Flexagons and Flexigation: An introduction to making and mapping simple flexagons.

This pdf provides a brief introduction to flexagons by looking at three simple tetraflexagons. It also explains several of my original flexagon treatments and provides a detailed introduction to the way in which flexagons can be mapped. This material was originally written in 2001 as introductory material for my leaflets on silverflexagons which have now been brought together and published as the Water Trade book Silverflexagons and the Flexitube. The material in Flexagons and Flexigation could not be included in this book for lack of space. I hope, however, that it will prove useful in a more general context.

This pdf also includes templates. They are included here for the sake of completeness, but they are probably too small to be of practical use. Larger, A4 size, versions of these templates can be downloaded from my website at www.origamiheaven.com/flexagons.htm.

So what is a flexagon, exactly?

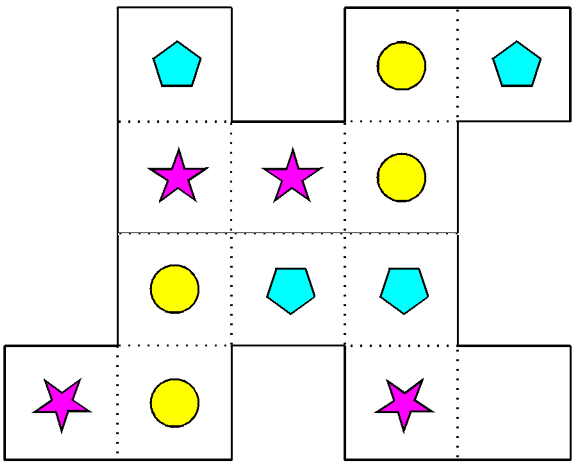
Since there is no widely agreed definition of what a flexagon is perhaps the best way to explain them is to say how they are made and how they behave. To make a flexagon you take a strip of paper, either straight or of some convoluted shape, divide it into a number of segments, usually of the same shape, by creases, fold it up / weave it into a polygonal shape and, if necessary, join the ends of the strip together. In the process some of the surfaces of some of the segments will have been hidden inside the weave. If the result is a flexagon it will now be possible to flex it along the lines of the creases between the segments so that the faces of the segments hidden inside the weave can be brought into view.

The behaviour of a flexagon largely depends on the shape of the segments the strip is divided into because this determines the angles at which the creases between them are set. Segments can be of many different shapes. Flexagons whose segments are hexagonal are called hexaflexagons. Flexagons whose segments are rectangular are called tetraflexagons.

Making the tri-tetraflexagon

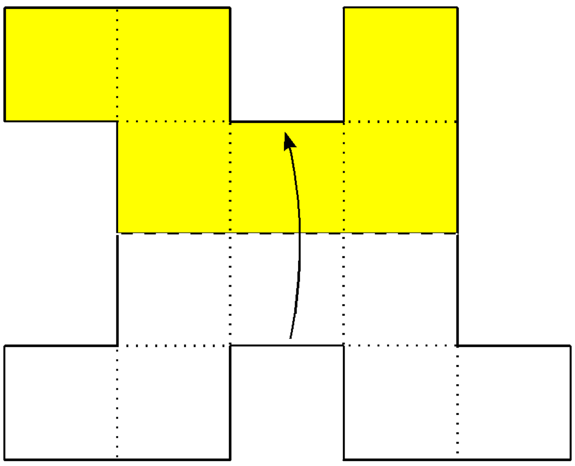
A template for the tri-tetraflexagon can be found on page 26.

1



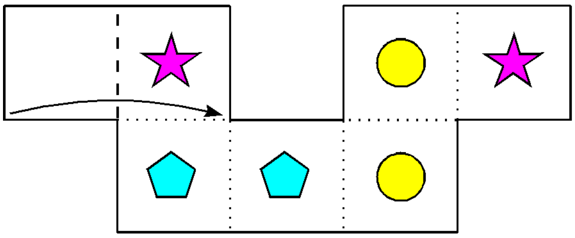
1. Cut out the template and fold carefully along all the dotted lines. Turn the template over sideways.

2



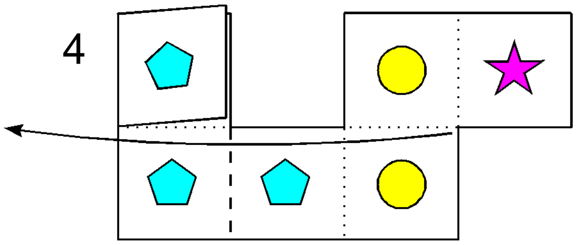
2. Apply glue to the yellow area. Fold the lower half of the template onto the upper half, making sure all the edges line up neatly, then press firmly together.

3



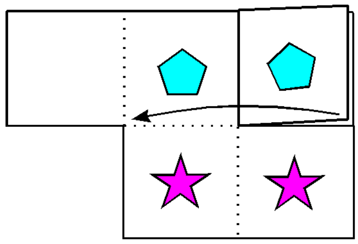
3. Fold the top left hand segment across to the right as shown

4



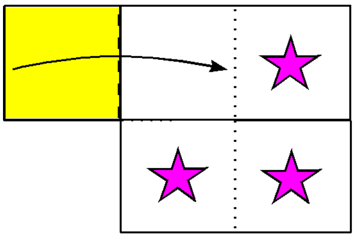
4. Fold the four right hand segments across to the left as shown.

5



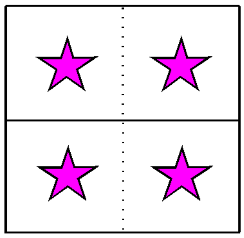
5. Fold the top right hand segment back into its original position.

6



6. Apply glue to the yellow area then fold the flap over onto the top of the blank segment to join the strip into a continuous band.

7

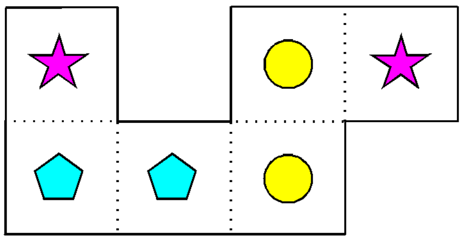


7. This is the result. The Tri-tetraflexagon is finished.

Mastering the jargon

Here is a quick explanation of what some of the terms used in this publication mean. Some of these are in common use. Others are my own invention.

8

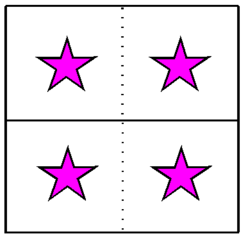


Picture 8 shows the strip of paper the Tri-tetraflexagon is made from. This strip is divided into sections by creases. From now on these sections will be called segments. The creases between them act as hinges, allowing adjoining segments to rotate backwards and forwards in relation to each other. Each segment has two surfaces, which, of course, correspond to the two surfaces of the paper the strip is cut from.

In the finished flexagon the segments are arranged in on top of each other stacks. There are four stacks of segments in the Tri-tetraflexagon.

The surfaces on the top and bottom of each stack are visible, those in the middle are hidden from view.

9



Each flat arrangement of the flexagon is known as a state.

A flex is a move that transforms the flexagon from one state to another by altering the arrangement of the segments in the stacks.

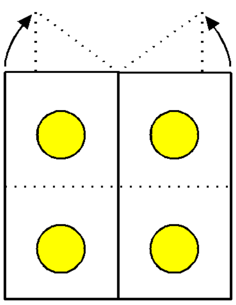
The front and back of the flexagon are called its faces. Each face of the Tri-tetraflexagon is formed from a combination of four surfaces. One such face is shown in picture 9.

The point where the segments touch each other corner to corner in the centre of the face is called the focus. The Tri-tetraflexagon has a single focus. Some complex flexagons have several.

Flexing the tri-tetraflexagon

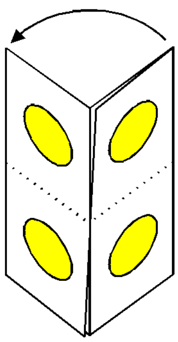
To learn how to do this you first need to turn the flexagon over sideways to find the face which is decorated with circles.

10



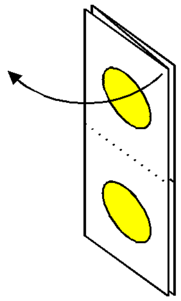
10. Begin to flex the flexagon by folding the left and right hand edges backwards.

11



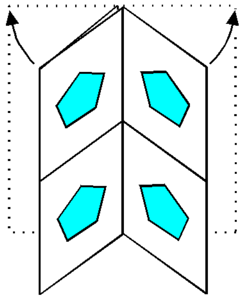
11. Continue folding the edges backwards until they meet behind.

12



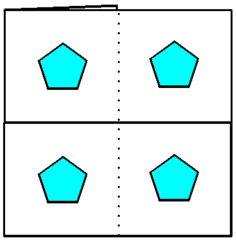
12. Separate the layers at the front of the flexagon and begin to pull them apart.

13



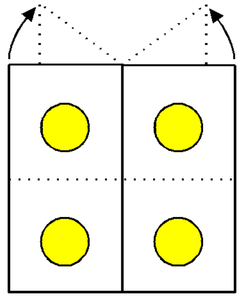
13. Continue opening the front of the flexagon to reveal the face decorated with pentagons.

14



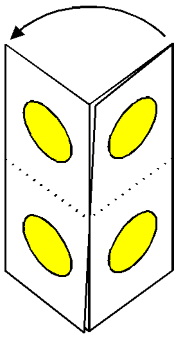
14. If you turn the flexagon over sideways you will find the face decorated with circles has moved to the back.

15



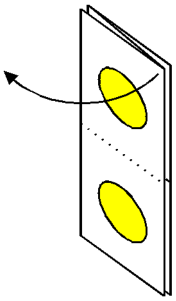
15. To return to the face decorated with stars first fold the left and right hand edges backwards.

16



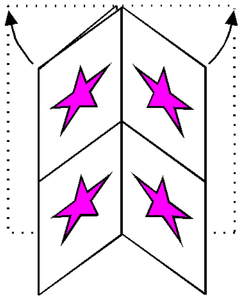
16. Continue folding the edges backwards until they meet behind.

17

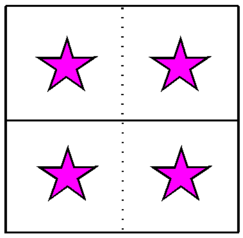


17. Separate the layers at the front of the flexagon and begin to pull them apart.

18



19



18. Continue opening the front of the flexagon to reveal the face decorated with stars.

19. You are back to the original state of the flexagon

Why tri-tetraflexagon?

This flexagon was discovered, and the name invented, in the early days of flexagon exploration. The tetra part means four and refers to the four sides of a rectangle. Tetraflexagon is a general name applied to flexagons made from strips divided into rectangular segments. The tri part means three and refers to the fact that this particular rectangular flexagon has three faces. So tri-tetraflexagon means three-faced rectangular flexagon.

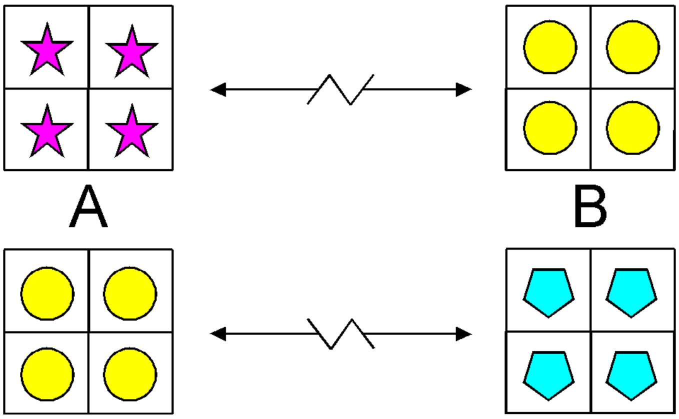
And flexigation? What does that mean?

Flexigation is the art of navigating a flexagon. Because the tri-tetraflexagon only has two different states it is easy to navigate. Some flexagons, however, have many, many more, some of which can be accessed by several different routes using different kinds of flex. Flexigation, of course, is much easier if you possess a map that shows you how to navigate your way through all the different states of a flexagon. Flexagon maps also make it easier to understand how different flexagons, and different types of flexagon, relate to each other.

How do I draw a flexagon map?

Glad you asked! A flexagon map must at the very least show each state of the flexagon and one way to navigate between them.

Here is a basic map of the tri-tetraflexagon that does exactly this:



Each state of the Tri-tetraflexagon is represented by a position on this map. Each position consists of a picture of each of the faces of the flexagon in that state and an identification letter placed between them. This is simply so that we can talk about the positions and states as position A or state B etc.

To use the map you first have to align the flexagon to it. To do this find the face decorated with stars and turn the flexagon around until the alignment of the stars matches the top picture of position A. To find the face decorated with circles (the lower picture of position A) all you need to do is turn the flexagon over.

This is not quite so simple as it sounds, since turning the flexagon over in different directions can lead to different results. In this map the pictures of the faces are arranged above and below each other. This tells you that you move from one face to the other by turning the flexagon over forwards. In maps of more complex flexagons it might be necessary to show the faces side by side, meaning you need to turn over sideways to move between them, or corner to corner, meaning you need to turn over diagonally.

Flexes are shown on the map as double headed arrows. The arrow is double-headed because a flex can always be reversed. Try this out for yourself by beginning with the face marked with pentagons on page 3 and reversing the moves to return to the face marked with circles.

There are many different kinds of flexes, which have different results, so the map also has to tell us what kind of flex is intended. This is done by adding a symbol to the middle of the flex-arrow. The zigzag symbol used in this map stands for the kind of flex explained on pages 4, 5 and 6 which we can call the zigzag flex. The symbol is particularly useful because it tells us exactly how each zigzag flex should be made.

To understand how this works find the face decorated with stars, align the flexagon to the map, then turn the flexagon over to find the face decorated with circles. Hold the flexagon in both hands and tip the top of the flexagon forward until you can only see the top edge. Now flex the flexagon in the way shown in pictures 10 to 14. When you begin to make the flex the edges form a V. When you pass the half-way point, and open up the front, the edges form an upside down V instead. Reverse the flex and you will of course get the upside down V before the normal V. The zigzag symbol between these two faces, the ones at the bottom of the map, gives you this information. If you want to flex from circles to pentagons the sequence is V then upside down V. If you want to flex back again the opposite is true.

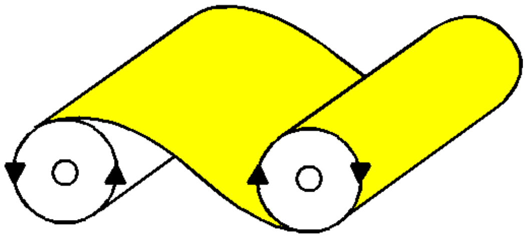
You can prove you have understood this by once more finding the face decorated with stars, aligning the flexagon to the map, and using it to find out how to flex to the face decorated with circles in state B.

A simple map of this kind is good enough for most purposes, but it does not provide all the information needed to study the behaviour of a flexagon in detail.

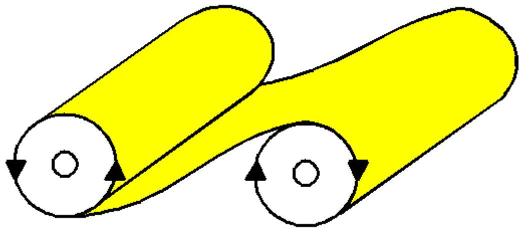
How does the tri-tetraflexagon work?

Flexing the tri-tetraflexagon alters the order and distribution of the segments and their surfaces within the stacks. This in turn alters the appearance of the faces.

You can visualise the tri-tetraflexagon as two rollers linked by two bands of cloth, like this:



One roller would look like this ...



... and the other like this.

The arrow heads show the direction in which the rollers are revolving. Although both right hand rollers are revolving clockwise and both left hand rollers anticlockwise, cloth wound round the rollers in the way shown would be transferred between them in opposite directions. In the case of the first roller cloth is being transferred from right to left and in the case of the second from left to right.

Because both sets of rollers are revolving in the same directions they could be linked together on the same spindles. Cloth would still be transferred between them in two opposite directions.

If the rolls of cloth are fixed to the rollers then at some point it will become impossible to turn the rollers any further in the same direction. They could however be reversed. The effect of this would be to reverse the direction of the transfer of cloth in both cases.

This is effectively what is happening when we flex the tri-tetraflexagon. While the whole flexagon is rotating as a unit, the top half of the flexagon is transferring a segment from one stack to another across the flexagon, say from right to left, while the bottom half of the flexagon is transferring a segment from one stack to the other in the opposite direction, say from left to right.

In the case of the tri-tetraflexagon this transfer of segments can only take place once before the direction of rotation has to be reversed. It should however be obvious that it will prove possible to construct more complex tetraflexagons in which it is possible to transfer several segments in turn in the same direction.

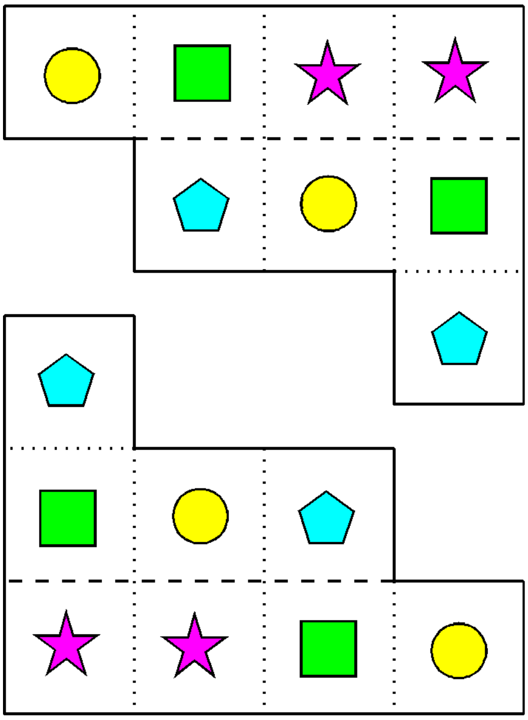
Making the 3-state linear tetraflexagon

As far as I know there is no traditional or widely accepted name for this flexagon. I have used various names for this flexagon in the past but will use the technically correct designation 3-state linear tetraflexagon here.

It would have been more correct to call the tri-tetraflexagon the 2-state linear tetraflexagon but I have refrained from doing so, until now, since the name tri-tetraflexagon is already so well established in the literature of flexagons.

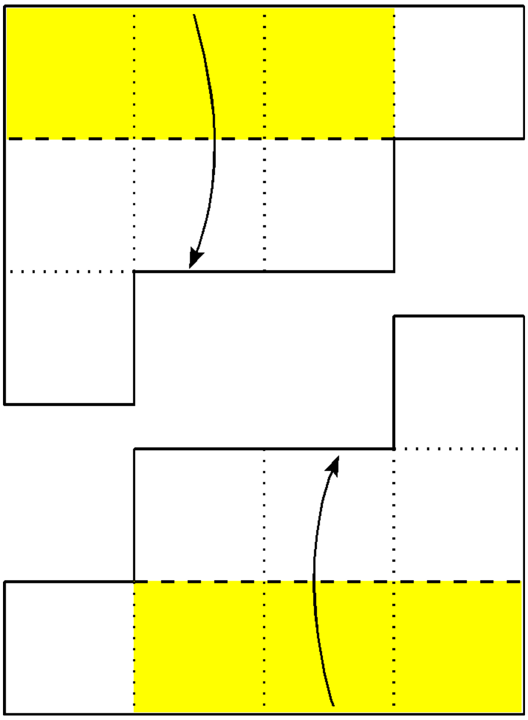
A template for this flexagon can be found on page 30.

20



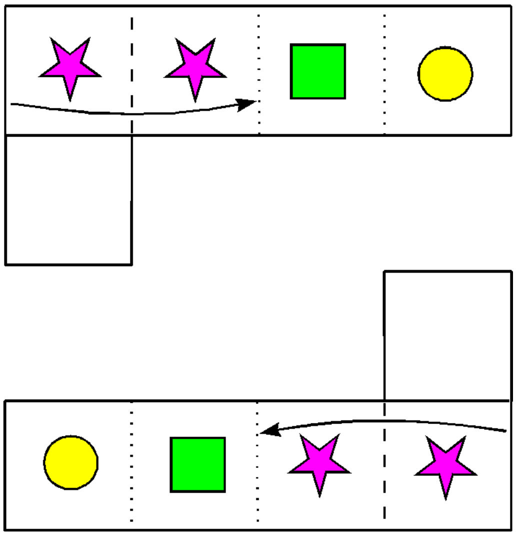
20. Cut out both parts of the template. Fold the templates carefully along all the dotted lines then turn them over sideways.

21



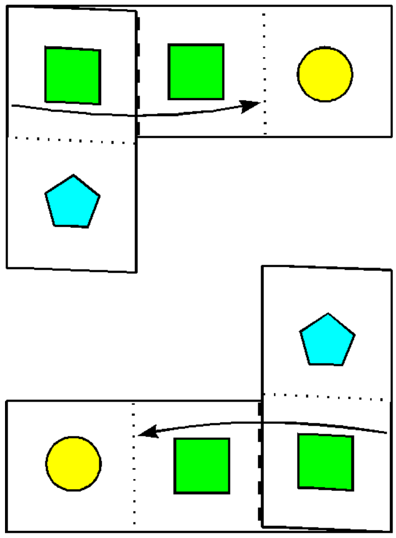
21. Apply glue to the yellow areas. Fold the templates as shown making sure all the edges line up neatly. Press the glued areas firmly together.

22



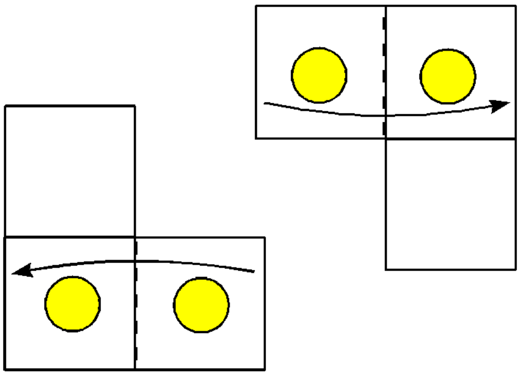
22. This is the result Fold the segments marked with stars face to face as shown.

23



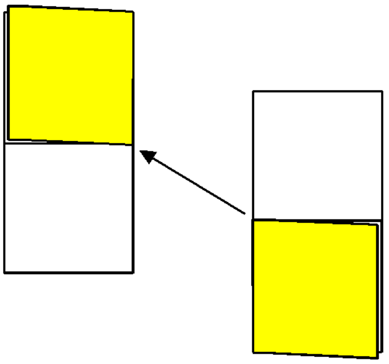
23. Fold the segments marked with squares face to face.

24



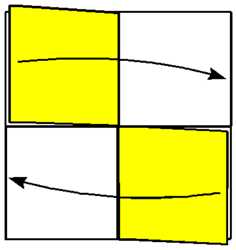
24. Fold the segments marked with circles face to face.

25

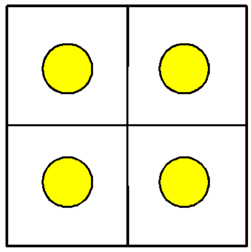


25. Apply glue to the yellow segments and bring the two halves of the flexagon together.

26



27

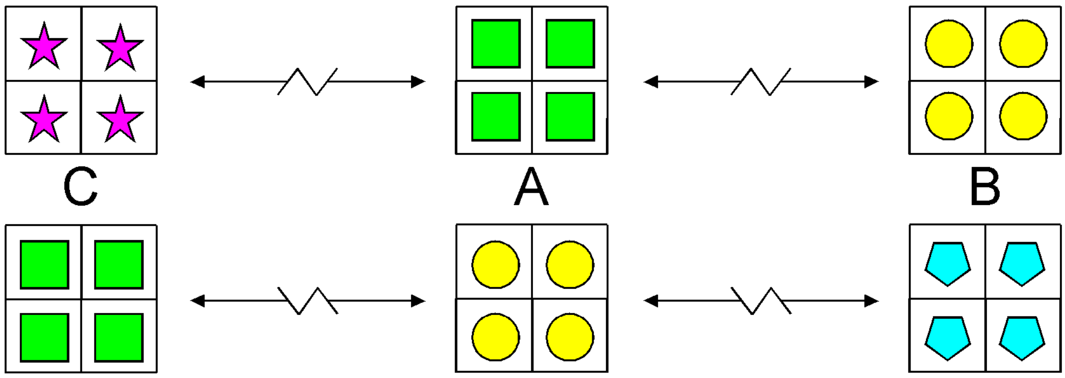


26. Carefully complete the flexagon by folding the connecting segments over as shown. Make sure all the edges line up accurately.

27. The 3-State linear tetraflexagon is finished.

Mapping the 3-state linear tetraflexagon

Here is a zigzag flex map of the 3-state linear tetraflexagon.



Position A is the home position of this flexagon i.e. the position corresponding to the state of the flexagon in which the segments are most symmetrically distributed between the stacks. When drawing flexagon maps it is a good idea to begin by finding the home position of the flexagon and locating it in the centre of the map, since the other positions will almost always be arranged symmetrically on either side of,and in more complex flexagons above and below, the home position.

Some flexagons have several home positions, or possibly no home position depending how you look at it. The tri-tetraflexagon is a good example of this, since, in terms of the distribution of segments between stacks, both states of the tri-tetraflexagon are the same.

Flexagon treatments

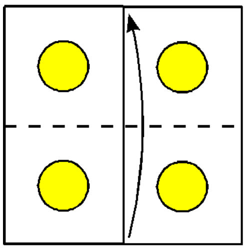
Flexagons are not ill, but sometimes they need treatment to make them better.

A flexagon treatment is a way of dressing up the appearance and handling of the flexagon so that it can be presented as a visual entertainment, a puzzle or a conjuring trick. The way in which the faces are decorated can be used to enhance the impact of the presentation or to disguise the way the puzzle is solved or the trick performed. Here are two simple treatments of the tri-tetraflexagon to show you what I mean.

The tri-tetraflexagon tube challenge

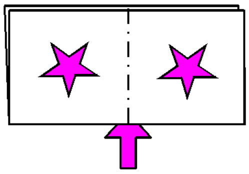
This is a simple treatment you can use to puzzle your friends. To set up the challenge follow steps 28 to 30 below. The face decorated with circles shown in picture 28 is the lower face of position A - see map on page 7.

28



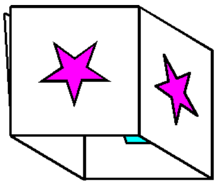
28. Fold the flexagon in half bottom to top. (Or top to bottom if you prefer. The result is the same.).

29



29. Insert your finger between the layers of the flexagon at the point shown and lift the top left hand segment towards you.

30



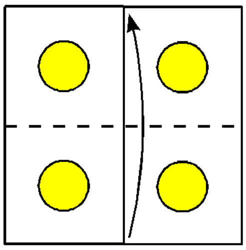
30. The result is a square section tube. Stand the tube upright on a flat surface.

You will see that the outside of the tube is decorated with stars and the inside with pentagons. Hand it to a friend in this position and challenge them to turn the tube completely inside out.

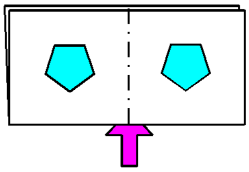
Solving the tri-tetraflexagon tube challenge

Reverse the instructions on page 13 to flatten the tube then open it out to display the face decorated with circles. Now use the map on page 7 to navigate to the other face decorated with circles and follow the Instructions below.

31



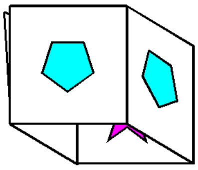
32



31. Fold the flexagon in half bottom to top. (Or top to bottom if you prefer. The result is the same)

32. Insert your finger between the layers of the flexagon at the point shown and lift the top left hand segment towards you.

33



33. If you compare this tube with the tube in picture 23 you will see that it has been turned completely inside out.

Making the challenge harder

If you want to make the puzzle harder try threading a piece of ribbon through the tube and challenging a friend to turn it inside out while you hold both ends of the ribbon. The ribbon actually makes no difference at all. The puzzle just appears more baffling. You can even change one tube into the other while it is threaded onto something more solid like a pencil or a magic wand.

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The 3-state linear tetraflexagon will also open up to form square section tubes. There are four different tubes to find altogether. They are stars outside/circles inside, circles outside/stars inside, squares outside/ pentagons inside and pentagons outside/squares inside. You may like to challenge yourself to change the stars outside/circles inside tube into the pentagons outside/squares inside tube, then thread a ribbon or a pencil etc through the tube and work the change in reverse. It appears impossible at first. But it can be done.

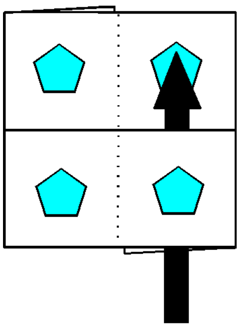
The tri-tetraflexagon ribbon release trick

To work this trick you need a length of fairly wide ribbon and a tri-tetraflexagon. The ribbon is threaded through the tri-tetraflexagon in the way shown below. Thread it through one side first, as shown in picture 27, then let someone examine it to make sure the flexagon cannot be easily removed. Then, to make sure the flexagon is really securely threaded on the ribbon (or at least that is what you tell your audience) thread the ribbon through the other side as well (pictures 35 and 36). In reality this actually unthreads the flexagon from the ribbon.

The trick is over. But no-one should have noticed yet. To make quite sure they do not notice before you want them to leave quite a big loop of ribbon free in the middle. Now it is up to you how you close the trick off. If you gently pull both ends of the ribbon the flexagon will just fall off the centre. Or you can get someone to hold the flexagon firmly by one edge while you just casually walk away with the ribbon. Or you can get them to hold the ribbon while you walk off with the flexagon - though this takes a little more practice.

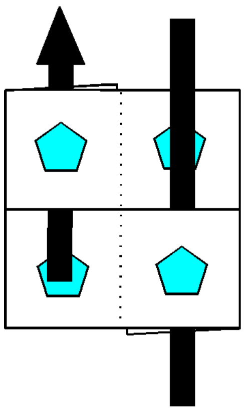
Play it whichever way appeals to you.

34



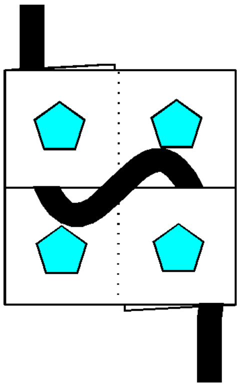
34. Thread the ribbon upwards in between the bottom right hand segments.

35



35. Now thread the top of the ribbon upwards in between the top left hand segments as well.

36



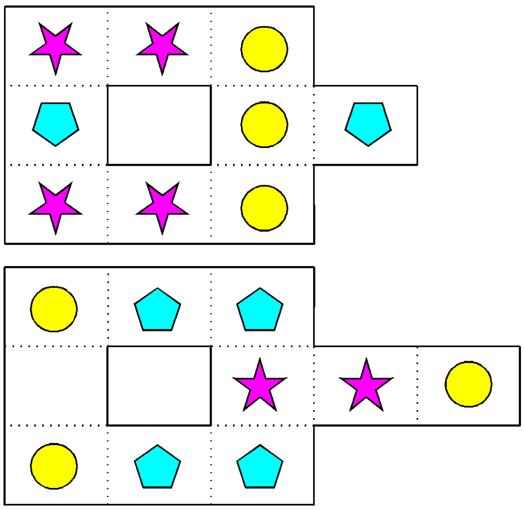
36. The ribbon should now be threaded like this. You are ready to perform the trick. Well, actually, you have already performed it!

Extending the tri-tetraflexagon

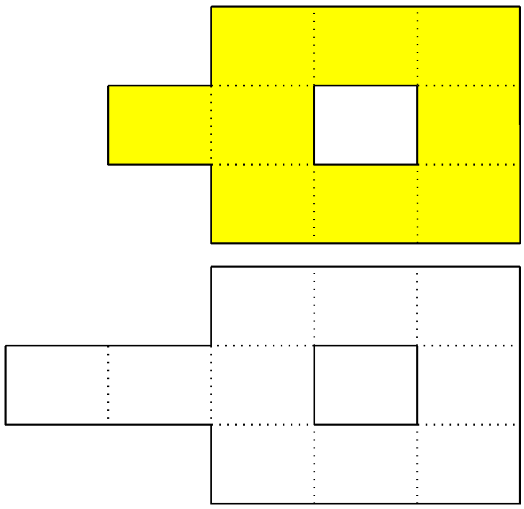
Because the tri-tetraflexagon is a linear tetraflexagon you can extend it by adding an extra row of segments to the bottom. A template for a 3-row tri-tetraflexagon can be found on page 28.

You make it like this:

37



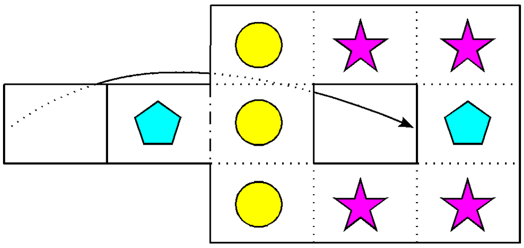
38



37. Cut out both templates. Fold both templates carefully along all the dotted lines then turn them over sideways.

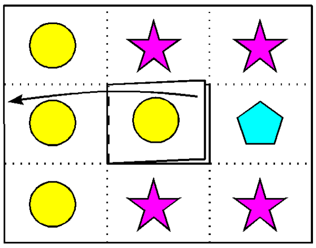
38. Apply glue to the whole of the blank side of the smaller template and glue it to the blank side of the larger template, making sure all the edges line up neatly.

39



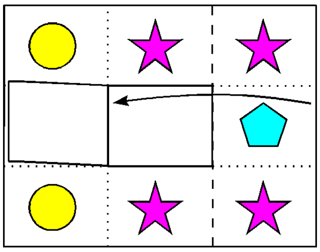
39. Swing the two left hand segments across behind the template so that a circle becomes visible in the window.

40



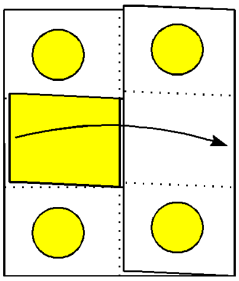
40. Fold the central segment through the window towards you and across to the left.

41



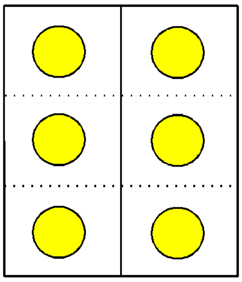
41. Fold the right hand column of segments across to the right.

42



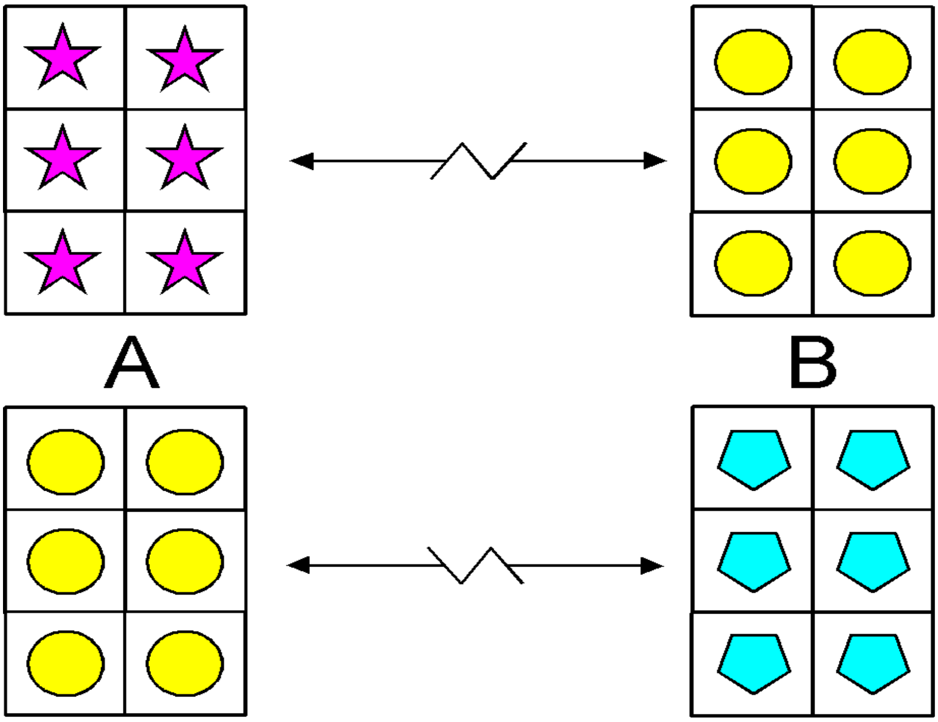
42. Apply glue to the yellow and press onto the middle segment in the right hand column, making sure the edges line up neatly.

43

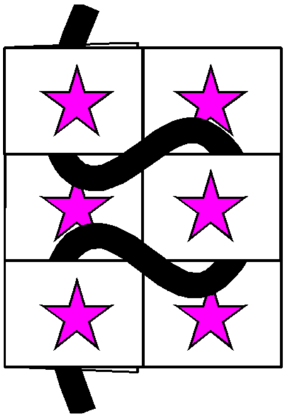


43. The 3-rpw extended tri-tetraflexagon is finished. Tri-tetraflexagons can be made with as many rows as you like in a similar way.

You can see from this map that the extended tri-tetraflexagon flexes in exactly the same way as the normal tri-tetraflexagon.



You can also work the ribbon release trick using the 3-row extended tri-tetraflexagon. The ribbon should be threaded like this:



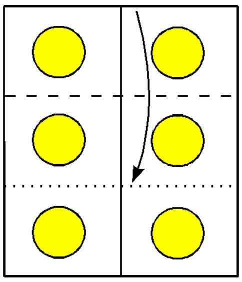
Because of the extra row of segments the effect is more surprising, but it is a little harder to release the ribbon smoothly.

The 3-row tri-tetraflexagon penetration puzzle

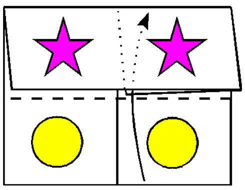
There is a hole right through the centre of the 3-row tri-tetraflexagon through which a surprisingly large object, such as, for instance, an egg or a table-tennis ball, will easily pass.

This hole is a square section tube of the kind we have already met when exploring the properties of the tri-tetraflexagon, but because the extended version has three rows of segments instead of just two the tube is more difficult to find. This is how you do it:

44



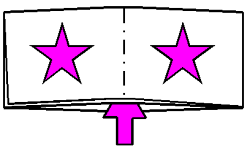
45



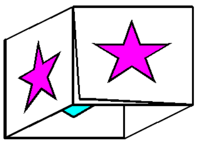
44. Begin from position A with the face marked with circles facing you. Fold down the top row of segments as shown.

45. Fold down the bottom row of segments upwards, tucking them in between the layers of the top row of segments as you do so. Gently persuade the flexagon to lie reasonably flat.

46



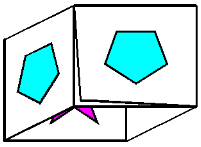
47



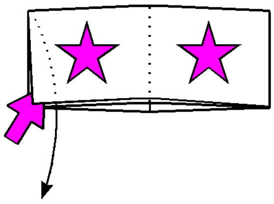
46. Insert a finger in between the layers as shown and open the flexagon up so that it forms a square section tube.

47. An egg or a table-tennis ball will pass quite easily through this tube.

48



49



48. As you would expect, the tube can be turned inside out by flexing to position A and opening up the other face decorated with circles in a similar way.

49. It can sometimes be difficult to escape from the tube. The secret is to flatten the tube in such a way that you can open up one corner and gently pull out the flap that lies inside.

Black holes

There is a whole class of extended tetraflexagons, which I have christened black holes, that can be opened to form tubes in this way.

Drawing more detailed maps

There are lots of different ways to draw maps of flexagons, and which one you choose will depend on how much detail you want to go into. The method of drawing a flexagon map given on page 7, which used a series of symbols to differentiate between the various faces of the flexagon, is fine as far as it goes, but it does not go very far. In order to draw a map that reveals all the properties of a particular flexagon we need to be able to tell each surface of each segment apart.

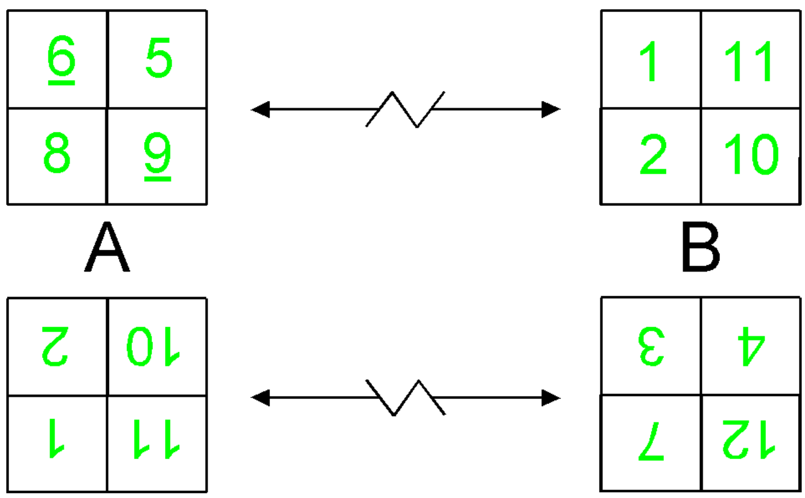
Since this is intended to be the first publication in a series about flexagons and flexigation it makes sense to map the tri-tetraflexagon using methods that can also be applied to mapping more complex flexagons, even if this makes the mapping process a little more complicated than it otherwise needs to be at this stage.

The tri-tetraflexagon strip is divided into six segments. Each of these has two surfaces, so the obvious way to mark the surfaces in order to tell them apart is to number them from 1 to 12. We could perfectly well do this at random, but since it would be useful to be able to draw a map of the Tri-tetraflexagon that could be used to compare the properties of the tri-tetraflexagon with the properties of more complex tetraflexagons, it is better to begin at one end of the strip and number the first segment 1, then work along the strip to the end using numbers 2 to 6. The reverse surface of the strip can be numbered using numbers 7 to 12, the reverse of surface 1 being numbered 7, the reverse of surface 2 being numbered 8, and so on along the strip to the end.

In more complex flexagons, making a flex can sometimes turn segments upside down (as can turning the flexagon over in a different direction) so we also need to be able to avoid confusing surface 6 with surface 9. The simplest way to do this is to underline them both.

A template that produces a tri-tetraflexagon strip numbered in this way can be found on page 27. You can make the flexagon by adapting the instructions on pages 2 and 3. It might be a good idea to do this now.

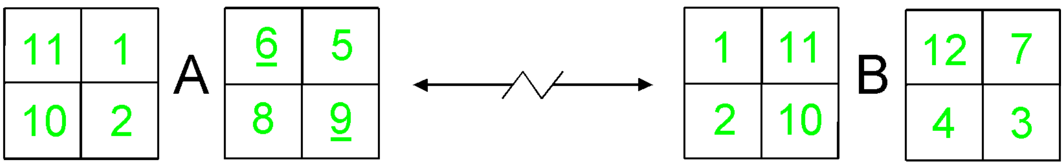
Here is a map made using this numbered version of the flexagon.



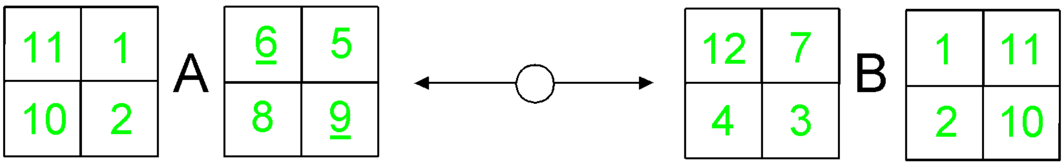
What can we learn from this map? Well, for one thing, we can learn that the name tri-tetraflexagon is misleading. As we have already seen, the name tri-tetraflexagon means three-faced tetraflexagon but this new map shows that while two of these faces are made up of the same four surfaces, the segments have been re-arranged during the flex in relation both to each other and the focus.

We could, it is true, at this simple level, agree to call them two versions of the same face, but maintaining this position becomes increasingly difficult as the flexagons you study become more complex. It is far better to think of flexagons in terms of states rather than faces. There are many similar errors in the literature of flexagons, some of which I have perpetrated or continued myself. As you find out more about flexagons you will discover that almost everything that was once thought to be true about flexagons and flexigation has had to be re-evaluated in the light of later discoveries, and that this process is still on-going. The same fate may well await what I have written here.

Can we simplify this map? Yes. If you can navigate from one face to another using one kind of zigzag flex you can always navigate between the two reverse faces using the alternative kind. Since this is always true we can simplify the map and only show the flex once. The simplified map would look like this.

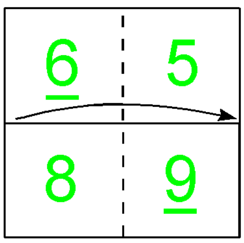


We could also draw a map of the tri-tetraflexagon like this.



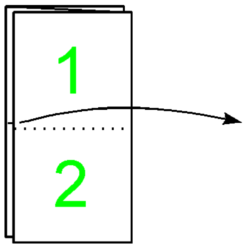
You will notice that the symbol on the flex arrow has changed. A circle symbol is used to stand for a rolling flex, which is a flex that can be made while the flexagon is lying on a table top, like this:

50

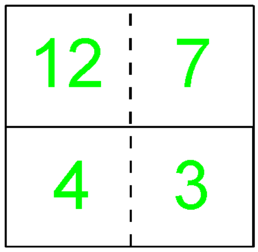


50. Begin with state A. Fold the flexagon in half left to right.

51



52



51. Open out, left to right, as shown.

52. This is state B.

Try this out, making sure you do not pick the flexagon up at any stage. Then reverse the flex, working right to left. This should bring you back to where you started from. Expressed in words the map says, ‘If the flexagon is in state A, and the 6/5/8/9 face is visible, you can reach the 12/7/4/3 face of state B by making a rolling flex to the right. When you have done this you can return to state A by making a rolling flex to the left.’

What the map does not tell you is that you can also move between these faces by making a rolling flex to the left (try it and see) and back again by making a rolling flex to the right. In fact if you make a whole series of rolling flexes in one direction or the other the flexagon will cycle between these two faces for as long as you have the patience to continue.

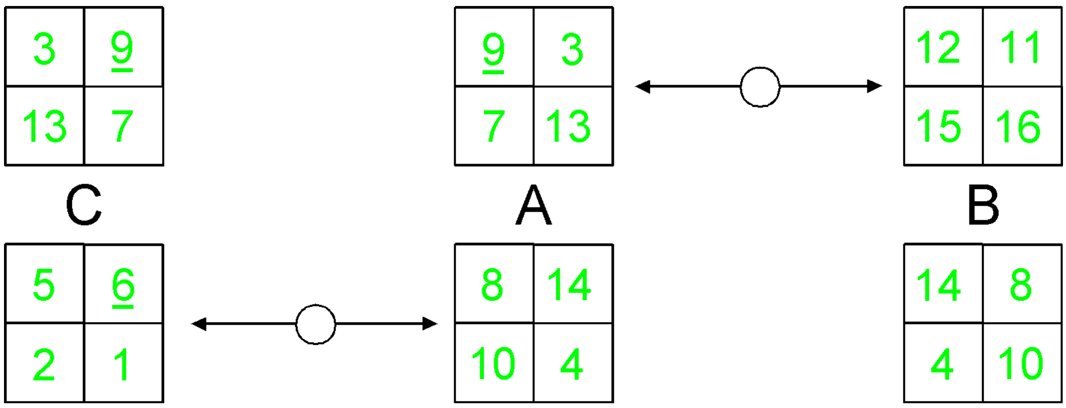
Why not record this on the map? Well, simply because it is not necessary to do so. A rolling flex can always be made in two opposite directions. So if one is shown, you know the other is equally possible. The basic principle of mapping flexagons, particularly complex flexagons, is to only show the information you need to show, otherwise the map quickly becomes too confusing to understand.

One final thing that is worth trying is to turn the tri-tetraflexagon over, lay it back down on the table top and try to make a rolling flex on the other side. You will find it is not possible. In more complex flexagons however you can often make a rolling flex from both faces of a position with totally different results.

It is important to remember that you should always try to map a flexagon using the kind of flexes that can be performed while the flexagon is lying on a table top, even though you may wish to demonstrate its properties in quite a different way (for instance by making a rolling flex behind the flexagon rather than in front).

A detailed map of the 3-state linear tetraflexagon can be drawn in a similar way. If you want to investigate the properties of this flexagon in detail you can do so using a version of the flexagon constructed using Template 6 on page 29. The surfaces of this template have been numbered using the same scheme that was used to number the surfaces of the tri-tetraflexagon, though, of course, since the 3-state linear tetraflexagon has more segments, more numbers are required.

Here is a rolling flex map of this numbered version.



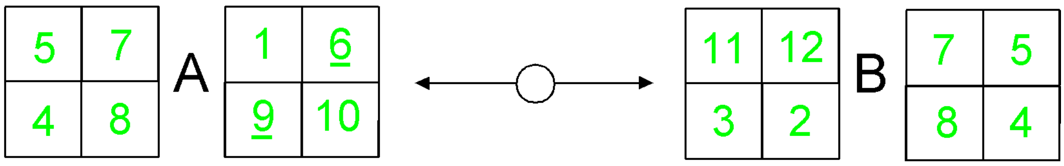
By studying this map in conjunction with the simple map on page 12 you should now be able to work out how to begin from either face of the home position and reach any other face in a single move.

The template on page 31 has been included so that you can, if you wish, draw an advanced map of the 3-row tri-tetraflexagons and investigate its many interesting properties further.

Mirror-image flexagons

It is possible to construct two versions of any tetraflexagon (and of many other flexagons as well) from the same strip of segments. In terms of the way in which they flex these flexagons are the mirror-image of each other, though whether they look like mirror-images will depend on how the strip is decorated.

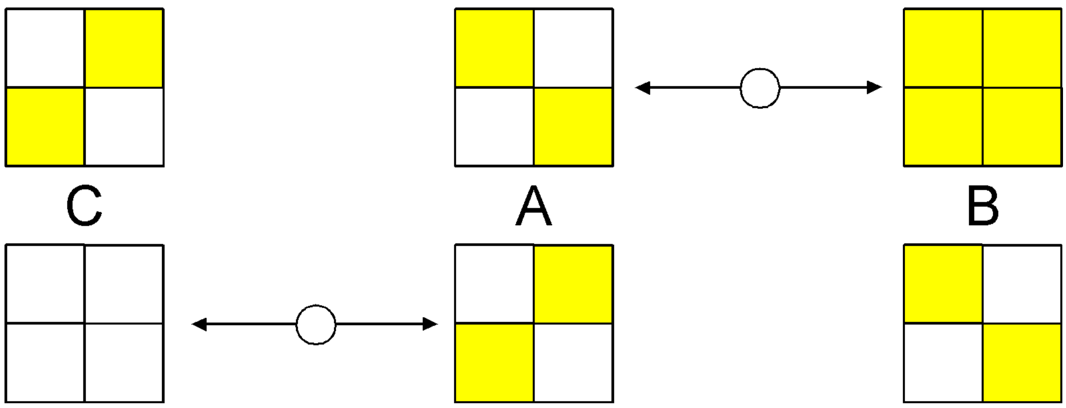
Here is a map of a mirror-image tri-tetraflexagon constructed from the template on page 27.



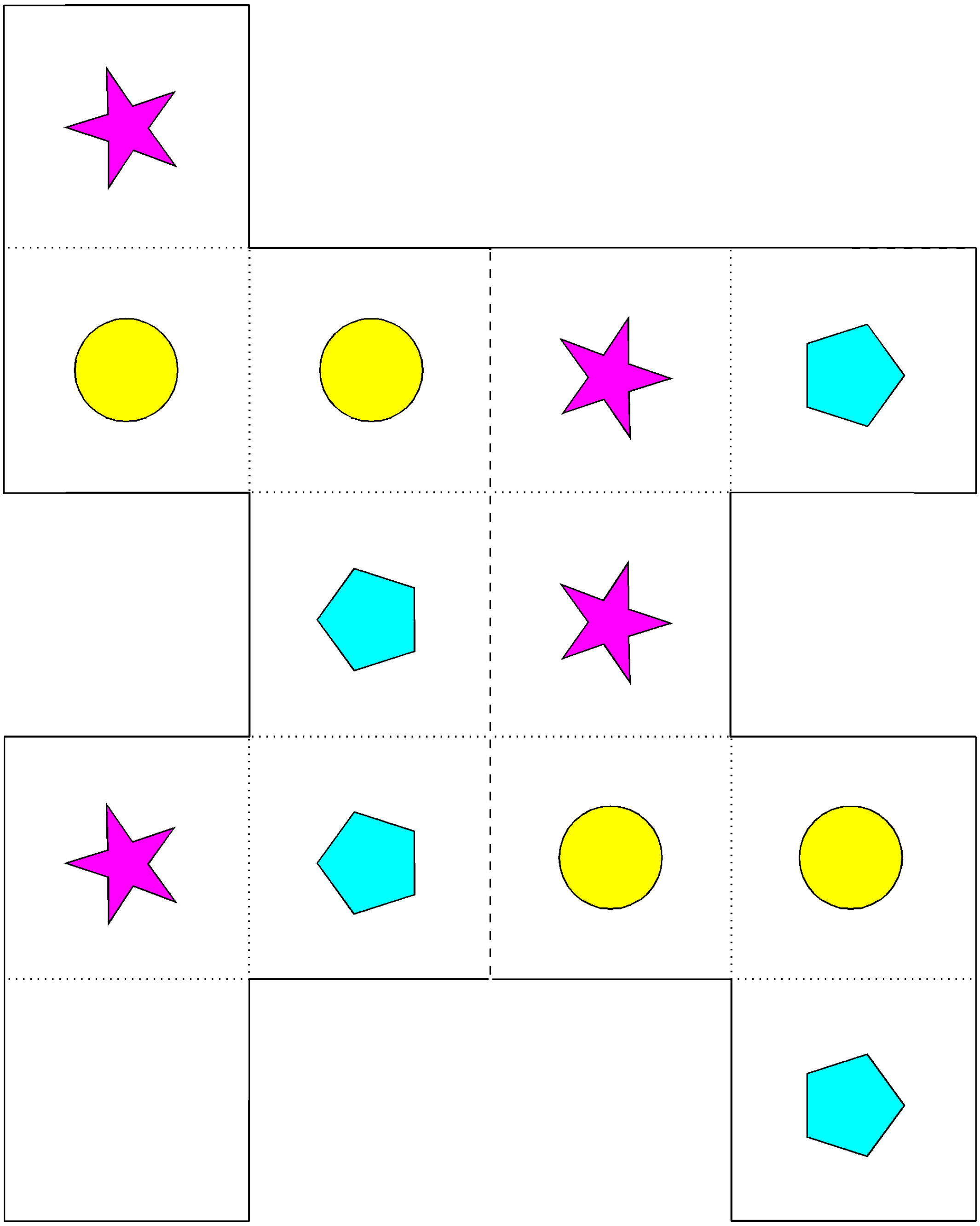
Making flexagons without using templates

Finally, you can, of course, make flexagons without using templates by cutting the necessary strip of segments from any kind of ordinary paper. Particularly interesting and attractive results can be obtained by using differentiated paper, that is paper which is either white on one surface and coloured on the other or plain on one surface and decorated on the other.

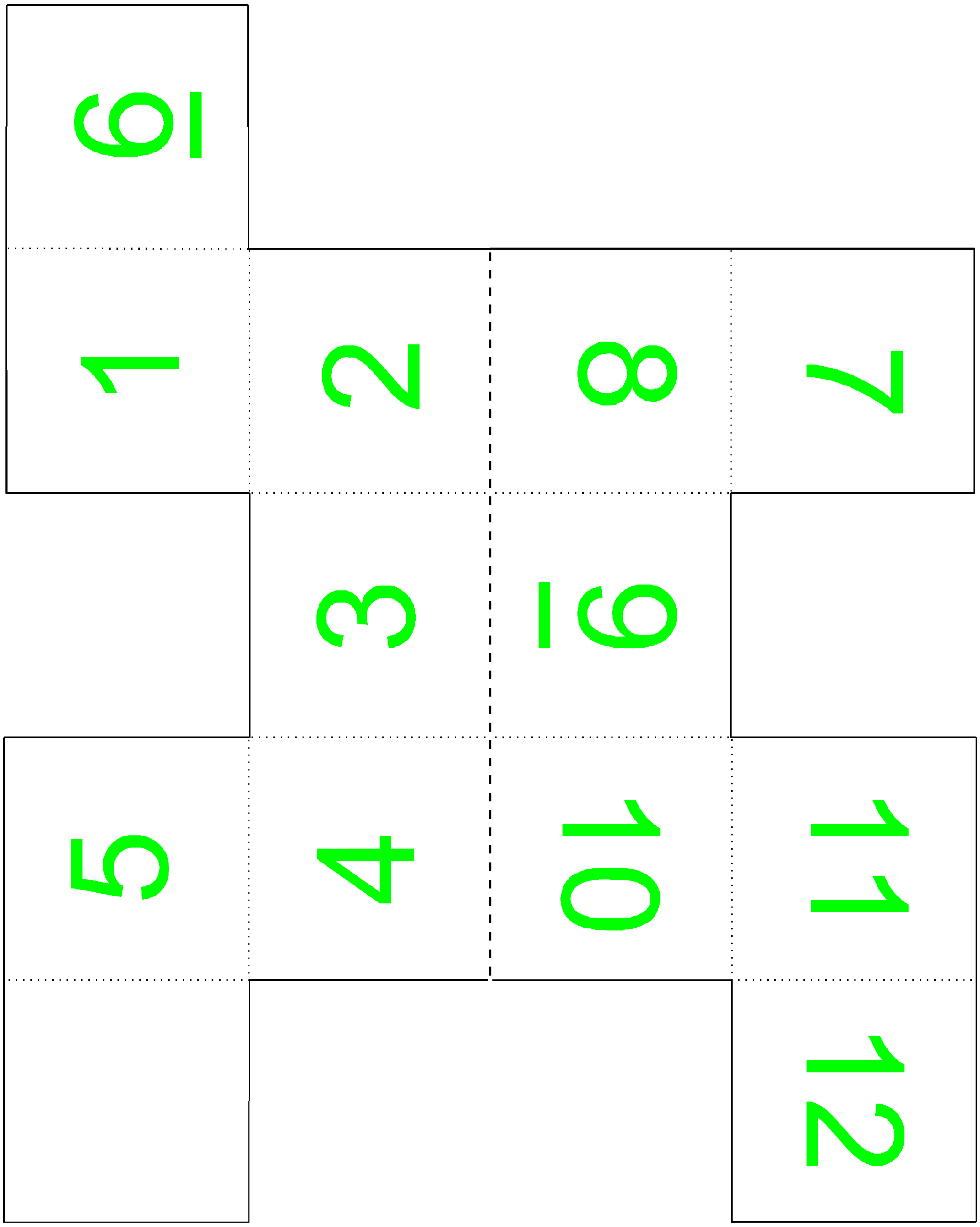
Here is a map of a 3-state linear tetraflexagon made from differentiated paper.



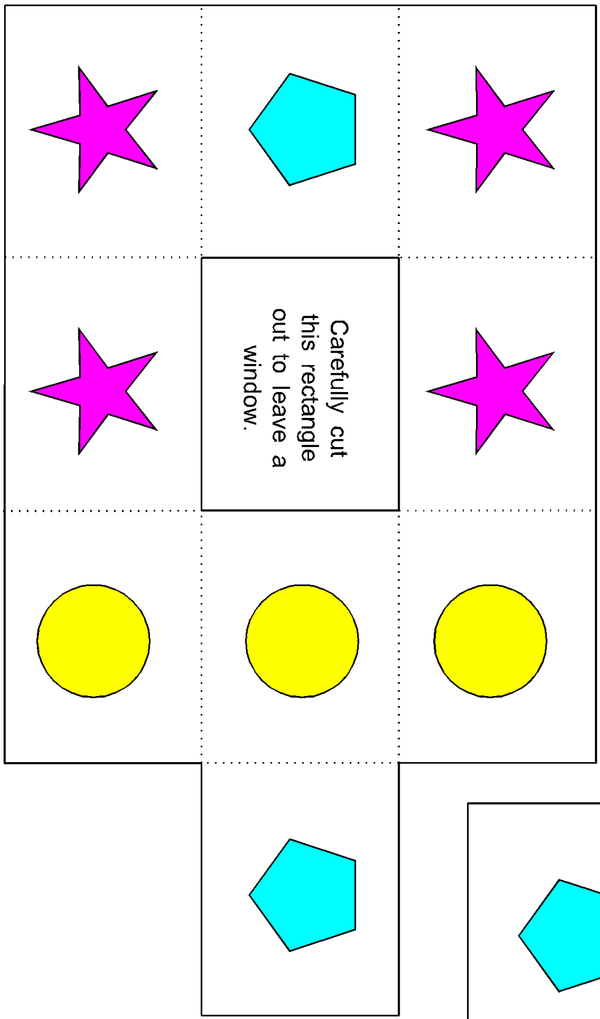
Template 1 - the tri-tetraflexagon

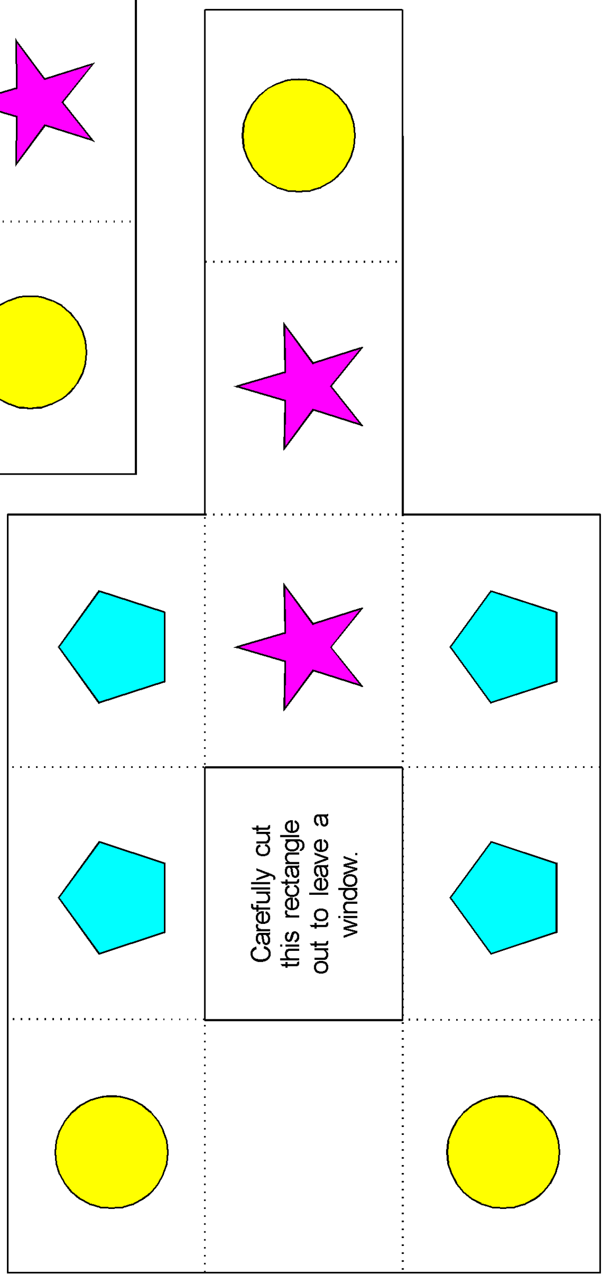


Template 2 - the tri-tetraflexagon

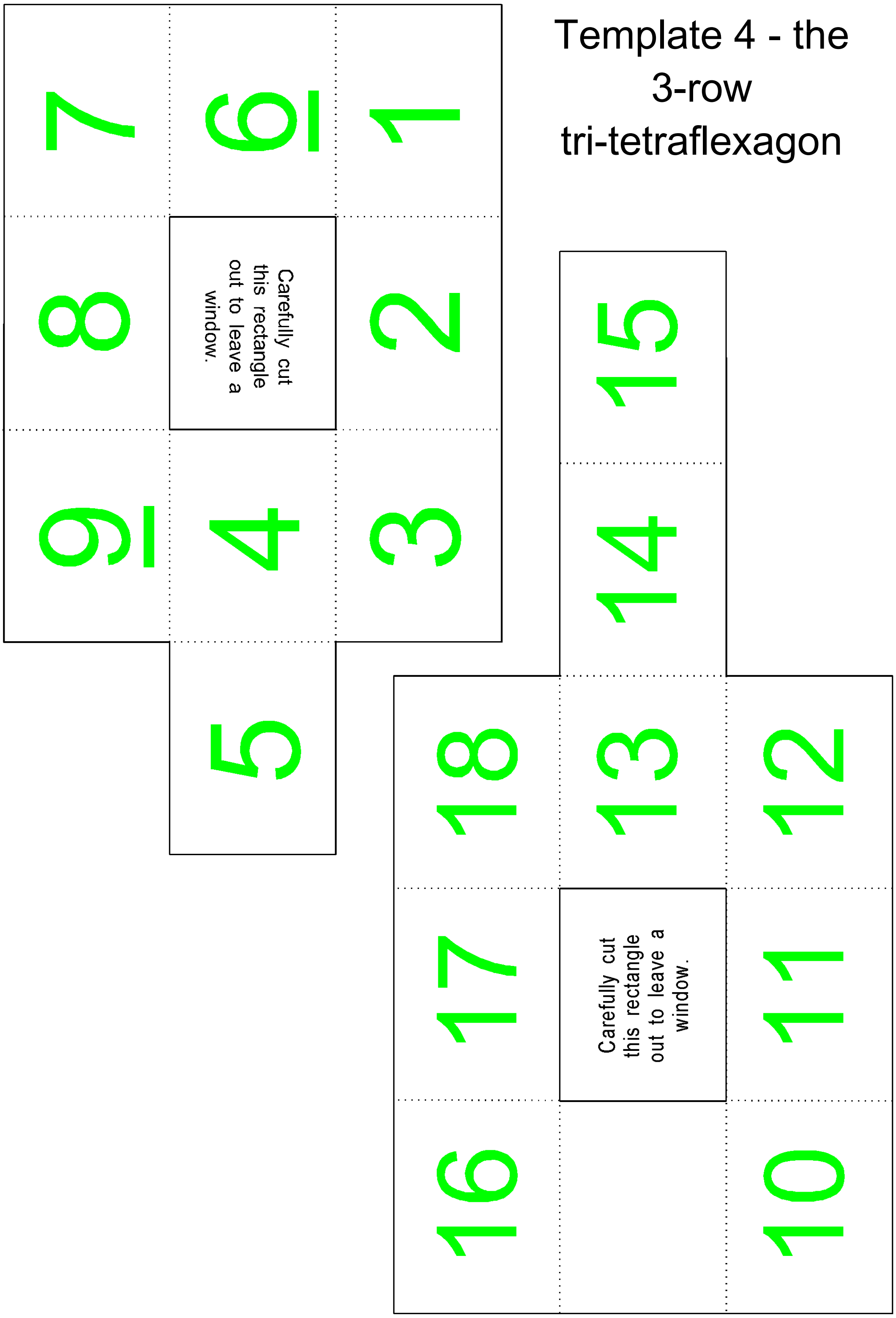


Template 3 - the 3-row tri-tetraflexagon

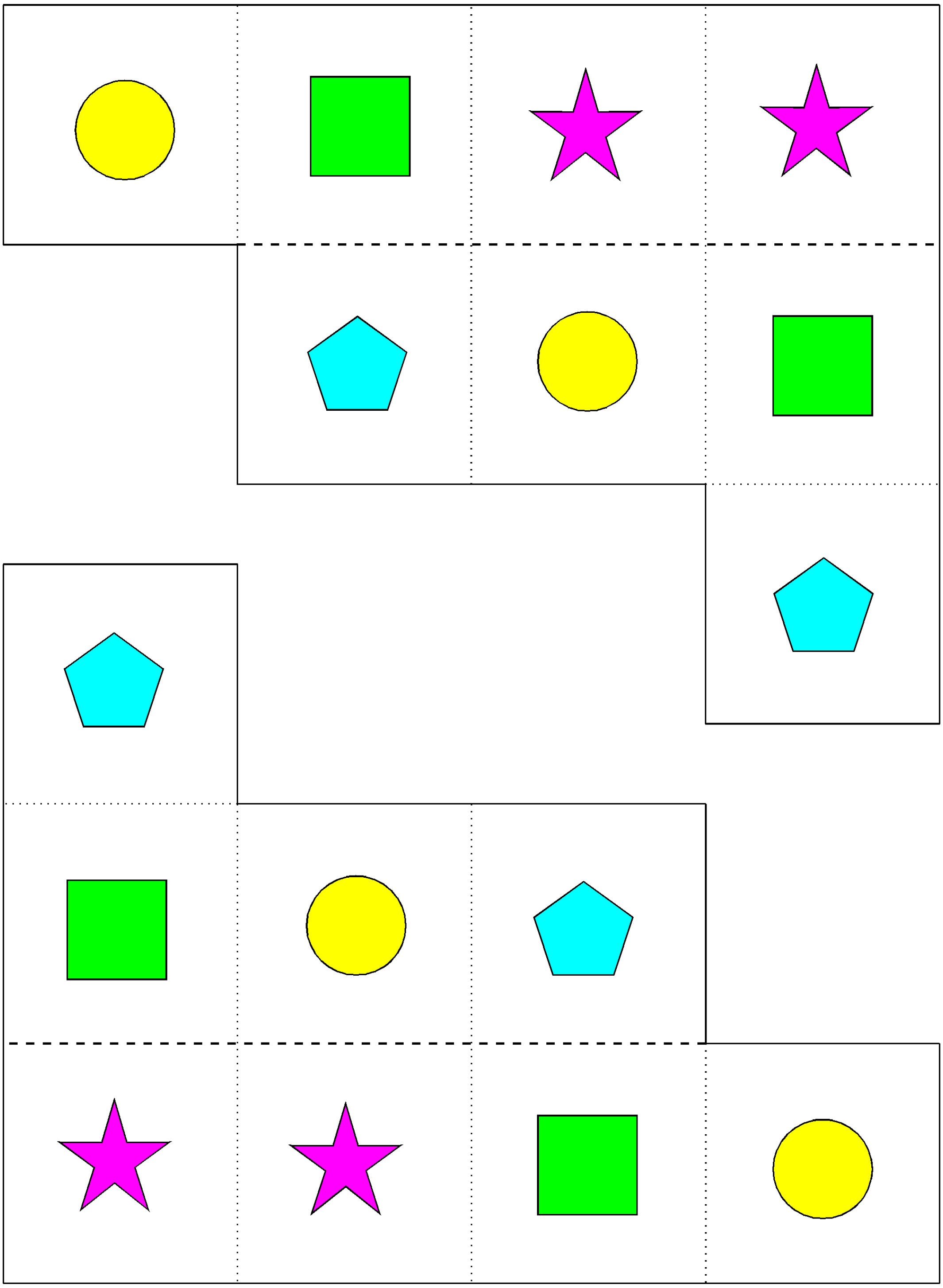




Template 4 - the 3-row tri-tetraflexagon



Template 5 - the 3-state linear tetraflexagon



Template 6 - the 3-state linear tetraflexagon