Finite Element Formulation for Plates
- Handout 3 -

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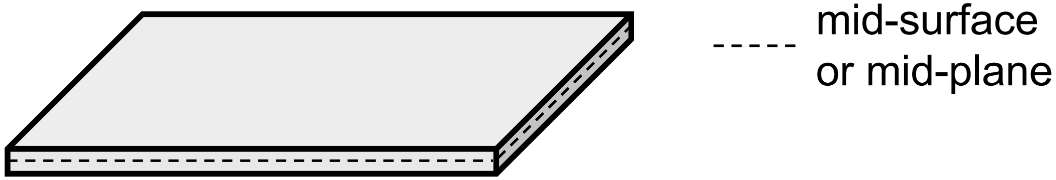
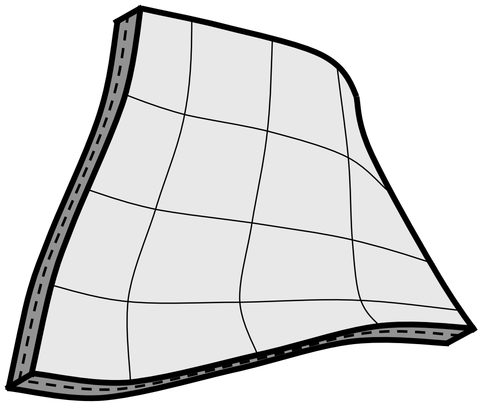
Completed Version

Definitions

- A plate is a three dimensional solid body with
  - one of the plate dimensions much smaller than the other two
  - zero curvature of the plate *mid-surface* in the reference configuration
  - loading that causes bending deformation

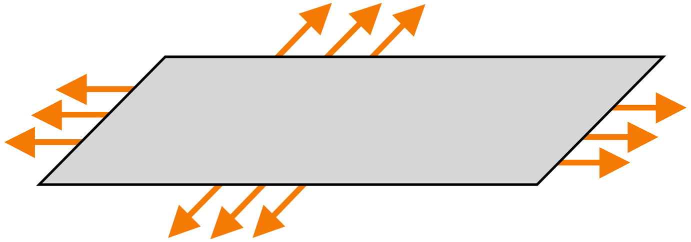


- A shell is a three dimensional solid body with
  - one of the shell dimensions much smaller than the other two
  - non-zero curvature of the shell *mid-surface* in the current configuration
  - loading that causes bending and stretching deformation

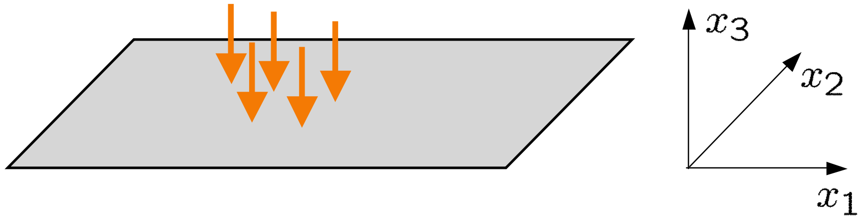


Membrane versus Bending Response

- For a plate membrane and bending response are decoupled



loading in the plane of the mid-surface
(membrane response active)



loading orthogonal to the mid-surface
(bending response active)

  - For most practical problems membrane and bending response can be investigated independently and later superposed
  - Membrane response can be investigated using the two-dimensional finite elements introduced in 3D7
  - Bending response can be investigated using the plate finite elements introduced in this handout

- For plate problems involving large deflections membrane and bending response are coupled
  - For example, the stamping of a flat sheet metal into a complicated shape can only be simulated using shell elements

Overview of Plate Theories

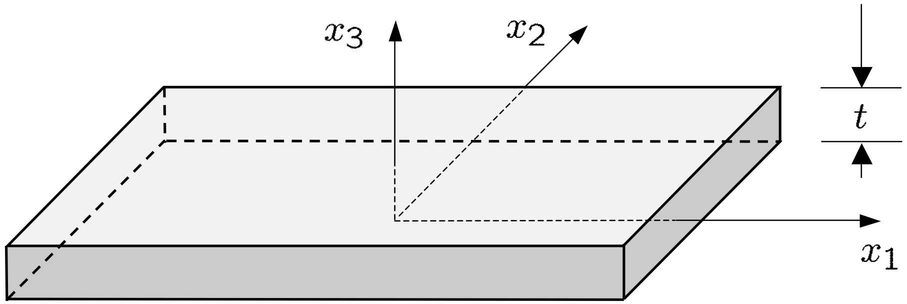
- In analogy to beams there are several different plate theories

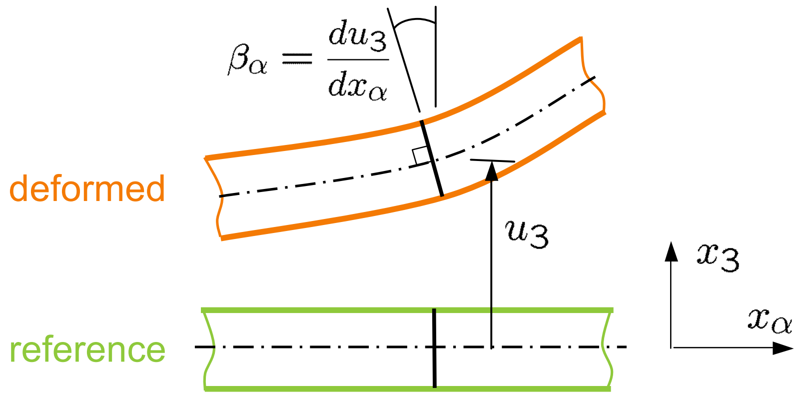
	thick	thin	very thin
Lengt / thickness	~5 to ~10	~10 to ~100	> ~100
physical characteristics	transverse shear deformations $\epsilon_{13} \neq 0$	negligible transverse shear deformations $\epsilon_{13} \approx 0$	geometrically non-linear

- The extension of the Euler-Bernoulli beam theory to plates is the Kirchhoff plate theory
  - Suitable only for thin plates
- The extension of Timoshenko beam theory to plates is the Reissner-Mindlin plate theory
  - Suitable for thick and thin plates
  - As discussed for beams the related finite elements have problems if applied to thin problems
- In very thin plates deflections always large
  - Geometrically nonlinear plate theory crucial (such as the one introduced for buckling of plates)

Kinematics of Kirchhoff Plate -1-

- Assumed displacements during loading





undeformed and deformed geometries
along one of the coordinate axis

- Kinematic assumption: Material points which lie on the mid-surface normal remain on the mid-surface normal during the deformation

- Kinematic equations
  - In-plane displacements

$$u_\alpha(x_1, x_2, x_3) = -\beta_\alpha(x_1, x_2)x_3 \quad \text{with } -\frac{t}{2} \leq x_3 \leq \frac{t}{2}$$

    - In this equation and in following all Greek indices take only values 1 or 2
    - It is assumed that rotations are small $(\sin(\beta_\alpha) \approx \beta_\alpha)$
  - Out-of-plane displacements

$$u_3(x_1, x_2, x_3) = u_3(x_1, x_2)$$

Kinematics of Kirchhoff Plate -2-

- Introducing the displacements into the strain equations of three-dimensional elasticity leads to

$$\left(\text{for 3d, } \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\right)$$

  - Axial strains and in-plane shear strain

$$\epsilon_{\alpha\gamma} = \underbrace{-\frac{1}{2}(u_{3,\alpha\gamma} + u_{3,\gamma\alpha})}_{\text{curvature matrix } \kappa_{\alpha\gamma}} x_3 = \kappa_{\alpha\gamma} x_3$$

  - All other strain components are zero
    - Out-of-plane shear

$$\epsilon_{\alpha 3} = 0$$

    - Through-the-thickness strain (no stretching of the mid-surface normal during deformation)

$$\epsilon_{33} = 0$$

Weak Form of Kirchhoff Plate -1-

- The plate strains introduced into the internal virtual work expression of three-dimensional elasticity

$$\int_\Omega \int_{-t/2}^{t/2} \sigma_{ij}\epsilon_{ij}\, dx_3 dx_\Omega = \int_\Omega \int_{-t/2}^{t/2} \sigma_{\alpha\gamma}\epsilon_{\alpha\gamma}(v)\, dx_3 d\Omega = \int_\Omega m_{\alpha\gamma}\kappa_{\alpha\gamma}(v)\, d\Omega$$

    - Note that the summation convention is used (summation over repeated indices)
    - Definition of bending moments $m_{\alpha\gamma} = \int_{-t/2}^{t/2} \sigma_{\alpha\gamma} x_3\, dx_3$

- External virtual work
  - Distributed surface load

$$\int_\Omega qv\, d\Omega$$

    - For other type of external loadings see TJR Hughes book

- Weak form of Kirchhoff Plate

$$\int_\Omega m_{\alpha\gamma}\kappa_{\alpha\gamma}(v)\, d\Omega = \int_\Omega qv\, d\Omega + \text{boundary terms}$$

    - Boundary terms only present if force/moment boundary conditions present

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Weak Form of Kirchhoff Plate -2-

- Moment and curvature matrices

$$m_{\alpha\beta} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad \kappa = \begin{bmatrix} -u_{3,11} & -u_{3,12} \\ -u_{3,21} & -u_{3,22} \end{bmatrix}$$

  - Both matrices are symmetric

Constitutive equation (Hooke's law)

- Plane stress assumption for thin plates $(\sigma_{33} = 0)$ must be used
- Hooke's law for three-dimensional elasticity (with Lamé constants)

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \qquad \text{for } i, j = 1, 2, 3$$

  - Through-the-thickness strain can be determined using plane stress assumption

$$\sigma_{33} = 0 = \lambda(\epsilon_{\alpha\alpha} + \epsilon_{33}) + 2\mu\epsilon_{33} \quad \Rightarrow \epsilon_{33} = \frac{-\lambda}{\lambda + 2\mu}\epsilon_{\alpha\alpha}$$

  - Introducing the determined through-the-thickness strain ϵ_{33} back into the Hooke's law yields the Hooke's law for plane stress

$$\sigma_{\alpha\gamma} = \frac{2\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\epsilon_{\gamma\gamma} + 2\mu\epsilon_{\alpha\gamma} \qquad \text{for } \alpha, \beta, \gamma = 1, 2$$

Weak Form of Kirchhoff Plate -3-

- Integration over the plate thickness leads to

$$\begin{bmatrix} m_{11} \\ m_{22} \\ m_{12} \end{bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix}$$

  - Note the change to Young's modulus and Poisson's ratio
  - The two sets of material constants are related by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

Finite Element Discretization

- The problem domain is partitioned into a collection of pre-selected finite elements (either triangular or quadrilateral)

- On each element displacements and test functions are interpolated using shape functions and the corresponding nodal values

$$u_3(x_1, x_2) = \sum_{K=1}^{NP} N^K(x_1, x_2) u_3^K$$

$$v(x_1, x_2) = \sum_{K=1}^{NP} N^K(x_1, x_2) v^K$$

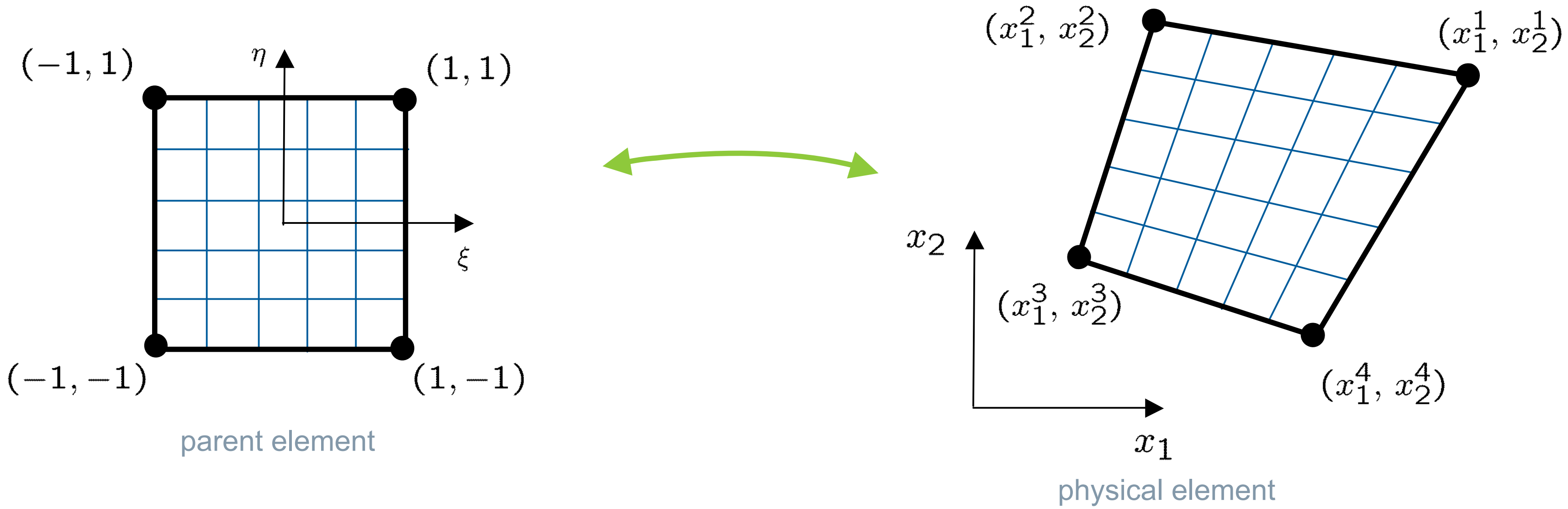
  - Shape functions N^K
  - Nodal values u_3^K, v^K

- To obtain the FE equations the preceding interpolation equations are introduced into the weak form
  - Similar to Euler-Bernoulli Beam the internal virtual work depends on the second order derivatives of the deflection u_3 and v virtual deflection
  - C^1-continuous smooth shape functions are necessary in order to render the internal virtual work computable

Review: Isoparametric Shape Functions -1-

- In finite element analysis of two and three dimensional problems the isoparametric concept is particularly useful

Isoparametric mapping of a four-node quadrilateral



- Shape functions are defined on the parent (or master) element
  - Each element on the mesh has exactly the same shape functions
- Shape functions are used for interpolating the element coordinates and deflections

$$x_\alpha = \sum_{K=1}^{NP} N^K(\xi,\eta) x_\alpha^K$$

Review: Isoparametric Shape Functions -2-

- In the computation of field variable derivatives the Jacobian of the mapping has to be considered

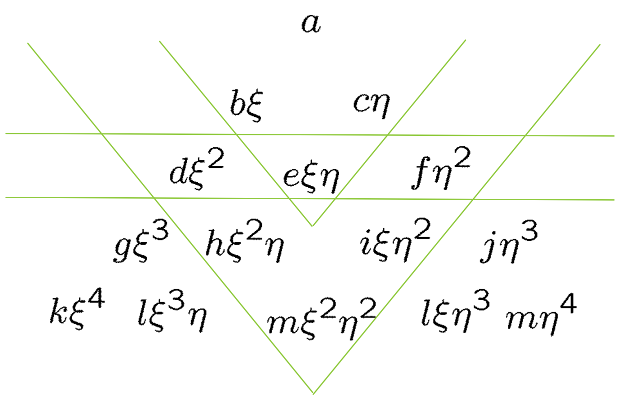
$$\begin{bmatrix} \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_3}{\partial x_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x_1} & \frac{\partial \eta}{\partial x_1} \\ \frac{\partial \xi}{\partial x_1} & \frac{\partial \eta}{\partial x_2} \end{bmatrix}}_{\boldsymbol{J}^{-1}} \begin{bmatrix} \frac{\partial u_3}{\partial \xi} \\ \frac{\partial u_3}{\partial \eta} \end{bmatrix} \qquad \text{(chain rule)}$$

  - The Jacobian is computed using the coordinate interpolation equation

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_2}{\partial \xi} \\ \frac{\partial x_1}{\partial \eta} & \frac{\partial x_2}{\partial \eta} \end{bmatrix}$$

Shape Functions in Two Dimensions -1-

- In 3D7 shape functions were derived in a more or less ad hoc way
- Shape functions can be systematically developed with the help of the Pascal's triangle (which contains the terms of polynomials, also called monomials, of various degrees)



Pascal's triangle
(with constants a, b, c, d, …)

- Triangular elements
  - Three-node triangle linear interpolation

$$u_3 = a + b\xi + c\eta$$

  - Six-node triangle quadratic interpolation

$$u_3 = a + b\xi + c\eta + d\xi^2 + e\xi\eta + f\eta^2$$

- Quadrilateral elements
  - Four-node quadrilateral bi-linear interpolation

$$u_3 = a + b\xi + c\eta + e\xi\eta$$

  - Nine-node quadrilateral bi-quadratic interpolation

$$u_3 = a + b\xi + c\eta + d\xi^2 + e\xi\eta + f\eta^2 + h\xi^2\eta + i\xi\eta^2 + m\xi^2\eta^2$$

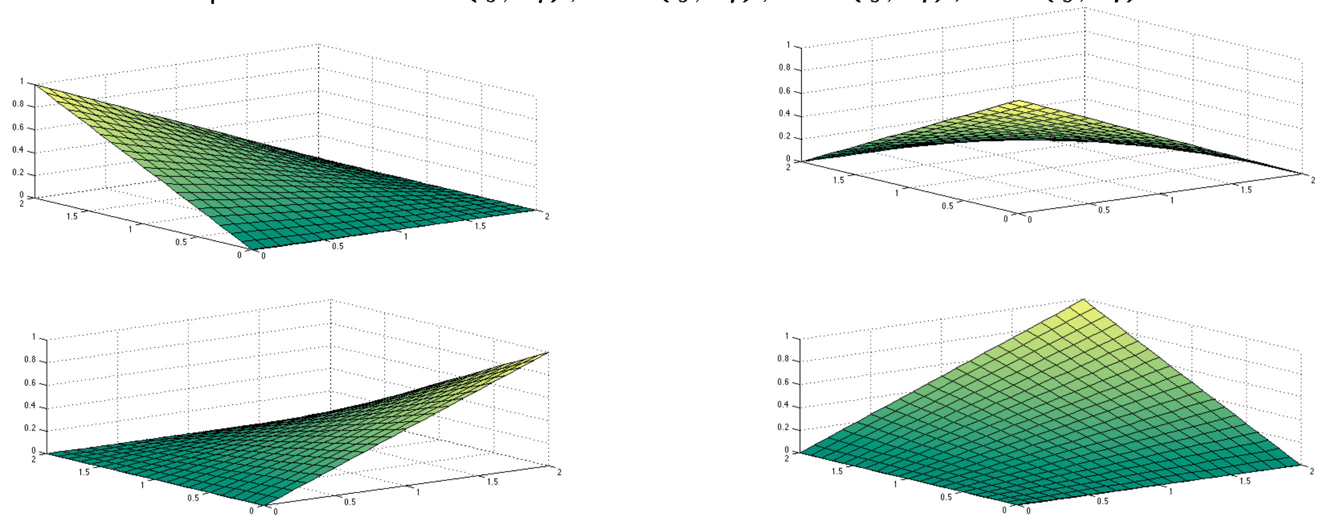
- It is for the convergence of the finite element method important to use only complete polynomials up to a certain desired polynomial order

Shape Functions in Two Dimensions -2-

- The constants a, b, c, d, e, … in the polynomial expansions can be expressed in dependence of the nodal values
  - For example in case of a a four-node quadrilateral element

$$u_3 = a + b\xi + c\eta + e\xi\eta \quad \Leftrightarrow \quad u_3 = N^1(\xi,\,\eta)u_3^1 + N^2(\xi,\,\eta)u_3^2 + N^3(\xi,\,\eta)u_3^3 + N^4(\xi,\,\eta)u_3^4$$

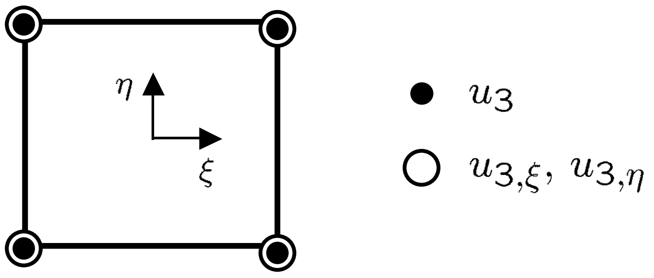
  - with the shape functions $N^1(\xi,\,\eta),\ N^2(\xi,\,\eta),\ N^3(\xi,\,\eta),\ N^4(\xi,\,\eta)$

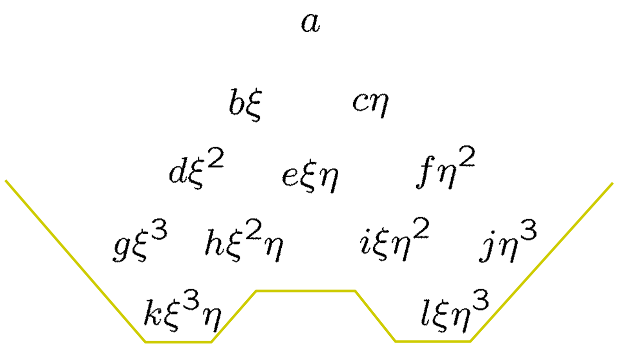


- As mentioned the plate internal virtual work depends on the second derivatives of deflections and test functions so that C^1-continuous smooth shape functions are necessary
  - It is not possible to use the shape functions shown above

Early Smooth Shape Functions -1-

- For the Euler-Bernoulli beam the Hermite interpolation was used which has the nodal deflections and slopes as degrees-of-freedom
- The equivalent 2D element is the Adini-Clough quadrilateral (1961)
  - Degrees-of-freedom are the nodal deflections and slopes
  - Interpolation with a polynomial with 12 (=3x4) constants





monomials

$$u_3 = a + b\xi + c\eta + d\xi^2 + e\xi\eta + f\eta^2 + g\xi^3 + h\xi^2\eta + i\xi\eta^2 + j\eta^3 + k\xi^3\eta + l\xi\eta^3$$

- Surprisingly this element does not produce C^1- continuous smooth interpolation (explanation on next page)

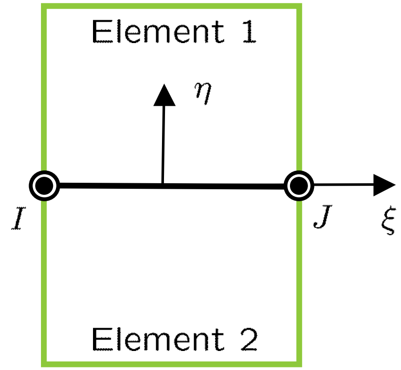
Early Smooth Shape Functions -2-

- Consider an edge between two Adini-Clough elements
  - For simplicity the considered boundary is assumed to be along the $\xi-$ axis in both elements
    - The deflections and slopes along the edge are

$$u_3|_{\eta=0} = a + b\xi + d\xi^2 + g\xi^3$$

$$u_{3,\xi}|_{\eta=0} = b + 2d\xi + 3g\xi^2$$

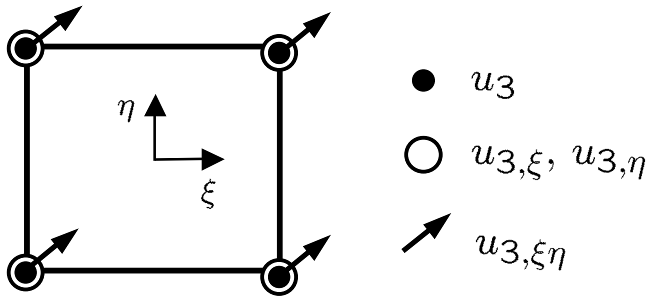
$$u_{3,\eta}|_{\eta=0} = c + e\xi + h\xi^2 + k\xi^3$$



    - so that there are 8 unknown constants in these equations
  - If the interpolation is smooth, the deflection and the slopes in both elements along the edge have to agree
  - It is not possible to uniquely define a smooth interpolation between the two elements because there are only 6 nodal values available for the edge (displacements and slopes of the two nodes). There are however 8 unknown constants which control the smoothness between the two elements.
- Elements that violate continuity conditions are known as “nonconforming elements”. The Adini-Clough element is a nonconforming element. Despite this deficiency the element is known to give good results

Early Smooth Shape Functions -3-

- **Bogner-Fox-Schmidt quadrilateral (1966)**
  - Degrees-of-freedom are the nodal deflections, first derivatives and second mixed derivatives



$$
\begin{array}{c}
a \\
b\xi \quad c\eta \\
d\xi^2 \quad e\xi\eta \quad f\eta^2 \\
g\xi^3 \quad h\xi^2\eta \quad i\xi\eta^2 \quad j\eta^3 \\
k\xi^3\eta \quad l\xi^2\eta^2 \quad m\xi\eta^3 \\
n\xi^3\eta^2 \quad o\xi^2\eta^3 \\
p\xi^3\eta^3
\end{array}
$$

monomials

  - This element is conforming because there are now 8 parameters on a edge between two elements in order to generate a C^1-continuous function

- **Problems**
  - Physical meaning of cross derivatives not clear
  - At boundaries it is not clear how to prescribe the cross derivatives
  - The stiffness matrix is very large (16x16)

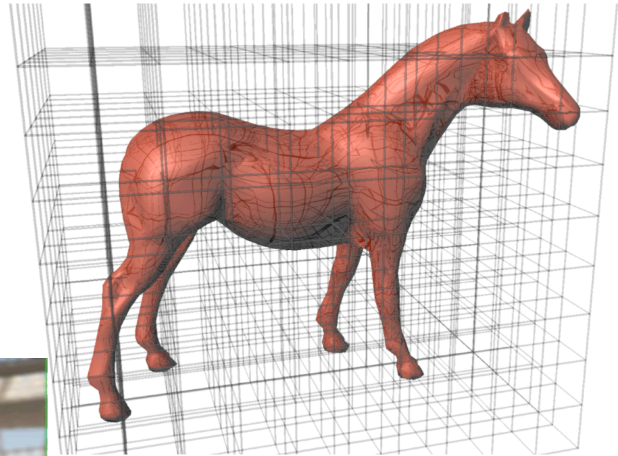
- **Due to these problems such elements are not widely used in present day commercial software**

New Developments in Smooth Interpolation

- Recently, research on finite elements has been reinvigorated by the use of smooth surface representation techniques from computer graphics and geometric design

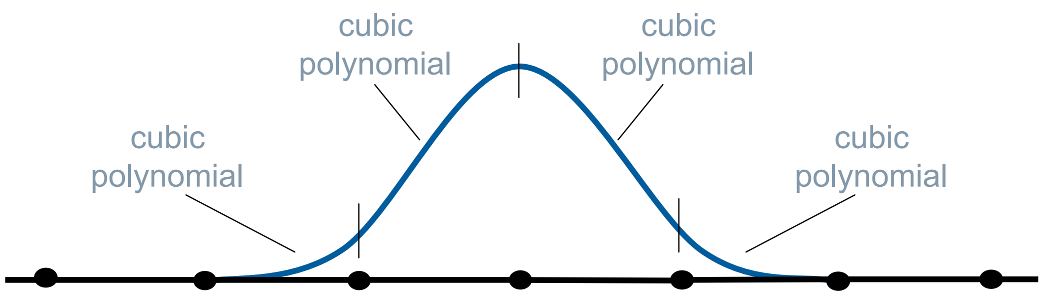
    - Smooth surfaces are crucial for computer graphics, gaming and geometric design

Fifa 07, computer game



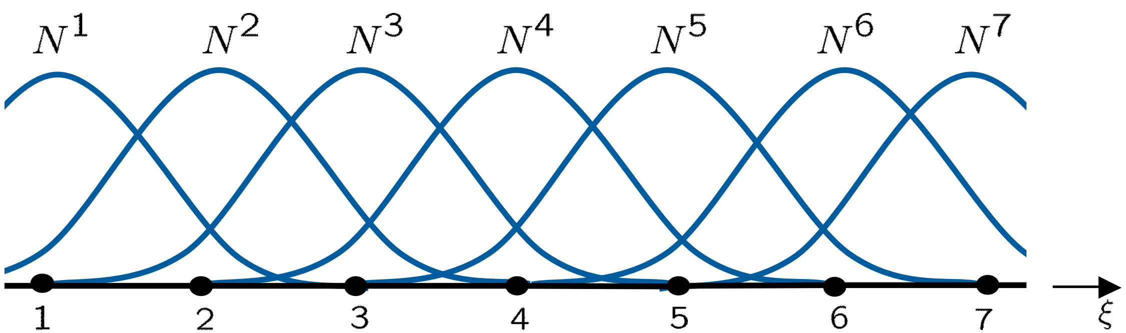
Splines - Piecewise Polynomial Curves

- Splines are piecewise polynomial curves for smooth interpolation
  - For example, consider cubic spline shape functions



- Each cubic spline is composed out of four cubic polynomials; neighboring curve segments are C^2 continuously connected (i.e., continuous up to second order derivatives)

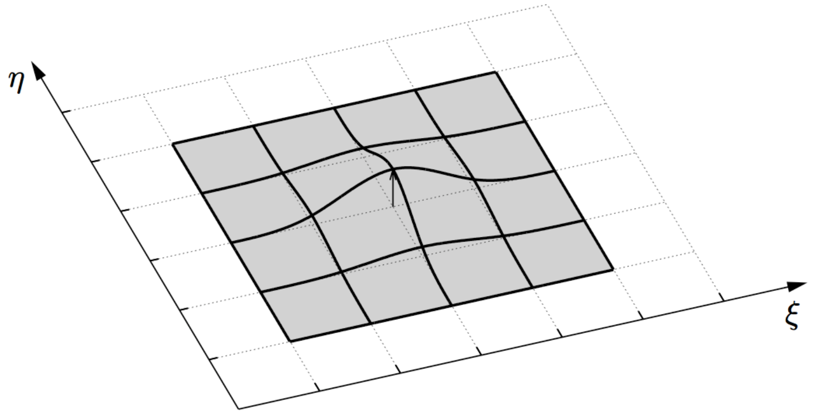
- An interpolation constructed out of cubic spline shape functions is C^2 continuous



$$u_3 = \sum_K N^K u_3^K$$

Tensor Product B-Spline Surfaces -1-

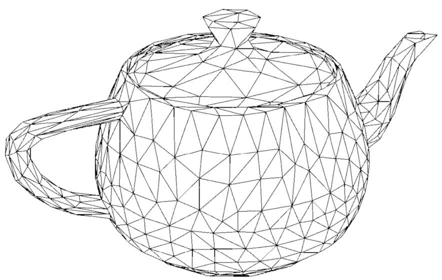
- A b-spline surface can be constructed as the “tensor-product” of b-spline curves



$$N(\xi,\eta) = N(\xi) \times N(\eta)$$

two dimensional — one dimensional — one dimensional

- Tensor product b-spline surfaces are only possible over “regular” meshes
- A presently active area of research are the b-spline like surfaces over “irregular” meshes
  - The new approaches developed will most likely be available in next generation finite element software



irregular mesh

spline like surface generated on irregular mesh