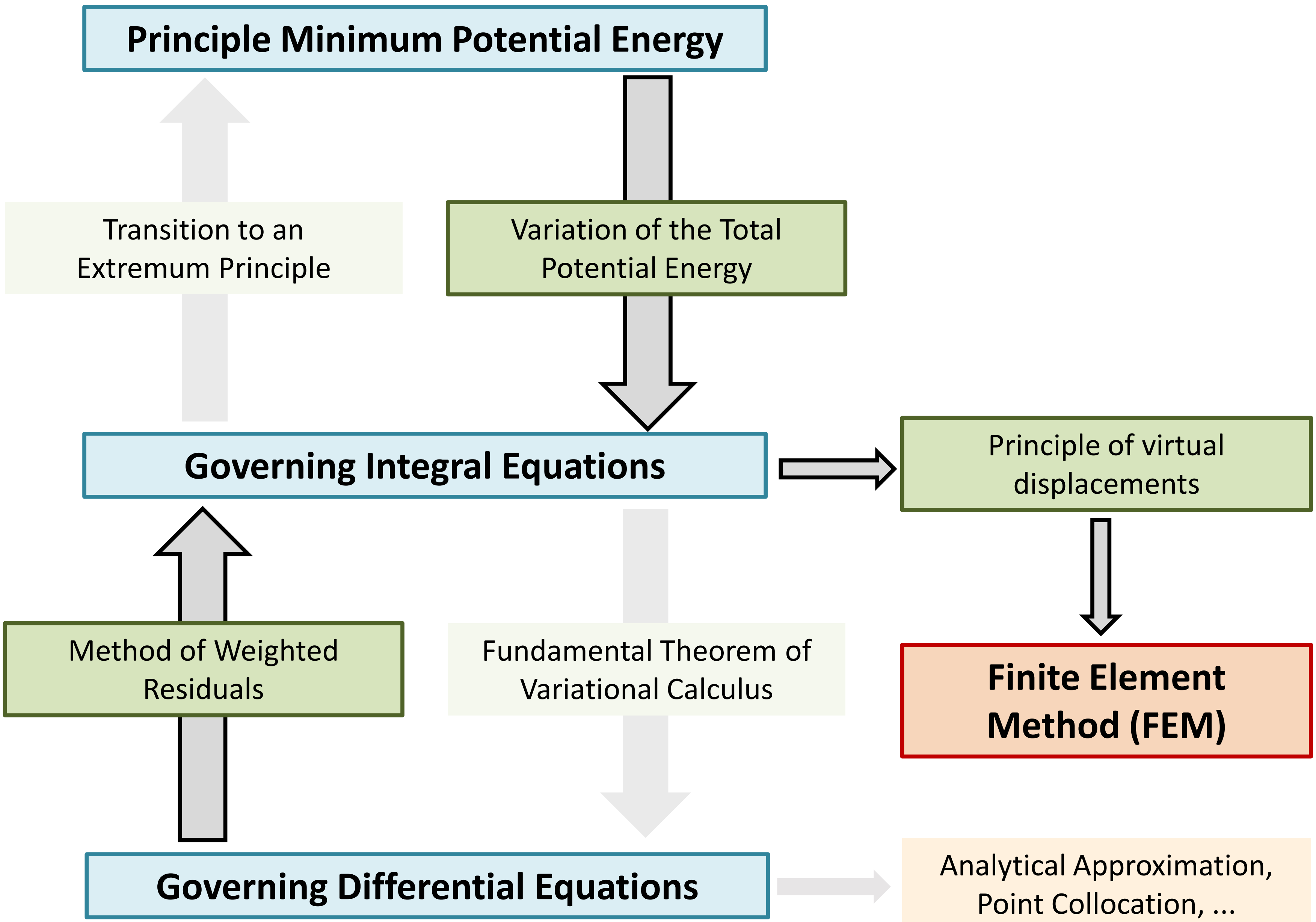
Finite Element Method

summary – statics



Principle Minimum Potential Energy
Transition to an Extremum Principle
Variation of the Total Potential Energy
Governing Integral Equations
Principle of virtual displacements
Method of Weighted Residuals
Fundamental Theorem of Variational Calculus
Finite Element Method (FEM)
Governing Differential Equations
Analytical Approximation, Point Collocation, ...

1. problem definition

- **analysis suited integral equations**

governing equations of the physical problem (heat, solid mechanics, ...)

- potential energy :
  applying the principle of the minimum of potential energy

- set of partial differential equations :
  applying the method of weighted residuals

⟹ both result in an analysis suited set of integral equations, e.g.

principle of virtual temperatures
principle of virtual displacements

governing differential equations

... summary

$$
\begin{array}{lcclccc}
\text{strain – displm.} & \boldsymbol{\epsilon} & = & \mathbf{D}\,\mathbf{u} & \mathbf{x} & \in & C & (1) \\
\text{stress – strain} & \boldsymbol{\sigma} & = & \mathbf{C}\,\boldsymbol{\epsilon} & \mathbf{x} & \in & C & (2) \\
\text{equilibrium} & 0 & = & \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{p}_V & \mathbf{x} & \in & C & (3) \\
\text{stress vector} & \mathbf{t} & = & \mathbf{S}\,\mathbf{n} & \mathbf{x} & \in & \partial C & (4)
\end{array}
$$

$$((1) \text{ in } (3)) \text{ in } (2):$$

$$
PDE \qquad
\boxed{
\begin{array}{lcl}
\mathbf{D}^T(\mathbf{C}\,\mathbf{D}\,\mathbf{u}) + \mathbf{p}_V & = & 0 \\
\sum_i \sum_m \frac{\partial \sigma_{im}}{\partial x_m} + \sum_i p_{Vi} & = & 0
\end{array}
}
$$

boundary conditions

... summary

Dirichlet (prescribed displacements)

$$\mathbf{x} \in \partial C \;\wedge\; \mathbf{u} \in C_u \;:\; u_i \;=\; u_{i0} \qquad (1)$$

Neumann (prescribed stresses)

$$\mathbf{x} \in \partial C \;\wedge\; \mathbf{t} \in C_t \;:\; t_i \;=\; t_{i0} \qquad (2)$$

C_u : set of prescribed displacements components

C_t : set of prescribed stress vector components

choice of a suited Ansatz (approximation for the physical variables)

- $\mathbf{u}(\mathbf{x}), \mathbf{x} \in C$ for displacements inside & on the surface
- $\mathbf{t}(\mathbf{x}), \mathbf{x} \in \partial C$ for stresses on the surface
- well suited: e.g. polynomial functions

$$u_i(\mathbf{x}) = \sum_k u_k \, s_k(\mathbf{x}) \qquad \mathbf{x} \in C$$

$$t_i(\mathbf{x}) = \sum_m t_m \, s_m(\mathbf{x}) \qquad \mathbf{x} \in \partial C$$

u_k, t_m free parameters (unknowns)

$s_k(\mathbf{x}), s_m(\mathbf{x})$ lin. independent polynomial coordinate functions

index i indicates i^{th} vector component substitution of Ansatz into governing equations

... residuals

$$
\begin{aligned}
\mathbf{r}_1 &= \boldsymbol{\epsilon} - \mathbf{D}\,\mathbf{u} & \mathbf{x} &\in C \\
\mathbf{r}_2 &= \boldsymbol{\sigma} - \mathbf{C}\,\boldsymbol{\epsilon} & \mathbf{x} &\in C \\
\mathbf{r}_3 &= \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{p}_V & \mathbf{x} &\in C \\
\mathbf{r}_4 &= \mathbf{t} - \mathbf{S}\,\mathbf{n} & \mathbf{x} &\in \partial C (= C_u + C_t) \\
r_5 &= u_i - u_{i0} & \mathbf{x} &\in C_u \\
r_6 &= t_i - t_{i0} & \mathbf{x} &\in C_t
\end{aligned}
$$

a priori conditions

- some residuals are satisfied by the choice of the Ansatz
- here: $\mathbf{r}_1 = 0$ and $\mathbf{r}_2 = 0$

application of the Galerkin method to r_3 to r_6

$$
\begin{aligned}
\mathbf{r}_3 &= \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{p}_V & \mathbf{x} &\in C \\
\mathbf{r}_4 &= \mathbf{t} - \mathbf{S}\,\mathbf{n} & \mathbf{x} &\in \partial C (= C_u + C_t) \\
r_5 &= u_i - u_{i0} & \mathbf{x} &\in C_u \\
r_6 &= t_i - t_{i0} & \mathbf{x} &\in C_t
\end{aligned}
$$

Method of Galerkin $\rightarrow$ integral form :

$$
\int_C \sum_i \sum_m (\delta u_i \frac{\partial \sigma_{im}}{\partial x_m})\, dv + \int_C \sum_i \delta u_i\, p_{Vi}\, dv + \int_{\delta C} \sum_i \delta u_i\, (t_i - \sum_m \sigma_{im}\, n_m)\, da + \int_{C_u} \sum_i \delta t_i\, (u_{i0} - u_i)\, da + \int_{C_t} \sum_i \delta u_i\, (t_{i0} - t_i)\, da = 0
$$

Gauss theorem transforms integral eqs. into governing integral form for numerical approximation:

- $(\mathbf{D}^T \boldsymbol{\sigma})$ contains derivatives of second order
- integration by parts reduces degree of derivatives
- extended form of the principle of virtual work/virtual displacements

Principle of virtual work :

$$\int_C \sum_i \sum_m (\delta \frac{\partial u_i}{\partial x_m} \sigma_{im})\, dv \;=\; \int_C \sum_i \delta u_i\, p_{Vi}\, dv + \int_{C_u} \sum_i \delta u_i\, t_i\, da + \int_{C_t} \sum_i \delta u_i\, t_{i0}\, da$$

$$\wedge \quad u_i \;=\; u_{i0} \quad \mathbf{x} \in \partial C_u$$

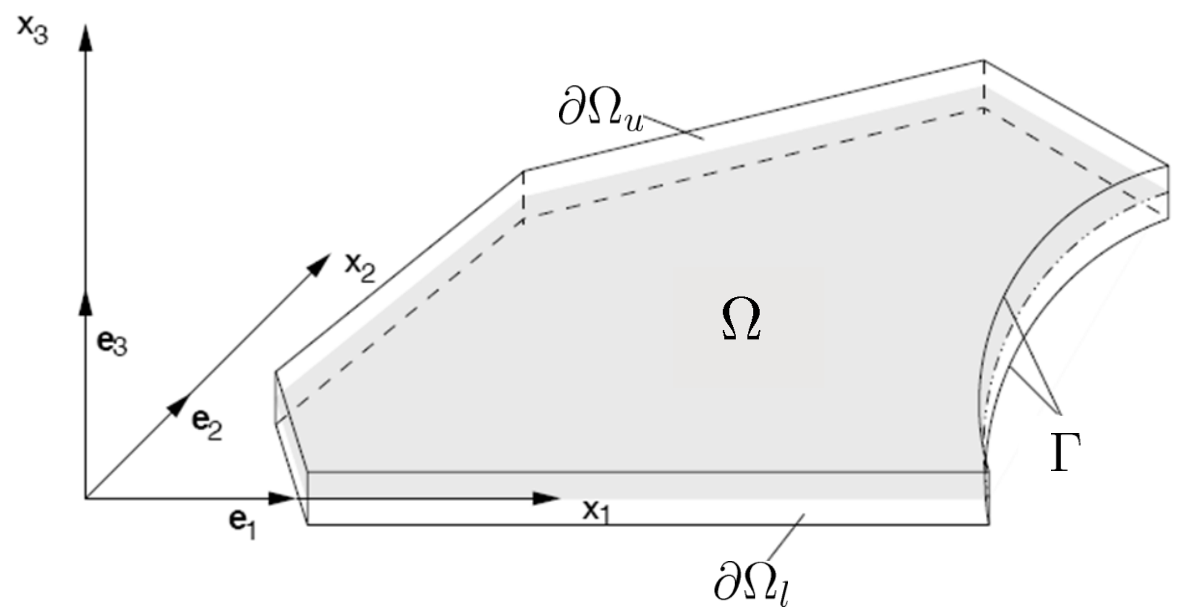
1. problem definition

- **analysis suited integral equations (3D formulation)**
- **idealization (e.g. 2D plane strain/stress, 1D bar/beam, 2D heat flow, ...)**

- considering structural properties
- introduction of specific hypotheses (Kirchhoff, Bernoulli, plane stress, ...)
  - introduction of additional equations
  - results in more equations than necessary to determine behavioral variables (3D theory is complete!)
  - leading to contradictions, cf. *Poisson* effect for plane stress

geometry – solution domain

- description of the body in a 3D Cartesian coordinate system
- body is considered plane if
  - at least one of the body dimensions is *constant* (little variation)
  - one of the *constant* dimension is significantly smaller than the others
  - e.g. plates: thickness is typically less than 10% of width/height
  - x_1-x_2 is chosen to be the mid-surface of the plane domain at $x_3 = 0$
  - here: assume unit thickness h=1



approximation in the physical behavior

- *plane state of stress* is assumed if
  - loading is restricted to the x_1-x_2 plane and is independent of x_3

$$p = p(x_1, x_2)$$

  - displacements are referred to the mid-plane and is independent of x_3

$$u = u(x_1, x_2)$$

  - strains/stresses are assumed to be *constant* over the thickness

$$\sigma_{11}, \sigma_{22}, \sigma_{12} \quad \text{are constant over } x_3$$

  - normal and shear stress in x_3 -direction is neglected

$$\sigma_{33} = \sigma_{13} = \sigma_{31} = 0$$

approximation in the physical behavior

- normal and shear stress in x_3 -direction is neglected

$$\sigma_{33} = \sigma_{13} = \sigma_{31} = 0$$

- BUT: constitutive equations give

$$\epsilon_{23} = \epsilon_{13} = 0$$

$$\epsilon_{33} \neq 0 = \frac{\nu}{1-\nu}(\epsilon_{11} + \epsilon_{22})$$

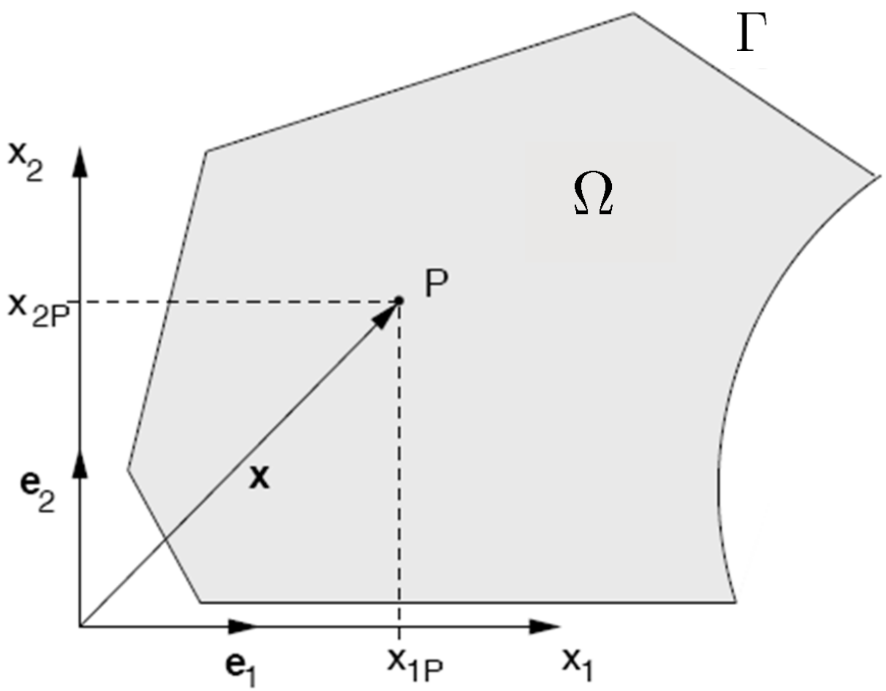
Poisson effect!

global coordinate system in the plane

- every point of the domain is specified by its location vector

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2$$

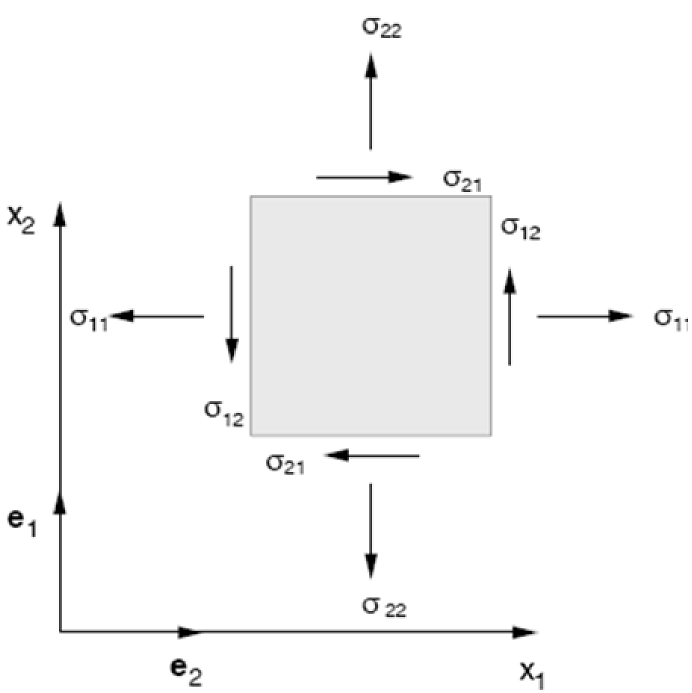
with $\mathbf{e}_i^T \mathbf{e}_j = \delta_{ij} \qquad i,j \in [1,2]$ basis vectors



state of stress

- stress tensor is symmetric
- stress components can be assembled in a vector (*Voigt notation*)

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

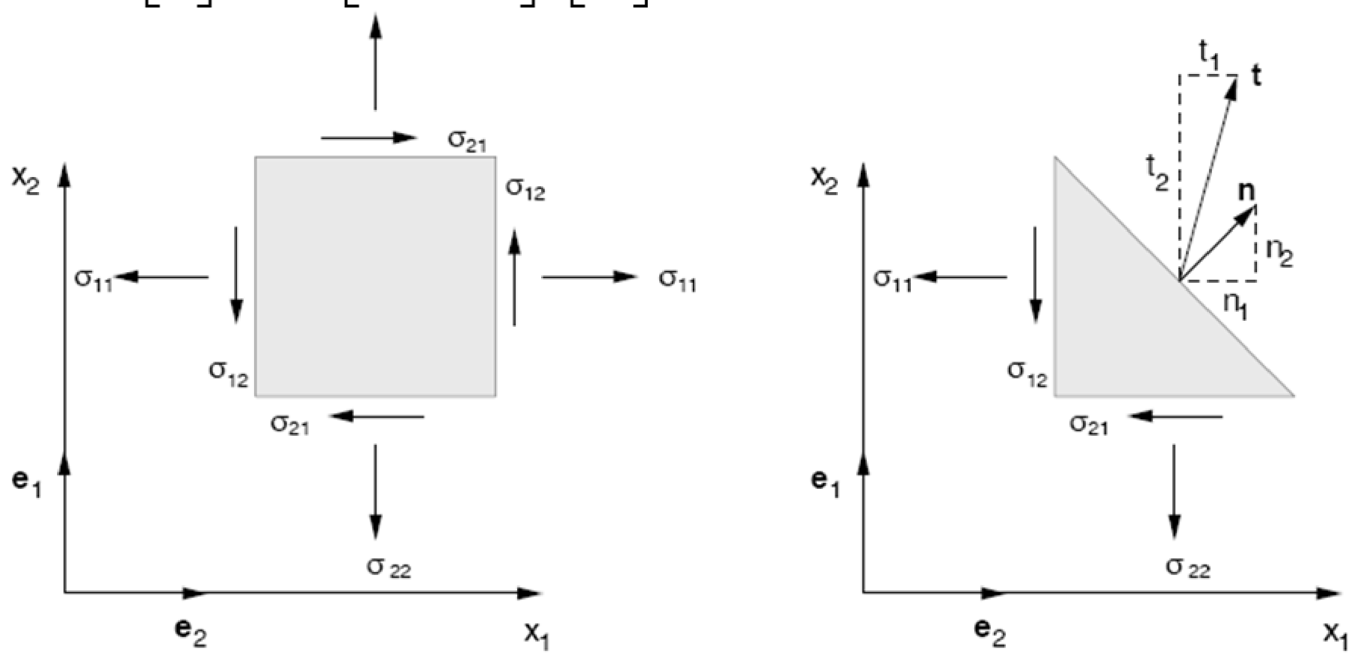


state of stress

- assume a known state of stress at point P
- the stress vector **t** on a cutting plane through P with normal vector **n** is obtained as

$$\mathbf{t} = \mathbf{S}\,\mathbf{n}$$

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$



state of strain

- transverse strain ϵ_{33} is computed from the normal strains ϵ_{11} and ϵ_{22}
- the strain coordinates

$$\epsilon_{ii} = \frac{\partial u_i}{\partial x_i} \qquad \epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad i \neq j \qquad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix}$$

are called *engineering strains* with origin in the mechanics of materials

- the strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \mathbf{E} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

is derived from continuum mechanics principles with a factor 0.5 for the shear strain coefficients

constitutive equations

- *assumption*: homogeneous, isotropic material properties

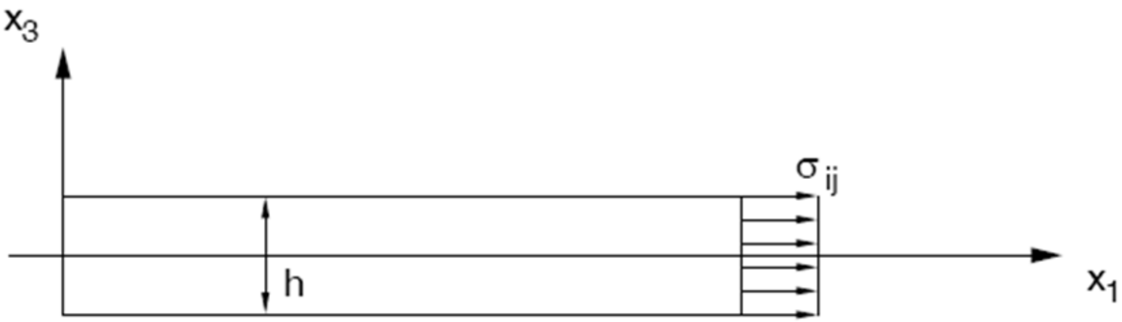
$$\begin{aligned}
\epsilon_{11} &= \frac{1}{E}(\sigma_{11} - \nu\,\sigma_{22}) \\
\epsilon_{22} &= \frac{1}{E}(\sigma_{22} - \nu\,\sigma_{11}) \\
\epsilon_{12} &= \frac{2(1+\nu)}{E}\sigma_{12}
\end{aligned}$$

- inversion gives the stress—strain relation

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbf{C}\,\boldsymbol{\epsilon} \\
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} &= \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix}
\end{aligned}$$

stress resultants

- *assumption*: constant stresses over the thickness



- integration over the thickness gives

$$\begin{aligned} s_{ij} &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{ij}\, dx_3 &&= h\,\sigma_{ij} \\ \mathbf{s} &= h\,\boldsymbol{\sigma} &&= h\,\mathbf{C}\,\boldsymbol{\epsilon} \end{aligned}$$

1. problem definition
   - analysis suited integral equations (3D formulation)
   - idealization (e.g. 2D plane strain/stress, 1D bar/beam, 2D heat flow, ...)

governing integral equations for the idealized system

here: plane stress

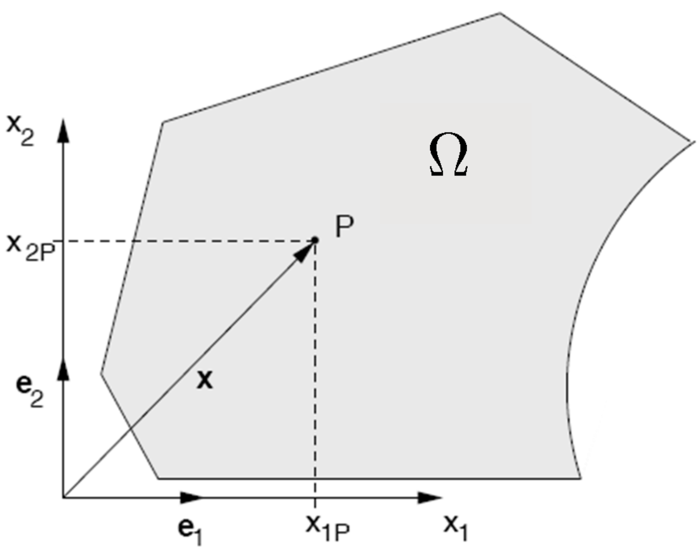
extended *Principle of Virtual Displacements*

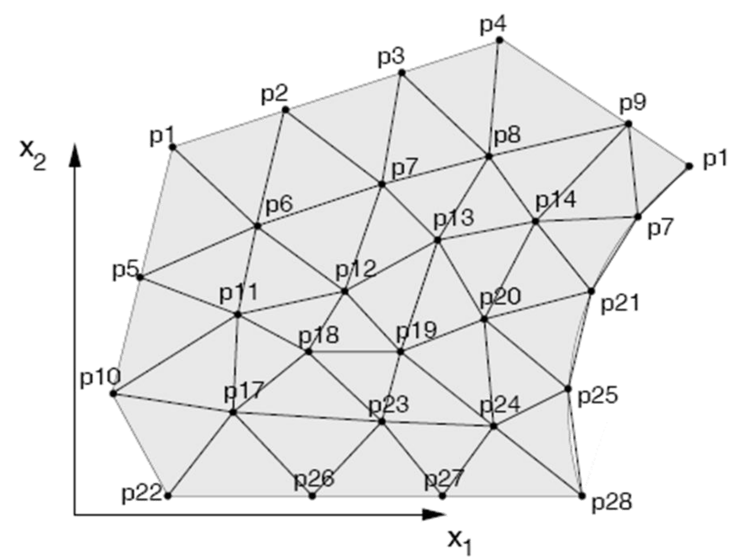
$$\int_\Omega \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, da \;=\; \int_\Omega \delta\mathbf{u}^T \mathbf{p}\, da + \int_{C_t} \sum_i \delta u_i\, t_{0i}\, ds + \int_{C_u} \sum_i \delta u_i\, t_i\, ds$$

$$\mathbf{x} \in C_u \;\Rightarrow\; u_i \;=\; u_{i0}$$

Ω	solution domain
C_u	set of points on Γ_u where displacements are prescribed
C_t	set of points on Γ_t where stresses are prescribed
$\delta\boldsymbol{\epsilon}$	variation of the state of strains
$\delta\mathbf{u}$	variation of the state of displacements
$\mathbf{p}$	load within the domain Ω
u_{0i}	prescribed displacement on Γ_u
t_{0i}	prescribed stress Γ_t

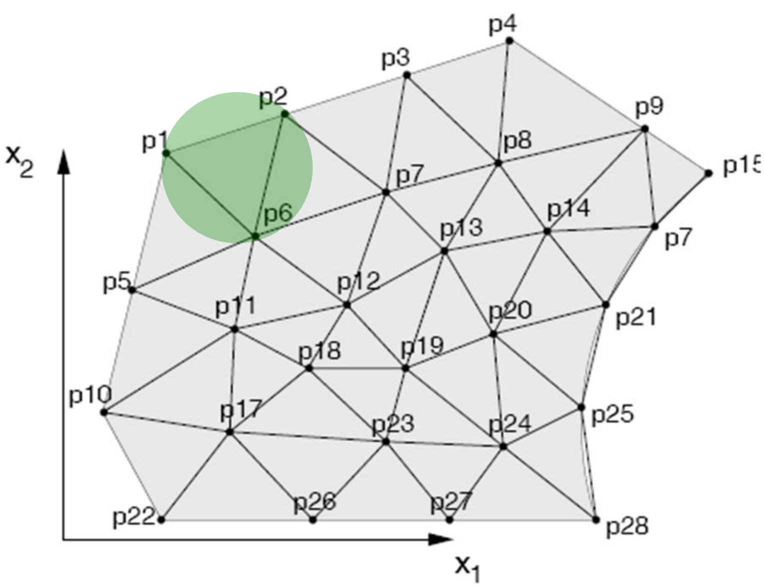
1. problem definition
2. discretization
   - introduction of a global coordinate system
   - definition of nodes, elements → generation of a consistent mesh
   - specification of degrees of freedom





discretization – topology

$$\begin{array}{ccc} \mathbf{u}_e & = & \mathbf{R}_e\,\mathbf{u}_s \\ (6\times 1) & & (6\times N)\,(N\times 1) \end{array}$$

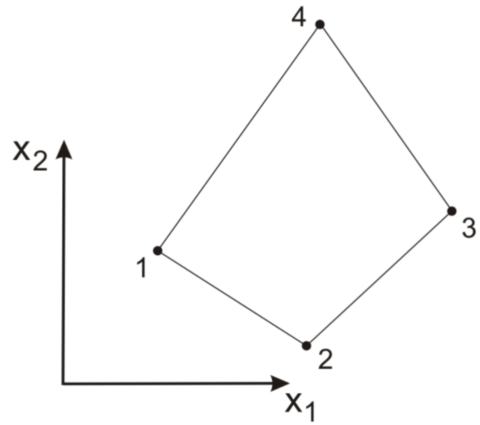


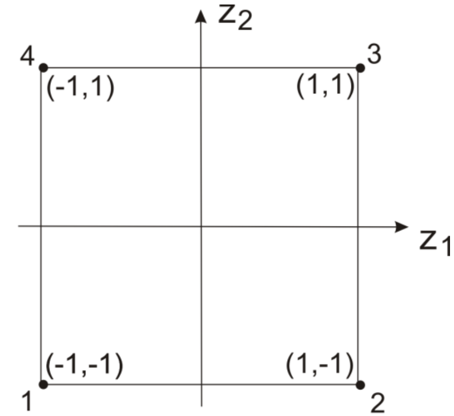
$$\begin{bmatrix} u_{1(p1)} \\ u_{2(p1)} \\ u_{1(p6)} \\ u_{2(p6)} \\ u_{1(p2)} \\ u_{2(p2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1(p1)} \\ u_{2(p1)} \\ u_{1(p2)} \\ u_{2(p2)} \\ \vdots \\ u_{1(p6)} \\ u_{2(p6)} \\ \vdots \\ u_{1(pN)} \\ u_{2(pN)} \end{bmatrix}$$

ROADMAP

1. **problem definition**
2. **discretization**
3. **element formulation**
   - **approximation of geometry & physical variables**
   - **derivation of algebraic equations for element and element load**

element properties – development of a bilinear element





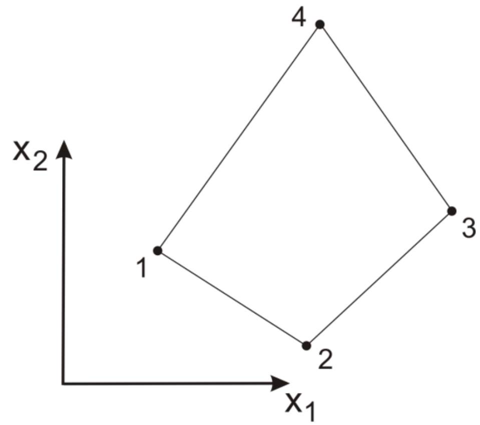
- approximation of the geometry

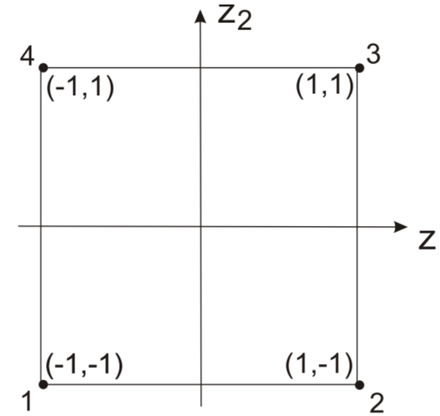
$$\mathbf{x} \quad = \quad \mathbf{x}_e^T \, \mathbf{N}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{1(1)} & x_{1(2)} & x_{1(3)} & x_{1(4)} \\ x_{2(1)} & x_{2(2)} & x_{2(3)} & x_{2(4)} \end{bmatrix} \frac{1}{4} \begin{bmatrix} (1-z_1)\,(1-z_2) \\ (1+z_1)\,(1-z_2) \\ (1+z_1)\,(1+z_2) \\ (1-z_1)\,(1+z_2) \end{bmatrix}$$

$$(-1 \;\leq\; z_1, z_2 \;\leq\; +1)$$

element properties





- approximation of the physics – displacements

$$\mathbf{u} \quad = \quad \hat{\mathbf{N}}^T \, \mathbf{u_e}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_{1(1)} \\ u_{2(1)} \\ u_{1(2)} \\ u_{2(2)} \\ u_{1(3)} \\ u_{2(3)} \\ u_{1(4)} \\ u_{2(4)} \end{bmatrix}$$

element properties

- approximation of the physics – strains

$$\boldsymbol{\epsilon} = \nabla \cdot \mathbf{u}$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- derivatives by application of the chain rule

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial z_k} \frac{\partial z_k}{\partial x_j} \qquad i, j, k \in \{1, 2\}$$

$$\frac{\partial u_i}{\partial z_1} \frac{\partial z_1}{\partial x_j} + \frac{\partial u_i}{\partial z_2} \frac{\partial z_2}{\partial x_j}$$

element properties

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} \qquad\qquad i,j,k \in \{1,2\}$$

$$\frac{\partial u_i}{\partial z_1}\frac{\partial z_1}{\partial x_j} + \frac{\partial u_i}{\partial z_2}\frac{\partial z_2}{\partial x_j}$$

- partial derivatives of the displacements w.r.t. *normalized* coordinates

$$\frac{\partial u_i}{\partial z_k} : \qquad \frac{\partial \mathbf{u}}{\partial z_k} = \frac{\partial}{\partial z_k}(\hat{\mathbf{N}}^T \, \mathbf{u_e}) = \hat{\mathbf{N}}^T_{,z} \, \mathbf{u_e}$$

- with

$$\mathbf{N}_{,z} = \begin{bmatrix} -(1-z_2) & -(1-z_1) \\ +(1-z_2) & -(1+z_1) \\ +(1+z_2) & +(1+z_1) \\ -(1+z_2) & +(1-z_1) \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{,z_1} & \mathbf{N}_{,z_2} \end{bmatrix}$$

element properties

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} \qquad i,j,k \in \{1,2\}$$

$$\frac{\partial u_i}{\partial z_1}\frac{\partial z_1}{\partial x_j} + \frac{\partial u_i}{\partial z_2}\frac{\partial z_2}{\partial x_j}$$

- partial derivatives of the displacements w.r.t. *normalized* coordinates

$$\frac{\partial u_i}{\partial z_k} : \qquad \frac{\partial \mathbf{u}}{\partial z_k} = \frac{\partial}{\partial z_k}(\hat{\mathbf{N}}^T\,\mathbf{u_e}) = \hat{\mathbf{N}}^T_{,z}\,\mathbf{u_e}$$

- partial derivative of the *normalized* coordinates w.r.t. *global* coordinates

$$\frac{\partial z_k}{\partial x_j} : \qquad \left(\frac{\partial \mathbf{x}}{\partial z_k}\right)^{-1} = \left(\mathbf{x}_e^T\,\frac{\partial}{\partial z_k}\mathbf{N}\right)^{-1} = \left(\mathbf{x}_e^T\,\mathbf{N}_{,z}\right)^{-1}$$

$$(\mathbf{X}_{,z})^{-1} := \mathbf{Z}_{,x}$$

element properties

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} \qquad i,j,k \in \{1,2\}$$

$$\frac{\partial u_i}{\partial z_1}\frac{\partial z_1}{\partial x_j} + \frac{\partial u_i}{\partial z_2}\frac{\partial z_2}{\partial x_j}$$

- partial derivatives of the displacements w.r.t. *normalized* coordinates

$$\frac{\partial u_i}{\partial z_k} : \qquad \frac{\partial \mathbf{u}}{\partial z_k} = \frac{\partial}{\partial z_k}(\hat{\mathbf{N}}^T \, \mathbf{u_e}) = \hat{\mathbf{N}}^T_{,z} \, \mathbf{u_e}$$

- partial derivative of the *normalized* coordinates w.r.t. *global* coordinates

$$\mathbf{X}_z(z_1, z_2) = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\mathbf{Z}_x(z_1, z_2) = \frac{1}{det\mathbf{X}_z} \begin{bmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{bmatrix} = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} \end{bmatrix}$$

element properties

- partial derivatives of the displacements w.r.t. *global* coordinates

$$
\begin{aligned}
\frac{\partial u_i}{\partial x_j} &= \frac{\partial u_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} \\
&= \mathbf{u_e}^T \, \hat{\mathbf{N}}_{,z} \, \mathbf{Z}_{,x} \\
&= \mathbf{u_e}^T \, \hat{\mathbf{N}}_{,x} \\
&= \begin{bmatrix} \mathbf{u}_{1_e}^T \\ \mathbf{u}_{2_e}^T \end{bmatrix} \begin{bmatrix} \mathbf{N}_{,x_1} & \mathbf{N}_{,x_2} \end{bmatrix}
\end{aligned}
$$

element properties

- interpolation of the engineering strains

$$\boldsymbol{\epsilon} \;=\; \nabla \cdot \mathbf{u} \qquad\qquad \rightarrow \;\; \mathbf{B}_e\,\mathbf{u}_e$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{N}^T_{,x_1} & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{N}^T_{,x_2} \\ \mathbf{N}^T_{,x_2} & \mathbf{N}^T_{,x_1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1_e} \\ \mathbf{u}_{2_e} \end{bmatrix}$$

analytic algebraic

- interpolation of the engineering stresses

$$\boldsymbol{\sigma} \;=\; \mathbf{C}\,\boldsymbol{\epsilon} \;\rightarrow\; \mathbf{C}_e\,\mathbf{B}_e\,\mathbf{u}_e$$

- element stiffness

$$\mathbf{K}_e = \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}\, \mathbf{B}_e \, da$$
$$= \int_{x_1} \int_{x_2} \mathbf{B}_e^T \mathbf{C}\, \mathbf{B}_e \, dx_2 \, dx_1$$
$$= \int_{z_1} \int_{z_2} \mathbf{B}_e^T \mathbf{C}\, \mathbf{B}_e \, det \mathbf{X}_{,z} \, dz_2 \, dz_1$$

- numerical integration (e.g. Gauss quadrature)

$$\mathbf{K}_e = \sum_n \sum_m \mathbf{B}_e^T(a_n, a_m)\, \mathbf{C}\, \mathbf{B}_e^T(a_n, a_m)\; det\, \mathbf{X}_{,z}(a_n, a_m)\; g_n\, g_m$$

a_n, a_m	coordinates of quadrature points
g_n, g_m	weights of quadrature points

- element load

$$\mathbf{p}_e = \int_{\Omega_e} \hat{\mathbf{N}}_e \, \mathbf{p}_0 \, da$$
$$= \int_{x_1} \int_{x_2} \hat{\mathbf{N}}_e \, \mathbf{p}_0 \, dx_2 \, dx_1$$
$$= \int_{z_1} \int_{z_2} \hat{\mathbf{N}}_e \, \mathbf{p}_0 \, det\mathbf{X}_{,z} \, dz_2 \, dz_1$$

- numerical integration (e.g. Gauss quadrature)

$$\mathbf{p}_e = \sum_n \sum_m \mathbf{N}_e(a_n, a_m) \, \mathbf{p}_0(a_n, a_m) \, det \, \mathbf{X}_{,z}(a_n, a_m) \; g_n \, g_m$$

a_n, a_m coordinates of quadrature points

g_n, g_m weights of quadrature points

ROADMAP

1. problem definition
2. discretization
3. element formulation
4. assembly – system formulation
   - relation between element and system degrees of freedom
     → topological relation
   - variation of Ansatz variables (displacements, strains, temperatures, ...)

system properties

- interpolation of the displacements in the solution domain

$$\mathbf{u} \;=\; \{\sum_{e} \hat{\mathbf{N}}^T \, \mathbf{R}_e\} \, \mathbf{u}_s$$

- interpolation of the strains in the solution domain

$$\boldsymbol{\epsilon} \;=\; \{\sum_{e} \mathbf{B}_e \, \mathbf{R}_e\} \, \mathbf{u}_s$$

system properties

- variation of the displacements in the solution domain

$$\delta \mathbf{u}_{s_k} \;:=\; \frac{\partial \mathbf{u}_s}{\partial u_k} \;=\; \{\sum_e \hat{\mathbf{N}}^T \, \mathbf{R}_e\} \, \frac{\partial \mathbf{u}_s}{\partial u_k} \;=\; \{\sum_e \hat{\mathbf{N}}^T \, \mathbf{R}_e\} \, \mathbf{e}_k$$

- variation of the strains in the solution domain

$$\delta \boldsymbol{\epsilon}_{s_k} \;:=\; \frac{\partial \boldsymbol{\epsilon}_s}{\partial u_k} \;=\; \{\sum_e \mathbf{B}_e \, \mathbf{R}_e\} \, \frac{\partial \mathbf{u}_s}{\partial u_k} \;=\; \{\sum_e \mathbf{B}_e \, \mathbf{R}_e\} \, \mathbf{e}_k$$

extended *Principle of Virtual Displacements*

$$\int_\Omega \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, da \;=\; \int_\Omega \delta\mathbf{u}^T \mathbf{p}\, da + \int_{C_t} \sum_i \delta u_i\, t_{0i}\, ds + \int_{C_u} \sum_i \delta u_i\, t_i\, ds$$

$$\mathbf{x} \in C_u \;\Rightarrow\; u_i \;=\; u_{i0}$$

governing algebraic equations

$$\int_\Omega \mathbf{e}_k^T (\sum_e \mathbf{R}_e^T \mathbf{B}_e^T)\, \mathbf{C}_e\, (\sum_e \mathbf{B}_e \mathbf{R}_e)\, \mathbf{u}_s\; da \;=\; \int_\Omega \mathbf{e}_k^T (\sum_e \mathbf{R}_e^T \hat{\mathbf{N}}_e)\, \mathbf{p}_0\; da + \mathcal{A}$$

$$\mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}_e\, \mathbf{B}_e\, da\; \mathbf{R}_e\}\; \mathbf{u}_s \;=\; \mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \int_{\Omega_e} \hat{\mathbf{N}}_e\, \mathbf{p}_0\; da\} + \mathcal{A}$$

$$\mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \mathbf{K}_e\, \mathbf{R}_e\}\; \mathbf{u}_s \;=\; \mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T\, \mathbf{p}_e\} + \mathcal{A}$$

$$\mathbf{K}_s\, \mathbf{u}_s \;=\; \mathbf{p}_s + \mathcal{A}$$

governing algebraic equations

$$\int_\Omega \mathbf{e}_k^T (\sum_e \mathbf{R}_e^T \mathbf{B}_e^T)\, \mathbf{C}_e\, (\sum_e \mathbf{B}_e \mathbf{R}_e)\, \mathbf{u}_s \; da \;=\; \int_\Omega \mathbf{e}_k^T (\sum_e \mathbf{R}_e^T \hat{\mathbf{N}}_e)\, \mathbf{p}_0 \; da + \mathcal{A}$$

$$\mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}_e \mathbf{B}_e \, da \, \mathbf{R}_e\} \, \mathbf{u}_s \;=\; \mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \int_{\Omega_e} \hat{\mathbf{N}}_e \, \mathbf{p}_0 \, da\} + \mathcal{A}$$

$$\mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \mathbf{K}_e \mathbf{R}_e\} \, \mathbf{u}_s \;=\; \mathbf{e}_k^T \sum_e \{\mathbf{R}_e^T \mathbf{p}_e\} + \mathcal{A}$$

$$\mathbf{K}_s \, \mathbf{u}_s \;=\; \mathbf{p}_s + \mathcal{A}$$

$\mathbf{K}_e$ element stiffness matrix (8×8)

$\mathbf{K}_s$ system stiffness matrix $(N \times N)$

$\mathbf{u}_s$ system primal vector $(N \times 1)$

$\mathbf{p}_s$ system load vector $(N \times 1)$

$\mathbf{e}_k$ k^{th} unit vector $(N \times 1)$

$\mathcal{A}$ surface load + concentrated node loads

1. problem definition
2. discretization
3. element formulation
4. assembly – system formulation
5. application of boundary conditions
   - essential boundary conditions (e.g. prescribed displacements)
   - natural boundary conditions (boundary loads)

preprocessing

1. **problem definition**
2. **discretization**
3. **element formulation**
4. **assembly – system formulation**
5. **application of boundary conditions**
6. **solution of the governing system of equations**
   - **linear system of equations**
   - **direct / iterative solution**

preprocessing

solution

1. problem definition
2. discretization
3. element formulation
4. assembly – system formulation
5. application of boundary conditions
6. solution of the governing system of equations
7. post-processing
   - interpolation of displacements, strains, stresses, ... based on the discrete solution
   - exploration of the model response, localization of stress concentrations, singularities, ...

preprocessing

solution

postprocessing

ROADMAP

1. problem definition
2. discretization
3. element formulation
4. assembly
5. boundary conditions

6. solution step

7. post-processing