FINITE ENERGY SOLUTIONS OF MIXED 3D DIV-CURL SYSTEMS

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Abstract. This paper describes the existence and representation of certain finite energy (L^2-) solutions of weighted div-curl systems on bounded 3d regions with C^2-boundaries and mixed boundary data. Necessary compatibility conditions on the data for the existence of solutions are described. Subject to natural integrability assumptions on the data, it is then shown that there exist L^2-solutions whenever these compatibility conditions hold. The existence results are proved by using a weighted orthogonal decomposition theorem for L^2-vector fields in terms of scalar and vector potentials. This representation theorem generalizes the classical Hodge-Weyl decomposition. With this special choice of the potentials, the mixed div-curl problem decouples into separate problems for the scalar and vector potentials. Variational principles for the solutions of these problems are described. Existence theorems, and some estimates, for the solutions of these variational principles are obtained. The unique solution of the mixed system that is orthogonal to the null space of the problem is found and the space of all solutions is described.

The second part of the paper treats issues concerning the non-uniqueness of solutions of this problem. Under additional assumptions, this space is shown to be finite dimensional and a lower bound on the dimension is described. Criteria that prescribe the harmonic component of the solution are investigated. Extra conditions that determine a well-posed problem for this system on a simply connected region are given. A number of conjectures regarding the results for bounded regions with handles are stated.

1. Introduction

The question to be studied here is: Given a Lebesgue-integrable real-valued function ρ, vector field ω, and a positive-definite matrix valued function ε, see (2.1) below, defined on a bounded region $\Omega \subset \mathbb{R}^3$, what can be said about the existence, and uniqueness, of weak solutions of the system

$$\operatorname{div}\big(\varepsilon(x)v(x)\big) = \rho(x) \quad \text{and} \tag{1.1}$$

$$\operatorname{curl} v(x) = \omega(x) \quad \text{for } x \in \Omega, \tag{1.2}$$

subject to the mixed boundary conditions (2.4)–(2.5) below? In particular what compatibility (necessary) conditions are required for this system to have a weak solution and how can finite energy (that is, L^2-) solutions be characterized? This is a system of four linear first order equations for three unknowns which requires some necessary conditions for solvability. A well-known condition is that $\operatorname{div}\omega \equiv 0$. In this paper it is shown that,

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subject to some natural assumptions, there is a necessary and sufficient condition for these *mixed boundary value problems* to have solutions.

The system (1.1)–(1.2) is fundamental in fluid mechanics and electromagnetic field theory. Maxwell's equations for an electromagnetic field are usually written in this form. When the normal, respectively tangential, components of the field are prescribed alone on the boundary this problem was studied in our recent paper [6]. In particular, that paper describes how, when the region has non-trivial topology, the boundary value problem may have non-unique solutions. To obtain a well-posed problem certain integrals, which have both physical and geometrical interpretations, of the solution must also be prescribed. There are many electromagnetic situations, where physical modeling requires that the normal component of the field be prescribed on part of the boundary and the tangential component of the field elsewhere. See, for example, the texts of Jackson [11, Section 3.12] or Hanson and Yakovlev [10, Chapter 6, Section 1] for discussions of this.

Section I of this paper describes the existence of finite-energy solutions of such problems. The main tool used here is a special orthogonal decomposition of the space $L^2(\Omega;\mathbb{R}^3)$ involving scalar and vector potentials and certain ε-harmonic vector fields. This is a special Hodge type representation that, when substituted in the boundary value problem, results in a decomposition of the problem into individual problems for the scalar and vector potentials. Variational principles for these problems are developed and studied. In particular Theorem 9.1 shows that condition (C2) in Section 6, is a necessary and sufficient condition for existence of solutions under natural conditions on the data. When the subspace $H_{DC\Sigma_\nu}(\Omega)$ of *mixed ε-harmonic vector fields* on Ω is non-zero there is non-uniqueness of solutions of the boundary value problem and Corollary 9.2 describes the set of all solutions of the boundary value problem. A priori bounds on a special solution of the problem are described, although there is no such bound on a general solution.

Section II is devoted to investigating this non-uniqueness and determining what extra functionals of a solution must be prescribed to have a well-posed problem. A first issue is to show that this space $H_{DC\Sigma_\nu}(\Omega)$ is finite dimensional. We are only able to prove this under some extra assumptions on the coefficients, although we conjecture that this holds more generally; see the end of Section 10. In Section 11, a lower bound on this dimension is obtained by constructing the subspace of gradient mixed ε-harmonic vector fields on Ω. This space is shown to have dimension M when the subset Σ_τ of $\partial\Omega$ where tangential boundary data is prescribed has $M+1$ connected components. When the region is not simply connected there are a number of open questions about the possible mixed ε-harmonic fields. Some conjectures about the geometrical interpretation of the dimension of this space is described in Section 12. These results enable us to describe extra criteria that guarantee well-posedness of this mixed div-curl boundary value problem when the region Ω is simply connected. This is done in Section 13. When Ω is not simply connected, a knowledge of the possible non-gradient mixed ε-harmonic fields on Ω is required before such well-posedness questions can be resolved.

The primary tools to be used here are variational methods for functionals on Sobolev spaces; for background on these topics see Zeidler [14] or Blanchard and Bruning [7].

Given the scientific and engineering importance of these equations, and the extensive mathematical development of these subjects, it is very surprising how little has been published about this general system. There is an extensive literature on special cases and two-dimensional models, as any review of texts on electromagnetic field theory will verify. A treatment of the analogous two-dimensional problem in bounded regions that is similar to the approach to be developed here may be found in our paper [5]. Other results for 3d mixed problems have recently been published by Fernandes and Gilardi [9] and also by Alonso and Valli [1]. The methods used here are quite different to theirs and our solvability results require different assumptions, including weaker conditions on the given data. The approach adopted here, based on the use of variational methods and potentials, may well provide a useful framework for computational simulations of these problems.

2. Assumptions and Notation

In this paper, we use similar definitions and notation to Auchmuty [3] or Auchmuty and Alexander [6]. In particular when a term is not defined here it should be taken in the sense used there. The requirements on the region Ω are the following:

Condition B1. *Ω is a bounded region in $\mathbb{R}^3$ and $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces; each surface having finite surface area.*

When (B1) holds and $\partial\Omega$ consists of $J+1$ disjoint, closed surfaces, then J is the *second Betti number* of Ω, or the dimension of the second de Rham cohomology group of Ω. Geometrically it counts the number of "holes" in the region Ω. The requirements for the coefficient matrix in (1.1) are usually:

Condition E1. *$\varepsilon(x) := \big(e_{jk}(x)\big)$ is a symmetric matrix valued function with each component e_{jk} continuous on $\bar{\Omega}$ and there are positive constants e_0 and e_1 such that,*

$$e_0\,|u|^2 \;\le\; \big(\varepsilon(x)u\big)\cdot u \;\le\; e_1\,|u|^2 \qquad \text{for all } \; x \in \Omega,\; u \in \mathbb{R}^3. \tag{2.1}$$

Define $L^2(\Omega;\mathbb{R}^3)$ to be the Hilbert space of all L^2-vector fields on Ω taking values in $\mathbb{R}^3$. The standard inner product on $L^2(\Omega;\mathbb{R}^3)$ is

$$\langle\, u, v\,\rangle := \int_\Omega u(x)\cdot v(x)\, d^3x. \tag{2.2}$$

We also use the weighted inner product

$$\langle\, u, v\,\rangle_\varepsilon \;:=\; \int_\Omega \big(\varepsilon(x)u(x)\big)\cdot v(x)\, d^3x. \tag{2.3}$$

The norm induced by this inner product is denoted $\|u\|_\varepsilon$. When ε satisfies (E1), this inner product and norm are equivalent to the standard one on $L^2(\Omega;\mathbb{R}^3)$. Two subspaces V, W of $L^2(\Omega;\mathbb{R}^3)$ are said to be *ε-orthogonal* when they are orthogonal with respect to (2.3).

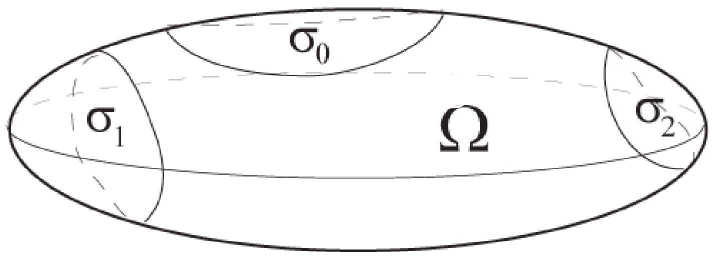

FIGURE 1. The topology of a domain Ω in R^3 supporting mixed div-curl problems. The regions σ_0, σ_1, σ_2 are connected subsets of the boundary $\partial\Omega$. We assume Condition B3 is satisfied. Then $\Sigma_\tau := \sigma_0 \cup \sigma_1 \cup \sigma_2$ has $M+1=3$ components and Σ_ν is connected. Here Ω is simply connected, and thus Corollary 13.4 states that the (affine) space of finite-energy fields satisfying the mixed div-curl problem (1.1)–(1.2) and (2.4)–(2.5) has dimension $M = 2$, where two degrees of freedom can be parametrized by the differences of a scalar potential on σ_1 and σ_0 and on σ_2 and σ_0.

Let Σ_ν and Σ_τ be disjoint, nonempty, open subsets of $\partial\Omega$, with $\bar{\Sigma}_\nu \cup \bar{\Sigma}_\tau = \partial\Omega$. The problem to be studied here is the solvability of the system (1.1)–(1.2) subject to prescribed *mixed boundary data* μ, η with

$$(\varepsilon v) \cdot \nu = \mu \quad \text{on } \Sigma_\nu, \quad \text{and} \tag{2.4}$$

$$v \wedge \nu = \eta \quad \text{on } \Sigma_\tau, \tag{2.5}$$

The set $\bar{\Sigma}_\nu \cap \bar{\Sigma}_\tau$ is the *transition set* for this boundary data. It is a closed set which is the (common) boundary of each of Σ_ν and Σ_τ and no boundary conditions are imposed on this interface.

We require the following integrability conditions on the data.

Condition C1. *$\rho \in L^{6/5}(\Omega)$, $\omega \in L^{6/5}(\Omega;\mathbb{R}^3)$, $\mu \in L^{4/3}(\Sigma_\nu)$ and $\eta \in L^{4/3}(\Sigma_\tau;\mathbb{R}^3)$ with $\eta \cdot \nu \equiv 0$ on Σ_τ.*

Sometimes μ and η are regarded as functions on $\partial\Omega$ in which case their extensions are taken to be identically zero outside their original domain. When v is a vector field on the boundary of the region Ω then its (outward) normal component is $v \cdot \nu$, and the field is said to be tangential on a subset Σ of $\partial\Omega$ if $v \cdot \nu \equiv 0$ on Σ.

Our results on the solvability of this problem depend on the topology of the sets Σ_ν and Σ_τ. The following conditions are required.

Condition B2. *Σ is a non-empty open subset of $\partial\Omega$ with $M+1$ disjoint nonempty open components $\{\sigma_0, \sigma_1, \ldots, \sigma_M\}$ and there is a positive d_0 such that $d(\sigma_j, \sigma_m) \geq d_0$ when $j \neq m$ and $M \geq 1$. The boundary of each σ_m is either empty or a $C^{1,1}$ curve.*

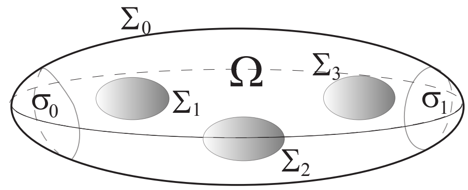

FIGURE 2. A second example. The boundary of Ω need not be connected. Here Ω is the region interior to the surface Σ_0 and exterior to the cavities bounded by Σ_1, Σ_2, Σ_3. As in Figure 1, σ_0 and σ_1 are connected subsets of Σ_0. The subset Σ_τ of $\partial\Omega$ is the union $\sigma_0 \cup \sigma_1 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, and hence has $M+1 = 5$ components. We assume Condition B3 is satisfied. Corollary 13.4 states that the (affine) space of finite-energy fields satisfying the mixed div-curl problem (1.1)–(1.2) and (2.4)–(2.5) has dimension $M = 4$; again the degrees of freedom are parametrized by differences in a scalar potential.

Here $d(\sigma_j, \sigma_m)$ is the Euclidean distance between the sets σ_j and σ_m. When (B2) holds, the complement $\partial\Omega \backslash \Sigma$ is a non-empty compact set consisting of a finite number of closed connected subsets of $\partial\Omega$. Let $\tilde{\Sigma}$ be the interior of $\partial\Omega\backslash\Sigma$; then $\tilde{\Sigma}$ also satisfies (B2), perhaps with a different number of components. Our standard assumption is:

Condition B3. *Ω and $\partial\Omega$ satisfy (B1), Σ_τ and Σ_ν satisfy (B2), possibly with different numbers of components.*

3. Sobolev Spaces of Functions and Fields

To treat problems with mixed boundary conditions we introduce Sobolev spaces of functions and fields which have specific boundary behavior defined by integral conditions. Throughout this Section $\Omega, \partial\Omega$ are assumed to satisfy (B1). Define $H_{DC}(\Omega)$ to be the space of fields $v \in L^2(\Omega; \mathbb{R}^3)$ whose weak divergence and curl are in $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^3)$ respectively. This is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{DC} \;:=\; \int_\Omega [u(x) \cdot v(x) \;+\; \operatorname{div} u. \operatorname{div} v \;+\; \operatorname{curl} u \cdot \operatorname{curl} v]\, d^3x. \tag{3.1}$$

This space is the intersection of $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$ and its properties are discussed in various texts including Dautray and Lions [8, Chapter 9]. In particular, fields in $H_{DC}(\Omega)$ have well-defined normal and tangential traces on the boundary. It is worth noting that there are vector fields in $H_{DC}(\Omega)$ which are not in $H^1(\Omega, \mathbb{R}^3)$. A field is in $H_{DC0}(\Omega)$ if it is in $H_{DC}(\Omega)$ and also

$$v \cdot \nu \;=\; 0 \qquad \text{and} \;\; v \wedge \nu \;=\; 0 \qquad \text{on} \;\; \partial\Omega.$$

To incorporate the boundary conditions we assume Σ is a nonempty open subset of $\partial\Omega$ which satisfies (B2) and $\tilde{\Sigma} := \partial\Omega \backslash \bar{\Sigma}$ is also a non-empty open subset of $\partial\Omega$. For such Σ, let $H^1_{\Sigma 0}(\Omega)$ be the space of all functions $\varphi \in H^1(\Omega)$, whose trace on Σ is zero. Define

$$G_\Sigma(\Omega) \;:=\; \{\nabla\varphi : \varphi \in H^1_{\Sigma 0}(\Omega)\}. \tag{3.2}$$

When $v \in H_{DC}(\Omega)$, we say that $v \cdot \nu = 0$ *weakly* on $\tilde{\Sigma}$ provided

$$\int_\Omega [\, v \cdot \nabla\varphi \;+\; \varphi \operatorname{div} v \,]\, d^3x \;=\; 0 \qquad \text{for all } \;\varphi \in H^1_{\Sigma 0}(\Omega). \tag{3.3}$$

Let $C^1_{\Sigma 0}(\Omega : \mathbb{R}^3)$ be the space of continuously differentiable vector fields on Ω which also are continuous on $\bar{\Omega}$ and satisfy

$$v \cdot \nu \;=\; 0 \qquad \text{on } \;\Sigma \qquad \text{and } \;\; v \wedge \nu \;=\; 0 \qquad \text{on } \;\tilde{\Sigma}. \tag{3.4}$$

We say that $v \wedge \nu = 0$ *weakly* on Σ provided

$$\int_\Omega [\, u \cdot \operatorname{curl} v - v \cdot \operatorname{curl} u \,]\, d^3x \;=\; 0 \qquad \text{for all } \;u \in C^1_{\Sigma 0}(\Omega : \mathbb{R}^3). \tag{3.5}$$

When v is continuous on $\bar{\Omega}$, continuously differentiable on Ω and in $H_{DC}(\Omega)$ then these definitions agree with the classical definitions as a consequence of the Gauss-Green Theorem.

Let $H_{DC\Sigma}(\Omega)$ denote be the closure of the space $C^1_{\Sigma 0}(\Omega : \mathbb{R}^3)$ with respect to the DC-inner product (3.1). Such fields are in $H_{DC}(\Omega)$ and satisfy the null boundary conditions (3.4) in the weak sense defined above. Many properties of this space were proved in Auchmuty [3] and a number of results from that paper are used here.

Lemma 3.1. *Assume (B1) holds and Σ is a non-empty open subset of $\partial\Omega$; then $H^1_{\Sigma 0}(\Omega)$ is a closed subspace of $H^1(\Omega)$. If $\nabla\varphi \in G_\Sigma(\Omega)$, then $\nabla\varphi \wedge \nu = 0$ weakly on Σ.*

Proof. Let $\{\varphi_n : n \geq 1\}$ be a sequence of functions in $H^1_{\Sigma 0}(\Omega)$ which converge to a limit φ in $H^1(\Omega)$. Then $\varphi_n \to \varphi \in L^2(\partial\Omega, d\sigma)$ from [9, Section 4.3, Theorem 1]. Moreover, $\int_{\partial\Omega} \varphi_n \psi \; d\sigma \;=\; 0$ for all ψ which are continuous on $\partial\Omega$ and have compact support in Σ since the trace of each $\varphi_n = 0$ on Σ. Let n increase to ∞ then, by continuity, the same is true for φ, and the subspace is closed. Suppose now that φ is C^1 on $\bar{\Omega}$ then the Gauss-Green Theorem yields that

$$\int_\Omega \nabla\varphi \cdot \operatorname{curl} A \; d^3x \;=\; \int_\Sigma A \cdot (\nabla\varphi \wedge \nu) \; d\sigma \;+\; \int_{\tilde{\Sigma}} A \cdot (\nabla\varphi \wedge \nu) \; d\sigma. \tag{3.6}$$

$$\;=\; \int_\Sigma A \cdot (\nabla\varphi \wedge \nu)\, d\sigma \qquad \text{for all } \;A \in \; C^1_{\Sigma 0}(\Omega : \mathbb{R}^3). \tag{3.7}$$

When φ and $\partial\Omega$ are smooth the first term on the right hand side is zero from classical calculus so the left hand side is zero. By density this then holds for all $\varphi \in H^1_{\Sigma 0}(\Omega)$, so the second sentence of the lemma follows upon substituting $\nabla\varphi$ for v in (3.5). □

Define the space

$$\operatorname{Curl}_{\varepsilon\Sigma}(\Omega) \;:=\; \{\varepsilon^{-1} \operatorname{curl} A : A \in H_{DC\Sigma}(\Omega)\}. \tag{3.8}$$

An essential result about this space is the following

Lemma 3.2. *Suppose* $\Sigma, \partial\Omega$ *satisfy (B1); then* $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$ *is a subspace of* $L^2(\Omega;\mathbb{R}^3)$ *that is* ε*-orthogonal to* $G_\Sigma(\Omega)$*. Moreover* $\mathrm{curl}\, A\cdot\nu \in H^{-1/2}(\partial\Omega)$ *and* $\mathrm{curl}\, A\cdot\nu = 0$ *weakly on* $\tilde{\Sigma}$*.*

Proof. When ε satisfies condition (E1), so does ε^{-1} and thus $\varepsilon^{-1}\,\mathrm{curl}\, A \in L^2(\Omega;\mathbb{R}^3)$ as $\mathrm{curl}\, A \in L^2(\Omega;\mathbb{R}^3)$ whenever $A \in H_{DC\Sigma}(\Omega)$. Suppose $w := \nabla\varphi \in G_\Sigma(\Omega)$ and $v := \varepsilon^{-1}\,\mathrm{curl}\, A \in \mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$; then

$$\langle\, v, w\,\rangle_\varepsilon \;=\; \int_\Omega \nabla\varphi\cdot\mathrm{curl}\, A\; d^3x.$$

When $A \in C^1_{\Sigma 0}(\Omega : \mathbb{R}^3)$, then (3.6) holds as above, so $\langle\, v, w\,\rangle_\varepsilon \;=\; 0$ from Lemma 3.1 and the boundary condition on A. For general $A \in H_{DC\Sigma}(\Omega)$, let $\{A^{(m)} : m \geq 1\}$ be a sequence of fields in $C^1_{\Sigma 0}(\Omega : \mathbb{R}^3)$ which converge to A in the DC-norm. For each $m \geq 1$, we have $\langle\, w, \varepsilon^{-1}\,\mathrm{curl}\, A^{(m)}\,\rangle_\varepsilon \;=\; 0$, so taking limits one finds $\langle\, v, w\,\rangle_\varepsilon \;=\; 0$ again and the spaces are ε-orthogonal as claimed. For any $\varphi \in H^1(\Omega)$ and a smooth field A on $\bar{\Omega}$, the divergence theorem yields

$$\int_\Omega \nabla\varphi\cdot\mathrm{curl}\, A\; d^3x \;=\; \int_{\partial\Omega} \varphi\,(\mathrm{curl}\, A\cdot\nu)\; d\sigma. \tag{3.9}$$

When $A \in H_{DC}(\Omega)$, this and the H^1-trace theorem imply that $\mathrm{curl}\, A\cdot\nu \in H^{-1/2}(\partial\Omega)$ as the left hand side is continuous. When $\varphi \in H^1_{\Sigma 0}(\Omega)$ and $A \in H_{DC\Sigma}(\Omega)$, this integral is zero from the first part of the lemma, so $\mathrm{curl}\, A\cdot\nu = 0$ on $\tilde{\Sigma}$ as (3.3) holds with v replaced by $\mathrm{curl}\, A$. □

Lemma 3.3. *Suppose* $\Sigma, \partial\Omega$ *satisfy (B1). A field* $v \in H_{DC}(\Omega)$ *is* ε*-orthogonal to* $G_\Sigma(\Omega)$ *if and only if, in a weak sense,*

$$\mathrm{div}(\varepsilon v) \;=\; 0 \qquad \text{on}\;\; \Omega \qquad \text{and}\;\; (\varepsilon v)\cdot\nu \;=\; 0 \qquad \text{on}\;\; \tilde{\Sigma}. \tag{3.10}$$

It is ε*-orthogonal to* $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$ *if and only if, in a weak sense,*

$$\mathrm{curl}\, v \;=\; 0 \qquad \text{on}\;\; \Omega \qquad \text{and}\;\; v\wedge\nu \;=\; 0 \qquad \text{on}\;\; \Sigma. \tag{3.11}$$

Proof. When v is ε-orthogonal to $G_\Sigma(\Omega)$ use of the Gauss-Green Theorem yields

$$\int_{\partial\Omega} \varphi((\varepsilon v)\cdot\nu)\, d\sigma \;-\; \int_\Omega \varphi\,\mathrm{div}(\varepsilon v)\; d^3x \;=\; 0 \qquad \text{for all}\;\; \varphi \in H^1_{\Sigma 0}(\Omega).$$

This is the weak form of (3.10) from (3.3). When $A \in H_{DC\Sigma}(\Omega),\; v \in H_{DC}(\Omega)$ then, from Gauss-Green,

$$\langle\, v, \varepsilon^{-1}\,\mathrm{curl}\, A\,\rangle_\varepsilon \;=\; \int_\Omega v\cdot\mathrm{curl}\, A\; d^3x \;=\; \int_{\partial\Omega} A\cdot(v\wedge\nu)\;\, d\sigma \;+\; \int_\Omega A\cdot\mathrm{curl}\, v\; d^3x.$$

This surface integral may be written

$$\int_{\partial\Omega} A\cdot(v\wedge\nu)\;\, d\sigma \;=\; \int_\Sigma A\cdot(v\wedge\nu)\; d\sigma \;+\; \int_{\tilde{\Sigma}} v\cdot(\nu\wedge A)\; d\sigma.$$

When v is ε-orthogonal to $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$, this yields

$$0 = \int_{\Sigma} A \cdot (v \wedge \nu)\, d\sigma + \int_{\Omega} A \cdot \mathrm{curl}\, v \; d^3x \qquad \text{for all } A \in H_{DC\Sigma}(\Omega). \tag{3.12}$$

This is the weak form of (3.11) and the result follows. □

Lemma 3.2 states that $G_{\Sigma}(\Omega)$ and $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$ are ε-orthogonal subspaces of $L^2(\Omega;\mathbb{R}^3)$; in general they do not span $L^2(\Omega;\mathbb{R}^3)$. Define $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ to be the class of all vector fields in $L^2(\Omega;\mathbb{R}^3)$ which are ε-orthogonal to $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$ and also to $G_{\Sigma}(\Omega)$. This definition guarantees that $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^3)$ and may be characterized explicitly as follows.

Lemma 3.4. *A vector field $h \in L^2(\Omega;\mathbb{R}^3) \in \mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ if and only if it satisfies*

$$\int_{\Omega} (\varepsilon h) \cdot \nabla\varphi \; d^3x = 0 \qquad \textit{for all } \varphi \in H^1_{\Sigma 0}(\Omega), \qquad \textit{and} \tag{3.13}$$

$$\int_{\Omega} h \cdot \mathrm{curl}\, A \; d^3x = 0 \qquad \textit{for all } A \in H_{DC\Sigma}(\Omega). \tag{3.14}$$

Proof. This is just a matter of rewriting the two ε-orthogonality conditions. □

In particular this states that a field is in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ if and only if it is a weak solution of (3.10) and (3.11). That is, the fields satisfy the boundary conditions

$$(\varepsilon h) \cdot \nu = 0 \quad \text{on } \tilde{\Sigma} \qquad \text{and} \quad h \wedge \nu = 0 \quad \text{on } \Sigma \tag{3.15}$$

in a weak sense.

A vector field $h \in L^2(\Omega;\mathbb{R}^3)$ is defined to be *ε-harmonic on* Ω provided it satisfies the system

$$\int_{\Omega} (\varepsilon h) \cdot \nabla\varphi \; d^3x = 0 \qquad \text{for all } \varphi \in H^1_0(\Omega), \qquad \text{and} \tag{3.16}$$

$$\int_{\Omega} h \cdot \mathrm{curl}\, A \; d^3x = 0 \qquad \text{for all } A \in H_{DC0}(\Omega). \tag{3.17}$$

Fields in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ are called *mixed ε-harmonic fields* since they satisfy *mixed boundary conditions* and are harmonic on Ω. When $\varepsilon \equiv I_3$, such fields are called *mixed (or $\tilde{\Sigma}$-) harmonic fields* and the corresponding spaces are denoted $\mathcal{H}_{\tilde{\Sigma}}(\Omega)$. Substituting Σ for $\tilde{\Sigma}$, this convention shows that $\mathcal{H}_{\Sigma}(\Omega) \subset H_{DC\Sigma}(\Omega)$.

I. SOLVABILITY of MIXED DIV-CURL SYSTEMS.

4. Mixed Weighted Orthogonal Decompositions

Our approach to studying the solvability of this mixed div-curl system is modeled on the method used in [6] for the cases of given normal, respectively tangential, components of the field. Namely we describe certain classes of scalar and vector potentials that provide an ε-orthogonal decomposition of finite energy (or L^2) fields of the form

$$v(x) \;=\; \nabla\varphi(x) + \varepsilon(x)^{-1}\,\mathrm{curl}\,A(x) + h(x). \tag{4.1}$$

Here h is an ε-harmonic vector field on Ω. Throughout this Section, we use the ε- inner product on $L^2(\Omega;\mathbb{R}^3)$. The representation result to be used here is the following generalization of the classical Hodge-Weyl decomposition. The usual Hodge-Weyl decomposition described in [2], [8, Chapter 9], or [6] and elsewhere corresponds to the case $\varepsilon(x) \equiv I_3$ and the choices $\Sigma_\tau = \emptyset$ or $\partial\Omega$ in the following analysis.

Theorem 4.1. *Assume (B3) and (E1) hold; then*

$$L^2(\Omega;\mathbb{R}^3) \;=\; G_{\Sigma_\tau}(\Omega) \oplus_\varepsilon \mathrm{Curl}_{\varepsilon\Sigma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega), \tag{4.2}$$

with $G_{\Sigma_\tau}(\Omega)$, $\mathrm{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$ and $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ defined as in Section 3.

Proof. The definitions (3.2) and (3.8) show that these spaces are subspaces of $L^2(\Omega;\mathbb{R}^3)$. They are orthogonal from Lemma 3.2 and the definition of $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. Thus the theorem is proved provided we can show that each of the spaces is closed. This is done below in Theorems 4.2 and 5.2 respectively. □

This result states that the scalar potential φ in (4.1) may be chosen to be in $H^1_{\Sigma_\tau 0}(\Omega)$. When the vector potential $A \in H_{DC\Sigma_\tau}(\Omega)$, the corresponding class of vector fields is ε-orthogonal from Lemma 3.2. The space $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ appearing here is the null space of our problem (1.1)–(1.2) and (2.4)–(2.5). Fields in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ are ε-harmonic fields that satisfy (3.15) in a weak sense with Σ_τ in place of Σ.

Given $v \in L^2(\Omega;\mathbb{R}^3)$, Riesz' Theorem states that the projection of v onto $G_{\Sigma_\tau}(\Omega)$ is given by $P_G(v) = \nabla\varphi_v$ where φ_v minimizes $\|v - \nabla\varphi\|_\varepsilon$ over all $\varphi \in G_{\Sigma_\tau}(\Omega)$. Equivalently φ_v minimizes $\mathcal{D}_v : H^1_{\Sigma_\tau 0}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}_v(\varphi) \;:=\; \int_\Omega [(\varepsilon\nabla\varphi)\cdot\nabla\varphi - 2(\varepsilon v)\cdot\nabla\varphi]\; d^3x. \tag{4.3}$$

The existence-uniqueness result for this variational problem may be stated as follows.

Theorem 4.2. *Assume (B3) and (E1) hold and $v \in L^2(\Omega;\mathbb{R}^3)$. Then there is a unique φ_v which minimizes $\mathcal{D}_v$ on $H^1_{\Sigma_\tau 0}(\Omega)$. A function φ_v minimizes $\mathcal{D}_v$ on $H^1_{\Sigma_\tau 0}(\Omega)$ if and only if it satisfies*

$$\int_\Omega [\varepsilon(\nabla\varphi - v)]\cdot\nabla\psi\, d^3x \;=\; 0 \qquad \textit{for all } \;\psi \in H^1_{\Sigma_\tau 0}(\Omega). \tag{4.4}$$

$G_{\Sigma_\tau}(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^3)$ and the projection P_G of $L^2(\Omega;\mathbb{R}^3)$ onto $G_{\Sigma_\tau}(\Omega)$ is given by $P_G v := \nabla\varphi_v$.

Proof. The functional $\mathcal{D}_v$ is continuous and convex from standard arguments, so it is weakly lower semi-continuous on $H^1_{\Sigma_\tau 0}(\Omega)$. Theorem 5.1 of [3] states that there is a $\lambda_1 > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 \, d^3x \;\geq\; \lambda_1 \int_\Omega |\varphi|^2 \, d^3x \qquad \text{for all } \; \varphi \in H^1_{\Sigma_\tau 0}(\Omega).$$

Substitute this, the Schwarz inequality and (2.1) in (4.3); then

$$2\mathcal{D}_v(\varphi) \;\geq\; \left[\int_\Omega e_0|\nabla\varphi|^2 \, d^3x \;+\; \lambda_1 \int_\Omega |\varphi|^2 \, d^3x\right] \;-\; 4\left\|\varepsilon v\right\|_2 \left\|\nabla\varphi\right\|_2 .$$

Thus $\mathcal{D}_v$ is coercive and strictly convex so it has a unique minimizer φ_v on $H^1_{\Sigma_\tau 0}(\Omega)$. Straightforward analysis shows that the Gateaux derivative of $\mathcal{D}_v$ is given by

$$\langle\, \mathcal{D}_v'(\varphi), \psi \,\rangle \;=\; 2\int_\Omega [\varepsilon(\nabla\varphi - v)] \cdot \nabla\psi \; d^3x \qquad \text{for } \; \varphi,\psi \in H^1_{\Sigma_\tau 0}(\Omega).$$

Thus φ_v is a minimizer of $\mathcal{D}_v$ on $H^1_{\Sigma_\tau 0}(\Omega)$ if and only if (4.4) holds. Since this variational problem has a solution for each $v \in L^2(\Omega;\mathbb{R}^3)$, [2, Corollary 3.3] yields the last sentence of the theorem. □

When φ_v, ε are sufficiently smooth applying the Gauss-Green Theorem to the integral in (4.4), yields

$$\int_{\partial\Omega} \psi[\varepsilon(\nabla\varphi_v - u)] \cdot \nu \; d\sigma \;+\; \int_\Omega \psi\nabla(\varepsilon(\nabla\varphi - v)) \; d^3x \;=\; 0 \qquad \text{for all } \; \psi \in H^1_{\Sigma_\tau 0}(\Omega).$$

Thus (4.4) may be regarded as a weak form of the system

$$\operatorname{div}(\varepsilon\nabla\varphi) \;=\; \operatorname{div}(\varepsilon v) \qquad \text{on } \; \Omega, \quad \text{subject to} \tag{4.5}$$

$$\varphi \;\equiv\; 0 \qquad \text{on } \; \Sigma_\tau \qquad \text{and} \quad \big(\varepsilon(\nabla\varphi)\big) \cdot \nu \;=\; (\varepsilon v) \cdot \nu \qquad \text{on } \; \Sigma_\nu. \tag{4.6}$$

This type of elliptic mixed boundary value problem has been extensively studied. See Stephan [12] for a treatment of such problems in three dimensional cases using boundary integral methods.

5. The Mixed Vector Potential

The vector field A in the representation (4.1) is called a *weighted vector potential of v associated with ε*. When $\varepsilon \equiv I_3$, such fields are just called *vector potentials* for v. For given v there is a large class of such (weighted) vector potentials. The analysis of our problems is simplified by a special choice of this vector potential. The following is a slight modification of [3, Proposition 7.2].

Theorem 5.1. *Suppose $\Sigma_\tau, \partial\Omega$ satisfy (B3) and $A \in H_{DC\Sigma_\tau}(\Omega)$. Then there is a unique $\hat{A} \in H_{DC\Sigma_\tau}(\Omega)$ such that*

(1) $\operatorname{div} \hat{A} = 0$, *and* $\operatorname{curl} \hat{A} = \operatorname{curl} A$ *on* Ω, *and*
(2) $\hat{A}$ *is* L^2*-orthogonal to* $\mathcal{H}_{\Sigma_\tau}(\Omega)$.

Proof. Given such an A, let $\hat{\varphi} \in H^1_{\Sigma_\nu 0}(\Omega)$ be the unique minimizer of

$$\mathcal{D}_A(\varphi) \ := \ \int_\Omega (|\nabla\varphi|^2 - 2A \cdot \nabla\varphi) \ d^3x$$

on $H^1_{\Sigma_\nu 0}(\Omega)$. This problem may be analyzed just as in Theorem 4.2 and $\hat{\varphi}$ is a weak solution of the system

$$\Delta\varphi \ = \ \operatorname{div} A \qquad \text{on} \ \ \Omega, \quad \text{subject to} \tag{5.1}$$

$$\varphi \ \equiv \ 0 \qquad \text{on} \ \ \Sigma_\nu \qquad \text{and} \ \ \nabla\varphi \cdot \nu \ = \ 0 \qquad \text{on} \ \ \Sigma_\tau. \tag{5.2}$$

This follows similar to equations (4.5)-(4.6) above. The vector field $\tilde{A} := A - \nabla\hat{\varphi} \in H_{DC\Sigma_\tau}(\Omega)$ and satisfies

$$\operatorname{div} \tilde{A} = 0 \qquad \text{and} \quad \operatorname{curl} \tilde{A} = \operatorname{curl} A \qquad \text{on} \ \ \Omega. \tag{5.3}$$

If $\mathcal{H}_{\Sigma_\tau}(\Omega) = \{0\}$, the result follows. When $\mathcal{H}_{\Sigma_\tau}(\Omega)$ is non-zero, let P_H be the orthogonal projection of $H_{DC\Sigma_\tau}(\Omega)$ onto this closed subspace. Then $\hat{A} := (I - P_H)\tilde{A}$ has properties (1) and (2). If B is another vector field which satisfies these conditions, then $\hat{A} - B$ is in $\mathcal{H}_{\Sigma_\tau}(\Omega)$. From property (ii) it is also L^2-orthogonal to $\mathcal{H}_{\Sigma_\tau}(\Omega)$, so $\hat{A} = B$. □

Define $Z_{\Sigma_\tau}(\Omega)$ to be the subspace of fields $A \in H_{DC\Sigma_\tau}(\Omega)$ that also satisfy

$$\operatorname{div} A \ = \ 0 \qquad \text{on} \ \ \Omega \quad \text{and are } L^2\text{-orthogonal to} \ \ \mathcal{H}_{\Sigma_\tau}(\Omega). \tag{5.4}$$

These conditions imply that $Z_{\Sigma_\tau}(\Omega)$ is a closed subspace of $H_{DC\Sigma_\tau}(\Omega)$. Define Q_C to be the orthogonal projection of $H_{DC\Sigma_\tau}(\Omega)$ onto $Z_{\Sigma_\tau}(\Omega)$. Thus Theorem 5.1 implies that (3.8) can be written as

$$\operatorname{Curl}_{\varepsilon\Sigma_\tau}(\Omega) \ = \ \{\varepsilon^{-1} \operatorname{curl} A : A \in Z_{\Sigma_\tau}(\Omega)\}; \tag{5.5}$$

i.e., fields in $\operatorname{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$ have a unique vector potential in $Z_{\Sigma_\tau}(\Omega)$. Moreover by reviewing the construction in this theorem, one sees that each field $A \in H_{DC\Sigma_\tau}(\Omega)$ has a representation

$$A \ = \ Q_C A \ + \ \nabla\varphi \ + \ k, \quad \text{with } \varphi \in H^1_{\Sigma_\nu 0}(\Omega) \text{ and } k \in \mathcal{H}_{\Sigma_\tau}(\Omega). \tag{5.6}$$

Let P_C be the ε-orthogonal projection of $L^2(\Omega; \mathbb{R}^3)$ onto the closure of $\operatorname{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$. To complete the proof of Theorem 4.1 and show that $\operatorname{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^3)$, we study the associated projection. Given $v \in L^2(\Omega; \mathbb{R}^3)$, Riesz' Theorem for projections states that $P_C v \ := \ \operatorname{curl} A_v$ where A_v minimizes $\|v - \varepsilon^{-1} \operatorname{curl} A\|_\varepsilon$ on $Z_{\Sigma_\tau}(\Omega)$. That is A_v minimizes

$$\mathcal{C}_v(A) \ := \ \int_\Omega [(\varepsilon^{-1} \operatorname{curl} A) \cdot \operatorname{curl} A - 2v \cdot \operatorname{curl} A] \ d^3x \tag{5.7}$$

on $Z_{\Sigma_\tau}(\Omega)$. The existence of solutions of this variational problem may be described as follows.

Theorem 5.2. *Assume (B3) and (E1) hold and* $v \in L^2(\Omega;\mathbb{R}^3)$. *Then there is a unique* A_v *which minimizes* $\mathcal{C}_v$ *on* $Z_{\Sigma_\tau}(\Omega)$. *A field* A *minimizes* $\mathcal{C}_v$ *on* $H_{DC\Sigma_\tau}(\Omega)$ *if and only if it satisfies*

$$\int_\Omega (\varepsilon^{-1}\,\mathrm{curl}\,A - v)\cdot \mathrm{curl}\,B\, d^3x \;=\; 0 \qquad \text{for all} \;\; B \in H_{DC\Sigma_\tau}(\Omega). \tag{5.8}$$

Moreover $\mathrm{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$ *is a closed subspace of* $L^2(\Omega;\mathbb{R}^3)$ *and the projection* P_C *of* $L^2(\Omega;\mathbb{R}^3)$ *onto* $\mathrm{Curl}_{\varepsilon\Sigma_\tau}(\Omega)$ *is given by* $P_C v := \mathrm{curl}\,A$ *where* A *is any minimizer of* $\mathcal{C}_v$ *on* $H_{DC\Sigma_\tau}(\Omega)$.

Proof. The functional $\mathcal{C}_v$ is convex and continuous on $Z_{\Sigma_\tau}(\Omega)$, so it is weakly l.s.c. From [3, equation (8.18)] there is a $\mu_1 > 0$ such that

$$\int_\Omega |\,\mathrm{curl}\,A|^2\; d^3x \;\geq\; \mu_1 \int_\Omega |A|^2\; d^3x \qquad \text{for all} \;\; A \in Z_{\Sigma_\tau}(\Omega).$$

This, Schwarz inequality and (2.1) applied to (5.7) yield

$$\mathcal{C}_v(A) \;\geq\; (2e_1)^{-1} \int_\Omega \left[|\,\mathrm{curl}\,A|^2 \;+\; \mu_1 |A|^2\right]\, d^3x \;-\; 2\,\|v\|_2\, \|\mathrm{curl}\,A\|_2$$

Thus $\mathcal{C}_v$ is coercive and strictly convex on $Z_{\Sigma_\tau}(\Omega)$ so it has a unique minimizer A_v on $Z_{\Sigma_\tau}(\Omega)$. The extremality condition (5.8) holds for all $B \in Z_{\Sigma_\tau}(\Omega)$ upon evaluation of the Gateaux derivative of the functional $\mathcal{C}_v$. Note that this functional may be extended to $H_{DC\Sigma_\tau}(\Omega)$ with the same formulae and the same minimal value, so (5.8) holds for all $B \in H_{DC\Sigma_\tau}(\Omega)$ and any minimizer in $H_{DC\Sigma_\tau}(\Omega)$. Since this variational problem has a solution for each $v \in L^2(\Omega;\mathbb{R}^3)$, [2, Corollary 3.3] yields the last sentence of the theorem. □

6. Necessary Conditions for Solvability

The div-curl boundary value problem (1.1)–(1.2) with (2.4)–(2.5) cannot have a solution for arbitrary fields ω, η. An obvious further condition is that ω should be solenoidal. In fact the following stronger criterion must hold.

Theorem 6.1. *(Necessary conditions) Assume (B2) and (C1) hold and* $v \in L^2(\Omega;\mathbb{R}^3)$ *is a weak solution of (1.1), (1.2) and (2.4)–(2.5). Then the data must satisfy condition* **[C2]:**

$$\int_\Omega \omega\cdot A\, d^3x \;+\; \int_{\Sigma_\tau} \eta\cdot A\; d\sigma \;=\; 0 \qquad \text{for any irrotational field} \;\; A \in H_{DC\Sigma_\tau}(\Omega). \tag{6.1}$$

Proof. Multiply (1.2) by $A \in H_{DC\Sigma_\tau}(\Omega)$ and integrate. Then the definition of A yields

$$\int_\Omega \omega\cdot A\, d^3x \;=\; \int_{\Sigma_\tau} v\cdot (A \wedge \nu)\; d\sigma \;+\; \int_\Omega v\cdot \mathrm{curl}\,A\, d^3x.$$

When A is irrotational then (C2) follows upon using (2.5) and the fact that $\eta \cdot \nu = 0$ on $\partial\Omega$. □

A field $A \in H_{DC\Sigma_\tau}(\Omega)$ is irrotational on Ω if and only if it has the form

$$A \;=\; h \;+\; \nabla\varphi \qquad \text{where } \;\varphi \in H^1_{\Sigma_\nu 0}(\Omega) \;\text{ and }\; h \in \mathcal{H}_{\Sigma_\tau}(\Omega) \tag{6.2}$$

from Auchmuty [3, Theorem 7.2]. Substitute $\nabla\varphi$ for A in (C2) to find

$$\int_{\Sigma_\tau} [\varphi(\omega\cdot\nu) \;+\; \eta\cdot\nabla\varphi]\; d\sigma \;-\; \int_\Omega \varphi \operatorname{div}\omega\, d^3x \;=\; 0 \qquad \text{for all }\; \varphi \in H^1_{\Sigma_\nu 0}(\Omega). \tag{6.3}$$

This implies that ω must be solenoidal on Ω and also that a weak form of a continuity equation holds on Σ_τ. Namely

$$\int_{\Sigma_\tau} [\varphi(\omega\cdot\nu) \;+\; \eta\cdot\nabla\varphi]\; d\sigma \;=\; 0 \qquad \text{for all }\; \varphi \in H^1_{\Sigma_\nu 0}(\Omega). \tag{6.4}$$

This equation relates ω and η on each component of Σ_τ and may be interpreted as a version of Kirchoff's law for currents. The condition (C2) also require the compatibility conditions

$$\int_\Omega \omega\cdot h\, d^3x \;+\; \int_{\Sigma_\tau} \eta\cdot h\; d\sigma \;=\; 0 \qquad \text{for }\; h \in \mathcal{H}_{\Sigma_\tau}(\Omega) \tag{6.5}$$

The number of independent conditions here is equal to the dimension of $\mathcal{H}_{\Sigma_\tau}(\Omega)$.

7. Variational Principles for the Scalar Potentials

When the potentials in the representation (4.1) are chosen as in Theorem 4.1, the equations for the scalar and vector potentials decouple. The equations for the scalar potential become

$$\operatorname{div}\big(\varepsilon(x)\nabla\varphi(x)\big) \;=\; \rho(x) \qquad \text{on }\; \Omega, \tag{7.1}$$

$$\big(\varepsilon(x)\nabla\varphi(x)\big)\cdot\nu(x) \;=\; \mu(x) \qquad \text{on }\; \Sigma_\nu \qquad \text{and }\; \varphi(x) \;=\; 0 \qquad \text{on }\; \Sigma_\tau. \tag{7.2}$$

A weak form of this equation is to find $\varphi \in H^1_{\Sigma_\tau 0}(\Omega)$ satisfying

$$\int_\Omega [(\varepsilon\nabla\varphi)\cdot\nabla\mathcal{X} \;+\; \rho\mathcal{X}]\; d^3x \;-\; \int_{\Sigma_\nu} \mu\mathcal{X}\; d\sigma \;=\; 0 \qquad \text{for all }\; \mathcal{X} \in H^1_{\Sigma_\tau 0}(\Omega). \tag{7.3}$$

The solution of this equation may be characterized as the minimizer of a natural variational principle. Consider the problem of minimizing $\mathcal{D} : H^1_{\Sigma_\tau 0}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}(\varphi) \;:=\; \int_\Omega \big[\big(\varepsilon(\nabla\varphi)\cdot\nabla\varphi\big) \;+\; 2\rho\varphi\big]\; d^3x \;-\; \int_{\Sigma_\nu} 2\mu\varphi\; d\sigma. \tag{7.4}$$

The existence and uniqueness result for this problem is the following

Theorem 7.1. *Assume (B3), (C1) and (E1) hold. Then $\mathcal{D}$ is bounded below on $H^1_{\Sigma_\tau 0}(\Omega)$ and has a unique minimizer $\hat{\varphi}$. This minimizer is the unique weak solution of (7.3).*

Proof. When Σ_τ obeys (B2), $H^1_{\Sigma_\tau 0}(\Omega)$ is a closed subspace of $H^1(\Omega)$ from Lemma 3.1. The proof that $\mathcal{D}$ is a continuous convex functional is standard. Theorem 5.1 of [3] provides a coercivity inequality of the form

$$\int_\Omega |\nabla\varphi|^2\, d^3x \;\geq\; \lambda_1 \int_\Omega |\varphi|^2\, d^3x \qquad \text{for all } \varphi \in H^1_{\Sigma_\tau 0}(\Omega). \tag{7.5}$$

Since (E1) holds, we find that $\mathcal{D}$ is strictly convex and coercive on $H^1_{\Sigma_\tau 0}(\Omega)$, so it is bounded below and attains a unique minimizer on this space. The quadratic functional $\mathcal{D}$ is Gateaux differentiable. Thus a function minimizes $\mathcal{D}$ on $H^1_{\Sigma_\tau 0}(\Omega)$ if and only if and it satisfies (7.3) from a standard result for convex minimization. □

As is usual in quadratic variational problems, the continuous dependence of the solutions on the data is quantified by bounds on the solutions. For this problem we have the following.

Theorem 7.2. *Assume (B3), (C1) and (E1) hold and $\hat{\varphi}$ is the unique solution of (7.3) in $H^1_{\Sigma_\tau 0}(\Omega)$. Then there are constants C_1, C_2 such that*

$$\|\nabla\hat{\varphi}\|_2 \;\leq\; C_1\|\rho\|_{6/5,\Omega} \;+\; C_2\|\mu\|_{4/3,\Sigma_\nu}. \tag{7.6}$$

Proof. The regularity condition (B1) is sufficient to ensure that the Sobolev imbedding theorem and the trace theorem holds for functions in $H^1(\Omega)$. Moreover $\|\nabla\varphi\|_2$ defines a norm on $H^1_{\Sigma_\tau 0}(\Omega)$ that is equivalent to the usual H^1-norm since (7.5) holds. Thus there are constants b_1, b_2 which depend only on $\Omega, \partial\Omega$ such that

$$\|\varphi\|_{6,\Omega} \;\leq\; b_1\|\nabla\varphi\|_2 \qquad \text{and} \quad \|\varphi\|_{4,\partial\Omega} \;\leq\; b_2\|\nabla\varphi\|_2. \tag{7.7}$$

Use this, (2.1) and Holder's inequality in (7.4); then

$$\mathcal{D}(\varphi) \;\geq\; e_0\|\nabla\varphi\|_2^2 - 2\left[b_1\|\rho\|_{6/5,\Omega} \;+\; b_2\|\mu\|_{4/3,\Sigma_\nu}\right]\|\nabla\varphi\|_2.$$

The value of this variational problem cannot be positive, so the minimizer satisfies an inequality of the form (7.6). □

8. Variational Principles for the Vector Potentials

Just as above, substitution of (4.1) in (1.1)–(1.2) and (2.4)–(2.5) leads to a system of equations for the vector potential. They can be written

$$\operatorname{curl}\big(\varepsilon(x)^{-1}\operatorname{curl} A(x)\big) \;=\; \omega(x) \qquad \text{on } \Omega, \tag{8.1}$$

$$\big(\varepsilon^{-1}(x)\operatorname{curl} A(x)\big)\wedge\nu(x) \;=\; \eta(x) \qquad \text{on } \Sigma_\tau, \qquad \text{and} \tag{8.2}$$

$$A(x)\wedge\nu(x) \;=\; 0 \qquad \text{on } \Sigma_\nu. \tag{8.3}$$

A weak form of this system is to find $A \in H_{DC\Sigma_\tau}(\Omega)$ satisfying

$$\int_\Omega [(\varepsilon^{-1}\operatorname{curl} A)\cdot\operatorname{curl} C - \omega\cdot C]\, d^3x \;-\; \int_{\Sigma_\tau} \eta\cdot C\, d\sigma \;=\; 0 \qquad \text{for all } C \in H_{DC\Sigma_\tau}(\Omega). \tag{8.4}$$

Consider the variational principle of minimizing $\mathcal{C} : H_{DC\Sigma_\tau}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{C}(A) \ := \ \int_\Omega \left[(1/2)(\varepsilon^{-1} \operatorname{curl} A) \cdot \operatorname{curl} A \ - \ \omega \cdot A\right] \, d^3x \ - \ \int_{\Sigma_\tau} \eta \cdot A \, d\sigma. \tag{8.5}$$

First note that this functional $\mathcal{C}(A)$ is convex and quadratic in $\operatorname{curl} A$ and linear in A itself. Thus $\mathcal{C}$ is linear on the subspace of irrotational fields. Thus if (C2) does not hold, $\mathcal{C}$ is unbounded below on $H_{DC\Sigma_\tau}(\Omega)$. So condition (C2) is a necessary condition for $\mathcal{C}$ to be bounded below. When (C2) holds, the definition of Q_C yields

$$\mathcal{C}(A) \ = \ \mathcal{C}(Q_C A) \qquad \text{for all} \ \ A \in H_{DC\Sigma_\tau}(\Omega). \tag{8.6}$$

This implies that

$$\inf_{A \in H_{DC\Sigma_\tau}(\Omega)} \mathcal{C}(A) \ = \ \inf_{A \in Z_{\Sigma_\tau}(\Omega)} \mathcal{C}(A). \tag{8.7}$$

The existence of a minimizer of the functional $\mathcal{C}$ on $Z_{\Sigma_\tau}(\Omega)$ may now be proved by using a coercivity estimate from [3].

Theorem 8.1. *Assume (B3), (C1), (C2) and (E1) hold. Then $\mathcal{C}$ is bounded below on $Z_{\Sigma_\tau}(\Omega)$ and there is a unique minimizer $\hat{A}$ of $\mathcal{C}$ on $Z_{\Sigma_\tau}(\Omega)$. There are constants C_3, C_4 such that*

$$\|\varepsilon^{-1} \operatorname{curl} \hat{A}\|_2 \ \leq \ C_3 \|\omega\|_{6/5,\Omega} \ + \ C_4 \|\eta\|_{4/3,\Sigma_\tau}. \tag{8.8}$$

Proof. $Z_{\Sigma_\tau}(\Omega)$ is a closed subspace of $H^1_{\Sigma_\tau 0}(\Omega)$ which has been studied in [3, Sections 7, 8]. In particular Theorem 8.3 and equation 8.15 say that there is a positive constant μ_1 which depends only on $\Omega, \partial\Omega$ and Σ_τ such that

$$\int_\Omega \|\operatorname{curl} A\|^2 \, d^3x \ \geq \ \mu_1 \int_\Omega \|A\|^2 \qquad \text{for all} \ \ A \in Z_{\Sigma_\tau}(\Omega). \tag{8.9}$$

Moreover this left hand side is an equivalent norm to the DC-norm on $Z_{\Sigma_\tau}(\Omega)$. Our assumptions imply that the conditions of [3, Theorem 3.4] hold, so the DC norm on $H^1_{\Sigma_\tau 0}(\Omega)$ is equivalent to the standard Sobolev H^1 product norm. Thus the Sobolev imbedding theorem and the trace theorem hold for these vector fields just as for scalar functions. In particular there are positive constants a_1, a_2 such that

$$\|A\|_{6,\Omega} \ \leq \ a_1 \|\operatorname{curl} A\|_2 \qquad \text{and} \ \ \|A\|_{4,\partial\Omega} \ \leq \ a_2 \|\operatorname{curl} A\|_2 \tag{8.10}$$

for all $A \in Z_{\Sigma_\tau}(\Omega)$. These imply that the functional $\mathcal{C}$ is continuous on $Z_{\Sigma_\tau}(\Omega)$. It is strictly convex as the quadratic term is also positive on $Z_{\Sigma_\tau}(\Omega)$. Use (2.1), (8.10) and Holders inequality to find that, for all $A \in Z_{\Sigma_\tau}(\Omega)$,

$$\mathcal{C}(A) \ \geq \ \|\operatorname{curl} A\|_2^2 / 2\varepsilon_1 - [a_1 \|\omega\|_{6/5,\Omega} \ + \ a_2 \|\eta\|_{4/3,\Sigma_\tau}] \|\operatorname{curl} A\|_2. \tag{8.11}$$

This implies that $\mathcal{C}$ is coercive on $H^1_{\Sigma_\tau 0}(\Omega)$, so it attains its minimum there. Since the minimum value cannot be strictly positive, this yields the inequality (8.8) for the minimizer. □

While there is uniqueness of the minimizers of $\mathcal{C}$ on this subspace $Z_{\Sigma_\tau}(\Omega)$, there is not uniqueness on $H_{DC\Sigma_\tau}(\Omega)$. In fact any field of the form

$$A \;=\; \hat{A} + \nabla\varphi + k \quad \text{for some } \varphi \in H^1_{\Sigma_\nu 0}(\Omega) \quad \text{and} \quad k \in \mathcal{H}_{\Sigma_\tau}(\Omega) \tag{8.12}$$

has $\mathcal{C}(A) = \mathcal{C}(\hat{A})$. These fields differ from $\hat{A}$ by an irrotational field so the estimate (8.8) remains valid for any such field.

9. Solvability of the Mixed div-curl problem

The results of the last two sections provide solutions of the original mixed div-curl boundary value problem. Assume the necessary condition (C2) holds, $\hat{\varphi}$ is the unique minimizer of $\mathcal{D}$ defined by (7.4) on $H^1_{\Sigma_\tau 0}(\Omega)$ and $\tilde{A}$ is a minimizer of $\mathcal{C}$ on $H_{DC\Sigma_\tau}(\Omega)$. Consider the vector field

$$\hat{v}(x) \;:=\; \nabla\hat{\varphi}(x) \;+\; \varepsilon(x)^{-1}\,\mathrm{curl}\,\tilde{A}(x). \tag{9.1}$$

Linearity, the equations (7.1)-(7.2), (8.1)–(8.3) and the results of Section 3 show that $\hat{v}$ is a solution of (1.1)–(1.2) subject to (2.4)–(2.5). Specifically, the following holds.

Theorem 9.1. *Assume (B3), (C1), (C2) and (E1) hold, then $\hat{v} \in L^2(\Omega;\mathbb{R}^3)$ defined by (9.1) is the unique solution of (1.1)–(1.2) subject to (2.4)–(2.5) that is ε-orthogonal to $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. Moreover there are positive constants C_1–C_4 such that*

$$\|\hat{v}\|_2 \;\le\; C_1\|\rho\|_{6/5,\Omega} \;+\; C_3\|\omega\|_{6/5,\Omega} \;+\; C_2\|\mu\|_{4/3,\Sigma_\nu} \;+\; C_4\|\eta\|_{4/3,\Sigma_\tau}. \tag{9.2}$$

Proof. The above construction ensures that each of $\nabla\hat{\varphi}$ and $\varepsilon(x)^{-1}\,\mathrm{curl}\,\tilde{A}$ is ε-orthogonal to $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ so $\hat{v}$ is. If $\tilde{v}$ is another such weak solution, then $\hat{v} - \tilde{v} \in \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. But $\hat{v} - \tilde{v}$ is ε-orthogonal to $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ so $\hat{v} = \tilde{v}$. The bound (9.2) follows from (7.6), (8.8) and (9.1). □

By inspection, the null space of this problem is $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. Consequently any field of the form

$$v(x) \;:=\; \hat{v}(x) \;+\; k(x) \qquad \text{with } k \in \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) \tag{9.3}$$

is a solution of this mixed *div-curl* problem. This may be stated explictly as follows.

Corollary 9.2. *Under the conditions of the theorem, the set of all solutions of (1.1)–(1.2) subject to (2.4)–(2.5) is given by (9.1) and (9.3).*

This implies that there cannot be uniqueness, or *apriori* bounds on the L^2-norm of solutions, when the space $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ is non-zero. That is the mixed boundary data (2.4)–(2.5) only prescribes the solution up to a field in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. The next part of this paper studies properties of the space $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ with a view to describing extra conditions that yield unique solutions.

II. NON-UNIQUENESS AND WELL-POSEDNESS

This last corollary shows that there may be considerable non-uniqueness of solutions to the mixed div-curl boundary value problem for (1.1)–(1.2). To have a well-posed problem further conditions must be imposed that determine the ε-harmonic component of a solution. That is, we must impose conditions that enable the determination of a unique component of the solution in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. For the planar case this was done in [5, Section 14] and for the cases where Σ is either $\emptyset$ or $\partial\Omega$, such results were developed in [6, Sections 10, 11]. In each case the extra conditions were the prescription of certain line integrals of the solution in addition to the boundary data. These line integrals had physical, and/or geometrical, interpretations as circulations and potential differences.

Here a similar analysis is developed for this mixed 3d div-curl problem. As is seen, the analysis is not as complete as in these previous cases—some isotropy is assumed—so a number of conjectures are posed whose solution would provide a better theory for this problem. However, the present results suffice for a large number of situations in fluid mechanics and electromagnetic theory.

10. Mixed Harmonic Fields

A first question is whether $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ is finite dimensional and, if so, how to describe its dimension? For the case of prescribed normal, or tangential, components on the whole boundary, such results have been known for a long time and are special cases of de Rham theory. The dimensions reflect topological properties of the region. For mixed boundary conditions, results for the planar case are described in [6, Section 14] and depend on the geometry of the boundary data as well as that of the region. For 3-dimensional problems, finite dimensionality results for the case where $\varepsilon(x) \equiv I_3$ are proved in [3, Section 8]. Here we obtain similar results for a general, isotropic, coefficient matrix $\varepsilon(x)$ obeying the following:

Condition E2. $\varepsilon(x) := e(x)I_3$ *with* e *continuous on* $\bar{\Omega}$*, in* $H^1(\Omega)$ *and such that there are positive constants* e_0 *and* e_1 *with*

$$0 \ < \ e_0 \ \leq \ e(x) \ \leq \ e_1 \qquad \text{for all} \ \ x \in \bar{\Omega}. \tag{10.1}$$

Let $H_{DC\tilde{\Sigma}}(\Omega)$ be the Sobolev space of vector fields defined as in Section 3. Theorem 3.4 of [3] proves that this space may be identified with the subspace of $H^1(\Omega;\mathbb{R}^3)$ of all fields which satisfy the mixed boundary conditions (3.4) in a weak sense:

$$v \cdot \nu \ = \ 0 \qquad \text{on} \ \ \tilde{\Sigma} \qquad \text{and} \ \ v \wedge \nu \ = \ 0 \qquad \text{on} \ \ \Sigma. \tag{10.2}$$

Define V to be the subspace of $H_{DC\tilde{\Sigma}}(\Omega)$ of all fields that are orthogonal to $G_\Sigma(\Omega)$. That is those fields u which satisfy

$$\int_\Omega e\,(u \cdot \nabla\varphi)\; d^3x \;=\; 0 \qquad \text{for all} \;\; \varphi \in H^1_{\Sigma 0}(\Omega). \tag{10.3}$$

Let B be the class of fields in V which satisfy

$$\int_\Omega [\, e\,|u|^2 \;+\; |\operatorname{curl} u|^2\,]\, d^3x \;\leq\; 1. \tag{10.4}$$

This may be regarded as a unit ball in V associated with an inner product on V.

Consider the variational principle of maximizing

$$\mathcal{E}(u) \;:=\; \int_\Omega e\,|u|^2\, d^3x \qquad \text{on} \;\; B, \tag{10.5}$$

and finding $\beta := \sup_{u\in B}\, \mathcal{E}(u)$. The following result describes the existence of solutions of this problem.

Theorem 10.1. *Assume (B3) and (E2) hold; then there are maximizers $\pm\,\hat{u}$ of $\mathcal{E}$ on B.*

Proof. When (B3) holds, [3, Theorem 3.4] shows that the spaces $H_{DC\tilde{\Sigma}}(\Omega)$ and the closed subspace of $H^1(\Omega;\mathbb{R}^3)$ of all fields satisfying (10.2) are equal. The imbedding of $H_{DC\tilde{\Sigma}}(\Omega)$ into $L^2(\Omega;\mathbb{R}^3)$ is compact, as this is true for the imbedding of $H^1(\Omega;\mathbb{R}^3)$ into $L^2(\Omega;\mathbb{R}^3)$ from Rellich's theorem. Thus the functional $\mathcal{E}$ defined by (10.5) is weakly continuous and convex on $H_{DC\tilde{\Sigma}}(\Omega)$, and on V, as it is continuous and convex on $L^2(\Omega;\mathbb{R}^3)$.

Now $\mathcal{E}(u) \leq 1$ for all $u \in B$ from the definitions of $\mathcal{E}$ and B, so $\beta \leq 1$. Let $\{u^{(m)} : m \geq 1\}$ be a maximizing sequence for $\mathcal{E}$ on B. The orthogonality condition (10.3) implies that

$$\operatorname{div}\, e u^{(m)} \;=\; 0 \qquad \text{weakly on } \Omega \qquad \text{for all} \quad m \geq 1.$$

Thus

$$e(x)\, \operatorname{div} u^{(m)}(x) \;=\; -(\nabla e(x) \cdot u^{(m)}(x)) \qquad \text{a.e. on } \Omega,$$

so

$$\|\operatorname{div} u^{(m)}\|_2 \;\leq\; {e_0}^{-1}\, \|\nabla e\|_2\, \|u^{(m)}\|_2 \qquad \text{for all} \;\; m \geq 1. \tag{10.6}$$

This and the definition of B show that the sequence $\{u^{(m)} : m \geq 1\}$ is bounded in $H_{DC\tilde{\Sigma}}(\Omega)$. Since B is closed and convex, this sequence has a weak limit point $\hat{u} \in B$ and the sequence converges strongly to $\hat{u}$ in $L^2(\Omega;\mathbb{R}^3)$ from Rellich's theorem. Thus $\mathcal{E}(\hat{u}) = \beta$ as $\mathcal{E}$ is continuous on $L^2(\Omega;\mathbb{R}^3)$ and there are maximizers as claimed. □

When $\beta = 1$, any maximizing field for $\mathcal{E}$ on B is a mixed ε-harmonic field in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ as such maximizing fields are irrotational and in V. Thus the following corollary follows.

Corollary 10.2. *Assume (B3) and (E2) hold. If $\beta < 1$, then $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega) = \{0\}$. If $\beta = 1$, then $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ is a proper finite dimensional subspace of V.*

Proof. When $\beta < 1$, then $v \neq 0$ in B has $\operatorname{curl} v \neq 0$, so $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega) = \{0\}$. Suppose $\beta = 1$ then there is at least one mixed ε-harmonic field on Ω. Let $\mathcal{F}$ be a set of fields in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ which are orthonormal with respect to the inner product on V defined by

$$\langle u, v \rangle_C \ := \ \int_\Omega \ [\, e\, u \cdot v \ + \ \operatorname{curl} u \cdot \operatorname{curl} v \,] \ d^3x \tag{10.7}$$

These fields are in B and are also orthonormal in $L^2(\Omega;\mathbb{R}^3)$ as they are irrotational. $\mathcal{F}$ cannot be infinite as B is compact in the weighted space $L^2(\Omega;\mathbb{R}^3)$. Hence the subspace $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ must be finite dimensional. □

This result shows that provided (B3) and (E2) holds, the corresponding space of weighted harmonic fields is finite dimensional. It would be of interest to show that $\dim \mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ is finite when we only require (B3) and (E1) to hold and we conjecture that this is the case. A further conjecture is that this dimension is independent of the coefficient matrix $\varepsilon(x)$ provided (B3) and (E1) hold.

11. Gradient Mixed ε-Harmonic Fields.

In this section, a lower bound on the dimension of $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ is described which depends on the number of connected components of the set Σ where the tangential boundary condition is imposed. Since the following analysis requires only that Condition (E1) holds, we treat this case. In Section 3, a field is defined to be in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ provided it is ε-orthogonal to both $G_\Sigma(\Omega)$ and $\mathrm{Curl}_{\varepsilon\Sigma}(\Omega)$. That is, they satisfy (3.13)-(3.14) and, from Lemma 3.3, are weak solutions of the system

$$\operatorname{div}(\varepsilon h) \ = \ 0 \qquad \text{and} \qquad \operatorname{curl} h \ = \ 0 \qquad \text{on } \ \Omega, \tag{11.1}$$

$$(\varepsilon h)\cdot \nu \ = \ 0 \qquad \text{on } \ \tilde{\Sigma} \qquad \text{and} \qquad h \wedge \nu \ = \ 0 \qquad \text{on } \ \Sigma. \tag{11.2}$$

The gradient solutions of this system may be described explicitly. We show that, when Σ has $M+1$ connected components, there are exactly M linearly independent such fields. Define $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega) \ := \ G(\Omega) \cap \mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ to be the subspace of gradient fields in $\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$. To construct a basis of $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ we need the following technical result.

Lemma 11.1. *Assume $\partial\Omega$ satisfies (B1) and σ_0, σ_1 are disjoint open subsets of $\partial\Omega$ with σ_0 connected and $d(\sigma_0, \sigma_1) \ = \ d_1 > 0$. Then there is a C^1 function $g : \partial\Omega \to [0,1]$ with $g(x) \ = \ 1$ on σ_0 and $g(x) \ = \ 0$ on σ_1.*

Proof. When σ_0, σ_1 are subsets of different components of $\partial\Omega$, take g to be identically 1 or 0 on the different components and satisfy Laplace's equation on Ω. The maximum principle, and regularity results, imply that g has the desired properties. When σ_0, σ_1 are subsets of the same component Σ_j of $\partial\Omega$, take g to be identically 0 on the other components. We construct the desired g on Σ_j. From Urysohn's Lemma there is a continuous function g_0 on Σ_j with the desired properties. Introduce local coordinates on Σ_j and convolve g_0 with a C^1 mollifier of sufficiently small support. The resulting function is the desired g. □

With σ_0, σ_1 as in this lemma, define $K(\sigma_0, \sigma_1)$ to be the subset of $H^1(\Omega)$ of functions whose trace is identically 1 on σ_0 and 0 on σ_1. This set has the following property.

Corollary 11.2. *With $\partial\Omega$, σ_0, σ_1 as above, $K(\sigma_0, \sigma_1)$ is a closed convex subset of $H^1(\Omega)$.*

Proof. The C^1 function constructed above has an extension to Ω from [13, Theorem 8.8], and the extension is in $K(\sigma_0, \sigma_1)$. If u_1, u_2 are two functions in $K(\sigma_0, \sigma_1)$, then $u_1 - u_2 \in H^1_{\Sigma 0}(\Omega)$ where $\Sigma = \sigma_0 \cup \sigma_1$. The result now follows from Lemma 3.1. ☐

Assume that $\Sigma := \{\sigma_0, \sigma_1, \ldots, \sigma_M\}$, $M \geq 1$ has more than one component. Given m with $0 \leq m \leq M$, let χ_m be the minimizer of $\mathcal{D}_0$ defined by (4.3) with $v \equiv 0$ on the set $K_m := K(\sigma_m, \Sigma \backslash \bar{\sigma}_m)$. This problem has a unique minimizer as K_m is a nonempty closed convex subset of $H^1(\Omega)$. The minimizer satisfies

$$\int_\Omega \varepsilon(\nabla\chi_m) \cdot \nabla\varphi \, d^3x \;=\; 0 \qquad \text{for all} \;\; \varphi \in H^1_{\Sigma 0}(\Omega). \tag{11.3}$$

This minimizing function χ_m is non-constant on Ω; so

$$\tilde{h}^{(m)}(x) \;:=\; \nabla\chi_m(x). \tag{11.4}$$

is a non-zero field on Ω. Each $\tilde{h}^{(m)} \in \mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ as the functions χ_m are H^1 solutions of the system (11.1) subject to

$$\chi(x) \;=\; \begin{cases} 1 & \text{for} \;\; x \in \sigma_m, \\ 0 & \text{for} \;\; x \in \Sigma \setminus \bar{\sigma_m}. \end{cases} \tag{11.5}$$

together with the natural boundary conditions

$$\big(\varepsilon(x)\nabla\chi(x)\big) \cdot \nu(x) \;=\; 0 \qquad \text{on} \quad \tilde{\Sigma}. \tag{11.6}$$

The subspace $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ may now be characterized as follows.

Theorem 11.3. *Assume ε satisfies (E1), (B1) holds and Σ has $M+1$ components with (B2) holding; then $\dim G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega) = M$. When $M \geq 1$, then $\{\tilde{h}^{(1)}, \ldots, \tilde{h}^{(M)}\}$ defined by (11.4) is a basis of $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$.*

Proof. Suppose $h = \nabla\psi \in G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ for some ψ in $H^1(\Omega)$. From (3.13), it satisfies

$$\int_\Omega (\varepsilon\nabla\psi) \cdot \nabla\varphi \, d^3x \;=\; 0 \qquad \text{for all} \;\; \varphi \in H^1_{\Sigma 0}(\Omega). \tag{11.7}$$

Now the second part of the boundary condition (11.2) implies that $\psi(x) \equiv c_m$ is constant on each component σ_m of Σ. When $M = 0$, then $\Sigma = \sigma_0$ and $\psi - c_0$ is in $H^1_{\Sigma 0}(\Omega)$. When Σ has positive surface measure, the uniqueness of solution of (11.7) implies that $\psi \equiv c_0$, so $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega) = \{0\}$. When $M \geq 1$, we may choose $c_0 = 0$ since the potential ψ is in $H^1(\Omega)$. The definition of the χ_m then implies that

$$\psi(x) \;-\; \sum_{m=1}^{M} c_m\chi_m(x) \;\equiv\; 0 \qquad \text{on } \Sigma.$$

Since ψ and each χ_m is a solution of (11.7), this right hand side is also a solution. It is identically zero on Ω since it is zero on Σ . Take gradients then h is a linear combination of $\{\tilde{h}^{(1)}, \dots, \tilde{h}^{(M)}\}$ so this set spans the subspace $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$. These fields are independent from the boundary conditions on the χ_m so the theorem follows. □

Suppose $M \geq 1$ and $h \in G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$ with

$$h(x) \;=\; \sum_{m=1}^{M} c_m \, \tilde{h}^{(m)}(x), \tag{11.8}$$

The coefficients c_m in this expansion can be determined using Fourier methods. Take ε-inner products of (11.8) with $\tilde{h}^{(k)}$; then

$$h_k \quad = \quad \sum_{m=1}^{M} h_{km}\, c_m \quad \text{where } h_{km} := \langle\, \tilde{h}^{(k)}, \tilde{h}^{(m)}\, \rangle_{\varepsilon}, \qquad \text{and} \tag{11.9}$$

$$h_k \quad := \quad \langle\, h, \tilde{h}^{(k)}\, \rangle_{\varepsilon} \;=\; \int_{\sigma_k} (\varepsilon h) \cdot \nu \; d\sigma \tag{11.10}$$

from the definitions of the $h^{(k)}$ using $\operatorname{div}(\varepsilon h) \equiv 0$ on Ω. This right hand side is the flux of εh through the component σ_k of Σ. The matrix (h_{km}) in (11.9) is a positive definite symmetric matrix which is the Grammian of a finite set of linearly independent fields. Thus there is a unique solution for the coefficients c_m determined by these fluxes of εh.

12. Other Mixed ε-Harmonic Fields .

When Ω is not simply connected then there may be ε-harmonic fields which are not gradients. Such fields are irrotational fields in Ω which have non-zero circulations around certain smooth closed curves in Ω. The usual examples of these fields described in de Rham theory satisfy zero flux boundary conditions and one can ask whether there are such ε-harmonic fields that satisfy mixed boundary conditions?

Such fields were described in the planar case in [5, Section 14]. For the 3-dimensional case, we have not been successful in finding a characterization of these fields that enables us to enumerate the possible independent fields that are not gradients. Let L_1 be the number of linearly independent fields in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ which are ε-orthogonal to fields in $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$. A further conjecture is that there is a geometric characterization of L_1 similar to the characterization given in [5, Section 14]. Roughly, L_1 counts the number of handles which are not encircled by some component of Σ_ν. To state this precisely, "handles" and "encircled" must be carefully defined. Specifically we conjecture that L_1 *is the rank of the image of the maps induced on relative homology groups* $H_2(\Omega, \bar{\Sigma}_\nu) \to H_2(\Omega, \partial\Omega)$ *by the inclusion of* $\bar{\Sigma}_\nu$ *into* $\partial\Omega$.

Another version of this question is whether $\dim \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) = M + L_1$? This is open even for the case $\varepsilon(x) \equiv I_3$.

13. Well-Posedness of the Mixed div-curl System.

A linear equation may be said to be well-posed provided it can be shown to have a unique solution. The results of Section 9 show that the mixed div-curl boundary problem is well-posed if and only if certain structural and compatibility conditions hold and $\dim \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) = 0$. The analysis of Section 11 shows that when Σ_τ has $M+1$ connected components, the space $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ has dimension at least M. Thus, when Σ_τ has 2 or more components, there is an affine subspace of solutions—assuming that there is 1 solution. This leads to the question "What extra conditions should be imposed on this problem, to guarantee uniqueness of the solution?" The description of the solution set in Corollary 9.2 shows that this requires that we impose conditions that select a unique component of the solution in the null space $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. This issue was resolved for the planar version of this problem in [5, Section 14]. Essentially it amounts to prescribing extra linear functionals of the solution that determine a unique ε-harmonic component k in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ of the solution 9.3.

To resolve these issues, some results are used about line integrals of continuous irrotational vector fields which satisfy

$$v \wedge \nu \;=\; 0 \qquad \text{on} \;\; \Sigma_\tau \tag{13.1}$$

Let Γ be a subset of $\partial\Omega$. A *curve* ξ in Ω, relative to Γ, is a piecewise C^1 map of an interval L into $\bar{\Omega}$ with endpoints in Γ. Such curves need not be simple. Let $\Xi_\Gamma(\Omega) = \Xi_\Gamma$ denote the set of such curves. A curve ξ is *closed* if its initial and final points are the same; it may be regarded as a curve with no endpoints. When $\Gamma' \subset \Gamma$, then $\Xi_{\Gamma'} \subset \Xi_\Gamma$. The smallest Γ is the empty set, in which case $\Xi_\Gamma = \Xi_\emptyset$ is the class of all closed curves. The largest Γ is $\partial\Omega$, and $\Xi_{\partial\Omega}$ is $\Xi_\emptyset$ together with the set of all curves with endpoints in $\partial\Omega$.

For continuous vector fields v on $\bar{\Omega}$, the line integral

$$\int_\xi v \;=\; \int_\xi v \cdot \tau \; ds \tag{13.2}$$

is a well-defined linear functional on the space of all such continuous fields. Here τ is the unit tangent field to ξ and s is a parametrization of ξ. In particular, we consider curves $\xi \in \Xi_\Gamma$, via (13.2), as linear functionals on spaces of ε-harmonic vector fields.

Two curves ξ_0 and ξ_1 in Ξ_Γ are *homotopic* if they can be continuously deformed into one another within the set Ξ_Γ. More precisely, suppose $\mathcal{E} : [0,1] \times [0,1] \to \bar{\Omega}$ is continuous with

$$\mathcal{E}(0,t) = \xi_0(t), \qquad \mathcal{E}(1,t) = \xi_1(t), \tag{13.3}$$

and for each s, the curve $\xi_s = \mathcal{E}(s,\cdot)$ is a C^1 curve in Ξ_Γ. In particular, for closed curves, $\mathcal{E}(s,0) = \mathcal{E}(s,1)$ for each s. It is permissible for a class of curves to have endpoints in Γ for some s and be closed for other s. Clearly homotopy is an equivalence relation on Ξ_Γ and the following result holds.

Theorem 13.1. *Let Γ be an open, or closed, subset of $\partial\Omega$, and u be a continuous, irrotational field on $\bar{\Omega}$ which satisfies $u \wedge \nu \;=\; 0$ on Γ. If ξ_0 and ξ_1 in Ξ_Γ are homotopic, then*

$$\int_{\xi_0} u \;=\; \int_{\xi_1} u. \tag{13.4}$$

This is proved in the usual manner of multivariable calculus. A detailed proof in the planar case is given in [5, Theorem 12]. This result enables us to characterize $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ when Ω is simply connected.

Theorem 13.2. *Assume ε satisfies (E1), Ω, Σ_τ, Σ_ν satisfy (B3) and Ω is simply connected. If Σ_τ has a unique component, then $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) = \{0\}$. If Σ_τ has $M+1$ components, then $\{\tilde{h}^{(1)}, \ldots, \tilde{h}^{(M)}\}$ defined by (11.4) is a basis of $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ and $\dim \mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) = M$.*

Proof. Take $\Sigma = \Sigma_\tau$ in the analysis of Section 11 and let h be any field in $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. It is smooth on Ω, and $\int_\xi h = 0$ for any simple closed curve $\xi \in \Omega$, as ξ is homotopic to a point. Hence from [6, Theorem 11.3], $h = \nabla\varphi$ for some φ in $H^1(\Omega)$. Thus $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega) = G\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ and the result follows from Theorem 11.3 of this paper. □

This result leads to our first well-posedness result, which follows directly from the first conclusion of this theorem.

Corollary 13.3. *Assume that Ω is simply connected, Σ_τ is connected and the conditions of Theorem 9.1 hold. Then $\hat{v}$ defined by (9.1) is the unique solution of (1.1)–(1.2) subject to (2.4)–(2.5).*

When Σ_τ has connected components σ_m with $0 \leq m \leq M$ and $M \geq 1$, the coefficients of the gradient harmonic fields may be identified by certain line integrals. Let $\xi_j \;:=\; \{x^{(j)}(t) : 0 \;\leq\; t \;\leq\; 1\}$ be a C^1 curve in $\bar{\Omega}$ with $x^{(j)}(0) \in \sigma_0$, $x^{(j)}(1)$ in σ_j and $x^{(j)}(t) \in \Omega$ for $0 < t < 1$. When $\tilde{h}^{(m)}$ is defined by (11.4) then, for $1 \;\leq\; j, m \;\leq\; M$,

$$\int_{\xi_j} \tilde{h}^{(m)} \;=\; \begin{cases} 1 & \text{if } m = j, \\ 0 & \text{if } m \neq j. \end{cases} \tag{13.5}$$

This is a consequence of the boundary condition (11.5) and the chain rule. Thus

$$c_m \;=\; \int_{\xi_m} h \qquad \text{for } \; 1 \;\leq\; m \;\leq\; M. \tag{13.6}$$

This shows that, with respect to this particular basis of $G\mathcal{H}_{\varepsilon\tilde{\Sigma}}(\Omega)$, the coefficients in (11.8) may be identified as "potential differences" of the field h between the components σ_0 and σ_m of Σ_τ. See Figures 1 and 2 in Section 2 for illustration of possible configurations. Also the projection of a field $u \in L^2(\Omega; \mathbb{R}^3)$ onto $\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$ is $\mathcal{P}_\varepsilon u$ where

$$\mathcal{P}_\varepsilon u \;:=\; u - \nabla\varphi_u - \varepsilon^{-1} \operatorname{curl} \hat{A}, \tag{13.7}$$

and φ_u, $\hat{A}$ are defined as in Theorems 4.2 and 5.2 respectively. These results may be combined to yield the following more general well-posedness result.

Corollary 13.4. *Assume that Ω is simply connected, Σ_τ has $M+1$ connected components and the conditions of Theorem 9.1 hold. Assume that the line integrals in equation (13.6) are prescribed for $1 \leq m \leq M$ where h is the field defined by (13.7) and the curves $\{\xi_m : 1 \leq m \leq M\}$ are defined as above. Then there is a unique solution of (1.1)–(1.2) subject to (2.4)–(2.5) and (13.6).*

Proof. From Corollary 9.2 and Theorem 13.2, the set of all solutions of (1.1)–(1.2) subject to (2.4)–(2.5) is given by (9.3) with $k \in G\mathcal{H}_{\varepsilon\Sigma_\nu}(\Omega)$. Call this ε-harmonic field h instead of k; then from equations (11.4) and (11.8), h is a smooth field on $\bar{\Omega}$ from elliptic regularity theory for the associated boundary value problem for the scalar potentials χ_m. The above analysis shows that the coefficients in the representation (11.8) are determined by the line integrals (13.6), so they determine a unique ε-harmonic component of the solution as claimed. □

Physically this result shows that a unique solution can be found provided one prescribes, in addition to the boundary data, M specific "potential differences" . That is by prescribing M special (equivalence classes of) line integrals of the field along paths joining different components of Σ_τ. The values of these line integrals are independent of the specific path ξ_m chosen as a consequence of Theorem 13.1. A similar result could be stated where instead of (13.6), the fluxes h_m defined by (11.10) are prescribed for the harmonic component of a field and for $1 \leq m \leq M$. When the region Ω has handles and Σ_τ has more than one component then the above functionals of the field determine the gradient component of the ε-harmonic field. It would be of considerable interest to know what further functionals are required to uniquely determine the ε-harmonic components of solutions in this case.

The conclusion of this analysis is that to have a well-posed mixed div-curl problem, extra functionals of the solution may need to be prescribed in addition to the boundary values. The number and type of these extra conditions depends on the differential topology of the domain Ω and of the sets Σ_τ and Σ_ν where the different types of boundary data are prescribed. A comprehensive theory has been described here for the case where Ω is simply connected. When the region Ω has handles, a number of open questions must be resolved before criteria for well-posedness can be described.

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