Foundations of Mathematics 120

Curriculum

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New Brunswick Department of Education and
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Curriculum Overview for Grades 10-12 Mathematics

BACKGROUND AND RATIONALE

Mathematics curriculum is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society.

It is essential the mathematics curriculum reflects current research in mathematics instruction. To achieve this goal, *The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol* has been adopted as the basis for a revised mathematics curriculum in New Brunswick. The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators and others.

The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP and the NCTM.

There is an emphasis in the New Brunswick curriculum on particular key concepts at each grade which will result in greater depth of understanding and ultimately stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

The intent of this document is to clearly communicate high expectations for students in mathematics education to all education partners. Because of the emphasis placed on key concepts at each grade level, time needs to be taken to ensure mastery of these concepts. *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM Principles and Standards, 2000).*

BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING

The New Brunswick Mathematics Curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice. These beliefs include:

- mathematics learning is an active and constructive process;
- learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates;
- learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking and that nurtures positive attitudes and sustained effort; and
- learning is most effective when standards of expectation are made clear with on-going assessment and feedback.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and aspirations.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best developed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial and symbolic representations of mathematics. The learning environment should value, respect and address all students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals for Mathematically Literate Students

The main goals of mathematics education are to prepare students to:

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success, and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Diverse Cultural Perspectives

Students come from a diversity of cultures, have a diversity of experiences and attend schools in a variety of settings including urban, rural and isolated communities. To address the diversity of knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of students, a variety of teaching and assessment strategies are required in the classroom. These strategies must go beyond the incidental inclusion of topics and objects unique to a particular culture.

For many First Nations students, studies have shown a more holistic worldview of the environment in which they live (Banks and Banks 1993). This means that students look for connections and learn best when mathematics is contextualized and not taught as discrete components. Traditionally in Indigenous culture, learning takes place through active participation and little emphasis is placed on the written word. Oral communication along with practical applications and experiences are important to student learning and understanding. It is important that teachers understand and respond to both verbal and non-verbal cues to optimize student learning and mathematical understandings.

Instructional strategies appropriate for a given cultural or other group may not apply to all students from that group, and may apply to students beyond that group. Teaching for diversity will support higher achievement in mathematics for all students.

Adapting to the Needs of All Learners

Teachers must adapt instruction to accommodate differences in student development as they enter school and as they progress, but they must also avoid gender and cultural biases. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom. The reality of individual student differences must not be ignored when making instructional decisions.

As well, teachers must understand and design instruction to accommodate differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Designing classroom activities to support a variety of learning styles must also be reflected in assessment strategies.

Universal Design for Learning

The New Brunswick Department of Education and Early Childhood Development's definition of inclusion states that every child has the right to expect that his or her learning outcomes, instruction, assessment, interventions, accommodations, modifications, supports, adaptations, additional resources and learning environment will be designed to respect his or her learning style, needs and strengths.

Universal Design for Learning is a "…framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged." It also "...reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient" (CAST, 2011).

In an effort to build on the established practice of differentiation in education, the Department of Education and Early Childhood Development supports *Universal Design for Learning* for all students. New Brunswick curricula are created with universal design for learning principles in mind. Outcomes are written so that students may access and represent their learning in a variety of ways, through a variety of modes. Three tenets of universal design inform the design of this curriculum. Teachers are encouraged to follow these principles as they plan and evaluate learning experiences for their students:

- **Multiple means of representation:** provide diverse learners options for acquiring information and knowledge
- **Multiple means of action and expression:** provide learners options for demonstrating what they know
- **Multiple means of engagement:** tap into learners' interests, offer appropriate challenges, and increase motivation

For further information on *Universal Design for Learning*, view online information at http://www.cast.org/.

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, and physical education.

NATURE OF MATHEMATICS

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this document. These components include: **change**, **constancy**, **number sense**, **patterns**, **relationships, spatial sense** and **uncertainty**.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as:

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Students need to learn that new concepts of mathematics as well as changes to already learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include:

- the area of a rectangular region is the same regardless of the methods used to determine the solution
- the sum of the interior angles of any triangle is 180°
- the theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Many important properties in mathematics do not change when conditions change. Examples of constancy include:

- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- the theoretical probability of an event.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p. 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. Evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing mathematically rich tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

ASSESSMENT

Ongoing, interactive assessment (*formative assessment*) is essential to effective teaching and learning. Research has shown that formative assessment practices produce significant and often substantial learning gains, close achievement gaps and build students' ability to learn new skills (Black & William, 1998, OECD, 2006). Student involvement in assessment promotes learning. Interactive assessment, and encouraging self-assessment, allows students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes:

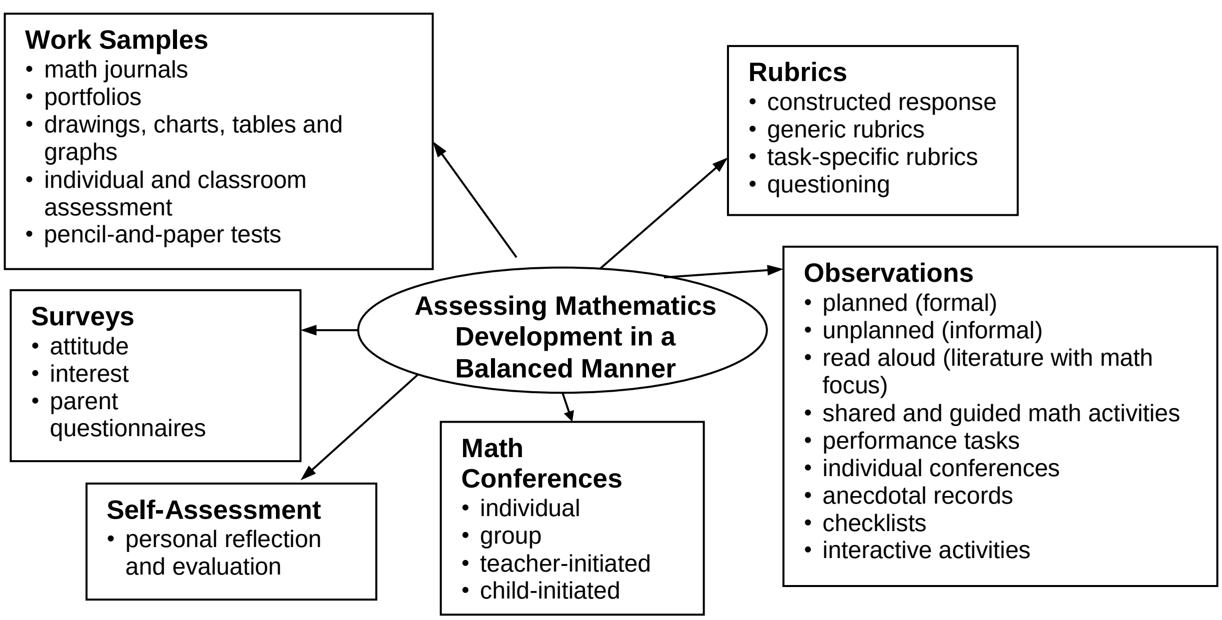
- providing clear goals, targets and learning outcomes
- using exemplars, rubrics and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies, 2000)

Formative assessment practices act as the scaffolding for learning which, only then, can be measured through summative assessment. *Summative assessment,* or assessment *of* learning, tracks student progress, informs instructional programming and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning and produce achievement gains.

Student assessment should:

- align with curriculum outcomes
- use clear and helpful criteria
- promote student involvement in learning mathematics during and after the assessment experience
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction

(adapted from: NCTM, Mathematics Assessment: A practical handbook, 2001, p.22)



CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.

GRADE TOPICS	10 11 12
The topics of study vary in the courses for grades 10–12 mathematics. Topics in the pathways include: • *Algebra* • *Financial Mathematics* • *Geometry* • *Logical Reasoning* • *Mathematics Research Project* • *Measurement* • *Number* • *Permutations, Combinations and Binomial Theorem* • *Probability* • *Relations and Functions* • *Statistics* • *Trigonometry*	**GENERAL OUTCOMES** **SPECIFIC OUTCOMES** **ACHIEVEMENT INDICATORS**

NATURE OF MATHEMATICS

Change
Constancy
Number Sense
Patterns
Relationships
Spatial Sense
Uncertainty

MATHEMATICAL PROCESSES:
Communication, Connections, Mental Mathematics and Estimation, Problem Solving, Reasoning, Technology, Visualization

MATHEMATICAL PROCESSES

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.
Students are expected to:

- communicate in order to learn and express their understanding of mathematics (Communications: C)
- develop and apply new mathematical knowledge through problem solving (Problem Solving: PS)
- connect mathematical ideas to other concepts in mathematics, to everyday
- experiences and to other disciplines (Connections: CN)

- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation: ME)
- select and use technologies as tools for learning and solving problems (Technology: T)
- develop visualization skills to assist in processing information, making connections and solving problems (Visualization: V).
- develop mathematical reasoning (Reasoning: R)

The New Brunswick Curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, knowledge, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication can help students make connections among concrete, pictorial, symbolic, verbal, written and mental representations of mathematical ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type, *How would you*...*?* or *How could you* ...*?,* the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learnings in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly look for, and engage in, finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences, and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"*Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding… Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*" (Caine and Caine, 1991, p. 5).

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental mathematics" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "*become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving*" (Rubenstein, 2001).

Mental mathematics "*provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers*" (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Technology [T]

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Calculators and computers can be used to:
- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense.

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world” (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills.
Measurement sense includes the ability to determine when to measure and when to estimate and involves knowledge of several estimation strategies (Shaw and Cliatt, 1989, p. 150).

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking.

Questions that challenge students to think, analyze and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, *Why do you believe that's true/correct?* or *What would happen if ….*

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

ESSENTIAL GRADUATION LEARNINGS

Graduates from the public schools of Atlantic Canada will be able to demonstrate knowledge, skills, and attitudes in the following essential graduation learnings. These learnings are supported through the outcomes described in this curriculum document.

Aesthetic Expression
Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship
Graduates will be able to assess social, cultural, economic, and environmental interdependence in a local and global context.

Communication
Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) as well as mathematical and scientific concepts and symbols to think, learn, and communicate effectively.

Personal Development
Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving
Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language, mathematical, and scientific concepts.

Technological Competence
Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems

PATHWAYS AND TOPICS

The Common Curriculum Framework for Grades 10–12 Mathematics on which the New Brunswick Grades 10-12 Mathematics curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics*. In New Brunswick all Grade 10 students share a common curriculum covered in two courses: *Geometry, Measurement and Finance 10* and *Number, Relations and Functions 10*. Starting in Grade 11, three pathways are available: *Finance and Workplace*, *Foundations of Mathematics*, and *Pre-Calculus*.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. Students are encouraged to cross pathways to follow their interests and to keep their options open. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all three pathways are to provide prerequisite attitudes, knowledge, skills and understandings for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of each pathway has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* and on consultations with mathematics teachers.

<u>Financial and Workplace Mathematics</u>

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, statistics and probability.

<u>Foundations of Mathematics</u>

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, statistics and probability.

<u>Pre-calculus</u>
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Students develop of a function tool kit including quadratic, polynomial, absolute value, radical, rational, exponential, logarithmic and trigonometric functions. They also explore systems of equations and inequalities, degrees and radians, the unit circle, identities, limits, derivatives of functions and their applications, and integrals.

Outcomes and Achievement Indicators

The New Brunswick Curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes and achievement indicators.

<u>General Curriculum Outcomes</u> (GCO) are overarching statements about what students are expected to learn in each strand/sub-strand. The general curriculum outcome for each strand/sub-strand is the same throughout the pathway.

<u>Specific Curriculum Outcomes</u> (SCO) are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as required for a given grade.

<u>Achievement indicators</u> are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome. In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for clarification and are not requirements that must be addressed to fully meet the learning outcome. The word *and* used in an outcome indicates that both ideas must be addressed to fully meet the learning outcome, although not necessarily at the same time or in the same question.

Instructional Focus

Each pathway in *The Common Curriculum Framework for Grades 10–12 Mathematics* is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful. Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches, and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning, assessment as learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.

Mathematics K-9

<u>Grade 10</u>

- *2 x 90 hr. courses; required to pass <u>both</u>*
- *May be taken in any order or in the same semester*

Geometry, Measurement and Finance 10

Number, Relations and Functions 10

<u>Grade 11</u>

- *3 x 90 hr. courses offered in 3 pathways*
- *Students are required to pass at least one of "Financial and Workplace Mathematics 11" **or** "Foundations of Mathematics 11".*
- *Pre-requisite Grade 10 course(s) must be passed before taking Grade 11 courses.*

Financial and Workplace Mathematics 110

<u>Pre-requisite</u>: Geometry, Measurement and Finance 10

Foundations of Mathematics 110

*<u>Pre-requisites</u>: Geometry, Measurement and Finance 1 **and** Number, Relations and Functions 10*

Pre-Calculus 110

<u>Pre-requisite or Co-requisite</u>: Foundations of Mathematics 110

<u>Grade 12</u>

- *5 x 90 hr. courses offered in 3 pathways*
- *Pre-requisite Grade 11 or Grade 12 course must be passed before taking Grade 12 courses.*

Financial and Workplace Mathematics 120

*<u>Pre-requisite</u>: Financial and Workplace Mathematics 110 **or** Foundations of Mathematics 110*

Foundations of Mathematics 120

<u>Pre-requisite</u>: Foundations of Mathematics 110

Pre-Calculus A 120

<u>Pre-requisite</u>: Pre-Calculus 110

Pre-Calculus B 120

<u>Pre-requisite or Co-requisite</u>: Pre-Calculus A 120

Calculus 120

*<u>Pre-requisites</u>: Pre-Calculus A 120 **and** Pre-Calculus B 120*

SUMMARY

The Conceptual Framework for Grades 10–12 Mathematics describes the nature of mathematics, the mathematical processes, the pathways and topics, and the role of outcomes and achievement indicators in grades 10–12 mathematics. Activities that take place in the mathematics classroom should be based on a problem-solving approach that incorporates the mathematical processes and leads students to an understanding of the nature of mathematics.

CURRICULUM DOCUMENT FORMAT

This guide presents the mathematics curriculum by grade level so that a teacher may readily view the scope of the outcomes which students are expected to meet during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings at a particular grade level are part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The heading of each page gives the General Curriculum Outcome (GCO), and Specific Curriculum Outcome (SCO). The key for the mathematical processes follows. A Scope and Sequence is then provided which relates the SCO to previous and next grade SCO's. For each SCO, Elaboration, Achievement Indicators, Suggested Instructional Strategies, and Suggested Activities for Instruction and Assessment are provided. For each section, the *Guiding Questions* should be considered.

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Mathematical Processes

[C] Communication [PS] Problem Solving [CN] Connections [ME] Mental Math [T] Technology [V] Visualization [R] Reasoning and Estimation

Scope and Sequence

Previous Grade or Course SCO's	Current Grade SCO	Following Grade or Course SCO's

Elaboration

Describes the "big ideas" to be learned and how they relate to work in previous Grades

Guiding Questions:
- *What do I want my students to learn?*
- *What do I want my students to understand and be able to do?*

Achievement Indicators

Describes observable indicators of whether students have met the specific outcome

Guiding Questions:
- *What evidence will I look for to know that learning has occurred?*
- *What should students demonstrate to show their understanding of the mathematical concepts and skills?*

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Suggested Instructional Strategies

General approach and strategies suggested for teaching this outcome

Guiding Questions
- *What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?*
- *What teaching strategies and resources should I use?*
- *How will I meet the diverse learning needs of my students?*

Suggested Activities for Instruction and Assessment

Some suggestions of specific activities and questions that can be used for both instruction and assessment.

Guiding Questions
- *What are the most appropriate methods and activities for assessing student learning?*
- *How will I align my assessment strategies with my teaching strategies?*

Guiding Questions
- *What conclusions can be made from assessment information?*
- *How effective have instructional approaches been?*
- *What are the next steps in instruction?*

Foundations of Mathematics 120

Specific Curriculum Outcomes

SCO: ***S1: Demonstrate an understanding of normal distribution, including standard deviation*** **and z-scores.** [CN, PS, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Statistics

S1: Demonstrate an understanding of normal distribution, including standard deviation and z-scores.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	S1: Demonstrate an understanding of normal distribution, including standard deviation and z-scores.

ELABORATION

In middle school, students were introduced to the concepts of mean, median and mode. In Grade 9 students collected and displayed data in the form of histograms and learned to distinguish between a sample and a population.

For this and the next outcome (S1 and S2) students will use and further develop these concepts. In this outcome they will consider full populations and will be introduced to the concepts of **normal distribution**, **standard deviation**, **z-scores** and **z-tables** to interpret sets of data.

To develop an understanding of these concepts it is important that students collect data to create different data sets that vary in size and in range of values. They should be able to determine the **mean**, **median** and **mode** of these data sets. This data can be sorted into a frequency distribution and graphed as a histogram (*Some resources define the interval for a frequency distribution to include all data points up to but not including the upper boundary of the interval (T1-83/84) while other resources include the upper boundary).*

As students graph increasing volumes of data they will discover that the distribution of random phenomena tends towards a **normal distribution**, also called a bell curve.

Standard deviation is a measure of the dispersion or scatter of data values in relation to the mean. A low standard deviation indicates that most data values are close to the mean, and a high standard deviation indicates that there is a wider range of data values, extending further from the mean. Students will explore this concept with real data that they have collected and that is relevant to them.

Before being introduced to formulas for the population mean and the standard deviation, students should demonstrate that they understand the concepts. They should be able to compare a variety of data sets graphically, pictorially, or in a table of values and be able to estimate means, and to determine which data sets have a greater or lower standard deviation.

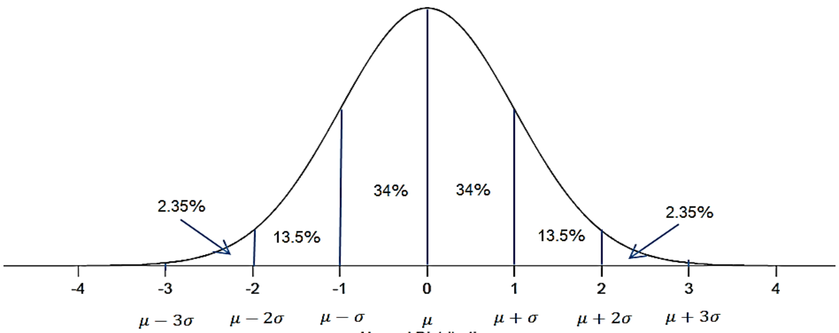
Once students have demonstrated that they understand these concepts, they will be introduced to the formula for the population mean and population standard deviation. These formulas assume that all points in the population are equally likely.

The population mean is calculated as: $\mu = \frac{\sum x}{N}$

The population standard deviation is calculated as : $\sigma = \sqrt{\frac{\sum(x-\bar{\mu})^2}{N}}$

SCO: ***S1: Demonstrate an understanding of normal distribution, including standard deviation*** **and z-scores.** [CN, PS, T, V]

For a **normal distribution** curve, approximately 68% of the data falls within 1 **standard deviation** of the mean, 95% falls within 2 **standard deviations** of the mean, and 99.7% within 3 **standard deviations** of the mean. These percentages should be related back to work on frequency distributions and polygons to calculate percentages of the population lying within a given class.



Normal Distribution

Students will also explore ***z*-scores** and ***z*-tables** (see the end of this document). A **z-score** is a standardized value that indicates the number of standard deviations (σ) that a data value (x) is above or below the mean (μ). Given the mean and standard deviation for a population, students should be able to find the value of a point given the ***z*-score.**

For example, if a point has a ***z*-score** of 1.5, this means that the data point is 1.5 standard deviations above the mean. Illustrating this concept with simple numbers, if $\mu = 10$ and $\sigma = 5$, the point is half way between 15 ($1\ SD$) and 20 ($2\ SD$), and so has a value of 17.5.

Once this understanding of ***z*-scores** is well established, to deal with more complex numbers students should use the following formula to calculate the ***z*-score**: $z = \frac{x-\mu}{\sigma}$. Finding a ***z*-score** for a point then allows the student to determine the percentage of values found above or below that point by using the normal distribution curve or by using a ***z*-score table** which provides a quick reference to percentages for ***z*-scores.**

For example, if a data point has a ***z*-score** of 1.0 it means that it is 1 **standard deviation** above the mean and 84% of the data points are lower than this point. This percentage can be determined by referring back to the normal curve diagram shown above (50%+34%) or by using the ***z*-score table** which gives a value of 0.8413 or $\sim 84\%$ for a ***z*-score** of 1.00.

Given the mean and standard deviation for a population, students should then be able to determine the ***z*-score** and the percentage of values above and below that point.

For example, the heights were determined for a population of women and it was found that $\mu = 64.5$" and $\sigma = 2.5$". One of women is 68" tall. This is more than one standard deviation above the mean ($68" > 64.5" + 2.5"$). The ***z*-score** was determined using the formula:
$z = \frac{x-\mu}{\sigma} = \frac{68-64.5}{2.5} = 1.4$

This ***z*-score** was found on the ***z*-table** (see end of document) at 0.9192. This tells us that this woman is taller than approximately 92% of the women in the population.

A positive **z-value** would indicate that a woman is taller than average, and a negative **z-value** would indicate that a woman is shorter than average.

For statistical calculations, students should learn effective use of a scientific calculator, a graphing calculator, or a statistical software program as appropriate.

SCO: ***S1: Demonstrate an understanding of normal distribution, including standard deviation*** **and *z*-scores.** [CN, PS, T, V]

ACHIEVEMENT INDICATORS

- Explain, using examples, the meaning of population standard deviation.
- Calculate, using technology, the population standard deviation of a data set recognizing that more typically a data set is regarded as a sample drawn from a population.
- Explain, using examples, the properties of a normal curve, including the mean, median, mode, standard deviation, symmetry and area under the curve.
- Determine if a data set approximates a normal distribution, and explain the reasoning.
- Compare the properties of two or more normally distributed data sets.
- Explain, using examples from multiple perspectives, the application of standard deviation for making decisions in situations such as warranties, insurance or opinion polls.
- Solve a contextual problem that involves the interpretation of standard deviation.
- Determine and explain, with or without technology, the *z*-score for a given value in a normally distributed data set.
- Solve a contextual problem that involves normal distribution.

Suggested Instructional Strategies

- Group work is very effective here and should be used extensively.
- Canadian data can be found at: http://www.statcan.gc.ca/start-debut-eng.html
- Have students collect data from their classmates on for example: heights, number of text messages sent in a day, distance of home from school (in minutes or kilometers), number of windows in their house, shoe size etc. Have them collect the same data but with different populations of students e.g. heights of Grades 9 to 12 students versus heights of only Grade 12 students in order to explore larger and smaller variations in data. Have them work with the data first to estimate values and then have them use formulas to determine *mean*, and *standard deviation* for the population, and then *z-values* for a given point and percentage of other data points above and below that value.
- For *standard deviation* students should first grasp the concept by examining data collected and determining how standard deviation for data sets compare. They should then calculate it by using the formula to help them understand how the value is determined and how it relates to an actual data set. Once that is understood they will need to learn how to use technology to determine the *standard deviation* of any set of data.
- To understand *normal curves*, students should repeat a controlled experiment many times, pool data and create progressive histograms. They should notice that the shape of the histograms becomes more bell-like as the number of data points represented in the distribution increases. They should come to understand that repeated measurements which are subject to accidental or random effects only, will produce bell-shaped distributions which exhibit the characteristics of a *normal curve*. A PDF of a normal curve is available at http://www.csic.cornell.edu/Elrod/normal_view.html for use in the classroom.
- When communicating the results of experiments, students would benefit from making classroom presentations and/or producing meaningful written reports.

SCO: ***S1: Demonstrate an understanding of normal distribution, including standard deviation*** **and z-scores.** [CN, PS, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Have students collect data on the heights of their classmates, and determine the *mean*, *median* and *mode* of the data. Have them use graphing technology to calculate the *standard deviation*. Continue with technology to construct a histogram with an appropriate bin width to determine if data is approximately normal. Students can use *z-tables* to determine what percentage of heights are expected to fall 2.3 standard deviations below the mean or 1.6 standard deviations above the mean.

Act Measure students' heights (in cm) and record them according to male and female. For each data set determine the measures of *central tendency* and the *standard deviation*. Assuming data is normally distributed, compare their properties and draw conclusions.

Q A manufacturer of washing machines has determined that the mean life of the machines is 60 months with a *standard deviation* of 15 months. What length of warranty should be offered if the manufacturer wants to restrict repairs to less than 15% of all washing machines sold?

$Answer$: $\frac{x-60}{15} = -1.04 \quad \therefore x = 44.4 \;\; or\ a\ 44\ month\ warranty$

Q Mr. Sweeney has a small class of 15 math students. He is interested in knowing the mean and standard deviation for their quiz. Their marks are:

$$82, 76, 65, 78, 81, 90, 52, 93, 50, 89, 70, 85, 59, 60, 52.$$

a) Explain why this group represents a population and not a sample.

b) Calculate, using technology, the mean and standard deviation of this data.

c) Create a set of data that has a smaller standard deviation and explain how you created it.

$Answers$: $a)\ all\ of\ the\ student\ results\ are\ being\ used \qquad b)\ \bar{x} = 72.1 \quad SD = 14.4$

$c)\ Answers\ will\ vary.\ \ Select\ points\ that\ are\ closer\ to\ the\ mean\ than\ for\ the\ original\ question$

$e.g. 15\ points\ such\ as: 69, 69, 70, 70, 70, 71, 71, 71, 71, 71, 71, 72, 72, 72, 73$

Q Scores on the SAT verbal test in recent years follow approximately a normal distribution with $\mu = 505$ and $\sigma = 110$. How high must a student score be in order to place in the top 10% of all students taking the SAT?

$Answer$: $\frac{x-\mu}{\sigma} = \frac{x-505}{101} = 1.28\ (z\ score\ from\ table) \qquad \therefore x = 645.8$

SCO: ***S1: Demonstrate an understanding of normal distribution, including standard deviation*** **and z-scores.** [CN, PS, T, V]

Q The mathematics department at a small university expects all students enrolling in Mathematics 1000 to take an entry test. The exam is marked out of 20. The results of all 40 students who wrote the exam were:

$9, 12, 9, 10, 15, 12, 11, 12, 10, 17,$
$7, 12, 12, 12, 14, 10, 11, 13, 5, 12,$
$12, 12, 11, 9, 12, 8, 14, 14, 11, 9,$
$16, 12, 11, 13, 14, 15, 12, 11, 11, 14.$

a) Construct a histogram and a frequency polygon.

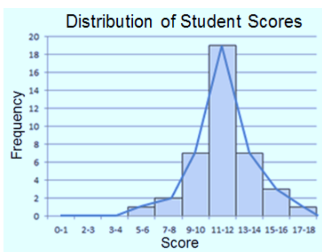
b) Does the distribution appear roughly “bell-shaped”? Explain.

c) Does this data set represent a sample or a population? Explain

d) Using technology, find the mean and standard deviation.

e) How many scores fall within one standard deviation of the mean? Express this as a percentage of all the scores.

$Answers$: $a)$



$b)$ $Yes -$ $more\ data\ is\ clustered\ in\ the\ middle\ of\ the\ graph$
$c)$ $Population\ as\ all\ student\ scores\ were\ used$
$d)$ $mean =\ 11.7 \quad standard\ deviation = 2.4$
$e)$ $29\ out\ of\ 40\ or\ 72.5\%\ of\ scores\ fall\ within\ 1\ SD\ of\ the\ mean$

Q The army reports that the distribution of head circumference among male soldiers is approximately normal with mean 22.8 inches and standard deviation 1.1 inches.

a) What percent of soldiers have head circumference greater than 23.9 inches?
b) What percent of soldiers have head circumference between 21.7 inches and 23.9 inches?
c) If 500 soldiers are selected at random, how many are expected to have a head circumference less than 23.9 inches?

$Answers$:

$a)$ $22.8 + 1.1 = 23.9\ is\ one\ SD\ above\ mean \quad 13.5\% + 2.5\% = 16\%$
$b)$ $from\ 21.7\ \ to\ 23.9\ is\ within\ one\ SD\ of\ the\ mean\ \ which\ is\ 68\%\ of\ soldiers$
$c)$ $100\% - 16\% = 84\%\ of\ 500\ soldiers\ or\ 480\ soldiers$

SCO: **S2: Interpret statistical data, using confidence intervals, confidence levels, margin of error.** [C, CN, R]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

S2: Interpret statistical data, using confidence intervals, confidence levels, margin of error.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	S2: Interpret statistical data, using confidence intervals, confidence levels, margin of error.

ELABORATION

It is often impractical to obtain data for a complete population. Instead, statistics for random samples of the population are used to make predictions about the population. In Grade 9 students were introduced to the idea of sampling, and learned the difference between random and non-random sampling.

There are many examples in electronic and print media of results of surveys, polls and studies indicating **confidence intervals (sample mean $\pm$ margin of error)**, **confidence level**, **sample size** and **population size**.

Although an understanding of how these values are calculated will help to deepen understanding, the intention of this outcome is not to do statistical calculations but rather for the student to be able to interpret and explain data presented to them.

Students should understand that a **confidence interval** is an interval of likely values for the population mean. Or put another way, if samples are taken repeatedly, the population mean is likely to fall within this interval at whatever the confidence level is set at e.g., 19 times out of 20 for a confidence interval of 95%.

The **confidence interval** is expressed as **sample mean $\pm$ the margin of error.** The **margin of error** is calculated with reference to a certain **confidence level.** The formulas for determining these values are shown below, and can be used to deepen understanding, but the focus should be on the ability to interpret results of a survey or study when they are expressed with reference to confidence intervals and levels (see Achievement Indicators).

The **sample mean** is the average of the values in the sample: $\bar{x} = \frac{\sum x}{n}$.

The **margin of error** = $\pm(z)\frac{s}{\sqrt{n}}$

$z = 1.65$ for a 90% **confidence level** (9 times out of 10)
$z = 1.96$ for a 95% **confidence level** (19 times out of 20)
$z = 2.58$ for a 99% **confidence level** (99 times out of 100)

$s =$ sample standard deviation $= \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$

$n =$ sample size

For example, if the **mean** is 54.0 and the **margin of error** is 3.5 at a 90% **confidence level**, the **confidence interval** would be 50.5 to 57.5, or with $\pm$ notation, expressed as 54.0 ± 3.5. This means that if samples are taken over and over again and

confidence intervals computed, 9 times out of 10, or 90% of the time these intervals would contain the **population mean**.

For this outcome students will interpret and explain these concepts from actual examples of data collected from real life, many of which can be found in print or electronic media. Statistical results of surveys or studies will be expressed as accurate within plus or minus a percentage or a point, at a certain confidence level.

For example, "The results of a survey of 300 people indicate that 65% of people regularly take their cars to work each day. The results are accurate within plus or minus 4 points, 19 times out of 20 and the total population is 34 500."

The student will explain that this means that the **confidence interval** is $65\% \pm 4\%$ or $22\,425 \pm 1380$ people, and the **confidence level** is 95%. Put another way, if the survey were conducted 100 times, for 95 of the samples $61\% - 69\%$ or $21\,045$-$23\,805$ people would respond that they regularly take their car to work each day.

Students will also explore the effect of sample size on the **margin of error** and the consequent **confidence interval** of a sample. Working with results from a variety of sources they will discover that as sample size increases, the confidence interval decreases. They should demonstrate that this is because with a greater sample size, the population will be more accurately represented.

Students will also understand that the sample size needs to increase in situations in which the **margin of error** is required to remain the same, but a greater **confidence level** is required.

For example, for a factory it is important to ensure that the weight of a baseball falls between 144.7 g and 145.3 g. A sample of 45 balls will give a confidence level of 90%. To increase the confidence level to 95% or 99% students should understand and be able to explain that an increase in sample size will be required.

A $\boldsymbol{z}$-score table (at the end of this document), can be used to find the z-score for a given **confidence level** by finding a percentage on the chart and reading the corresponding z-score. **Confidence levels** include only the data between $+z$ and $-z$. However, the chart provided in this document gives the total area under the curve to the left of the z-value, including the 'tail' beyond the $-z$ score. To use this chart, this value will need to be subtracted from the value given.

Using the chart, to find z for a 90% confidence level, 0.05 or 5% will be subtracted from 0.9505 given on the chart for $z = 1.65$. For a 95% confidence level, 0.025 or 2.5% will be subtracted from 0.9750 given on the chart for $z = 1.96$. For a 99% confidence level, 0.005 or 0.5% will be subtracted from 0.9951 given on the chart for $z = 2.58$.

There should also be some discussion with students with reference to cases that may require different **levels of confidence**. For example, most polls regarding preferences for political parties are reported as accurate "19 times out of 20" (or with 95% confidence). On the other hand, the medical profession would require a 99% or greater confidence level, and likely a very large sample size, when examining the side effects of a new drug.

SCO**: S2: Interpret statistical data, using confidence intervals, confidence levels, margin of error.** [C, CN, R]

ACHIEVEMENT INDICATORS

- Explain, using examples, the meaning of a confidence interval, margin of error or confidence level.
- Explain, using examples, how confidence levels, margin of error and confidence intervals may vary depending on the size of the random sample.
- Make inferences about a population from sample data, using given confidence intervals, and explain the reasoning.
- Provide examples from print or electronic media in which confidence intervals and confidence levels are used to support a particular position.
- Interpret and explain confidence intervals and margin of error, using examples found in print or electronic media.
- Support a position by analyzing statistical data presented in the media

Suggested Instructional Strategies

- The best way to introduce and determine confidence intervals is through the use of real-life examples
- Provide examples from print or electronic media in which confidence intervals and confidence levels are used to support a particular position.
- Interpret and explain confidence intervals and margin of error, using examples found in print or electronic media.
- Support a position by analyzing statistical data presented in the media

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Of 200 people responding to a telephone poll, 30% indicated that they regularly composted their garbage. These results are accurate within 5 percentage points, 9 times out of 10 and the city population is 54 760.

a) What is the confidence interval given in % points and in population?

b) What is the confidence level and what does this mean?

c) What steps would you take to increase the accuracy of the poll? How would this steps affect the margin of error and the confidence level?

Answers:

a) The confidence interval is $30\% \pm 5\%$ *or* $16\,428 \pm 2738$ *people*

b) The confidence level is 90% *meaning that if you were to take* 100 *samples, in* 90 *of those samples,* 25% *to* 35% *of people would indicate that they regularly compost their garbage.*

c) Increasing the sample size would increase the accuracy of the poll. This would decrease the margin of error and increase the confidence level.

Q A sample is taken from a normal population. Based on this sample, Lenna claims she is 90% confident that the population mean is between 45.6 and 50.2. Based on the same sample, Phillip claims he is 95% confident that the population mean is between 46.6 and 49.2. How do you know that someone has made a calculation error?

Answer: For Lenna the confidence interval is 4.6 *and for Phillip the confidence interval is on* 2.6*. However for the same sample to increase the confidence level to* 95% *the interval should increase.*

SCO**: S2: Interpret statistical data, using confidence intervals, confidence levels, margin of error.** [C, CN, R]

Q A botanist collects a sample of 50 iris petals and measures the length of each. It is found that $\bar{x} = 5.55\ cm$ and $s = 0.57cm$. He then reports that he is 95% confident that the average petal length is between $5.39\ cm$ and $5.71cm$.

a) Identify the confidence interval, and the confidence level.

b) Explain what information the confidence interval gives about the population of iris petal length.

c) How would the length of a 99% confidence interval be different from that of a 95% confidence interval?

d) If you did not know the point estimate but were still given that the confidence interval is between 5.39 cm and 5.71cm, how could you determine the point estimate?

Answers:
$a)\ The\ point\ estimate\ \bar{x}$
$= 5.55cm, confidence\ interval 5.55\ cm \pm 0.57\ cm\ or\ 5.39\ to\ 5.71\ confidence\ level\ 95\%$
$b)\ If\ you\ were\ to\ take\ 100\ samples, 95\ times\ out\ of\ 100\ the\ interval\ would\ capture\ the\ population\ mean.$
$c)\ There\ would\ be\ a\ greater\ spread\ in\ the\ interval.$
$d)\ You\ could\ determine\ the\ point\ estimate\ by\ taking\ the\ midpoint\ of\ the\ interval$

Q The following data shows a random sample of the ages of football players in a large European soccer league consisting of 1000 players:
$26, 24, 25, 36, 26, 32, 31, 34, 32, 27,$
$32, 23, 24, 29, 30, 30, 29, 33, 25, 32,$
$24, 25, 28, 22, 22, 24, 23, 33, 32, 31,$
$26, 28, 32, 25, 22, 28, 25, 29, 25, 27.$

The mean $\bar{x} = 27.775$, the sample standard deviation $s = 3.799$, $n = 40$

a) Determine a 99% confidence interval for the mean age of soccer players in this league, and explain the meaning of this confidence interval.
b) What is the point estimate?
c) What is the margin of error of your confidence interval? Explain the meaning of the margin of error.
d) Based on this data, what, if anything, can you conclude about the mean age of soccer players in a Canadian soccer league? Explain.

Answers:
$a)\ A\ 99\%\ confidence\ interval\ is\ found\ between$
$\pm\ 2.58(z) of\ the\ mean.\ This\ means\ that\ if\ you\ were\ to\ take\ 100\ samples,$
$99\ times\ out\ of\ a\ hundred\ the\ mean\ would\ fall\ in\ this\ interval.$
$b)\ point\ estimate = the\ mean = 27.78$
$c)\ margin\ of\ error = \pm z\left(\frac{s}{\sqrt{n}}\right) = \pm 2.58\left(\frac{3.80}{\sqrt{40}}\right) = 1.55 \therefore the\ confidence\ interval\ is\ 27.78 \pm 1.55$
$or\ 26.23 - 29.33\ at\ a\ 99\%\ confidence\ level$
$d)\ It\ would\ be\ reasonable\ to\ conclude\ that\ the\ mean\ age\ of\ soceer\ players\ in\ the\ Canadian\ Soccer\ League$
$is\ less\ than\ 30\ years\ of\ age.$

SCO**: LR1: Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.** [CN, ME, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Logical Reasoning

LR1: Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
LR2: Analyze puzzles and games that involve numerical reasoning, using problem-solving strategies.	**LR1:** Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

ELABORATION

This outcome should be developed throughout the course and not treated as a "stand alone" topic. It is suggested that students engage in at least one activity related to this outcome every week or two as time permits.

In Grade 10, the focus was on spatial puzzles and games, and in Grade 11 on numerical puzzles and games. The focus in Grade 12 is on analyzing numerical and logical reasoning and problem-solving strategies. Students should be able to explain and verify a strategy to solve a puzzle or win a game. Mathematical or logical concepts, typical conventions and rules from previous courses can be applied, placing the emphasis on reasoning.

It is important to note that some games primarily involve **logical reasoning** (such as Chess and Sudoku) while others also involve **numerical reasoning** (like Kakuro and Slitherlink). Both types of games should be equally examined.

ACHIEVEMENT INDICATORS

(It is intended that this outcome be integrated throughout the course by using games and puzzles such as chess, Sudoku, Nim, logic puzzles, magic squares, Kakuro and cribbage.)

- Determine, explain and verify a strategy to solve a puzzle or to win a game; e.g.,
  - guess and check
  - look for a pattern
  - make a systematic list
  - draw or model
  - eliminate possibilities
  - simplify the original problem
  - work backward
  - develop alternative approaches.
- Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.
- Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

SCO**: LR1: Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.** [CN, ME, PS, R]

Suggested Instructional Strategies

- Before assigning games to individual students try games together as a group, as instructions to games are not always clearly understood.
- Find simple versions of the games and increase the difficulty (see example below).
- Have students look for patterns and then develop a strategy to fit these patterns.
- Have students develop a game for classmates to play.
- Change a rule or parameter to a well-known or familiar game and explain how it affects the outcome of the game.
- Find a game online and critique the quality of the game.
- Do not confine yourself to paper-and-pencil games and puzzles; try to have at least a few options that are more "hands-on" such as chess, Rubik's cubes, Chinese checkers, backgammon. Another option is to use online flash games that require strategies.
- Have students consider various games, and determine which is better - guess and check or eliminating possibilities?
- Have students create a strategy journal organized into sections for individual or group, and number or logic games and puzzles. They should use this journal to track strategies and solutions used throughout the course.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Solve the following puzzles:

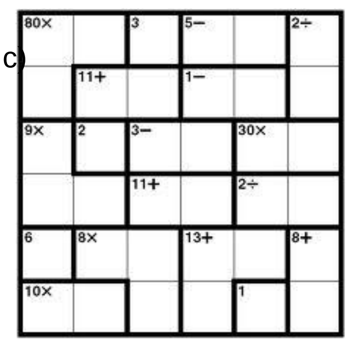
a)

	2	4	
1			3
4			2
	1	3	

b)

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

c)



Answers:

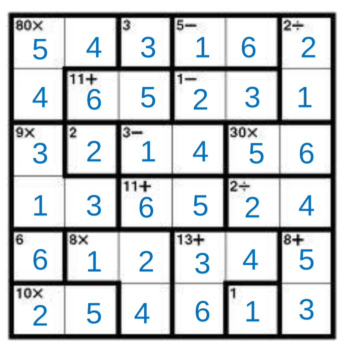
a)

3	2	4	1
1	4	2	3
4	3	1	2
2	1	3	4

b)

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

c)



SCO**: LR1: Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.** [CN, ME, PS, R]

Act Have students access interactive online links such as the following to explore a variety of reasoning puzzles and games:

http://www.kidsmathgamesonline.com/logic.html

http://nlvm.usu.edu/

http://www.squidoo.com/Different-Chess-Games

http://samgine.com/free/number-puzzles/

http://www.fibonicci.com/numeracy/number-sequences-test/medium/

http://www.mindjolt.com

http://education.jlab.org/nim/index.html

http://www.thelogiczone.plus.com

http://www.brainbashers.com/logic.asp

http://www.brocku.ca/caribou/games/

Act For the game, Nim, change the number of rows and have students attempt to make a connection to arithmetic patterns.

Q During a recent police investigation, the police chief interviewed five local villains to try and identify who stole Ms. Goody's cake from the bake sale. Each villain made one true and one false statement. Can you determine who stole the cake? Explain.

Below is a summary of their statements:

Arnold: It wasn't Edward. It was Brian.

Brian: It wasn't Charlie. It wasn't Edward.

Charlie: It was Edward. It wasn't Arnold

Derek: It was Charlie. It was Brian.

Edward: It was Derek. It wasn't Arnold

Answer:

- *Considering Brian's statement, if one statement is true and one false, either Charlie or Edward did it.*
- *Considering Derek's statement, if one statement is true and one false, either Charlie or Brian did it.*

Therefore Charlie committed this terrible crime, confirmed by the other statements.

SCO**: LR1: Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.** [CN, ME, PS, R]

Q Four women like to stay fit and active. Last week each cycled, jogged and swam on three different days; and none did the same thing on the same day as anyone else. Can you discover the days on which every woman exercised?

1. Anne jogged the day after Beth swam.
2. The woman who cycled on Tuesday went swimming earlier in the week than Cathy swam.
3. The woman who swam on Wednesday and jogged on Thursday went cycling the day before Doris cycled.
4. Beth cycled the day before Doris jogged.

		cycled				jogged				swam			
		Mon	Tues	Thurs	Fri	Tues	Thurs	Fri	Sat	Mon	Wed	Thurs	Fri
	Anne												
	Beth												
	Cathy												
	Doris												
swam	Mon												
	Wed												
	Thurs												
	Fri												
jogged	Tues												
	Thurs												
	Fri												
	Sat												

Answer:
Ann cycled on Friday, jogged on Saturday and swam on Thursday.
Beth cycled on Thursday, jogged on Tuesday, and swam on Friday.
Cathy cycled on Monday, jogged on Thursday, and swam on Wednesday.
Doris cycled on Tuesday, jogged on Friday and swam on Monday.

		cycled				jogged				swam			
		Mon	Tues	Thurs	Fri	Tues	Thurs	Fri	Sat	Mon	Wed	Thurs	Fri
	Anne				●				●			●	
	Beth			●		●							●
	Cathy	●					●				●		
	Doris		●					●		●			
swam	Mon		●					●					
	Wed	●					●						
	Thurs				●				●				
	Fri			●		●							
jogged	Tues			●									
	Thurs	●											
	Fri		●										
	Sat				●								

SCO**: LR2: Solve problems that involve the application of set theory.** [CN, PS, R, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

LR2: Solve problems that involve the application of set theory.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	LR2: Solve problems that involve the application of set theory.

ELABORATION

In previous grades students will have studied number sets, will have gained an understanding of basic set notation with reference to the domain and range of a function, and will have worked with Venn diagrams.

For this outcome, students begin to formalize their knowledge of Venn diagrams and other graphic organizers. They will examine different ways to represent sets and learn how to use set notation to describe sets and subsets.

Students will examine the relationships between sets and subsets. They will differentiate between finite and infinite sets, and disjoint and overlapping sets. Students will determine if two sets intersect and when they do, they will identify the number of elements in each region, and what each region represents.

Students will use sets and set notation to solve problems and will discover practical applications of set theory, such as conducting Internet searches, solving puzzles, and solving problems that involve three sets.

It is important to note that this is not meant to be a study in abstract algebra. As described by the achievement indicators, the goal of this outcome is to have students be proficient in organizing data based on characteristics and recognizing relationships between elements. The various notations should be the vehicle for communicating the connections in a mathematically valid way, but should be not the main focus of this outcome.

New terminology will include:

Element: an item or object that is part of a set e.g., 5 is an element in the set of prime numbers.

Empty set: a set that contains no elements e.g., the set of dogs with six legs. An empty set is represented as { } or $\emptyset$.

Disjoint sets or mutually exclusive sets: two sets that share no elements. The intersection produces a null set, represented as $A \cap B = \emptyset$.

Subset: a set in which all the elements of the set are also the elements of another larger set, represented as $A \subset B$

Universal set: a set that contains everything including the set itself, represented as U.

Compliment: the set of all elements that are not in a defined set, represented as $\bar{A}$, A' or A^c. If a set A is defined and has a certain number of elements, set $\bar{A}$ contains all elements that are not in set A.

Union - the set of all elements of two or more sets. The union of set A and set B would be represented by $A \cup B$.

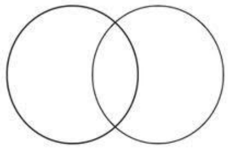
SCO: **LR2: Solve problems that involve the application of set theory.** [CN, PS, R, V]

Intersection - the set of all elements common to two or more sets. The intersection of set A and set B would be represented as $A \cap B$.

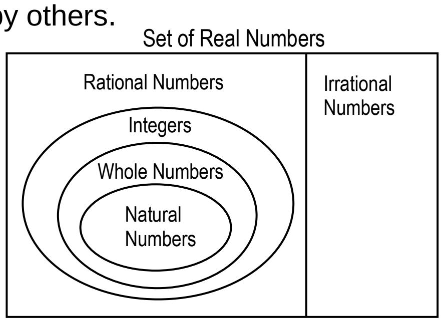
Carroll Diagrams - a yes/no diagram, two-condition diagram

	Prime	Not Prime
Even	2	4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26
Not Even	3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41	1, 9, 15, 21, 25, 27, 33, 35, 39, 41, 45, 49

Venn Diagrams – sets are represented by intersecting circles.



Euler Diagrams – are a less restrictive form of a Venn diagram. Sets may be wholly contained by others.



ACHIEVEMENT INDICATORS

- Provide examples of the empty set, disjoint sets, subsets and universal sets in context, and explain the reasoning.
- Organize information such as collected data and number properties, using graphic organizers, and explain the reasoning.
- Explain what a specified region in a Venn diagram represents, using connecting words (and, or, not), or set notation.
- Determine the elements in the complement, the intersection or the union of two sets.
- Explain how set theory is used in applications such as Internet searches, database queries, data analysis, games and puzzles.
- Identify and correct errors in a given solution to a problem that involves sets.
- Solve a contextual problem that involves sets, and record the solution, using set notation.

SCO**: LR2: Solve problems that involve the application of set theory.** [CN, PS, R, V]

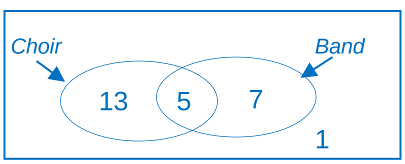
Suggested Instructional Strategies

- Many of the achievement indicators can be combined with the P2 outcomes.
- Have permanent (reusable) examples of Carroll, Euler and Venn Diagrams on the wall in your classroom or a set of reusable diagrams for student use.
- It may be beneficial to start with a discussion about the number systems (natural, whole, integer, etc.) as these would be familiar, and serve as a good example of using an Euler diagram.
- Give students a set of playing cards and have them examine and sort the cards based on their set properties. They should then create graphic organizers and use set notation to describe the relationships.
- Have groups interview other students on various characteristics (on either/or items, such as owning a dog or having a sister) and create a Venn Diagram representing the various sets. Then have a different group analyse the intersections and unions of the sets.
- Cross-curricular strategies may include having the students analyse dichotomous keys (e.g., from Biology) and turning them into Carroll or Euler Diagrams.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

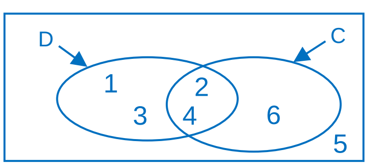
Q Of 26 people in a class, 18 students sing in the chorus, 12 play in the jazz band, and 5 students are in both the chorus and the band. Create a Venn diagram and then determine how many students neither sing in the chorus nor play in the jazz band.

$Answer\text{: } Only\ 1\ person\ does\ not\ participate\ in\ the\ music\ program.$



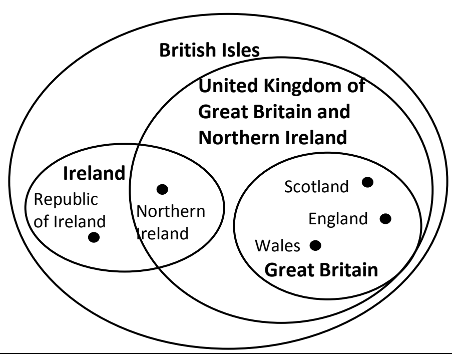
Q Show three number sets from a die as a Venn diagram:
Set C: the set of even numbers,
Set D: the set of numbers less than 5,
Universal set of all the numbers on the die.

$Answer\text{: } \quad C = \{2, 4, 6\} \quad D = \{1, 2, 3, 4\} \quad U = \{1, 2, 3, 4, 5, 6\}$



Q Based on the diagram shown, which of the following statements are true?

- Scotland is part of Great Britain, The United Kingdom and The British Isles $(true)$
- Northern Ireland is part of Great Britain, the United Kingdom and the British Isles. $(false)$
- Ireland is part of the United Kingdom. $(false)$



SCO: **LR3: Solve problems that involve conditional statements. [C, CN, PS, R]**

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

LR3: Solve problems that involve conditional statements.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	LR3: Solve problems that involve conditional statements.

ELABORATION

This outcome is the first exposure students have had to **conditional statements** which are statements of the form "If P, then Q", where P is a condition and Q is a consequence. e.g. "If I eat, then I am full." The following new terminology is defines various types of **conditional statements**.

A **converse** statement is one written from an "$if\text{-}then$" statement. The converse of "$If\ P, then\ Q$" is "$If\ Q, then\ P$". The condition of the original statement becomes the consequence of the converse, and vice-versa.

An **inverse** statement is one that is written from a conditional statement using negatives. The inverse of "$If\ P, then\ Q$" is "$If\ not\ P, then\ not\ Q$."

A **contrapositive** statement is one that is written from a conditional statement using both converse order **and** inverse structure. The contrapositive of "$If\ P, then\ Q$" is "$If\ not\ Q, then\ not\ P$."

A **counterexample** is an example which makes a statement false e.g., a possible counterexample of "$If\ a\ shape\ has\ four\ sides, then\ it\ is\ a\ rectangle$" could be "$a\ rhombus$."

A **biconditional statement** is one with the structure "$P\ if\ and\ only\ if\ (iff)\ Q$" that is true if both a conditional statement and its converse are true e.g., both "$If\ a\ polygon's\ internal\ angles\ sum\ to\ 180°, then\ the\ polygon\ is\ a\ triangle$" is true, and its converse $If\ a\ polygon\ is\ a\ triangle, then\ it\ is\ a\ polygon\ with\ an\ internal\ angle\ sum\ of\ 180°$" is true, therefore the biconditional statement is true, "$A\ polygon\ has\ an\ internal\ angle\ sum\ of\ 180°\ if\ and\ only\ if\ (iff)\ it\ is\ a\ triangle$."

A **truth table** is a graphic organizer used to identify the type of logical statement. ***At this level, the truth table is only used to identify a statement as biconditional or not***.

It is critical that students understand that the truth of any theorem does not necessarily imply the truth of its converse. In some cases a converse is also true, in other cases it is not, so students should always test the truth of a converse. For example, they might write the converse of "$If\ an\ angle\ is\ inscribed\ in\ a\ semicircle, then\ it\ is\ a\ right\ angle$," and see that the converse is not true in general. The discussions arising out of the examination of these types of examples encourage logical thinking.

SCO: **LR3: Solve problems that involve conditional statements. [C, CN, PS, R]**

ACHIEVEMENT INDICATORS

- Analyze an “if-then” statement, make a conclusion, and explain the reasoning.
- Make and justify a decision, using “what if?” questions, in contexts such as probability, finance, sports, games or puzzles, with or without technology.
- Determine the converse, inverse and contrapositive of an “if-then” statement; determine its veracity; and, if it is false, provide a counterexample.
- Demonstrate, using examples, that the veracity of any statement does not imply the veracity of its converse or inverse.
- Demonstrate, using examples, that the veracity of any statement does imply the veracity of its contrapositive.
- Identify and describe contexts in which a bi-conditional statement can be justified.
- Analyze and summarize, using a graphic organizer such as a truth table or Venn diagram, the possible results of given logical arguments that involve bi-conditional, converse, inverse or contrapositive statements.

Suggested Instructional Strategies

- Vary the subject matter of the statements; it is important to use both mathematical and non-mathematical statements to help students make connections to other subject areas.
- Have permanent, reusable graphic organizers on the wall that can be used often while teaching and referred to during individual or group work.
- Use a variety of graphic organizers in conjunction, filling them out simultaneously as a class during instruction. This will deepen student understanding of how each representation works.

SCO: **LR3: Solve problems that involve conditional statements. [C, CN, PS, R]**

Questions (Q) and Activities (Act) for Instruction and Assessment

Act In groups of four, give each student a cue card. Each person begins by writing a conditional statement on the card. It is then passed to the person on their left, who analyzes the original statement as to whether it is true or false. The card is passed again, and the third person to receive it writes the converse (regardless of the outcome at step two). The fourth person analyzes the converse as to whether it is true or false, as the second person did for the original statement. The activity is completed with the original owner of the card writing a bi-conditional statement, if both the original statement and the converse were determined to be true.

Act When playing the games and puzzles associated with outcome LR1, have students write the conditional statements that result from various moves e.g., when playing chess, have students come up with conditional statements such as "If I move my pawn here, then it can be taken by my opponent's bishop."

Q Write the converse of the following statement: "If a triangle has three equal angles, then it is an equilateral triangle."
Answer: If a triangle is an equilateral triangle, it has three equal angles.

Q Determine if the following statements are valid biconditional statements:

"*A polygon is a square if and only if it has four right angles and four equal sides*
Answer: yes

"*An animal is a spaniel iff it is a dog*."
Answer: no

SCO: P1: Interpret and asses the validity of odds and probability statements.
[C, CN, ME]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Probability

P1: Interpret and assess the validity of odds and probability statements.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P1: Interpret and assess the validity of odds and probability statements.

ELABORATION

Students had exposure to calculating probabilities in middle school and in Grade 9. However, this is the first time that probability will be explored in depth as its own topic.

Students will determine probability and odds in various situations, and will interpret and assess probability and odds statements. Teachers should have various materials on hand, such as newspapers, magazines, and textbooks. Outcomes P1, P2, P3 and P4 should be considered collectively when planning.

Odds compares the number of times a favourable outcome will occur to the number of times an unfavorable outcome will occur (part : part). **Probability** compares the number of times a favourable outcome will occur to the number of times all outcomes will occur (part : whole).

For example, the odds that a randomly chosen day of the week is Sunday are 1 **to** 6 $(1\!:6)$. However, the probability of it being Sunday is 1 **in** 7, or $\frac{1}{7}$, and the outcome of interest is indicated first. However, sometimes the outcome of interest is stated second, as in: "$The\ odds\ of\ that\ happening\ are\ 1000\ to\ 1$", meaning it would be highly unlikely that this would occur. It is important to be consistent in vocabulary used, with odds using **"to",** and probability using **"in".**

Students will be required to express both the odds <u>for</u> an outcome and the odds <u>against</u> an outcome, and be clear on which they are describing.

ACHIEVEMENT INDICATORS

- Provide examples of statements of probability and odds found in fields such as media, biology, sports, medicine, sociology and psychology.
- Explain, using examples, the relationship between odds (part-part) and probability (part-whole).
- Express odds as a probability and vice versa.
- Determine the probability of, or the odds for and against, an outcome in a situation.
- Explain, using examples, how decisions may be based on probability or odds and on subjective judgments.
- Solve a contextual problem that involves odds or probability.

SCO: P1: Interpret and asses the validity of odds and probability statements.
[C, CN, ME]

Suggested Instructional Strategies

- Students should be able to work comfortably with probabilities and odds interchangeably and in various forms (fractions, decimals, ratios, etc.). Start by having students compare like forms such as fractions with fractions. As they become more familiar with the idea, give problems with different expressions and forms such as fractions and percentages.
- Use ordering (e.g., least likely to most likely) as a way of familiarizing students with the various expressions of odds and probability.
- Incorporate discussion about interpretations. For example, ask "Does this mean that the most likely event will happen, or that the least likely event will not happen?".
- The outcome focuses on how valid odds and probability statements are. Make it a common part of your teaching to ask "Is this valid?". This should incorporate discussion about the source of the statement, and if that source is reliable and reputable.
- Incorporate discussion about the nature of subjective judgment. Why do people see that there is a 60% chance of rain, yet still decide to go to the beach?

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Give students information in the form of odds or probabilities and have them interpret the information in a written description. E.g., give them the odds on a series of boxing matches and have them come up with a "story" to go with them.

Q Express the following in the requested form:

a) There is a 0.25 probability of drawing a heart from a standard deck of cards. What are the odds of this happening?

$Answer: \quad odds(heart) = 0.25 : 0.75 \ \ or \ \ 1 : 3$

b) The odds against rolling a number less than 3 on a die are 2:1. What is the probability of rolling a number less than 3?

$Answer: \ Odds\ against\ (rolling\ a\ number < 3) = 2:1$

$P(rolling\ a\ number < 3) = \dfrac{1}{3}$

Q A bag of candy has three yellow, four red and five green jellybeans. What is the probability of drawing a jellybean at random that is:

a) green?

b) not red?

c) yellow or red?

$Answers: \quad a)\ P(G) = \dfrac{5}{12} \qquad b)\ \ P(R) = \dfrac{8}{12} = \dfrac{2}{3} \qquad c)\ P(Y\ or\ R) = \dfrac{7}{12}$

SCO**: P2: Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events. [**CN, PS, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

P2: Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P2: Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.

ELABORATION

Mutually exclusive events are events that cannot happen concurrently. For example, you cannot flip a coin and get "head" and "tail" at the same time – these are mutually exclusive. The intersection of two mutually exclusive events in terms of their set spaces is a null set (this can be discussed in conjunction with outcome LR2).

Non-mutually exclusive events are events in which at least one outcome is in common. For example, if event A is rolling an even number on a six-sided die, and event B is rolling a number less than 3 on a six-sided die, then A and B are non-mutually exclusive because in the case of rolling a 2, both outcomes would be satisfied.

Two events are said to be **complementary** events if one or the other must happen. For example, getting "heads" on a coin flip and getting "tails" on a coin flip are complementary because one or the other must happen, and to not get one result is to get the other.

The notation for complementary events, is shown as: $P(A)$ for the probability of event A, and $P(\bar{A})$ for the probability of the complement of event A. In all cases: $P(A) + P(\bar{A}) = 1$

Any event, even those with more than two outcomes, can be stated in a complementary manner. For example, if event A is selecting a red jellybean, the compliment of A would be not selecting a red jellybean, regardless of how many additional colours of jellybeans are available for selection.

ACHIEVEMENT INDICATORS

- Classify events as mutually exclusive or non–mutually exclusive, and explain the reasoning.
- Determine if two events are complementary, and explain the reasoning.
- Represent, using set notation or graphic organizers, mutually exclusive (including complementary) and non–mutually exclusive events.
- Solve a contextual problem that involves the probability of mutually exclusive or non-mutually exclusive events.
- Solve a contextual problem that involves the probability of complementary events.
- Create and solve a problem that involves mutually exclusive or non–mutually exclusive events

SCO**: P2: Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events. [**CN, PS, R, V]

Suggested Instructional Strategies

- Visualization of the events as sets will help some students understand the difference between mutually exclusive and non-mutually exclusive pairs of events. It will be useful to have a permanent, reusable Venn diagram or Carroll diagram of both mutually exclusive and non-mutually exclusive events (see outcome LR2).
- Have students use manipulatives such as cards or multilink cubes during class time to simulate events as a way of helping them discover the differences between mutually exclusive and non-mutually exclusive.
- Incorporate set notation into your teaching as the material is presented, as a way of helping students meet the achievement indicators and to make connections with other GCOs (see outcome LR2).
- Use a simulator program or website to generate large sets of data on typical probability events, and have students discuss what happens with large sets of data. This is a good activity to use when discussing both probability and statistics (see outcome S1).

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Joey is asked to draw a card from a standard deck of 52, without looking at it. What is the probability of each of the following occurrences?

a) his card being a Jack

b) his card being a Queen or a King

c) his card being a Queen or a heart

d) his card being an Ace or not being an Ace

Answers:

$a)\ \ P(J) = \frac{4}{52} = \frac{1}{13}$

$b)\ \ P(Q\ or\ K) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

$c)\ \ P(Q\ or\ heart) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

$d)\ \ P(A\ or\ not\ A) = \frac{4}{52} + \frac{48}{52} = \frac{52}{52} = 1$

SCO**: P3: Solve problems that involve the probability of two events.** [CN, PS, R]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

P3: Solve problems that involve the probability of two events.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P3: Solve problems that involve the probability of two events.

ELABORATION

This outcome focuses on multiple events, and how the occurrence of one event does or does not affect subsequent events.

Dependent events are affected by previous events, and the number of favourable events and possible events is reduced as items or options are removed. For example: You are eating a bag of jellybeans one candy at a time. At a given point, there are twelve jellybeans: three yellow, four green and five red. To determine the probability of picking a red jellybean next, eating it, and then picking a yellow jelly bean after that, you would determine the probability of each event and multiply them together:

$$P(red, yellow) = \frac{5}{12} \times \frac{3}{11} = \frac{15}{132} = \frac{5}{44}$$

The number of possible jellybeans to choose from on the second draw has been reduced because the red jellybean was not replaced.

Independent events are events in which subsequent events are not affected by previous events and probabilities remain the same from trial to trial for the same event. For example: You have a bag of 12 coloured marbles with three yellow, four green and five red. To find the probability of drawing a red marble, replacing it, then drawing a yellow marble, you would determine the probability of each event and then multiply them together:

$$P(red, yellow) = \frac{5}{12} \times \frac{3}{12} = \frac{15}{144} = \frac{5}{48}$$

The number of possible marbles has not changed because the first marble drawn was replaced.

ACHIEVEMENT INDICATORS

- Compare, using examples, dependent and independent events.
- Determine the probability of an event, given the occurrence of a previous event.
- Determine the probability of two dependent or two independent events.
- Create and solve a contextual problem that involves determining the probability of dependent or independent events.

SCO: **P3: Solve problems that involve the probability of two events.** [CN, PS, R]

Suggested Instructional Strategies

- Have students use manipulatives (marbles, cards etc.) during class time to simulate events as a way of helping them discover the differences between dependent and independent events.
- This outcome's achievement indicators focus on determining probabilities, and as such, incorporating practice into class time should be a priority.
- Have students sort independent events from dependent events, without calculating the probabilities.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Determine whether the situation describes a series of dependent events or independent events.

a) A caramel chocolate is taken out of a box and eaten. A coconut cream chocolate is taken out of the box and eaten.

b) A deck of cards is cut, and the top card which is the Queen of Clubs is given to the first player. The deck of cards is cut a second time and this time the 5 of Clubs is the top card and is given to the second player.

c) A deck of cards is cut, and a red card shows. The deck of cards is cut a second time and a card with an odd number shows.

d) I vote for the Liberal candidate. You vote for an NDP candidate.

Answers: a) Dependent – the second person is less likely to get a caramel chocolate.
b) Dependent c) Independent – all cards remain in the deck
d) Independent assuming you are not affected by who I vote for

Q There are 300 numbered parking spaces in a parking garage. Parking spaces are randomly assigned by a parking officer as you enter. Joey, Alex and Katie pull into the garage, in that order, and are the first to arrive that morning. If Joey is assigned space #56, what is the probability that Alex, then Katie, are also assigned one of the first 100 spaces?

$Answer:\ P(Alex)\times P(Katie)=\frac{99}{299}\times\frac{98}{298}=\frac{9702}{89102}\cong 0.11$

Q Micah and Callie go with their parents to an ice cream shop. If there are 29 flavours to choose from, what is the probability that Micah chooses chocolate and Callie chooses banana-nut?

$Answer:\ \frac{1}{29}\times\frac{1}{29}=\frac{1}{841}$

Q A bag contains 10 blue marbles and 6 red marbles. What is the probability of randomly selecting two red marbles in a row, provided that the marbles are not replaced after being selected?

$Answer:\ \frac{6}{16}\times\frac{5}{15}=\frac{30}{240}=\frac{1}{8}$

SCO: P4: **Solve problems that involve the fundamental counting principle. [**PS, R, V,]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

P4: Solve problems that involve the fundamental counting principle.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P4: Solve problems that involve the fundamental counting principle**.**

ELABORATION

Students were introduced to the role of probability in various situations in Grade 9 but have not calculated probability since grade 8. For this outcome, students will use graphic organizers, such as tree diagrams, to visualize and calculate sample space. From patterns observed, they will formulate an understanding of the **Fundamental Counting Principle**.

This **Fundamental Counting Principle**, also known as the **Multiplication Principle**, states that if one event has m possible outcomes and a second independent event has n possible outcomes, there will be a total of $m \times n$ possible outcomes for these two outcomes occurring together. This enables finding the number of outcomes without listing and counting each one.

ACHIEVEMENT INDICATORS

- Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.
- Represent and solve counting problems, using a graphic organizer.
- Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.
- Solve a contextual problem by applying the fundamental counting principle.

Suggested Instructional Strategies

- Begin instruction with having students construct tree diagrams to represent different outfits (tops, pants, socks, shoes) or different meal combinations served in a restaurant (appetizers, main course, desserts, beverages), and then count the total number of different arrangements/combinations in each situation. Ask students to write a journal entry summarizing any shortcuts they observe as to how they can calculate the total number of outcomes. Verify the results obtained by applying the *Fundamental Counting Principle* to each situation.

SCO: P4: **Solve problems that involve the fundamental counting principle. [**PS, R, V,]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q A restaurant offers "select-your-own" sundaes choosing one item from each of three categories:

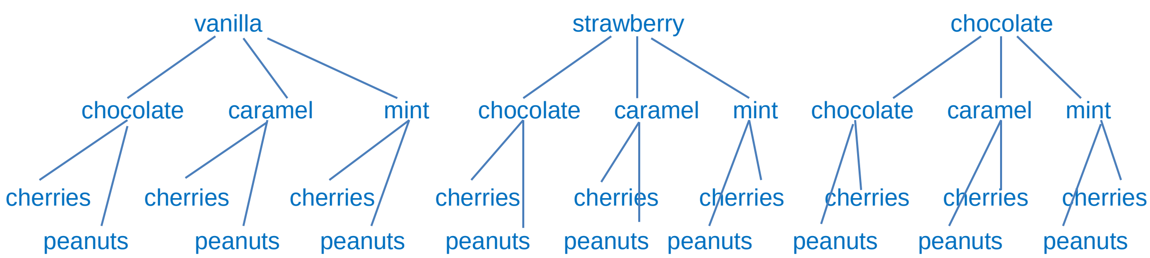
Ice Cream	**Sauce**	**Extras**
vanilla	chocolate	cherries
strawberry	caramel	peanuts
chocolate	mint	

a) Using a tree diagram, list all possible desserts that can be ordered.

b) Would you expect the choices of a dessert to be equally likely for most customers? Why or why not?

c) If the probability of selecting chocolate ice cream is 40%, and vanilla is 10%, chocolate sauce is 70%, and cherries 20%, what is the dessert with the highest probability of being selected.

Answer: *a*)



b) *Answers will vary.*

c) *Ice cream* – *chocolate* 40%, *vanilla* 10%, *strawberry* 50%

Sauce – *chocolate* 70%, *caramel and mint* 30% *total*

Extras – *cherries* 20%, *peanuts* 80%

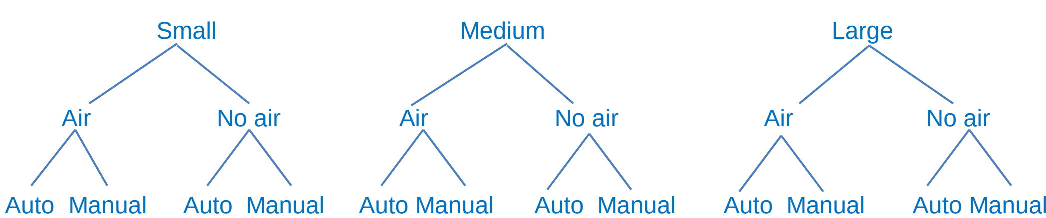
∴ *Strawberry ice cream with chocolate sauce and peanuts has the highest probability of being selected*

Q A certain model car can be ordered with one of three engine sizes, with or without air conditioning, and with automatic or manual transmission.

a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.

b) Suppose you want the car with the smallest engine, air conditioning, and manual transmission. A car agency tells you there is only one of the cars on hand. What is the probability that it has the features you want, if you assume the outcomes to be equally likely?

Answer: *a*)



b) $P(Smallest) = \frac{1}{3}$ $P(AC) = \frac{1}{2}$ $P(Manual) = \frac{1}{2}$ $P(all\ three\ features) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$

SCO: P4: **Solve problems that involve the fundamental counting principle. [**PS, R, V,]

Q In a restaurant there are 4 kinds of soup, 12 different entrees, 6 desserts, and 3 kinds of drinks.

a) How many different four-course meals can a patron choose from?

b) If 4 of the 12 entrees are chicken and two of the desserts involve cherries, what is the probability that someone will order wonton soup, a chicken dinner, a cherry dessert and milk?

$Answer$: $a)$ $14 \cdot 12 \cdot 6 \cdot 3 = 864$ $different\ four\ course\ meals.$

$b)$ $\frac{1}{4} \cdot \frac{4}{12} \cdot \frac{2}{6} \cdot \frac{1}{3} = \frac{1}{108}$

SCO: **P5: Solve problems that involve permutations.** [ME, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

P5: Solve problems that involve permutations

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P5: Solve problems that involve permutations.

ELABORATION

This outcome should be taught in conjunction with ***P6*** *which involves combinations.*

As students create tree diagrams and determine the number of possible outcomes using the fundamental counting principle (P6), they should learn to recognize and use **n! (n factorial)** to represent the number of ways to arrange n distinct objects.

n factorial or **n!** is the product of all positive integers less than or equal to n.
In general, $n! = n(n-1)(n-2)\ldots(3)(2)(1),\ where\ n \in N\ and\ 0! = 1$.
e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

When order matters, factorials can be used to find the number of possible arrangements or **permutations** for a given number of people or objects.

For example, $3!$ will give the number of permutations for three people standing in a line. There are three people to choose from for the first position, two people left to choose from for the second position in the line, and only one person to choose for the end of the line, or $3! = 3 \times 2 \times 1 = 6$ possible arrangements.

Factorials are also used when there are a limited number of positions for a greater number of people or objects, and the order matters. However, in this case the number of permutations is determined with reference to both the total number of elements and the number of positions available.

For example, if 1st, 2nd and 3rd prizes are to be awarded to a group of 8 trumpeters, there is a choice of 8 people for the 1^{st} prize, 7 people for the 2^{nd} prize, and 6 people for the 3^{rd} prize. Therefore there are $8 \times 7 \times 6 = 336$ ways to award 3 prizes to 8 people (assuming no difference between performances!), or more formally stated, there are 336 permutations of 8 objects taken 3 at a time.

This is expressed ${}_8P_3 = 8 \times 7 \times 6$.

It is equivalent to ${}_8P_3 = \frac{8!}{5!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$.
The general form is ${}_nP_r$ for n objects taken r at a time, so ${}_nP_r = \frac{n!}{(n-r)!}$

Another situation is when some objects of a set are the same. In this case there are fewer permutations because some arrangements are identical. To avoid counting the permutations of letters more than once, the number of permutations will be equal to $\frac{n!}{n_1, n_{2,} \ldots n_r}$ when n is the total number of objects, and $n_1, n_2, \ldots\ n_r$ are numbers of identical objects, with $= n_1 + n_2 + \ ... + n_r$. For example, for the word BANANA there is one B, ($n_1 = 1$), two N's ($n_2 = 2$) and three A's ($n_3 = 3$), so the total number of permutations of the letters for the word BANANA will be $P = \frac{6!}{1!2!3!} = 60$.

Students will also learn to use permutations to solve **probability questions**, first determining the number of permutations of n objects taken r at a time, ${}_nP_r = \frac{n!}{(n-r)!}$, and then relating that to the probability of one particular case.

For example, to determine the probability that A, K, Q of Spades will be dealt in that order as the first three cards, from a well shuffled deck of 52 cards, the number of permutations for 52 cards taken 3 at a time, would be determined as ${}_{52}P_3 = \frac{52!}{(52-3)!} = \frac{52!}{49!} = 52 \times 51 \times 50 = 132\ 600$. The probability of one particular sequence is only 1 out the total number of possibilities, or $P(A, K, Q) = \frac{1}{132\ 600}$.

ACHIEVEMENT INDICATORS

(It is intended that circular permutations not be included.)

- Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.
- Represent the number of arrangements of n elements taken n at a time, using factorial notation.
- Determine, with or without technology, the value of a factorial.
- Simplify a numeric or algebraic fraction containing factorials in both the numerator and denominator.
- Solve an equation that involves factorials.
- Determine the number of permutations of n elements taken r at a time.
- Determine the number of permutations of n elements taken n at a time where some elements are not distinct.
- Explain, using examples, the effect on the total number of permutations of n elements when two or more elements are identical.
- Generalize strategies for determining the number of permutations of n elements taken r at a time.
- Solve a contextual problem that involves probability and permutations.

SCO: **P5: Solve problems that involve permutations.** [ME, PS, R, T, V]

Suggested Instructional Strategies

- Using their prior knowledge, allow students to work in groups to solve a simple problem similar to the following:

  Adam, Marie, and Brian line up at a banking machine. In how many different ways could they order themselves?

  Using a systematic list students might come up with the following solution: $AMB, ABM, MBA, MAB, BAM, BMA$.

  Repeat with a similar type question, asking students to look for patterns which involve the use of factorials. Have them generalize the formula to apply for n objects selected r at a time.

- When students are comfortable determining the number of permutations, introduce the use of permutations to determine wanted probabilities.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Arrange the following in order of magnitude.

a) $6!$ b) $11!$ c) $\frac{15!}{13!}$ d) $3!\,4$

e) $\frac{9!}{2!}$ f) $\frac{10!}{4!}$ g) $\frac{100!}{57!}$ h) $4! - 3!$ i) $\frac{8!}{7!}$

Pick three of the above expressions and create a problem in which these symbols would be used in the solution.

Answer: In order from least to greatest

$i)\ \frac{8!}{7!}\quad h)\ 4!-3!\quad d)\ 3!\,4\quad c)\ \frac{15!}{13!}\quad a)\ 6!\quad f)\ \frac{10!}{4!}\quad e)\ \frac{9!}{2!}\quad b)\ 11!\quad g)\ \frac{100!}{57!}$

Q Write each as a ratio of factorials.

a) $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
b) $35 \times 34 \times 33 \times 32 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
c) $14 \times 13 \times 8 \times 7 \times 3 \times 2 \times 1$
d) $25 \times 24 \times 23 \times 5 \times 4 \times 3 \times 2 \times 1$

Answers: $a)\ 7!\quad b)\ \frac{35!\,6!}{31!}\quad c)\ \frac{14!\,8!\,3!}{12!\,6!}\quad d)\ \frac{25!\,5!}{22!}$

Q Consider the word COMPUTER and the ways you can arrange its letters using each letter only once.

a) One possible permutation is PUTMEROC. Write five other possible permutations.
b) Use factorial notation to represent the total number of permutations possible. Write a written explanation to explain why your expression makes sense.

Answer:
a) Answers will vary and may include $UTERCOMP, PMOCRETU, OCRETUMP, MURPCOTE, TOCERUMP$
b) $8! = 40\,320$

SCO: **P5: Solve problems that involve permutations.** [ME, PS, R, T, V]

Q The electronic lock to a house has six buttons. To open, a four-button combination has to be entered in sequence and can be tried only once before the lock freezes.

(a) If none of the buttons is repeated, what is the probability of randomly entering the correct combination?
(b) If you are allowed to repeat buttons, what is the probability of randomly entering the correct combination?

$Answers$: $a)\ \frac{1}{6\times5\times4\times3}=\frac{1}{360}$ $\quad b)\ \frac{1}{6\times6\times6\times6}=\frac{1}{1296}$

Q Determine the number of ways four different graduation scholarships can be awarded to 30 students under each of the following conditions:

a) No student may receive more than one scholarship.
b) Any student may receive any number of scholarships.

$Answers$: $a)\ 30\times29\times28\times27=657\,720$
$b)\ 30\times30\times30\times30=810\,000$

Q Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random.

(a) In how many ways could the 9 positions be filled
(b) What is the probability that one particular person (e.g., Sebastien) will be the pitcher?
(c) What is the probability that one of three people (e.g., George, Sandy or Errol) will be first baseman?
(d) What is the probability that one of three people (e.g., Carolyn, Ashley, or Sandy) will be first baseman, and that one of two other people (e.g., Sam or Marg) will be pitcher?

$Answers$: $a)\ 9!=362\,880$ $\quad b)\ \frac{1}{9}$ $\quad c)\ \frac{3}{9}=\frac{1}{3}$ $\quad d)\ \frac{3}{7}\times\frac{2}{8}=\frac{6}{72}=\frac{1}{12}$

Q Raffle tickets for a new bicycle are given out at a fundraiser. Each number is three digits, and the first digit cannot be zero.

(a) What is the probability of ticket number 514 winning the bicycle? What assumption did you make?
(b) What is the probability that a ticket with three as a second digit will win the bicycle?

$Answers$: $a)\ \frac{1}{9\times10\times10}=\frac{1}{900}$ $\quad b)\ \frac{9\times1\times10}{9\times10\times10}=\frac{1}{10}$

SCO: **P6: Solve problems that involve combinations.** [ME, PS, R, T, V]	

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

P6: Solve problems that involve combinations.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P6: Solve problems that involve combinations**.**

ELABORATION

For this outcome, students will investigate **combinations,** where the order of the selection is not important.

For example , if a committee of 3 people is selected from a group of 5 individuals, it is not important in what order they are named, the committee still includes the same 3 people. However, the number of permutations of choosing ABC from the larger set of 5 or ${}_5P_3$, will include all $3!$ permutations of ABC. To eliminate counting each permutation as unique, the total number of possible permutations must be divided by $3!$. Thus the number of combinations, denoted as ${}_nC_r$, or $\binom{n}{r}$ or n choose r, would be ${}_nC_r = \frac{{}_nP_r}{r!}$ which simplifies to $\frac{n!}{r!(n-r)!}$. For this example ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3\times2\times1}{(3\times2\times1)(2\times1)} = 10$, so there are 10 possible ways to select 3 people from a group of 5.

In the case of selecting two groups from the same larger group, the number of possible combinations will combine both groups.

For example, if a committee of 4 people and a committee of 3 people are selected from a group of 10 people, and no person is assigned to both committees the combinations for each committee will be determined first.

For the first committee of 4 from 10 people, ${}_{10}C_4 = \binom{10}{4} = 210$, so there are 210 ways to form the committee. There are just 6 people left, so for the second committee of 3 from 6 people, ${}_6C_3 = \binom{6}{3} = 20$, and there are 20 ways to form this committee. Combining the two committees, there are $210 \times 20 = 4200$ ways to form both committees from 10 people. Selecting the smaller committee first will yield the same result.

Combinations are sometimes used along with other counting techniques. Student should be comfortable using technology to determine combinations, permutations and factorials.

For example, a 17 member student council at the high school consists of 9 girls and 8 boys, and one of the committees has 4 council members, and must have 2 girls and 2 boys. There are ${}_9C_2 = \frac{9!}{(2!)(7!)} = 36$ ways of selecting the 2 girls, and ${}_8C_2 = \frac{8!}{(2!)(6!)} = 28$ ways of selecting 2 boys.

Because the committee must include 2 girls and 2 boys, there are $36 \times 28 = 1008$ ways of forming the committee. If the four committee members are selected at random there are ${}_{17}C_4 = \frac{17!}{4!(13!)} = 2380$ possible combinations. Therefore the probability that the committee will consist of 2 boys and 2 girls is $\frac{1008}{2380} \approx 0.424$.

SCO**: P6: Solve problems that involve combinations.** [ME, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Explain, using examples, why order is or is not important when solving problems that involve permutations or combinations.
- Determine the number of combinations of n elements taken r at a time.
- Generalize strategies for determining the number of combinations of n elements taken r at a time.
- Solve a contextual problem that involves combinations and probability.

Suggested Instructional Strategies

- Group work is effective here. Have groups of students use systematic lists, and to look for patterns as they solve problems. Having worked previously with permutations and the formula involving factorials, now extend the formula. Have them work towards generalizing the formula to apply for n objects selected r at a time.
- When students are comfortable determining the number of combinations, introduce the use of combinations to determine wanted probabilities.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q There are three black marbles and two white marbles in a box. Without looking in the box, choose two of the five marbles. How many ways can:

a) Two marbles that are the same colour be selected?
b) Each marble be a different colour?
$Answers:\ a)\ 4 \quad b)\ 6$

Q If a coin is flipped five times, what is the probability of flipping four heads and one tail?
$Answer:\ a)\ \frac{5C_1}{2^5} = \frac{5}{32}$ *or this could be solved using a tree diagram*

Q A scratch-and-win game has nine prize boxes. You are allowed to scratch three boxes and if the three pictures show identical objects, you win the object.

a) If the ticket has three pictures of a bicycle, and the other six pictures are of six other things, what is the probability that you will win the bicycle?
b) If the ticket has four pictures of a coffee, and the other five pictures show five different things, what is the probability that you will win a coffee?
c) If the ticket has three pictures of a coffee, three pictures of a donut and three pictures of bicycle, what is the probability that you will win one of these three things?

$Answers:\ a)\ \frac{1}{9C_3} = \frac{1}{84} \quad b)\ \frac{4C_3}{9C_3} = \frac{4}{84} = \frac{1}{21} \quad c)\ \frac{1}{9C_3} + \frac{1}{9C_3} + \frac{1}{9C_3} = \frac{3}{9C_3} = \frac{3}{84}$

SCO**: P6: Solve problems that involve combinations.** [ME, PS, R, T, V]

Q Fran bought $20\ \ 6/49$ tickets when she heard that the jackpot was $\$10\,000\,000$. Her friend Anna told her that she was wasting her money, but Fran responded that, "If I only buy one ticket I only have one chance of winning, but if I buy 20 tickets I have 20 chances." How should Anna explain to Fran that degree to which buying 20 tickets instead of 1 ticket actually increases her chances of winning?

Answer: The probability of winning with one ticket is $\frac{1}{49C_6} = \frac{1}{13\,983\,816}$.

The probability of winning with 20 tickets is $\frac{20}{49C_6} = \frac{5}{34\,954\,954}$.

The difference between the two is negligable.

Q A piggy bank contains 5 dimes and 10 quarters. It is shaken until two coins drop out.

a) What is the probability that two quarters will fall out?
b) What is the probability that two dimes will fall out?
c) What is the probability that one of each type of coin will fall out?

$$Answers:\ a)\ \frac{10C_2}{15C_2} = \frac{3}{7} \quad b)\ \frac{5C_2}{15C_2} = \frac{2}{21} \quad c)\ \frac{5C_1 \cdot 10C_1}{15C_2} = \frac{10}{21}$$

Q A committee of five students is chosen randomly from a group of seven ten-year olds and nine eleven-year olds. What is the probability of each of the following events?

a) Two of the students chosen are ten, and three are eleven.
b) Four of the students chosen are ten, and one is eleven.

$$Answers:\ a)\ \frac{7C_2 \cdot 9C_3}{16C_5} = \frac{21}{52} \quad b)\ \frac{7C_4 \cdot 9C_1}{16C_5} = \frac{15}{208}$$

Q A golf bag contains 12 green balls, 4 red balls, and 18 white balls. Four balls are picked at random, one for each of the four people about to tee off.

a) What is the probability of removing 4 white balls?
b) What is the probability of removing 3 white balls?
c) What is the probability of removing 2 green balls, 1 red ball, and 1 white ball?

$$Answers:\ a)\ \frac{18C_4}{34C_4} = \frac{45}{682} \quad b)\ \frac{18C_3}{34C_4} = \frac{6}{341} \quad c)\ \frac{12C_2 \cdot 4C_1 \cdot 18C_1}{34C_4} = \frac{54}{527}$$

SCO: **P7: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).** [CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

P7: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	P7: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

ELABORATION

This outcome will introduce new material to students, though they will build on their knowledge of multiplying binomials (This outcome is also in Pre-Calculus B 120 as PCB4). For $(a + b)(c + d) = ac + bc + ad + bd$, students should notice that each term in the expansion has one factor from $(a + b)$ and one factor from $(c + d)$. So, for example, ac has two factors a and c. The a is from $(a + b)$ and the c is from $(c + d)$. Thus the number of terms in the expansion is four since there are two choices from $(a + b)$ and two choices from $(c + d)$. Students should also notice that since there are two factors, $(a + b)$ and $(c + d)$, there are two factors in each term of the expansion.

The product of one binomial and itself will also follow the same pattern, $(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy$, but the multiplication would be completed by collecting the like terms and using exponents, $x^2 + 2xy + y^2$.

Students should explore patterns as they expand $(x + y)^n$ to develop an understanding of Pascal's triangle and the binomial theorem, building on their knowledge of combinations.

For example,

for $(x + y)^5 = (x + y)(x + y) \ldots (x + y) \rightarrow xxxxx + xxxxy + \cdots + yyyyx + yyyyy$.
Each term is made up of five factors and using exponents, as in $x^a y^b$ where $a + b = 5$.
So, for term 1, $xxxxx \rightarrow x^5 \rightarrow x^5 y^0 \rightarrow 5 + 0 = 5$.
Term 2, $xxxxy \rightarrow x^4 y \rightarrow x^4 y^1 \rightarrow 4 + 1 = 5$.
Term 3, $xxxyy \rightarrow x^3 y^2 \rightarrow 3 + 2 = 5$ etc.

Combinations can be used to determine the coefficients for each term, by determining the number of ways different terms occur.

For example,
The number of times x^5 or y^5 occurs, can be determined as ${}_5C_5 = 1$.
The number of times x^4 or y^4 occurs would be ${}_5C_4 = 5$,
The number of times x^3or y^3occurs is ${}_5C_3 = 10$.
Students should then link the 5th row of Pascal's triangle, 1 5 10 10 5 1, to the binomial theorem.

SCO**: P7: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).** [CN, R, V]

ACHIEVEMENT INDICATORS

- Explain the patterns found in the expanded form of $(x+y)^n,\ n \le 4\ and\ n \in N$, by multiplying n factors of $(x+y)$.
- Explain how to determine the subsequent row in Pascal's triangle, given any row.
- Relate the coefficients of the terms in the expansion of $(x+y)^n$ to the $(n+1)$ row in Pascal's triangle.
- Explain, using examples, how the coefficients of the terms in the expansion of $(x+y)^n$ are determined by combinations.
- Expand, using the binomial theorem, $(x+y)^n$.
- Determine a specific term in the expansion of $(x+y)^n$.

Suggested Instructional Strategies

- For the binomial, $(x+y)$ have students find simplified expressions for $(x+y)^2$, $(x+y)^3, (x+y)^4, etc.$, look for patterns in their coefficients, and find a connection between the expansion power and that same row in the Pascal's Triangle with respect to the coefficient values. $((x+y)^0)=1$ is the top row $(row\ 0)$.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q What is the coefficient of the x^4y^2 term?

a) $(x\ +\ y)^6$

$Answer\colon\ (x+y)^6 = 6C_2(x^4y^2) = \frac{6!}{4!(6-4)!} = \frac{6\times5}{2\times1} = 15$

$\boldsymbol{or}\ \ (x+y)^6 = x^6 + 6x^5\mathrm{y} + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6\mathrm{x}y^5 + y^5$

$The\ coefficient\ is\ \mathbf{15}.$

b) $(x\ +\ 2y)^6$

$Answer\colon\ (x+2y)^6 = 6C_2x^4(2y)^2 = 6C_2(x^4 4y^2) = \frac{(4)6!}{4!\,(6-4)!} = 60$

$\boldsymbol{or}\ \ (x+2y)^6 = x^6 + 6x^5(2\mathrm{y}) + \mathbf{15x^4(2y)^2} + 20x^3(2y)^3 + 15x^2(2y)^4 + 6\mathrm{x}(2y)^5 + (2y)^6$

$The\ coefficient\ is\ \mathbf{60.}$

c) $(x-\ y)^6$

$Answer\colon\ (x-y)^6 =\ {}_6C_2[x^4(-y)^2] = 6C_2(x^4y^2) = \frac{6!}{4!(6-4)!} = \frac{6\times5}{2} = 15$

$\boldsymbol{or}\ \ (x-y)^6 = x^6 + 6x^5(-\mathrm{y}) + \mathbf{15x^4(-y)^2} + 20x^3(-y)^3 + 15x^2(-y)^4 + 6\mathrm{x}(-y)^5 + (-y)^5$

$The\ coefficient\ is\ \mathbf{15.}$

SCO**: P7: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).** [CN, R, V]

Q From left to right, find the specified term in each expansion.

a) $10th$ term in $(x+\ y)^{12}$

b) $20th$ term in $(x+y)^{19}$

c) $8th$ term in $(a+b)^{10}$

$Answers$: $a)\ 12\ C_9 = \frac{12!}{9!(12-9)!} = \frac{12\times11\times10}{3\times2} = 220$ 10th term is $220x^3y^9$

$b)\ 19\ C_{19} = \frac{19!}{19!(19-19)!} = 1$ 20th term is y^{19}

$c)\ 10\ C_7 = \frac{10!}{7!(10-7)!} = \frac{10\times9\times8}{3\times2} = 1$ 8th term is $120a^3b^7$

Q a) Find the sum of the elements in each row, for the first six rows of Pascal's Triangle.

b) Rewrite the sums as exponents with the base 2.

c) Compare the exponents of the sums to the number of elements in the corresponding row. What do you notice?

$Answers$: $a)$

1	$sum = 1$
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64

$b)$

sum	Sum as 2^x	# elements in row
1	2^0	1
2	2^1	2
4	2^2	3
8	2^3	4
16	2^4	5
32	2^5	6
64	2^6	7

$c)$ *The number of elements in a row* $=$ *exponent* $+1$

Q Betty Lou missed math class today. Helen phoned her at night to tell her about how combinations are helpful when expanding binomials. Write a paragraph or two about what Helen would have told her.

Answer: Combinations are helpful when expanding binomials because it is a quick way to find the coefficient of a term.

SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.** [C, CN, PS, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Relations and Functions

RF1: **Represent data using polynomial functions (of degree ≤ 3), to solve problems.**

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
RF2: Demonstrate an understanding of the characteristics of quadratic functions, including: vertex, intercepts, domain and range, axis of symmetry.	**RF1:** Represent data using polynomial functions (of degree ≤ 3), to solve problems.

ELABORATION

The approach to graphing in this outcome is similar to that for quadratics in *Foundations of Mathematics 110*. The intention of the *Foundations Pathway* is to address the needs of students going on to the academic arts and social science programs and thus the primary goal is to understand the characteristics of the graphs and their relationship to applications.

During the intermediate grades students have been encouraged to translate between different representations of a relationship. For example, from tables they should be able to describe relationships and make equations and graphs; from graphs of situations they should be able to describe the situations in words and with equations; from a written description they should be able to sketch a graph or make a table.

Students have analyzed and applied linear functions and have come to understand that a linear relation represents a constant growth rate. Students have also worked with quadratic functions and their properties: vertex, intercepts, domain, and range. Understanding the connections among various representations of relationships will facilitate students' learning (e.g., connecting slope to rate of change or initial values to intercepts).

Students should, through a variety of experiences with functions, come to recognize the elements in a real-world problem that suggest a particular model. For example, area, accelerated motion, and trajectory suggest quadratic functions.

Cubic functions will be new to most students. Traditionally, polynomials with degree 3 or more are called higher degree polynomials. Students will learn to identify the characteristics of a polynomial function with real coefficients including the maximum number of x-intercepts (identifying coordinates from graphs and factored form of equations), the y-intercept, the domain, the range, the possible number of turning points (local maxima or minima), and the end behaviour.

SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.**
[C, CN, PS, T, V]

In this section students will discover how to make the connections between the equations of polynomials and their graphs, in order to predict where different features will occur.

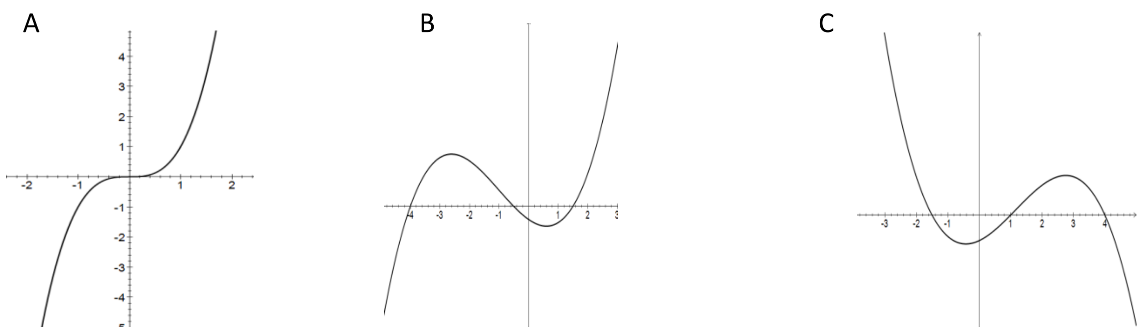
For example:

$y = 5$	constant polynomial (zero degree)
$y = 5x + 2$	linear polynomial (1^{st}degree, 1 x-intercept)
$y = 5x^2 + 2x - 1$	quadratic polynomial (2^{nd} degree, max 2 x-intercepts)
$y = 5x^3 - x + 3$	cubic polynomial (3^{rd}degree, max 3 x-intercepts)

Students should be able to identify the degree of any polynomial graph by looking at its shape.

For example, any 3^{rd}degree polynomial has one of the shapes shown below. Graph A is the graph of the function $y = x^3$. Although it only crosses the x-axis once, it actually has a triple root of 0. Also, it has no local maximum or minimum.

Graphs B and C are of the general cubic equation $y = ax^3 + bx^2 + cx + d$. Graph B shows a function in which a is positive, and Graph C a function in which a is negative. Both have 3 roots.



Students will not be expected to factor cubic polynomials to find the x-intercepts. However, they should be able to determine the y-intercept from the equation, and determine x-intercepts from the factored form of the equation.

Students should be able to explain how the degree of the function and the leading coefficient will indicate the end behaviour of the graph. They should also be able to describe how changing the constant term will affect the number of x-intercepts.

Students should be able to predict which type of polynomial function would best model a set of data. They can then use technology to run a regression to test their prediction.

Although some graphs can be plotted using tables of values, in most cases students should use a graphing calculator or similar tool to view the graph of a function, or should be given a paper copy of the graph.

SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.**
[C, CN, PS, T, V]

ACHIEVEMENT INDICATORS

- Describe, orally and in written form, the characteristics of polynomial functions by analyzing their graphs.
- Describe, orally and in written form, the characteristics of polynomial functions by analyzing their equations.
- Match equations in a given set to their corresponding graphs.
- Graph data and determine the polynomial function that best approximates the data.
- Interpret the graph of a polynomial function that models a situation, and explain the reasoning.
- Solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.

Suggested Instructional Strategies

- At this point a quick review of slope and equation of a line would be advisable. Students could be presented with graphs of linear equations and be required to state the slope, intercepts, domain and range.
- Use the interactive quadratic function grapher at “Math is Fun” to explore relationships between quadratic equations and their graphed parabolas: http://www.mathsisfun.com/algebra/quadratic-equation-graph.html
- Another interactive graphing activity can be found at: http://illuminations.nctm.org/ActivityDetail.aspx?ID=215
- Have students create graphs or tables of values to represent relationships that are linear, quadratic, or cubic. Then have them exchange with a partner and ask the partner to determine which it is and explain why.
- Have students investigate the maximum number of x-intercepts for linear, quadratic, and cubic, functions using graphing technology.
- Have students describe and compare the key features of the graphs of the functions $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$.

SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.** [C, CN, PS, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Have groups of students create sets of cards, some with equations and others with the matching graphs. Have groups mix them up and challenge other groups to match them.

Q It costs $216 to make a digital master of your school crest. Thereafter the student council can buy crested T-shirts for $7 each. If the student council sells the T-shirts for $16 each, how many must be sold to reach the break-even point.

a) Write the equation for the cost function and sketch it.

b) Write the equation for the profit function and sketch it on the same set of axes.

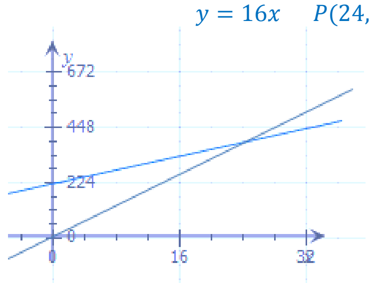
c) Find the intersection point algebraically or with graphing technology, to determine the break-even point.

Answers: $a)\ C = 7x + 216$ $\quad b)\ P = 16x$

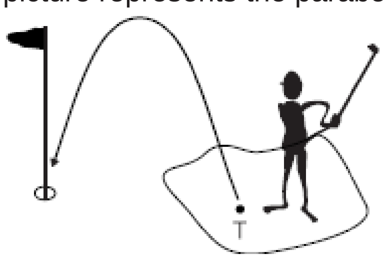
$c)\ 7x + 216 = 16x \quad x = 24$

or graph $y = 7x + 216$

$y = 16x \quad P(24, 384)$



Q This picture represents the parabolic path of a golf ball as it flies through the air.



a) Describe how the height of the golf ball changes from the start to the finish of its path.

b) Sketch and explain a graph to illustrate your description.

c) When a golf ball travels through the air (goes up and then back down to the ground) do you think it maintains the same speed at all times? Explain.

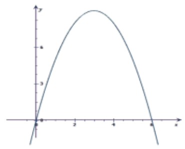
d) Where in its flight is the speed of the ball the slowest? Explain.

e) After retrieving a ball from the roof of my house I threw it up into the air towards the street. Sketch a picture of the flight of the ball. How is it the same as the flight of the golf ball in question? How is it different?

SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.** [C, CN, PS, T, V]

Answers: *a) The height increases until it reaches its maximum and then it decreases until it lands.*

b)



c) No, it travels faster near the start and near the finish, and slower near the maximum.

d) It's speed is slowest near the maximum, where the slope is zero before falling back down.

e) It starts higher, but still goes up and then back down again to ground level.

Q A box with no top is being made out of a $20\ cm$ by $30\ cm$ piece of cardboard by cutting equal squares of side length x from the corners and folding up the sides. The table of values below shows the volume of the box, $V(x)$, based on side length, x.

x	1	2	3	6	8
$V(x)$	504	832	1008	864	448

a) Graph the data and, based on the shape of the graph, determine the type of polynomial function (linear, quadratic, or cubic) that will best approximate the data.

b) Use technology to find the polynomial function that best models the situation.

c) Use the function to determine the volume of a box created by cutting out squares of side length $4.8\ cm$.

d) Use the graph to determine the side length that would result in a volume of $1000cm^3$ by interpolating within the data set given.

Answers: *a) The plotted points are not symmetrical enough to be quadratic, so the function is probably cubic.*

b) $y = 4x^3 - 100x^2 + 600x$

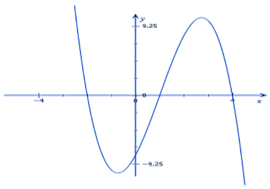
c) $y = 4(4.8)^3 - 100x^2 + 600(4.8) = 1018.4cm^3$

d) $x = 2.93$

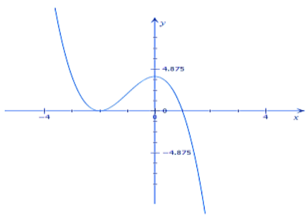
SCO**: RF1: Represent data using polynomial functions (of degree ≤ 3), to solve problems.** [C, CN, PS, T, V]

Q Sketch the graphs of $f(x) = -(x-1)(x+2)(x-4)$ and $g(x) = -(x-1)(x+2)(x+2)$ and compare their shapes and the number of x-intercepts. *(These window dimensions work well: Xmin=* -6*, Xmax=* 6*, Ymin=* -12*, Ymax=* 12*.)*

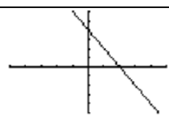
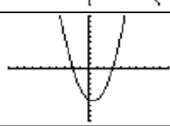
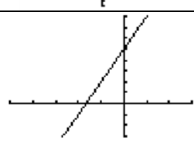
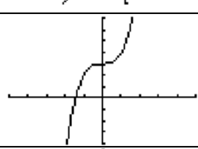
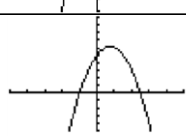
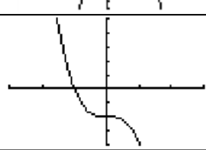
$Answer\colon\ y = -(x-1)(x+2)(x-4) \quad a\ is\ negative, 2\ turning\ points, 3\ \ x\ intercepts$



$Answer\colon$
$y = -(x-1)(x+2)(x+2)$
$a\ is\ negative, 2\ turning\ points, double\ root\ of\ 0, and\ 1\ x\ intercept$



Q Fill in the following table.

Graph	Type of Function	Possible Equation of the Function $(answers\ will\ vary)$
	$Linear$	$y = -2x + 8$
	$Quadratic$	$y = x^2 - x - 6$
	$Linear$	$y = 2x + 5$
	$Cubic$	$y = 4x^3 - 3x^2 + 6$
	$Quadratic$	$y = -2x^2 + 4x + 6$
	$Cubic$	$y = -x^3 - x - 2$

SCO: **RF2: Represent data using exponential and logarithmic functions, to solve problems.** [C, CN, PS, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF2: Represent data using exponential and logarithmic functions, to solve problems.

Scope and Sequence of Outcomes:

Grade 10	Foundations of Mathematics 120
AN3: Demonstrate an understanding of powers with integral and rational components (NRF10) **N4:** Demonstrate an understanding of simple and compound interest (GMF10)	**RF2:** Represent data using exponential and logarithmic functions, to solve problems.

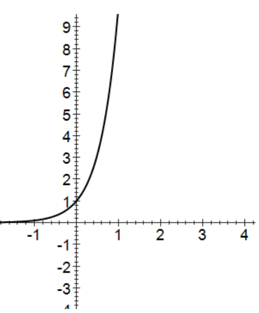
ELABORATION

In this outcome students are given the opportunity to explore data sets modeled by **exponential** and **logarithmic** functions. Although they will have heard the term exponential growth and decay with reference to topics such as population, compound interest, radioactivity, and value depreciation, this will be the first time for them to formally explore these types of functions.

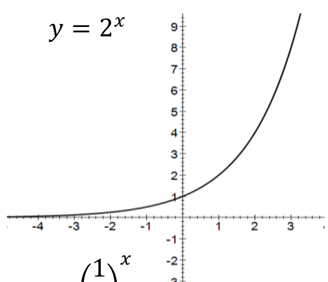
The intention of this outcome is to focus on the graphing of data sets. Students will learn to recognize exponential and logarithmic functions as different from other polynomial graphs they have seen.

Exponential functions have the general form $f(x) = a^x$ where $a > 0$ and $a \neq 1$. By graphing data sets and using a graphing calculator or similar tool, students will discover that **exponential functions** have no x intercepts, have a y-intercept at $(0,1)$, are increasing functions when $a > 1$, and decreasing functions when $0 < a < 1$. This will be an introduction to the concept of **asymptotes**, which for **exponential functions** is the x-axis.

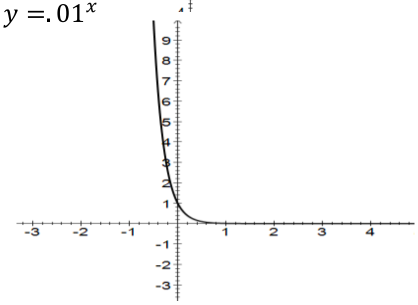
$y = 10^x$



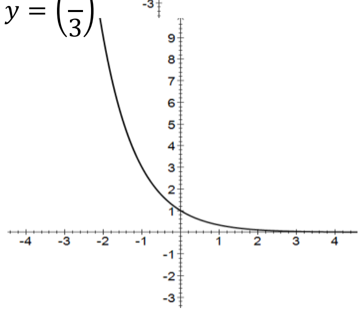
$y = 2^x$



$y = .01^x$



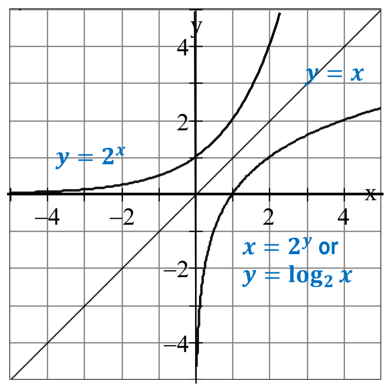
$y = \left(\frac{1}{3}\right)^x$



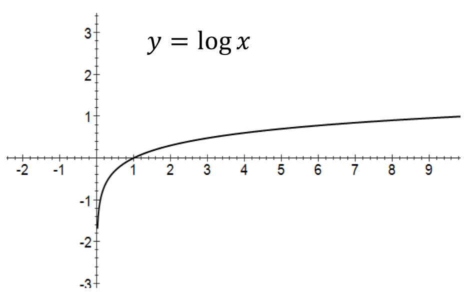
SCO**: RF2: Represent data using exponential and logarithmic functions, to solve problems.** [C, CN, PS, T, V]

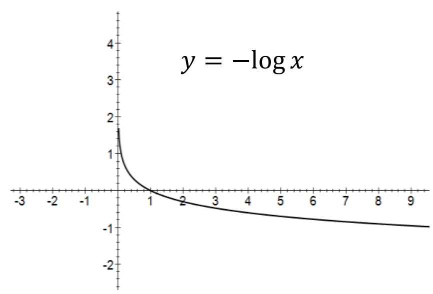
Using graphing technology, students will learn to recognize exponential patterns in data sets. If a scatter plot of the data appears to be exponential, students will run an exponential regression to give an algebraic model of the data set.

As they explore further, students should compare the graphs of $y = a^x$, $y = \log_a x$, and the inverse graph of $y = a^x$ in which x and y are switched, $x = a^y$. They will discover that $y = \log_a x$ is the same graph as $x = a^y$ and that both are the reflection of $y = a^x$ across the $x = y$ line.



For **logarithmic functions**, $a > 1$, $a \neq 1$. The asymptote is the y-axis, and the intercept is $(1{,}0)$. When the log is > 0 the graph will increase, when < 0 the graph will decrease.





Exponential equations can be expressed in logarithmic form and vice versa.

Logarithmic form　　　　　　　　Exponential form

exponent

$y = \log_a x$　　　　　　　　$x = a^y$

base

Common logarithms have a base of 10 and can be written with or without the base as $\log_{10} x$ or simply as $\log x$. Students should be familiar with this convention.

For **exponential** and **logarithmic functions** students should be able to determine if the function is increasing or decreasing, the end behaviour, the asymptotes, the x- or y-intercept, and the domain and range.

SCO**: RF2: Represent data using exponential and logarithmic functions, to solve problems.** [C, CN, PS, T, V]

ACHIEVEMENT INDICATORS

- Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their graphs.
- Describe, orally and in written form, the characteristics of exponential or logarithmic functions by analyzing their equations.
- Match equations in a given set to their corresponding graphs.
- Graph data and determine the exponential or logarithmic function that best approximates the data.
- Interpret the graph of an exponential or logarithmic function that models a situation, and explain the reasoning.
- Solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential or logarithmic functions, and explain the reasoning.

Suggested Instructional Strategies

- Explore the inverse of $y = 10^x$. On a graphing calculator graph $y_1 = 10^x$
  a) On the same graph, using a window of $[-2.35, 2.35, 1]$ by $[-2,2,1]$ graph $y_2 = log\ x$. What do you notice?
  b) Look at the TABLE for the two functions. Especially look at the y values when $\mathrm{x} = 1$ and $\mathrm{x} = 10$
- Use the online interactive graph generator to explore various exponential and logarithmic functions: http://illuminations.nctm.org/ActivityDetail.aspx?ID=215

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Have students explore characteristics of exponential functions as they work through the following steps:

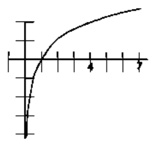
1. Graph $y = 2^x$and explain why the graph (from left to right) curves slowly at first, then much more quickly.
2. Graph $y = 3^x$ and compare its growth rate with the $y = 2^x$.
3. Work in pairs, each with a graphic calculator to explore the behaviour of these two graphs using the window settings $-10 \leq x \leq 2$, and range $-1 \leq y \leq 10$.
4. Describe the curving behaviour and explore the behaviour at the extreme left and extreme right.
5. Explore the behaviour of these two graphs using the window settings $-5 \leq x \leq -1$ and $-0.1 \leq y \ \leq 1$.
6. Change the domain to $-10 \leq x \leq -5$ and the range to $-0.01 \leq y \leq 0.01$ and the.
7. Discuss the behaviour of the graph as it approaches the x-axis.
8. Explain why the graphs on one screen look so similar to the graph on the other screen.
9. Ask the students if they think either of the two graphs will ever intersect the x-axis and to explain their answers.

SCO: **RF2: Represent data using exponential and logarithmic functions, to solve problems.** [C, CN, PS, T, V]

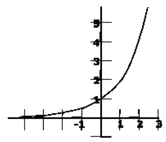
Q Match the following equations with the graphs below.

a) $y = \left(\frac{1}{2}\right)^x$ b) $y = 2^x$ c) $y = -2^x$ d) $y = log_2 x$

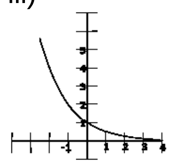
i)



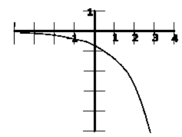
ii)



iii)



iv)



Answer: i) and d), ii) and b), iii) and a), iv) and c)

Q When Jose and Terri carried out a coffee cooling experiment, they obtained the results given in the table below. Here *t* stands for the time in minutes since the experiment began and T for the temperature of the water in degrees Celsius. Room temperature was $20^\circ C$.

t	0	2	4	6	8	10	12	14	16	18	20
T	89.0	83.2	78.0	73.7	70.4	67.4	64.8	62.6	60.0	58.0	55.8

a) Graph the data. What do you think will happen to the water temperature if you wait long enough?

b) Add another row to the table showing the difference between the water temperature and room temperature. How would you test whether an exponential model would fit these numbers?

c) The first three numbers do not appear to fit the same pattern as the rest. Can you suggest a reason for this?

d) Find an exponential function that fits the data from b) (remember to add the $20°$ that was subtracted).

e) According to this model, what temperature would you expect the water to be after 30 minutes?

Answer: a) The water will cool to room temperature of $20℃$

b)

diff	*69*	*63.2*	*58*	*53.7*	*50.4*	*47.4*	*44.8*	*42.6*	*40.0*	*38.0*	*35.8*

To test if this is an exponential function, graph the data or perform an exponential regression.

c) The water will cool more quickly at the beginning.

d) $y = 66.5(0.97^x) + 20$

e) $45.5℃$

SCO**: RF2: Represent data using exponential and logarithmic functions, to solve problems.** [C, CN, PS, T, V]

Q Examine the following tables and indicate which one(s) are suggesting an exponential relationship. Explain your thinking.

$a)$

x	y
0	0.093
1	0.1875
2	0.375
3	0.75
4	1.5
5	3

$b)$

x	y
-7	20 000
-6	200
-5	2
-4	0.02
-3	0.0002
-2	0.000002

$c)$

x	y
-6	-3456
-4	-1504
-3	-828
-2	-352
1	-124

$Answers:$
$a)\ exponential, doubling$ $b)\ exponential, decreasing\ by\ a\ factor\ of\ 100$ $c)\ not\ exponential$

Q Given that $f(x) = (1.2)^{x-1} - 3$.

a) What are the domain and the range of the function?
b) For what values of x does f increase? decrease?
c) What is the approximate zero of ?
d) Describe f as a combination of transformations of $y = (1.2)^x$.

$Answers:$
$a)\ Domain\ x \in R, Range\ \ y > -3$ $b)\ the\ function\ always\ increases$
$c)\ -2$ $d)\ 1\ unit\ right, 3\ units\ down$

SCO**: RF3: Represent data using sinusoidal functions, to solve problems.** [C, CN, PS, T, V]

[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math and Estimation

RF3: Represent data, using sinusoidal functions, to solve problems.

Scope and Sequence of Outcomes:

Foundations of Mathematics 110	Foundations of Mathematics 120
	RF3: Represent data, using sinusoidal functions, to solve problems.

ELABORATION

In Grade 10 students studied right angle trigonometric ratios, including sine. Representing sinusoidal functions graphically will be new to them.

As with the previous two outcomes, the intention in this outcome is to have students gain a general understanding of a function and the applications it models. In this case students will explore the characteristics of sine functions and how they model real life situations. They will also learn to recognize sinusoidal graphs and the data sets on which they are based.

Sine functions are **periodic**, meaning they repeat over a specific **period**. They can be used to describe oscillating events such as the tides coming in and out over time, the height of a person on a Ferris Wheel going up and down, or the height of a point on a rolling object as it rolls along a horizontal distance. A sine graph can be describe as a **sinusoidal** curve.

Sinusoidal curves should be introduced slowly and include discussions about angles with measures greater than $180°$. Students will have encountered angles larger than 180^0 in *Geometry, Measurement, and Finance 10* (G5) but this will have been in the context of right, obtuse, straight, and reflex angles. Their work with trigonometry previous to this has involved right angle trigonometry, sine law, cosine law, or geometry. They will not be familiar with angles in standard position, quadrantal angles or negative angles. As students work with sine functions, both as graphs and as equations, they should become comfortable switching back and forth between representations.

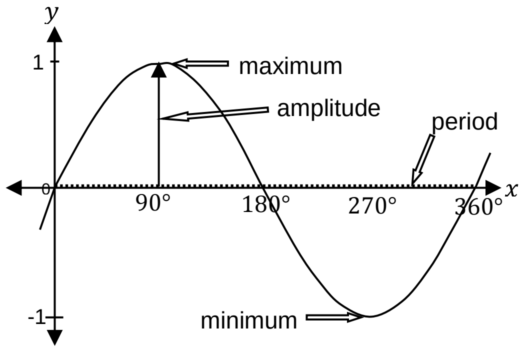
The **sinusoidal axis** of a sine curve (also called the midline) is the horizontal central axis of the curve, halfway between the maximum and minimum values and is defined as: $y = \frac{maximum\ value + minimum\ value}{2}$.

The **amplitude** is the maximum vertical distance of the graph of a sinusoidal function, above and below the sinusoidal axis of the curve measured as half the distance between the minimum and maximum values, $y = \frac{maximum\ value - minimum\ value}{2}$.

The **period** of the graph is the horizontal distance for one cycle of the graph. It can also be defined as the horizontal distance between two corresponding points on the graph.

The emphasis for this outcome, should be on the determination and comparison of these characteristics from given graphs.

SCO**: RF3: Represent data using sinusoidal functions, to solve problems.** [C, CN, PS, T, V]



The graph of the function, $y = \sin x$ is shown above. It is periodic, continuous, has a domain $\{x|x \in R\}$, and range $\{y|-1 \leq y \leq 1, y \in R\}$, has a maximum of $+1$, and a minimum of -1, has an amplitude of 1, a period of $360°$ or 2π radians, and has a y-intercept of 0.

Students will not have seen radians prior to this. They should be shown why 360^0 is equal to 2π or $2(3.14) = 6.28$ radians. The approach should be simplified as much as possible and should use proportional reasoning to show that a radian is a little less than 60^0.

As students explore applications of sine functions, they will see functions that are transformations of $y = \sin x$. Students will have some familiarity with vertical stretches and reflection across the x-axis from their study of quadratic functions in the form $y = ax^2 + bx + c$ in *Foundations of Mathematics 1100*. However, they will not have experience with horizontal stretches.

Students can explore the effect of changing the value of a, b, c, or d in a sine function one at a time to develop an understanding of their effects on a graph of a sinusoidal function.

For $y = a\sin\big(b(x - c)\big) + d$

- a represents the amplitude, or the vertical stretch. Amplitude $= |a|$
- b affects the length of the period, or the horizontal stretch. Period $= \frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$
- c is the value of the horizontal translation, right if $c > 0$, or left if $c < 0$.
- d is the value of the vertical translation up or down, and also shifts the sinusoidal axis, up if $d > 0$, or down if $d < 0$.

This exploration should extend only to the level needed to recognize characteristics of a sinusoidal function from graphs or equations that are given. Applications of these functions to model real-life situations should also be used to increase understanding of the effects of a, b, c, and d.

SCO**: RF3: Represent data using sinusoidal functions, to solve problems.** [C, CN, PS, T, V]

ACHIEVEMENT INDICATORS

- Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their graphs.
- Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their equations.
- Match equations in a given set to their corresponding graphs.
- Graph data and determine the sinusoidal function that best approximates the data.
- Interpret the graph of a sinusoidal function that models a situation, and explain the reasoning.
- Solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning.

Suggested Instructional Strategies

- Have students create sinusoidal data sets by rolling and measuring objects, and recording their height and distance along a surface.
- Match cards with graphs and equations for both sine and cosine functions.
- Use interactive software, which allows the manipulation of the coefficients (a,b,c & d) from the equation to be demonstrated graphically. This is offered on various online sites such as: http://illuminations.nctm.org/ActivityDetail.aspx?ID=174 or http://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php
- In your classroom, post a labeled sine and cosine graph with their corresponding equations.
- As students make connections between graphs and their equations, a good interactive summation tool can be used, such as the one found at: http://illuminations.nctm.org/ActivityDetail.aspx?ID=215

SCO: **RF3: Represent data using sinusoidal functions, to solve problems.** [C, CN, PS, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

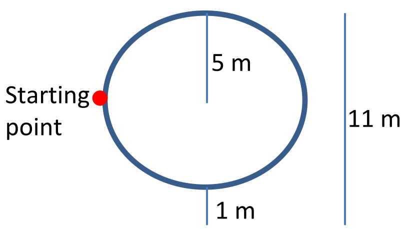
Q A Ferris wheel has a maximum height of $11\ m$, a radius of $5\ m$ (which allows a $1.0\ m$ clearance at the bottom) and rotates once every $40\ sec$. As the ride begins you are half way to the top of the wheel.

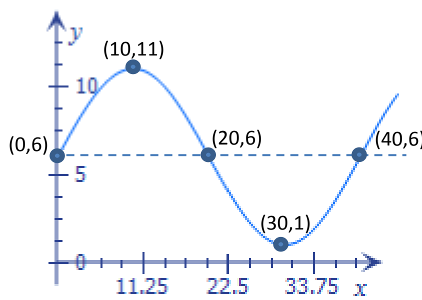
a) Sketch a graph that shows height above the ground as a function of time using a sine function.

b) What is the lowest you go as the wheel turns? Explain why this must be a positive number.

c) How high will you be after 2.5 minutes?

Answer: $a)\quad y = 5\sin 9x + 6$





$b)$ *The lowest point is* 1 *metre above ground, which is therefore positve.*
$c)$ 2.5 *minutes is* 150 *seconds,* 3 *full cycles plus* 30 *seconds, which is at the lowest point or* $1\ m.$

Q Plot the following points and use regression to determine the values for a,b,c and d for the function, $y = a\sin\big(b(x-c)\big) + d$. State the vertical distance between the maximum and local minimum.

x	0	1	2	3	4	5	6	7	8
y	-1.976	-1.794	-1.5	-1.206	-1.024	-1.024	-1.206	-1.5	-1.796

Answer: $y = 0.5\sin(0.63x - 1.257) - 1.5$ *The vertical distance is* $2 \times 0.5 = 1$

Q A spring oscillates in height according to the function $y = \left(\frac{1}{2}\right) sin(180t - 40)) + 5$ where t represents the time in seconds and y represents the height in metres.

a) What is the amplitude of the oscillation?

b) What is the period of oscillation?

c) When is the sinusoidal axis?

d) What is the maximum height it reaches?

e) At what times, for $0 < t < 10$, is the object at its minimum height?

Answer: $a)\ \frac{1}{2}$ $\quad b)\ \frac{360}{180} = 2$ $\quad c)\ 5$ $\quad d)\ 5 + \frac{1}{2} = 5.5$

$e)\ \frac{3}{4},\ 2\frac{3}{4},\ 4\frac{3}{4},\ 6\frac{3}{4},\ 8\frac{3}{4}\quad \left\{\frac{3}{4} + 2K, 0 \le K \le 4\right\}$

SCO**: RF3: Represent data using sinusoidal functions, to solve problems.** [C, CN, PS, T, V]

Q Tommy has a tree swing near the river in his backyard. The swing is a single rope hanging from a tree branch. When Tommy swings, he goes back and forth across the shore of the river. One day his mother (who was taking an adult math course) decided to model his motion using her stopwatch. She finds that, after $2\ sec$, Tommy is at one end of his swing, $4\ m$ from the shoreline while over land. After $6\ sec$, he reaches the other end of his swing, $5.2\ m$ from the shoreline while over the water.

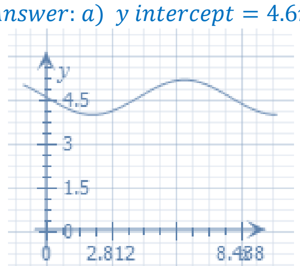
a) Sketch a graph of this sinusoidal function.

b) Write the equation expressing distance from the shore versus time.

c) Predict the distance when
 i) time is $6.8\ sec$
 ii) time is $15\ sec$
 iii) time is $30\ sec$

d) Where was Tommy when his mother started the watch?

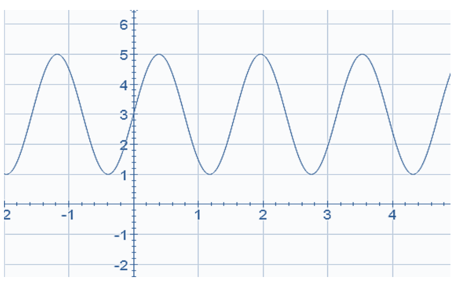
Answer: $a)\ y\ intercept = 4.6m,\quad y\ max = 5.2m, y\ min = 4m\quad period = 8\ sec$



$b)\ d = (0.6\sin(45(t-4))) + 4.6\qquad c)\ i)\ \sim 5.085m\ \ ii)\ \sim 5.024m\ \ iii)\ \sim 5.2m$

$d)$ *Tommy was in the exact middle of his swing going towards the shore.*

Q Write the sinusoidal equation for the following graph.



Answer: $2\sin 225\,x + 3$

Act Have students place a piece of masking tape on the bottom edge of an empty can. Roll the can along a metre stick a short distance on the floor, measuring the horizontal distance traveled and the height of the masking tape at various intervals during its rotation. Record at least 10 measurements within the first two revolutions. Have the students plot the coordinate pairs on a graph. They should draw connections between the diameter and circumference of the can and amplitude and period of the graph. A more detailed explanation of this activity with follow-up questions can be found at:
http://www.ed.gov.nl.ca/edu/k12/curriculum/documents/mathematics/math2204/3.pdf

SUMMARY OF CURRICULUM OUTCOMES

Foundations of Mathematics 120

[C] Communication, **[PS]** Problem Solving, **[CN]** Connections, **[R]** Reasoning,
[ME] Mental Mathematics and Estimation, **[T]** Technology **[V]** Visualization

Statistics

General Outcome: Develop statistical reasoning.

Specific Outcomes

S1. Demonstrate an understanding of normal distribution, including standard deviation, *z*-scores. [CN, PS, T, V]

S2. Interpret statistical data, using confidence intervals, confidence levels, margin of error. [C, CN, R]

Logical Reasoning

General Outcome: Develop logical reasoning.

Specific Outcomes

LR1. Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies. [CN, ME, PS, R]

LR2. Solve problems that involve the application of set theory. [CN, PS, R, V]

LR3. Solve problems that involve conditional statements. [C, CN, PS, R]

Probability

General Outcome: Develop critical thinking skills related to uncertainty.

Specific Outcomes

P1. Interpret and assess the validity of odds and probability statements. [C, CN, ME]

P2. Solve problems that involve the probability of mutually exclusive and non–mutually exclusive events. [CN, PS, R, V]

P3. Solve problems that involve the probability of two events. [CN, PS, R]

P4. Solve problems that involve the fundamental counting principle. [PS, R, V]

P5. Solve problems that involve permutations. [ME, PS, R, T, V]

P6. Solve problems that involve combinations. [ME, PS, R, T, V]

P7. Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]

Relations and Functions

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

RF1. Represent data, using polynomial functions (of degree ≤ 3), to solve problems. [C, CN, PS, T, V]

RF2. Represent data, using exponential and logarithmic functions, to solve problems. [C, CN, PS, T, V]

RF3. Represent data, using sinusoidal functions, to solve problems. [C, CN, PS, T, V]

z-score Chart

For a normal distribution, to determine the percent of data values that are found at or below a given z-score value, use the chart below to find the z-score and the corresponding percentage. For example, to determine the percentage of data found at or below a z-score of -1.26, go to the -1.2 row and the 0.06 column. The value of 0.1038, found where the row and column intersects, indicates that 10.38% of the data is found at or below 1.26 standard deviations below the mean.

Negative z-scores

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.0
-3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
-2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
-2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
-2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
-2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
-2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
-2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
-2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
-2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
-2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
-2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
-1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
-1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
-1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
-1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
-1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
-1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
-1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
-1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
-1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
-1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
-0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
-0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
-0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
-0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
-0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
-0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
-0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
-0.2	0.3829	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
-0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
-0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

Positive z-scores										
z	**0.0**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0..6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	00.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

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