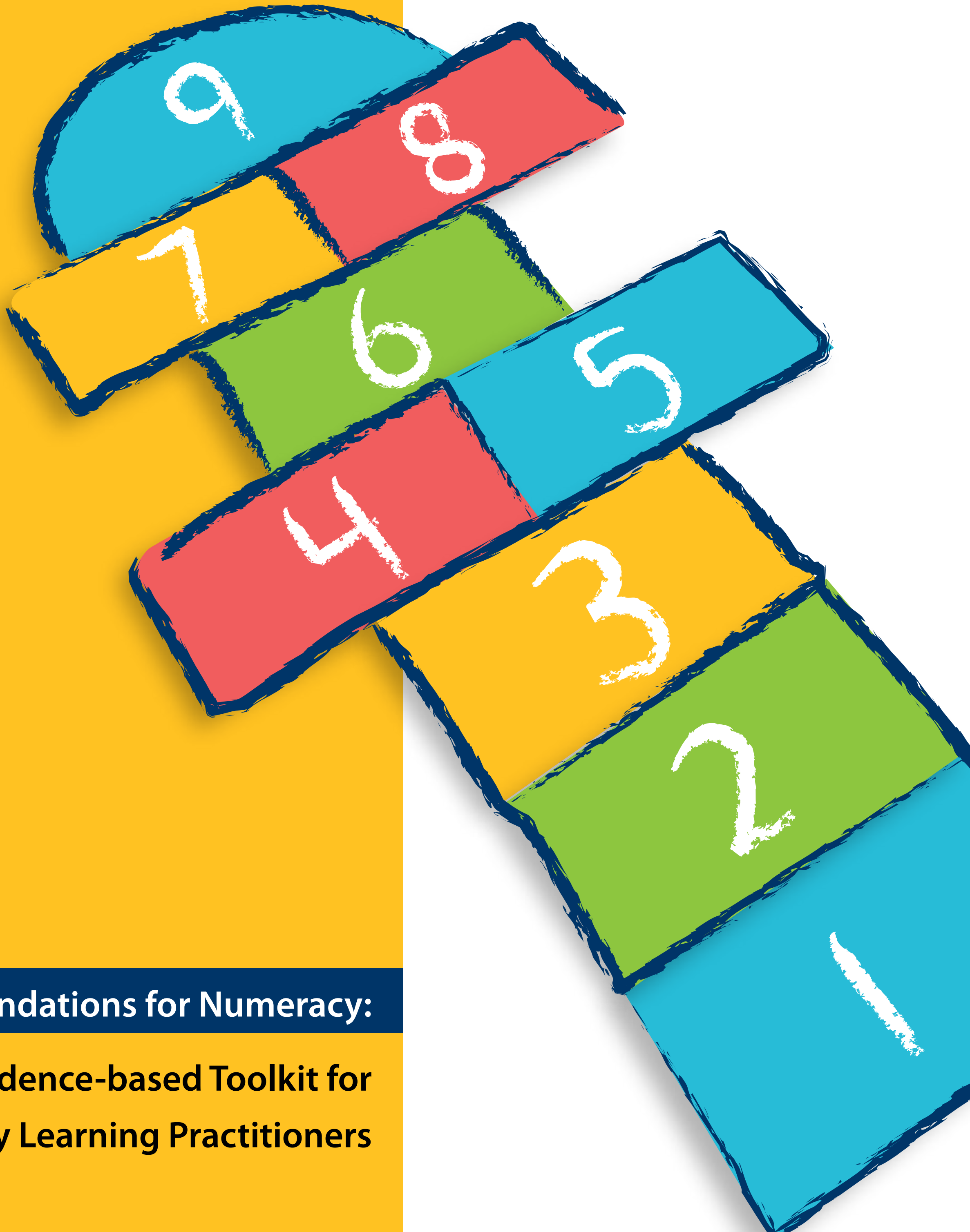

Foundations for Numeracy:

An Evidence-based Toolkit for Early Learning Practitioners

Réseau canadien de recherche sur le langage et l'alphabétisation
CANADIAN LANGUAGE & LITERACY RESEARCH NETWORK
Partager la science. Éveiller les esprits. *sharing the science. opening minds.*

Canadian Child Care Federation
Fédération canadienne des services de garde à l'enfance

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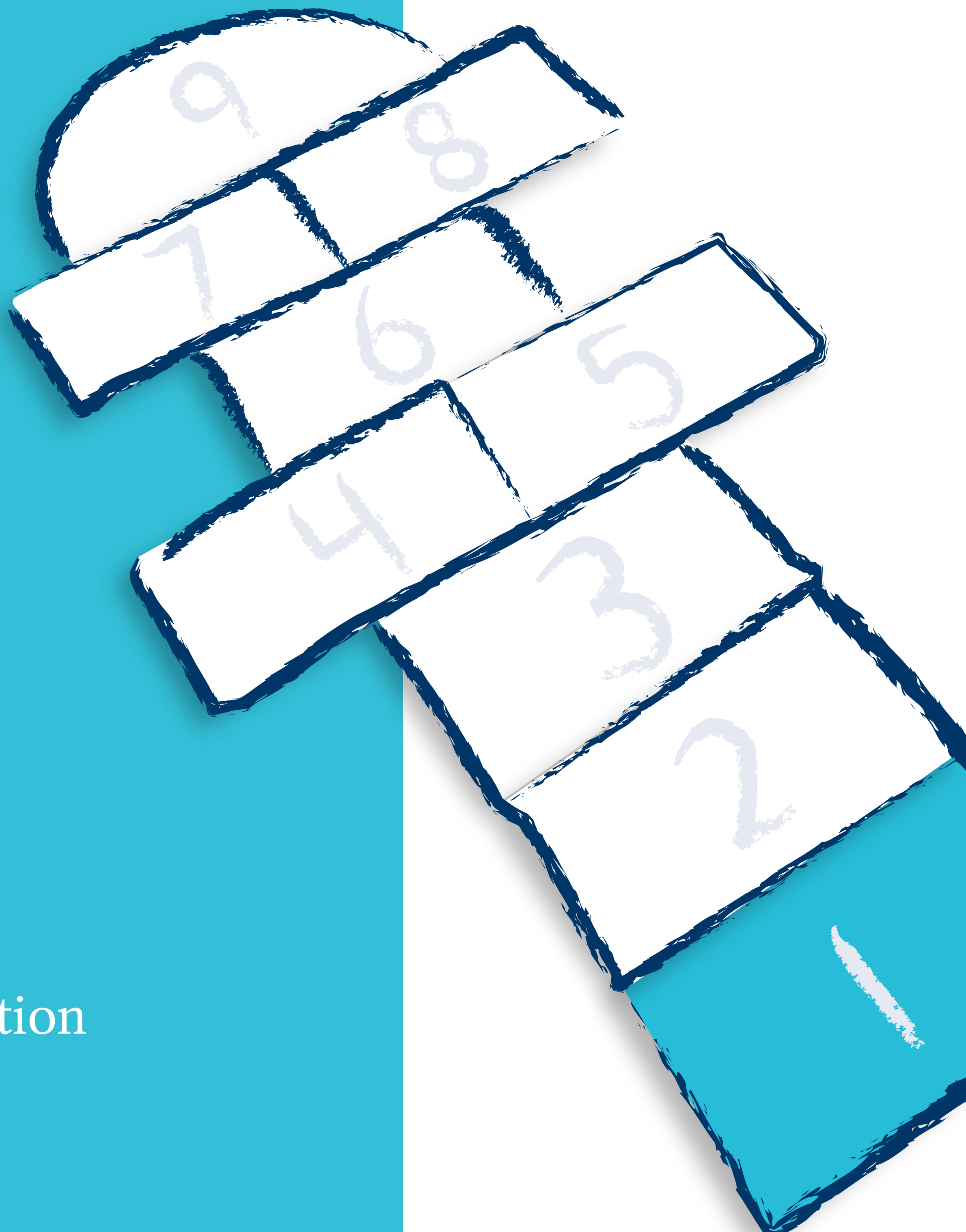
Introduction



INTRODUCTION

This resource kit was created for early learning practitioners and teachers to help them support the development of numeracy skills of children in their care. The information presented in the kit is based on a comprehensive review of recent well-designed research studies on the learning and teaching of mathematics. The findings of these studies are communicated in an accessible format, making this resource an effective reference tool that can be used in daily practice.

The kit is divided into two volumes: one for early learning and child care practitioners and the other for elementary school teachers. Each volume includes a research summary and several additional components. The current volume, which is intended for practitioners that work with children 0 to 5 years of age, includes the following components:

- Ages & Stages of Numeracy Development (a resource for early childhood educators on developmental milestones)
- Creating a Math-Rich Environment (tips for early childhood educators on how to make their classrooms more inviting and conducive to mathematics learning)
- Activity Cards (learning activities for children 3-5 years of age)
- Resources for Child Care Practitioners (a list of both print and online resources on supporting early numeracy)
- *Math with Kids is Fun!* and *Ages and Stages of Numeracy Development* (two resource sheets from the Canadian Child Care Federation)
- Glossary of Terms (definitions of technical terms related to early numeracy)

The kit is intended to supplement and enhance early childhood educators' previous knowledge, as well as to introduce new information on mathematical concepts. It allows educators to stay up-to-date on the latest advances in mathematics teaching and learning, and helps educators to identify the most effective approaches that can be used in early learning environments. It is a useful professional development resource for those working with young children and a learning resource for practitioners in training. The terms early learning practitioner, early learning and child care practitioner, child care practitioner, early childhood educator and educator are used interchangeably throughout this kit and refer to those individuals who are working with children aged 0 to 5 years and their families in early learning and child care environments.

Numeracy during the Early Years

The everyday world for a young child is full of opportunities to engage with number and quantity. From the first few days of life, infants pay special attention to expressions of quantity in their environments. Babies' everyday experiences provide the foundation for more advanced math concepts that develop throughout early childhood and beyond. As babies grow into toddlers, their knowledge of counting and quantity has the potential to improve very quickly. Preschoolers are capable of thinking about arithmetic and can solve math problems in meaningful ways.

Clearly, however, not all children are the same, and they develop differently. Most differences in children's numeracy skills are a result of the opportunities they are given to think and talk about number in their home and early learning environments. Children who are provided with opportunities to engage in numeracy activities when they are young are more prepared to face the types of numeracy activities they will encounter in school. This, in turn, means that they will be more likely to succeed not only in math, but academically.

Children enjoy numeracy activities and are highly motivated to work with numbers. They are eager to imitate the counting words used by adults. For instance, children often label their toys with number words before they even know what these words mean. From observing children's play, it is clear that they are naturally attracted to mathematical features in their environments. For example, they spontaneously compare the size of objects, they use number words often, they make attempts at counting, and they pay attention to characteristics of patterns and shape, including symmetry, when they build towers with blocks.

Toddlers and preschoolers have enormous mathematical potential. Realizing this potential is an important element of school readiness. Quality early learning environments must therefore provide encouragement and opportunities for children to think and talk about numbers and math in ways that connect to the real world that surrounds them.

2

The Research

RESEARCH SUMMARY

To effectively support children's learning, educators need information based on existing research evidence. They can then integrate this knowledge with their professional experience and their understanding of children's needs. This research summary draws on a variety of sources related to the learning and teaching of mathematics and summarizes their findings. Educators will find information here about current thinking on the principles that underlie learning and development, particularly as they relate to mathematics.

In Part 1, we will focus on the cognitive processes that influence mathematics learning and achievement. We begin with a discussion of what the research tells us about three levels of cognition: information processing, mental representations, and metacognitive processes (thinking about thinking). Following this, we discuss research findings on the social and emotional factors that influence learning, in particular children's learning goals, motivation, beliefs about learning, and the influence of math anxiety on achievement.

In Part 2, the focus shifts to the development of mathematical concepts from the early years (preschool) through the transition to school (Kindergarten) and into the elementary years (Grades 1 to 6). From a child's early mathematical abilities and skills to the more formalized sets of rules and strategies learned in school, this section will focus on some of the key underlying concepts and widely applicable skills. These include numerosity, cardinality, ordinality, problem solving, the mental number line, fractions, estimation, arithmetic, and proportional reasoning.

PART 1: THE PROCESSES UNDERLYING CHILDREN'S LEARNING

What can educators learn from cognitive science that they can apply to their learning environments and classrooms? Cognitive scientists study every type of human learning and can provide insight into many of the underlying processes that guide children's learning:

- **information processing** (e.g., attention, working memory, and the retrieval, transfer, and retention of information);
- **mental representations** (e.g., conceptual and procedural knowledge); and
- **metacognitive processes** (e.g., processes that control mental operations, such as strategy selection and self-monitoring behaviours).

These processes can be considered as the "cognitive building blocks" of children's achievement. We will discuss each in turn in order to better understand how children learn and how educators can best support their learning (National Mathematics Advisory Panel [NMAP], 2008).

Cognitive factors are not the only ones that contribute to children's achievement in mathematics. A child's motivation, capacity for self-regulation, and anxiety about math can all have a strong effect on cognitive processing, and thus on achievement. Good teaching takes all of these factors into account, recognizing that social and emotional factors, as well as children's goals and beliefs about learning, are critical components of the learning process.

COGNITIVE PROCESSES

Information Processing

The first step in many types of learning or processing of information is to **focus our attention**. However, as we are well aware, there is a limit to how many things we can pay attention to at once. Attention changes with age: under many conditions younger children are less attentive than adults, and thus more prone to distractions (Cowan, Elliott, & Saults, 2002). However, our ability to attend to information is not entirely out of our control; it can also be improved with practice (Baumeister, 2005; Gailliot, Plant, Butz, & Baumeister, 2007; Muraven, Baumeister, & Tice, 1999).

Once we focus our attention on information, it is encoded in our **working memory**.

Working memory refers to the ability to keep information active in your mind while you use that information to perform an operation. For instance, if we ask a child to solve the problem "3 plus 5" without writing anything down, she must keep the information "3 plus 5" active, decode the meaning of the individual symbols, and then carry out a number of operations to reach the answer. In a way, the function of working memory in problem solving is like learning how to drive a standard transmission car. For a new driver, focussing on traffic and shifting is very demanding, and – just as with math processing – sometimes there are accidents, or errors.

In this driving analogy, working memory can be described as the "attention-driven control of information" (Baddeley, 1986, 2000; Engle, Conway, Tuholski, & Shisler, 1995). Depending on the type of incoming information, working memory stores information in one of three systems: the language-based phonetic buffer (e.g., remembering a phone number), the visuospatial sketch pad (e.g., remembering a visual pattern), or the episodic buffer (where information from long-term memory and the world is combined). Working memory has been strongly associated with academic learning, including in mathematics. A deficient working memory is one source of learning problems encountered by children with learning disabilities in mathematics. Conversely, a strong working memory is a major factor behind the accelerated learning shown by gifted children (NMAP, 2008, p. 4-5).

Attention and working memory ability increase with age and there are ways to improve children's working memory at any age. The most effective way to improve working memory is to help children achieve the quick, easy, and effortless retrieval of information from long-term memory, particularly of basic skills, facts, and procedures (Schneider & Shiffrin, 1977; Shiffrin & Schneider, 1977).

This quick retrieval, called "automaticity," is only achieved through practice (e.g., Cooper & Sweller, 1987). For most types of learning, automaticity of basic skills frees up working memory for more complex aspects of problem solving, such as creating mental pictures of the information, analyzing the problem, choosing and employing a strategy, and checking the answer obtained (NMAP, 2008). Again, we can make a comparison to learning to drive a standard transmission car. With practice, skills like changing gears and checking the blind spot are mastered. When these once demanding tasks become automatic, the driver's attention can be directed to the road ahead.

MENTAL REPRESENTATIONS

Types of Knowledge

There are three types of knowledge relevant to math:

- **factual knowledge** is information that can be learned by memorization and repetition (i.e., rote learning), such as knowing that 2 + 2 = 4. It also refers to memory of specific events and information.
- **conceptual knowledge** is the knowledge of why and how a procedure works, and includes general knowledge and understanding of a subject (Hiebert & Lefevre, 1986). It is information stored in long-term memory, acquired through thoughtful reflection over a long period. For example, knowing that when we count, the last number we say represents how many items are in the set.
- **procedural knowledge** describes the implicit memory for cognitive and motor sequences and skills, in short, knowing how to complete an activity or a task. For example, knowing how to solve the problem 2 + 3 by continuing to count "3, 4, 5…" (Hunt & Ellis, 1994; NMAP, 2008).

These three types of knowledge mutually support each other to facilitate learning and understanding (NMAP, 2008). Conceptual and procedural knowledge in particular have been shown to be positively correlated: when one increases, so does the other (Rittle-Johnson & Siegler, 1998). For instance, research has shown that an early measure of the degree to which elementary students have conceptual understanding predicts not only their procedural ability in the same unit, but also procedural skill in the future (Hiebert & Wearne, 1996). Conceptual, procedural, and factual knowledge are all important for success in mathematics: "conceptual understanding of mathematical operations, fluent execution of procedures, and fast access to number combinations together support effective and efficient problem solving" (NMAP, 2008, p. 26).

Although researchers have previously debated as to which of these skills was the most important, today most take a more nuanced view, assuming that conceptual and procedural knowledge enhance each other. That is, as a child's conceptual knowledge grows, his procedural skill improves, and vice versa. The exact relationship between the two may vary across mathematical topics, but conceptual and procedural knowledge are both important and function together to contribute to a child's mathematical knowledge (Baroody, 2003; Rittle-Johnson & Siegler, 1998).

METACOGNITIVE PROCESSES

Metacognition can be defined loosely as "thinking about one's own thinking." Most theories distinguish between two types:

- **metacognitive knowledge** – what we know about our own thinking; also, how, when, and why to use particular strategies and resources; and
- **metacognitive regulation** – how we use what we know to regulate and control our thinking (Schraw & Moshman, 1995).

Thus, children engage in many metacognitive processes when they analyze problems, select appropriate strategies to solve them, regulate their problem-solving process, and check the validity of their answers.

Metacognitive processes are touched on throughout this research summary. We will focus particularly on self-regulation: the ability to set goals, plan, self-monitor, evaluate, learn adjustments, and choose a strategy (NMAP, 2008). Efforts to improve a child's self-regulation skills include prompting children to check their answers, set goals for improvement, and chart their daily progress. These efforts have been shown to also improve mathematics learning (e.g., Fuchs et al., 2003).

Self-Regulation

Self-regulation involves both motivation and cognitive processes, and is related to children's use of strategies for problem solving. As Siegler (1996) stated in the "overlapping waves" theory of development, children know and tend to use a variety of strategies for solving problems. Individual children choose different strategies for particular problems or in particular situations depending on differences in their knowledge of answers to problems and also their degree of perfectionism. Children who are not able to self-regulate effectively often have a poor knowledge of how to use strategies and may guess at the answer to a problem. As a result, they may be more likely to be labelled as "mathematics disabled" or to fail a grade (Siegler, 1988; Kerkman & Siegler, 1993). Expert problem solvers with good self-regulation skills, on the other hand, "spend more time analyzing problems before initiating solutions, reflect more frequently on their problem solving, and alter their approach more flexibly" (Fuchs et al., 2003, p. 307).

Consider the following example:
Marie-Eve and Emily are playing a game with connecting squares and a die. The aim of the game is to create the longest chain by taking turns rolling the die and adding the rolled number of squares to the chain. On her second turn, Marie-Eve has a chain

of five squares and rolls a four. She pauses, looks at her chain, then adds a chain of four. Realizing that she already has five links, she counts them as "5" and counts on "6, 7, 8, 9" to find out how many she has in total. Emily rolls a six and quickly adds six more squares to her chain of three. She then proceeds to count all the squares: "1, 2, 3, 4, 5, 6, 7, 8, 9."

In this simple illustration, Marie-Eve has a more efficient strategy: counting on from a number she already knows instead of counting all of her tokens over again. In contrast, Emily has used a less efficient strategy, even though she arrives at the correct solution. Siegler (2000) suggests that earlier strategies like Emily's persist in children's repertoire because children may not be able to carry out more efficient methods. In addition, children who do not take the time to reflect on a problem may fail to see the inefficiency of a particular solution.

SOCIAL AND MOTIVATIONAL INFLUENCES

Children's Learning Goals

One of the most widely adopted theories of motivation for learning describes people based on their reasons for pursuing challenges and facing obstacles. Their goals may be either mastery-oriented or performance-oriented (Ames, 1992). Children with mastery-oriented goals tend to choose more difficult materials in order to challenge themselves. Their focus is inward, on their own learning, rather than outward, on a comparison to other people's performance. They attribute any problems to their own lack of effort and try harder the next time they face a challenge. In contrast, children who have performance-oriented goals focus primarily on comparing their abilities to others' and tend not to seek out challenges for themselves. If these children experience difficulty with a problem, they are more likely to give up. They tend to blame failure on their own lack of ability and avoid difficult problems in the future (Ames, 1992; Ames & Archer, 1988). Children with mastery-oriented goals perform significantly better in math than do students with performance-oriented goals (e.g., Gutman, 2006; Linnenbrink, 2005; Wolters, 2004).

So how can we, as educators, support children's learning if their own motivations affect their learning so strongly? Fortunately, children are not born with an unchangeable orientation to either mastery or performance goals. Like working memory, a child's goals can be acquired and encouraged through certain kinds of learning situations. For example, the following conditions tend to foster mastery-oriented goals:

- providing meaningful reasons for engaging in a task and understanding it;
- promoting high interest and intermediate challenge;
- emphasizing gradual skill improvement; and
- arranging for novelty, variety, and diversity (Ames, 1992).

Motivation

Related to mastery- and performance-oriented goals is the concept of intrinsic and extrinsic motivation. A child with intrinsic motivation to learn has "the desire to learn for no reason other than the sheer enjoyment, challenge, pleasure, or interest of the activity," while children who have extrinsic motivation for learning make efforts in the hope of some external reward (NMAP, 2008, pp. 4-12; also Berlyne, 1960; Hunt, 1965; Lepper, Corpus, & Iyengar, 2005; Walker, 1980). Several studies have shown that intrinsic motivation is associated with academic achievement and learning (e.g., Lepper et al., 2005; Gottfried, Fleming, & Gottfried, 2001). Most children, in fact most people, have a mixture of intrinsic and extrinsic motivational factors that drive their learning goals. Awareness of the importance of both types of motivation can be helpful in guiding educators' choices of activities and rewards.

Children's Beliefs About Learning

A child's academic goals and motivation for learning both play an important role in their mathematics education, but there is also a great deal of research on how children's beliefs influence their academic success, in particular their beliefs about mathematics and the source of their own success in mathematics (e.g., Leder, Pehkonen, & Torner, 2002; Muis, 2004). If children develop positive beliefs about mathematics and math education, they will develop a productive "mathematical disposition," that is, they will see math as making sense, as useful and worthwhile. They will feel that putting effort into their studies of math will pay off (National Research Council [NRC], 2001).

Without a positive and productive mathematical disposition, children are likely to believe that they "can't do math," that they are not naturally mathematically minded, and thus that they will never succeed in math, regardless of the amount of effort they put in. Children who believe that effort is necessary to do well in math will persist longer on complex tasks than children who believe that success depends on having innate ability (NMAP, 2008). In general, children who believe that intelligence is malleable and who put in effort academically tend to do better in school than those children who believe intelligence cannot be changed (Dweck, 1999). It is important for educators to pay attention to children's beliefs about the nature of intelligence since, fortunately, these beliefs can be changed. Greater emphasis on the importance of effort leads to greater engagement in math and thus to improved achievement (Blackwell, Trzesniewski, & Dweck, 2007; NMAP, 2008).

Self-Efficacy

The term "self-efficacy" refers to the set of beliefs one has about one's own ability to succeed at difficult tasks (Bandura, 1997). Self-efficacy correlates significantly with performance in mathematics for students from elementary school to university (e.g., Pajares & Miller, 1994; Kloosterman & Cougan, 1994). In their early years and the primary grades, children's beliefs about mathematics are not related to their achievement, since most children at this age see themselves as able to do math. This confidence decreases over the years, however. By Grade 6, student beliefs are correlated with their achievement: children do as well or as poorly as they believe they are capable of doing, and low achievers start to dislike mathematics (Kloosterman & Cougan, 1994; Wigfield et al., 1997). Ability is important for success in mathematics, but feelings of self-efficacy influence how ability is expressed in actual performance.

MATH ANXIETY

Some people experience "math anxiety," an emotional reaction in situations that involve numbers, ranging from a mild apprehension to a genuine fear or dread (NMAP, 2008). Not only is math anxiety stressful, it is also related to low performance in mathematics, avoidance of more advanced studies in math, and poor scores on standardized tests. Little is known, however, about how math anxiety begins or what the contributing factors are (NMAP, 2008). Although conventional wisdom says that girls are more anxious about mathematics than boys, research has found that gender has little effect on math anxiety overall (e.g., Ashcraft & Ridley, 2005). In some studies, girls in all grades have reported higher levels of anxiety about mathematics, but their anxiety does not seem to translate to either mathematics performance or the degree to which they avoid math (Hembree, 1990). It has been suggested that girls may simply be more willing to admit anxiety (Ashcraft & Ridley, 2005).

Recent research on math anxiety has shifted from an investigation of contributing factors to a more process-oriented approach. Studies have tried to understand the cognitive consequences of math anxiety. It was discovered that people with math anxiety may have a difficulty with working memory. The hypothesis is that their working memory capacity is occupied with managing their anxiety, instead of trying to solve the mathematics problems (Ashcraft & Kirk, 2001; LeFevre, DeStefano, Coleman, & Shanahan, 2005). When children's anxiety is reduced, often through some kind of cognitive therapy, their math achievement improves, often more than either the children or their teachers expected. It appears that their ability had been depressed by their own anxiety (NMAP, 2008).

Changes can be made in the classroom to reduce math anxiety. Some changes, such as providing calculators, have not proved effective. On the other hand, a review of basic skills and a focus on the relationship between good study habits and performance have shown positive effects for students with math anxiety (Hutton & Levitt, 1987). Further, if students are encouraged to attribute success to controllable factors, like hard work and test preparation, they tend to work more persistently and their performance improves (Dweck, 1975).

Some students may be more likely than others to become anxious about mathematics. Risk factors include low mathematics aptitude, low working memory capacity, concern over public embarrassment, gender, and negative attitudes in significant adults (educators and parents). In addition, social and intellectual support from peers and teachers is associated with better performance in mathematics for all students, regardless of whether they have math anxiety (NMAP, 2008).

TRANSFER OF LEARNING

To achieve success in mathematics, it is essential to be able to **transfer** skills from one type of problem to another. This means being able "to correctly apply one's learning beyond the exact examples studied to superficially similar problems (near transfer) or to superficially dissimilar problems (far transfer)" (NMAP, 2008, p. 7). Children are more likely to achieve transfer if they have a deeper conceptual understanding of the material, which is often achieved through work with more difficult problems. Challenging material requires children to apply more attention and effort to process the information, which leads to better retention (NMAP, 2008). Abstract representations of information can also benefit transfer of learning to more concrete examples (e.g., Sloutsky, Kaminski, & Heckler, 2005; Uttal, 2003). However, children need to start by working with less challenging material in order to get an initial understanding. Only then will work with more challenging material allow them to deepen their understanding and improve their ability to transfer learning.

INTERWOVEN SKILLS FOR MATHEMATICAL COMPETENCE

As we have seen, there are a number of cognitive and social/emotional factors that contribute to achieving success in mathematics. All these factors interact and affect one another as the process of mathematics learning develops over time. Some models have been put forward by researchers to frame how this process happens. For instance, the Competence, Learning, Intervention, and Assessment (CLIA) model states that mathematical competence can be reached only if children gain five skills:

- **mathematical knowledge** (e.g., facts, symbols, algorithms[a], concepts, and rules)
- **heuristic methods** (e.g., systematic strategies for problem solving)
- **metaknowledge** (e.g., thinking about one's own thinking, emotions, and motivation)
- **self-regulatory skills** (e.g., planning, monitoring)
- **self-efficacy beliefs** (e.g., thinking about oneself in relation to mathematics)

[a] In mathematics, an algorithm is a set of precise step-by-step instructions for how to arrive at an answer. It refers to a formal procedure, usually one that is explicitly taught.

These skills all develop concurrently, not one after the other (De Corte & Verschaffel, 2006).

It has been suggested that children need to have math made relevant to them; in particular, they need to have opportunities to use the knowledge and skills they have learned to solve problems. However, exposure and practice are not enough: they must also want to use the knowledge and skills they have learned. All of these conditions are influenced by the child's beliefs, not only about what he finds interesting, but also about what counts as a mathematical context (Perkins, 1992). Thus, in addition to skills and strategies, one's beliefs and attitudes are also important.

THE IMPORTANCE OF GOOD MATHEMATICS TEACHING

An important predictor of children's achievement is the quality of the early years educator and of the classroom teacher (e.g., Darling-Hammond, 2000; Darling-Hammond & Youngs, 2002; Hanhushek, Kain, & Rivkin, 1998). Studies on school-aged children have shown that effective teaching can account for the greatest differences between more and less effective schools (Clotfelter, Ladd, & Vigdor, 2007; Klecker, 2007). The National Mathematics Advisory Panel[b] stated that "teachers are crucial for creating opportunities for students to learn mathematics" and that in a single elementary school year, "differences in the quality of teaching [can] account for between 12 and 14% of the total variability in students' mathematics achievement gains" (NMAP, 2008, p. 35). This effect is compounded when students have several either effective or ineffective teachers one after the other.

Truly effective mathematics teaching brings together four required components:

- an appreciation of the discipline of mathematics itself, and of what it means to do mathematics
- an understanding of how children learn
- the provision of a problem-solving environment for learning
- the integration of assessment into teaching to enhance both learning and instruction (National Council of Teachers of Mathematics [NCTM], 1989).

A variety of educator characteristics can positively affect children's performance in mathematics. First and foremost, the educator's own knowledge of the subject significantly influences children's learning; this relationship appears to be particularly true for mathematics (Wayne & Youngs, 2003). In addition, math teachers who had continued to study mathematics after high school had students with greater mathematical gains than those students whose teachers did not hold advanced degrees, whether these degrees were in mathematics or not (Hill, Rowan, & Ball, 2005).

Although researchers agree that educators' knowledge of their subject matter contributes to children's learning, it takes more than knowledge to be effective. The way educators put their knowledge into action plays a vital role in the development of children's understanding of mathematics. For example, an educator's mathematical behaviours – such as their level of explanation, their choice of representation, and their interactions with students' mathematical thinking – all influence children's mathematics performance (Hill et al., 2005). Experience as an educator is also a factor, but its impact is influenced by the qualities and abilities of the individual teacher (Kukla-Acevedo, 2009). The importance of strong mathematics educators, in preschool and at all grade levels, cannot be overstated.

SUMMARY

This section has provided information on some of the mental processes underlying children's achievement: attention, working memory, and the retrieval, transfer, and retention of information; factual, conceptual and procedural knowledge; strategy selection and use; and self-monitoring behaviours. Knowledge of these processes can help educators understand how children learn and thus how they can best support that learning. While these cognitive building blocks are important, other factors also contribute to children's mathematics achievement. Motivation, self-regulation, and mathematics anxiety all can have a strong effect on children's cognitive processing, and thus on their achievement. Effective educators must take all of these factors into account.

PART 2: THE DEVELOPMENT OF MATHEMATICS CONCEPTS

EARLY MATHEMATICAL ABILITIES

Young children have a natural desire to understand the world around them. They are "active, resourceful individuals who can construct, modify, and integrate ideas by interacting with the physical world and with peers and adults" (NCTM, 2000, p.75). Mathematics is one means by which we understand the world, and children engage with math long before they begin school (Bryant, 1997).

[b] In 2006, the National Mathematics Advisory Panel (NMAP), a panel of 24 distinguished mathematics researchers, was convened to advise the President of the United States and the U.S. Secretary of Education on ways to foster increased mathematics performance using research-based instructional methods (NMAP, 2008). The panel produced their report in 2008. One of the strongest of the panel's recommendations concerned the importance of applying what is known from research on how children learn to the teaching of mathematics. In particular, the panel noted that "a) there are great advantages for children who have a strong start in mathematics, b) conceptual understanding, procedural fluency, and quick, effortless (i.e., "automatic") recall of facts are related in a mutually reinforcing and beneficial way, and c) effort, not just inherent talent, is a vital component of achievement in mathematics" (NMAP, 2008, p. 11).

Clements (2004) asserts that even before Kindergarten, "children have the interest and ability to engage in significant mathematical thinking and learning" (p. 11). During the early years, children explore the mathematical dimensions of their world, comparing quantities, finding patterns, navigating their environment, and tackling real problems (National Association for the Education of Young Children [NAEYC] & NCTM, 2002). Knowledge of quantity emerges early in life and develops significantly during a child's first three years. Research has shown that infants can tell the difference between small quantities, for instance, between two items as opposed to three items (Starkey, Spelke, & Gelman, 1990). Toddlers typically learn their first number word (usually "two") at around twenty-four months. By age four, children are able to compare quantities and use words like "more" and "less." As children get more experience with counting, they begin to count larger collections, count on from a given number, and learn number patterns. Children also explore shape, space, and measurement. A child building a tower out of blocks is using knowledge about shape (which blocks are best for the base of the tower), space (where best to place the blocks to ensure a sturdy tower), and measurement (how many blocks can be placed on the tower before it is taller than the builder). Pre-Kindergarten children are also keen to recognize and analyze patterns – the beginnings of algebraic thinking (Clements, 2004).

"Research suggests that children's early mathematical experiences play an enormous role in the development of their understanding of mathematics, serve as a foundation for their cognitive development, and can predict mathematics success in the high school years" (Shaklee, O'Hara, & Demarest, 2008, p. 1). The National Council of Teachers of Mathematics (NCTM) also maintains that the foundation for children's mathematical development is established in the early years (2000). Moreover, NCTM, in a joint positional statement with NAEYC, asserted that children aged three to six require high quality, challenging and accessible math education in order to build a strong foundation for their future mathematics learning (2002).

Children's mathematics ability at the beginning of Kindergarten is a strong predictor of later academic success, even stronger than their early reading ability (Duncan et al., 2007). Mathematics ability is, in turn, based on knowledge accumulated during the years before Kindergarten. Children learn by building on prior knowledge, extending as far back as early childhood. A theory of "overlapping waves" of learning and development describes the gradual, incremental processes that occur as children grow and learn (Siegler, 1996). Studies observing children at play reveal that young children naturally engage in a significant amount of mathematical activity (Clements & Sarama, 2005; Ginsburg, Inoue, & Seo, 1999; Seo & Ginsburg, 2004). Even before they begin elementary school, children can reason and solve problems (Gopnik, Meltzoff, & Kuhl, 1999; NMAP, 2008).

In deciding what is "developmentally appropriate," we need to look beyond a child's age or grade. Both NRC and NMAP document the finding that what children are ready to learn is largely a result of their prior opportunities to learn (Duschl, Schweingruber, & Shouse, 2007). Claims that children are either too young, in the wrong stage, or not ready to learn something have been shown time and again to be wrong (NMAP, 2008). A significant body of research has shown that young children are more competent than was previously thought. Moreover, this research suggests that without adequate attention to math in the early years, a child may be at risk for later school failure (Lee & Ginsburg, 2007).

There is strong evidence for the importance of a well-built foundation in mathematics, just as there is for reading. Sarama and Clements (2004) argue that a complete mathematics program may also contribute to children's later learning of other subjects, especially literacy. Much of the recent research has reported that mathematics does in fact support the development of literacy.

Numerosity and Ordinality

Children are born with some abilities necessary for processing quantities, abilities that have also been noted in rats, pigeons, and other primates. They can make decisions about which quantity is more or less, and can, to some degree, understand processes such as "taking away, resulting in less." Researchers disagree about the connection between these abilities and actual mathematical understanding; nonetheless, it is clear that infants and very young children are capable of more than was once assumed. For instance, studies of six-month-old infants have shown that they can tell the difference between larger and smaller quantities. They can do this both with objects they see and with sounds they hear. However, this ability is limited and they are more accurate with smaller quantities. When they are asked to compare two sets that both contain a large number of items, they can only recognize the difference when the larger set contains at least double the number of items as the smaller set (Brannon, Abbott, & Lutz, 2004; Lipton & Spelke, 2003; Xu & Spelke, 2000). When dealing with smaller numbers, babies aged four- to seven-and-a-half months can discriminate between a set of two and a set of three objects, but not between a set of four and a set of six objects (Starkey & Cooper, 1980).

The technical term for "the ability to discriminate arrays of objects based on the quantity of presented items" is numerosity (Geary, 2006, p. 780). Sensitivity to numerosity has been demonstrated many times using one to three objects, and sometimes four, with infants, even as early as the first week of life (e.g., Antell & Keating, 1983; Starkey, 1992; Starkey, Spelke, & Gelman, 1983, 1990; van Loosbroek & Smitsman, 1990). These findings suggest that even as infants, we have an intuitive sense of approximate magnitude (i.e., how much there is) called ordinality (Dehaene, 1997; Gallistel & Gelman, 1992). This sense of more and less emerges in a very basic form around ten months of age (Brannon, 2002; Feigenson, Carey, & Hauser, 2002).

Arithmetic

As we discussed above, two important cognitive factors that affect learning are the mental representation of information and the memory for information. Research on these factors has been done with children aged one-and-a-half to four. The findings show that at age two, children can mentally represent and remember one, two, and sometimes three items. By two and a half, their representation of and memory for up to three items is more consistent (Starkey, 1992). By age three or three-and-a-half, up to four items can be represented and remembered. In the same study, the children's addition and subtraction abilities were also examined, using a nonverbal calculation task. The youngest children, who were one-and-a-half, understood addition and subtraction with numbers less than or equal to two (e.g., 1 + 1; 2 - 1), but not with larger numbers. Two-year-olds were accurate with values up to three, and none of the children (even up to age four) were accurate with values of four or five (Starkey, 1992). Research such as this suggests that between ages two and three, children are not just aware of the concept of small numbers, but can also begin to learn how to solve simple nonverbal calculations involving one and two items. By the time they are four, many children can solve problems involving three (and sometimes four) items (Jordan, Huttenlocher, & Levine, 1994).

These basic arithmetic problems can be made slightly more complex for older preschool children by introducing the concept of the inverse relation between addition and subtraction. For example, when starting with two items, if one item is added and one item is taken away, there are still two items left; adding one and taking away one cancel each other out. In the example "2 + 1 – 1 = ?", if this inverse relation is understood, no adding or subtracting needs to be performed to know that the answer is 2. In studies, some four-year-olds used a procedure based on the inverse relation between addition and subtraction to solve problems like this. At this young age, they demonstrated at least a basic understanding of this fundamental principle of arithmetic (Klein & Bisanz, 2000).

Number Concepts

Sometime between ages two and three, children begin to map the number words of their language and culture onto their knowledge of numerosity and systems of magnitude, beginning with counting (Spelke, 2000; Gelman & Gallistel, 1978). Children appear to know very early that the number words all represent different quantities and that the sequence in which they say these number words is important (Gelman & Gallistel, 1978). At the same time, they also understand that number words are different from other descriptive words, such as "big" or "red" (Geary, 2006). Children may know certain qualities of numbers before they are able to apply and use that knowledge fully. For instance, by age two and a half, children can tell the difference between a set of three items and a set of four items. They also know that "4" is more than "3." However, they still may not be able to consistently connect number words with quantities in order to label sets as containing three or four items (Bullock & Gelman, 1977). It has been argued that at least a year of counting experience, usually from age two to age three, is necessary for children to both associate number words to their mental representations of quantities and use that knowledge in counting (Wynn, 1992). Quantities of four and above seem to be more difficult for preschool children.

Counting Procedures

Five implicit principles are thought to guide a preschool child's development of counting procedures (Gelman & Gallistel, 1978):

- **Stable order** refers to the fact that the number words are always used in the same order (e.g., counting in the order of "1, 2, 4" is incorrect).
- **One-to-one correspondence** means that one and only one number word can be assigned to each counted object in the set (e.g., an item in a set that has been assigned "3" cannot also be assigned "5").
- **Cardinality** refers to the fact that the value of the last number word used when counting indicates the quantity of items in the set (e.g., counting "1, 2, 3, 4" means there are four items in the set).
- **Abstraction** means that any set of items can be counted (e.g., a book, two bananas, and three pencils can be counted together as a set of six items).
- **Order irrelevance** means that items can be counted in any order (e.g., counting from right to left, left to right, or in no particular sequence at all will result in the same total number of items).

The first three principles are the basic "how to count" rules, which set the initial structure for children's developing knowledge of counting (Gelman & Meck, 1983). Children refine their understanding of counting and add to these basic principles as they observe and think about counting. For awhile, children assume that some aspects of counting are essential when in fact they are just conventions. For instance, by habit, we may always count from left to right. Observing this, children may believe that counting must necessarily be done in a standard direction. We also, by habit, tend to move from one item to the item next to it when we count. Children may gather from this that counting must be done in this way to be accurate, and that adjacency is an essential element of counting. By age five, most children know the essential features of counting but many continue to believe that adjacency is mandatory (LeFevre et al., 2006).

By age five, most children's knowledge of the essential principles is quite good, though they still make some mistakes. By the end of Kindergarten, many children can count sets that contain a quantity of items for which they know the number words: if they know the numbers up to twelve, they can accurately count a set with twelve items. However, quite a few children are still struggling even in higher grades. Those who are not proficient counters and who do not know the essential principles by the time they enter Grade 1 may be at risk for difficulties with mathematics (Geary, 2003).

Geometry and Measurement

Geometry and measurement have been called the second most important area of mathematical learning. According to some authors, "one could [even] argue that this area – including spatial thinking – is as important as number" (Sarama & Clements, 2009, p. 159). Geometry and measurement are important partly because they make real-world connections: "Geometry, measurement, and spatial reasoning are important… because they involve 'grasping'… that space in which the child lives, breathes, and moves… that space that the child must learn to know, explore, and conquer in order to live, breathe, and move better in it" (NCTM, 1989, p. 48).

Geometry and measurement also contribute to the foundation for learning in math and other subjects (Clements, 2004). For example, spatial thinking is essential to the development of number quantification, non-routine problem-solving ability, and mathematical reasoning (Sarama & Clements, 2009). In addition, spatial thinking provides a real-life application for number and arithmetic (Clements & Stephan, 2004).

Children in preschool and Kindergarten should be given rich opportunities to explore shape (Clements, 2004; Clements & Sarama, 2000; Clements & Stephan, 2004). They can also benefit from activities that encourage them to consciously reflect on the properties and attributes of shapes (Orton, Orton, & Frobisher, 2005). Such activities may include sorting, finding examples of shapes in the environment, combining shapes to create new ones, and constructing and altering shapes.

When they enter school, children commonly have a working knowledge of shape, congruence, and symmetry; instruction should be designed to "build on this knowledge and move beyond it" (Clements, 2004, p. 285). Children's formal geometric knowledge and skills will benefit from being exposed to basic geometric shapes, names, and other concepts; however, by this point, mere exposure is insufficient. Children "must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations" (NMAP, 2008, p. 29).

THE TRANSITION TO SCHOOL

Around the age of four or five, children may be receiving more formal education in mathematics in preschool or Kindergarten. They bring with them their intuitive understanding of quantities and accumulated experiences with mathematical concepts in their daily life. At first, the more formal mathematics lessons tend to be disconnected from children's intuitive understandings, but gradually, they will achieve integration of these two systems, their formal and informal learning.

Number Concepts and Counting

By the time they reach Kindergarten at age four or five, most children can use number words to solve simple addition and subtraction problems with small numbers (Baroody & Ginsburg, 1986; Groen & Resnick, 1977; Saxe, 1985; Siegler & Jenkins, 1989). At this stage, they often solve problems by using concrete objects (including fingers) to help them count (Geary, 2006). These tools serve to connect numerosities in the world to internal representations, they reduce the load on memory, and they help make sure the procedures are carried out correctly (Siegler & Shrager, 1984).

Cardinality and Ordinality

As mentioned above, cardinality refers to the fact that the last number counted is the total number of items in the set. Ordinality, at its most basic level, is the concept of more and less. A child's sense of ordinality develops into an understanding that higher numbers are associated with more items and lower numbers with fewer items. Children need to understand both cardinality and ordinality to become competent in math. Without cardinality, counting would not provide any meaningful information, and "without [ordinality], distinct numerosities such as 'one' and 'four' bear no more relation to one another than do cows and blenders" (Brannon & Van De Walle, 2001, p. 54).

When children know the sequence of number words (1, 2, 3, 4...), they can then develop more precise mental representations of numbers beyond three or four, which in turn enhance their knowledge of cardinality and ordinality. In fact, children seem to have an implicit understanding of both cardinality and ordinality even before they learn the sequence of number words (Bermejo, 1996; Brainerd, 1979; Brannon & Van de Walle, 2001; Cooper, 1984; Huntley-Fener & Cannon, 2000; Ta'ir, Brezner, & Ariel, 1997; Wynn, 1990, 1992). However, it is not enough to have an intuitive sense of these concepts. To become mathematically competent, it is essential to have a more mature sense of cardinality and ordinality in which they are connected to the counting sequence, and this takes time.

Here is an example of how a preschooler who has an immature understanding of cardinality might act. Imagine that an educator counts aloud the fingers on a child's hand and asks him how many fingers he has. He may need to count his fingers again before answering. Only a child with a mature grasp of cardinality will be able to simply repeat the last number word that the adult said. By five years old, most children have a good grasp of cardinal value for quantities of ten and under, so they are able to do this (Bermejo, 1996; Freeman, Antonucci, & Lewis, 2000). Once a child has learned the verbal counting system, their abilities improve rather dramatically toward a mature appreciation of cardinality (Brannon & Van De Walle, 2001).

Counting

As we have seen with respect to children's understandings of ordinality and cardinality, counting is a foundational component of children's early work with number (NCTM, 2000). It takes some time to master the verbal counting system. Typically, it begins to emerge in children around the age of four and solidifies by age five or six, by which time children generally make few errors and have a good grasp of the essential counting principles (Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990, 1992). By the age of five, most children have acquired the basic foundational skills of numeracy: they can match sets that contain the same number of items, label small numerosities, and use counting to determine cardinality (e.g., Bermejo, 1996; Bermejo & Lago, 1990; Fuson, 1988; Gelman & Gallistel, 1978; Huttenlocher et al., 1994; Mix, 1999; Mix, Sandhofer, & Baroody, 2005; Wynn, 1990). When children start school, their knowledge of number is informal, but nonetheless rich and varied (Baroody, 1992; Fuson, 1998; Gelman, 1994). During the early elementary years, teachers help students to strengthen their sense of number by moving them from basic counting techniques to a more sophisticated understanding of numbers (NCTM, 2000).

Commutative and Associative Properties

A solid sense of numbers includes understanding that numbers are sets of smaller numbers that can be decomposed and recombined. For example, 12 can be decomposed into 2 + (4 + 6) and recombined into (2 + 4) + 6, or decomposed into 2 x (3 x 2) and recombined into (2 x 2) x 3.

Decomposition and recombination are related to two properties of the operations of addition and multiplication.

- the **commutative property** refers to the fact that you can change the order in which you add or multiply two numbers without changing the answer. For example, a child who understands the commutative property of arithmetic knows that the sum of 4 + 3 is the same as the sum of 3 + 4. In the same way, 5 x 8 and 8 x 5 have the same product. More generally, the commutative property can be stated as "a + b = b + a" and "a x b = b x a."
- the **associative property** is similar to the commutative property, but deals with more numbers. It states that the order in which three numbers are added or multiplied does not affect the sum or product. This can be stated as "a + (b + c) = (a + b) + c" and "a x (b x c) = (a x b) x c."

Some work has been done on the associative property of addition (Canobi, Reeve, & Pattison, 1998, 2002); however, most of the research in this area has focussed on the commutative property of addition (Baroody, Ginsburg, & Waxman; 1983; Resnick, 1992). Resnick (1992) proposed that knowledge of the commutative property is built on conceptual steps. First, in preschool or Kindergarten, children go through a pre-numerical stage during which they solve problems by manipulating physical objects. They find out that it doesn't matter in what order objects are combined into a set, because the total will still be the same (Gelman & Gallistel, 1978; Resnick, 1992). Next, around four or five years old, children begin to map specific quantities onto this action, for example, five cars plus three trucks equals three trucks plus five cars (Canobi et al., 2002; Sophian, Harley, & Martin, 1995). Following this, when they are in Grade 2 or 3, children move away from a reliance on physical objects and begin to use only numbers: 5 + 3 = 3 + 5 (Baroody et al., 1983). Finally, children achieve a formal knowledge of the commutative property as an arithmetic principle (a + b = b + a); the exact timing of this last stage is not certain (Resnick, 1992).

Understanding of the associative property of addition is acquired in much the same way, beginning in Kindergarten with physical objects and moving to an implicit understanding in Grade 1 or 2. However, children do not come to understand the associative property as an arithmetic principle until they have an implicit understanding of the commutative principle (Canobi et al., 1998, 2002).

The Mental Number Line

Mathematics involves cognitive processes that require the dual coding of imagery and language. Imagery is fundamental to the process of thinking with numbers because it allows us to create mental representations for mathematical concepts (Bell & Tuley, 2003). One of the most important of these representations is the **mental number line**. Some of the central achievements of formal mathematics depend on understanding the relationship between number and space; fundamental to this is the arrangement of numbers on a line (de Hevia & Spelke, 2008).

Learning the concept of number itself appears to be related to a child's ability to generate a mental number line (Dehaene, 1997). A mental number line is an imaginary horizontal line with numbers along it in ascending order. It is, of course, a metaphor, not an actual structure in the brain. This number line is a mental image that reflects our knowledge, a tool we use to represent numbers and relative magnitudes.

Forming a mental number line requires the ability to visualize and abstract number to order numbers by quantity, to locate a given number along a line, and to generate any portion of the number line that may be required for problem solving (Gervasoni, 2005). It is related to learning addition and subtraction, as well as the estimation of the magnitude of numbers (Siegler & Booth, 2005).

There are three main areas in which the number line is particularly useful for young children's mathematical development (Griffin, Case, & Siegler, 1994). First, a mental number line allows children to respond to questions about relative magnitude without referring to concrete objects. Second, mental number lines support the acquisition of the increment rule, which describes how addition or subtraction alters the cardinal value of the set and therefore moves that value up or down on the number line. Third, children who have developed a mental number line can also determine the relative position of a number on that line, which is useful for determining relative quantity when it cannot be determined more directly (Gervasoni, 2005).

The ability to use the mental number line to represent specific quantities only emerges with formal education, after the transition to school (e.g., Siegler & Opfer, 2003). Research indicates that young children have difficulty making estimates of the position of a number on the number line (e.g., placing 84 on a number line from 1 to 100), but that this skill improves over the elementary years (Siegler & Booth, 2004; Siegler & Opfer, 2003). Their initial difficulty may occur because young children tend to see the distance between 1 and 2 as larger and more certain than the distance between 51 and 52; the numbers get "squished up" towards the right end of the number line (Dehaene, 1997; Gallistel & Gelman, 1992). By Grade 6, most children have a correct, linear sense of the number line and of the fact that numbers are spaced evenly along it (Siegler & Opfer, 2003).

THE ELEMENTARY YEARS

After the transition to the more formal education system of elementary school, children solidify the knowledge gained earlier and deepen their conceptual understanding of mathematics. They face new challenges, such as fractions, and they use new skills, such as estimation, problem-solving strategies and algorithms. They also achieve understanding of new concepts, such as arithmetic operations, proportion, reversibility, and commutative and associative properties. Because of the breadth and extent of foundational skills that need to be mastered, academic success in mathematics can be challenging.

Biologically Primary and Secondary Knowledge

As we have discussed, for very young children mathematics-related thinking is primarily made up of inherent types of cognition, such as language and early quantitative competencies. These have been called biologically primary abilities because they typically emerge with little or no formal instruction. They appear universally, across all cultures. Once children have reached school age, however, they build on these biologically primary abilities to learn skills that need to be formally taught. Some of the information learned in school is considered to be a "cultural invention," with arbitrary symbols such as number words. This knowledge is referred to as biologically secondary (Geary, 1994, 1995).

An example of socially constructed, biologically secondary information is the base-10 system, an essential component of mathematics. A child who does not grasp the fundamentals of this system will have difficulty understanding other concepts (Geary, 1995). As Geary (2006) states: "Many children require instructional techniques that explicitly focus on the specifics of the repeating decade structure of the base-10 system and [techniques] that clarify often confusing features of the associated notational system" (p. 791). Children who speak certain European languages may need more help with this than children who speak Asian languages. In Chinese, for instance, the base-10 system is made obvious by the number words for 11, 12, and 13, which translate as "ten-one, ten-two, ten-three." This contrasts with the English "eleven, twelve, thirteen," which make no reference to base-10 (see, for example, Fuson & Kwon, 1991). This transparent connection between the number word, the Arabic digit, and the magnitude represented gives children who speak Asian languages an initial advantage over English-speaking children in understanding the base-10 concept (Miller et al., 2005). It may also enhance their conceptual understanding of arithmetic (Miura, 1987), although children who speak non-Asian languages appear to catch up quickly. Teachers can benefit from knowing about possible difficulties in linking numbers to corresponding words. Children in French immersion classes, for example, may be confused about the numbers between 70 and 100 (e.g., compare soixante-quinze [translates as sixty-fifteen] to seventy-five) even when they have mastered the labels in English (Seron & Fayol, 1994).

Fractions

To build their knowledge and understanding of fractions, elementary school children need to already have a firm base of skills and concepts. They need to have learned and practised certain basic arithmetic facts until they come automatically. They must be able to perform mathematical procedures with whole numbers and possess a deep understanding of core mathematical concepts (NMAP, 2008). Procedural and conceptual skills also influence a child's ability to estimate, make computations, and to find the solution to word problems.

Children have considerable difficulty learning the conceptual and procedural aspects of fractions (Geary, 2006). Research has focussed on these aspects of fractions (Clements & Del Campo, 1990; Hecht, 1998; Hecht, Close, & Santisi, 2003) and on the mechanisms that influence their acquisition (Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). At first, when children begin to learn the formal features of fractions, such as the numerator and denominator system, they tend to rely on what they already know about whole number counting and arithmetic (Gallistel & Gelman, 1992).

Although fractions are considered biologically secondary information in a formal mathematics context, children already have some understanding of part/whole relationships based on their experience with physical objects (Mix, Levine, & Huttenlocher, 1999). In preschool and early elementary school, children already understand simple fractional relationships – they know whether a cookie is being divided equally, or if one person is receiving a larger share. It is not known yet if the ability to visualize parts of a whole is a biologically primary ability (Geary, 2006).

Research has focussed on older elementary school children's computational skills, conceptual understanding, and ability to solve word problems that involve fractions (Byrnes & Wasik, 1991; Rittle-Johnson et al., 2001). Once a child has conceptual knowledge of fractions, that knowledge will likely have an effect on problem-solving performance. As with whole numbers, procedural knowledge will also inform conceptual knowledge when learning fractions (NMAP, 2008). Children's procedural ability has been shown to predict computational skills, and computational skills in turn predict accuracy at solving word problems with fractions and estimation skills (Hecht, 1998). In addition, the acquisition of conceptual knowledge of fractions and basic arithmetic skills was related to children's working memory capacity and to the amount of time spent on the task in class (Hecht, 2003).

Number Sense

Number sense can be broadly defined as the understanding of number and operations, the ability to use this understanding to learn and develop strategies for handling numbers and operations, and the ability to use numbers as a way of communicating and dealing with information (McIntosh, Reys, & Reys, 1992). (Definitions vary slightly in curriculum documents of different Canadian provinces.) More specifically, **number sense** encompasses three subcomponents:

- knowing about and using numbers (e.g., number order, multiple representations, relative and absolute magnitude)
- knowing about and using operations (e.g., mathematical properties, such as the commutative and associative properties, and relationships between operations)
- knowing about and using numbers and operations in computational settings (e.g., use of estimation, knowing that multiple strategies exist for the solution of any problem, efficient use of problem-solving methods, reviewing and checking one's answer) (McIntosh et al., 1992).

As many mathematical skills, number sense is not achieved all at once, but rather is a process that unfolds over years, developing with age and experience. For older elementary students, the third subcomponent is the most relevant. We will therefore examine number sense as it relates to estimation, problem solving, and word problems.

Estimation

Estimation may not be a formal subject in elementary school, but it is a skill that people use frequently, both in and out of school. To estimate is to approximate the value of something, often when it is difficult or unnecessary to determine an exact answer. We also use estimation to check whether our calculation of an answer is reasonable. Sowder (1992) identifies three forms of estimation: computational (e.g., estimating the answer to a word problem), measurement (e.g., estimating the area of the classroom), and numerosity (e.g., estimating the number of people at a soccer game). Siegler and Booth (2004) added a fourth form, number line estimation (e.g., placing numbers 0-100 on a number line).

Research on estimation has focussed on computational arithmetic (Case & Okamoto, 1996; Dowker, 1997, 2003; LeFevre, Greenham, & Waheed, 1993; Lemaire & Lecacheur, 2002) and on work with the number line (Siegler & Booth, 2004, 2005; Siegler & Opfer, 2003). These studies have shown that children, and some adults, find it hard to make reasonable estimates. The skill of estimation appears only with formal schooling and requires practice. For all types of estimation, both children and adults use a variety of strategies. Their skills improve in efficiency, sophistication, and adaptivity with age and experience (De Corte & Verschaffel, 2006).

Problem Solving

Research in the fields of both mathematics education and cognitive science has shown the benefits of a standards-based curriculum for mathematics instruction (e.g., NCTM, 2000). Although traditional direct instruction techniques are helpful, students also benefit from developing their own strategies for problem solving. They are required to "convey their personal understandings of [a] problem so that they can choose between the relative merits of different strategies that they invent" (Moseley & Brenner, 2009, p. 2).

In this section, we will examine what the research says about problem solving with both arithmetic and word problems.

Arithmetic Problems

There is not just one way to solve a problem, and no student will use only one strategy to solve all problems. An individual child will use a variety of arithmetic strategies, even within the same day or for the same type of problem (see Siegler, 1998, for a review). It is common for people, from young children up to adults, to have multiple and flexible strategies when learning arithmetic – addition, subtraction, multiplication, and division (LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003).

Older elementary school children sometimes use the algorithms and strategies taught in school when they do multi-digit arithmetic. However, researchers have found that they may also use varied, informal strategies that differ from what they have been taught (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Reys, Reys, Nohda, & Emori, 1995). Both children and adults use this kind of invented strategy.

A research project tracked children's arithmetic problem solving for three years, and identified five categories of invented strategies, of which three were most frequently seen:

- **combining units** strategies, wherein the 100s, 10s, and units are dealt with separately, for example $37 + 38 = 30 + 30$, then $7 + 8$
- **sequential** strategies, wherein the value of the second number is counted up or down from the first number, for example $37 + 38$ is solved by $37 + 30 = 67$, $67 + 8 = 75$
- **compensating** strategies, wherein the numbers are adjusted to simplify the arithmetic, for example, $37 + 38 = (35 + 35) + 2 + 3 = 75$) (Carpenter et al., 1998).

Researchers noted that students who tended to use their own methods would typically sample from all three of these strategy categories. In general, these students could transfer their knowledge to new and different problems better than those who followed only the standard step-by-step procedures they had been taught.

Children's choice of strategy often depends on the amount of conceptual knowledge they have – for instance, their knowledge of addition, units, grouping by tens, and the properties of the four basic operations (Ambrose, Baek, & Carpenter, 2003). This is another instance of the way conceptual and procedural knowledge influence each other.

Regardless of the type of strategy used, whether invented or taught, the flexible and adaptive use of multiple strategies is a characteristic of expertise with multi-digit arithmetic (De Corte & Verschaffel, 2006). As discussed in the section on students' beliefs about learning, the way students feel about math is an important factor in their success or failure in the subject. Research shows that the way students approach problem solving is also a factor. For instance, it is unproductive to "stubbornly" use standard step-by-step procedures in cases where mental arithmetic would be more appropriate, for example, to calculate the problem 4,002 – 3,998 (e.g., Buys, 2001). Students' fear of taking risks in problem solving will also influence their ability to succeed (Thompson, 1999).

Word Problems

Students in the early elementary grades encounter three general types of one-step word problems:

- **change problems** contain some event that changes the value of a quantity: Robin has 5 pencils and Carly gives him 3 more; how many does Robin have now? Change problems can be subdivided into two categories, depending on whether the quantity increases or decreases.
- **combine problems** describe two parts that are considered separately or in combination: Robin and Carly have 8 pencils all together; Carly has 3 pencils; how many does Robin have?
- **compare problems** contain two amounts to be compared for the difference between them: Robin has 5 pencils and Carly has 3 pencils; how many fewer pencils does Carly have than Robin? There are also two categories of compare problems, depending on whether the question is which has more or which has fewer.

Most children in the early elementary grades can use modelling to solve simple one-step problems, such as combine problems for which the answer is a whole number. For instance, they can represent the objects in the problem with manipulatives, tally marks, or their fingers and count to get the answer. As they get better at problem solving, they replace such cumbersome strategies with shorter, internalized ones that make the process more efficient. They also generalize their strategies so that they can apply them to new problems with a similar underlying mathematical structure (De Corte & Verschaffel, 2006). Proficient problem solving can be defined as the ability to represent a problem, decide on a solution procedure, and carry out that procedure. Predictably, children become proficient at addition and subtraction relatively quickly, while multiplication and division problems take longer to master (Anghileri, 2001; Clark & Kamii, 1996).

Children do not reach expert problem solving status without a few quirks, however. One interesting phenomenon that has been observed is the "suspension of sense making." Children seem to suffer a sort of logical oversight that prevents them from realizing when problems are false or absurd. For instance, when researchers gave students in Grades 1 and 2 the problem: "There are 26 sheep and 10 goats on a ship. How old is the captain?", the majority gave a numerical answer, most often 36 (Carpenter, Lindquist, Matthews, & Silver, 1983). Older elementary students are not immune to the effects of suspension of sense making. Students in Grade 8 were given the problem: "An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many busses are needed?" The majority of students correctly divided 1,128 by 36, but less than a third used the remainder (12) to conclude that an extra bus was needed for these "left-over" individuals (Carpenter et al., 1983). While these older students were not as easily confused by absurd questions, they still did not apply their knowledge of the real world to their answers. They effectively suspended their sense-making abilities, resulting in very few "realistic" responses or comments on word problems such as this one. Research regularly identifies this effect and finds that it is strong and resistant to change (for a review, see Verschaffel, Greer, & De Corte, 2000).

Students whose problem-solving skills are still developing also tend to demonstrate a lack of strategic approaches, not to be confused with a lack of problem-solving strategies. When faced with a problem, children do not spontaneously respond by analyzing the problem, making a drawing of it, breaking it down into more manageable units, or other valuable strategies. That is, they rarely step back and consider the problem's context and elements before they attempt to solve it by applying a procedure. Even when given encouragement to take these steps, they do not significantly improve their performance (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). This phenomenon is particularly common in students who have weak problem-solving skills (e.g., Hegarty, Mayer, & Monk, 1995). Unlike students with strong problem-solving skills, they tend to rely on superficial methods rather than on building a mental representation and carefully analyzing the problem.

Students' lack of strategic approaches is directly related to a lack of metacognitive activity during the problem-solving process, such as self-regulation, self-monitoring, and reflection. Good problem solvers self-regulate more often than poor problem solvers do, and this is true both of younger and older children (Carr & Biddlecomb, 1998; Garofalo & Lester, 1985). In addition to conceptual understanding and computational fluency, students need to know how to approach problem solving strategically in order to succeed. They also must use self-regulation strategies while they work on the problem.

Proportional Reasoning

Proportionality is an important concept not just in mathematics and science, but also in everyday life, for instance to halve or double a recipe. A cake will not rise if we increase the other ingredients without increasing the baking powder by the same proportion. In mathematics, proportionality describes multiplicative relationships between rational quantities and is the basis for rational number operations, basic algebra, and problem solving in geometry (e.g., Fuson & Abrahamson, 2005; Saxe, Gearhart, & Seltzer, 1999; Sophian, Garyantes, &

Chang, 1997). "The ability to reason proportionally develops in students [between] Grades 5 [and] 8. It is of such great importance that it merits whatever time and effort that must be expended to assure its careful development" (NCTM, 1989, p. 82). Some of the mathematical concepts relating to ratio and proportion include direct and indirect relations, linearity, rate of change, and scaling.

Proportional reasoning can be seen as analogical reasoning with quantities – both conceptual analogies and quantitative proportions require students to analyze the relations between relations (Boyer, Levine, & Huttenlocher, 2008). Although proportional reasoning is generally thought to develop in the later elementary grades, there has been disagreement in the research literature as to the age at which children are first able to use proportional reasoning successfully. It is a complex construct that varies according to number structures and context. Some studies have shown evidence for (somewhat modified) proportional reasoning in the early elementary years (e.g., Goswami, 1989; Sophian & Wood, 1997), but other studies support proportional reasoning only as a later achievement, after age eleven (e.g., Fujimura, 2001; Schwartz & Moore, 1998). Younger children can reason proportionately if the quantities involved are continuous rather than discrete (Spinillo & Bryant, 1999; Jeong, Levine, & Huttenlocher, 2007).

One type of strategy for dealing with problems involving proportional reasoning is multiplicative: the terms in the ratio are related multiplicatively. The first ratio is determined to be a:b, where b is a multiple of a. This relation is then extended to the second ratio. This is what we do when we double a recipe: everything is multiplied by two. This classical comparison of ratios underpins almost all the number-related concepts that are studied in school, including fractions, percentages, ratios, proportion, rates, similarity, trigonometry, and rates of change (Mitchelmore, White, & McMaster, 2007). Another strategy is called building-up and involves establishing the relationship of one ratio and extending that relationship to the second ratio by addition. This strategy is the dominant one observed in the majority of elementary students (Tourniaire & Pulos, 1985). Students apply both correct and incorrect strategies when attempting problems involving proportional reasoning.

CONCLUSION

Children begin their exploration of mathematics with a natural desire to discover the world around them. At a young age, they are curious, creative, and inquisitive risk-takers who use mathematics as a means to understand their surroundings.

Research has shown that young children require high quality, challenging, and accessible math education experiences in order to build a strong foundation for their future learning. What children are ready to learn in mathematics depends largely on their previous opportunities. In general, children learn by building on prior knowledge, and mathematics is particularly additive in nature. Concepts build on one another, so that early misunderstandings will impede further learning. A strong foundation in mathematics means a bright future: children's mathematics ability at the beginning of Kindergarten is a strong predictor of later academic success, even stronger than their early reading ability.

When children transition to school, they integrate their own intuitive understanding of mathematics with the new information from the more formal education system. As they progress through the elementary grades, children solidify the knowledge they have gained in the early years and deepen their conceptual understanding. They face new challenges, use new skills, and achieve understanding of new concepts.

Mathematics educators who have a good knowledge of their subject and who can put this knowledge into action in the classroom will be able to guide, support and augment children's developing understanding of mathematics. Educators' knowledge, behaviours, and attitudes related to mathematics are vital for student success. Educators have a great responsibility to provide children with a strong foundation in mathematics and thus enhance their chances for later academic success.

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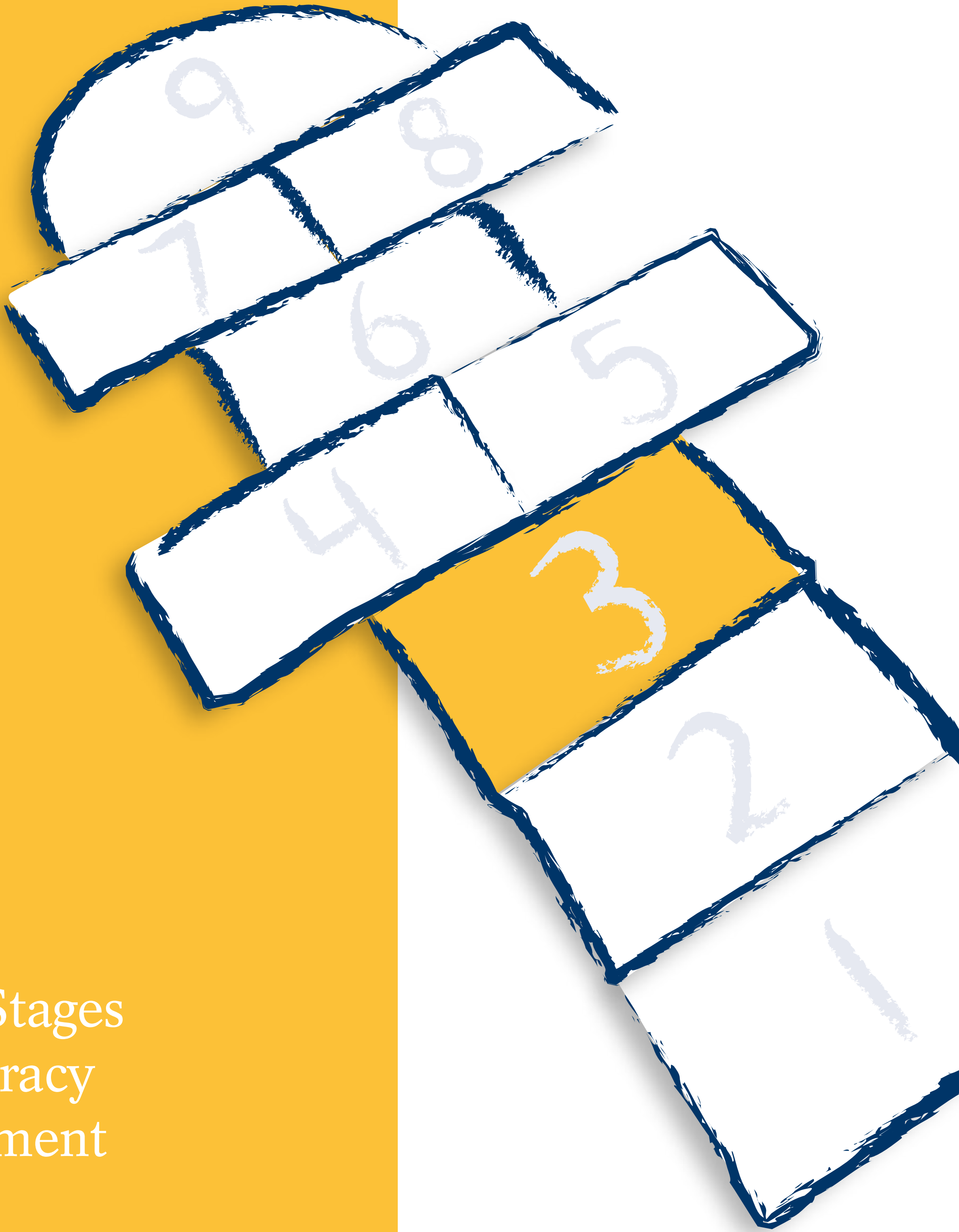


3

Ages & Stages of Numeracy Development

AGES & STAGES OF NUMERACY DEVELOPMENT

Although all children follow similar developmental paths, every child is different and develops numeracy and quantity skills in different ways and at different times. Most children follow a path similar to that of Sara, the child highlighted in the description below.

Birth to About 12 Months Old

As an infant, Sara naturally shows the beginnings of what will later develop into an understanding of numeracy and quantity. Less than a day after birth, Sara has a certain awareness of small quantities and can tell the difference between a picture containing two dots and a picture containing three dots. Sara can also quickly "see" up to three items and knows that a group of three toys is different from a group of two toys or one toy. As she grows, her understanding of quantity also grows. By the time Sara is five months old, she knows that a bottle that is half full of juice is somehow different from a full bottle of juice. Sara is also surprised to see three toys if she didn't see someone add one to the two she had before. Her ability to perceive larger quantities also improves. Sara knows that a group of eight toys is different from a group of 16 toys, even though she doesn't yet know what the difference actually is.

About 12 to 24 Months Old

Around her first birthday, Sara's ability to perceive large quantities gets even better. Now Sara can tell that a group of eight toys is different from a group of ten toys, two sets that are almost the same size. She also understands that number words are important and is attracted to play and speech that involve counting and number words. Sara learns her first number words and spontaneously uses those number words when she plays. For example, Sara starts to label her toys with words like "two," even though she doesn't understand the mathematical meaning of the word "two." Sara can also show "one" and "two" on her fingers.

About Two to Three Years Old

Sara's interest in number words continues to increase throughout her preschool years. Her vocabulary of number words quickly increases, and Sara produces strings of number words such as "one, two, three, five." Even though she doesn't use them in the correct sequence, Sara understands that order is an important aspect of number words, and she consistently says them in the same order when she counts. She can also identify the "first" and "last" child in line. By three years old, Sara learns how to recite the number words from 1 to 10, even though she may not understand the significance of these words. Sara has an intuitive feel for basic arithmetic with small numbers. For example, she knows that if one candy is put together with two candies, there should be three candies, and if one candy is taken away from two candies, only one candy should be left. Sara can also divide up eight toys equally between her friend and herself.

About Three to Four Years Old

Towards four years of age, Sara can count up to 30 and backwards from 5. In general, her counting sequence is longer and more accurate. Sara realizes that the last number word used to count a group of objects tells how many objects there are. She can show the numbers 1 to 5 with her fingers and uses words such as "first," "second," and "third" more often. Sara can now divide ten toys equally into a larger number of shares, such as among five children. She also knows that if you add sand to a pile, the pile should look bigger, and she can recognize which of two piles has had more sand added to it. Sara starts exploring length and can measure length by comparing two objects when they are placed next to each other.

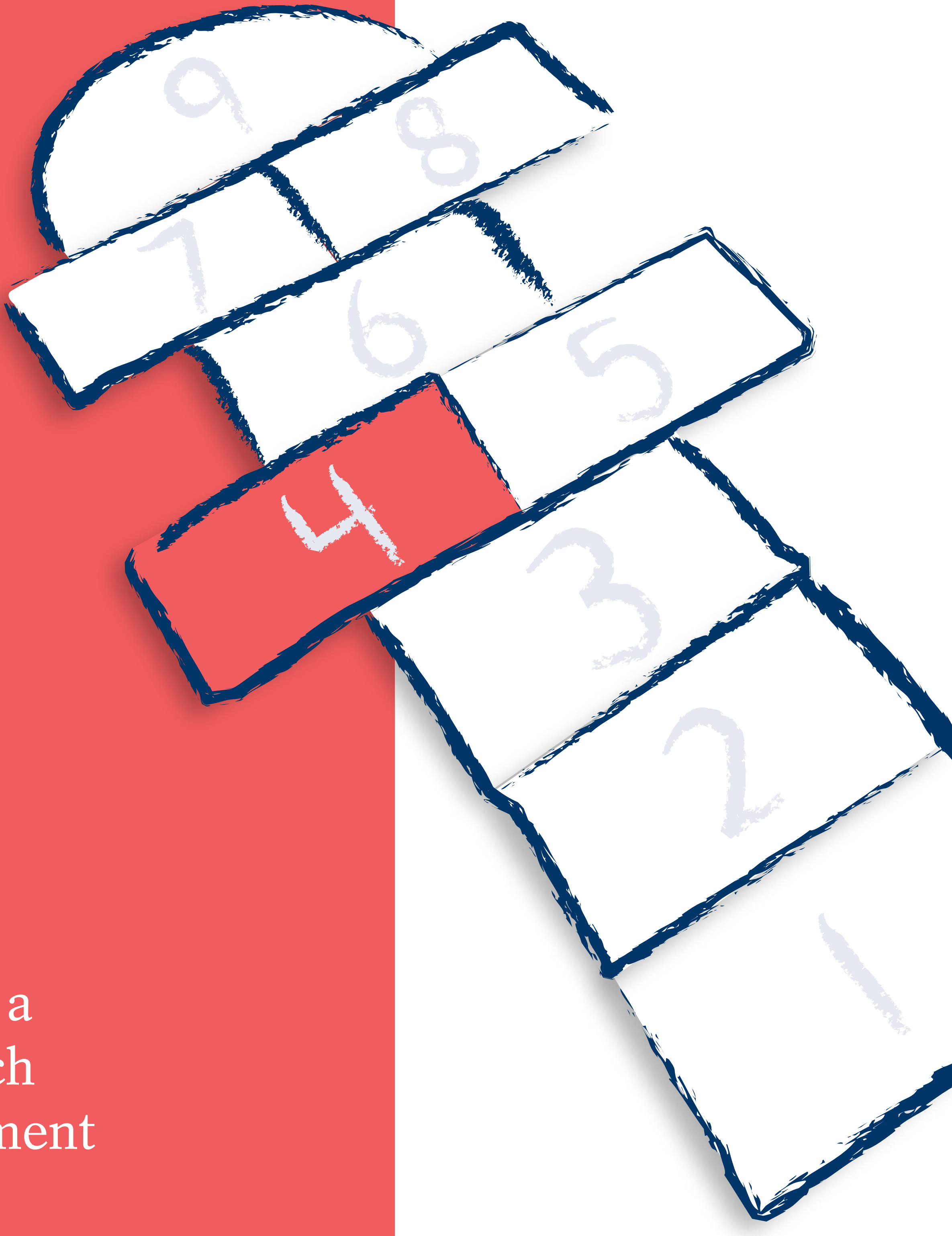
About Four to Six Years Old

Sara's counting and quantity skills continue to develop quickly and become more abstract. She can now count up to 100, skip count first by 10s (10, 20, 30…) and later by 5s and then 2s. She also starts counting up starting with numbers other than 1 ("8, 9, 10…"). She is starting to think about the relationship between individual items (like popsicle sticks) and groups of ten items (e.g., a bundle of ten popsicle sticks). She is forming the foundation for learning place value and other math concepts when she gets to school. Sara reliably produces the correct answers to addition and subtraction word problems with sums to about 5. For example, one problem Sara could solve would be: "I had three dinosaurs and I got two more for my birthday. Now how many do I have?" She knows the doubles up to 10 (1 + 1 is 2, 2 + 2 is 4…) and can divide 100 things equally among ten children. Sara can quickly tell that there are five objects on the table without counting them, and she recognizes common patterns up to 10, like dots on number cubes or dominoes. She can use several different ways to measure, compare, and reproduce the length of objects. Sara also talks about her comparisons of quantities by using words such as "taller," "shorter," "skinnier," "fatter," and "wider."

What Sara's Parents and Child Care Practitioners Did to Help

Sara's development of quantity and numeracy skills is typical of most children. She has intuitive understanding of quantity, as well as basic concepts of arithmetic, such as addition, subtraction and division. She doesn't necessarily know the formal ways to represent these concepts (such as 2 + 2 = 4), but she knows that two dolls added to the two dolls she already has gives her four dolls. Her understanding of many quantity concepts like measuring and counting is similarly informal and intuitive.

Sara's parents and child care practitioners supported her development by talking about numbers and quantities in the real world. They engaged her in conversations about towers that are taller than others, days that are hotter than others, and trees that are closer than others. They talked about guessing the number of people on the bus, about knowing that  on a number cube represents four dots, and about identifying the third person in line. In their conversations with her, they used rich language that helped her to think about such attributes as height, temperature and distance, and to compare quantities in the world around her. Also, Sara's parents had a lot of board games, as well as dominoes and playing cards at home. These games showed Sara that numbers can be represented in a variety of ways. By talking about these representations with her parents, Sara was able to make important connections between number and quantity.



4

Creating a Math-Rich Environment

CREATING A MATH-RICH ENVIRONMENT

As early childhood educators, we don't need to look far when we want to create a math-rich environment, since math is all around us. The beauty of math is that we do not need any special materials to engage children in the counting process and other activities related to mathematical concepts. What we do need to do is look at the environment in a numerical way. Then we discover that a math-rich environment is at our fingertips: chairs can be counted, tables represent shapes, toys can be grouped into sets based on similar characteristics, each child requires one plate and one cup for lunch… and the list goes on. By looking at our surroundings in a different way, a numerical way, we can encourage a natural and holistic approach to numeracy.

The best way for educators to approach math with young children is to make it a meaningful part of their day. Mathematical concepts can be interwoven with routines and transitions to provide opportunities throughout the day for involving children in the use of number concepts.

Arrival at the Program

When the children arrive at their early learning and care program, you can take attendance by counting out loud the number of children present and those who are absent. You could involve the children by using name cards: the names of children present would go in one pile and of those absent in a second pile. You could further develop the idea by classifying the cards of the absent children by gender, to see how many girls and how many boys are away. You can use this opportunity to explore concepts of more or less, meaningful counting, sets of present children and absent children, and even compare the set of absent children in the current week to that of the previous week.

Preparing for Lunch

The process of getting ready for lunch provides an excellent opportunity to involve children in practising counting, estimation, sets, and one-to-one correspondence. After counting how many children are present on a given day, the children must determine how many cups, plates, napkins, and sets of cutlery are needed. Be sure to give children time to decide what will be needed and let them use whatever strategy they prefer. If they have made an error, you don't need to point it out. Let them discover through trial and error. When they sit down to eat, if something is missing, you can ask the children to help solve the problem.

They can also practise estimating how much food will be required. For example you might say, "We usually have four bananas for our group. Today we have two fewer children. I wonder how many bananas we will need?"

When the food is being passed around, you can ask the children to count out three carrots each and then pass the dish on to the next person. If rice is in a large bowl, each child can measure out two heaping spoonfuls onto his or her plate. You could also facilitate a discussion on the amount of food available. "It looks like we have run out of milk. Ali, could you please go to the kitchen with Ella and ask for one litre of milk?"

If it is possible to do cooking activities with the children, they will have opportunities to measure the ingredients, count the spoonfuls and divide up the completed dish into equal portions at the end. You can introduce the idea of patterns by making fruit kabobs. Put pieces of fruit on a stick in a repeating order, for instance, apple, grape, banana, apple, grape, banana. Then ask the children to copy your pattern to make their kabob, or make their own pattern. The more children are actively and physically involved in the process, the more they will learn and remember in a meaningful way.

Organization of the Play Environment

When planning the learning environment, you can integrate specifically designed math activities. Some examples are pegs and peg boards, abacuses, unit blocks, Lego, and very simple board games for four year olds. In addition, you can organize a process to help children control the number of participants at the different learning centres. Using a chart with symbols, children can tell if an area is full. For a special activity, you could perhaps implement a wait list with a child's name and number indicating their preferred area.

Children can learn to solve simple practical problems in the play room. For example, you could ask them, "Let's move the computer table to the quiet corner. Before we move it, how do we know if it will fit?" Four year olds can figure out how to use a string or other object to solve this measurement problem.

Here is a scene that shows how an educator can make concepts of whole, half, and equal a natural part of play time for toddlers:

> At the play dough table, David is sitting with a group of three toddlers. David knows that children learn best by playing with materials first hand. He holds a large ball of blue play dough and asks the children if they would each like some to play with and make cookies. The children eagerly watch as David divides the play dough into three equal parts. Each child rolls the play dough flat and then back into a ball and repeats the process. David comments on what the children are doing, Then he asks them if they think they each have enough play dough to make two balls.

In this example, David talks to the children about what they are doing. You help children to grasp and use mathematical concepts when you give them words. These words don't need to be complicated. For instance, here are some everyday words related to the mathematical concepts of measurement and comparison: large, small; big, little; long/tall, short; heavy, light; fast, slow; thick, thin; wide, narrow; near, far; high, low; more, less/fewer. Children will learn to use this vocabulary to describe mathematical relationships they see around them.

Organizing Time

You can use a calendar to help children learn about measuring time. Make the calendar meaningful by marking special days, like holidays and children's birthdays. You can point to the squares on the calendar as you count off the days till the special event. Young children find it difficult to tell time on a clock, but you can use a timer to measure short periods. For instance, you could ask a child to turn the timer to line the arrow up with the number "30" (they will need help with this) and say, "We will have lunch when the timer rings, in 30 minutes. That's as long as… (name a favourite video or TV show)."

Tidying Up

Tidy-up time is an opportune moment to emphasize classification and order as well as problem solving. For instance, when the children are putting away blocks, you can use visual cues to help them learn where unit blocks belong. This is another opportunity for them to understand that it takes two small square blocks to equal the same size as the rectangular block. You can count the books together as they go back onto the shelf (more or less!). Children can also learn to create sets if you ask them to pick up all the blue blocks, for instance, or all the trucks, or all the balls. Vary the criteria you use for the sets: by colour, by size, by shape, by texture, etc.

Songs, Books, and Finger Plays

Songs, books and finger plays are commonly used to introduce and reinforce numbers with young children. Number words are part of many children's songs. The finger play, "Five little monkeys" reinforces counting backwards and subtraction. Children use their fingers to represent the monkeys and eliminate one at a time as the crocodile eats them up. Although very young children do not necessarily understand the logic of subtraction, they become familiar with names of numbers and at some point will make the link to subtraction.

Group Time

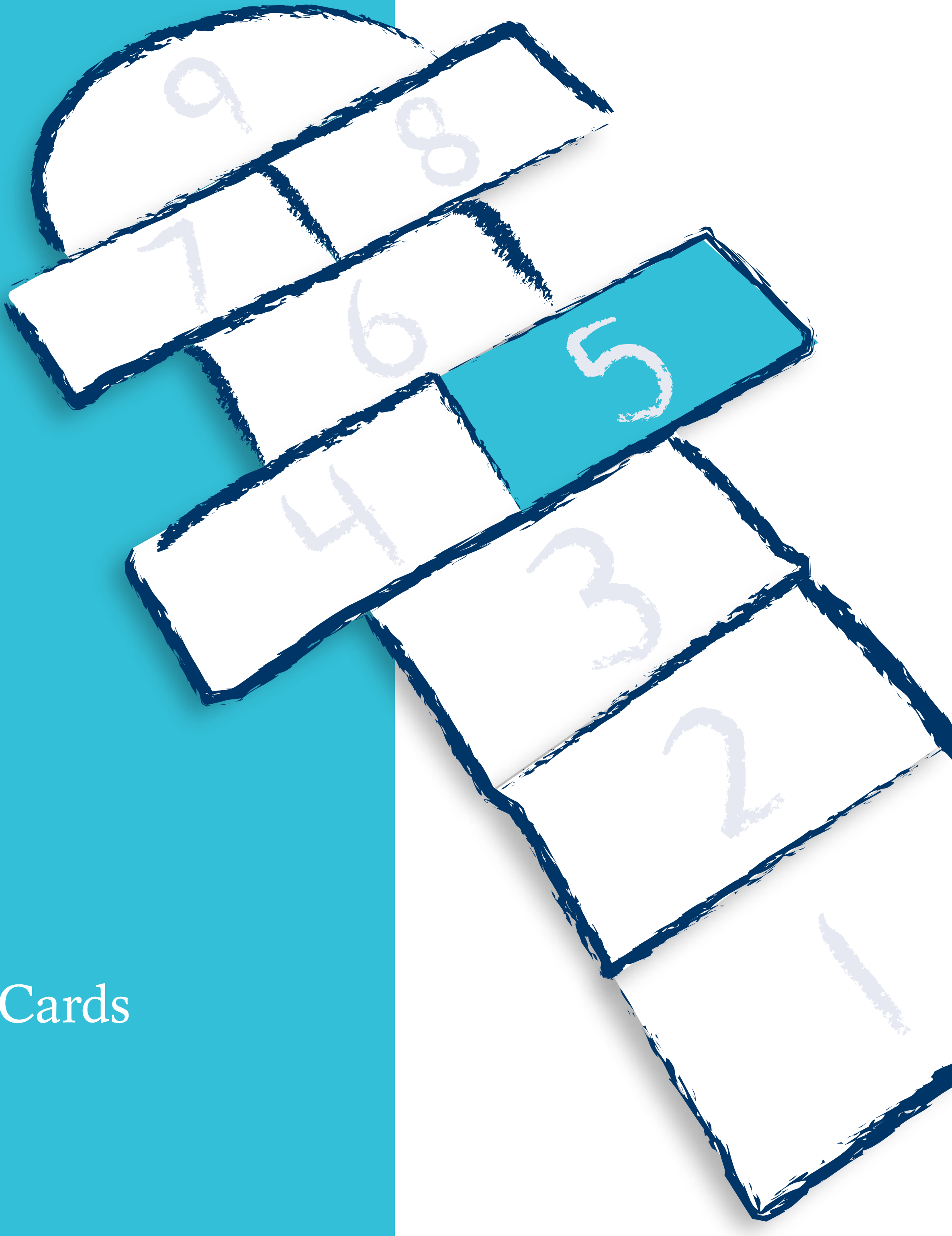
Graphing can easily be discussed and implemented during small group time. You can draw simple bar graphs to illustrate children's likes, their dislikes, colour of hair, kind of pet and favourite colour. You scaffold the learning by asking questions: "Which group is the largest?" "Which group is the smallest?" "Are any groups the same size?" "Since Rosa and Licheng are absent today, what might happen when they add their choices to the graph?" "Would that change which group has the most?"

The Role of the Educator

The best way to ensure that children have confidence in their abilities in math is to convey the message that you yourself embrace math and think that numbers are fun. Encourage children to approach problems in different ways and resist the urge to solve their problems for them. Your role is to instil in children the desire to be flexible thinkers and find different ways of coming to solutions. How the children come to their answers is more important than the end solution. Even when children make an error, they have still learned far more through the process of trying than if an adult imposes a particular strategy. When they make an error, you can talk with them about what may have happened. Let children tell you how they went about solving the problem. Often the error shows up when they put their strategy into words. They are far more likely to remember this information when it has been reinforced through their own actions.

Preparing for Future Success

When children have many experiences like these behind them, mathematical concepts such as wholeness, half, equal, addition, subtraction, and patterns become second nature to them as they move through the preschool years and toward kindergarten. Because they have had the opportunity to work with the concepts in everyday activities, these concepts simply make sense.



5

Activity Cards

ACTIVITY CARDS

The following are some activities to support learning about mathematical concepts. They can be done in a group learning setting with children aged approximately three to five. After you have completed the activity, take a moment for reflection on how you presented it, how it was received and what possible changes or extensions you may try next time. There are some suggested extensions of the activities written on the reverse of each card.

The cards have been designed to remain in the resource or if you prefer, they can be cut on the perforated edge and used as individual cards.

Enjoy!

NAME OF ACTIVITY

Big to Little

I

MATH CONCEPT
Seriation

OBJECTIVES
Order objects from little to big within a variety of categories

MATERIALS
Three or more objects of different sizes in each category. Possible categories include: dolls, pencils, pots, spoons, stuffed animals, coins, hats, cups, cans, books, trucks, jars, etc.

METHOD OF PRESENTATION
Give the children three different sized items from one category and ask them to arrange the items from largest to smallest. If a child has difficulty with three items, ask him or her to compare two items from the same category. When the children have practised a few times and are successful, vary the activity by asking them to put the same items in order, heaviest to lightest. You can also ask them to come up with their own criterion for ordering the items.

NAME OF ACTIVITY

Three Billy Goats Gruff

II

MATH CONCEPT
Topology, ordinal position, geometry

OBJECTIVES
Size discrimination; understanding first, second and third; understanding rectangular and triangular shapes

MATERIALS
A storybook of "The Three Billy Goats Gruff"

METHOD OF PRESENTATION
Have the children sit in a circle with their feet in front of them and knees bent. Read the "Three Billy Goats Gruff". While reading, ask the children to make sound effects that correspond to the size of the goats. For example, for the small goat, they could tap their feet lightly on the floor once. For the middle goat, they could make two louder thumps. For the largest goat, they could stomp their feet loudly three times. Make the words "first," "second," and "third" stand out. Ask the children which billy goat they think would be the loudest and which one would be the quietest.

NAME OF ACTIVITY

Grapes for Everyone

III

MATH CONCEPT
Whole numbers and fractions; estimation

OBJECTIVES
Recognize separate parts of a whole object

MATERIALS
A bunch of grapes on the stem

METHOD OF PRESENTATION
Show the children a bunch of grapes on a stem. Talk about the different parts of the bunch. Ask the children, "How is one grape separated from another? Do you think we can each have one grape? Do you think there is enough for two each?" Invite the children to take turns pulling grapes off the stem. Find out if each child has two grapes. Reinforce vocabulary such as "parts" (grapes) and "whole" (a bunch of grapes on a stem).

** For children under 5 years, grapes should be cut lengthwise before eating*

EXTENSION IDEAS
As children become competent using three items at a time, increase the number of items and use less obvious ones. Depending on the characteristics of the items, you can vary the criterion for sorting (e.g., shortest to tallest, thickest to thinnest, etc).

Big to Little

EXTENSION IDEAS
Build a bridge with large hollow blocks of different shapes.
Use problem-solving techniques with the children to build the bridge so that it slopes upward at the start and downward at the end. Discuss the geometric shapes of the blocks – rectangles, triangles, and squares. Give the children an opportunity to hear the words related to ordinality (first, second and third) as they play the part of each billy goat.

Three Billy Goats Gruff

EXTENSION IDEAS
Have the children count the total number of grapes. You can also have the children estimate the total number of grapes in a bunch or the number of grapes each person will have. Another idea would be to cut a whole fruit (e.g., banana or pineapple) into pieces and have the children estimate the size of each child's piece. Ask the children, "If we divide this whole banana so that each child at our table gets a piece, how many pieces do we need? How big should each piece be?" An unsliced loaf of bread can also be used to investigate the same concepts.

Grapes for Everyone

NAME OF ACTIVITY

Circle Song

IV

MATH CONCEPT
Geometry

OBJECTIVES
Recognize differences between a circle, a square and a triangle

MATERIALS
None

METHOD OF PRESENTATION
Sing the words below to the tune of "Farmer in the Dell." At the same time, make large circle motions in the air with your arms.

The circle goes around,
The circle goes around,
A great big ball, it never stops,
The circle goes around.

Sing this song several times and ask the children to follow you as you make smaller circles, then bigger circles. Pause and ask the children to think about what they do with their arms to make circles. Ask the children, "Do your arms stop or do they always keep going? Do they make corners?" Sing the song and make the arm circles again, then ask the children what they have observed.

NAME OF ACTIVITY

Printing Patterns

V

MATH CONCEPT
Patterning; geometry

OBJECTIVES
To detect the rule in a pattern and then follow the rule to continue the pattern (i.e., an arrangement of items in an order that repeats itself)

MATERIALS
Paint, different shapes of cookie cutters, paper

METHOD OF PRESENTATION
Demonstrate a pattern by using two cookie cutters of different shapes dipped in red paint. Print a line of shapes, alternating between the two. After several repetitions of the pattern, ask the children which cookie cutter you should use next. Then show a more complicated pattern using three different shapes of cookie cutters.

NAME OF ACTIVITY

Number Scavenger Hunt

VI

MATH CONCEPT
Numeral recognition; number recognition; one-to-one correspondence; cardinality

OBJECTIVES
Recognize the numerals 1 to 5, understand one-to-one correspondence

MATERIALS
Numerals 1 to 5, each written on a separate sheet of paper and laminated; small laminated pictures of items that could be found in a park, such as pine cones, leaves, sticks, stones (15 pictures of each type of item)

METHOD OF PRESENTATION
Day 1: Show the children laminated numerals and ask them to say the numeral names (e.g., one, two, three). Show the children two individual pictures of stones. Have the children count the number of stones and place the stones on the back of the corresponding numeral (numeral "2"). Repeat the same process with each numeral and the corresponding number of pictures of different items. Ask the children questions about how many pictures are needed to match each numeral. Get them to choose the correct number from the pile of pictures.

Continue at the back on this card

EXTENSION IDEAS
Try varying this activity by singing a verse about squares and a verse about triangles.

Squares have four straight sides,
Squares have four straight sides,
Over, down, across and up,
Squares have four straight sides.

Triangles have three sides,
Triangles have three sides,
Up the hill, then down and back,
Triangles have three sides.

While singing, ask the children to follow you as you trace the shape in the air with your arm. You can also have children lie on the floor to make the shape (four children for a square, three for a triangle) and walk around them while you sing. Ask children to notice the characteristics of the shape as they make it, either with their arm movements or by walking. Exaggerate the turning of the corners and make the number of sides stand out when you sing.

Circle Song

EXTENSION IDEAS
Incorporate geometry into the activity by using cookie cutter shapes (e.g., triangle, circle, rectangle, etc.). The pattern can be varied by using the same shape of cookie cutter, but changing the colour of paint. Have the children try making their own patterns. Another idea is to use an overhead projector and geometric shapes cut out of cellophane. Ask the children to make a pattern by placing the cellophane shapes on the glass of the projector, then project the pattern on the wall. Discuss each child's pattern.

Printing Patterns

Day 2: Repeat the game from the day before to reinforce one-to-one correspondence and cardinality. One-to-one correspondence means that when we count, we assign one number to each item and do not go back to any item twice. Cardinality refers to the idea that the last number word in a count is the answer to the question, "How many are there?" The more practice children have with these concepts, the more confident they will become and the better prepared for the actual scavenger hunt.

Day 3: On the back of each laminated numeral, attach the corresponding number of pictures of one item. Hide the pages in the area (playground, park, etc.) where you will be taking the children. Once you arrive, ask the children to find the hidden numerals. When they find the numeral, have them identify the item on the back. The children must look for the same number of real objects in the park. Finish the game by counting the items with the whole group.

EXTENSION IDEAS
When children become confident with the numerals 1 to 5, you can add 6 and 7. Vary the items that you associate with each numeral.

Number Scavenger Hunt

NAME OF ACTIVITY

Broad Jump

VII

MATH CONCEPT
Estimation and measurement

OBJECTIVES
Introducing estimation and measuring in metres and centimetres

MATERIALS
Measuring tape; an open space where children can safely perform a standing broad jump, preferably in a large sandy area

METHOD OF PRESENTATION
Demonstrate a standing broad jump to the children and show them how to use a measuring tape to measure the length of your jump. Get the children to take turns jumping and measuring their jumps. On the next turn, before measuring, ask the children to estimate how far they have jumped. Write down the children's estimations and then measure the actual length. Compare the estimated length to the actual measured length of the jump.

NAME OF ACTIVITY

Fish in a Bucket

VIII

MATH CONCEPT
Numeral recognition and meaningful counting

OBJECTIVES
Match the numeral name to the written numeral

MATERIALS
Five fish cut out of lightweight cardboard and numbered 1 to 5 on both sides; two paper clips attached to the mouth end of each fish; five paper cups also marked 1 to 5; a "fishing pole" (tie a string to the end of a stick and attach a magnet to the end of the string)

METHOD OF PRESENTATION
Show the child how to use the "fishing pole" to catch the fish by touching the magnet at the end of the string to the paper clips on the fish. Ask the child to catch the numbered fish and put them in the cup that has the same numeral written on it. Afterwards, review the activity with the child and encourage him or her to name the numeral on each cup and fish.

NAME OF ACTIVITY

Graphing

IX

MATH CONCEPT
Data representation

OBJECTIVES
Represent data using concrete objects

MATERIALS
Two types of food, for instance, apple slices and orange slices

METHOD OF PRESENTATION

Let each child taste each type of food, for instance, give them a small piece of apple and a small piece of orange. Ask the children which fruit they prefer. Put an apple on one plate and an orange on the other. Put each plate on a chair and have the chairs side by side. Ask the children to line up in single file in front of their favourite fruit. Try to have the children stand the same space apart in each line. With the two lines standing side by side, the children have made the equivalent of a bar graph. Ask, "Are the lines the same length? Are there more people in one line than in the other? Do more children like the apple or the orange? Do we need to count each person to know which fruit is more popular?"

EXTENSION IDEAS

Show the children how to represent measurements graphically. Use the lengths of their jumps to make a vertical or horizontal bar graph. Put the children's names under (or next to) their bar and write the measurement in metres along the other dimension.

Broad Jump

EXTENSION IDEAS

Say the numeral name and ask the child to pick up the corresponding fish. As children become proficient, you can make fish and cups numbered from 1 to 10. Try varying the activity by taking away one or two fish before the child starts fishing. When the child has finished, ask him or her to say the numeral names of the fish that are missing. If ask the child does not know at first, ask him or her to look at the empty cups.

Fish in a Bucket

EXTENSION IDEAS

Show the children how to make a bar graph on paper to chart the same information about their preferences. At the bottom edge of a large sheet of flip chart paper, draw a picture of an apple and a picture of an orange, side by side. Above each picture, draw columns to the top of the page. Make small paper rectangles that will fit into the columns and mark them with the children's initials. Give the children their rectangle and ask them to take turns sticking the rectangles in the column above the fruit that they prefer. Encourage them to take the time to carefully place the rectangles end to end and between the column guidelines. Talk about how this graph is the same as their lines in front of the plates of fruit.

Graphing

6

Supporting Early Numeracy: Resources for Child Care Practitioners

Recommended Online Resources

A Good Start to Numeracy: Effective Numeracy Strategies from Research and Practice in Early Childhood
By: Brian Doig, Barry McCrae, and Ken Rowe
Available at: www.dest.gov.au/sectors/school_education/publications_resources/profiles/good_start_to_numeracy.htm

This 63-page literature review was carried out for the Australian government. It gives an overview of the research and practice in early childhood numeracy on children between birth and eight years of age. The emphasis is on examining the research literature for effective strategies and practices. Summaries of these, along with suggested activities for child care and classroom settings, are presented at the end of each section.

Building Blocks
By: Douglas H. Clements and Julie Sarama
Available at: http://gse.buffalo.edu/org/buildingblocks/index_2.htm

Building Blocks is a research-based mathematics program that uses technologically-enhanced materials for young children in pre-K through Grade 2. The program focuses on two main topical areas: (a) spatial and geometric competencies and concepts and (b) numeric and quantitative concepts. The name *Building Blocks* was chosen because the technology emphasizes the key ideas, or building blocks, upon which mathematical knowledge is built. Accordingly, the program's computer activities situate key mathematical ideas in everyday situations and engage children in mathematical procedures that are connected to these concepts.

Creative Pathways to Math: Development of Mathematical Concepts
By: Douglas H. Clements and Julie Sarama
Available at: http://www2.scholastic.com/content/collateral_resources/pdf/ECTonline/developmentofmathematicalconcepts_1_2_03.pdf

This developmental chart outlines what children can understand about numbers, shapes and measurement at ages three, four, and five.

Encyclopedia of Language and Literacy Development
Available at: http://literacyencyclopedia.ca/

This web-based resource developed by the Canadian Language and Literacy Research Network (CLLRNet) helps provide answers to questions about children's language, literacy, and numeracy development. Early learning childcare practitioners can draw on the *Encyclopedia* for reliable, evidence-based information to support their daily practices. The *Encyclopedia* includes an extensive section on numeracy development, with contributions from leading numeracy researchers around the world. For sample entries, see chapters on *Acquisition of Early Numeracy and Basic and Environmental Processes Underlying Numeracy Acquisition.*

Fostering Early Number Sense
By: Arthur J. Baroody
Available at: www.excellence-earlychildhood.ca/documents/Arthur_BaroodyANG.pdf

In this keynote presentation that took place at the 40th Annual Banff International Conference on Behavioural Science *Effective Early Learning Programs: Research, Policy and Practice* in March 2008, the author concludes that verbal skills are important in fostering early understanding of numbers.

Math Play: How Young Children Approach Math
By: Douglas H. Clements and Julie Sarama
Available at: http://www2.scholastic.com/browse/article.jsp?id=3747373

The authors present the case that mathematical experiences for very young children should build largely upon their play and the natural relationships between learning and life in their daily activities, interests, and questions. They give examples and make concrete suggestions on how to put these principles into practice in the early childhood classroom. Look for more ideas in other articles on this website by the same authors.

Number Worlds
By: Sharron Griffin
Available at: http://clarku.edu/numberworlds/index.htm

Number Worlds is a research-based mathematics program for young children developed by Sharon Griffin, a researcher at Clark University in the United States. This program teaches specific math concepts and skills that are the foundation for later mathematical learning. It provides hands-on games and activities that encourage children to construct their own mathematical meanings. The website for *Number Worlds* includes sample activities, videos of classroom practice, as well as sample assessment tools.

Numerical Knowledge in Early Childhood
By: Catherine Sophian
Available at: www.child-encyclopedia.com/pages/PDF/SophianANGxp.pdf

This article in the online *Encyclopedia of Early Childhood Development* discusses the variability and malleability of young children's numerical thinking, which indicates the potential for early childhood instructional programs to contribute substantially to children's growing knowledge about numbers.

PreKorner™ Early Childhood Numeracy Resources
Available at: www.designedinstruction.com/prekorner/early-numeracy.html

This section is part of the website of *Designed Instruction,* an educational consulting firm in Texas. This website provides background information on early numeracy, with links to research reports and suggestions for age-appropriate, play-based activities.

Recommended Online Activities

Esso Family Math
By: Nancy Chapple, Judi Waters and Linda Adams
Available at: www.edu.uwo.ca/essofamilymath

The *Esso Family Math Project* is a community-based initiative for families who want to help their children experience success in mathematics. It is a research-based program that was developed at the University of Western Ontario. Families learn to use everyday activities and materials to foster learning of mathematical concepts. These activities can be adapted for children in child care settings. Two books can be downloaded in PDF format, one for use with four- to six-year-olds and the other for seven- to ten-year-olds. They include lists of suggested books.

Early Childhood: Where Learning Begins: Mathematics
By: Carol Sue Fromboluti and Natalie Rinck
Available at: www.ed.gov/pubs/EarlyMath/title.html

This booklet, published by the U.S. Department of Education, National Institute on Early Childhood Development and Education in 1999, explains mathematical concepts and gives examples of activities to use with children two to five years old. The emphasis is on integrating activities into play and daily routines. The publication is addressed to parents but ideas are easily adaptable to the child care setting.

Family Math Fun!
By: Kate Nonesuch
Available at: www.nald.ca/library/learning/familymath/cover.htm

This ready-to-use manual of family numeracy activities can easily be adapted for a child care setting. Activities include recipes, rhymes, games and crafts. This manual was created as a result of the collaboration between 30 parents in the Cowichan Valley and Kate Nonesuch, an instructor at Vancouver Island University. The manual can be downloaded in PDF format.

KinderArt
Available at: www.kinderart.com/littles/littles_numbers.shtml

This website offers some free ideas for simple math-related rhymes, craft and kitchen activities, including variations for different ages, illustrated by videos.

Math Dance
Available at: www.mathdance.org

This site focuses on the relationship between movement and learning math. Practitioners will find activities that develop awareness of space, laterality and sequencing.

PBS Parents, Early Math
Available at: http://www.pbs.org/parents/earlymath/

This site, from the U.S. Public Broadcasting System, is aimed at parents, but includes many activity ideas that can easily be adapted to child care settings. Suggestions for creative and fun activities are grouped by age, starting with infants and toddlers. The site also includes simple online games and a list of math-related books for children.

Recommended Print Resources

Canadian Language and Literacy Research Network (CLLRNet), & Canadian Child Care Federation (CCCF). (2009). *Foundations for numeracy: An evidence-based toolkit for the effective mathematics teacher.* London, ON: CLLRNet.

This resource kit is designed for elementary school teachers. It includes a summary of research on the development of mathematics skills, along with practical suggestions/tools to help children succeed in mathematics in Grades K-6.

Charner, K., Murphy, M., & Clark, C. (Eds.) (2007). *The giant encyclopedia of math activities for children 3 to 6.* Beltsville, MD: Gryphon House.

This book is one of a popular series advertised as "written by teachers, for teachers." It describes age-appropriate activities, complete with materials, lists and detailed instructions. An index of math concepts makes it easy to choose activities geared to a broad range of elements that make up numeracy.

Copley, J. M., Jones, C., & Dighe, J. (2007). *Mathematics: The creative curriculum approach.* Washington, DC: Teaching Strategies, Inc.

This very complete manual is part of a comprehensive curriculum for children aged three to five. Readers will find age-appropriate activities based on research into how young children learn mathematical concepts and skills. Activities are both child initiated and teacher led. Some are set up as focused lessons and others are integrated into daily routine and play. Tools to assess learning are also included.

Katzen, M. (2005). *Salad people and more real recipes: A new cookbook for preschoolers and up.* New York: Tricycle Press.

Cooking teaches children about measuring and sequencing. This preschooler friendly cookbook presents healthy and tasty recipes in both words and drawings for non-readers.

Martin, J., & Milstein, V. (2007). ***Integrating math into the early childhood classroom: Activities and research-based strategies that build math skills, concepts, and vocabulary into classroom routines, learning centers, and more.*** **Markham, ON: Scholastic.**

This resource recognizes that three- to five-year-olds learn math concepts best through repetition and practice in daily activities. It offers easy and natural ways to fit math-rich experiences into the daily activities and routines in early learning and child care settings.

Simpson, J. (2005). Circle-time poetry: ***Math: Delightful poems with activities that help young children build phonemic awareness, oral language, and early math skills.*** **Markham, ON: Scholastic.**

Each of the 20 poems in this book is related to a math concept (counting, symmetry, shape, measuring, etc.) and is presented with related movement, craft and language activities, as well as suggested books to extend the learning.

Books for Children

Early learning practitioners are encouraged to ask a children's librarian to suggest suitable books to help facilitate early math skills. Book lists are available online from the *Canadian Association of Young Children* at www.cayc.ca/backissues/promolit.pdf and in the *Esso Family Math and Family Math Fun!* publications noted above. The following titles are just a few examples, with emphasis on award winners and Canadian authors.

Counting Books

For children ages 2 to 6:

Bellfontaine, K. (2008). *Canada 1, 2, 3.* Toronto: Kids Can Press.

For children ages 3 to 6:

McFarlane, S. (2002). *A pod of orcas: A seaside counting book.* Toronto: Fitzhenry & Whiteside.

For children ages 3 to 8:

Organ, B. (2004). *My Newfoundland and Labrador counting book.* St. John's, NL: Creative Book Publishing.

Taylor, C. (2005). *Out on the prairie: A Canadian counting book.* Markham, ON: Scholastic.

For children ages 4 to 8:

Thornhill, J. (1990). *The wildlife 1, 2, 3: A nature counting book.* New York: Simon & Schuster.

For children ages 5 to 8:

Brookes, D. (1990). *Passing the peace: A counting book for kids.* Manotick, ON: Penumbra Press.

Kusugak, M. (1996). *My Arctic 1, 2, 3.* Toronto: Annick Press Ltd.

Story Books

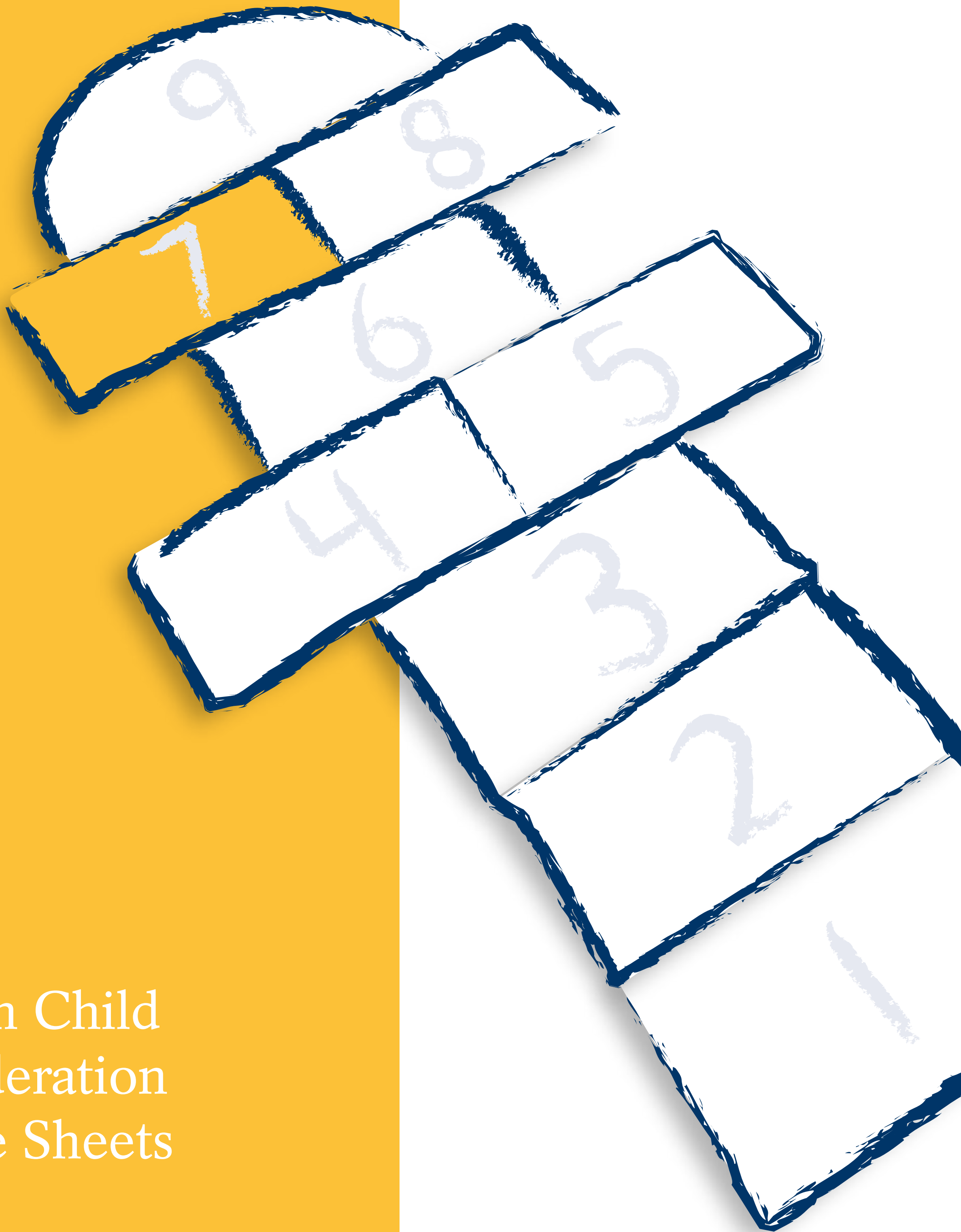
Early learning practitioners should look for stories that involve comparing sizes and arranging items in order, for instance *Goldilocks and the Three Bears and The Three Billy Goats Gruff.* All of these books can be read to children, told as a story, or presented as a puppet play. Older children could be instructed to act out these stories. Some books play with the idea of shapes while they tell a story, for instance, *The Greedy Triangle* by Marilyn Burns and *The Wing on a Flea* by Ed Emberley.

Songs and Rhymes

Rhymes, songs, finger plays and clapping games often include counting, both normally and in reverse (e.g., Five little monkeys jumping on the bed). Because they usually involve lots of repetition, they also teach children about patterns and sequences.

Recommended Materials for Early Numeracy

Apart from general craft materials and toys, graduated blocks are high on the list of recommended materials for discovering and exploring math concepts. Other math related items can be found in the catalogues of suppliers of products for child care and early learning settings. Even with a small budget, early learning practitioners can find many opportunities to practice math throughout the day by using different shapes, sizes, numbers and patterns. It is easy to introduce math concepts to children!



7

Canadian Child
Care Federation
Resource Sheets

Canadian Child Care Federation
Leaders in Early Learning

#94

Math with Kids is Fun!

Have you ever sung a counting song with your baby? Asked your toddler, "Which tower of blocks is higher?" Said, "One for you, one for your sister and one for me, as you passed out apple slices?" If so, you've been preparing your child for future success in studying mathematics at school.

Literacy means being able to read and understand words. Numeracy is an understanding of numbers and an ability to reason with them. Like literacy, numeracy starts in the very early years. Infants as young as six months can tell the difference between a pile of 12 toys and a pile of 24 toys. As a parent, you build on this understanding when you introduce words like "more" and "less." In an informal way, you are laying the groundwork for the concepts of addition and subtraction.

Basic principles

Here are some principles to keep in mind when introducing children to numbers.

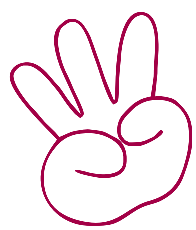
- Children learn through play. Keep an **attitude of play**, and follow what the child is interested in.
- Children learn **through their senses**. Use **real objects** they can see and touch.
- **Repetition** is the key to understanding. Take advantage of events that happen in **everyday routines** to make children aware of numbers and shapes that are all around them.
- Children's abilities develop **slowly over time**, and each child develops at a **unique pace**. **Wait till a child is ready** before introducing more complex concepts.

A good foundation

You can use the following activities and opportunities to build a foundation that will prepare children for school. You don't need any complicated equipment. You can count anything, starting with your child's two hands!

Vocabulary - Children need to know the words for mathematical ideas, and not just the numbers (one, two, three....). Talk to them about size (a big truck, a small ball), about quantity (a full cup, an empty bowl) and order (your turn first, my turn second). Songs and finger plays are fun ways to repeat these words over and over.

Counting - A four year old might be able to say the numbers up to 30, but chances are he can only think *logically* about five objects. It takes practice for children to learn that counting means assigning one number to each object and that the last number named is the number of objects in the group. Start early to develop this awareness with a game of "Simon Says": "Simon says, take two steps forward. One. Two."? When you read a picture book, point to similar items on the page: "I see three trees. One. Two. Three. How many birds do you see?" For older children, cooperative board games give practice in moving a marker as many squares as the dots on a die.



Shape recognition - Craft activities are a chance to talk about geometric shapes: "Here's a circle for the face. Can you choose two circles for the eyes?" Help children get familiar with the shape of number symbols by using play dough to make the numbers from one to five for them to trace with their finger. When you take a walk together, point out the numbers on the houses you pass.

Comparison - Get children interested in comparisons by talking about *them*: "Your fingers are longer than the baby's." "Your pants are longer than your shorts." "You and I have the same number of toes. Let's count them."

Sequence - Putting things in order is an important mathematical skill. Your children can practise this by doing simple clapping games. For instance, take turns setting a short pattern of slow and quick claps. The other person must repeat the same pattern. You can also practise putting things in order of size. For example, making a row of cans from the tallest to the shortest.

Matching and grouping - You can combine matching and grouping with household chores. Get your children to help sort socks into pairs. When it's time to put away toys, suggest they put all the blocks into one box and the toy cars into another.

Measuring - At first, children can measure things with their bodies. "How many times can you put your hands across the book?" Show them how to place the second hand next to the first hand, not on overlapping. Cooking together provides lots of opportunities for measuring, though you might want to have your preschooler put that spoonful of salt into a small bowl before adding it to your sauce, just in case his measurement skills aren't yet accurate!

Get inspired and make up your own activities to enrich playtime and your family routine. With the attitudes that math is fun, your children will be on the road to future success with handling the mathematics of daily life.

Prepared by Elizabeth (Betsy) Mann for the Canadian Child Care Federation and the Canadian Language and Literacy Research Network. Permission is not required to make photocopies for public education purposes. Photocopies may not be sold. To purchase this resource sheet, contact the Canadian Child Care Federation, 201-383 Parkdale Ave., Ottawa, Ontario, K1Y 4R4. Phone (613) 729-5289 or 1 800 858-1412. Fax (613) 729-3159. Email: info@cccf-fcsge.ca. Website: www.qualitychildcarecanada.ca.

Ages & Stages of Numeracy Development

Newborn to 4 months old

- Can tell the difference between a picture of two dots and a picture of three dots.
- Can immediately “see” that there are two or three dots on a page, even though the ability to count is not yet developed.
- Shows surprise when a puppet jumps more times than they are used to seeing.

5 – 6 months old

- Can tell that a jar that is half full of juice is different from a jar that that is full
- Shows surprise at three toys when there are only supposed to be two toys.
- Can tell the difference between two large sets of toys if one of the sets is at least twice as large as the other; for example, can see that a set of 12 toys is different from a set of 24 toys.

9 – 12 months old

- Can tell the difference between two large sets of toys even if the sets are almost the same size; for example can see that a set of eight toys is different from a set of ten toys.

12 – 18 months old

- For small sets of blocks, can learn to pick the smaller of the two sets.

2 years old

- Can learn some number words.
- Knows that number words are important.
- Labels toys with number words.

2 – 3 years old

- Knows that when one candy is taken away from two candies, one candy is left.
- Knows that when one candy is added to two candies, there should be three candies altogether.
- Tries to count using number names even though the number names are often not in the correct order.
- Uses number words in the same order every time when counting objects, even though the number words are not necessarily in the correct order.
- Can learn to recite the number words 1 to 10.
- Can represent 1 and 2 with finger patterns.
- Can divide up eight toys between two children by using a “one-for-me, one-for-you” strategy.
- Learns to pick out the “first” and “last” person in a line.

3 – 4 years old

- When counting objects, knows that the last number word spoken answers the question “how many are there?”
- By the age of three and a half, reliably gives correct answers to addition and subtraction problems involving small quantities, for example 1 + 2 and 3 - 2, by using concrete objects

(manipulatives) or by pointing to a picture of the correct answer; for example, when given ▲ ▲ joined to ▲, can point to ▲ ▲ ▲.

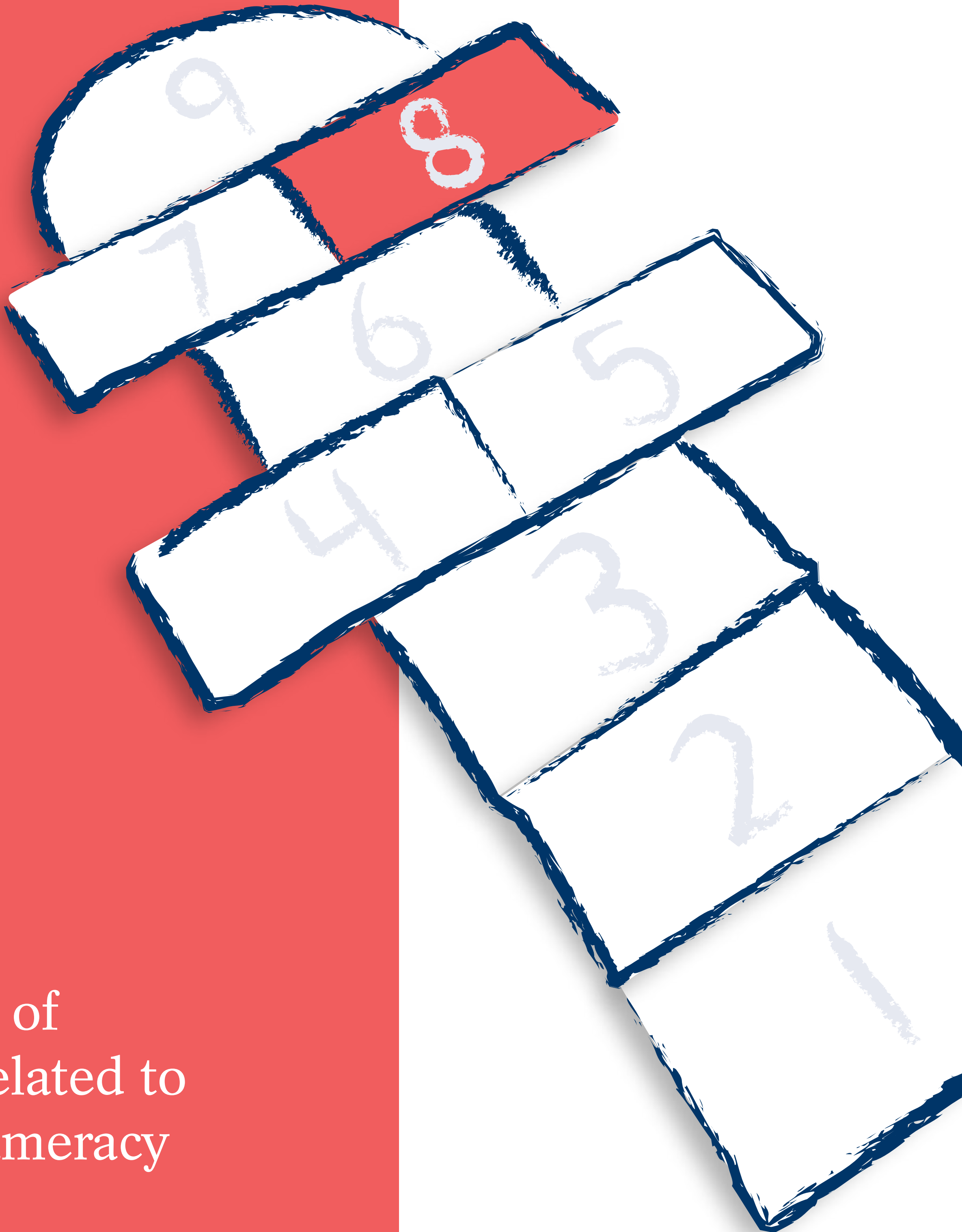
- Knows that a pile of sand should look bigger when more sand has been added to it.
- Recognizes one-digit numbers.
- Can share ten toys equally among five children and knows that each child has an equal share.
- Can learn to count from 1 to 30.
- Measures length by directly comparing two objects, for example, “This book is as long as my arm.”
- Represents 5 using a finger pattern.

4 to 5 years old

- Learns to count backwards from 5.
- Understands and uses ordinal terms: “first,” “second,” “third,” “fourth,” and “fifth.”
- Using manipulatives, can find the answer to simple addition and subtraction word problems that total up to 5, and later up to 10; for example, “I had three dolls and I got four more for my birthday. How many dolls do I have now?”
- Learns to count backwards from 10.
- Learns to skip counts by 10s (10, 20, 30...), and later by 5s and 2s.
- Can learn to write one-digit numerals.
- Can learn to start counting up from numbers other than one, for example, “7, 8, 9, 10.”

5 – 6 years old

- Can divide up large sets (20 items and more) equally among five people.
- Knows what number comes next up to the number 9.
- Knows that the distance between two objects doesn’t change unless the objects are moved.
- Can learn to count backwards from 20.
- Knows that if Mary is taller than Josie, and Josie is taller than Fred, then Mary is also taller than Fred.
- Knows that a bundle of ten popsicle sticks is the same as ten individual popsicle sticks.
- Compares the length of two objects using string.
- Represents up to 10 using finger patterns.
- Understands and uses the ordinal terms “first,” “second,” … up to “tenth.”
- Knows the doubles up to 10, for example, 2 and 2 is 4, 3 and 3 is 6.
- Can learn to count up to 100.
- Recognizes that there are five toys in a set without counting them.
- Can learn to recognize patterns of up to ten items and connects the patterns with the quantity indicated, for example, “∶ ∶ means there are 4 dots.”
- Measures things using other objects placed end-to-end, for example, “My book is ten paperclips long.”
- Names, discusses, and compares objects using words such as “taller,” “shorter,” “skinnier,” “fatter,” “wider,” and “longer.”
- Writes two-digit numerals.
- Reads number words up to 10, for example, can read “one,” “two,” and so on.
- Can learn to start the counting sequence from any number between 2 and 18, for example, “13, 14, 15, 16, 17,…”
- Understands that a bundle of 18 popsicle sticks is the same as a bundle of ten popsicle sticks plus eight individual popsicle sticks.
- Can label shares of 1/2, 1/3, 1/4, and 1/5 using the words “half,” “third,” “fourth,” and “fifth.”
- Can learn to measure length of objects using centimetres and metres.
- Using manipulatives, can create a straight road that is “just as far to walk” as a given road with a bend in it.
- Can divide up to 100 items equally among ten children.



8

Glossary of Terms Related to Early Numeracy

A

Abstraction: One of the principles of counting: any group of objects can be counted, regardless of individual item type. For example, one orange, two pencils and three blocks can be counted (from 1 to 6) to find the total number of items.

Algorithm: In mathematics, an algorithm is a set of precise step-by-step instructions for how to arrive at an answer to a given problem; a formal procedure that is usually explicitly taught.

Associative Property: In addition and multiplication, the order in which three numbers are added or multiplied does not affect the sum or product. This is not true for subtraction and division. For example, (1 + 2) + 3 is the same as 1 + (2 + 3). (See also: Commutative Property.)

Automaticity: The quick, easy, and effortless (that is, the "automatic") retrieval of facts or procedures from long-term memory.

B

Base-10 System: The number system most commonly used in North America, based on grouping in tens. Ten is the base, and each place to the left is 10 times greater. For example, 100 = 10 times greater than 10. Each place value to the right of base-10 is one tenth of it (1/10); for example, 1 is 1/10, or one tenth, of 10.

Biologically Primary Knowledge: Inherent types of cognition, such as language and early quantitative competencies; usually emerge with little to no formal instruction, across all cultures.

Biologically Secondary Knowledge: Skills that build on biologically primary abilities and are cultural inventions (e.g., base-10 arithmetic).

C

Cardinality: One of the principles of counting and initial "how to count" rules: the value of the last number word used while counting indicates the quantity of items in the set. For example, counting "1, 2, 3" means that there are three items in the set.

Cardinal Numbers: The counting numbers (1, 2, 3, 4, 5, 6…) used to measure the size, or cardinality, of a set.

Change Problems: A type of word problem that contains some event that changes the value of a quantity. For example, "Robin has 5 pencils and Carly gives him 3 more. How many does Robin have now?"

Classification: Grouping items according to a characteristic. For example, putting all the blocks in one bucket and all the balls in another.

Combine Problems: A type of word problem that describes two parts that are considered separately or in combination. For example, "Robin and Carly have 8 pencils all together. Carly has 3 pencils. How many does Robin have?"

Combining Units Strategy: A strategy used to solve arithmetic problems wherein the 100s, 10s, and units are dealt with separately. For example, solving 37 + 38 by 30 + 30 then 7 + 8.

Commutative Property: In addition and multiplication, the order in which two numbers are added or multiplied does not affect the sum or product. For example, the sum of 4 + 3 is the same as the sum of 3 + 4. (See also: Associative Property.)

Compare Problems: A type of word problem that contains two amounts to be compared for the difference between them. For example, "Robin has 5 pencils and Carly has 3 pencils. How many fewer pencils does Carly have than Robin?"

Compensating Strategy: A strategy used to solve arithmetic problems wherein the numbers are adjusted to simplify the arithmetic. For example, solving 37 + 38 as (35 + 35) + 2 + 3 = 75.

Conceptual Knowledge: Knowledge of why and how a mathematical procedure works, as well as general mathematical knowledge and understanding. For example, knowing that when counting, the last number stated represents the quantity of items in the set.

Conservation: Refers to the principle that rearranging the elements in a set (changing their order, moving them farther apart, turning them upside down, etc.) does not change the total quantity of items. The principle also applies to weight and volume. For example, pouring water from a tall skinny glass into a wide bowl does not change the amount of water.

Counting On: The ability to start at any number in the number sequence and continue counting from that number onward. (See also: Number-after Skill.)

Counting Skills: The ability to recite numbers in order. Children may be able to recite the number words in the correct order without understanding the underlying meaning.

D

Data: Information used as the basis of calculation.

Digit: The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For example, 4093 has four digits: 4, 0, 9, and 3.

E

Episodic Buffer: One of the three systems in the model of working memory; this memory storage system can integrate information across domains to form visual, spatial, and verbal information with time sequencing.

Estimation: An approximation of the exact value of an operation.

Extrinsic Motivation: The effort to learn made in the hope of some type of external reward such as good grades, a teacher or parent's praise, a sticker, etc.

F

Factual Knowledge: Knowledge of information that can be learned through memorization and repetition (e.g., 2 + 2 = 4), as well as memory of specific events and information.

Fraction: Any part of a whole, number, or group. For example, ¼ or ½.

Frequency: How often something occurs; more specifically, the number of times a particular item appears in a set of data. For example, the following question relates to frequency: "There are ten children here today, how many are girls?"

G, H, I

Geometry: A branch of mathematics that involves the study of shapes and configurations (e.g., straight lines, circles, etc).

Graph: A visual representation of data, often used to make it easier to quickly compare quantities.

Heuristic Methods: The systematic strategies that one uses for problem solving. (See also: Algorithm.)

Intrinsic Motivation: The desire to learn for the sheer enjoyment, challenge, pleasure, or interest of the activity.

L, M

Language-based Phonetic Buffer: One of the three systems in the model of working memory, this system is also known as the Phonological Loop. It temporarily stores the phonological, or auditory, information of language.

Manipulatives: Objects that children can handle (manipulate) to understand and work out simple arithmetic problems; for example, beans, buttons, or blocks. Children build their understanding of math with concrete objects before they move on to abstract number concepts.

Mastery-Oriented Goals: Students with mastery-oriented goals seek to master that which they are learning, focusing on their own achievement and attributing their success to effort. These students tend to challenge themselves and persist in the face of difficulty.

Math Anxiety: An emotional reaction, ranging from mild apprehension to fear or dread, in academic and everyday situations that deal with numbers.

Mental Number Line: A mental representation of numbers and relative magnitudes; requires the ability to visualize and abstract number, to order numbers by quantity, to locate a given number along an imaginary line, and to generate any portion of the number line that may be required for problem solving.

Metacognitive Knowledge: Also known as Metaknowledge; the knowledge of how, when, and why to use specific strategies or resources; what an individual knows about his or her own thinking.

Metacognitive Regulation: How one's knowledge is used to regulate and control one's own thinking.

N

Non-Standard Measurement: Before using standard units of measure such as centimetres or grams, children measure objects by comparing them to other items in their environment. For example, hand widths, block lengths, etc.

Number-After Skill: The ability to state the next number in the counting sequence when one starts at any number. For instance, knowing that five is the next number after four without needing to count up from one. (See also: Counting On.)

Number Line: A horizontal line on which numbers are written in order from left to right. Once children have understood ordinality, they can look at the line and see that numbers farther to the right represent larger quantities than those on the left. As children learn about counting and about the concept of number itself, they start to generate a mental picture of the number line.

Number Recognition: The ability to state the number of items in a particular group (e.g., 'There are 2 dogs').

Number Sense: The understanding of number and operations; encompasses three subcomponents: 1) knowing about and using numbers; 2) knowing about and using operations; and 3) knowing about and using numbers and operations in computational settings.

Numeracy: A broad term that includes knowledge of number, arithmetic, procedures, problem solving, and measurement.

Numeral Recognition: The ability to name the number when you see its numeral representation (e.g., 2 = two).

Numerals: The written system for expressing numbers. There are many different numeral systems across the world (e.g., Hindu-Arabic, East Asian, Alphabetic, etc.).

Numerosity: An approximate sense of number that babies as well as non-human animals (e.g., rats, lions, primates) have.

O

One-to-One Correspondence: One of the principles of counting and initial "how to count" rules: one, and only one, word can be assigned to each counted object. For example, an item in a set that has been assigned "3" cannot also be assigned "5".

Order-Irrelevance: One of the principles of counting: items in a set can be counted in any sequence and still reach the same total. For example, counting from right to left, left to right, or in no particular sequence at all will result in the same total number of items.

Ordinality: At its most basic level, the concept of more and less; develops to an understanding that higher numbers are associated with more items, and lower numbers with fewer items.

Ordinal Number: The number that refers to place or position (e.g., 1st, 2nd, 3rd).

P

Pattern: Any repeated design or recurring sequence. For example, the sequence apple, orange, pear, apple, orange, pear or the flowers on wallpaper are both patterns.

Performance-Oriented Goals: Students with performance-oriented goals are focused outward, comparing their performance and learning to that of others; they tend to attribute success to ability, avoid challenging themselves, and give up when dealing with a difficult problem.

Procedural Knowledge: Knowledge about how to complete an activity or task, including the motor sequences and skills needed. For example, knowing how to solve the problem 2 + 3 by continuing to counting on from 3 to get the sum – "4, 5."

Proportionality: Refers to the multiplicative relationships between rational quantities; the basis for rational number operations, basic algebra, and problem solving in geometry.

R, S

Range: The difference between the smallest and largest numbers in a group. For instance, the age range of the students in our classroom is from 3 to 5.

Self-Efficacy: The set of beliefs one holds about one's own ability to succeed at difficult tasks.

Self-Regulation: A combination of motivation and cognitive processing; includes goal setting, planning, self-monitoring, evaluation, learning adjustments, and strategy choice.

Sequence: An ordered set of objects, numbers, shapes, etc. that are arranged according to a rule. For example, arranging dolls in order based on height, tallest to shortest.

Sequential Strategy: A strategy used to solve arithmetic problems wherein the value of the second number is counted up or down from the first number. For example, 37 + 38 is solved by 37 + 30 = 67, then 67 + 8 = 75.

Seriation: (or ordering) The ability to arrange objects in order by size (e.g., arranging balls from largest to smallest).

Set: A collection of items that are grouped together; members of a set are called elements.

Stable Order: One of the principles of counting and initial "how to count" rules: the order in which number words are used to count objects is always the same. For example, counting in the order of "1, 2, 3" is correct, but "1, 2, 4" is incorrect.

T, V

Topology: A branch of geometry that studies such concepts as space, dimension, shape and transformation.

Transfer: Also known as Learning Transfer; the ability to apply the skills and concepts used to solve one type of problem to another type of problem; learning can be applied beyond problems studied to both similar problems (Near Transfer) and to dissimilar problems (Far Transfer).

Visuospatial Sketch Pad: One of the three systems in the model of working memory; this system enables short term storage and manipulation of visual or spatial information.

W

Working Memory: Attention-driven control of information represented in the brain in one of three content-specific systems: the language-based phonetic buffer, the visuospatial sketch pad, and the episodic buffer.