DIFFERENTIAL EQUATIONS with TI-89
ABDUL HASSEN and JAY SCHIFFMAN

We will assume that the reader is familiar with the calculator's keyboard and the basic operations. In particular we have assumed that the reader knows the functions of the **SECOND, APLHA,** and **GREEN DIAMOND** keys. Thus we will simply say "display the **Y=**editor" assuming that the reader will first press the **DIAMOND** key and then **F1.** We have use **bold** face upper case letters to refer to the calculator's commands or keys. When we say "You can access the command **and** from **Math 8 8**", we mean first press **MATH**. (which is **2nd** then 5) and then press **8** twice. Note that most menus have several submenus, which in turn may have many options. To display and select an option we may need to use the curser keys.

Most built-in functions can be found by pressing **CATALOG** followed by the first letter of the desired command. One can then select the command by scrolling down, if necessary, using the cursor key. To clear the home screen, use **F1 8.** To go back to the home screen, use **HOME** or **ESC** or **QUIT**. To obtain an approximate value (in decimals) use the **GREEN DIAMOND** followed by **ENTER.** To draw a graph use **GREEN DIAMOND** followed by **F3**.

A. Direction Fields and Graphs of Differential Equations

Example 1: Draw the direction filed of the differential equation $y' = x^2 - y$
Solution:
1. Press the **MODE** key and from the **GRAPH** mode select **6: DIFF EQUATIONS**.
2. Display the **Y=** editor and enter your differential equation. Use t for independent variable and $y1$ for y. For the prime notation for the derivative use **2ND =.** Draw the graph.

The figures below show the above steps.
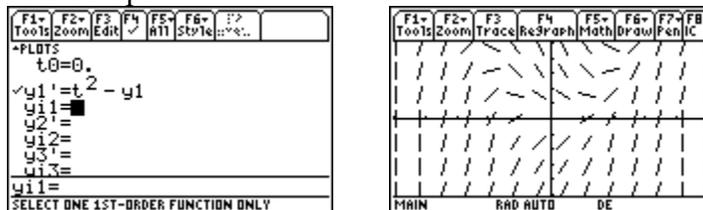
Example 2: Draw the graph of the solution of $y' = x^2 - y$ that passes through (1,-2)

Solution: Enter your differential equation as in Example 1.
1. Use the cursor key to move up to the initial value of t and press **F3.** Then type 1 and **ENTER.**
2. Type –2 for the y-value in the line $y_{i1} =$. Draw the graph.

The first figure below shows the inputs while the second shows the graph.
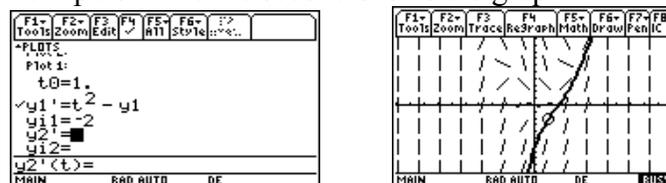
Remarks:
1. To draw the graph of the solution without the direction field, from the **Y=** editor press **F1 9** to display the **GRAPH FORMAT**. From the **Fields** option (the last row) use the right cursor key to choose **3:FLDOFF** and press **ENTER.** Now draw the graph.

2. You can obtain solutions passing through other points by simply changing the initial conditions while you are in the direction field/graph screen. To do this press **F8 (**which is **2ND F3.)** and type the initial condition for t and **ENTER** and then type the initial value for y and **ENTER.**

3. There are several styles for the graph of the solution through a given point. From the **Y=** editor screen press **F6** (which is **2nd F1**) and choose any of the styles and see what happens.

The figures below show **GRAPH FORMAT** and the graph of the equation $y' = x^2 - y$ with different initial values using **F8** (from the graph display).
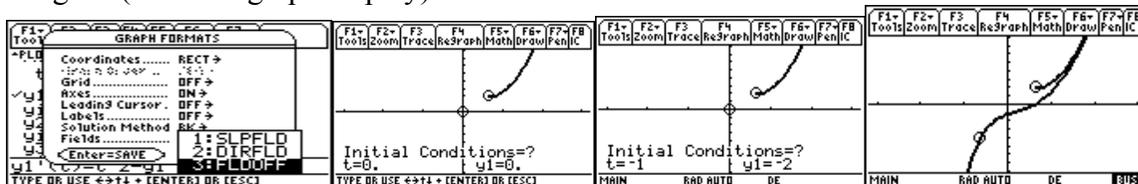
Example 3:   Draw the direction filed for the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 5x + 3y - 2t \\ \dfrac{dy}{dt} = 4x - 3y - t^2 \end{cases}$$

Solution: In the **Y=** editor use *y1* for *x* and *y2* for *y*. From the **Fields** option of the **graph format**, choose **2:DIRFLD.** (See Remark 1 above.) The first two figures show the inputs and the direction field, respectively. The last two figures show a graph of an initial value problem for systems of equation. (Write the IVP.)
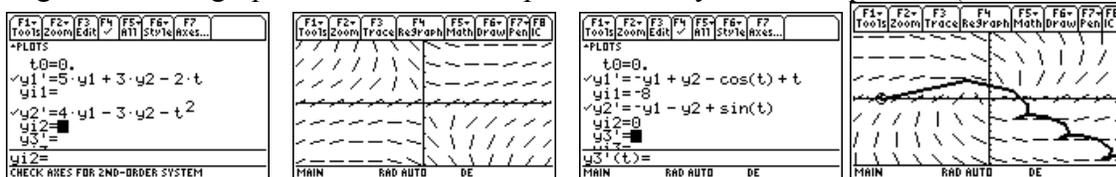
Example 4:   Draw the direction filed for $y'' + 3y' - y = x$.

Solution.   TI-89 draws direction fields only for first order and systems of first order differential equations. Thus we need to convert this second order equation in to systems of first order equations. As before we use t for the independent variable and y_1 for *y*. We let $y_2 = y_1'$ Then the given equation is equivalent to the system

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = t + y_1 - 3y_2 \end{cases}$$

Proceed as in Example 3. The main point of this example is that we can use this technique for higher order differential equations. (See Section **D** below)

B.   Solving First and Second Order Differential Equations

The command for solving first and second differential equations is **deSolve(** which can be accessed by **F3 C**. The format for this command is

> **deSolve(*the differential eqn, independent variable, dependent variable*)**

Example 5:   Solve a) $y' = y + x^2$ b) $y'' + 5y' - 6y = 0$

Solution: In the input line enter **deSolve(** $y' = y + x^2$, **x, y) ENTER** . The figures tell the story!
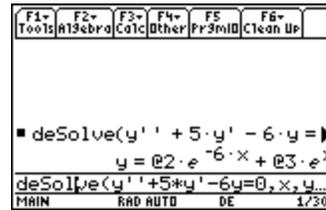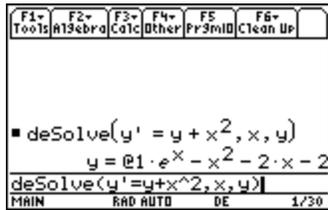
Note that TI – 89 is giving you the general solution for a) as $y = @1 \cdot e^x - x^2 - 2x - 2$. Thus @1 stands for an arbitrary parameter (that we usually write as c or c_1.)

Example 6: Solve the initial value problem (IVP) $y'' + 2y' - 3y = e^x$, $y(0) = 1$ and $y'(0) = -1$

Solution: In the input line enter **deSolve(** $y'' + 2y' - 3y = e^x$ **and** $y(0) = 1$ **and** $y'(0) = -1$ **, x, y) ENTER** You can access **and** from **Math (** which is **2nd 5) 8 8**. You could also type it. Use **ALPHA (-)** for space.

Example 7: Draw the graph of the solution of the IVP $y'' + 2y' + 5y = e^x$, $y(0) = 1$ and $y'(0) = -1$
Solution: First solve the IVP as in Example 6. Then from the input line use **F4 1** to **Define y1(x)** as the solution. You can get the solution by using the upward cursor key and pressing **ENTER**. Make sure to delete **y**. This will automatically enter the solution as *y1* on the **Y**=editor. Here is the solution and the graph. (We have used [-5,8] by [-10,30] for the graph window.) Make sure that the **MODE** is on **Functions** not on **Diff Equations**.
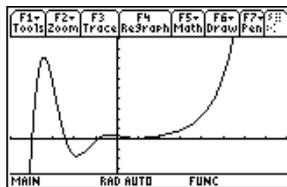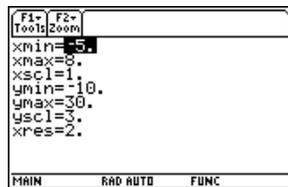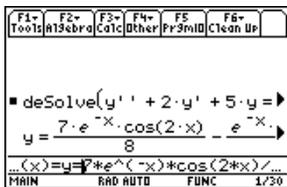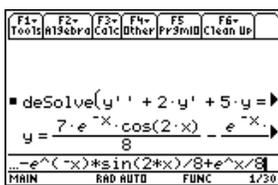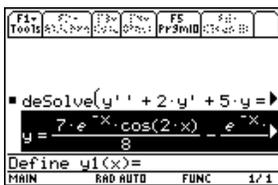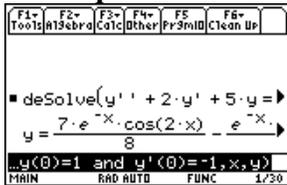
C. Euler and Runge-Kutta Methods

The TI 89 can generate numerical solutions using the Euler and Runge-Kutta methods. The command for this is **BldData *name*.** Here **BldData** is the command for building the table of values and ***name*** is the name of the data. We show the details in the following example.
*Example 8:* Consider the IVP: $y' = x - y$, $y(0) = 1$
         a)   Use Euler's method to construct a numerical solution for the IVP.
         b)   Use Runge-Kutta method to construct a numerical solution for the IVP
         c)   Solve the IVP
         d)   Construct a table to compare the two numerical methods and the exact solution.

Solutions: a)   Here are the steps to build the data for the numerical solution using Euler's method.
     1.   Enter the differential equation in the **Y=** editor and delete any other equations.
     2.   From the **graph format,** select **EULER** for **Solutions Method** and **FLDOFF** for **Fields**.
     3.   Press **HOME** then **CATALOG** and **b.** Select **BldData** and **ENTER**. Type **eu** for the name and **ENTER**. (You could also type **blddata eu** on the input line after your press **HOME**)
     4.   Open **eu** using **APPS 6 2** and then select **eu** for **Variable**.
   b)   For the Runge-Kutta method from the **graph format**, select **RK** for **Solutions Method**. Press **HOME** and type **blddata rk** and **ENTER**.

c)   Solve the IVP using **deSolve(** and **Define y(x)** as the solution.(See Example 7.)

d)   To compare the two methods and the exact solution, follow these steps.

            1.   Use **APPS 6 3** to create a new data, call it **comp**.

            2.   Press **F4** and type **eu[1]** and **ENTER**.(**eu[1]** refers to the first column of the data called **eu.**)

            3.   Move to the second column and press **F4**. Type **eu[2]** and **ENTER**.

            4.   Move to the third column and press **F4**. Type **rk[2]** and **ENTER**.

            5.   Move to the fourth column and press **F4**. Type **y(c1)** and **ENTER**.

Here are the figures showing the results of the above steps. If your table is different from ours, change the value of *tstep* in the **WINDOWS** to *0.1*.
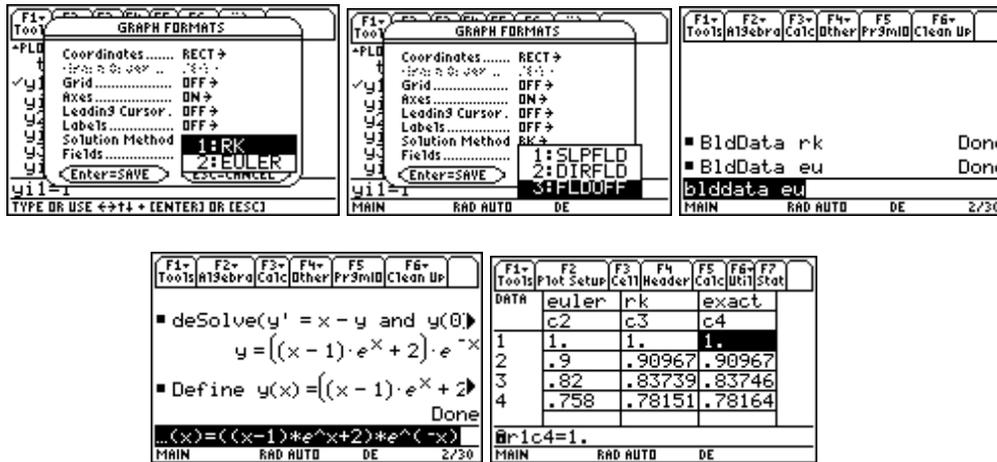
D.   Solving Third and Higher Order Differential Equations

Remark: TI 89 does not solve 3rd and higher order differential equations. To obtain the graph of a solution of third and higher order equation, we convert the equation into systems of first order equations and draw the graphs.(See Example 4 above.) However, we can utilize the TI 89 capability to solve polynomial equations with complex roots to solve linear differential equations of higher order with constant coefficients. Here are some examples.

Example 9:   Solve $y''' + 3y'' - y' - 3y = 0$

Solution: The auxiliary equation is $t^3 + 3t^2 - t - 3 = 0$ and using the **cSolve(** (which can be accessed by **F2 A 1**) command for solving equations with complex roots, we obtain $t = 1$ or $t = -1$ or $t = -3$. Thus the general solution is given by $y = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}$.

Example 10:   Solve $y''' + 3y'' + 28y' + 26y = 0$.

Solution: The auxiliary equation is $t^3 + 3t^2 + 28t + 26 = 0$ and using **cSolve(** $t^3 + 3t^2 + 28t + 26 = 0$, *t*) we get $t = -1 + 5i$ or $t = -1 - 5i$ or $t = -1$. Thus the general solution is $y = c_1 e^{-x} + c_2 e^{-x} \cos(5x) + c_3 e^{-x} \sin(5x)$

Example 11:   Solve the IVP $y''' - y'' - 4y' + 4y = 0$, $y(0) = -4$, $y'(0) = -1$, and $y''(0) = -19$

Solution: The auxiliary equation here is $t^3 - t^2 - 4t + 4 = 0$ and **cSolve(** $t^3 - t^2 - 4t + 4 = 0$, *t*) yields the solutions $t = 2$ or $t = 1$ or $t = -2$. Now use **F4 1** to define the general solution as $y = a \cdot e^{2x} + b \cdot e^x + c \cdot e^{-2x}$. To solve for *a, b, c* using the initial condition, we could **ENTER**

             Solve($(y | x = 0) = -4$ *and* $(d(y, x) | x = 0) = -1$ *and* $(d(y, x, 2) | x = 0) = -19$, *a* **)**

For the derivative, use **F3 1** or **2nd 8**. Here are the steps.
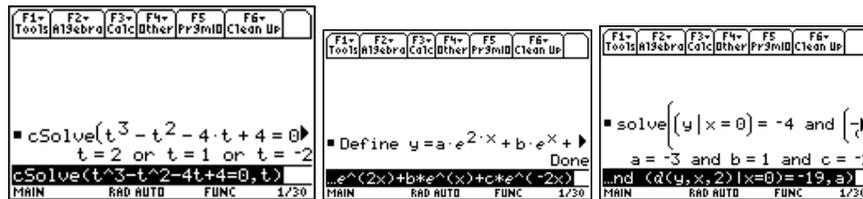
Thus the solution of the IVP is $y = -3e^{2x} + e^x - 2e^{-2x}$.

E.   Solving Systems of Differential Equations

In Section **A** we have discussed how to obtain the graph of a solution of a system of differential equations. Here we will solve systems with constant coefficients using the theory of eigenvalues and eigenvectors.

Example 12:   Solve the system of equations given by $X' = AX$ where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Solution: It is now recommended that you clear all the single variables you might have used earlier. **F6 1 ENTER** will accomplish this. Here are the relevant steps.

     1.   The first task will be to enter the matrix **A**. Use **APPS 6 3** and **Type** choose **2:matrix**. Use the down cursor key to go to the variable box and type **a** for the name of the matrix. For both row and col dimensions type 2.(Again use the down cursor key after you typed the inputs.)Now **ENTER** and type the entries of the matrix.(The first row must be filled in first.) Press **HOME** and **CLEAR**

     2.   Find the eigenvalues of the matrix by using **Math 4 9 a) ENTER** or by typing **eigvl(a)**

     3.   Find the eigenvectors of the matrix by using **Math 4 A a) ENTER** or by typing **eigvc(a)**

The figures below are the result of the above steps. The last figure shows the eigenvalues and vectors.
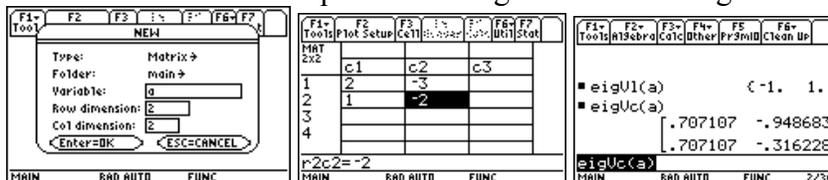
Thus the general solution of the equation is given by

$$X = c_1 \begin{bmatrix} .707107 \\ .707107 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -.968683 \\ -.316228 \end{bmatrix} e^t$$

Remark:

1.   The first number given by **eigVl(a)** is the first eigenvalue which in this case is -1 and second eigenvalue is 1. The first column of the **eigVc(a)** is an eigenvector corresponding to the first eigenvalue of **a**. Note that TI 89 is normalizing the vectors, that is the eigenvectors are unit vectors.

2.   For our purposes and easier notations, it is convenient to rewrite the eigenvectors with integer entries. This is usually possible. One possible method is to replace the smallest number in the columns by 1 and divide the other entries in that column by the smallest value you just replaced. Use the command **eigVc(a)[j,k]** to refer to the j-k entry of the matrix **eigVc(a)**.It is clear that the first columns are equal thus for the fist eigenvector we may take $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The second one may not be clear so we replace -.316228 by 1. Note then that -.96683/-.316228 is 3.062. Thus it is highly recommended that you compute **eigVc(a)[2,1]/ eigVc(a)[2,2]**. We find that this is 3. Thus we may take $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as the second eigenvector.

Thus the general solution could also be given by $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t$.

3. Note that we can express the above solution as $X = \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. The matrix $b = \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$ is

sometimes referred to as the *fundamental matrix* of the equation $X' = AX$.

Example 13: Solve $X' = AX$, $X(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Solution: As in Example 12 we solve $X' = AX$ and express the answer in the form given in Remark 3
above. All we have to do now is solve the system of equations $\begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}_{|t=0}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

To this end we enter the matrix $\begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$ into the calculator as **b**. and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as **d**. The we compute the
command **rref(augment((b|t=0),d))**. The last column of this row reduced echelon form matrix gives the
solution for c_1 and c_2. **rref** and **augment** can be accessed from **Math(2ⁿᵈ 5) 4 4** and **Math 4 7**, respectively.
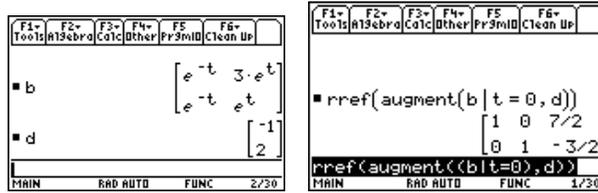
Example 14: Solve $X' = AX + F$, where $F = \begin{bmatrix} e^{t} \\ t \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Solution: Let **b** be as in Example and let $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then the general solution to the system is given by

$X = b \cdot c + b \cdot \int (b^{-1} \cdot f)dt$. Enter F as **f** and execute the command $b \cdot \int (b^{-1} \cdot f)dt$. For $\int$, use **F3 2** or **2ⁿᵈ 7**.
Here are the figures for these steps.
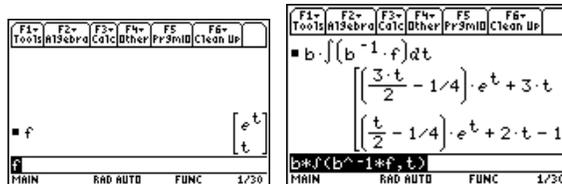
Therefore the general solution is given by

$$X = \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \left(\dfrac{3t}{2} - \dfrac{1}{4}\right)e^{t} + 3t \\ \left(\dfrac{t}{2} - \dfrac{1}{4}\right)e^{t} + 2t - 1 \end{bmatrix}$$

Example 15: Solve $X' = AX + F$, $X(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where $F = \begin{bmatrix} e^{t} \\ t \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Solution: We will use the notations of Examples 13 and 14. The solution is then given by the formula:

$$X = b \cdot (b^{-1}(0)) \cdot d + b \cdot \int_0^t (b^{-1}(s) \cdot f(s))ds$$

We now need to **Define b^(-1)** and **f** as functions of **s** rather than as functions of **t**. We will use **e** and **g**, for **b^(-1)** and **f**, respectively. Here is a partial picture.
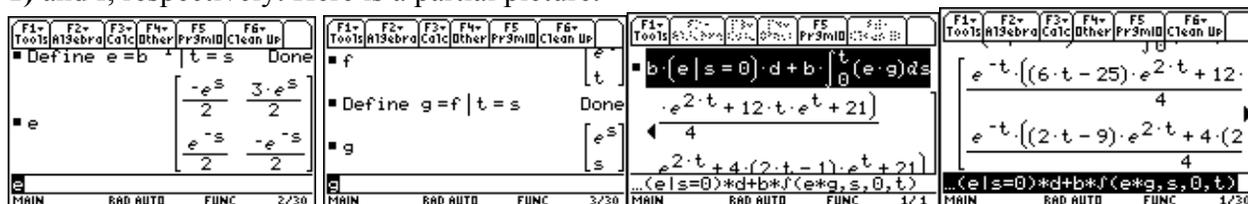
The solution is $X = \begin{bmatrix} \dfrac{3te^{t}}{2} - \dfrac{25e^{t}}{4} + \dfrac{21e^{-t}}{4} + 3t, & \dfrac{te^{t}}{2} - \dfrac{9e^{t}}{4} + \dfrac{21e^{-t}}{4} + 2t - 1 \end{bmatrix}$