and A = $5.0 \times 10^{-4} m^2$.

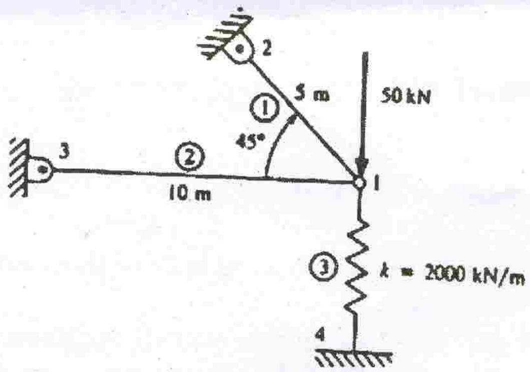


Figure 12 (a)

Or

(b) A concentrated load P = 50 kN is applied at the centre of a fixed beam of length 3 m, depth 200 mm and width 120 mm. Calculate the deflection and slope at the midpoint. Assume E = 2×10^5 N/mm²

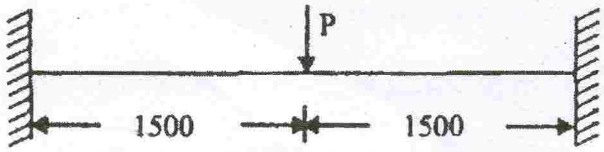


Figure 12 (b)

13. (a) Calculate displacements and stress in the given triangular plate, fixed along one edge and subjected to concentrated load at its free end. Take E = 70 GPa, thickness of the plate = 10 mm and poisson's ratio = 0.3

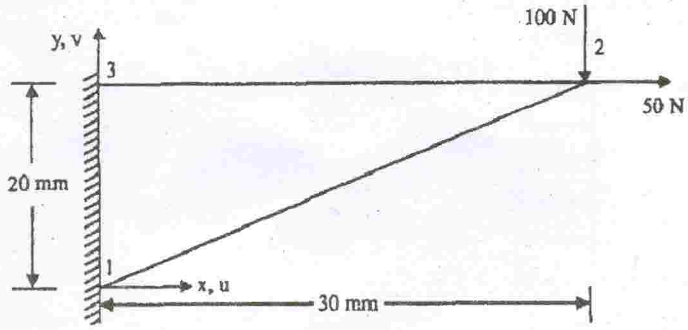


Figure 13(a)

Or

(b) A circular fin of 40 mm diameter is fixed to a base maintained at 50°C as shown in figure 13(b). The fin is insulated on the surface except the end face which is exposed to air at 25°C. The length of the pin is 1000 mm, the fin is made of metal with thermal conductivity of 37W/m K. If the convection heat coefficient with air is 15 W/m^2 K. Find the temperature distribution at 250,500,750 and 1000 mm from base.

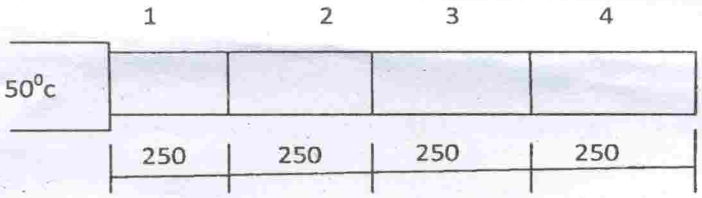


Figure 13(b)

14. (a) For the plane strain elements shown in figure 14 (a), the nodal displacements are given as $u_1 = 0.005$ mm, $v_1 = 0.002$ mm, $u_2 = 0.0$, $v_2 = 0.0$, $u_3 = 0.005$ mm, $v_3 = 0.30$ mm. Determine the element stresses and the principle angle. Take E = 70 GPa and Poisson's ratio = 0.3 and use unit thickness for plane strain. All coordinates are in mm.

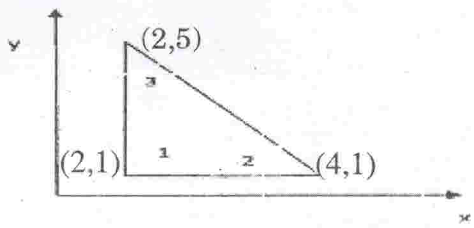


Figure 14 (a)

Or

(b) Establish the traction force vector and estimate the nodal forces corresponding to a uniform radial pressure of 7 bar acting on an axisymmetric element as shown in figure 14 (b). Take E = 200 GPa and Poisson's ratio = 0.25

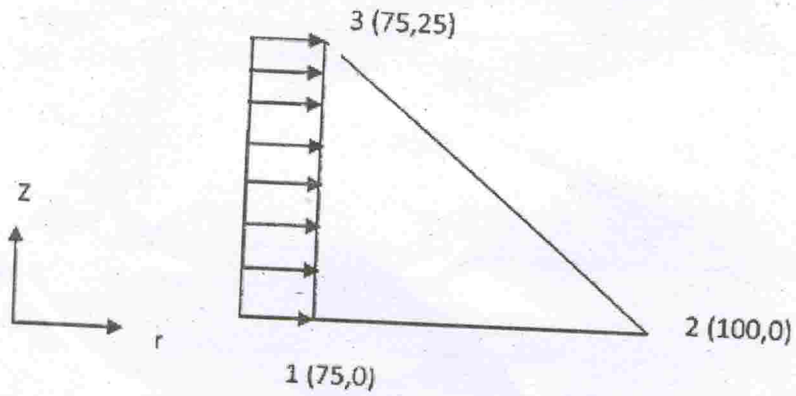


Figure 14(b)

15. (a) Derive the element characteristics of a four node quadrilateral element.

Or

(b) Evaluate the intergrals

(i) $I = \int_{-1}^{1} [x^2 + \cos(\frac{x}{2})]\, dx$

(ii) $I = \int_{-1}^{1} [3^x - x]\, dx$

Using appropriate Gaussian Quadrature.

Reg. No. : | | | | | | | | | | | |

Question Paper Code : D 2308

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2010.

Seventh Semester

Mechanical Engineering

ME 1401 — INTRODUCTION OF FINITE ELEMENT ANALYSIS

(Common to Automobile Engineering and Mechatronics Engineering)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A — $(10 \times 2 = 20$ marks)

1. What is the limitation of using a finite difference method?

2. List the various methods of solving boundary value problems.

3. Write down the interpolation function of a field variable for three-node triangular element.

4. Highlight at least two rules to guide the placement of the nodes when obtaining approximate solution to a differential equation.

5. List the properties of the global stiffness matrix.

6. List the characteristics of shape functions.

7. What do you mean by the terms : c^0, c^1 and c^n continuity?

8. Write down the nodal displacement equations for a two dimensional triangular elasticity element.

9. List the required conditions for a problem assumed to be axisymmetric.

10. Name a few boundary conditions involved in any heat transfer analysis.

PART B — (5 × 16 = 80 marks)

11. (a) Discuss the following methods to solve the given differential equation :

$$EI\frac{d^2y}{dx^2} - M(x) = 0$$

with the boundary conditions $y(0) = 0$ and $y(H) = 0$

(i) Variational method

(ii) Collocation method.

Or

(b) For the spring system shown in Figure 1, calculate the global stiffness matrix, displacements of nodes 2 and 3, the reaction forces at node 1 and 4. Also calculate the forces in the spring 2. Assume, $k_1 = k_3 = 100$ N/m, $k_2 = 200$ N/m, $u_1 = u_4 = 0$ and P = 500 N.

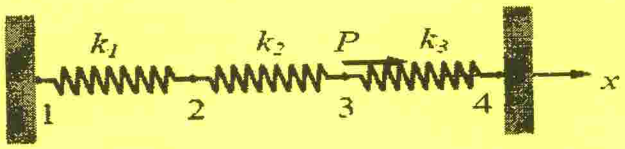


Figure 1 Spring System Assembly

12. (a) Determine the joint displacements, the joint reactions, element forces and element stresses of the given truss elements.

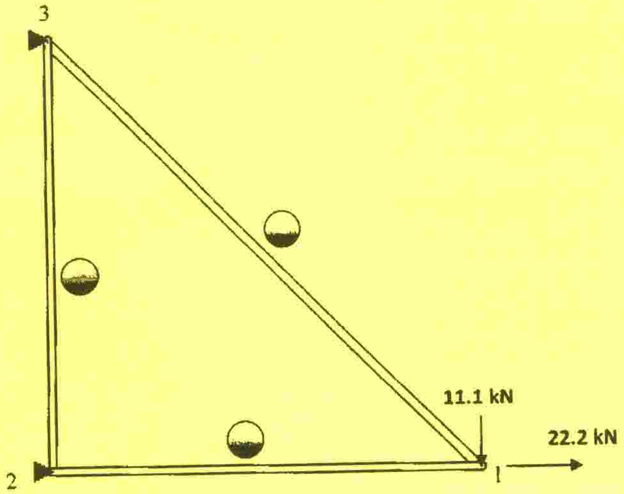


Figure 2 Truss with applied load

Table 1 : Element Property Data

Element	A cm^2	E N/m^2	L m	Global Node Connection	α Degree
1	32.2	6.9e10	2.54	2 to 3	90
2	38.7	20.7e10	2.54	2 to 1	0
3	25.8	20.7e10	3.59	1 to 3	135

Or

(b) Derive the interpolation function for the one dimensional linear element with a length 'L' and two nodes, one at each end, designated as 'i' and 'j'. Assume the origin of the coordinate system is to the left of node 'i'.

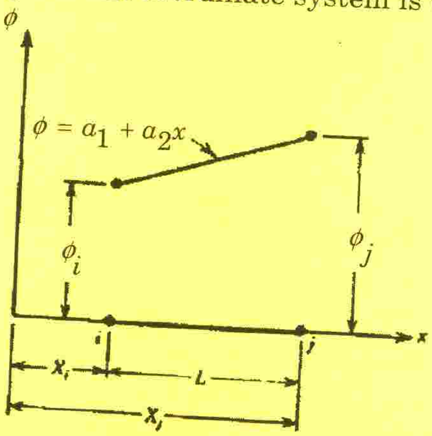


Figure 3 the one-dimensional linear element

13. (a) Determine three points on the 50°C contour line for the rectangular element shown in the Figure 4. The nodal values are $\Phi_i = 42^\circ\text{C}$, $\Phi_j = 54^\circ\text{C}$, $\Phi_k = 56^\circ\text{C}$ and $\Phi_m = 46^\circ\text{C}$.

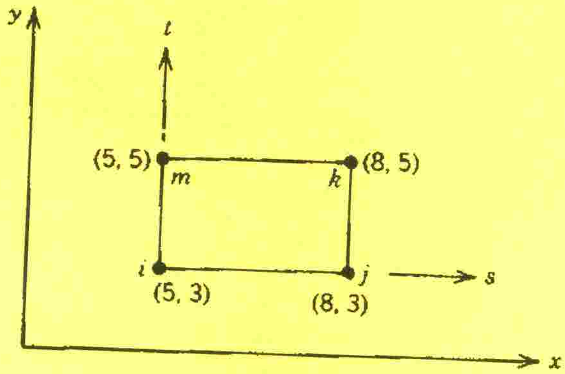


Figure 4 Nodal coordinates of the rectangular element

Or

(b) The simply supported beam shown in Figure 5 is subjected to a uniform transverse load, as shown. Using two equal-length elements and work-equivalent nodal loads obtain a finite element solution for the deflection at mid-span and compare it to the solution given by elementary beam theory.

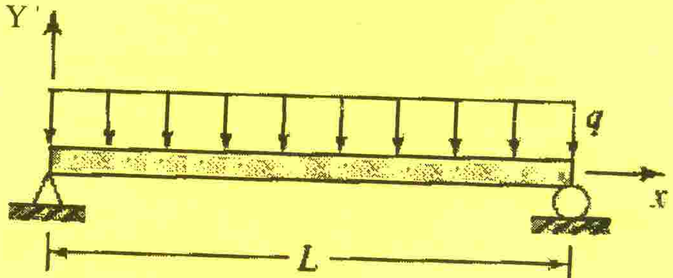


Figure 5 uniformly loaded beam

14. (a) For the plane strain element shown in the Figure 6, the nodal displacements are given as : $u_1 = 0.005$ mm, $u_2 = 0.002$ mm, $u_3 = 0.0$ mm, $u_4 = 0.0$ mm, $u_5 = 0.004$ mm, $u_6 = 0.0$ mm. Determine the element stresses. Take E = 200 Gpa and $\gamma = 0.3$. Use unit thickness for plane strain.

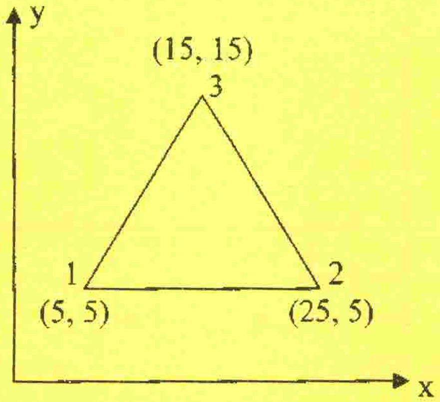


Figure 6 Triangular Element

Or

(b) Determine the element stiffness matrix and the thermal load vector for the plane stress element shown in Figure 7. The element experiences 20°C increase in temperature. Take E = 15e6 N/cm², $\gamma = 0.25$, t = 0.5 cm and a = 6e–6/°C.

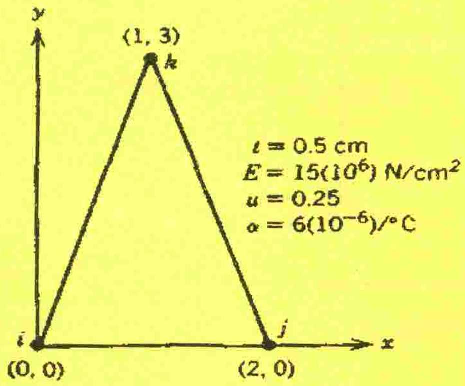


Figure 7 Triangular elastic elements

15. (a) Use Gaussian quadrature to obtain an exact value of the integral.

$$I = \int_{-1}^{1}\int_{-1}^{1} (r^3 - 1)(s - 1)^2 \, dr\, ds .$$

Or

(b) Define the following terms with suitable examples :

(i) Plane stress, Plane strain

(ii) Node, Element and Shape functions

(iii) Iso-parametric element

(iv) Axisymmetric analysis.