Forward and inverse wave problems: Quantum billiards and brain imaging

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Courant Institute

The one-minute talk : two areas

Simple linear 2^{nd}-order PDEs

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I. Scaling method for Helmholtz eigenproblem

- *waves:* elliptic PDE, time-independent
- short-wavelength limit $\rightarrow$ numerically hard
- quantum chaos explains fast new method

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Simple linear 2^{nd}-order PDEs

I. Scaling method for Helmholtz eigenproblem

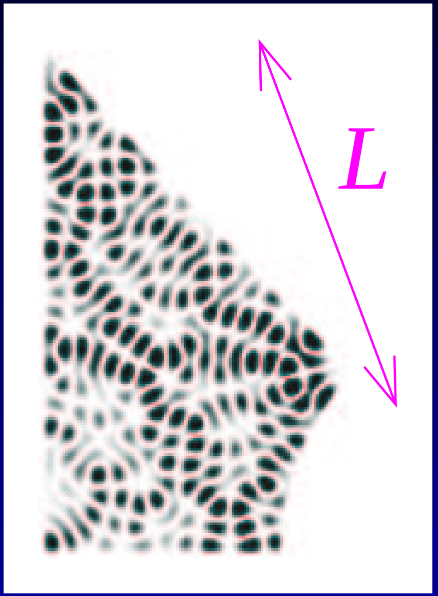
- *waves:* elliptic PDE, time-independent
- short-wavelength limit $\rightarrow$ numerically hard
- quantum chaos explains fast new method

II. Brain imaging with diffuse optical tomography

- *diffusion:* parabolic PDE, time-dependent
- ill-posed inverse problem, messy 3D geometry
- clinical and functional neuroimaging

I. Scaling method

with Cohen (Ben-Gurion), Heller (Harvard)



Domain in $d \geq 2$

$$\begin{aligned} (\nabla^2 + k^2)\psi(\mathbf{r}) &= 0 \quad \text{inside domain} \\ \psi(\mathbf{r} \in \text{boundary}) &= 0 \quad \text{Dirichlet} \end{aligned}$$

Want spectrum k_μ, eigenfunctions ψ_μ

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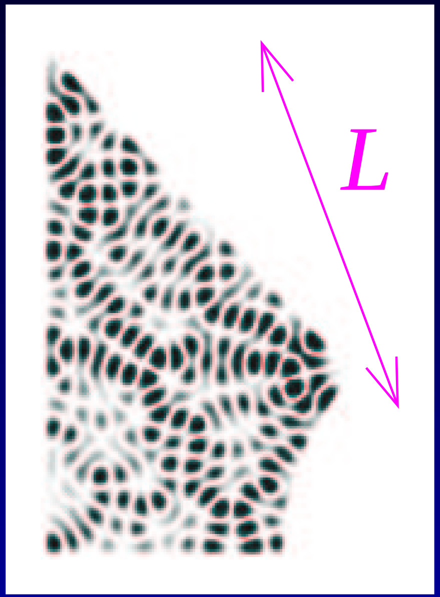

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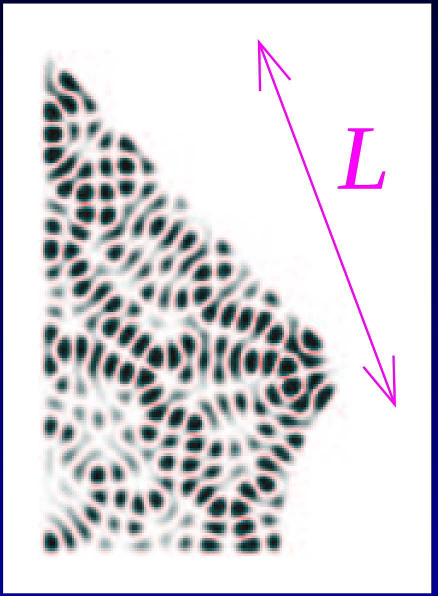

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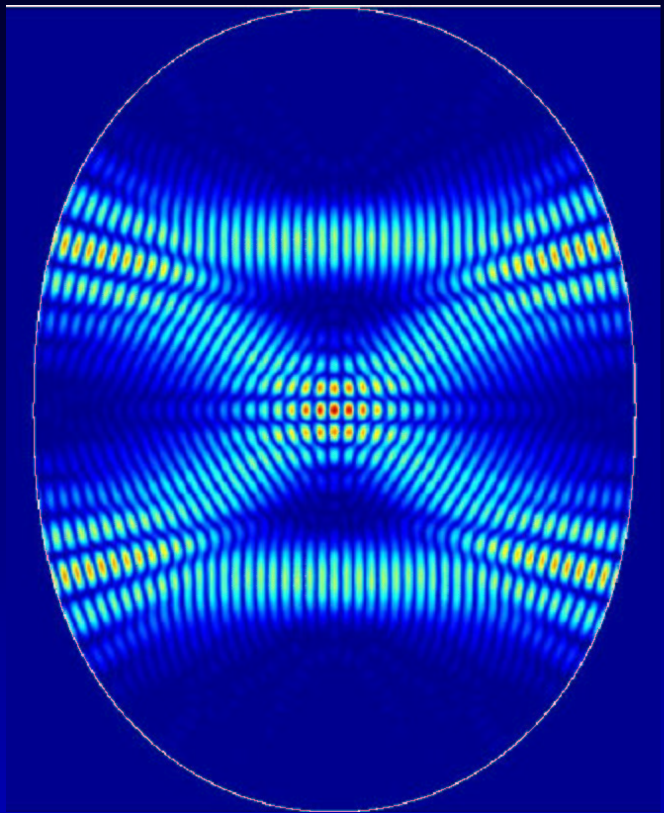
Often care about $kL \gg 1$

E.g. spectral statistics as $kL \to \infty$ (asympotics).

Physical examples



dielectric laser
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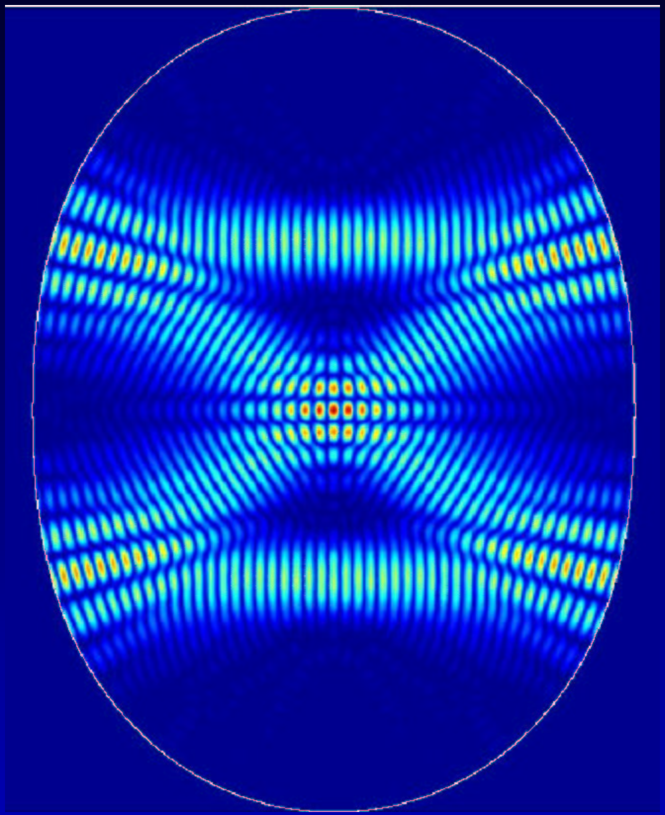

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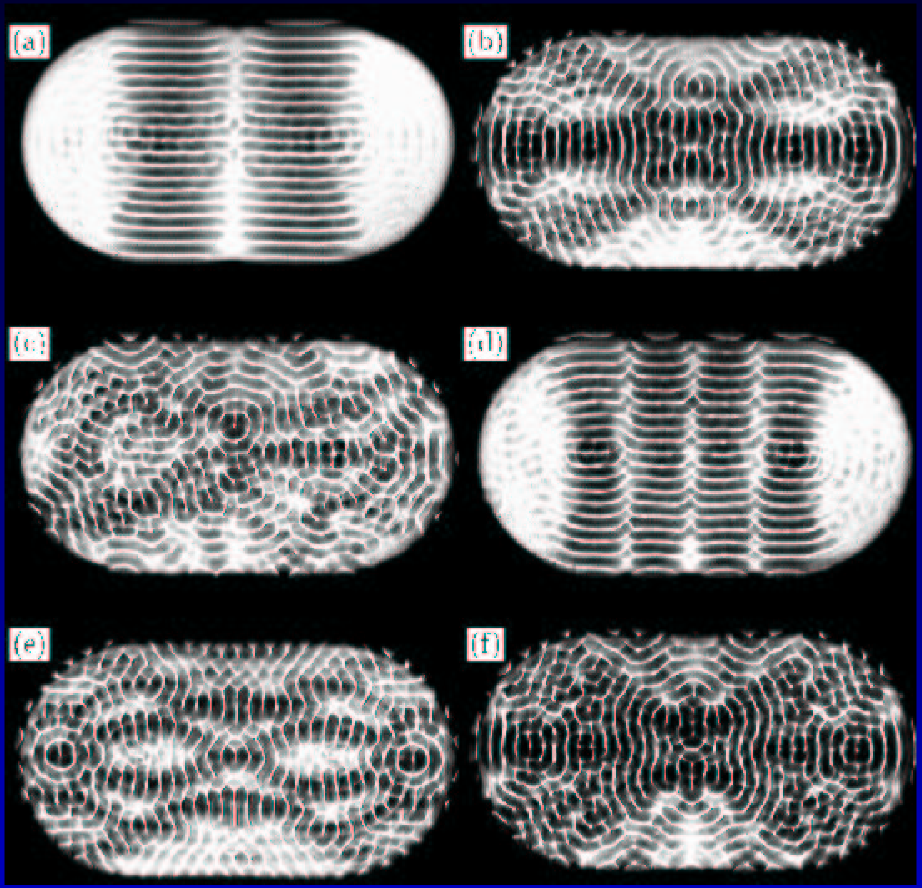


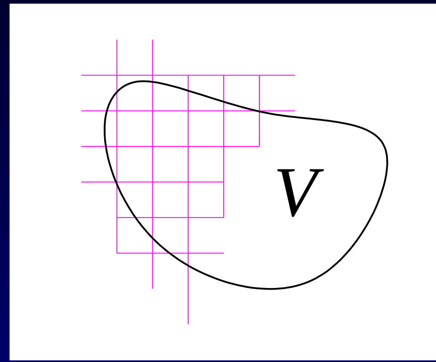
liquid surfaces Kudrolli

3 approaches

1. Finite Element (FEM) type

basis funcs obey BCs, not PDE

basis size $N \gg$ (#λ-sized patches in V).



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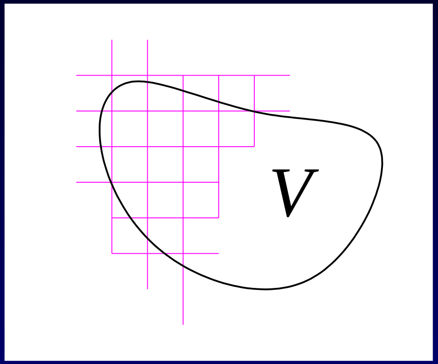

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Greens func known $\Rightarrow$ use as basis

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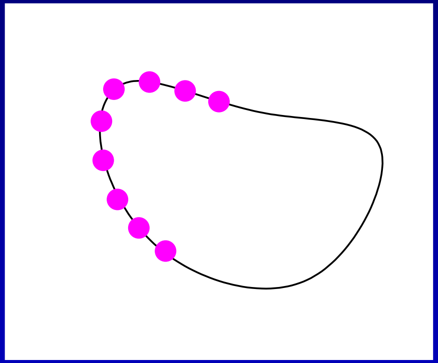

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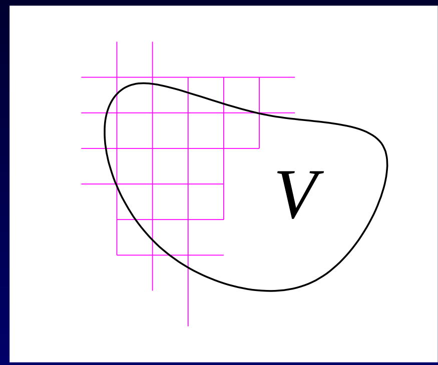

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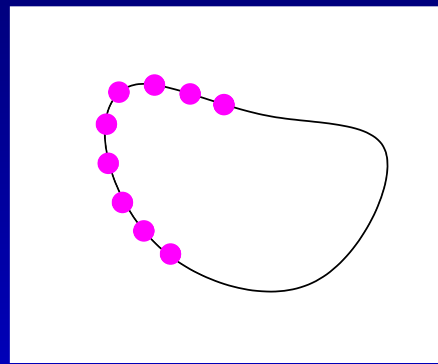

3. Measure resonances of *real system*

- microwaves cavities, etc. . . $kL \leq$ Q-factor, painful!

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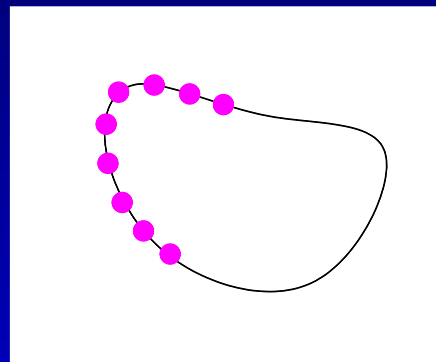

Scaling sketch

quadratic functional $F_k[\psi] \equiv \oint d\mathbf{s}\ (\mathbf{r}\cdot\mathbf{n})^{-1}\ \psi^2$

is nearly diagonal in the basis :

$\{\psi_\mu\}$ spatially rescaled to *same* wavenumber k.

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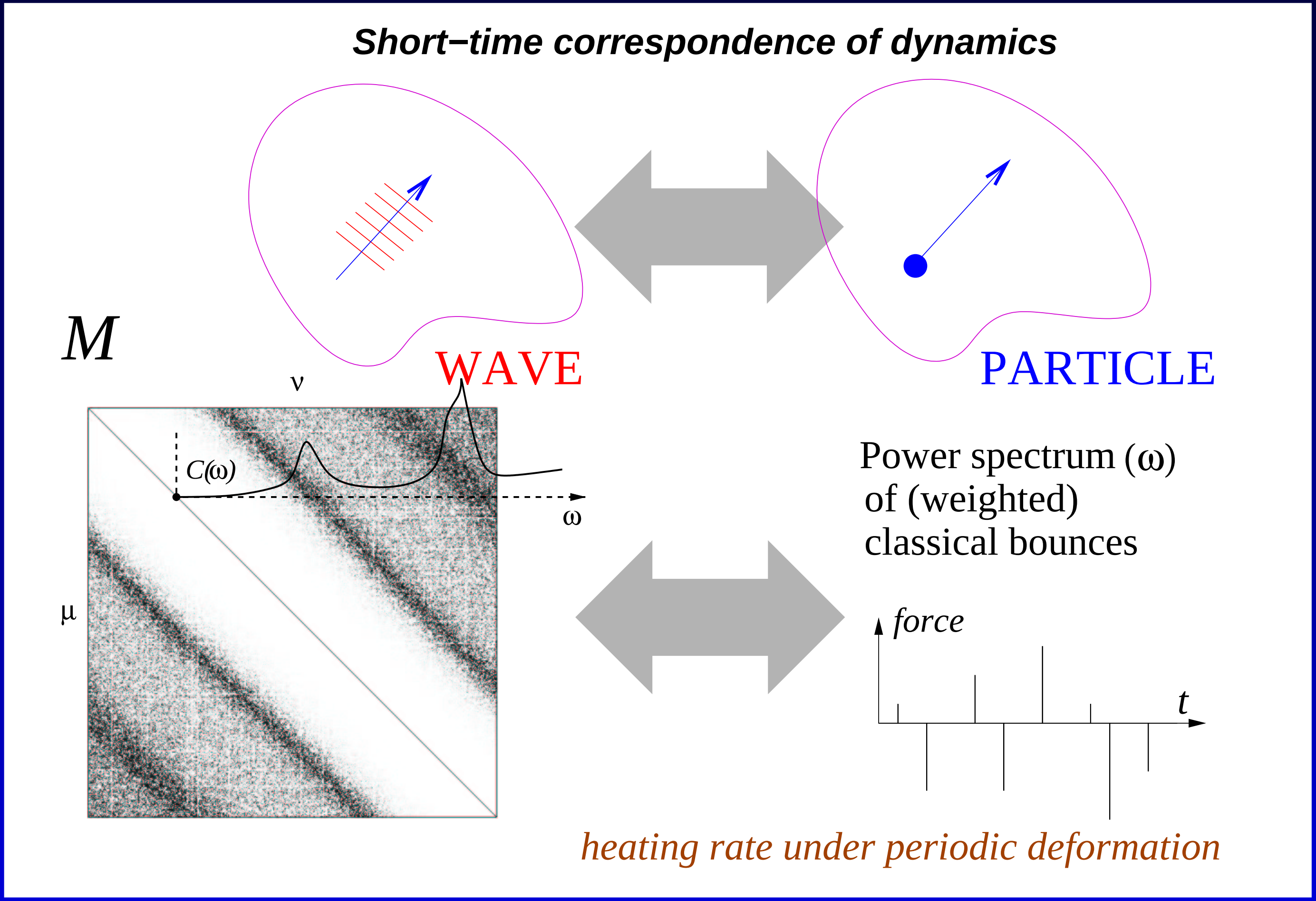
Special F relies on boundary overlap of ψ_μ's …

Quasi-orthogonality sketch

$$M_{\mu\nu} \equiv \oint ds\, (\mathbf{r}\cdot\mathbf{n})\, \partial_n\psi_\mu\, \partial_n\psi_\nu \;\approx\; \delta_{\mu\nu}.$$



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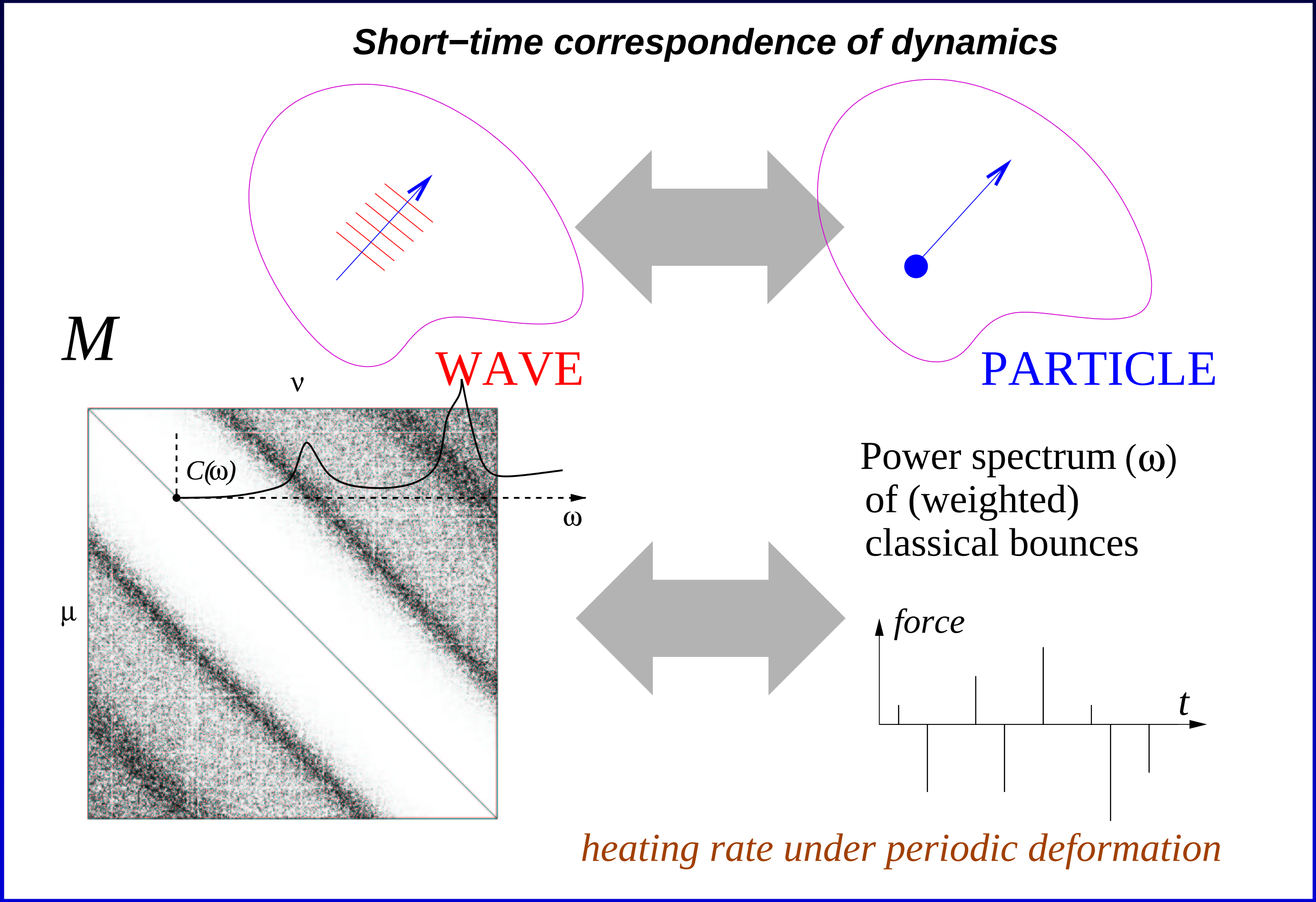
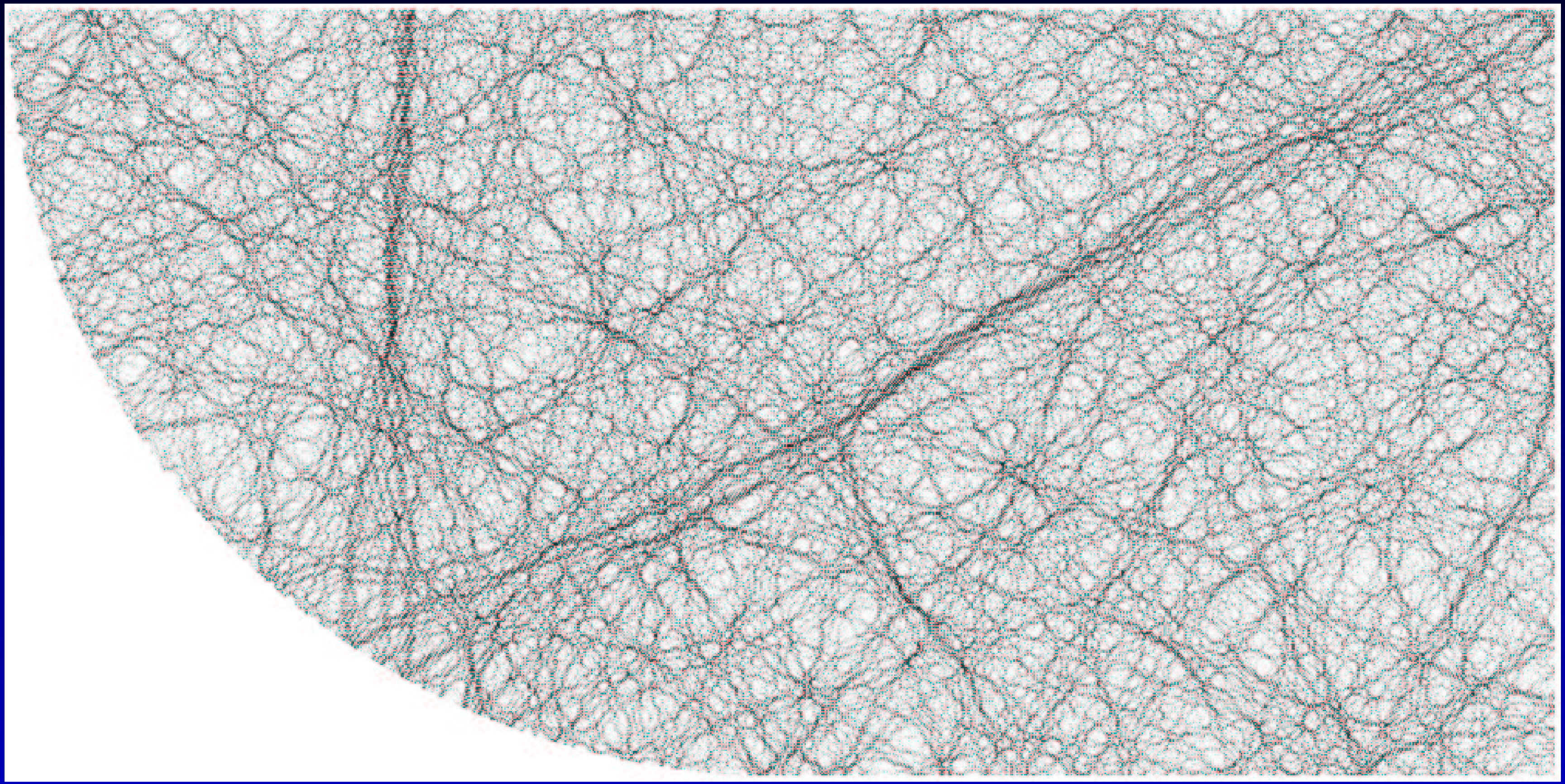


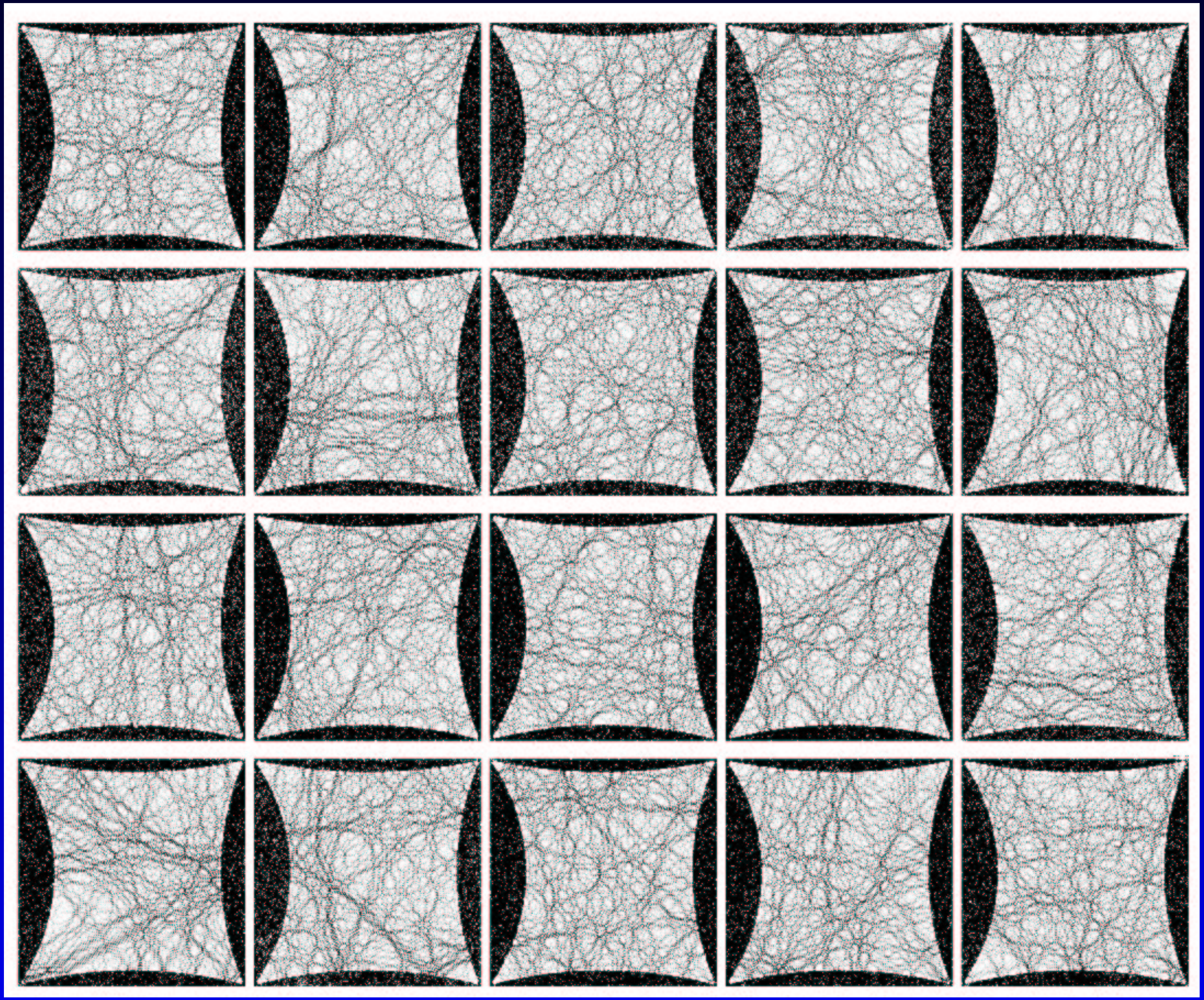
Special deformations : no heating as $\omega \to 0$.

Results ($d = 2$)



plane-wave basis, $kL \approx 2000$
speed: 100 such ψ_μ found per minute

New basis for nonconvex



new singular basis, $kL \approx 400$

Directions

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- Application to spectral statistics

II. Diffuse Optical Tomography

with Boas et al. (NMR Center, MGH / Harvard)

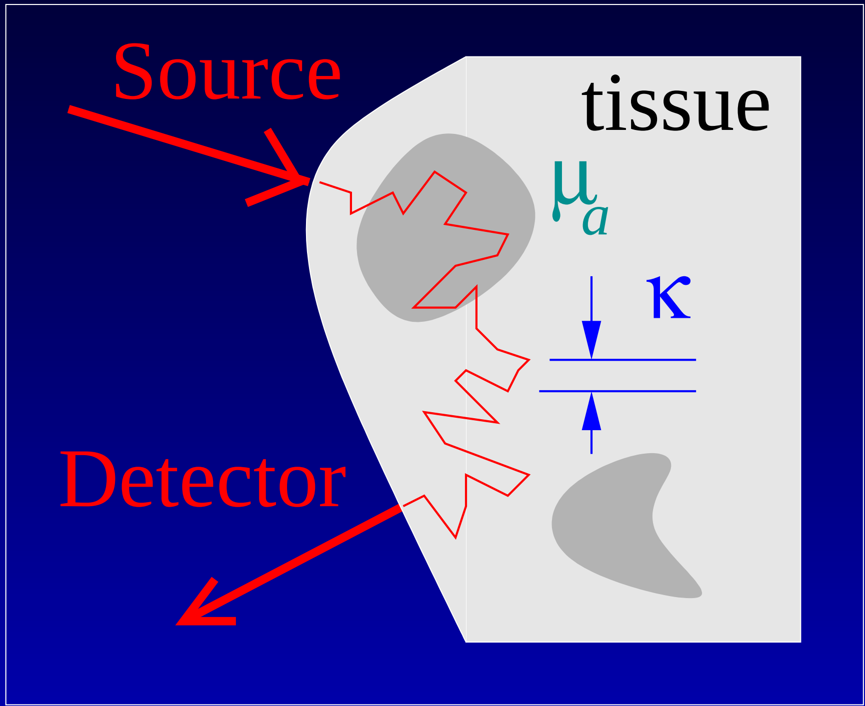


Image inside diffusive media?

scattering length κ
absorption μ_a

Learn about $\mu_a(\mathbf{r})$, $\kappa(\mathbf{r})$

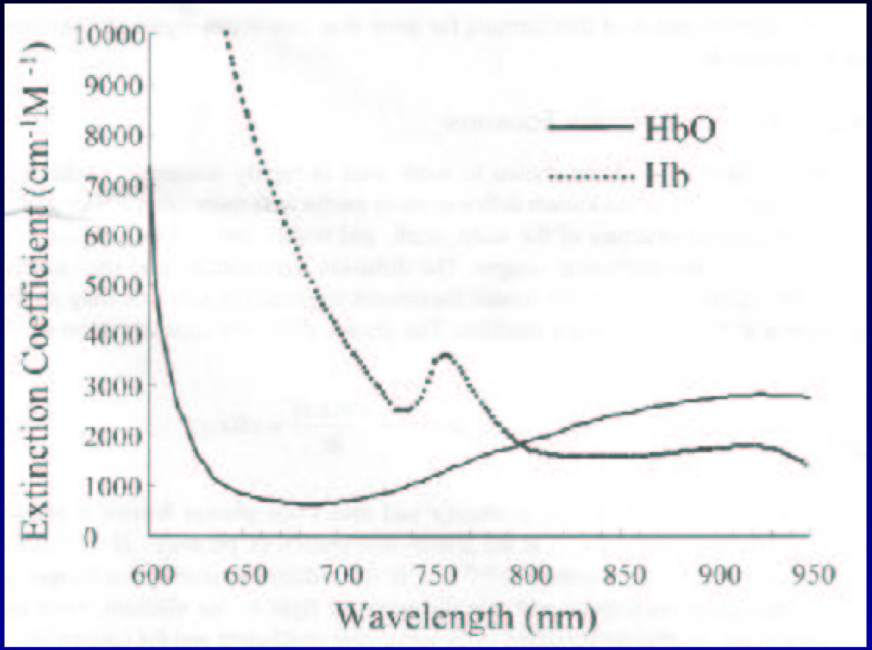
λ	$\ll$	κ	$\ll$	depth
1μm		1mm		few cm

It’s all about blood

Near infrared: μ_a small

Hemoglobin dominates

Hb - deoxy
HbO - oxy



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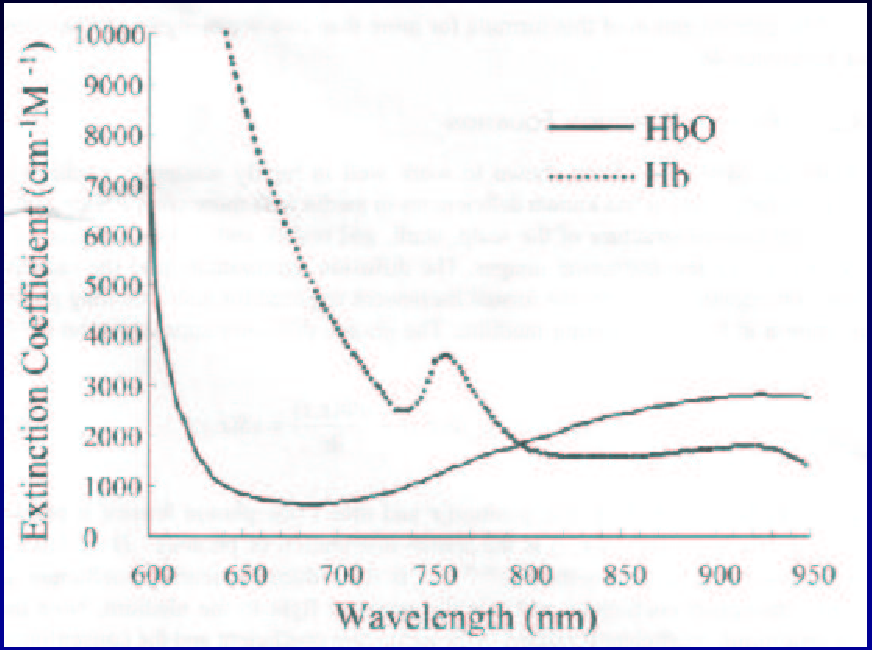

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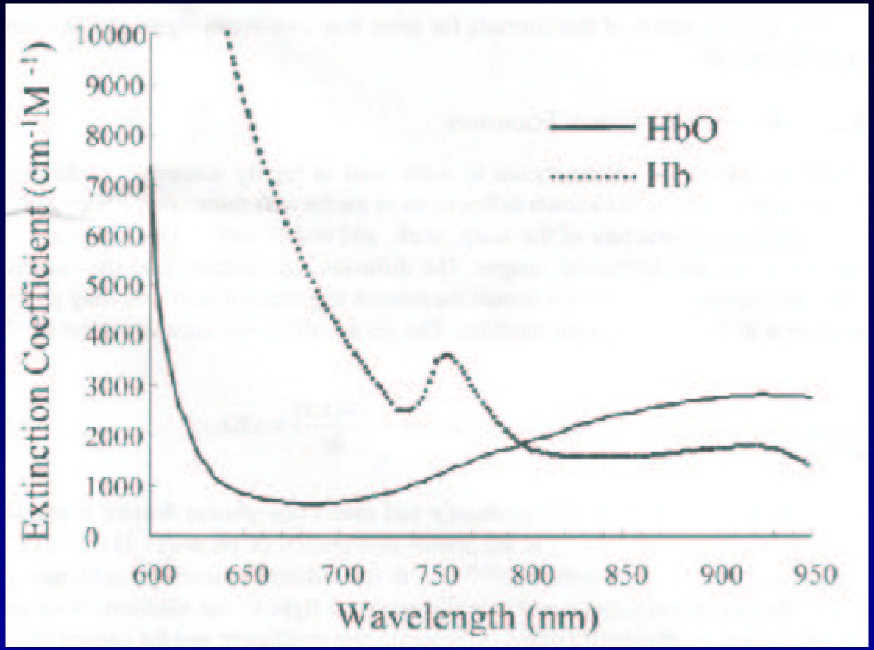

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Clinical: stroke, trauma, babies, breast tumors…

Neuronal activation $\rightarrow$ Hb, HbO changes

Last decade: imaging the brain in action!

DOT equipment

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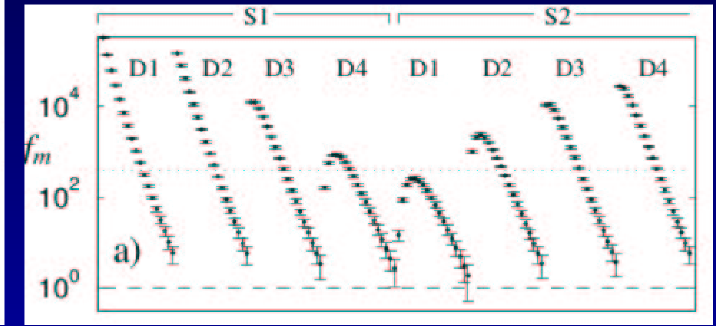
signals:



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- 10^{-12} s light pulse $\rightarrow$ photon count vs time

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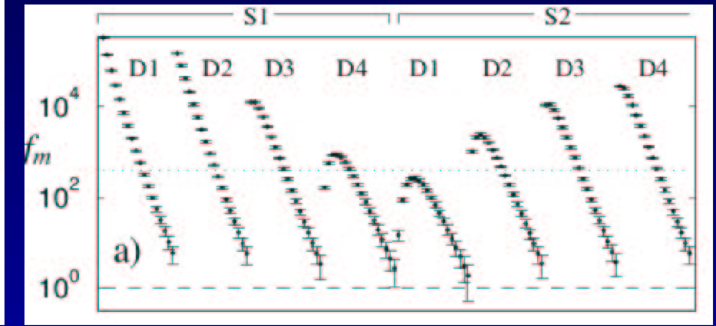

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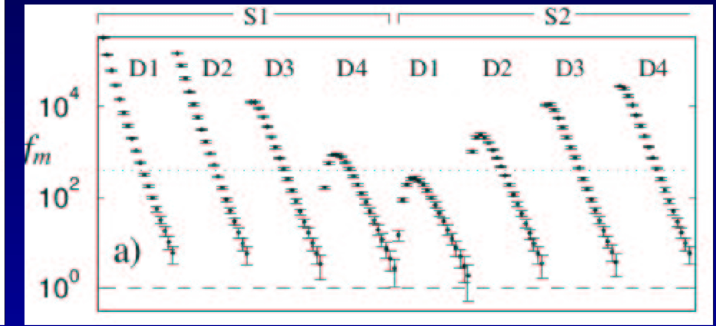

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DOT: 1-2 cm, 10-100 ms, $\$10^5$, portable, Hb & HbO.

Forward model

$$\mathbf{x} \equiv \{\mu_a(\mathbf{r}), \kappa(\mathbf{r})\} \quad \xrightarrow{\mathbf{f}} \quad \mathbf{y} = \mathbf{f}(\mathbf{x})$$

parameter vector — expected signal vector

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Incoherent waves $\to$ transport equation $\to$ diffusion:

$$\frac{1}{v}\frac{\partial}{\partial t}\phi \;=\; \nabla(\kappa(\mathbf{r}) \cdot \nabla\phi) - \mu_a(\mathbf{r})\phi + q(\mathbf{r}, t)$$

$\phi =$ fluence, Robin BCs $\frac{\partial \phi}{\partial n} \propto \phi$.

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Finite-Difference Time-Domain in 3D ($\sim$ 2mm lattice)

- $O(\Delta t)$ accuracy, for now…

Inverse problem

$$\mathbf{x} \overset{?}{\longleftarrow} \mathbf{y}_{\text{measured}}$$

Ill-posed : *many* $\mathbf{x}$ have $\mathbf{f}(\mathbf{x}) \approx \mathbf{y}_{\text{measured}}$

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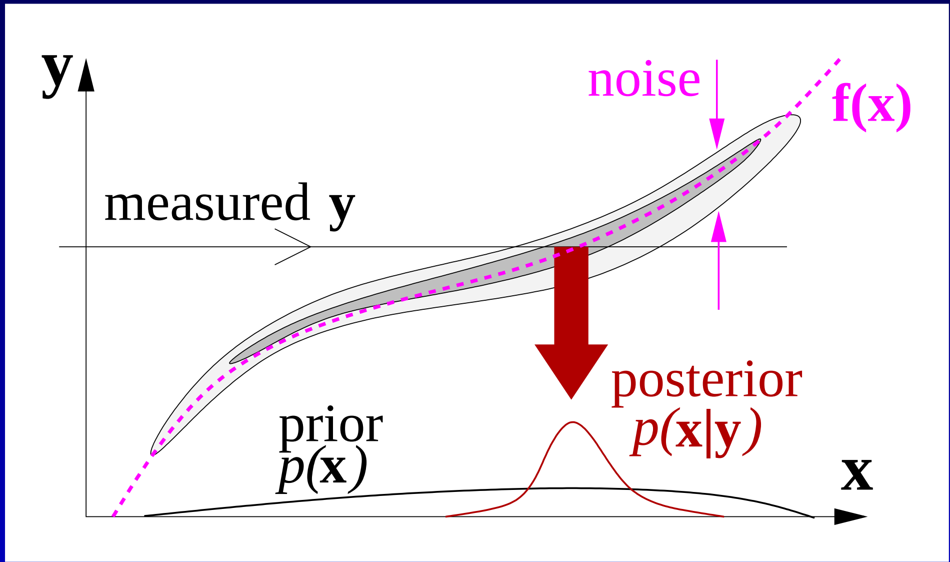

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posterior likelihood prior

Embraces Ill-posedness

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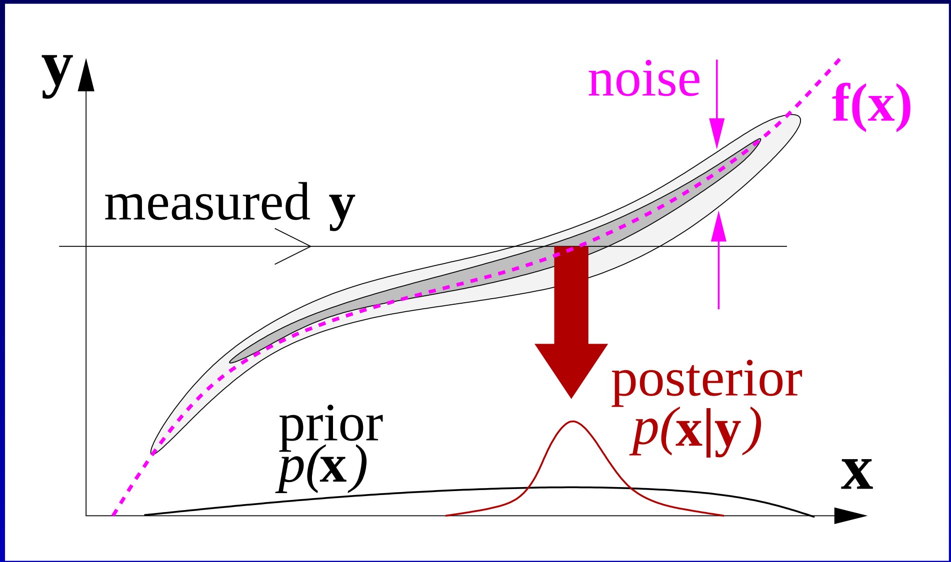

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Use realistic noise model: { Poisson photon stats; forward model error }

Baseline meas. with MRI help

Use geometry from MRI : $\dim(\mathbf{x}) = 10^5 \;\to\; 6$

$\mathbf{x} \equiv \{\mu_a, \kappa\}$ for skull, scalp, brain.

Q: How well can measure absolute brain μ_a, κ ?

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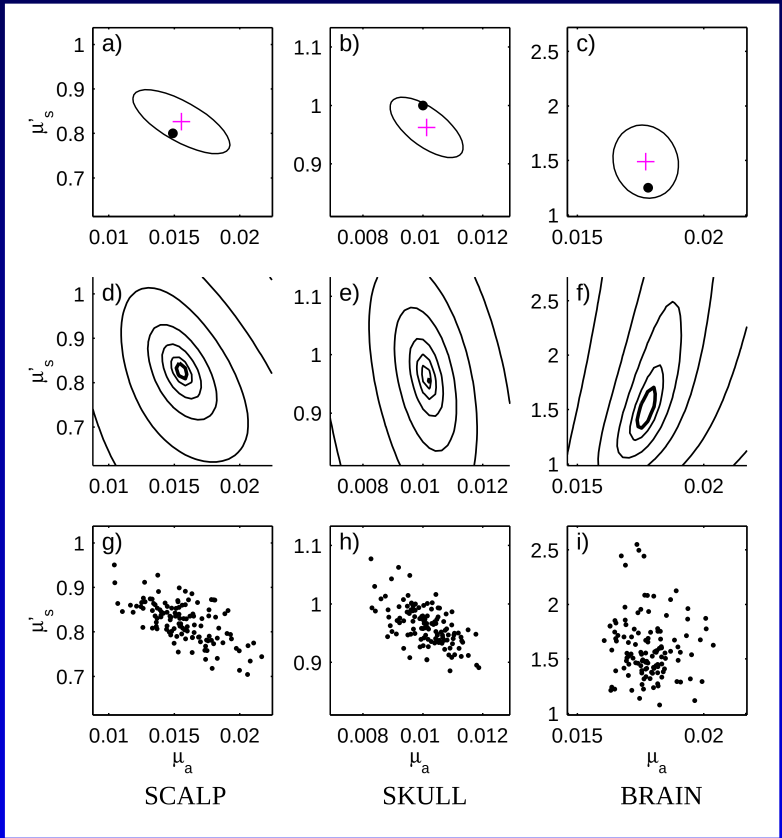

posterior PDF $\rightarrow$ errorbars

- Gaussian approx to PDF
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Show 10^6 detected photons gives 5% in μ_a, 20% in κ

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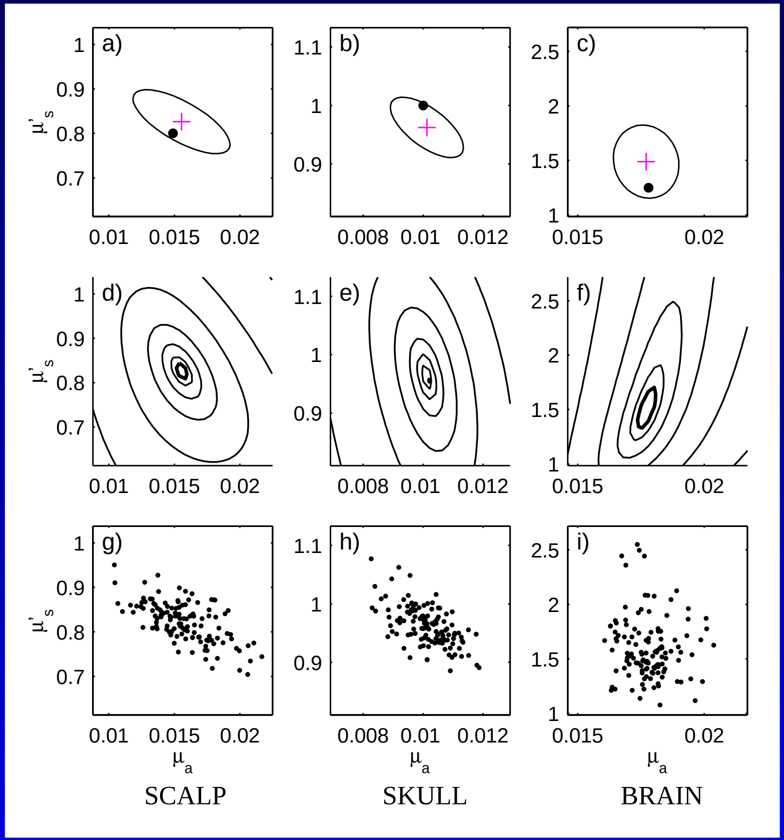

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f(**x**) is *expensive* $\Rightarrow$ want fewest evaluations

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- AI / Optimization: explore high-dim PDFs