The Finite Element Method for Fluid Mechanics

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Chapter 1

Finite Elements for Coercive Problems

1.1 Some notions of Functional Analysis

1.1.1 Sobolev spaces

Spaces $L^2(\Omega)$ and $H^1(\Omega)$. Unless otherwise stated, Ω always denotes in these lecture notes an open bounded set in $\mathbb{R}^d$, $d = 1, 2$ or 3 and has a smooth boundary $\partial\Omega$. The outward normal to $\partial\Omega$ is denoted by $\boldsymbol{n}$.

The space $L^2(\Omega)$ denotes the space of functions whose square is integrable for the Lebesgue measure. It is equipped with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx,$$

and the associated norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2\, dx\right)^{1/2}.$$

The space $\Big(L^2(\Omega), \|\cdot\|_{L^2}\Big)$ is a Hilbert space.

$L^2_0(\Omega)$ is defined as the space of $L^2(\Omega)$ functions with zero mean value

$$L^2_0(\Omega) = \Big\{u \in L^2(\Omega), \int_\Omega u dx = 0\Big\}. \tag{1.1}$$

When u is a sufficiently smooth function, it satisfies

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int_\Omega \varphi(x)\frac{\partial u}{\partial x_i}(x)\, dx, \quad \forall \varphi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with a compact support in Ω, and therefore φ vanishes on $\partial\Omega$. However, not all functions of interest to this course are necessarily smooth. Therefore, the notion of a "weak derivative" is next introduced.

A function $u \in L^2(\Omega)$ is said to be *weakly differentiable* if there exist functions $w_i \in L^2(\Omega)$, $i = 1..d$, such that

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int_\Omega w_i(x)\varphi(x)\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

In that case, the functions w_i are called the *weak partial derivatives* of u and are denoted here by $\dfrac{\partial u}{\partial x_i}$. A practical way to check if a function $u \in L^2(\Omega)$ is weakly differentiable is to check whether it can be proven that there exists a constant $C > 0$ such that for all $\varphi \in C_0^\infty(\Omega)$ and for $i = 1..d$,

$$\left| \int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \right| \leq C \|\varphi\|_{L^2(\Omega)}.$$

The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \text{ such that } \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1..d \right\}.$$

Introducing the scalar product

$$(u, v)_{H^1(\Omega)} = \int_\Omega u(x)v(x)\, dx + \int_\Omega \boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} v(x)\, dx$$

and the corresponding norm

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u(x)|^2\, dx + \int_\Omega |\boldsymbol{\nabla} u(x)|^2\, dx \right)^{1/2}$$

leads to the definition of the Hilbert space $\Big(H^1(\Omega), \|\cdot\|_{H^1}\Big)$.

Spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and Poincaré inequality. In general, the trace on $\partial\Omega$ of a function $u \in L^2$ – that is, the value of u on the boundary $\partial\Omega$ — cannot be defined (for example, consider the function $u(x) = sin(1/x)$ in $\Omega = [0,\ 1]$; its value at $x = 0$ cannot be determined). On the contrary, the trace of a function $u \in H^1(\Omega)$ always exists and is denoted here by $u|_{\partial\Omega}$.

The space of $H^1(\Omega)$ functions with a vanishing trace on $\partial\Omega$ is denoted by $H_0^1(\Omega)$. The quantity

$$|v|_1 = \left(\int_\Omega |\boldsymbol{\nabla} u(x)|^2\, dx \right)^{1/2} = \|\boldsymbol{\nabla} u\|_{L^2(\Omega)},$$

is a norm in $H_0^1(\Omega)$. However, it is not a norm in $H^1(\Omega)$ since, for example, any constant function $u \neq 0$ is such that $|u|_1 = 0$. Furthermore, $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $H_0^1(\Omega)$. This is a consequence of the *Poincaré inequality* which states that in a bounded domain Ω, there exists a constant $C_\Omega > 0$ such that for all $v \in H_0^1(\Omega)$,

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\boldsymbol{\nabla} v(x)\|_{L^2(\Omega)}. \tag{1.2}$$

The above inequality plays a major role in the remainder of this course.

The space $\left(H_0^1(\Omega), |\cdot|_1\right)$ is a Hilbert space. Its *dual space*[1] is denoted by $H^{-1}(\Omega)$. An element T of $H^{-1}(\Omega)$ is a *continuous linear form* on $H_0^1(\Omega)$. The action of $T \in H^{-1}(\Omega)$ on $v \in H_0^1(\Omega)$ is usually denoted by $\langle T, v\rangle$. In general, an element T of $H^{-1}(\Omega)$ is not a function but a distribution. For example, if I is the open interval $(-1,\ 1) \subset \mathbb{R}$, then the Dirac measure defined by

$$\langle \delta, v\rangle = v(0),$$

is an element of $H^{-1}(I)$ (this is not true however in the case of a higher dimension $d > 1$!). Any $L^2(\Omega)$ function resides in $H^{-1}(\Omega)$ and if $T \in L^2(\Omega)$, then

$$\langle T, v\rangle = \int_\Omega T(x) v(x)\, dx.$$

Throughout this course, $H^{-1}(\Omega)$ is heavily used because it is a very handy space. For example, abstract theorems in Hilbert spaces X involving dual spaces X' will be often applied in this course and therefore when $X = H_0^1(\Omega)$, $X' = H^{-1}(\Omega)$ naturally arises. Nevertheless, if one does not feel comfortable with $H^{-1}(\Omega)$, one can almost always replace it by $L^2(\Omega)$ and replace the duality pairing by the L^2 scalar product.

Notation. Throughout these lecture notes, the following notation is used for the Sobolev norms. Let $K \subset \Omega$,

$$\|v\|_{0,K} = \|v\|_{L^2(K)}, \quad \|v\|_{1,K} = \|v\|_{H^1(K)},$$

$$|v|_{1,K} = \left(\int_K |\boldsymbol{\nabla} v(x)|^2\, dx\right)^{1/2} = \left(\sum_{i=1}^{d} \left\|\frac{\partial v}{\partial x_i}\right\|_{0,K}^2\right)^{1/2},$$

$$\|v\|_{\infty,K} = \sup_{x\in K} |v(x)|.$$

When K is omitted, the norm is taken over Ω. For example, $|u|_1$ is a notation for $|u|_{1,\Omega}$. The indices in $\|\cdot\|_0$, $\|\cdot\|_1$, $|\cdot|_1$ refer to the order of the derivatives except of course in $\|\cdot\|_\infty$.

[1] The reader is reminded that if X is a Banach space, its dual space is denoted by X' and is defined as the space of the *continuous* linear forms on X. The dual norm is defined by $\|f\|_{X'} = \sup_{u\in X} \dfrac{\langle f, u\rangle}{\|u\|_X}$.

Spaces $H^m(\Omega)$. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ be a multi-index, let $|\alpha|$ denote its length defined here as $\sum\limits_{i=1..d} \alpha_i$, and let

$$\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

The space $H^m(\Omega)$ is defined by

$$H^m(\Omega) = \left\{ v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), \forall \alpha \in \mathbb{N}^d \text{ such that } |\alpha| \leq m \right\},$$

and the H^m norm and semi-norm are given by

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha v\|^2_{L^2(\Omega)} \right)^{1/2} \quad \text{and} \quad |u|_{H^m(\Omega)} = \left(\sum_{|\alpha| = m} \|\partial^\alpha v\|^2_{L^2(\Omega)} \right)^{1/2},$$

respectively. The space $\left(H^m(\Omega), \|\cdot\|_{H^m}\right)$ is a Hilbert space.

Let $K \subset \Omega$. Sometimes, the shorter notations $\|\cdot\|_{m,K}$ and $|\cdot|_{m,K}$ are used for $\|\cdot\|_{H^m(K)}$ $\|\cdot\|_{m,K}$ and $|\cdot|_{H^m(K)}$, respectively. In particular, the notation

$$|u|_{2,K} = \left(\sum_{i,j=1}^{d} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{0,K} \right)^{1/2}$$

is often used and when K is omitted, it is implied that $K = \Omega$.

1.1.2 Green formulae.

All Green formulae (or "integration by parts" formulae) can be deduced from the following one

$$\int_\Omega \frac{\partial v}{\partial x_i}\, dx = \int_{\partial\Omega} v\, n_i\, d\gamma. \tag{1.3}$$

Indeed, let $\boldsymbol{f} = (f_1, \ldots, f_d)$. Applying the above formula to f_i and summing the results for $i = 1$ to d gives

$$\int_\Omega \operatorname{div} \boldsymbol{f}\, dx = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\, d\gamma. \tag{1.4}$$

Next, using this formula and the result $\operatorname{div}(u\boldsymbol{f}) = \boldsymbol{\nabla} u \cdot \boldsymbol{f} + u \operatorname{div} \boldsymbol{f}$ leads to

$$\int_\Omega (\boldsymbol{\nabla} u \cdot \boldsymbol{f} + u \operatorname{div} \boldsymbol{f})\, dx = \int_{\partial\Omega} u\boldsymbol{f} \cdot \boldsymbol{n}\, d\gamma. \tag{1.5}$$

Finally, applying the above formula with $\boldsymbol{f} = \boldsymbol{\nabla} v$ yields

$$\int_\Omega (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + u\Delta v)\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{n}}\, d\gamma, \tag{1.6}$$

where $\dfrac{\partial v}{\partial \boldsymbol{n}}$ is a notation for $\boldsymbol{\nabla} v \cdot \boldsymbol{n}$.

Other formulae, for example those involving the **curl** operator, can also be easily deduced from (1.3).

Given a vector function $\boldsymbol{u} = (u_1, \dots, u_d)$, the gradient of this vector is defined as the matrix

$$\boldsymbol{\nabla u} = \left[\frac{\partial u_i}{\partial x_j}\right]_{i,j=1..d}.$$

The quantity $\sum_{i,j=1..d} \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial v_i}{\partial x_j}$ is denoted by $\boldsymbol{\nabla u} : \boldsymbol{\nabla v}$. Hence,

$$\int_\Omega (\boldsymbol{\nabla u} : \boldsymbol{\nabla v} + \boldsymbol{u} \cdot \Delta \boldsymbol{v})\, dx = \int_{\partial\Omega} \boldsymbol{u} \cdot (\boldsymbol{\nabla v} \cdot \boldsymbol{n})\, d\gamma,$$

where $\boldsymbol{\nabla v} \cdot \boldsymbol{n}$ denotes the product of the matrix $\boldsymbol{\nabla v}$ by the vector $\boldsymbol{n}$.

1.1.3 Weak formulations

To introduce the concept of a weak formulation, the following algebraic problem is first presented. Let A be an $n \times n$ non singular matrix and B be an n-long vector. Consider a search for the solution U$\in \mathbb{R}^n$ of the linear system

$$\mathrm{AU{=}B}. \tag{1.7}$$

Let $(\cdot,\cdot)$ denote the Euclidian scalar product in $\mathbb{R}^n$. Since the only vector orthogonal to any other vector of $\mathbb{R}^n$ is the zero vector, problem (1.7) is equivalent to searching for U$\in \mathbb{R}^n$ such that

$$(\mathrm{AU}, \mathrm{V}) = (\mathrm{B}, \mathrm{V}), \qquad \forall \mathrm{V} \in \mathbb{R}^n. \tag{1.8}$$

For partial differential equations, the counterpart of (1.7) is called a *strong formulation*, whereas the counterpart of (1.8) is called a *weak formulation*. By abuse of language and for a reason that is discussed in the next Section, the weak formulation is also often called a *variational formulation*.

Consider next the following model problem (known as the *Poisson problem*)

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.9}$$

where, for example, $f \in L^2(\Omega)$. The role played by the matrix A in the algebraic example is played here by the operator $-\Delta$. Using the $L^2(\Omega)$ scalar product, one can formally write

$$\int_\Omega -\Delta u\, v\, dx = \int_\Omega f\, v\, dx,$$

for all functions v in some appropriate space. This is the counterpart of (1.8) where the functions v are called *test functions*. Using Green's formula (1.6), the above equation can be transformed into

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\gamma = \int_\Omega f v \, dx.$$

The key point is to choose an appropriate function space to set the above problem. To this effect, it is first noted that since $u = 0$ on $\partial\Omega$, it is reasonable to choose test functions vanishing on $\partial\Omega$. In particular, this allows to get rid of the integral over $\partial\Omega$ in the above equality. Then, the Cauchy-Schwarz inequality

$$\int_\Omega \nabla u \cdot \nabla v \, dx \leq \|\nabla u\|_0 \|\nabla v\|_0.$$

is invoked. Hence, if ∇u and ∇v are in $L^2(\Omega)$, the terms of the weak formulation make sense and therefore a natural functional space for the considered problem is $\Big(H_0^1(\Omega), \|\cdot\|_1\Big)$. One could also think of $\Big(\boldsymbol{C}^1(\Omega), \|\cdot\|_1\Big)$; however, this space is not complete and therefore this choice would prevent the application of some convenient results about Hilbert spaces such as the Lax-Milgram theorem.

In summary, the model problem (1.9) can be reformulated as follows: search for $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx. \tag{1.10}$$

More generally, one could choose $f \in H^{-1}(\Omega)$. In this case, the right side of the weak formulation can be written as $\langle f, v\rangle$. This term makes sense for $v \in H_0^1(\Omega)$ since, by definition, $H^{-1}(\Omega)$ is the space of continuous linear forms on $H_0^1(\Omega)$.

1.1.4 Lax-Milgram theorem

Let $\Big(X, \|\cdot\|_X\Big)$ be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on $X \times X$.

Proposition 1.1 (Continuity)
The bilinear form $a(\cdot,\cdot)$ is continuous *if and only if:*

$$\exists M > 0, \forall u, v \in X, |a(u,v)| \leq M \|u\|_X \|v\|_X. \tag{1.11}$$

For a continuous bilinear form, $\|a\|$ denotes the smallest constant M satisfying (1.11)

$$\|a\| = \sup_{u,v \in X} \frac{|a(u,v)|}{\|u\|_X \|v\|_X}.$$

Definition 1.1 (Coercivity) . The bilinear form $a(\cdot,\cdot)$ is said to be *coercive*[2] if

$$\exists \alpha > 0, \forall v \in X, a(v,v) \geq \alpha \|v\|_X^2. \tag{1.12}$$

The largest α satisfying this relation is called the *coercivity constant*. A coercive bilinear form with a coercivity constant α is said to be α-*coercive*.

Theorem 1.1 (Lax-Milgram)
Let $\Big(X, \|\cdot\|_X\Big)$ be a Hilbert space. Let $f \in X'$ and let $a(\cdot,\cdot)$ be a bilinear form on $X \times X$ that is continuous and coercive with a constant of coercivity α. There exists a unique $u \in X$ such that

$$a(u,v) = \langle f, v\rangle, \forall v \in X. \tag{1.13}$$

Also, u satisfies

$$\|u\|_X \leq \frac{\|f\|_{X'}}{\alpha}$$

The latter result shows that the linear application which associates f to u is continuous. Therefore, problem (1.13) is *well-posed*.

Remark 1.1 . *Consider the operator $A : X \to X'$ defined by*

$$\langle Au, v\rangle = a(u,v), \quad \forall\, (u,v) \in X \times X. \tag{1.14}$$

Under the hypotheses of the Lax-Milgram theorem, operator A is a continuous isomorphism from X onto X'.

Remark 1.2 . *When $a(\cdot,\cdot)$ satisfies the hypotheses of the Lax-Milgram theorem and is* symmetric*, the space X can be equipped with the norm $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ which is equivalent to $\|\cdot\|_X$. In this case (and only in this case), the Lax-Milgram theorem is simply the Riesz representation theorem stating that the solution $u \in X$ is the representer of $f \in X'$ for the scalar product $a(\cdot,\cdot)$.*

1.1.5 Minimization problems

Theorem 1.2 (Energy minimization)
Let $\Big(X, \|\cdot\|_X\Big)$ be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on $X \times X$. Assume that $a(\cdot,\cdot)$ is continuous, symmetric $\big($i.e. $a(u,v) = a(v,u)$, $\forall u, v \in X\big)$ *and positive* $\big($i.e. $a(v,v) \geq 0$, $\forall v \in X\big)$*. Let $f \in X'$. Then, the two following assertions are equivalent:*

(i) $u \in X$ is such that $a(u,v) = \langle f, v\rangle$, $\forall v \in X$.

(ii) $u \in X$ minimizes the energy functional $J(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle$ *on X — that is,*

$$J(u) = \min_{v\in X} J(v).$$

[2] Coercivity is also sometimes referred to as *ellipticity* in the literature.

Note that Theorem 1.2 is not an existence result. However, if it is further assumed that (1.12) is satisfied, the Lax-Milgram theorem proves the existence and uniqueness of the solution.

Thus in the case where $a(\cdot,\cdot)$ is positive and symmetric, the "weak formulation" $\big($point (i) of Theorem 1.2$\big)$ is indeed equivalent to a "variational formulation" $\big($point (ii)$\big)$. Although this equivalence holds only in the positive symmetric case, it explains the abuse of language consisting of calling the weak formulation a variational formulation.

1.2 Galerkin method and finite elements

1.2.1 Abstract framework

Let $\Big(X, \|\cdot\|_X\Big)$ be a Hilbert space, $f \in X'$, and let $a(\cdot,\cdot)$ be a bilinear form on $X \times X$. Consider the problem of searching for $u \in X$ such that

$$a(u, v) = \langle f, v\rangle, \forall v \in X. \tag{1.15}$$

Some discretization methods for partial differential equations (*e.g* finite differences or finite volumes) consist of modifying the differential operators themselves. In contrast, the Galerkin method does not affect the operators but only the functional space. More precisely, it simply consists of replacing the space X by a *finite dimensional subspace* $X_h \subset X$. Then, the discretized problem becomes

$$a(u_h, v_h) = \langle f, v_h\rangle, \forall v_h \in X_h. \tag{1.16}$$

Sometimes, the same idea is applied but with a modified operator $a_h(\cdot,\cdot)$. In such a case, the method is refered to as a *generalized Galerkin method* (examples of such approaches are discussed later in these lecture notes). Some other times, the finite dimensional space X_h is not a subspace of X. Then, the method is said to be *non conformal* (this case will not be encountered in this course).

For coercive problems, the following result shows that the approximation error of a Galerkin method is controlled by the error associated with the approximation of X by X_h and a stability constant that involves the continuity and coercivity constants.

Theorem 1.3 (Céa)
Let $a(\cdot,\cdot)$ be a continuous and α-coercive bilinear form. If u is the solution of problem (1.15) and u_h is the solution of the Galerkin approximation (1.16), then

$$\|u - u_h\|_X \leq \frac{\|a\|}{\alpha} \inf_{v_h \in X_h} \|u - v_h\|_X. \tag{1.17}$$

If in addition $a(\cdot,\cdot)$ is symmetric, then

$$\|u - u_h\|_X \leq \sqrt{\frac{\|a\|}{\alpha}} \inf_{v_h \in X_h} \|u - v_h\|_X. \tag{1.18}$$

1.2.2 Lagrange finite elements

It is assumed here that the reader is familiar with the general concepts of finite elements and meshes. A few results that are important for this course are reminded.

Only *polyhedral* domains Ω in $\mathbb{R}^d$, $d = 1, 2$ or 3, and *simplicial* meshes $\mathcal{T}_h$ are considered throughout this course. Thus if $K \in \mathcal{T}_h$, then K is a simplex, *i.e.* K is a segment if $d = 1$, a triangle if $d = 2$, and a tetrahedron if $d = 3$. The measure of K (length if $d = 1$, area if $d = 2$, volume if $d = 3$) is denoted by $\text{meas}(K)$. It will always be assumed that $\text{meas}(K) \neq 0$.

The diameters of K and that of the largest ball included in K are denoted by h_k and ρ_k, respectively. The ratio of these two quantities is denoted by σ_K. Hence,

$$h_K = \text{diam}(K), \qquad \rho_K = \sup\{\text{diam}(S), S \text{ ball } \subset K\}, \qquad \sigma_K = \frac{h_K}{\rho_K}.$$

Note that $\sigma_K > 1$. For a family of meshes $\{\mathcal{T}_h\}_{h>0}$, the parameter h refers to

$$h = \max_{K \in \mathcal{T}_h} h_K.$$

Any element K of a mesh $\mathcal{T}_h$ can be viewed as the image of a unique *reference element* $\hat{K}$ by an affine map $\mathcal{F}_K(\hat{x}) = M_K \hat{x} + b_K$, where M_K is a $d \times d$ matrix with $\det M_K > 0$, and b_K is a vector in $\mathbb{R}^d$. This can be written as

$$K = \mathcal{F}_K(\hat{K}).$$

If v is a function defined on K, $\hat{v}$ is defined on $\hat{K}$ by

$$\hat{v} = v \circ \mathcal{F}_K. \tag{1.19}$$

If $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$ as well as the associated matrix norm, then

$$\det M_K = \frac{\text{meas}(K)}{\text{meas}(\hat{K})}, \qquad |M_K| \leq \frac{h_K}{\rho_{\hat{K}}}, \qquad |M_K^{-1}| \leq \frac{h_{\hat{K}}}{\rho_K}. \tag{1.20}$$

Let $m \geq 0$. There exists $C > 0$ such that for all K and all $\hat{v} \in H^m(\hat{K})$,

$$|v|_{m,K} \leq C \|M_K^{-1}\|^m |\det M_K|^{1/2} |\hat{v}|_{m,\hat{K}}, \tag{1.21}$$

where v is defined by (1.19).

Approximation in K. Let $k \geq 0$ and $\mathbb{P}^k(K)$ denote the space of polynomial functions defined in K and with a degree at most equal to k. The Lagrange interpolation operator of degree k on K is denoted by $\mathcal{I}_K^k$. By definition, for any function $v \in \boldsymbol{C}^0(\overline{K})$ (space of continuous functions), $\mathcal{I}_K^k v$ is the polynomial function of degree k which takes the same values as v on all the vertices $(a_i)_{i=1..d}$ of K

$$\mathcal{I}_K^k(v)(a_i) = v(a_i), \qquad \mathcal{I}_K^k(v) \in \mathbb{P}^k(K).$$

The following interpolation result (see *e.g.* [6], Theo. 1.103, p. 59) is important. There exists a constant $C > 0$ such that for all $m \in \{0, \ldots, k+1\}$, for all K, for all $v \in H^{k+1}(K)$,

$$|v - \mathcal{I}_K^k v|_{m,K} \leq C h_K^{k+1-m} \sigma_K^m |v|_{k+1,K}. \tag{1.22}$$

In dimension $d = 1$, $\sigma_K = 1$. In dimension $d = 2$,

$$\sigma_K \leq \frac{2}{\sin\theta_K} \tag{1.23}$$

where θ_K is the smallest angle of the triangle K.

Approximation in Ω. Let $\mathcal{T}_h$ be a simplicial mesh of Ω. Consider the Lagrange finite element space of degree k

$$X_h^k = \{v \in \boldsymbol{C}^0(\Omega), v|_K \in \mathbb{P}^k(K)\}. \tag{1.24}$$

Denote by $\mathcal{I}_h^k$ the Lagrange interpolation operator of degree k on Ω. For all vertices $(a_i)_{i=1..n}$ of the mesh $\mathcal{T}_h$ and all $v \in \boldsymbol{C}^0(\bar{\Omega})$

$$\mathcal{I}_h^k(v)(a_i) = v(a_i), \qquad \mathcal{I}_h^k(v) \in X_h^k.$$

To infer global approximation properties over Ω from the local approximation properties (1.22) over K, an additional assumption on the asymptotic behavior of the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is needed.

Definition 1.2 (shape-regularity) . A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be *shape-regular* if there exists σ such that

$$\forall h > 0, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma. \tag{1.25}$$

For example, in two dimensions, it follows from (1.23) and (1.25) that a triangle cannot become too flat as h goes to zero.

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of simplicial meshes of Ω. Consider the problem of approximation in X_h^k, the space of Lagrange finite elements of degree k. There exists $C > 0$ such that for $1 \leq l \leq k$, for all $h > 0$ and all $v \in H^{l+1}(\Omega)$ (see *e.g.* [6], Cor. 1.109, p.61)

$$\|v - \mathcal{I}_h^k v\|_{0,\Omega} \leq C h^{l+1} |v|_{l+1,\Omega}, \tag{1.26}$$

and

$$|v - \mathcal{I}_h^k v|_{1,\Omega} \leq C h^l |v|_{l+1,\Omega}. \tag{1.27}$$

The above inequalities show that increasing the degree of the finite element is meaningful only if the solution can be expected to be sufficiently regular. For example, if a second-order finite element approximation ($k = 2$) of the solution is chosen, an optimal convergence rate can be achieved only if the solution is at least in $H^3(\Omega)$.

Inverse inequalities. It is well-known that in infinite dimension, all the norms are not equivalent. For example, the L^2 norm can be upper-bounded by the H^1 norm; however, the reverse is not true and therefore the L^2 and H^1 norms are not equivalent. In contrast, all norms are equivalent in finite dimensions. For example, in a finite element space X_h^k, the H^1 norm can be upper-bounded by the L^2 norm. Such an inequality is called an *inverse inequality* because it is "inverted" with respect to the inequality which occurs in the infinite dimension case. However, the constant of this inequality blows up as h goes to 0 because X_h^k tends to the infinite dimensional space when h goes to 0.

The following *local inverse inequality* can be proved (see *e.g.* [6], lemma 1.138, p. 75): let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes in $\mathbb{R}^d$, there exists a constant C such that for all $h > 0$, $K \in \mathcal{T}_h$ and $v \in \mathbb{P}^k(K)$

$$\|v\|_{1,K} \leq C h_K^{-1} \|v\|_{0,K}. \tag{1.28}$$

To obtain a global inverse inequality (*i.e.* an inequality not only valid in K but in whole Ω), the concept of quasi-uniform family of meshes is needed.

Definition 1.3 (Quasi-uniformity) A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be *quasi-uniform* if it is shape-regular and there exists $\tau > 0$ such that

$$\forall h > 0, \forall K \in \mathcal{T}_h, h_K \geq \tau h. \tag{1.29}$$

Then, the following result can be proved (see *e.g.* [6], corollary 1.141, p. 76): let $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of affine meshes in $\mathbb{R}^d$, there exists a constant C such that for all $h > 0$, $K \in \mathcal{T}_h$ and $v \in \mathbb{P}^k(K)$

$$\|v\|_{1,\Omega} \leq C h^{-1} \|v\|_{0,\Omega}. \tag{1.30}$$

1.2.3 Algebraic aspects

The reader is reminded that $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^n$ as well as the associated matrix norm and that $(\cdot,\cdot)$ denotes the Euclidian scalar product.

Definition 1.4 (Condition number) . The *condition number* of a non singular square matrix A is defined by

$$\kappa(A) = |A|\,|A^{-1}|. \tag{1.31}$$

Proposition 1.2
Let A be a symmetric positive definite matrix. Let λ_{max} and λ_{min} be respectively the maximal and minimal eigenvalues of A. Then,

$$\kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}.$$

A linear system associated with a matrix with a "large" condition number is ill-conditioned. This means that the solution may be very sensitive to perturbations in the data. In such a case, *iterative methods* (Jacobi, conjugate gradient, GMRES, *etc.*) are less efficient than otherwise. It is therefore useful to have an idea of the condition number of the matrices obtained with the finite element method. The following technical result (which can be found, for example, in [9]), is useful.

Proposition 1.3
Let A and P be two $n \times n$ symmetric positive definite matrices such that there exist $m_1, m_2 > 0$ for which

$$m_1(PX, X) \leq (AX, X) \leq m_2(PX, X), \forall X \in \mathbb{R}^n.$$

Then

$$\kappa(P^{-1}A) \leq \frac{m_2}{m_1}.$$

Consider a family of simplicial meshes $\{\mathcal{T}_h\}_{h>0}$ and a Lagrange finite element space X_h built on $\mathcal{T}_h$. Let $(\phi_i)_{i=1..n}$ be a basis of X_h. The following matrices can be defined

$$\mathrm{M} = \left[\int_\Omega \phi_j \phi_i \, dx\right] \qquad \text{(mass matrix)},$$

$$\mathrm{K} = \left[\int_\Omega \boldsymbol{\nabla}\phi_j \boldsymbol{\nabla}\phi_i \, dx\right] \qquad \text{(stiffness matrix)}.$$

If $\mathrm{U} = (U_i)_{i=1..n}$ denotes the coordinates of a function $u_h \in X_h$ in the basis $(\phi_i)_{i=1..n}$ — that is,

$$u_h = \sum_{i=1}^n U_i \phi_i,$$

then

$$(\mathrm{MU}, \mathrm{U}) = \|u_h\|^2_{L^2(\Omega)}.$$

It can be proven ([9]) that there exist $m_1, m_2 > 0$ such that

$$m_1 h^d |\mathrm{U}|^2 \leq \|u_h\|^2_{L^2(\Omega)} \leq m_2 h^d |\mathrm{U}|^2.$$

Thus

$$m_1 h^d (\mathrm{U}, \mathrm{U}) \leq (\mathrm{MU}, \mathrm{U}) \leq m_2 h^d (\mathrm{U}, \mathrm{U}).$$

Hence, according to Proposition 1.3, there exists a constant $C > 0$ independent of h such that:

$$\kappa(\mathrm{M}) \leq C.$$

On can also prove ([9]) that there exist $m_1, m_2, m_3 > 0$ such that

$$m_1 h^d (\mathrm{U}, \mathrm{U}) \leq (\mathrm{KU}, \mathrm{U}) \leq m_2 h^d \left(1 + \frac{m_3}{h^2}\right) (\mathrm{U}, \mathrm{U}).$$

Hence, according to Proposition 1.3, there exists a constant $C > 0$ independent of h such that

$$\kappa(\mathrm{K}) \leq \frac{C}{h^2}.$$

The above results show that the condition number of the mass matrix M is independent of h, whereas the conditioning of the stiffness matrix K can grow like $1/h^2$. Thus in practice, a linear system associated with the mass matrix can be solved very efficiently by an iterative solver. On the other hand, there is a dilemma for the case of the stiffness matrix: the smaller is h, the better is the accuracy, but the worse is the condition number. Thus, a linear system associated with a stiffness matrix must be *preconditioned* — that is, it must be pre-multiplied by a suitable regular matrix P^{-1} in order to have $\kappa(\mathrm{P}^{-1}\mathrm{A}) \ll \kappa(\mathrm{A})$.

1.3 Limitations of the coercive framework

The material presented so far in this chapter constitutes the basic framework of the finite element method. This course focuses however on those cases where this framework is inadequate. This occurs, for example, when the coercivity assumption is not fulfilled (*e.g.* the Stokes and Darcy problems) or when the stability constant $\|a\|/\alpha$ in (1.17) is very large (*e.g.* quasi-incompressible material and advection dominated advection-diffusion problems).

1.3.1 Poisson's equations in mixed form and Darcy's equations

Consider the Poisson problem

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $f \in H^{-1}(\Omega)$. As shown above, this problem can be written in a weak form (1.10). It is exactly the kind of problem (1.13) that can be studied with the Lax-Milgram theorem. Indeed, choosing $X = H_0^1(\Omega)$, $\|\cdot\|_X = \|\cdot\|_1$, it is easy to verify (1.11) using the Cauchy-Schwarz inequality, and (1.12) using the Poincaré inequality (1.2). Note that in this case, one can equivalently choose $\|\cdot\|_X = |\cdot|_1$. Thus, the Lax-Milgram theorem reveals that the Poisson problem is well-posed.

The coercivity issue depends on the choice of the variational formulation and that of the functional spaces. For example, defining $\boldsymbol{j} = -\boldsymbol{\nabla} u$, solving the Poisson problem can be reformulated as finding $(\boldsymbol{j}, u)$ such that

$$\begin{cases} \boldsymbol{j} + \boldsymbol{\nabla} u &= 0 \quad \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{j} &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.32}$$

The above formulation of the Poisson problem is not a mathematical curiosity. Indeed, assume for example that u is an electrical potential. Then, $\boldsymbol{j}$ is a current density (up to some physical constants). Thus, in a problem where the electrical current is a relevant unknown, it may be more appropriate to discretize this formulation rather than the original one.

Assume that $f \in L^2(\Omega)$. As before, on can search for u in $H^1_0(\Omega)$. Since $\operatorname{div} \boldsymbol{j} \in L^2$, a natural space for $\boldsymbol{j}$ is

$$H(div, \Omega) = \Big\{ \boldsymbol{k} \in L^2(\Omega)^d, \operatorname{div} \boldsymbol{k} \in L^2(\Omega) \Big\}.$$

Equipping the above space with the norm

$$\|\boldsymbol{k}\|_{div,\Omega} = \left(\|\boldsymbol{k}\|^2_{0,\Omega} + \| \operatorname{div} \boldsymbol{k}\|^2_{0,\Omega} \right)^{1/2},$$

leads to the Hilbert space $\Big(H(div,\Omega), \| \cdot \|_{div,\Omega} \Big)$.

A possible variational formulation for problem (1.32) is then:

Find $(\boldsymbol{j}, u) \in H(div,\Omega) \times H^1_0(\Omega)$ *such that for all* $(\boldsymbol{k}, v) \in H(div,\Omega) \times H^1_0(\Omega)$

$$\begin{aligned} \int_\Omega \boldsymbol{j} \cdot \boldsymbol{k}\, dx - \int_\Omega u \operatorname{div} \boldsymbol{k}\, dx &= 0 \\ \int_\Omega v \operatorname{div} \boldsymbol{j}\, dx &= \int_\Omega f v\, dx. \end{aligned} \tag{1.33}$$

Let $\mathcal{X}$ denote the space $H(\operatorname{div}, \Omega) \times H^1_0(\Omega)$ equipped with the norm $\|(\boldsymbol{k}, v)\|^2_{\mathcal{X}} = \|\boldsymbol{k}\|^2_{div} + |v|^2_1$. The following form

$$\Psi((\boldsymbol{j}, u), (\boldsymbol{k}, v)) = \int_\Omega \boldsymbol{j} \cdot \boldsymbol{k}\, dx - \int_\Omega u \operatorname{div} \boldsymbol{k}\, dx + \int_\Omega v \operatorname{div} \boldsymbol{j}\, dx,$$

is a bilinear form on $\mathcal{X} \times \mathcal{X}$, and the following form

$$\langle F, (\boldsymbol{k}, v) \rangle = \int_\Omega f v\, dx.$$

is a linear form in $\mathcal{X}'$. Problem (1.33) can be reformulated as finding $(\boldsymbol{j}, u) \in \mathcal{X}$ such that for all $(\boldsymbol{k}, v) \in \mathcal{X}$,

$$\Psi((\boldsymbol{j}, u), (\boldsymbol{k}, v)) = \langle F, (\boldsymbol{k}, v) \rangle.$$

If one attempts to prove the coercivity of Φ, one gets

$$\Phi((\boldsymbol{j}, u), (\boldsymbol{j}, u)) = \int_\Omega |\boldsymbol{j}|^2\, dx.$$

Note that u does not appear in the right side of the above identity. Therefore, there is no lower bound of the form $\alpha\|(\boldsymbol{j},u)\|^2_{\mathcal{X}}$. Thus, the bilinear form Φ is non coercive on $\mathcal{X}\times\mathcal{X}$ and the Lax-Milgram theorem cannot be used to study formulation (1.33).

Another example of a similar problem is provided by Darcy's equations. Indeed, given $f \in R^d$, $g \in R^d$, and a symmetric positive definite matrix $K \in R^{d\times d}$, consider the problem of finding u and p such that

$$\left\{\begin{array}{rcll} K^{-1}\boldsymbol{u} + \boldsymbol{\nabla} p &=& \boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &=& g & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.34}$$

Variants of the above governing equations govern flows in porous media and therefore are popular in oil recovery applications. In such cases, K denotes the permeability tensor of the porous medium, $\boldsymbol{u}$ is a velocity and p is a pressure.

Should one apply the div operator to the first of equations (1.34), one would obtain a Poisson problem in the unknown p which could be studied with the Lax-Milgram theorem. However, doing so, one would eliminate the unknown $\boldsymbol{u}$ which is relevant from the physical viewpoint. At the continuous level, one could still recover $\boldsymbol{u}$ from p. However after discretization, such an operation is likely to deteriorate the accuracy. Hence, formulation (1.34) is interesting for the discretization purpose. Nevetheless, as for the case of formulation (1.33), it cannot be studied directly with the Lax-Milgram theorem.

Problems like (1.32) and (1.34) are said to be in *mixed form* because they involve two different unknowns (this is more a usage than a definition). As shown above, such problems cannot be studied with the Lax-Milgram theorem, because of their lack of coercivity. At the continuous level, these problems are equivalent to their coercive counterparts which can be studied with the Lax-Milgram theorem. However, this is not the case at the discrete level.

1.3.2 Stokes problem

The Stokes equations are often used to model a viscous incompressible fluid. The Stokes problem consists of searching for two functions $\boldsymbol{u} \in H^1(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$\left\{\begin{array}{rcll} -\Delta\boldsymbol{u} + \boldsymbol{\nabla} p &=& \boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &=& 0 & \text{in } \Omega,\\ \boldsymbol{u} &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.35}$$

where $\boldsymbol{f}$ is a given distribution in $H^{-1}(\Omega)^d$ (to simplify, one can assume that $\boldsymbol{f}$ is a given function in $L^2(\Omega)^d$). The equation $\operatorname{div}\boldsymbol{u} = 0$ results from the incompressibility of the fluid.

Coercive formulation. A first variational formulation of this problem is derived by formally multiplying $(1.35)_1$ by $\boldsymbol{v} \in V = \{\boldsymbol{w} \in H^1_0(\Omega)^d, \operatorname{div}\boldsymbol{w} = 0\}$ to obtain

Find $\boldsymbol{u} \in V$ *such that, for all* $\boldsymbol{v} \in V$,

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{1.36}$$

Note that the incompressibility constraint is enforced in the space where the solution is looked for and that the pressure term has disappeared because $\boldsymbol{v}$ is divergence free. Using the Lax-Milgram theorem, it can be easily shown that this problem is well-posed (the continuity of the solution results from the Cauchy-Schwarz inequality, and the coercivity results from the Poincaré inequality). Indeed, the above formulation in V of the Stokes problem is a vectorial Poisson problem.

Formulation (1.36) is very convenient from a theoretical viewpoint. Nevertheless, it has two major flaws from a practical viewpoint: first it does not provide the pressure, and second, it is based on a space which is not convenient in a classical finite element framework (the usual finite element spaces typically provide approximations of $H^1(\Omega)^d$ not of V).

Therefore, it is more natural to consider in practice the following variational formulation of the Stokes problem.

Mixed formulation. Multiplying the first of equations (1.35) by $\boldsymbol{v} \in H_0^1(\Omega)^d$ and the second of these equations by $q \in L_0^2(\Omega)$ leads to the following problem.

Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ *such that for all* $(\boldsymbol{v}, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \, dx - \int_\Omega p \operatorname{div} \boldsymbol{v} \, dx & = & \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ \displaystyle\int_\Omega q \operatorname{div} \boldsymbol{u} \, dx & = & 0. \end{array} \right. \tag{1.37}$$

In contrast to (1.36), the above formulation provides the pressure and is set in spaces that can be naturally approximated by finite elements. Nevertheless, if one tries to apply the Lax-Milgram theorem, one discovers the same difficulty with formulation (1.37) as with the Darcy problem. Indeed, let $\mathcal{X}$ denote the space $H_0^1(\Omega)^d \times L_0^2(\Omega)$ equipped with the norm $\|(\boldsymbol{v}, q)\|_{\mathcal{X}}^2 = \|\boldsymbol{v}\|_1^2 + \|q\|_0^2$. Introduce in $\mathcal{X} \times \mathcal{X}$ the bilinear form

$$\Psi((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \, dx - \int_\Omega p \operatorname{div} \boldsymbol{v} \, dx + \int_\Omega q \operatorname{div} \boldsymbol{u} \, dx.$$

Introduce also $F \in \mathcal{X}'$ defined by

$$\langle F, (\boldsymbol{v}, q) \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle.$$

Then, formulation (1.37) can be rewritten as follows.

Find $(\boldsymbol{u}, p) \in \mathcal{X}$ *such that for all* $(\boldsymbol{v}, q) \in \mathcal{X}$

$$\Psi((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \langle F, (\boldsymbol{v}, q) \rangle.$$

To make an attempt at proving the coercivity of $\Psi(\cdot,\cdot)$, one computes

$$\Psi((\boldsymbol{u}, p), (\boldsymbol{u}, p)) = \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 \, dx,$$

which cannot be lower-bounded by $\|(\boldsymbol{u}, p)\|_{\mathcal{X}}^2$ since $\|p\|_0$ is absent in the above identity. Hence, the mixed formulation (1.37), which is the most convenient from the practical viewpoint, cannot be studied in the usual framework of the Lax-Milgram theorem.

1.3.3 Linear elasticity

While incompressible fluid mechanics is the main focus application of these lecture notes, some of the issues addressed by these notes arise in other engineering problems. For example, consider the linear elasticity equations in the case of an almost incompressible material with shear modulus G and bulk modulus κ. Let $\boldsymbol{f}$ be an external force, $\boldsymbol{v}$ a displacement, and $\boldsymbol{\epsilon}^D(\boldsymbol{v}) = \boldsymbol{\epsilon}(\boldsymbol{v}) - \frac{\operatorname{div} \boldsymbol{v}}{3} \boldsymbol{Id}$ the corresponding deviatoric strain tensor. The energy of the system is given in this case by

$$J(\boldsymbol{v}) = G \int_\Omega |\boldsymbol{\epsilon}^D(\boldsymbol{v})|^2 \, dx + \frac{\kappa}{2} \int_\Omega (\operatorname{div} \boldsymbol{v})^2 \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx. \tag{1.38}$$

For simplicity, assume that the displacement is zero on the boundary of Ω. Minimizing the above energy is then equivalent to finding $\boldsymbol{u} \in H_0^1(\Omega)^d$ such that for all $\boldsymbol{v} \in H_0^1(\Omega)^d$

$$2G \int_\Omega \boldsymbol{\epsilon}^D(\boldsymbol{u}) : \boldsymbol{\epsilon}^D(\boldsymbol{v}) \, dx + \kappa \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx. \tag{1.39}$$

Using the Korn inequality (see, for example, [6], Theo. 3.77, p. 156), it can be shown that the above problem is coercive. Therefore, this problem can be studied with the Lax-Milgram theorem. The continuity constant involves $\max(G, \kappa)$ whereas the coercivity constant involves $\min(G, \kappa)$. The larger the bulk moduli κ is, the more incompressible is the material. The limit $\kappa = \infty$ corresponds to a fully incompressible material. Thus for an almost incompressible material, the ratio of the continuity constant and coercivity constant tends to infinity. In particular, the constant in the Céa Lemma (1.17) is very large. In corresponding numerical simulations, one observes a phenomenon known as *locking*. The abstract theory presented in Chapter 2 for studying the Stokes equations will enable the explanation in Chapter 3 of this phenomenon and will provide a guideline for avoiding it.

1.3.4 Advection-diffusion

Let Ω a bounded domain of $\mathbb{R}^d$, μ a positive constant, $\boldsymbol{b}$ a divergence free vector field in $\left(L^\infty(\Omega)\right)^d$ and $f \in L^2(\Omega)$. Consider the stationary advection-diffusion equation

$$\left\{ \begin{array}{rcll} \boldsymbol{b} \cdot \boldsymbol{\nabla} u - \mu \Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \right.$$

It can be proven that solving the above problem is equivalent to finding $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle F, v \rangle, \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega (\mu \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \boldsymbol{b} \cdot \boldsymbol{\nabla} u v) \; dx \quad \text{ and } \quad \langle F, v \rangle = \int_\Omega f v \, dx.$$

Note that in this problem, the bilinear form is non-symmetric and therefore the solution of this problem is not related to a minimization problem. The bilinear form $a(\cdot, \cdot)$ is continuous and coercive on $V \times V$. Indeed, it can be proven that

$$|a(u, v)| \leq \max(C_\Omega \|\boldsymbol{b}\|_\infty, \mu) |u|_1 |v|_1,$$

and

$$a(v, v) \geq \mu |v|_1^2.$$

From the Lax-Milgram theorem, it follows that the above problem is well-posed. Nevertheless, when the flow regime is dominated by the advection, *i.e.* when $\|\boldsymbol{b}\|_\infty$ becomes much greater than μ (up to some constants to get the correct physical unit) and the constant in the Céa inequality becomes very large. In this case, very poor numerical results are observed. Later, it will be shown that *stabilization methods*, which are generalized Galerkin methods, help in this case improving the accuracy of the numerical results.

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