Foundations of Mathematics

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Preface

This set of notes is an outline of the concepts, structures, and theorems that serve as the point of departure for *Mathematical Physics*. The notes are intended to be used in conjunction with a companion volume entitled: Foundations Problem Set .

Contents

Introduction

The phrase *Mathematical Physics* refers to the mathematical models that are used to understand the physical environment. These models are founded on the two main ideas of *Calculus*: *Differentiation* and *Integration*. In turn, the framework of *Number*, *Function*, and *Limit* underlies calculus. The objective here is to develop this framework to the point of functional literacy in mathematics.

We introduce the various themes of mathematical reasoning:

Logic and Set Theory

We begin with the syntactic and semantic issues of the language of mathematics. Syntax refers to axiom systems and semantics refers to models of axiom systems.

Algebraic Structures and Homomorphisms

Structures with *Binary Operations*, and the *Preservation of these Operations* by *Homomorphisms*, is the subject matter of *Algebra*. Example The *Natural Logarithm* is a function mapping the set of all positive real numbers onto the set of all real numbers: $Ln : (0, \infty) \to \mathbb{R}$. The *Logarithm Addition Theorem* : $Ln\ a\,b = Ln\ a + Ln\ b$, asserts that the natural logarithm is a *Homomorphism* from the positive reals, relative to multiplicative structure, onto the set of all reals, relative to additive structure.

Topological Structures and ContinuousMaps

The *Limit Process* is the subject matter of *Topology*. The issues are *Convergence* in structures, and the *Preservation of Convergence* by functions. A prototype for a limit process is the convergence of the infinite sequence of multiplicative inverses of positive integers to the number *zero*. A *Continuous Function* is a function that preserves convergence; the natural logarithm is also an example of a continuous function.

Analysis and the Class of Elementary Functions

The class of algebraic and transcendental functions is the subject matter of *Analysis*. In the case of real number variables, the main theorem (and focal point of the notes) is the assertion that Integration and Differentiation are inverse operations on functions.

We emphasize a nomenclature and methodology in which the different themes of mathematical reasoning are unified; and, for each branch of mathematics, there is a natural lead–in for subsequent work. The collection of theorems is a problem set in the *Moore School* tradition. The notes are intended to be used with a *Back and Forth* approach; the first step is a run–through to get a sense of where things are located.

I. Logic and Set Theory

1. Outline of the Language of Mathematics

(1) Syntax : Finitary Combinatorics of Strings of Formal Symbols

(i) Classification of Symbols

Logical Symbol

Variable $x_1, x_2, \ldots, x_n, \ldots$

Connective Symbol

Negation	$\neg$	
Conjunction	$\wedge$	
Disjunction	$\vee$	
Conditional	$\rightarrow$	
Biconditional	$\leftrightarrow$	

Quantifier

Existential	$\exists$	There Exists
Universal	$\forall$	For All

Syntactic Equality Symbol $\approx$

Parametric Symbol

Function Symbol Including Constant Symbols

Predicate Symbol

(ii) Classification of Strings

Term	Name of an Individual Syntactical Point
Formula	Atomic Formula, Free versus Bound Variable, Prenex Normal Form
Sentence	Each Variable occurs within the Scope of a Quantifier

(iii) Syntactical Proof and Syntactical Theorem

Axiom Classified as *Logical* or *Parametric*

Rule of Inference

(2) Semantics : Assignment of Meaning to Syntax

(i) Structure and Variable Assignment

(ii) Models of Axiom Systems

(3) Metatheoretic Concepts

Set–Theoretic Concepts

Set *Membership* Symbol**:** $\in$ *(Primitive Concept)*

Set *Inclusion* Symbol**:** $\subseteq$ Definition**:** $A \subseteq B \Leftrightarrow \forall x \; [x \in A \Rightarrow x \in B]$

Equality of *Ordered Pairs***:**

$$\forall x \; \forall y \; \forall u \; \forall v \; \left[\langle x, y \rangle = \langle u, v \rangle \Leftrightarrow x = u \;\; \& \; y = v \right]$$

Relative to the ordered pair**:** $\langle x, y \rangle$**,** the *First Component (Coordinate)* is x**,** and the *Second Component (Coordinate)* is y**.**

Symbol Correspondence

Correspondence from syntactic to semantic symbols**:**

(i) Logical Symbols

Connectives	$\neg$	$\neg$
	$\wedge$	$\&$
	$\vee$	*or*
	$\rightarrow$	$\Rightarrow$
	$\leftrightarrow$	$\Leftrightarrow$
Quantifiers	$\exists$	$\exists$
	$\forall$	$\forall$
Equality	$\approx$	$=$

(ii) Parametric Symbols

Bold–Face $\mapsto$ Light–Face

We require a *Formal Language* in which to frame our various axiom systems**,** so that we then have the flexibility of assigning any number of different *Interpretations or Meanings* to the symbols**;** for instance**,** a function symbol $\mathfrak{F}$ in a formal language is typically assigned to infinitely many different set–theoretic functions f as the application of the axiom system varies**.**

(4) Additional Quantifiers

$\forall_{ae}$ **:** For All but Finitely Many (Almost Everywhere)

$\exists$ **:** There Exists at Least One

$\exists\dagger$ **:** There Exists at Most One

$\exists!$ **:** There Exists a Unique (One and only One)

$\exists_{den}$ **:** There Exist Countably Infinitely Many (Denumerably Many)

$\exists_{unc}$ **:** There Exist Uncountably Many

2. Function Concept

Dynamic Viewpoint of Transformation

Hypothesis

Suppose that each of A and B is a nonempty set, and that $f : A \to B$.

This notation means that f is a *Function from* A *into* B, which in turn, means that f can be viewed as a method of associating, mapping, or transforming each point of A into a unique (one and only one) point of B.

The function concept is officially defined in terms of the ordered pair membership of the graph as a subset of the cartesian product of the *Domain* crossed with the *Target*.

(1) Binary Relation (and its Inverse Relation)

A *(Binary) Relation* is defined to be a set R such that each element of R is an ordered pair:

$$R \text{ is a binary relation } \Leftrightarrow \forall u \left[u \in R \Rightarrow \exists x \ \exists y \ \left(u = \langle x, y \rangle \right) \right]$$

The *Relation Inverse* of a binary relation R is defined by:

$$R^{-1} = \left\{ \langle y, x \rangle : \langle x, y \rangle \in R \right\}$$

(2) Function Definition in its Dynamic Form

Cross (Cartesian) Product

$$A \times B = \left\{ \langle x, y \rangle : x \in A \ \& \ y \in B \right\}$$

Dynamic Function Definition

$$f : A \to B \ \Leftrightarrow \ f \subseteq A \times B \ \& \ \forall x \in A \ \exists ! \, y \in B \ \left[\langle x, y \rangle \in f \right]$$

($A \times B$ is a *Graph Ambient Set* for f)

(3) Forward and Reverse Action of a Function

(i) Domain and Target

In the case of a dynamic function definition: $f : A \to B$, the *Domain* of f is A and the choice of *Target (Codomain)* for f is B:

$$Domain\ f = A \qquad Target\ f = B$$

(ii) Function Evaluation (Forward Direction)

If f is a function, then the notation: $f(x)=y$, is defined by: $\langle x,y\rangle \in f$. Therefore:

$$f(x)=y \Leftrightarrow \langle x,y\rangle \in f$$

We use the terminology: *Evaluation at the Argument* $x \in A$ *returning the Value* $y \in B$.

(iii) Range (Forward Direction)

$$Range\, f = Image\, f = \{y \in B : \exists x \in A\ [f(x)=y]\}$$

The cross product: $A \times Range\ f$, is the smallest (relative to set inclusion) possible graph ambient set.

(iv) Point–Inverse Set (Fiber) (Reverse Direction of a Pull–Back)

$$\forall y \in B\ \left[f^{-1}(y)=\{x \in A : f(x)=y\}\right]$$

The fiber $f^{-1}(y)$ is the *Solution Set* in A to the equation: $f(x)=y$

(4) Distinguished Classes of Functions

(i) Surjective Function

The function f is *Surjective (Onto its Target)* provided that:

$$Range\, f = Target\, f = B$$

The function f is *Surjective* iff *each point–inverse set contains at least one point*:

$$\forall y \in B\ \exists x \in A\ [f(x)=y]$$

(ii) Injective Function

The function f is *Injective (One–to–One)* provided that:

$$\forall x,y \in A\ [x \neq y \Rightarrow f(x) \neq f(y)]$$

The function f is *Injective* iff *each point–inverse set contains at most one point*:

$$\forall y \in B\ \exists \dagger x \in A\ [f(x)=y]$$

(iii) Bijective Function

The function f is *Bijective* provided that f is both injective and surjective.

The function f is *Bijective* iff *each point–inverse set contains exactly one point*:

$$\forall y \in B\ \exists ! x \in A\ [f(x)=y]$$

(5) Functional Inverse

Suppose that $f : A \to B$. Then, $f \subseteq A \times B$ is a binary relation, and in turn, it is always possible to construct the relation inverse:

$$f^{-1} = \left\{ \langle y, x \rangle \subseteq B \times A : f(x) = y \right\}$$

The relation inverse is a well–defined single–valued function: $f^{-1} : B \to A$, iff $f : A \to B$ is a bijection of A onto B.

If $f^{-1} : B \to A$ exists as a single–valued function, then f^{-1} is necessarily a bijection of B onto A.

(6) Set Image and Inverse Image

Suppose that $f : A \to B$, and that each of $C \subseteq A$ and $D \subseteq B$ is a nonvoid subset.

Image

Then, under the forward direction of f:

The *Image of* C *under* f is:

$$f[C] = \left\{ f(x) : x \in C \right\} \subseteq B$$

Inverse Image (Pull–Back)

Then, under the reverse (inverse) direction of f:

The *Inverse Image*, or *Preimage, of* D *under* f is:

$$f^{-1}[D] = \left\{ x \in A : f(x) \in D \right\} \subseteq A$$

(7) Function Restriction and Extension

Suppose that $f : A \to B$, and that $C \subseteq A$ is a nonvoid subset.

Then, the *Restriction of* f *to* C, denoted $f|C$, is defined by:

$$g = f|C \quad g : C \to B \quad g(x) = f(x) \quad (x \in C)$$

Terminology: *g Restricts f* *f Extends g*

3. Stationary (Graph) Viewpoint of the Function Concept

Stationary Definitions

(i) As mentioned above, a *(Binary) Relation* is a set of ordered pairs. A *Function* is a relation f such that no two distinct ordered pairs in f have the same *First Component* :

$$f \text{ is a function } \Leftrightarrow f \text{ is a binary relation}$$
$$\& \ \forall x \ \forall y \ \forall z \ \left[\langle x, y\rangle \in f \ \& \ \langle x, z\rangle \in f \Rightarrow y = z\right]$$

$(Abstract\ \ Vertical\ Line\ Test)$

(ii) If f is a function, then f is moreover an *Injective (One–to–One)* function provided that no two distinct ordered pairs in f have the same *Second Component* :

$$\forall a \ \forall b \ \forall y \ \left[\langle a, y\rangle \in f \ \& \ \langle b, y\rangle \in f \ \Rightarrow \ a = b\right]$$

$(Abstract\ \ Horizontal\ \ Line\ Test)$

(iii) The *Domain* of f is the set of all first components of ordered pairs in f :

$$Domain\ f = \left\{x : \exists y \ \left[\langle x, y\rangle \in f\right]\right\}$$

$\left(Projection\ \ of\ \ Graph\ f\ \ to\ the\ First\ \ Component\ \ Set\right)$

(iv) The *Range (Image)* of f is the set of all second components of ordered pairs in f :

$$Range\ f = Image\ f = \left\{y : \exists x \ \left[\langle x, y\rangle \in f\right]\right\}$$

$\left(Projection\ \ of\ \ Graph\ f\ \ to\ the\ Second\ \ Component\ \ Set\right)$

Remark

From the stationary viewpoint of function construction, the issue of a choice of target does not arise.

Power Set

Suppose that X is a set. Then, the *Power Set* of X is defined to be the collection of all subsets of X (including the void set):

$$Power\ Set\ (X) = PS(X) = \{S : S \subseteq X\}$$

Covers and Partitions

Suppose that X is a nonvoid set, and $Y \subseteq Power\ Set\ (X) = \{S : S \subseteq X\}$. Then:

$$Y\ is\ a\ Cover\ of\ X \text{ iff } \forall p \in X\ \ \exists\ S \in Y\ [p \in S]$$

$$Y\ is\ a\ Partition\ of\ X \text{ iff } Y\ is\ a\ pairwise\ disjoint\ cover\ of\ X\ by\ nonvoid\ subsets$$

Alternatively, a collection Y of nonvoid subsets of X is a partition of X provided that:

$$\forall p \in X\ \ \exists!\ S \in Y\ [p \in S]$$

Thus, a partition is a cover without overlap. The *Cells of a Partition* are the partition elements.

Fiber Partition induced by a Function

Suppose that each of A and B is a nonempty set, and that $f : A \to B$. Then, the collection of all the fibers of f is a partition of $A = Domain\, f$:

$$The\ fiber\ partition\ induced\ by\ f$$

$$= Fiber\ Partition\ (f) = \left\{ f^{-1}(y) : y \in Range\, f \right\} \subseteq PS(A)$$

Thus, one method of constructing functions from A into B is to first decompose A into a partition, and then in the second step, associate with each cell of the partition a unique element of B. If different cells are associated with different points of B, then the fibers of the function coincide with the cells of the partition constructed in the first step.

4. Transitive Binary Relations

(1) Axiom Systems for a Binary Predicate Symbol

Suppose P is a binary predicate symbol. The universal quantification closure of each of the following formulas is a *First–Order Binary Predicate Axiom on* P $($*Polish Notation*$)$:

(i)	$P x\ x$	*Reflexivity*
(ii)	$\neg P x\ x$	*Irreflexivity*
(iii)	$P x\ y \rightarrow P y\ x$	*Symmetry*
(iv)	$P x\ y \wedge P y\ x \rightarrow x \approx y$	*Anti–Symmetry*
(v)	$P x\ y \vee P y\ x \vee x \approx y$	*Comparability*
(vi)	$P x\ y \leftrightarrow \neg P y\ x \wedge \neg\ x \approx y$	*Trichotomy*
(vii)	$P x\ y \wedge P y\ z \rightarrow P x\ z$	*Transitivity*

A structure for a binary predicate symbol is an ordered pair $\langle A, R\rangle$ where A is a nonvoid set, and $R \subseteq A \times A$ is a subset of the cross product of A with itself.

Hierarchy of Order Relations

(1) A *Reducibility* is a structure $\langle A, R\rangle$ for P which moreover is a model of the axioms:

Reflexivity *Transitivity*

(2) A *Partial Order* is a structure $\langle A, R\rangle$ for P which moreover is a model of the axioms:

Reflexivity *Anti – Symmetry* *Transitivity*

(3) A *Strict Partial Order* is a structure $\langle A, R\rangle$ for P which moreover is a model of theaxioms:

Irreflexivity $($*Anti – Symmetry*$)$ *Transitivity*

$($Irreflexivity & Transitivity $\Rightarrow$ Anti–Symmetry$)$

(4) A *Strict Total Order* is a structure $\langle A, R\rangle$ for P which moreover is a model of the axioms:

Trichotomy *Transitivity*

(Trichotomy $\Leftrightarrow$ Irreflexivity & Anti–Symmetry & Comparability)

Equivalence Class Partition Theorem

Suppose A is a nonvoid set, and $R \subseteq A \times A$ is an *Equivalence Relation* on A; this means that $\langle A, R \rangle$ is a model of: *Reflexivity* *Symmetry* *Transitivity*

For each $a \in A$, the *Equivalence Class* of a with respect to R is defined by:

$$[a]_R = \{ b \in A : \langle a, b \rangle \in R \}$$

Then, the collection of all equivalence classes: $\{ [a]_R : a \in A \}$, is a partition of A.

Least Element Axiom The universal quantification closure of the following formula is the *Second–Order Least Element Binary Predicate Axiom on* P:

$$[P x y \leftrightarrow \neg P y x \wedge \neg \; x \approx y] \wedge [P x y \wedge P y z \rightarrow P x z]$$
$$\wedge \;\; \forall X \; [\exists u \, X u \rightarrow \exists v \; (X v \wedge \forall u [X u \rightarrow u \approx v \vee P v u])]$$

In this formula, X is a *Second–Order Variable* that varies over subsets of the universe of a structure for the language, as opposed to the usual first–order variables that vary over the points of the universe of a structure. The first two conjuncts of the formula are the first–order Trichotomy and Transitivity axioms of a Strict Total Order.

A *Well–Ordering* is a structure $\langle A, R \rangle$ for P which moreover is a model of the second–order Least Element Axiom.

(2) Peano Axiomatization of the Natural Numbers

Peano Axioms

Suppose that c is a constant symbol, and that σ is a unary function symbol.

First–order *Peano Successor Axioms*:

$$\forall x \; [\neg \; c \approx \sigma x] \qquad \forall x \, \forall y \; [\neg \; x \approx y \rightarrow \neg \; \sigma x \approx \sigma y]$$

Second–order *Peano Induction Postulate*:

$$\forall X \; [X c \wedge \forall u \, (X u \rightarrow X \sigma u) \rightarrow \forall u \; X u]$$

Natural Number Characterization Theorem

As a structure for the language consisting of c and σ, the *Natural Number System* $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ where the constant symbol c is interpreted as 0, and the function symbol σ is interpreted as the natural number successor function $n \mapsto n + 1$, is characterized as the unique model for which each of the Peano Axioms is satisfied.

5. Cardinality

(1) Cardinality Operator

Cardinality and Similarity

Suppose that each of A and B is a nonempty set. Then:

$$Card(A) = Card(B) \text{ iff } \exists f \ [f : A \to B \ \textit{is a bijection}]$$

The set A is *Similar* to the set B, denoted by $A \ Sim \ B$, provided that the cardinality of A is equal to that of B: $A \ Sim \ B \Leftrightarrow Card(A) = Card(B)$

Remark

If $f : A \to B$ *is a bijection*, then $f^{-1} : B \to A$ *is a bijection*.

(2) Cardinality Domination

Suppose that each of A and B is a set. Then, the set A is *Dominated* (respectively, *Strictly Dominated*) by the set B iff $A \lesssim B$ (respectively, $A \ll B$).

(i) $A \lesssim B$ iff $\exists f \ [f : A \to B \ \textit{is an injection}]$

(ii) $A \lesssim B$ iff $\exists f \ [f : B \to A \ \textit{is a surjection}]$ (Equivalent Characterization)

(iii) $A \ll B$ iff $\exists f \ [f : A \to B \ \textit{is an injection}] \ \& \neg \exists f \ [f : B \to A \ \textit{is an injection}]$

Cardinality Comparison Theorem

Suppose that each of A and B is a set. Then: $A \lesssim B$ *or* $B \lesssim A$

(3) Set Classification Definition by Cardinality Comparison to the Natural Numbers

Recall that $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ denotes the set of all natural numbers.

(i) A set A is *Finite* iff $A \ll \mathbb{N}$.

(ii) A set A is *Infinite* iff $\mathbb{N} \lesssim A$.

(iii) A set A is *Countably Infinite (Denumerable)* iff $\mathbb{N} \lesssim A$ & $A \lesssim \mathbb{N}$.

(iv) A set A is *Countable* iff $A \lesssim \mathbb{N}$.

(v) A set A is *Uncountable* iff $\mathbb{N} \ll A$.

(4) Theorems on Set Cardinality

Suppose that each of A and B is a nonvoid set. Then:

(i) Theorem on Finite Sets

$$A \text{ is finite iff } \forall f : A \to A \; [f \text{ is injective } \Leftrightarrow f \text{ is surjective}]$$

(ii) Theorem on Infinite Sets

$$A \text{ is infinite iff } A \text{ can be bijectively mapped onto a proper subset of } A$$

(iii) Cantor's Theorem on the Power Set

$$A \ll Power\ Set(A)$$

(iv) Cantor's Theorem on the Cross Product

$$A \text{ is infinite} \Rightarrow Card(A) = Card(A \times A)$$

(v) Bernstein–Cantor–Schröder Theorem

$$\exists f \; [f : A \to B \text{ is an injection}] \;\&\; \exists g \; [g : B \to A \text{ is an injection}]$$
$$\Rightarrow \exists\, h \; [h : A \to B \text{ is a bijection}]$$

Equivalently:

$$\exists f \; [f : A \to B \text{ is a surjection}] \;\&\; \exists g \; [g : B \to A \text{ is a surjection}]$$
$$\Rightarrow \exists\, h \; [h : A \to B \text{ is a bijection}]$$

Equivalently:

$$A \lleq B \;\&\; B \lleq A \Rightarrow A \text{ Sim } B$$

(5) Theorem on the Cardinality of Number Systems (Chapter II)

(a) $Card(\mathbb{N}) = Card(\mathbb{Z}) = Card(\mathbb{Q})$

(b) $Card(\mathbb{P}) = Card(\mathbb{R}) = Card(\mathbb{C}) \quad = Card(Power\ Set(\mathbb{N}))$

(5) Proof Constructions

Cantor's Theorem on the Power Set

For every set A: $\quad A \ll Power\ Set(A)$

(1) Suppose there exists a surjective function $f : A \to PS(A)$ mapping A onto its power set.

(2) Let $X = \{a \in A : a \notin f(a)\} \in PS(A)$.

(3) Since f is surjective, and X is an element of the target collection, we can choose $u \in A$ such that $f(u) = X$.

(4) Then:

$$u \in f(u) \;\Rightarrow\; u \notin X \;\Rightarrow\; u \notin f(u)$$

$$u \notin f(u) \;\Rightarrow\; u \in X \;\Rightarrow\; u \in f(u)$$

In each case: The first conditional is by the *Defining Property* of X; and, the second conditional is by the choice of u.

(5) Thus, each of the cases: $u \in f(u)$ and $u \notin f(u)$, is untenable, thereby contradicting the hypothesis on the existence of a surjective f.

Bernstein–Cantor–Schröder Theorem

Suppose that each of A and B is a nonempty set, and that each of $f : A \to B$ and $g : B \to A$ is an injection. Then, we can construct a bijection $h : A \to B$ of A onto B.

(1) By simultaneous recursion, we construct two nested decreasing (set inclusion) sequences of sets:

$$\begin{array}{ll} C : \mathbb{N} \to PS(A) & D : \mathbb{N} \to PS(B) \\ C_0 = A & D_0 = B \\ C_{n+1} = g[D_n] \subseteq A & D_{n+1} = f[C_n] \subseteq B \end{array}$$

(2) Define $h : A \to B$ by cases:

$$\begin{array}{lll} h(x) = f(x) & x \in C_{2n} \setminus C_{2n+1} & (n \in \mathbb{N}) \\ h(x) = g^{-1}(x) & x \in C_{2n+1} \setminus C_{2n+2} & (n \in \mathbb{N}) \\ h(x) = f(x) & x \in \bigcap\limits_{n \in \mathbb{N}} C_n & \end{array}$$

II. Real and Complex Number Systems

Algebra

1. Function Composition and Permutation Groups

(1) Function Composition

Suppose that each of $A, B, C,$ *and* D is a nonvoid set; suppose further that $f : A \to B$ and $g : C \to D$. Then, the *Composition of* f *followed by* g, denoted $h = g \circ f$, is the function defined by:

Dynamic

$$h : E \to D \qquad h(x) = g(f(x)) \qquad (E = \varnothing \ \Rightarrow \ h \text{ is undefined})$$

$$E = \textit{Domain } h = \{x \in \textit{Domain } f : f(x) \in \textit{Domain } g\} \subseteq A$$

Stationary

$$h = \{\langle a, d\rangle : \exists x \ [\langle a, x\rangle \in f \ \& \ \langle x, d\rangle \in g]\} \neq \varnothing$$

(2) Axiomatic Theory of Groups

A *Group* is defined to be a model of the following axiom system of first–order logic; that is, a group is a structure for the language of the system in which each of the axioms below is satisfied. The language consists of one binary function symbol: $\bullet$, and one constant symbol: $\mathcal{C}$.

We use *Infix Notation* for binary function symbols.

The universal quantification closure of each of the following formulas is an axiom:

$$x \bullet (y \bullet z) \approx (x \bullet y) \bullet z \qquad \textit{Associative Law}$$

$$\mathcal{C} \bullet x \approx x \bullet \mathcal{C} \approx x \qquad \textit{Identity Existence}$$

$$\exists y \ [y \bullet x \approx x \bullet y \approx \mathcal{C}] \qquad \textit{Inverse Existence}$$

(3) Group Homomorphism

Suppose that each of $\langle G, \Delta, \mathfrak{a}\rangle$ and $\langle H, \nabla, \mathfrak{b}\rangle$ is a group. Then, the functionality and the membership of constants satisfy:

$$\Delta : G \times G \to G \quad (\mathfrak{a} \in G) \qquad \nabla : H \times H \to H \quad (\mathfrak{b} \in H)$$

Suppose further that $\varphi : G \to H$. Then, φ is moreover a *Group Homomorphism*, relative to the group operations Δ *and* ∇ , provided that the universal quantification closure of the following equation is satisfied:

$$\varphi(x \Delta y) = \varphi(x) \nabla \varphi(y)$$

If φ is a *Group Homomorphism*, relative to the group operations Δ *and* ∇ , and φ is an injective function, then φ is moreover a *Group–Theoretic Isomorphism*, relative to the group operations Δ *and* ∇ .

The group $\langle G, \Delta, \mathfrak{a} \rangle$ is *Isomorphic* to the group $\langle H, \nabla, \mathfrak{b} \rangle$ provided that there exists a group–theoretic isomorphism $\varphi : G \to H$ that is moreover surjective.

(4) Subgroup

Suppose that $\langle G, \Delta, \mathfrak{a} \rangle$ is a group, and that $H \subseteq G$. Then, $\langle H, \Delta | H \times H , \mathfrak{a} \rangle$ is a *Subgroup* of $\langle G, \Delta, \mathfrak{a} \rangle$ provided that:

(i) $Range(\Delta | H \times H) \subseteq H$

(ii) $\mathfrak{a} \in H$

(iii) $\forall x \in H \ \exists y \in H \ [x \Delta y = \mathfrak{a}]$

(5) Permutation Group

Permutation

A *Permutation* on a nonvoid set A is a bijection from A onto A :

$$Perm(A) = \{ f : f : A \to A \textit{ is a bijection} \}$$

Symmetric Group

The *Symmetric Group* on a nonvoid set A , denoted by $Sym(A)$, is the model of the group–theoretic axiom system where the universe is the collection of all permutations on A , the binary operation symbol $\bullet$ is interpreted as function composition $\circ$, C is interpreted as the identity permutation, and the inverse element is interpreted as the function inverse:

$$Sym(A) = \langle Perm(A), \circ, Id_A \rangle \qquad Id_A : A \to A \quad Id_A(a) = a$$

Example of a Group for which the CommutativeLaw Fails

If a , b , and c are three distinct symbols, then there exist $x, y \in Sym(\{a, b, c\})$ such that: $x \circ y \neq y \circ x$

2. Commutative Rings with Multiplicative Identity

(1) Language and Axiom System

A *Commutative Ring with Multiplicative Identity* is defined to be a model of the following axiom system of first–order logic; by this we mean a structure for the language of the system in in which each of the axioms below is satisfied.

The language consists of two binary function symbols: $+$ *and* $\bullet$, and two constant symbols: 0 *and* 1.

The universal quantification closure of each of the following formulas is an axiom:

(i) *Axioms of Ring Addition*

$x+y \approx y+x$	*Commutative Law*
$x+(y+z) \approx (x+y)+z$	*Associative Law*
$0+x \approx x+0 \approx x$	*Identity Existence*
$\exists y\ [\, y+x \approx x+y \approx 0\,]$	*Inverse Existence*

(ii) *Axioms of Ring Multiplication*

$x \bullet y \approx y \bullet x$	*Commutative Law*
$x \bullet (y \bullet z) \approx (x \bullet y) \bullet z$	*Associative Law*
$1 \bullet x \approx x \bullet 1 \approx x$	*Identity Existence*

(iii) *Distributive Law* (*Multiplication distributes over Addition*)

$$x \bullet (y+z) \approx (x \bullet y) + (x \bullet z)$$

(iv) *Distinct Identity Elements* : $\neg\,[0 \approx 1]$

Remark

The Distributive Law is the only axiom that relates the two binary operations.

(2) Canonical Initial Model

The *Integer Number System* as an extension of the *Natural Number System* is a prototype:

$$\mathbb{Z} = \{\ldots\ , -3, -2, -1, 0\ , 1, 2, 3, \ \ldots\}$$

The *Natural Number System*, with point–set: $\mathbb{N} = \{0,1,2,\ldots,n,\ldots\}$, is initially characterized as the unique model of the Peano Axioms; addition and multiplication operations are then defined by the universal quantification closure of the following *Recursion Equations*:

(i) $\quad x + 0 = x$ *(Ground Equation)*

$\quad x + Successor(y) = Successor(x + y)$ *(Iteration Equation)*

(ii) $\quad x \bullet 0 = 0$ *(Ground Equation)*

$\quad x \bullet Successor(y) = x \bullet y + x$ *(Iteration Equation)*

We construct the *Integer Number System* as the unique extension of the natural number system such that the following sentence is satisfied:

$$\forall n \in \mathbb{N}\ \left[n + (-n) = 0 \right]$$

(3) Quotient Models

Congruence relative to the Integers

The *Congruence Relation with Modulus* $n \in \mathbb{N}$ *with* $n \geq 2$, on the Integer Number System, is defined by:

$$x \equiv y\ \ Mod\ n \ \Leftrightarrow\ \exists\, k \in \mathbb{Z}\ \left[x - y = k\, n \right]$$

Integer Congruence Theorem

For each $n \in \mathbb{N}$ *with* $n \geq 2$, Congruence Modulo n is an equivalence relation on $\mathbb{Z}$, with $n-$ *many* equivalence classes, that is moreover *Compatible with the Algebraic Operations*:

This means that we can form a new commutative ring with multiplicative identity by taking the corresponding partition of $\mathbb{Z}$ as the point–set for the new system and by defining the new operations as follows:

$$[x]+[y]=[x+y] \qquad [x]\,[y]=[x\,y]$$

In order for the operations to be *Well–Defined*, it must be shown that the values of each operation, which are equivalence classes, depend only upon the operation arguments, which again are equivalence classes, and do not depend upon the choice of class representatives inherent in the execution of each operation.

3. Fields

(1) Language and Axiom System

A *Field* is defined to be a model of the following axiom system of first–order logic; that is, a field is a structure for the language of the system in which each of the axioms below is satisfied.

The language consists of two binary function symbols: $+$ *and* $\bullet$, and two constant symbols: 0 *and* 1.

The universal quantification closure of each of the following formulas is an axiom:

(i) *Axioms of Field Addition*

$x+y \approx y+x$	*Commutative Law*
$x+(y+z) \approx (x+y)+z$	*Associative Law*
$0+x \approx x+0 \approx x$	*Identity Existence*
$\exists y\ [\,y+x \approx x+y \approx 0\,]$	*Inverse Existence*

(ii) *Axioms of Field Multiplication*

$x \bullet y \approx y \bullet x$	*Commutative Law*
$x \bullet (y \bullet z) \approx (x \bullet y) \bullet z$	*Associative Law*
$1 \bullet x \approx x \bullet 1 \approx x$	*Identity Existence*
$\exists y\ \left[\neg\,(x \approx 0) \rightarrow\ y \bullet x \approx x \bullet y \approx 1\right]$	*Inverse Existence*

(iii) *Distributive Law* (*Multiplication distributes over Addition*)

$$x \bullet (y+z) \approx (x \bullet y) + (x \bullet z)$$

(iv) *Distinct Identity Elements* : $\neg\,[0 \approx 1]$

Remark The first two parts of the system are the axioms of a *Commutative Group* (with a special exclusionary clause on *zero* in the second inverse axiom) for each of the languages $+$ *and* 0, and, $\bullet$ *and* 1 individually. There is redundancy in each of the Identity Existence and Inverse Existence Axioms; the redundancy disappears (in part) for the axiomatization of *Arbitrary Groups* in which the Commutative Law is omitted.

(2) Derived Concepts

We introduce unary function symbols for the additve and multiplicative inverses (after providing the appropriate uniqueness proofs):

$$x \mapsto -x \qquad\qquad x \mapsto x^{-1} \quad \neg\,(x \approx 0)$$

Then, the field operations *Subtraction* and *Division* are defined by the equations:

$$x - y \approx x + (-y) \qquad\qquad x \div y \approx x \bullet \left(y^{-1}\right) \quad \neg\,\left(y \approx 0\right)$$

4. Fields of Characteristic Zero

(1) Language and Axiom System

The axiom system for *Fields of Characteristic Zero* is obtained from the Field Axioms above by appending the following infinite axiom schema extending the last field axiom on distinct identity elements:

$$\neg\,[\,0 \approx 1\,]$$

$$\neg\,[\,0 \approx 1+1\,]$$

$$\neg\,[\,0 \approx 1+1+1\,]$$

$$\vdots$$

In fields of characteristic *zero*, we define the following (partial) substructures:

(i) *Natural Number System*: $\mathbb{N}$ is the *Operation Closure* of $\{0,1\}$ under the addition operation.

(ii) *Integer Number System*: $\mathbb{Z}$ is the *Operation Closure* of $\{0,1\}$ under the addition operation and the additive inverse operation.

(iii) *Rational Number System*: $\mathbb{Q}$ is the *Operation Closure* of $\{0,1\}$ under all of the field operations.

These are the *Canonical (Partial) Substructures* of a field of characteristic *zero*; the substructure $\mathbb{Q}$ is isomorphic to the *Prime Subfield* of the total field, which means that an isomorphic copy of $\mathbb{Q}$ is the smallest (in the sense of set inclusion) subfield of the total field.

If each of F *and* K is a field of characteristic *zero*, then at each level of closure, the corresponding canonical (partial) substructures are isomorphic. Thus, each of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ is invariant as we range over all fields of characteristic *zero*.

Remark Congruence over the integers with a prime modulus is an example of a field that has *nonzero* characteristic.

Structure of the Real Number System

5. Ordered Fields

(1) Axiom System with a Binary Predicate Symbol as Primitive

The axiom system for *Ordered Fields* is obtained by first enlarging the language by introducing a binary predicate symbol $<$, and then, by appending to the Field Axioms the following *Linear and Algebraic Order Axioms* of first–order logic.

The universal quantification closure of each of the following formulas (infix notation) is an axiom:

(i)	$\neg\,[\,x < x\,]$	*Irreflexive Law*
	$x < y \;\wedge\; y < z \;\rightarrow\; x < z$	*Transitive Law*
	$x < y \;\vee\; y < x \;\vee\; x \approx y$	*Comparability Law*
(ii)	$x < y \;\rightarrow\; x + z < y + z$	*Translation Law*
	$x < y \;\wedge\; 0 < z \;\rightarrow\; x \bullet z < y \bullet z$	*Positive Slope Law*

Remark

The first two axioms of the first part of the system specify the axioms for a *Strict Partially Ordered Structure*.

In combination with the third axiom, we obtain a *Strict Totally Ordered Structure*.

Remark

The second part of the system relates the ordering to the algebra.

Remark

The terminology for the last axiom is suggested by the fact that a linear real–valued function of one real variable is increasing iff the slope of the graph is positive.

Trichotomy

For each element a of an ordered field, exactly one of the following conditions is satisfied:

$$a < 0 \qquad a = 0 \qquad 0 < a$$

An element a of an ordered field is defined to be *Positive* iff $0 < a$; and, an element a is defined to be *Negative* iff $a < 0$.

Corollary Every ordered field has characteristic *zero*.

(2) Axiom System with a Unary Predicate Symbol as Primitive

Ordered Field Characterization by Positive Elements

Suppose that Pos is a unary predicate symbol *Axiomatized* by (i) and (ii) :

$$(i) \quad \forall x \left[Pos\ x \leftrightarrow \neg\ (Pos\ -x) \wedge \neg\ (x \approx 0) \right]$$

$$(ii) \quad \forall x\ \forall y \left[Pos\ x \wedge Pos\ y \to Pos\ x+y \wedge Pos\ x \bullet y \right]$$

Then, axiomatize the ordering by universally closing: $x < y \leftrightarrow Pos\ \ y - x$

Positive Sequence Theorem

Each term of the sequence: $\langle 1,\ 1+1,\ 1+1+1,\ \ldots \rangle$, is positive.

Order Topology (Chapter III – Section 1 – Items (1), (2) and (3))

We automatically associate with each ordered field F the *Order Topology*; the *Canonical Global Base* for the order topology is the collection of all open intervals:

$$\{(a,b):\ a < b\}\ where\ (a,b) = \{x:\ a < x < b\}$$

A function whose domain is $\mathbb{N}$ is a *Sequence*. A sequence $p:\mathbb{N} \to F$ *Converges to a Point* $q \in F$ in the order topology of F means that:

$$\forall a,b \in F\ with\ a < q < b\ \ \exists\, m \in \mathbb{N}\ \ \forall n \geq m\ \ \left[a < p(n) < b \right]$$

Archimedean Property

An ordered field F has the *Archimedean Property* provided that:

The strictly decreasing sequence of multiplicative inverses of the positive elements of $\mathbb{N}$ converges to *zero* in the order topology of F.

Archimedean Characterization Theorem

In an ordered field, each of the following is equivalent to the Archimedean Property :

(i) The set $\mathbb{N}$ is *Unbounded in the Ordering* of the ordered field F :

$$\forall a \in F\ \ \exists\, n \in \mathbb{N}\ \ [a < n]$$

(ii) The set $\mathbb{Q}$ is *Everywhere Dense in the Order Topology* of the ordered field F :

$$\forall a,b \in F\ with\ a < b\ \ \exists\, q \in \mathbb{Q}\ \ [a < q < b]$$

6. Order–Complete Ordered Fields

(1) Order–Completeness Axiom

The last step is to append to the Ordered Field Axioms, all of which are axioms of first–order logic, the *Order–Completeness Axiom of Second–Order Logic*, in which, X is a second–order variable that varies over unary predicates in structures for the language:

$$\forall X\ \exists \lambda\ \big(\ \exists u\, X u\ \wedge\ \exists v\, \forall u\, [X u \rightarrow u \leq v]$$
$$\longrightarrow\ \ \forall u\ [X u \rightarrow u \leq \lambda]\ \ \wedge\ \forall v\ [\forall u\, (X u \rightarrow u \leq v)\ \rightarrow\ \lambda \leq v]\ \big)$$

In a structure for the language, the hypothesis asserts that X is a name for a nonempty set bounded from above, and the conclusion asserts that λ is a *Least Upper Bound* for the set; thus, the effect of the axiom in a model (in conjunction with the effect of the first–order axioms), is that every nonempty subset with an upper bound has a least upper bound. The existence of at most one least upper bound is proved as a theorem.

Abbreviations

Lub $\equiv$ *Least Upper Bound* $\equiv$ *Sup* $\equiv$ *Supremum*

Glb $\equiv$ *Greatest Lower Bound* $\equiv$ *Inf* $\equiv$ *Infimum*

(2) Real Numbers

Real Number Theorem

There exists a unique order–complete ordered field; we define the *Real Number System*: $\langle \mathbb{R}\ , +\ , \bullet\ , 0\ , 1\ , < \rangle$, to be the structure in question.

Irrational Numbers

The set of all *Irrational Numbers* is defined by: $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$

Open Subsets of the Real Line

A subset $U \subseteq \mathbb{R}$ is defined to be an *Open Set in the usual Euclidean Topology* provided:

$$\exists \Gamma \subseteq PS(\mathbb{R})\ \left[\Gamma \textit{ is a collection of open intervals } \ \&\ U = \bigcup \Gamma\right]$$

Open Component Theorem

Every open subset $U \subseteq \mathbb{R}$ in the usual Euclidean topology is the union of a collection of *Pairwise Disjoint* open intervals and open rays.

(3) Characterizations of Order–Completeness

Connected Line Theorem

An ordered field F is order–complete iff F is *Connected in the Order Topology*:

This means that for every proper subset $S \subseteq F$, there exists $a \in F$ such that every open interval about a contains both a point of S and a point of $F \setminus S$.

Embedding Theorem (Using Concepts from Chapters III and IV)

For every Archimedean ordered field F, there is a unique order–isomorphism of F onto a dense subfield of $\mathbb{R}$ such that each element of $\mathbb{Q}$ is held fixed.

The fact that the image of F is a subfield of $\mathbb{R}$ follows from the *Continuity of the Binary Operations*. With respect to the order topology in $\mathbb{Q}$ and the *Product Topology* in $\mathbb{Q} \times \mathbb{Q}$, the restriction of each of the algebraic binary field operations is continuous:

$$+ \ : \ \mathbb{Q} \times \mathbb{Q} \ \rightarrow \ \mathbb{Q} \qquad \bullet \ : \ \mathbb{Q} \times \mathbb{Q} \ \rightarrow \ \mathbb{Q}$$

(Regardless of the original field, these restrictions are the same.)

Moreover, each restriction continuously extends, for each field F and $\mathbb{R}$ individually, to the total product space in the domain, and to the original field in the range.

Absolute Value and Distance

In an ordered field, the *Absolute Value* of an element a is defined by cases:

$$|a| = a \quad a \textit{ is positive or } a = 0 \qquad |a| = -a \quad a \textit{ is not positive and } a \neq 0$$

In an Archimedean ordered field, the *Distance* between two points a *and* b, is defined to be the absolute value of the difference: $d(a,b) = |a - b|$

Remark In the Archimedean case, the absolute value is in turn uniquely identified with a real number by the Embedding Theorem.

Cauchy Sequence

A sequence $p : \mathbb{N} \rightarrow F$ taking values in an Archimedean ordered field F is *Cauchy* provided that:

$$\forall \varepsilon > 0 \ \ \exists n \in \mathbb{N} \ \ \forall s,t \geq n \ \ \left[d\left(p_s, p_t\right) < \varepsilon \right]$$

Cauchy Convergence Theorem

An ordered field F is order–complete iff it is Archimedean, and every Cauchy sequence taking values in F converges to a point of F in the order topology.

Structure of the Complex Number System

7. Algebraically Closed Fields of Characteristic Zero

(1) Language and Axiom System

An *Algebraically Closed Field of Characteristic Zero* is a model of the following axiom system. We return to the language that consists of two binary function symbols: $+$ *and* $\bullet$, and two constant symbols: 0 *and* 1.

We extend the axioms for fields of characteristic *zero* by appending the universal quantification closure of each of the infinitely many formulas of the following schema:

$$\exists x\ [\neg\ a \approx 0 \rightarrow a \bullet x \bullet x + b \bullet x + c \approx 0\ ]$$

$$\exists x\ [\neg\ a \approx 0 \rightarrow a \bullet x \bullet x \bullet x + b \bullet x \bullet x + c \bullet x + d \approx 0\ ]$$

$$\vdots$$

Root Existence Theorem

In a model, every nonconstant polynomial has at least one root.

(2) Complex Numbers

Complex Number System

The *Complex Number System*: $\langle \mathbb{C}, +, \bullet, 0, 1 \rangle$, is defined by extending the *Real Number Field* according to the following rules.

The point–set for the system is $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ (so the geometry of the point–set is that of a plane), and the operations are defined by extending those for real numbers:

$$(a + b\,i) + (c + d\,i) = (a + c) + (b + d)\,i$$

$$(a + b\,i) \bullet (c + d\,i) = (a \bullet c - b \bullet d) + (a \bullet d + b \bullet c)\,i$$

$$a, b \in \mathbb{R} \qquad i = \langle 0, 1 \rangle \qquad b\,i = \langle 0, b \rangle \qquad a + b\,i = \langle a, b \rangle \qquad i^2 = -1$$

Complex Number Theorem

The *Complex Number System* is an algebraically closed field of characteristic *zero*; moreover, the *Complex Number System* is the *Unique* algebraically closed algebraic field extension of the *Real Number Field*.

III. Convergence Concepts

1. Topological Spaces

(1) Axioms for Abstract Topological Spaces

Axioms

A *Topological Space* is an ordered pair $\langle X, \tau \rangle$ such that X is a nonvoid set, and τ is a collection of subsets of X for which the following axioms are satisfied:

$$\tau \subseteq PS(X)$$

(i) Both the empty set $\varnothing$ and the total set X are elements of τ : $\varnothing, X \in \tau$

(ii) For every (nonvoid) finite collection of sets where each set is an element of τ, the intersection of the collection is an element of τ :

Set–Theoretic Closure of τ under the *Finite Argument Intersection Operator*

$$\forall \mathit{Nonvoid\ Finite}\ \Gamma \subseteq \tau\ \left[\bigcap \Gamma \in \tau\right]$$

$$\left(\bigcap \Gamma = \left\{x \in X : \forall U\ \left[U \in \Gamma \Rightarrow x \in U\right]\right\}\right)$$

(iii) For every (arbitrary) collection of sets where each set is an element of τ, the union of the collection is an element of τ :

Set–Theoretic Closure of τ under the *Union Operator*

$$\forall \Gamma \subseteq \tau\ \left[\bigcup \Gamma \in \tau\right]$$

$$\left(\bigcup \Gamma = \left\{x \in X : \exists U\ \left[U \in \Gamma \,\&\, x \in U\right]\right\}\right)$$

Point–Set and Topology of a Space

The total set X (of *Indivisible Elements or Points*) is the *Point–Set* for the space; and the collection τ is the *Topology* for the space. The members of τ are the *Open Sets* for the space.

Extreme Examples If $\tau = PS(X)$, then τ is the *Discrete Topology* on X; and, if $\tau = \{\varnothing, X\}$, then τ is the *Indiscrete Topology* on X.

Remark Informally, a topology on a set is a *Notion of Relative Closeness* among the various points that belong to the set.

Remark Different topologies can be chosen for the same point–set; and, as either the topology or point–set varies, the associated topological space varies. In fact, the point–set is always implicit in the topology as its largest member with respect to set inclusion (alternatively, the point–set is the union of the collection of all open sets).

(2) Topological Base and Axioms of Base Countability

Global and Local Basis Concepts

For a topological space $\langle X, \tau \rangle$, a subcollection $\mu \subseteq \tau$ is a *Global Base* for τ iff each open set in τ is the union of a subcollection of open sets in μ:

$$\forall p \in X \ \forall U \in \tau \ \exists V \in \mu \ \left[p \in U \Rightarrow p \in V \subseteq U \right]$$

Therefore, $\mu \subseteq \tau$ is a (global) base for τ iff the *Set–Theoretic Closure* of μ under the *Union Operator* returns τ .

A subcollection $\mu \subseteq \tau$ of the topology τ is a *Local Base* for τ at $p \in X$ iff each open set in τ that contains p is the union of a subcollection of open sets in μ:

$$\forall U \in \tau \ \exists V \in \mu \ \left[p \in U \Rightarrow p \in V \subseteq U \right]$$

Thus, we pass from *Global to Local*, by transforming $p \in X$ from a universally bound variable to a free variable in the respective defining formulas.

Basis Axioms

(i) A topological space $\langle X, \tau \rangle$ satisfies the *First Axiom of (Base) Countability* iff $\forall p \in X \ \exists \mu \subseteq \tau \ \left[\mu \ is \ a \ local \ base \ for \ \tau \ at \ p \ \ \& \ \ \mu \ is \ countable \right]$.

(ii) A topological space $\langle X, \tau \rangle$ satisfies the *Second Axiom of (Base) Countability* iff $\exists \mu \subseteq \tau \ \left[\mu \ is \ a \ global \ base \ for \ \tau \ \ \& \ \ \mu \ is \ countable \right]$.

(3) Euclidean Spaces and Topological Product

Remark The real line is the prototype for *Point–Set Theory*.

Order Topology In the case of an *Order Topology*, the collection of all open intervals relative to a totally ordered structure provides a global base.

(i) Reals Let $\mathbb{R}$ denote the *Real Number System* with the *Usual Topology*:

The collection of all open intervals is the *Canonical Global Base* for the usual Euclidean topology of the real line.

(ii) Products of Reals Let $\mathbb{R}^n$ $\left(n \in \mathbb{N}^+ \right)$ denote $n-$ *Dimensional Euclidean Space*. By this, we mean the *Topological Product* of $n-$ *many* factors of the topological space $\mathbb{R}$:

The point–set is the $n-$ dimensional cartesian product of lines; basic open sets for the product space are cartesian products of open sets, one from each factor.

(4) Topological Subbase

Open Sets by Iterated Closure For a space $\langle X, \tau \rangle$, a subcollection $\nu \subseteq \tau$ is a *Global Subbase* for τ iff a base for τ is obtained from ν by taking for the elements of the base the intersection of each (nonvoid) finite collection of open sets in ν.

The *Set–Theoretic Closure* of ν under the *Finite Argument Intersection Operator* returns a base for τ:

$$\forall p \in X \ \ \forall U \in \tau \ \ \exists n \in \mathbb{N} \ \ \exists V_0, \ldots, V_n \in \nu$$
$$\left[p \in U \Rightarrow p \in V_0 \cap \cdots \cap V_n \subseteq U \right]$$

Therefore, $\nu \subseteq \tau$ is a subbase for τ iff the *Set–Theoretic Closure* of ν under both the *Finite Argument Intersection Operator* and the *Union Operator* returns τ itself. To formulate the local concept, omit the first quantification in the above formula.

Euclidean Product Subbase $\left\{ R^{+}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^{n} \right\} \cup \left\{ R^{-}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^{n} \right\}$

The set of $n-$dimensional *Unbounded Rectangular Corner Regions* is a subbase for $\mathbb{R}^{n}$ $(n \in \mathbb{N}^{+})$; where $R^{+}(\mathbf{x})$ (respectively, $R^{-}(\mathbf{x})$) is the product of open rays (a, ∞) (respectively, $(-\infty, a)$) for which the endpoints are the coordinates of $\mathbf{x}$.

Canonical Subbase for a Product

The *Canonical Subbase* for a product is the collection of all inverse images of open subsets of the factor spaces relative to the various projection functions: $\mathbf{x} \mapsto x_i$.

(5) Topological Subspace

Inheriting a Topology Suppose $\langle X, \sigma \rangle$ is a topological space. Every nonvoid subset $Y \subseteq X$ is automatically transformed into a *Topological Subspace*:

The subspace is the topological space where the chosen subcollection Y is the point–set, and the topology τ is *Inherited* from the total space by intersecting each of the original open sets with the new point–set: $\tau = \left\{ V \subseteq Y : \exists U \in \sigma \ \left[V = U \cap Y \right] \right\}$

Euclidean Subspaces Let $\mathbb{Q}$ and $\mathbb{P}$ denote the topological spaces for which the set of all *Rationals* and the set of all *Irrationals* are the respective point–sets, and basic open sets for the topology in each case are obtained by intersecting open intervals on the real line with the new point–set. Thus, each of $\mathbb{Q}$ and $\mathbb{P}$ is a topological subspace of $\mathbb{R}$ (with the usual topology).

In turn, products of subspaces of $\mathbb{R}$, such as $\mathbb{Q}^{n}$ $\left(n \in \mathbb{N} \textit{ with } n \geq 2 \right)$, form the class of all *Productive Subspaces* of higher dimensional Euclidean Spaces.

(6) Topological Quotient

Partitions and Quotients

Suppose that $\langle X,\sigma \rangle$ is a topological space, and that $Y \subseteq PS(X)$ is a *Partition* of X. This means that Y is a pairwise disjoint cover of X by nonvoid subsets, where a *Cover* is in turn defined by: $\forall p \in X \ \exists S \in Y \ [p \in S]$

Then, the *Quotient Topology* τ on Y is: $\forall V \subseteq Y \ [V \in \tau \iff \bigcup \mathrm{V} \in \sigma]$

The new space $\langle Y,\tau \rangle$ is a *Quotient Space* on the original space $\langle X,\sigma \rangle$.

Quotients of Euclidean Spaces

Partition each of the line and the plane by taking for the cells of the partition all *Rigid Motion Additive Translations* of the integers: $\mathbb{Z}$, and the cross product of the integers with itself: $\mathbb{Z} \times \mathbb{Z}$, respectively. Then, the induced quotient spaces are topologically equivalent (the same topological properties) to the circle and the torus respectively, where the torus is the topological product of the circle with itself.

(7) Closed Sets

For a space $\langle X,\tau \rangle$, a subset $A \subseteq X$ is defined to be *Closed* (relative to the given topology τ) iff the complement $X \setminus A$ is open, that is, $X \setminus A \in \tau$.

Therefore, the closed sets are obtained from the open sets by applying the set–theoretic *Complementation Operator*; conversely, the open sets can be obtained from the closed sets by the same operator. The symmetry on this issue suggests the important alternative point of view of taking the closed sets as primitive.

2. Metric Spaces

(1) Distance

Let $\mathbb{R}$ denote the *Real Number System*. A *Metric Space* is an ordered pair $\langle X, d\rangle$ such that X is a nonvoid set, and $d : X \times X \to \mathbb{R}$ is a function for which the universal quantification closure of each of the following axioms is satisfied:

(i)	$d(p,q) \geq 0$ & $[d(p,q) = 0 \Leftrightarrow p = q]$	*Positive Definiteness*
(ii)	$d(p,q) = d(q,p)$	*Symmetry*
(iii)	$d(p,q) \leq d(p,r) + d(r,q)$	*Triangle Inequality*

The function d is a *Distance Function* or *Metric* on the set X.

Since the quantification in each of the axioms is entirely universal, it follows that for each $Y \subseteq X$, the restriction $d \,|Y \times Y$ also satisfies the axioms, and is, therefore, a distance function on Y: $\langle Y, d \,|Y \times Y\rangle$ is a metric space.

(2) Basic Neighborhood Function and the Induced Topology

Suppose that $\langle X, d\rangle$ is a metric space. A *Basic Open Neighborhood* relative to d is a set–value of the *Basic Neighborhood Function* $N = N(d)$ (d is a parameter):

$$N : X \times \mathbb{R}^+ \to PS(X) \qquad N_\varepsilon(p) = \{q \in X : \, d(p,q) < \varepsilon\}$$

$$(p \in X \text{ \textit{is the neighborhood center and} } \varepsilon > 0 \text{ \textit{is the neighborhood radius}})$$

$$\mathbb{R}^+ = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\} \qquad PS(X) = \textit{Power Set } (X)$$

(i) The *Topology* τ *Induced by* d is defined by:

$$U \in \tau \;\Leftrightarrow\; \forall p \in U \;\exists \varepsilon > 0 \;[N_\varepsilon(p) \subseteq U]$$

(ii) Thus, a metric space is automatically a topological space: $\langle X, \tau\rangle$; and, a reference to $\langle X, d\rangle$ as a topological space automatically refers to $\langle X, \tau\rangle$.

(iii) The range of the basic neighborhood function N is the *Canonical Base* for τ.

Local Basis Theorem for Metric Spaces

(Corollary of the Archimedean Property of the Real Line)

Every metric space satisfies the First Axiom of Base Countability.

(3) Euclidean Spaces

Each Euclidean space: $\mathbb{R}^n \ \left(n \in \mathbb{N}^+\right)$, is automatically transformed into a metric space by the *Euclidean Distance Function* $\big($derived from the Pythagorean Theorem$\big)$:

$$d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \left[(p_1-q_1)^2+\cdots+(p_n-q_n)^2\right]^{\frac{1}{2}}$$

(4) Diameter and Set Distance in Metric Spaces

Let $\mathbb{R}^e = \mathbb{R} \cup \{-\infty,\infty\}$ denote the *Extended Real Number System* in which both a smallest and a largest point are adjoined to $\mathbb{R}$ (we are *not* extending the algebra).

Suppose that $\langle X, d\rangle$ is a metric space. Then:

(i) The *Diameter Function* $Diam : PS(X) \to \mathbb{R}^e$ is defined by:

$$Diam(S) = Lub\{d(p,q):\ p,q \in S\} \qquad (Diam(\varnothing)=0)$$

A subset $S \subseteq X$ is *Bounded* iff $Diam(S) < \infty$ is finite.

(ii) We extend the distance function d to nonempty subsets of X as follows:

$$d(p,S) = Glb\{d(p,q):\ q \in S\} \qquad (p \in X,\ S \subseteq X)$$

$$d(S,T) = Glb\{d(p,q):\ p \in S \ \&\ q \in T\} \qquad (S,T \subseteq X)$$

(5) Limit Points and Boundary Points in Metric Spaces

Suppose that $\langle X, d \rangle$ is a metric space. Then, the *Deleted Basic Neighborhood* with center $p \in X$ and radius $\varepsilon > 0$ is defined by:

$$N_\varepsilon^\circ(p) = \{q \in X : d(p,q) < \varepsilon \ \& \ q \neq p\} = N_\varepsilon(p) \setminus \{p\}$$

Suppose $S \subseteq X$ and $p \in X$ (we are neutral as to whether p is in S). Then:

(i) $p \in X$ is a *Limit Point* (or *Point of Accumulation*) of $S \subseteq X$ provided that:

$$\forall \varepsilon > 0 \ \exists q \in S \left[q \in N_\varepsilon^\circ(p) \right]$$

(ii) $p \in X$ is *Isolated From* $S \subseteq X$ iff p is not a limit point of S:

$$\exists \varepsilon > 0 \left[N_\varepsilon^\circ(p) \cap S = \varnothing \right]$$

(iii) $p \in X$ is a *Boundary Point* of $S \subseteq X$ provided that:

$$\forall \varepsilon > 0 \left[N_\varepsilon(p) \cap S \neq \varnothing \ \& \ N_\varepsilon(p) \cap (X \setminus S) \neq \varnothing \right]$$

(6) Theorem on Vanishing Set Distance in MetricSpaces

Suppose that $\langle X, d \rangle$ is a metric space, that $S \subseteq X$ is nonvoid, and $p \in X \setminus S$. Then, the following conditions are equivalent:

$$(i) \ p \in Bd(S) \quad (ii) \ p \in Lp(S) \quad (iii) \ d(p, S) = 0$$

Corollary Every closed subspace of a metric space can be represented as the intersection of a countable collection of open sets; the dual result is that every open subspace can be represented as the union of a countable collection of closed sets.

3. Operators on Topological Spaces

(1) Limit Points and Boundary Points in Topological Spaces

Suppose that $\langle X, \tau \rangle$ is a topological space, $S \subseteq X$ and $p \in X$. Then:

(i) $p \in X$ is a *Limit Point* (or *Point of Accumulation*) of $S \subseteq X$ provided that:

$$\forall U \in \tau \ \ \exists q \in U \ \left[p \in U \ \Rightarrow \ q \in S \ \& \ q \neq p \right]$$

(ii) $p \in X$ is *Isolated From* $S \subseteq X$ iff p is not a limit point of S:

$$\exists U \in \tau \ \left[p \in U \ \& \ U \cap S \subseteq \{p\} \right]$$

(iii) $p \in X$ is a *Boundary Point* of $S \subseteq X$ provided that:

$$\forall U \in \tau \ \left[p \in U \ \Rightarrow \ U \cap S \neq \varnothing \ \& \ U \cap (X \setminus S) \neq \varnothing \right]$$

(2) Operators

An informal, and occasional, mathematical usage of the word *Operator* is to designate a function for which the domain and target coincide.

Whenever an argument of an operator is equal to its value, the point in question is a *Fixed Point* of the operator.

(3) Topological Operators

Suppose that $\langle X, \tau \rangle$ is a topological space. The domain and target of each of the following operators is $PS(X) = Power\ Set\ (X)$. Each operator depends upon τ. Let $S \subseteq X$.

(i) Interior The *Interior Operator* is defined by:

$$Int(S) = \{ p \in S : \exists U \in \tau \ [p \in U \subseteq S] \} \qquad Int = Int_\tau$$

(ii) Boundary The *Boundary Operator* is defined by:

$$Bd(S) = \{ p \in X : p \ is\ a\ boundary\ point\ of\ S \} \qquad Bd = Bd_\tau$$

(iii) Exterior The *Exterior Operator* is defined by:

$$Ext(S) = \{ p \in X \setminus S : \exists U \in \tau \ [p \in U \subseteq X \setminus S] \} \qquad Ext = Ext_\tau$$

(iv) Derived Set The *Derived Set Operator* is defined by:

$$Lp(S) = \{ p \in X : p \ is\ a\ limit\ point\ of\ S \} \qquad Lp = Lp_\tau$$

(v) Isolated Set The *Isolated Set Operator* is defined by:

$$Iso(S) = \{ p \in S : p \ is\ an\ isolated\ point\ of\ S \} \qquad Iso = Iso_\tau$$

(vi) Topological Closure

The *Topological Closure Operator* is defined by: $Cl = Cl_\tau$

$$Cl(S) = \bigcap \{ A : X \setminus A \in \tau \ \& \ S \subseteq A \}$$

(4) Operator Theorem

Suppose that $\langle X, \tau \rangle$ is a topological space, and $S \subseteq X$. Then:

(i) Equations of Symmetry

$$Bd(S) = Bd(X \setminus S) \qquad Lp(S) \setminus S = Bd(S) \setminus S$$

$$X = Int(S) \cup Bd(S) \cup Ext(S)$$

(ii) Open Set Characterization

A subset $U \subseteq X$ is open iff $U = Int(U)$ (U is disjoint from its boundary).

(iii) Balanced (Regular) Open Set Characterization

A *Balanced (Regular) Open Set* is a fixed point of the composition operator below; here, the fixed point set coincides with the range of the composition:

$$Int \circ Cl : PS(X) \to PS(X)$$

Suppose $U \in \tau$. Then: $U = Int(Cl(U)) \iff Bd(U) = Bd(X \setminus Cl(U))$

(iv) Closed Set Characterization

(a) Each of $Bd(S)$ and $Cl(S)$ is a closed set; in a metric space, $Lp(S)$ is also closed.

(b) $Cl(S) = S \cup Lp(S) = S \cup Bd(S) = Lp(S) \cup Iso(S)$

(c) $Cl(S)$ is the smallest (in the sense of set inclusion) closed set that includes S as a subset.

(d) The following conditions on a subset $A \subseteq X$ are equivalent:

(i) A *is closed* $\quad (ii)$ $Lp(A) \subseteq A \quad (iii)$ $Bd(A) \subseteq A \quad (iv)$ $A = Cl(A)$

Remark

A set is open iff it is disjoint from its boundary.

Remark

A set is closed iff it includes its boundary.

Remark

A closed set is the disjoint union of its (possibly void) interior and its (possibly void) boundary; in the case of a void boundary, the set is both open and closed.

(4) Dense and Nowhere Dense Sets

Suppose that $\langle X, \tau \rangle$ is a topological space.

(i) A set $D \subseteq X$ is *(Everywhere) Dense* in X iff D intersects each nonvoid open set:

$$\forall U \in \tau \left[U \neq \varnothing \Rightarrow D \cap U \neq \varnothing \right]$$

Equivalently, $D \subseteq X$ is *(Everywhere) Dense* in X provided that: $Cl(D) = X$

(ii) A set $S \subseteq X$ is *Nowhere Dense* in X iff $Cl(S)$ has a void interior: $Cl(S) = Bd(S)$

Dual Statement : $Cl(S) = Bd(S) \Leftrightarrow X \setminus Cl(S)$ *is open everywhere dense*

(5) Examples of Everywhere Dense Subsets

Real Line Each of $\mathbb{Q}$ and $\mathbb{P}$ is everywhere dense in the real line $\mathbb{R}$.

Euclidean Spaces For each $n \in \mathbb{N}^+$: $\mathbb{Q}^n$ is everywhere dense in $\mathbb{R}^n$.

4. Alternative Characterizations of Topological Spaces

Remark

The fundamental duality between open and closed sets is indicated below. For each type of operator here, the range and the fixed point set coincide.

(1) Abstract Topological Interior Operator

A *Topological Interior Operator*: $Int : PS(X) \to PS(X)$, on a nonvoid set X is an operator for which the universal closure of each of the following axioms is satisfied:

(i) $Int(\varnothing) = \varnothing$ *and* $Int(X) = X$

(ii) $Int(U) \subseteq U$

(iii) $Int(U \cap V) = Int(U) \cap Int(V)$

(iv) $Int(Int(U)) = Int(U)$

A subset $U \subseteq X$ is defined to be an open set iff U is a fixed point of the topological interior operator: $U = Int(U)$; closed sets are defined by complementation.

(2) Abstract Topological Closure Operator

A *Topological Closure Operator*: $Cl : PS(X) \to PS(X)$, on a nonvoid set X is an operator for which the universal closure of each of the following axioms is satisfied:

(i) $Cl(\varnothing) = \varnothing$ *and* $Cl(X) = X$

(ii) $A \subseteq Cl(A)$

(iii) $Cl(A \cup B) = Cl(A) \cup Cl(B)$

(iv) $Cl(Cl(A)) = Cl(A)$

A subset $A \subseteq X$ is defined to be a closed set iff A is a fixed point of the topological closure operator: $A = Cl(A)$; open sets are defined by complementation.

5. Sequential Convergence

(1) Sequences and Subsequences

(i) A *Sequence* is a function whose domain is the set of all natural numbers. (More generally, a sequence is a function whose domain is a well–ordered set.)

(ii) A sequence $s : \mathbb{N} \to \mathbb{N}$ is *Strictly Increasing* provided that:

$$\forall n \in \mathbb{N}\ \left[s(n+1) > s(n) \right]$$

(iii) Suppose that X is a nonempty set, and $f : \mathbb{N} \to X$ is a sequence taking values in X. Then, a sequence $g : \mathbb{N} \to X$ is a *Subsequence* of f provided that:

$$\exists\, s\ \left[s : \mathbb{N} \to \mathbb{N}\ \ \&\ \ s \textit{ is strictly increasing}\ \ \&\ \ g = f \circ s \right]$$

In this relationship, $s : \mathbb{N} \to \mathbb{N}$ is the *Auxiliary (Strictly Increasing) Sequence* in the construction of the subsequence g from the original sequence f.

(2) Sequential Convergence

Hypothesis

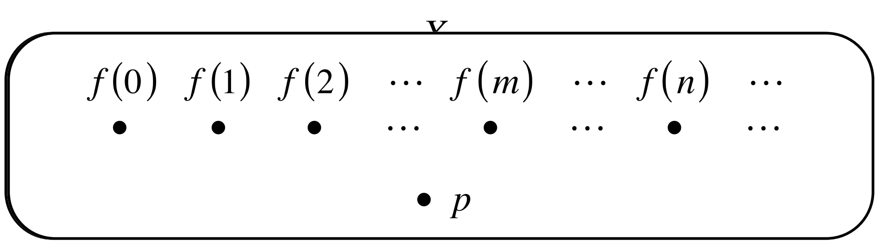
Suppose that $\langle X, d \rangle$ is a metric space, that $f : \mathbb{N} \to X$ is a sequence taking values in the target X, and that $p \in X$.

We visualize the sequence $f : \mathbb{N} \to X$ from two points of view.

Graph of the Sequence:

$$\begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & m & \cdots & n & \cdots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots \\ f(0) & f(1) & f(2) & \cdots & f(m) & \cdots & f(n) & \cdots \end{array}$$

Range of the Sequence: $\quad Range\, f \subseteq X$



Initial and Tail Ends

Each $m \in \mathbb{N}$ splits the natural number set into a finite *Initial Segment*: $\{n \in \mathbb{N} : n < m\}$, and an infinite *Tail End*: $\{n \in \mathbb{N} : n \geq m\}$.

Sequential Convergence

The sequence f *Converges* to the point p (relative to d) provided that:

$$\forall \varepsilon > 0 \quad \exists m \in \mathbb{N} \quad \forall n \in \mathbb{N} \textit{ with } n \geq m \quad \left[d\left(f(n), p\right) < \varepsilon \right]$$

The role of various parts of the quantifier prefix of the definition corresponds in order to:

(1) Specification of a Basic Neighborhood centered at p

(2) Specification of a Tail End of $\mathbb{N}$

(3) Tail End Membership

The formula matrix of the definition specifies:

Neighborhood Membership of $f(n)$

We visualize the specification of a tail end, and tail end membership, in the context of the picture for the sequence graph; whereas, we visualize the specification of a neighborhood, and neighborhood membership in the context of the sequence range as a subset of the ambient (surrounding) space X.

An equivalent characterization of sequential convergence is provided by:

$$\forall \varepsilon > 0 \;\; \forall_{ae}\, n \in \mathbb{N} \quad \left[d\left(f(n), p\right) < \varepsilon \right]$$

(3) Neighborhoods and Convergence

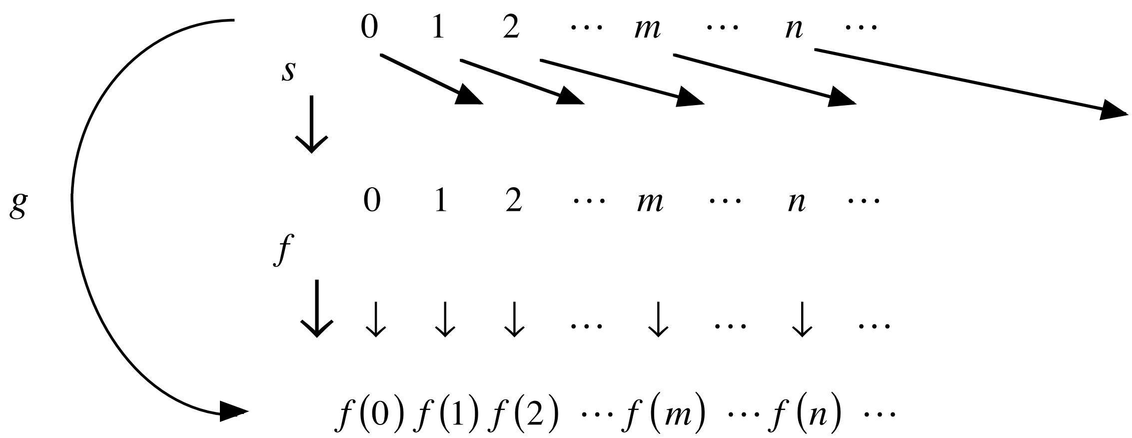
(i) Suppose that $\langle X, d \rangle$ is a metric space, and that $f : \mathbb{N} \to X$ is a sequence. Then:

$$f : \mathbb{N} \to X \textit{ Converges to a Point } p \in X$$
$$\Leftrightarrow \; \forall \varepsilon > 0 \;\; \exists m \in \mathbb{N} \;\; \forall n \geq m \quad \left[f(n) \in N_{\varepsilon}(p) \right]$$

(ii) Suppose that $\langle X, \tau \rangle$ is a topological space, and that $f : \mathbb{N} \to X$ is a sequence. Then:

$$f : \mathbb{N} \to X \textit{ Converges to a Point } p \in X$$
$$\Leftrightarrow \; \forall U \in \tau \textit{ with } p \in U \;\; \exists m \in \mathbb{N} \;\; \forall n \geq m \quad \left[f(n) \in U \right]$$

(4) Visualizing Subsequences



The official definition of *Subsequence* given above captures the intuitive idea of first listing the original sequence, and then building a subsequence by taking an infinite walk, and step by step, marking off relevant points as indicated in the following picture:

$$f(0)\,\overset{\bullet}{f(1)}\,f(2)\,\overset{\bullet}{f(3)}\,\overset{\bullet}{f(4)}\,f(5)\,f(6)\,\overset{\bullet}{f(7)}\,\cdots f(m)\,\cdots \overset{\cdots}{f(n)}\,\cdots$$

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(5) Limit of a Sequence

Initial Theorem on Sequential Convergence in Metric Spaces

(i) A sequence in a metric space can converge to at most one point.

(ii) In a metric space, every subsequence of a convergent sequence is itself convergent and converges to the same point as that of the original sequence.

Suppose $\langle X, d\rangle$ is a metric space with induced topology τ, and $f:\mathbb{N}\to X$ is a sequence that converges to $p\in X$.

Then, p is the *Limit of the Sequence* f, denoted: $\underset{n\to\infty}{Lim}\ f(n)=\underset{n\to\infty}{Lim}\ f_n=p$

Remark

The limit process depends upon the <u>topology τ</u>.

Remark

Our introductory use of the symbol f to denote a sequence is intended to emphasize that a sequence is first of all a function. We now switch to more conventional notation; the switch aids in tracking the *Type Hierarchy of Functions*.

(6) Limit of a Sequence as related to Limit Point of a Set

Ambient Theorem on Sequences taking values in a Metric Space

Suppose that $\langle X, d \rangle$ is a metric space, and $A \subseteq X$. Then:

(i) The set A is closed iff for every convergent sequence $p : \mathbb{N} \to A$ taking values in A, the limit of the sequence is a point in A: $\underset{n \to \infty}{Lim}\ p(n) \in A$.

(ii) A point $q \in X$ is a limit point of A iff there exists a one–to–one convergent sequence $p : \mathbb{N} \to A \setminus \{q\}$ taking values in $A \setminus \{q\}$ such that: $\underset{n \to \infty}{Lim}\ p(n) = q$.

(iii) A point $q \in X$ is a limit point of the range of a sequence $p : \mathbb{N} \to X$ iff there exists an injective (one–to–one) subsequence of p that converges to q.

(7) Universal Sequences

A sequence of points $p : \mathbb{N} \to X$ is *Universal* with respect to τ provided that:

$$\forall \mathit{nonvoid}\ U \in \tau\ \ \exists n \in \mathbb{N}\ \ \left[p(n) \in U \right]$$

Thus, a sequence $p : \mathbb{N} \to X$ is universal with respect to τ iff *Range* p is everywhere dense. A topological space $\langle X, \tau \rangle$ is *Point–Separable* provided that the space admits a universal sequence.

(8) Universal Sequences in the context of Metric Spaces

Suppose that $\langle X, d \rangle$ is a metric space, and τ is the topology induced by d.

(i) Convergence Theorem on Universal Sequences (in Metric Spaces)

If $p : \mathbb{N} \to X$ is a universal sequence with respect to τ, then for every $q \in X$, there exists a subsequence of p that converges to q.

(ii) Global Basis Theorem on Universal Sequences (in Metric Spaces)

There is a countable global base for the topology τ iff there exists a universal sequence with respect to τ.

Remark Thus, for metric spaces, the Second Axiom of Base Countability is characterized by the existence of a universal sequence.

(iii) Theorem on the Existence of Euclidean Universal Sequences

For each $n \in \mathbb{N}^{+}$: There exists a universal sequence $p : \mathbb{N} \to \mathbb{R}^{n}$ with respect to the $n-$ dimensional Euclidean topology.

6. Compactness

(1) Compact Subsets of Topological Spaces

Covering Definition of Compactness

A subspace $A \subseteq X$ of a topological space $\langle X, \tau \rangle$ is *Compact* iff from every open cover of A, it is possible to extract a finite subcollection that is itself a cover of A.

Briefly, $A \subseteq X$ is *Compact* iff every open cover of A has a finite subcover:

$$\forall \Gamma \subseteq \tau \ \exists \Delta \subseteq \Gamma \ \left[A \subseteq \bigcup \Gamma \ \Rightarrow \ A \subseteq \bigcup \Delta \ \ \& \ \ \Delta \textit{ is finite} \right]$$

(In the context: $A \subseteq X$, members of a cover of A by subsets of X are not necessarily subsets of A.)

(2) Sequential Convergence Characterization for Metric Spaces

Sequence Theorem

Suppose that $\langle X, d \rangle$ is a metric space, and that $A \subseteq X$.

Then, $A \subseteq X$ is a compact subspace iff for every sequence $f : \mathbb{N} \to A$ taking values in A, there exists a subsequence $g = f \circ s$ of f that converges to a point of A:

$$s : \mathbb{N} \to \mathbb{N} \textit{ is strictly increasing} \quad g : \mathbb{N} \to A \quad Lim_{n \to \infty} g_n = p \in A$$

(3) Compactness Theorem for Euclidean Subspaces

Closed and Bounded Consequence Theorem (in Metric Spaces)

Suppose that $A \subseteq X$ is a compact subspace of a metric space $\langle X, d \rangle$.

Then: (i) *A is closed* (ii) *A is bounded*

Remark In Euclidean spaces, the converse holds:

Heine–Borel Theorem

A subspace of a Euclidean space is compact iff it is closed and bounded.

Constructions for the Consequence Theorem

Suppose N is the neighborhood function and τ is the topology induced by d.

(i) (1) Let $p \in Lp(A)$.

(2) Let $U : \mathbb{N} \to \tau$ defined by: $U_n = X \setminus Cl\left(N_\varepsilon(p) \right) \qquad \varepsilon = \dfrac{1}{n+1}$

(3) Let $\Gamma = Range\ U$.

Claim If $p \notin A$, then Γ is an open cover of A for which there does not exist a finite subcollection that covers A.

(ii) (1) Let $p \in X$.

(2) Let $U : \mathbb{N} \to \tau$ defined by: $U_n = N_{n+1}(p)$

(3) Let $\Gamma = Range\ U$.

Claim The collection Γ is an open cover of X such that for each finite subcollection, the union of the subcollection is a bounded subset of X.

Heine–Borel Theorem

Let $n \in \mathbb{N}^+$. Suppose that $A \subseteq \mathbb{R}^n$ is a subset of $n-$*dimensional* Euclidean space. Then, $A \subseteq \mathbb{R}^n$ is a compact subspace iff A is closed and bounded.

Construction for showing that the Closed Unit Interval is a Compact Subspace

(1) Let Γ be an open cover of closed unit interval $\mathbb{I} = [0,1] \subseteq \mathbb{R}$ as a subspace of the line.

(2) Define A by: $A = \left\{ t \in [0,1] : \exists \textit{ Finite Subcollection } \Delta \subseteq \Gamma \ \left(\Delta \textit{ covers } [0,t] \right) \right\}$

(3) Let $L = Lub\ A$.

Claim $L = 1$

7. Connectedness

(1) Connected Subsets of Topological Spaces

Mutually Separate Sets and Open–Closed Sets in the Subspace Topology

In a topological space, two sets A and B are *Mutually Separate* iff each of A and B is nonvoid, A and B are disjoint, and neither set contains a limit point of the other set.

In turn, a subset $C \subseteq X$ of a topological space $\langle X, \tau \rangle$ is a *Connected Subset* iff C can not be partitioned as the union of two mutually separate subsets included in C.

Every nonvoid $C \subseteq X$ inherits a topology from τ. The *Subspace Topology* for C is inherited from the total space by intersecting C with each total space open set: $\{U \cap C : U \in \tau\}$.

Two mutually separate sets with union C are nonvoid disjoint *Open–Closed* sets in the subspace topology for C. Thus, C is connected iff C can not be partitioned into two nonvoid disjoint open sets belonging to its subspace topology.

(2) Theorem on the Connectedness of Euclidean Spaces

Real Line Order–Completeness in $\mathbb{R}$ implies that the real line $\mathbb{R}$ is connected.

Construction

(1) Suppose that $\{U, V\}$ is a partition of $\mathbb{R}$, $a \in U$, $b \in V$ and that $a < b$.

(2) Let $A = \{ t \in [a,b] : [a,t] \subseteq U \}$.

(3) Let $L = Lub\ A$.

Claim

The point L witnesses that U and V are not mutually separate.

Fact on Intermediate Points

A subset $C \subseteq \mathbb{R}$ is connected iff the following condition is satisfied:

$$\forall a,b \in C\ \forall x \in \mathbb{R}\ [a \le x \le b \ \Rightarrow\ x \in C]$$

Euclidean Spaces For each $n \in \mathbb{N}^+$: $\mathbb{R}^n$ is connected.

Construction Tool

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{u} \neq \mathbf{v}$. Then, the *Line Segment Joining* $\mathbf{u}$ and $\mathbf{v}$ is defined by:

$$Seg(\mathbf{u},\mathbf{v}) \ = \ \{ t\,\mathbf{u} + (1-t)\,\mathbf{v} : t \in [0,1] \} \ \subseteq \mathbb{R}^n$$

8. Cauchy Completeness in Metric Spaces

(1) Cauchy Sequences

Suppose that $\langle X, d \rangle$ is a metric space.

Then, $p : \mathbb{N} \to X$ is a *Cauchy Sequence* (relative to d) provided that:

$$\forall \varepsilon > 0 \ \ \exists n \in \mathbb{N} \ \ \forall s, t \geq n \ \ \left[d\left(p_s, p_t\right) < \varepsilon \right]$$

Cauchy Theorem on Sequential Convergence in Metric Spaces

(i) Every convergent sequence in a metric space is a Cauchy sequence.

(ii) If a subsequence of a Cauchy sequence converges, then the Cauchy sequence is itself convergent and converges to the same limit as that of the subsequence.

(2) Cauchy Complete Metric Spaces

A metric space $\langle X, d \rangle$ is *(Cauchy) Complete* iff for every Cauchy sequence $p : \mathbb{N} \to X$ taking values in X , p converges to a point of X .

Remark

In a complete metric space, the Cauchy condition is both necessary and sufficient to determine whether a sequence converges. The defining condition is entirely internal to the sequence, so we have a test for convergence that does not require a global survey of candidates for the limit.

Theorem on Cauchy Completeness in Metric Spaces

(i) Compact Metric Spaces

Every compact metric space is Cauchy complete.

(ii) Euclidean Spaces

For each $n \in \mathbb{N}^+$: $\mathbb{R}^n$ is Cauchy complete.

Construction in the Real Line

Suppose that $p:\mathbb{N}\to\mathbb{R}$ is a Cauchy sequence (relative to the usual Euclidean distance function).

(1) Choose a strictly increasing sequence $c:\mathbb{N}\to\mathbb{N}$ such that:

$$\forall n\in\mathbb{N}\ \ \forall s,t\geq c(n)\ \left[d\left(p_s,p_t\right)<\frac{1}{n+1}\right]$$

(2) Construct a nested decreasing (inclusion) sequence of closed intervals in the real line:

$$A_n=\left\{p_m : m\geq c(n)\right\}$$

$$G_n=Glb\ A_n\qquad L_n=Lub\ A_n$$

$$\left\langle\left[G_n,L_n\right]:n\in\mathbb{N}\right\rangle$$

Claim

$$Lim_{n\to\infty}\ p_n=\bigcap_{n\in\mathbb{N}}\left[G_n,L_n\right]$$

IV. Mapping Concepts

1. Continuous Functions

(1) Continuity

Continuity Definition

Suppose that each of $\langle X, \sigma \rangle$ and $\langle X, \tau \rangle$ is a topological space, and that $f : X \to Y$ is a function of X into Y. Then:

(i) f is *Continuous* at $q \in X$ (relative to σ *and* τ) provided that:

$$\forall V \in \tau \;\; \exists U \in \sigma \;\left[f(q) \in V \;\Rightarrow\; q \in U \;\&\; f[U] \subseteq V \right]$$

(ii) f is *Continuous (Everywhere)* (relative to σ *and* τ) provided that:

$$\forall V \subseteq Y \;\left[V \in \tau \;\Rightarrow\; f^{-1}[V] = U \in \sigma \right]$$

$$(\textit{Inverse images of open sets are open})$$

The dual formulation of global continuity is the requirement that the inverse image of each closed subset of the target space is a closed subset of the domain space.

The relevant set–theoretic property is the pull–back conditional on partitions:

$$\{S, T\} \textit{ is a partition of } Y \;\Rightarrow\; \left\{ f^{-1}[S], f^{-1}[T] \right\} \textit{ is a partition of } X$$

(2) Characterizations of Continuity

(i) In the above Definition of Continuity, the replacement of τ by either a Global Base or a Global Subbase for τ yields an equivalent characterization.

The relevant set–theoretic equations are:

$$f^{-1}[\cup\Gamma] = \cup \left\{ f^{-1}[S] : S \in \Gamma \right\} \qquad f^{-1}[\cap\Gamma] = \cap \left\{ f^{-1}[S] : S \in \Gamma \right\}$$

$$\left(\Gamma \subseteq PS(Y),\; \Gamma \neq \varnothing \right)$$

(ii) Suppose the First Axiom of Base Countability is satisfied in both the domain and target spaces. Then: A function is continuous iff it preserves sequential convergence.

Remark

A continuous function is completely determined by the values that it assumes on a dense set of arguments of its domain.

Composition

The equation below underlies that a composition of continuous functions is itself continuous:

$$(f \circ g)^{-1}[V] = g^{-1}\left[f^{-1}[V]\right]$$

(4) Characterization of Continuity in the context of Metric Spaces

Hypothesis

Suppose that each of $\langle X, d\rangle$ and $\langle Y, e\rangle$ is a metric space, that σ *and* τ are the respective induced topologies, and that $f : X \to Y$ is a function of X into Y.

Global Continuity

Each of the following is a necessary and sufficient condition for the global continuity of f (relative to σ *and* τ):

Closeness

$$\forall q \in X \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall p \in X \;\; \left[d(p,q) < \delta \;\Rightarrow\; e\big(f(p), f(q)\big) < \varepsilon \right]$$

Convergence

$$\forall q \in X \;\; \forall p : \mathbb{N} \to X \;\; \left[\underset{n\to\infty}{Lim}\, d(p_n, q) = 0 \;\Rightarrow\; \underset{n\to\infty}{Lim}\, e\big(f(p_n), f(q)\big) = 0 \right]$$

$$\forall q \in X \;\; \forall p : \mathbb{N} \to X \;\; \left[\underset{n\to\infty}{Lim}\, p_n = q \;\Rightarrow\; \underset{n\to\infty}{Lim}\, f(p_n) = f(q) \right]$$

In the first matrix conditional for preservation of convergence, in both the antecedent and the consequent, sequential convergence in the real line is at issue.

In the second conditional, the issue in the antecedent and in the consequent, is sequential convergence in $\langle X, \sigma\rangle$, and sequential convergence in $\langle Y, \tau\rangle$, respectively.

Continuity at a Point

For a necessary and sufficient condition for the continuity of f at $q \in X$, omit the quantification on q in each of the preceding formulas.

Remark

In comparison to the definitions for the general topological setting, the neighborhood determined by ε is the analogue of V, and the neighborhood determined by δ is the analogue of U; ε and V are *Value Closeness Constraints*.

Oscillation

The *Oscillation* of f, relative to d and e, is defined by:

$$Osc_f : X \to \left\{ a \in \mathbb{R}^e : a \geq 0 \right\}$$

$$Osc_f(q) = Glb\left\{ Diam\, f\left[N_\delta(q) \right] : \delta > 0 \right\}$$

$$\left(q \in X,\ N = N(d),\ Diam = Diam(e) \right)$$

Oscillation Theorem

Then: f is continuous at $q \in X$ iff $Osc_f(q)$ is *zero*.

2. Uniform Continuity

(1) Contrast between Global and Uniform Continuity

Suppose that each of $\langle X, d\rangle$ and $\langle Y, e\rangle$ is a metric space, and that $f : X \to Y$ is a function of X into Y.

Then, f is *Uniformly Continuous* (relative to d *and* e) provided that:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall p, q \in X$$

$$\left[d(p,q) < \delta \;\Rightarrow\; e\big(f(p), f(q)\big) < \varepsilon \right]$$

Contrast the quantification above with that of global continuity for metric spaces. It is apparent that uniform continuity implies global continuity, and that in the case of uniform continuity, δ depends only upon ε : $\delta = \delta(\varepsilon)$, as opposed to both ε and the choice of q for the weaker property: $\delta = \delta(q, \varepsilon)$. This fact also explains the terminology.

(2) Theorem on Uniform Continuity from Continuity on a Compact Domain

Suppose that each of $\langle X, d\rangle$ and $\langle Y, e\rangle$ is a metric space, that σ *and* τ are the respective topologies, and that $f : X \to Y$ is continuous (relative to σ *and* τ).

Suppose further that $\langle X, \sigma\rangle$ is compact. Then, f is uniformly continuous.

Construction Let $\varepsilon > 0$.

(1) Using global continuity: For each $q \in X$, choose a radius $r(q) > 0$ for a basic neighborhood centered at q for which the image under f is included as a subset of the basic neighborhood centered at $f(q)$ with radius $\varepsilon \div 2$.

(2) Using compactness, choose a finite subcover of the following open cover of X :

$$\left\{ N_t(q) : q \in X \;\&\; t = r(q) \div 2 \right\}$$

(N is the basic neighborhood function induced by d)

(3) Let δ be the minimum value of the radius of a neighborhood belonging to the finite subcover.

Claim Our construction returns a value of δ that satisfies the uniform continuity constraint imposed by the initial (and arbitrary) choice of ε.

V. Analysis

1. Real–Valued Functions of One Real Variable
Class of Elementary Functions

Function – Building Operations

The following methods are used to construct new functions from given functions:

(i) *Linear Combinations* (*Addition and Multiplication by Constants*)
(ii) *Algebraic Combinations* (*Arithmetical Operations and Extractions of Roots*)
(iii) *Composition of Functions*
(iv) *Taking the Functional Inverse* (*Rigid–Motion Reflection about* $y = x$)
(v) *Definition by Cases* (*Including Absolute Value and Step Functions*)
(vi) *Geometric Transformations*
(vii) *Rigid – Motion Horizontal and Vertical Translations* (*Additive Constant*)
$(viii)$ *Rigid – Motion Reflection about the lines* : $x = 0 \quad y = 0$ (*Minus Sign*)
(ix) *Horizontal and Vertical Expansions and Contractions* (*Multiplicative Constant*)

Basic Functions

Basic Algebraic Functions

(1) Polynomials

The *Polynomial of Degree* n *with Coefficients* $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$, where $n \neq 0$ implies $a_n \neq 0$, is the function $f : \mathbb{R} \to \mathbb{R}$ defined by the following equation:

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_m x^m + \cdots + a_n x^n$$

Each *Exponent* m, $0 \le m \le n$, must be a natural number; thus, a polynomial is a *Linear Combination of Power Functions* x^m with natural number exponents $m \in \mathbb{N}$:

$$y = x^m = 1 \bullet x \bullet x \bullet \cdots \bullet x \quad (m - \textit{many factors of } x)$$

(2) Rational Functions

A *Rational Function* is a quotient (ratio) of two polynomials for which the denominator is not the constant polynomial function taking the value *zero* everywhere.

(A polynomial is a special case of a rational function where the denominator is the constant polynomial function taking the value *one* everywhere.)

(3) Extraction of Roots

Suppose that $n \in \mathbb{N}$ with $n > 1$. Then, the *Root Function of Degree* n, denoted by $y = \sqrt[n]{x} = x^{1 \div n}$, is the functional inverse of $y = x^n$ in the case where n is odd, and is the functional inverse of the restriction of $y = x^n$ to $x \in [0, \infty)$ for n even. A root function is a power function where the exponent is a multiplicative inverse of a natural number $n > 1$.

(4) Algebraic Power Functions

An *Algebraic Power Function* with positive exponent is a function f defined as a composition of a root function, and a power function with a positive integer exponent:

$$f(x) = x^a = x^{m \div n} = \left(x^{1 \div n}\right)^m = \left(x^m\right)^{1 \div n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$$

$$\left(x \geq 0 \textit{ varies}; \ a = m \div n \in \mathbb{Q} \textit{ is fixed}; \ m, n \in \mathbb{N}^+\right)$$

The domain of f extends to all $x \in \mathbb{R}$ in the case $a = m \div n > 0$ where the denominator $n \in \mathbb{N}$ is an odd natural number. For the exponent $-a$, the power function is defined to be the composition of f above followed by the multiplicative inverse function.

Remark For a power function, the exponent is held fixed, whereas the *Base* varies with the argument. If the exponent is irrational, then the function is a *Transcendental Power Function*, and as such, falls outside of the current classification.

Auxiliary Functions

Projection Functions

The functions $Proj_1 : \mathbb{R}^2 \to \mathbb{R}$ and $Proj_2 : \mathbb{R}^2 \to \mathbb{R}$ denote the projection functions to the first and second coordinates respectively:

$$Proj_1(x, y) = x \qquad Proj_2(x, y) = y$$

Wrapping Function

On the right unit semi–circle, Θ varies over signed arc length:

$$g(y) = \int_0^y \frac{1}{\sqrt{1 - t^2}} \, dt = \Theta \qquad (-1 < y < 1)$$

$$g \textit{ continuously extends to } 1 \textit{ and } -1$$

Definition of π: $\pi = 2g(1) = -2g(-1)$

Define $W:\mathbb{R}\to\mathbb{R}^2$ by:

$$y=g^{-1}(\Theta) \qquad x=\sqrt{1-y^2} \qquad \left(\, g(-1)=-2^{-1}\pi\le\Theta\le 2^{-1}\pi=g(1)\,\right)$$

$$W(\Theta)=W(\Theta+2n\,\pi)=\langle x,y\rangle \quad W\left(\Theta+(2n+1)\pi\right)=\langle -x,-y\rangle \quad (n\in\mathbb{Z})$$

The function $W:\mathbb{R}\to\mathbb{R}^2$ uniformly wraps the real line around the unit circle with *zero* taken to the base point $\langle 1,0\rangle$, the positive number ray mapped counter–clockwise, the negative number ray mapped clockwise, and with every closed number interval of length 2π being mapped once around the unit circle with overlap at the endpoints (and only at the endpoints).

(See Chapter VI – Section 5 .)

Basic Transcendental Functions

(1) Trigonometric Functions

We define the *Cosine* and *Sine* functions to be the following compositions:

$$Cosine=Proj_1\circ W \qquad Sine=Proj_2\circ W$$

The other four basic trigonometric functions are defined algebraically from the cosine and sine:

$$Secant(x)=\frac{1}{Cos(x)} \qquad Cosecant(x)=\frac{1}{Sin(x)}$$

$$Cotangent(x)=\frac{Cos(x)}{Sin(x)} \qquad Tangent(x)=\frac{Sin(x)}{Cos(x)}$$

Restrictions of Trigonometric Functions to their Principal Parts (or Branches)

Domains of the Principal Parts:

Cosine:	$[0,\pi]$	Sine:	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
Secant:	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$	Cosecant:	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\ \frac{\pi}{2}\right]$
Cotangent:	$(0,\pi)$	Tangent:	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(2) Arc Trigonometric Functions

Each principal part is an injective function; the *Arc Trigonometric Functions* are the functional inverses of the Principal Parts of the Trigonometric Functions.

(3) Natural Logarithm and the Exponential Functions

$$Ln\, x = \int_1^x t^{-1} dt \quad (x > 0)$$

$$\text{Definition of } e\text{:} \quad \int_1^e t^{-1} dt = 1$$

The function $y = Ln\ x$ is the *Natural Logarithm*; let Exp denote the functional inverse of the natural logarithm. The *Exponential Function with Base* a is the function f defined by composing the linear function, with slope $Ln\ a$, with Exp:

$$f(x) = Exp\big((Ln a)\ x \big) \quad f(x) \textit{ is denoted by } a^x = e^{x\, Ln\, a}$$

$$\big(x \in \mathbb{R} \textit{ varies as the argument};\ a \textit{ is the fixed base};\ a > 0,\ a \neq 1 \big)$$

(4) Logarithmic Functions

All exponential functions are injective.

The *Logarithmic Function with Base* $a \neq e$ is defined to be the functional inverse of the exponential function with base $a \neq e$.

(5) Transcendental Power Functions

The *Transcendental Power Function with Exponent* a is the function f defined by:

$$f(x) = x^a = e^{a\, Ln\, x}$$

$$\big(x > 0 \textit{ varies};\ a \in \mathbb{P} \textit{ is a fixed irrational number} \big)$$

Remark

Transcendental functions *Transcend* algebraic operations.

Remark

A real or complex number is defined to be *Algebraic* iff it is a root of a polynomial with rational coefficients; a complex number that is not algebraic is defined to be *Transcendental*.

The numbers e and π are real transcendental numbers; the number i is an *Imaginary* (that is, nonreal complex) algebraic number.

Definition of the Class of Elementary Functions

An *Elementary Real–Valued Function of One Real Variable* is defined to be either one of the basic functions listed above, or any function that can be constructed from the basic functions by applying the function–building operations listed above some finite number of times. If all of the basic functions involved in the construction are algebraic, then the end result is algebraic also; otherwise, the combination function is a transcendental function.

That is, if any transcendental component function appears in a function construction, then the end result is itself defined to be a transcendental function.

Thus, the class of elementary functions is partitioned into two disjoint subclasses:

Algebraic and Transcendental

This means that every elementary function belongs to one of the two classes, but no function belongs to both.

2. Limits and Continuity

Hypothesis and Notation

Suppose that each of $\langle X, d \rangle$ and $\langle Y, e \rangle$ is a metric space. Suppose further that $A \subseteq X$, $q \in X$ is a limit point of A, and that $f : A \to Y$ is a function of A into Y.

Finally, let $L \in Y$ be an arbitrary choice of a point in the target space.

(1) Function Limit

The limit concept for a metric space variable argument is defined by:

$$\textit{Limit of } f(p) \textit{ as } p \textit{ converges to } q\text{, } p \in A \setminus \{q\}\text{, } \textit{equals } L$$

$$Lim_{p \to q} f(p) = L$$

$$\text{iff } \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall p \in A \;\; \left[0 < d(p, q) < \delta \;\Rightarrow\; e\big(f(p), L\big) < \varepsilon \right]$$

The constraint: $p \in \textit{Domain}\, f \textit{ with } p \neq q$, is implicit in the notation: $p \to q$.

(2) Characterization of Continuity by Function Limit

The function f is continuous at q iff each of the conditions below is satisfied:

(i)	$q \in A = \textit{Domain } f$	*Existence of a Function Value*
(ii)	$Lim_{p \to q} f(p) = L$	*Existence of a Function Limit*
(iii)	$f(q) = L$	*Value of the Function and value of the Function Limit Coincide*

(3) Removable and Essential Discontinuities

The discontinuities of f are classified as follows:

(i) If $Lim_{p \to q} f(p) = L$, but either f is undefined at q, or $f(q) \neq L$, then q is a *Removable Discontinuity* of f.

(ii) If $Lim_{p \to q} f(p)$ does not exist, then q is an *Essential Discontinuity* of f.

3. Differentiation

Algebraic Approximation: Slope of a Secant $\equiv$ Average Rate of Change

Limit Process: Slope of the Tangent $\equiv$ Instantaneous Rate of Change

Hypothesis Suppose that $u, v \in \mathbb{R}$ with $u < v$, $x \in (u, v)$, and that $f : (u, v) \to \mathbb{R}$. Let (a, b) denote the open interval centered at *zero* such that:

$$|a| = |b| = Min\{\ |x-u|,\ |x-v|\ \}$$

Then:

(i) The *Difference Quotient* for f with *Fixed Point* x is defined by:

$$DQ : (a, b) \setminus \{0\} \to \mathbb{R} \qquad DQ(h) = \frac{f(x+h) - f(x)}{h}$$

In the definition of the function DQ: h is the independent variable, and, x is being held fixed as a parameter.

(ii) If it exists, the *Simplified Difference Quotient* for f with *Fixed Point* x is defined to be the *Unique Extension* of DQ to a function that is defined and continuous at *zero* :

$$SDQ : (a, b) \to \mathbb{R} \qquad \forall h \in (a, b) \setminus \{0\}\ \left[SDQ(h) = DQ(h) \right]$$

$$SDQ \textit{ is defined and continuous at zero}$$

$$\left(DQ \textit{ has zero as a removable discontinuity} \right)$$

(iii) If it exists, the *Derivative* of f at x, denoted: $f'(x)$, is defined by:

$$f'(x) = Lim_{\,h \to 0}\ DQ(h) = Lim_{\,h \to 0}\ SDQ(h) = SDQ(0)$$

After the evaluation: $SDQ(0)$, only the parameter x occurs as an unknown. The new function f' is the *Derivative* of f; the original function f is an *Antiderivative* of f'.

Remark In the preceding three–step construction, there are (potentially) four distinct functions: *Original Function*, *Difference Quotient*, *Simplified Difference Quotient*, *Derivative*

The independent variable of the original function and its derivative is x; whereas, h is the independent variable of each difference quotient.

4. Integration

Algebraic Approximation: Riemann Sum ≡ Sum Signed Rectangle Areas

Limit Process: Definite Integral ≡ Signed Area of the Planar Region

Hypothesis Suppose that $a,b \in \mathbb{R}$ with $a \neq b$, and that $f:[u,v] \to \mathbb{R}$ is a continuous function, where $u = Min\{a,b\}$, and $v = Max\{a,b\}$.

(1) Partition (Orientation, Delta Function, and Mesh)

(i) A *Partition* of $[u,v]$ *Oriented* from u to v is a strictly increasing finite sequence x such that: $x:\{0,1,\ldots,n\} \to [u,v] \quad x_0 = u \quad x_{i-1} < x_i \quad x_n = v \quad \left(n \in \mathbb{N}^+,\ 1 \le i \le n\right)$

$$(\textit{Traverse the interval from } u \textit{ to } v)$$

(ii) A *Partition* of $[u,v]$ *Oriented* from v to u is a strictly decreasing finite sequence x such that: $x:\{0,1,\ldots,n\} \to [u,v] \quad x_n = u \quad x_i < x_{i-1} \quad x_0 = v \quad \left(n \in \mathbb{N}^+,\ 1 \le i \le n\right)$

$$(\textit{Traverse the interval from } v \textit{ to } u)$$

(iii) Regardless of the orientation, define the *Change in x Function* (Δx) by:

$$(\Delta x):\{1,\ldots,n\} \to \mathbb{R} \qquad (\Delta x)_i = x_i - x_{i-1}$$

The values of the change in x function: (Δx), are either entirely positive (increasing case), or entirely negative (decreasing case).

(iv) The *Mesh* of the partition x is defined to be the largest subinterval length:

$$Mesh(x) = Max\left\{ \left|(\Delta x)_i\right| : 1 \le i \le n \right\}$$

Remark Thus, the effect of a partition is to both orient the original interval, and to subdivide the original interval into finitely many subintervals.

(2) Sample Points

A choice of *Intermediate Arguments or Sample Points* for a partition x with domain $\{0,1,\ldots,n\}$, is defined to be a finite sequence t such that:

$$t:\{1,2,\ldots,n\} \to [u,v]$$

$$\forall i \in \{1,2,\ldots,n\} \ \left[x_{i-1} \le t_i \le x_i \ (\textit{Increasing}) \ or \ x_i \le t_i \le x_{i-1} \ (\textit{Decreasing})\right]$$

(3) Riemann Sum

The *Riemann Sum* corresponding to the integrand f, a partition x with domain $\{0,1,\ldots,n\}$, and a choice of sample points t for x, is defined by:

$$\sum_{i=1}^{n} f(t_i)(\Delta x)_i = f(t_1)(\Delta x)_1 + f(t_2)(\Delta x)_2 + \cdots + f(t_n)(\Delta x)_n$$

(4) Definite Integral

After the existence proof as indicated below, the *Definite Integral* of f with *Lower Limit* a and *Upper Limit* b is the real number defined by the following construction:

Suppose that $R : \mathbb{N} \to \mathbb{R}$ is a sequence of Riemann sums for which each of the two conditions below is satisfied:

(i) Each of the underlying partitions is oriented from a to b.

(ii) For each Riemann sum R_n $(n \in \mathbb{N})$, let $Mesh(R_n)$ denote the mesh of the underlying partition of R_n. Then, the sequence of mesh values converges to *zero* :

$$Lim_{n\to\infty}\ Mesh(R_n) = 0$$

Then, the definite integral of f from a to b is the number defined by the equation:

$$\int_a^b f(t)\,dt = Lim_{n\to\infty}\ R_n$$

Meaning of the Expression : Existence of the Definite Integral

The definite integral of f from a to b exists iff $Lim_{n\to\infty}\ R_n$ exists, and the value of the limit depends only upon the orientation of the partitions, and the requirement that the sequence of mesh values, of the underlying partitions, converges to *zero*.

Thus, we need to prove the existence of the definite integral for a continuous function defined on a closed interval.

In the case of coincidence of lower and upper limits of integration, we take the definite integral to be *zero* by definition:

$$a = b \ \Rightarrow\ \int_a^b f(t)\,dt = 0$$

5. Theorems on Continuous Functions

Background

(i) A subspace of a Euclidean space is compact iff it is closed and bounded.

(ii) A continuous function mapping a compact subspace of a metric space into a metric space is uniformly continuous.

Hypothesis for the Theorems

Suppose $a, b \in \mathbb{R}$ with $a \neq b$; let $\alpha = Min\{a,b\}$, and let $\beta = Max\{a,b\}$. Suppose further that $f : [\alpha, \beta] \to \mathbb{R}$ is a continuous function.

Conclusions

(i) Continuous Images

(1) Intermediate Value Theorem (IVT)

The image of a connected set under a continuous function is itself connected:

Range f is connected

(2) Compact Range Theorem (CRT)

The image of a compact set under a continuous function is itself compact:

Range f is compact

(ii) Uniform Continuity from Continuity and Compactness of Domain

(3) Uniform Continuity Theorem (UCT)

The function f is uniformly continuous.

(iii) Continuity and Definite Integration

(4) Existence of the Definite Integral (EDI)

The definite integral of f from a to b: $\int_a^b f(t)\,dt$, exists.

(5) Mean Value Theorem of Integral Calculus (MVT–IC)

There exists $c \in (\alpha, \beta)$ such that: $f(c)(b-a) = \int_a^b f(t)\,dt$

6. Theorems on Differentiable Functions

(1) Fundamental Theorem of Calculus–Part I (FTC–I)

Hypothesis and Terminology

Continuous Integrand

Suppose that $u, v \in \mathbb{R}$ with $u < v$, and that $f : (u, v) \to \mathbb{R}$ is a continuous function.

Antidifferentiation on Open Sets

A function for which the derivative is defined on an open set including the domain of f and for which the derivative is equal to f on $Domain\ f$, is an *Antiderivative* of f.

Indefinite Integration Suppose that $a, z \in \mathbb{R}$ with $a \in (u, v)$, and that each individual value of the function g is constructed by a definite integration as indicated below:

$$g : (u, v) \to \mathbb{R} \qquad g(x) = \int_a^x f(t)\,dt \ + z$$

(The argument x determines which definite integral by specifying the upper limit of integration; regardless of x, the lower limit is held fixed at the value of the parameter a.)

Then, the function g is the *Indefinite Integral* corresponding to the parameters:

Integrand f *Fixed Lower Limit* a *Additive Constant* z

Conclusion

(i) First Formulation of the Conclusion of FTC – I Indefinite Integration

Then, g is everywhere differentiable, and, for each $x \in (u, v)$: $g'(x) = f(x)$

The rate of change in the signed area of the planar region bounded by the integrand is the signed vertical line segment length from the $x-$ *axis* to the graph of the integrand.

(ii) Second Formulation of the Conclusion of FTC – I Antiderivatives

Then, the indefinite integral g is an antiderivative of f; thus, every continuous function defined on an open interval has an antiderivative on the open interval.

Remark Integration followed by differentiation returns the original continuous integrand f (integration and differentiation are inverse operations on functions).

Remark The hypothesis of the preceding theorem requires only a continuous function, whereas the conclusion asserts that indefinite integration of a continuous integrand returns a differentiable function.

Remark Our proof below of FTC – I is an application of MVT – IC.

(2) Mean Value Theorem of Differential Calculus (MVT – DC)

Suppose that $a, b \in \mathbb{R}$ with $a < b$, and that $f : [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$, and that f is a differentiable function on the open interval (a,b).

Then, there exists $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Equivalently, there exists a line tangent to *Graph* f, at a point over the interior of the domain of f, that is parallel to the secant joining the graph endpoints.

(3) Antiderivative Comparison Theorem (ACT) Corollary of MVT – DC

Suppose that $s, u, v, t \in \mathbb{R}$ with $s < u < v < t$, and that $f : [u,v] \to \mathbb{R}$ is a continuous function. Suppose further that each of $g : (s,t) \to \mathbb{R}$ and $h : (s,t) \to \mathbb{R}$ is a differentiable function such that: $g' = h' = f$ on $[u,v]$ (each of g and h is an antiderivative of f on $[u,v]$). Then, g and h differ by an additive constant on $[u,v]$:

$$\exists z \in \mathbb{R}\ \forall x \in [u,v]\ \left[g(x) = h(x) + z \right]$$

Equivalently, each of *Graph* $g \big| [u,v]$ and *Graph* $h \big| [u,v]$ is a rigid motion vertical translation of the other.

(4) Fundamental Theorem of Calculus–Part II (FTC – II)

(i) First Formulation of FTC – II

Hypothesis Suppose that $a, b \in \mathbb{R}$ with $a < b$, and that $f : [a,b] \to \mathbb{R}$ is a continuous function. Suppose further that on an open interval, h is an antiderivative of f.

The assertion on h means that there exist $u, v \in \mathbb{R}$, with $u < a < b < v$, such that:

$$h : (u,v) \to \mathbb{R} \text{ is differentiable} \qquad h'(t) = f(t) \ \ (t \in [a,b])$$

(Since f is continuous, and $h' = f$ on $[a,b]$: h' is continuous on $[a,b]$.)

Conclusion $\displaystyle \int_a^b f(t)\,dt = \int_a^b h'(t)\,dt = h(b) - h(a)$

Remark In contrast to $\text{FTC} - \text{I}$, we now differentiate first, and then integrate.

(ii) Second Formulation of $\text{FTC} - \text{II}$

Continuously Differentiable Functions

Suppose that $u, v \in \mathbb{R}$ with $u < v$, and that $f:(u,v) \to \mathbb{R}$ is a differentiable function for which the derivative f' is itself a continuous function on (u,v). In the case of a continuous derivative, the original function f is *Continuously Differentiable*.

Suppose that each of a and b is a point in the interval (u,v).

Then, regardless of orientation: $\int_a^b f'(t)\,dt = f(b) - f(a)$

In turn, this implies: $f(x) = \int_a^x f'(t)\,dt \; + z \quad z = f(a) \quad x \in (u,v)$

An antiderivative of a continuous integrand is an indefinite integral of the integrand.

Remark Differentiation followed by integration returns the original function.

Remark Thus, the second formulation of $\text{FTC} - \text{II}$ is simply the extension to functions that follows from the first formulation.

Remark We give below two proofs of $\text{FTC} - \text{II}$ with the alternative strategies:

Corollary of $\text{FTC} - \text{I}$ and ACT Application of MVT – DC

Fundamental Theorem of Calculus (Consolidation)

Let $\mathfrak{C}$ denote the collection of all globally continuous real–valued functions with an open interval domain in the real line; and, let $\mathfrak{CD}$ denote the collection of all globally continuously differentiable real–valued functions with an open interval domain in the real line.

Then, the condition below is sufficient for the Differentiation – Translation operator: $f \mapsto \langle f', f(a) \rangle$, to be a bijection from $\mathfrak{CD}$ onto $\mathfrak{C} \times \mathbb{R}$ for which Indefinite Integration is the operator inverse:

$$f(x) = \int_a^x f'(t)\,dt \; + f(a) \qquad x \in (u,v) = \mathit{Domain}\, f \qquad a = u + \frac{|u - v|}{2}$$

Extension The Fundamental Theorem extends as follows:

First, we can extend from *Open Interval* domain to *Open Connected Set* domain; we can then extend from *Open Connected Set* domain to simply *Open Set* domain. (The second step uses the Open Component Theorem of Chapter II – Section 6 – (2).)

7. Retrospectus

There are two main ideas, each involving an application of a limit process to infinitely many algebraic approximations:

Differentiation: Limit of the Difference Quotient.

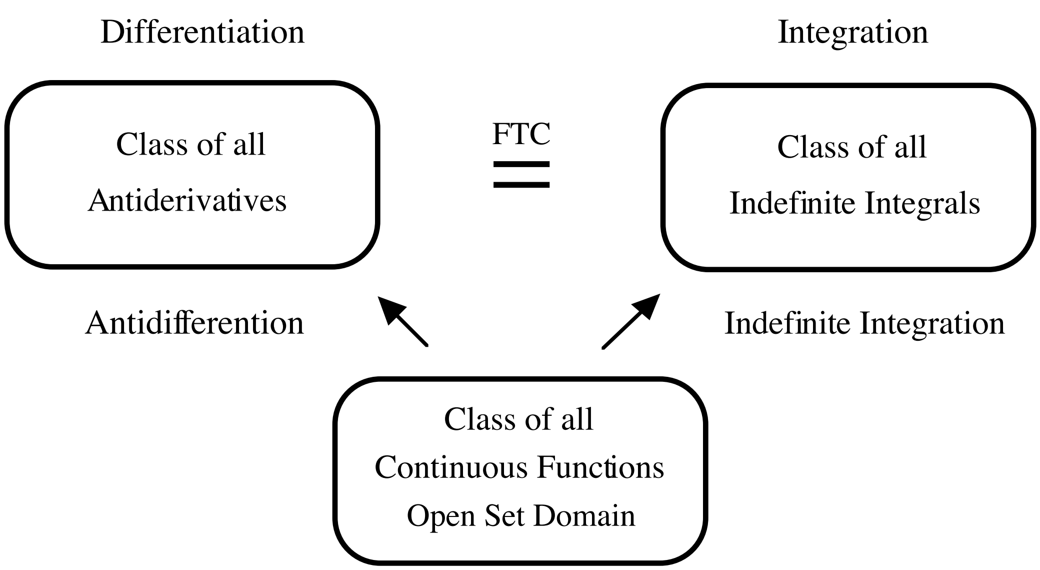
Definite Integration: Limit of an infinite sequence of Riemann Sums.

Indefinite integration is repeated definite integration thereby generating a function. Passing from *Definite* to *Indefinite* refers to the fact that the upper limit of the definite integration varies with the argument for the indefinite integral function.

Antidifferentiation is simply a change in perspective from that of differentiating an initial function to the alternative view where the initial function is the result of a differentiation.

Each of Indefinite Integration and Antidifferentiation can be represented as an infinite–valued operator acting on continuous functions.

The main theorem is the assertion that integration and differentiation are inverse operations on functions: FTC – I is integration followed by differentiation; FTC – II is differentiation followed by integration. Unfortunately, it is often the case that the indefinite integral is confused with the *General Antiderivative* at the point of definition, as opposed to a relationship between distinct concepts established by proving the main theorem:



FTC – I: Class of all Indefinite Integrals $\subseteq$ Class of all Antiderivatives

FTC – II: Class of all Antiderivatives $\subseteq$ Class of all Indefinite Integrals

Each of the operators Antidifferentiation and Indefinite Integration transforms the class of all *Continuous Functions* into the class of all *Continuously Differentiable Functions*.

VI. Proof Constructions

1. Constructions Underlying the Fundamental Theorem

Hypothesis

For the Theorems on Continuity, we assume that $\alpha = a < b = \beta$; and therefore, $f : [a,b] \to \mathbb{R}$ is a continuous function with domain $[a,b]$.

For the two theorems on definite integration: Items (3) and (4) below, the case $b < a$ is by symmetry.

(1) Intermediate Value Theorem

Proof Strategy

(1) Let $u, v \in Range\, f$ with $u < v$. Let $w \in (u, v)$.

By the Fact on Intermediate Points, we need to show that $w \in Range\, f$.

(2) Assume otherwise: $w \notin Range\, f$; from this assumption, we now derive a contradiction.

Construction

$$U = f^{-1}\left[(-\infty, w)\right] \qquad V = f^{-1}\left[(w, \infty)\right]$$

Claim

U *and* V *are mutually separate sets such that* $[a,b] = U \cup V$

Verification

(1) By the definition of a function:

A partition of the target pulls back to a partition of the domain.

(2) By the continuity of f:

The pull–back of an open subset of the target is an open subset of $Domain\, f = [a,b]$; in turn, an open subset of $[a,b]$ is the intersection of $[a,b]$ with an open subset of $\mathbb{R}$.

Conclusion

The connectedness of the interval $[a,b]$ has been contradicted.

(2) Compact Range Theorem

Suppose that $f:[a,b] \to \mathbb{R}$ $(a,b \in \mathbb{R}, a < b)$ is a continuous function with compact domain $[a,b]$. Then, $Range\, f \subseteq \mathbb{R}$ is a compact subspace of the real line.

Proof Strategy

Using the Sequence Theorem, we need to show that each sequence taking values in $Range\, f$ has a subsequence that converges to a point of $Range\, f$.

Construction

(1) Let $y:\mathbb{N} \to Range\, f$ (we are choosing the sequential terms in the vertical axis).

(2) We now pull back to the domain (in the horizontal axis).

Let $x:\mathbb{N} \to [a,b]$ such that: $y = f \circ x \quad f(x_n) = y_n \quad (n \in \mathbb{N})$

(3) By the Heine–Borel Theorem, $[a,b]$ is a compact subspace of $\mathbb{R}$.

(4) Therefore, by the Sequence Theorem, we can choose a subsequence $t = x \circ s$ of x such that t converges to a point $c \in [a,b]$:

$$s:\mathbb{N} \to \mathbb{N} \textit{ is strictly increasing} \quad t:\mathbb{N} \to [a,b] \quad c = Lim_{n\to\infty}\, t_n$$

(5) As a continuous function, f preserves sequential convergence, and therefore:

$$Lim_{n\to\infty}\, f(t_n) = Lim_{n\to\infty}\, (f \circ t)_n = f(c)$$

Claim

The sequence $y \circ s$ is a convergent subsequence of y; $y \circ s$ converges to $f(c)$.

Verification

(1) By the associative law for composition of functions:

$$y \circ s = (f \circ x) \circ s = f \circ (x \circ s) = f \circ t$$

(2) In turn, by the continuity of f as noted above:

$$Lim_{n\to\infty}\, (y \circ s)_n = Lim_{n\to\infty} (f \circ t)_n = f(c)$$

---

(3) Existence of the Definite Integral

The definite integral of f from a to b: $\int_a^b f(t)\,dt$, exists.

Fixed Partition

Suppose that $x:\{0,1,\ldots,n\}\to[a,b]$ $(n\in\mathbb{N})$ is a partition from a to b.

Lower Riemann Sum for the Partition

By the Compact Range Theorem, we can choose a sequence of sample points $m:\{1,2,\ldots,n\}\to[a,b]$ for the partition x such that for each i, $1\le i\le n$:

$f(m_i)$ is the absolute minimum value of f on the subinterval $[x_{i-1},x_i]$

Then, the *Lower Sum* for the partition x is the Riemann Sum:

$$\textit{Lower Sum}_f\,(x) = \sum_{i=1}^{n} f(m_i)(\Delta x)_i$$

Upper Riemann Sum for the Partition

By the Compact Range Theorem, we can choose a sequence of sample points $M:\{1,2,\ldots,n\}\to[a,b]$ for the partition x such that for each i, $1\le i\le n$:

$f(M_i)$ is the absolute maximum value of f on the subinterval $[x_{i-1},x_i]$

Then, the *Upper Sum* for the partition x is the Riemann Sum:

$$\textit{Upper Sum}_f\,(x) = \sum_{i=1}^{n} f(M_i)(\Delta x)_i$$

Remark

For the definite integral of f from b to a, the lower sum is returned by the maximum values and the upper sum is returned by the minimum values.

Order Relation

The lower sum for a fixed partition x is less that or equal to the upper sum for x.

We show below that this extends to where the partition of the lower sum is distinct from that of the upper sum.

Common Refinement

A *Common Refinement* of two partitions is a partition for which the set of subdivision points includes the subdivision points of each of the original partitions.

Lower and Upper Sum Refinement Lemma

Every lower sum for f *on* $[a,b]$ $\leq$ *Every upper sum for* f *on* $[a,b]$

(Here, the partitions of the lower sums vary independently from those of the upper sums.)

The lemma follows immediately from the following observation:

The lower sum for a common refinement is greater than or equal to each of the original lower sums; and, the upper sum is less than or equal to each of the originals.

Proof Strategy

Let L denote the least upper bound for the collection of all lower sums for f on $[a,b]$; and, let G denote the greatest lower bound for the collection of all upper sums for f on $[a,b]$.

By the Refinement Lemma , $L \leq G$.

We need to show that:

$$L = \int_a^b f(t)\,dt = G$$

Moreover, we need to show that the upper and lower sums of a partition can be forced arbitrarily close by simply choosing the partition with a sufficiently small mesh.

Construction

(1) Let $\varepsilon > 0$. Choose $\delta > 0$ according to the uniform continuity of f on $[a,b]$ relative to the value closeness constraint $\varepsilon(b-a)^{-1}$ (Uniform Continuity Theorem).

(2) Let $x:\{0,1,\ldots,n\}\to[a,b]$ $\left(n\in\mathbb{N}^{+}\right)$ be a partition from a to b for which the mesh is less than δ.

(3) Let $m:\{1,2,\ldots,n\}\to[a,b]$ be a choice of sample points so that the Lower Sum for the partition x is the Riemann Sum :

$$\textit{Lower Sum}_f\ (x) = \sum_{i=1}^{n} f(m_i)(\Delta x)_i$$

(4) Let $M:\{1,2,\ldots,n\}\to[a,b]$ be a choice of sample points so that the Upper Sum for the partition x is the Riemann Sum:

$$\textit{Upper Sum}_f\ (x) = \sum_{i=1}^{n} f(M_i)(\Delta x)_i$$

(5) Then: $\left(\sum_{i=1}^{n} f(m_i)(\Delta x)_i\right) \le L \le G \le \left(\sum_{i=1}^{n} f(M_i)(\Delta x)_i\right)$

Sum Closeness Verification

$$\left(\sum_{i=1}^{n} f(M_i)(\Delta x)_i\right) - \left(\sum_{i=1}^{n} f(m_i)(\Delta x)_i\right)$$

$= \sum_{i=1}^{n} \left[f(M_i) - f(m_i)\right] (\Delta x)_i$ Distributive Law

$< \sum_{i=1}^{n} \varepsilon (b-a)^{-1} (\Delta x)_i$ By the choice of x and δ: $|M_i - m_i| < \delta$,

and thus, $f(M_i) - f(m_i) < \varepsilon (b-a)^{-1}$

$= \varepsilon (b-a)^{-1} \sum_{i=1}^{n} (\Delta x)_i$ Distributive Law

$= \varepsilon (b-a)^{-1} (b-a)$ Sum of the changes in x is the Interval Length

$= \varepsilon$ Cancellation

(4) Mean Value Theorem of Integral Calculus

Construction

(1) Apply the Compact Range Theorem to obtain the absolute minimum value m and absolute maximum value M assumed by the function f, and, let each of p and q be a point in $[a,b]$ such that $f(p)=m$ and $f(q)=M$.

(2) Then: $m(b-a) \le \int_a^b f(t)\,dt \le M(b-a)$

(3) Apply the Intermediate Value Theorem to the linear function $y=(b-a)x$ restricted to the domain $[m,M]$:

$$\textit{Choose } z \in [m,M] \textit{ such that } \left[\int_a^b f(t)\,dt = (b-a)z\right]$$

Remark The slope of the linear function: $y=(b-a)x$, is $(b-a)$.

(4) If the integrand f is constant, then arbitrarily choose $c \in (a,b)$.
Then: $m = z = M = f(c)$

(5) Suppose that f is not constant. Then: $p \neq q$; and in turn, by the continuity of f, the inequality $m < z < M$ holds.

By the Intermediate Value Theorem applied to f restricted to the closed interval joining p and q, we can choose $c \in [p,q]$ such that $z = f(c)$.

(6) Arguing by cases, we can further require a choice of $c \in (p,q)$ strictly in between p and q such that $z = f(c)$.

Conclusion

$$f(c)(b-a) = \int_a^b f(t)\,dt \qquad c \in (a,b)$$

Remark Thus, after the initial set–up using the Compact Range Theorem, the Mean Value Theorem of Integral Calculus follows from two applications of the Intermediate Value Theorem.

(5) Fundamental Theorem ofCalculus–Part I   Application of MVT–IC

Construction

(1) $g'(x) = Lim_{\,h\to 0}\ h^{-1}\left[g(x+h) - g(x)\right]$

(2) $\qquad = Lim_{\,h\to 0}\ h^{-1}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$

(3) $\qquad = Lim_{\,h\to 0}\ h^{-1}\left[\int_x^{x+h} f(t)\,dt\right]$

(4) $\qquad = Lim_{\,h\to 0}\ h^{-1}\left[f(c)h\right]$

for some choice of c *strictly in between* x *and* $x+h$

(5) $\qquad = Lim_{\,h\to 0}\ f(c)$

(6) $\qquad = Lim_{\,c\to x}\ f(c)$

(7) $\qquad = f(x)$

Verification

(1) Definition of the derivative $g'(x)$

(2) Definition of g

(3) Definite Integral Splitting Lemma   (Chapter VI – Section 2 – (1))

(4) Mean Value Theorem of Integral Calculus

(5) Simplified Difference Quotient

(6) As $h \to 0$, $x+h \to x$, and therefore: $c \to x$ (since $x < c < x+h$)

(7) Continuity of f *at* x

(6) Mean Value Theorem of Differential Calculus

Construction

(1) Let $g:[a,b]\to\mathbb{R}$ defined by:

$$g(x)=f(x)\ -\ \left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\ +\ f(a)\right]$$

(2) For each argument x, $\left|g(x)\right|$ is the *Vertical Distance* between the graph of f and the graph of the secant joining the endpoints of $Graph\ f$:

$$y=\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\ +\ f(a)$$

$\left(g \textit{ subtracts the value of the secant from the value of } f\right)$

(3) Using the rules of differentiation:

$$g'(x)=f'(x)\ -\ \left(\frac{f(b)-f(a)}{b-a}\right)$$

(4) Since $g(a)=g(b)=0$, g is either constant, or we can use the Compact Range Theorem to choose an interior point $c\in(a,b)$ at which g assumes an extremum.

(5) Apply the Turning Point Lemma (Chapter VI – Section 2 – (2)): $g'(c)=0$

Conclusion

$$g'(c)=0\ \ \&\ \ \left[g'(c)=0\ \Rightarrow\ f'(c)=\frac{f(b)-f(a)}{b-a}\right]$$

(7) Antiderivative Comparison Theorem Corollary of MVT–DC

Construction

We claim $k = g - h$ is constant on $[u, v]$. Otherwise, we can choose a and b, with $u \le a < b \le v$, such that: $k(a) \ne k(b)$.

In turn, by the Mean Value Theorem of Differential Calculus, applied to the function k restricted to $[a, b]$, we can then choose $c \in (a, b)$ such that:

$$k'(c) = \frac{k(b) - k(a)}{b - a} \ne 0$$

Conclusion

This result contradicts the hypothesis that each of g and h is an antiderivative for f:

$$k'(c) = (g - h)'(c) = g'(c) - h'(c) = f(c) - f(c) = 0$$

(8) Fundamental Theorem of Calculus–Part II

(i) First Formulation Corollary of FTC–I and ACT

Construction

(1) Continuously extend f to the open interval (u,v).

(2) By the Fundamental Theorem of Calculus–Part I, the following indefinite integral g (where the additive constant is *zero*) is an antiderivative for f on (u,v):

$$g:(u,v)\to\mathbb{R} \qquad g(x)=\int_a^x f(t)\,dt \qquad g'(x)=f(x)$$

(3) Each of g and h is an antiderivative for f on $[a,b]$. By the Antiderivative Comparison Theorem, we can now choose the translation constant that relates g and h.

Let $z\in\mathbb{R}$ such that: $g(x)=h(x)+z \quad x\in[a,b]$

Conclusion $\displaystyle\int_a^b f(t)\,dt = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$$= g(b)-g(a)=\left[h(b)+z\right]-\left[h(a)+z\right]=h(b)-h(a)$$

(ii) Second Formulation Application of MVT–DC

Construction

(1) Assume $a\neq b$, and let $u=Min\{a,b\}$, and $v=Max\{a,b\}$. Let $n\in\mathbb{N}^+$, and suppose that $x:\{0,1,\ldots,n\}\to[u,v]$ is a partition of $[u,v]$ from a to b.

(2) By $n-many$ applications of the Mean Value Theorem of Differential Calculus, we can choose a finite sequence c of intermediate arguments for x such that:

$$c:\{1,2,\ldots,n\}\to[u,v] \qquad f(x_i)-f(x_{i-1})=f'(c_i)(\Delta x)_i \qquad (1\le i\le n)$$

(3) Then: $\displaystyle f(b)-f(a) = \sum_{i=1}^{n} f(x_i)-f(x_{i-1}) = \sum_{i=1}^{n} f'(c_i)(\Delta x)_i$

Conclusion

Since the partition x is arbitrary: $\displaystyle f(b)-f(a)=\int_a^b f'(t)\,dt$

2. Splitting and Turning Point Lemmas

(1) Definite Integral Splitting Lemma

Suppose that $u, v \in \mathbb{R}$ with $u < v$, that each of a, b, and c is a choice of a point in the open interval (u,v), and that $f:(u,v) \to \mathbb{R}$ is a continuous function.

Then, regardless of whether the choices a, b, c are distinct and regardless of the ordering of the choices a, b, c in the interval:

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt \quad \textit{and} \quad \int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$$

(2) Differentiation Turning Point Lemma

(With more accuracy: Differentiation Extremum Lemma)

Suppose that $a, b \in \mathbb{R}$ with $a < b$, and that $g:[a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$, and that g is a differentiable function on the open interval (a,b).

Suppose further that $c \in (a,b)$ is an interior point at which g locally assumes an extreme value. Then, $g'(c) = 0$.

Construction

Suppose that g assumes a local minimum at c (the argument for the case of a local maximum is by symmetry).

Let DQ denote the difference quotient for g with fixed point c.

In the case of a local minimum, for $h \neq 0$ sufficiently close to *zero*, the numerator of the difference quotient is always *zero* or positive: $g(c+h) - g(c) \geq 0$.

Therefore, for all $h \neq 0$ in a sufficiently small neighborhood of *zero* :

$$h < 0 \Rightarrow DQ(h) \leq 0 \quad \& \quad h > 0 \Rightarrow DQ(h) \geq 0$$

Conclusion

$$g'(c) = \ Lim_{\,h \to 0^-} \ DQ(h) \ \leq 0$$

$$g'(c) = \ Lim_{\,h \to 0^+} \ DQ(h) \ \geq 0$$

3. Differentiability Hierarchy

(1) Differentiability implies Continuity

Suppose that $u, v \in \mathbb{R}$ with $u < v$, $a \in (u, v)$, and that $f : (u, v) \to \mathbb{R}$. Suppose further that f is differentiable at a. Then, f is continuous at a.

By hypothesis: $\exists b \in \mathbb{R} \left[f'(a) = Lim_{\,h \to 0} \dfrac{f(a+h) - f(a)}{h} = b \right]$

In order for a fraction to converge to a constant as the denominator converges to *zero*, it is necessary for the numerator to converge to *zero* also:

$$Lim_{\,h \to 0}\ f(a+h) - f(a) = 0$$

The limit equation for the numerator asserts that f is continuous at a.

(2) Hierarchy of Oscillation Examples

(i)	$f : \mathbb{R} \to \mathbb{R}$	$f(x) = Sin\left(x^{-1}\right)$	$x \in \mathbb{R} \setminus \{0\}$	$f(0) = 0$
(ii)	$f : \mathbb{R} \to \mathbb{R}$	$f(x) = x\ Sin\left(x^{-1}\right)$	$x \in \mathbb{R} \setminus \{0\}$	$f(0) = 0$
(iii)	$f : \mathbb{R} \to \mathbb{R}$	$f(x) = x^2\ Sin\left(x^{-1}\right)$	$x \in \mathbb{R} \setminus \{0\}$	$f(0) = 0$

In each case, f is continuously differentiable on $\mathbb{R} \setminus \{0\}$: f' exists and is itself a continuous function on $\mathbb{R} \setminus \{0\}$.

(i) The first example has an essential discontinuity at *zero*:

As $x \to 0$, the graph of f oscillates between the graphs of the horizontal lines $y = 1$ and $y = -1$. For each of lines $y = 1$ and $y = -1$, there is an infinite sequence that converges to *zero* such that the tangent to the graph of f over each argument of the sequence is the horizontal line in question:

Define: $x : \mathbb{N} \to \mathbb{R} \setminus \{0\}$ $\qquad x_n = \left(2n\ \pi + \dfrac{\pi}{2} \right)^{-1}$

$Lim_{\,n \to \infty}\ x_n = 0$ $\qquad f \circ x$ *is constant with the value* 1

Define: $y : \mathbb{N} \to \mathbb{R} \setminus \{0\}$ $\qquad y_n = \left((2n+1)\ \pi + \dfrac{\pi}{2} \right)^{-1}$

$Lim_{\,n \to \infty}\ y_n = 0$ $\qquad f \circ y$ *is constant with the value* -1

(ii) The second example is continuous at *zero*, but not differentiable at *zero* :

(a) The function f is continuous at *zero*:

The graph of f oscillates between the graphs of the linear functions $y = x$ and $y = -x$. As a result, $Lim_{x \to 0}\ f(x) = 0$.

(b) The function f is not differentiable at *zero*:

Let DQ denote the difference quotient for f with fixed point *zero*. We use the same twoinput sequences converging to *zero* as in the first example.

Define: $h : \mathbb{N} \to \mathbb{R} \setminus \{0\} \quad h_n = \left(2n\ \pi + \frac{\pi}{2} \right)^{-1}$

$Lim_{n \to \infty}\ h_n = 0 \qquad (DQ) \circ h$ *is constant with the value* 1

For each n, the secant through the origin corresponding to the argument h_n is $y = x$.

Define: $k : \mathbb{N} \to \mathbb{R} \setminus \{0\} \quad k_n = \left((2n+1)\ \pi + \frac{\pi}{2} \right)^{-1}$

$Lim_{n \to \infty}\ k_n = 0 \qquad (DQ) \circ k$ *is constant with the value* -1

For each n, the secant through the origin corresponding to the argument k_n is $y = -x$.

(iii) The third example is differentiable at *zero*, but not continuously differentiable at *zero*. The derivative is defined at *zero*, but has *zero* as an essential discontinuity:

(a) The derivative is defined at *zero* :

The graph of f oscillates between the graphs of the parabolas $y = x^2$ and $y = -x^2$. As a result, as $h \to 0$, secants passing through the origin converge to the $x-axis$: $f'(0) = 0$

(b) The derivative f' has *zero* as an essential discontinuity:

$$Lim_{x \to 0}\ f'(x) = Lim_{x \to 0} \left[-Cos\left(x^{-1}\right) + 2x\ Sin\left(x^{-1}\right) \right] \quad Does\ Not\ Exist$$

As $x \to 0$: $2x\ Sin\left(x^{-1}\right) \to 0$, whereas $-Cos\left(x^{-1}\right)$ oscillates between $y = 1$ and $y = -1$.

4. Examples of Uniform Continuity

(1) Bounded Derivative Theorem

Suppose that $f: U \to \mathbb{R}$ is everywhere differentiable on a connected open set $U \subseteq \mathbb{R}$, and that its derivative f' is a function with a bounded range.

Then, f is uniformly continuous.

(The theorem refers to the Euclidean distance function: $d(p,q) = |p - q|$.)

Construction

(1) By hypothesis, we can choose a positive $b \in \mathbb{R}$ such that: $\forall x \in U \; \left[\; |f'(x)| \leq b \; \right]$

(2) Let $\varepsilon > 0$. Define δ by: $\delta = \varepsilon \, b^{-1}$

(3) Let $p, q \in U$ such that $p < q$, and, $d(p,q) = |p - q| < \delta$.

(4) By hypothesis, *Domain* f is connected, and therefore, $[p, q] \subseteq$ *Domain* f.

(5) By the Mean Value Theorem of Differential Calculus, we can choose $c \in (p, q)$ such that: $f'(c)(p - q) = f(p) - f(q)$

Closeness Verification

$d(f(p), f(q))$	$= \| f(p) - f(q) \|$	Euclidean Distance
	$= \| f'(c) \| \; \| p - q \|$	Application of MVT–DC†
	$\leq b \; \| p - q \|$	Choice of b
	$< b \; \delta$	Choice of p and q
	$= b \; \varepsilon \; b^{-1}$	Choice of δ
	$= \varepsilon$	Cancellation

† We are also using: $\forall x, y \in \mathbb{R} \; \left[\; |x \, y| = |x| \; |y| \; \right]$

(2) Sufficient Conditions for Uniform Continuity

Suppose that $D \subseteq \mathbb{R}$ is nonempty, and that $f : D \to \mathbb{R}$.

Relative to the Euclidean distance function and induced topology, each of the conditions below is individually sufficient for the uniform continuity of f:

(i) The function f is continuous on D, and, the domain $D \subseteq \mathbb{R}$ is compact.

(ii) The domain $D \subseteq \mathbb{R}$ is a connected open set, the function f is differentiable on D, and, the range of the derivative f' is bounded.

(3) Applications

(i) The function below is uniformly continuous by the first condition but not the second:

$$f : [-1, 1] \to \mathbb{R} \qquad f(x) = Arcsin\ x$$

(ii) The function below is uniformly continuous by the second condition but not the first:

$$h : \mathbb{R} \to \mathbb{R} \qquad h(x) = \frac{f(x)}{g(x)}$$

Each of f *and* g *is a polynomial for which* $Degree\, f \le Degree\ g$

Real Roots of $g = \{x \in \mathbb{R} : g(x) = 0\} = \varnothing \quad (Roots\ of\ g \subseteq \mathbb{C} \setminus \mathbb{R})$

$$\text{Example} \qquad h(x) = \left(1 + x^2\right)^{-1}$$

(4) Vertical Asymptotes and Vertical Tangents

Example A rational function with a vertical asymptote is not uniformly continuous.

Example Whereas a vertical asymptote precludes uniform continuity for each $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$, vertical tangent lines are a possibility:

(i) The function $Arcsine$ has vertical tangent lines $x = -1$ and $x = 1$.

(ii) The power function $f(x) = x^a$ for which $0 < a < 1$ (or the continuous extension of the power function to *zero* in the case of an irrational exponent) is uniformly continuous, and has the $y-axis$ as a vertical tangent line.

Remark The uniform continuity of the power function follows from an application of the two conditions mentioned above used in combination.

5. Arc Length

(1) Arc Length Formula

Suppose that $u, v \in \mathbb{R}$ with $u < v$, that $f : (u, v) \to \mathbb{R}$ is a continuously differentiable function on (u, v), and that $a, b \in (u, v)$ with $a < b$. Then, the *Length of the Function Graph* from $\langle a, f(a) \rangle$ to $\langle b, f(b) \rangle$ is defined by the following definite integral:

$$Arc\,Length_f(a,b) \;=\; \int_a^b \sqrt{1 + f'(t)^{\,2}}\;dt$$

Remark   We need to show that the above Arc Length Equation is the *Correct Definition* of the geometric generalization of the Pythagorean Theorem from line segments to arcs.

Suppose $n \in \mathbb{N}^+$, and $x : \{0, 1, \ldots, n\} \to [a,b]$ is a partition of the interval $[a,b]$ from a to b. The partition induces an algebraic approximation of the arc length:

$$\sum_{i=1}^{n} \sqrt{(\Delta x)_i^{\,2} + (\Delta y)_i^{\,2}}$$

$$(\Delta x)_i = x_i - x_{i-1} \qquad (\Delta y)_i = f(x_i) - f(x_{i-1}) \qquad (1 \le i \le n)$$

By the Mean Value Theorem of Differential Calculus $(n - \textit{many}$ applications$)$, we can choose a finite sequence $c : \{1, \ldots, n\} \to [a,b]$ such that:

$$(\Delta y)_i \;=\; f(x_i) - f(x_{i-1}) \;=\; f'(c_i)(\Delta x)_i \qquad x_{i-1} \le c_i \le x_i \qquad (1 \le i \le n)$$

Then:
$$\sum_{i=1}^{n} \sqrt{(\Delta x)_i^{\,2} + (\Delta y)_i^{\,2}}$$
$$= \sum_{i=1}^{n} \sqrt{(\Delta x)_i^{\,2} + f'(c_i)^2 (\Delta x)_i^{\,2}}$$
$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\;(\Delta x)_i$$

Thus, the approximation is a *Riemann Sum* for the function: $s = \sqrt{1 + f'(t)^{\,2}}$

As the mesh converges to *zero*, we obtain the arc length integral as the value of the limit.

(2) Application to the Unit Circle

Let $f:[-1,1]\to\mathbb{R}$ defined by $f(t)=\sqrt{1-t^2}$ **.** Then**:**

$$1+f'(t)^2=1+\left(\frac{-t}{\sqrt{1-t^2}}\right)^2=1+\frac{t^2}{1-t^2}=\frac{1}{1-t^2}\qquad(-1<t<1)$$

Using the Arc Length Formula **,** the length of the arc on the unit circle from the apex $\langle 0,1\rangle$ to $\langle x,y\rangle$, $y=\sqrt{1-x^2}$ **,** is the absolute value of the indefinite integral defined by**:**

$$Arcsin\ x=\int_0^x\frac{1}{\sqrt{1-t^2}}\,dt=\Theta\qquad(-1<x<1)$$

The function $\Theta = Arcsin\ x$ continuously extends to the endpoints**;** we define π (redundantly) by the following equations**:**

$$Arcsin\ (1)=2^{-1}\pi\qquad Arcsin\ (-1)=-2^{-1}\pi$$

The tangent lines are vertical at the endpoints**,** so the derivative of the $Arcsine$ exists only on the interior of the interval**.**

VII. Geometric Structure of Function Graphs in the Plane

1. Smoothness

Hypothesis Suppose that f is a real–valued function defined on an open subset of the real line, and that f possesses global first and second derivatives.

(1) Rates of Change

Suppose that $a, b \in Domain\, f$ with $a < b$ and $[a,b] \subseteq Domain\, f$.

Chord (Line Segment Joining Two Distinct Points of a Function Graph)

For all arguments $x, y \in [a,b]$ with $x < y$, let $LS(x, y)$ denote the line segment joining the points $\langle x, f(x) \rangle$ and $\langle y, f(y) \rangle$ on the graph of f.

Then, each of the concepts below is defined as the universal quantification closure of the indicated property as x and y vary over the interval $[a,b]$ under the constraint $x < y$:

(i) f is *Strictly Increasing* on $[a,b]$ iff $LS(x, y)$ *has a Positive Slope*

(ii) f is *Strictly Decreasing* on $[a,b]$ iff $LS(x, y)$ *has a Negative Slope*

(iii) f is *Concave Up* on $[a,b]$ iff $LS(x, y)$ *lies above* $Graph\, f$

(iv) f is *Concave Down* on $[a,b]$ iff $LS(x, y)$ *lies below* $Graph\, f$

Turning Point

A point of $Graph\, f$ at which there is a change in the (strictly) increasing–decreasing behavior of f is a *Turning Point* of $Graph\, f$. For instance, in the case where f assumes a local maximum at $x \in (a,b)$, the conditional for asserting the presence of a turning point is:

$$f \text{ is strictly increasing on } [a,x] \;\&\; f \text{ is strictly decreasing on } [x,b]$$
$$\Rightarrow \langle x, f(x) \rangle \text{ is a turning point of } Graph\, f$$

Inflection Point

A point of $Graph\, f$ at which there is a change in the concavity of f is a *Point of Inflection* of $Graph\, f$.

Concavity Theorem

Application of the Mean Value Theorem of Differential Calculus

(i) $f' > 0$ & $f'' > 0$

$\Rightarrow f$ is *Increasing Concave Up* $\quad$ (f is increasing at an increasing rate)

(ii) $f' > 0$ & $f'' < 0$

$\Rightarrow f$ is *Increasing Concave Down* $\quad$ (f is increasing at a decreasing rate)

(iii) $f' < 0$ & $f'' > 0$

$\Rightarrow f$ is *Decreasing Concave Up* $\quad$ (f is decreasing at an increasing rate)

(iv) $f' < 0$ & $f'' < 0$

$\Rightarrow f$ is *Decreasing Concave Down* $\quad$ (f is decreasing at a decreasing rate)

Here, *Increasing* and *Decreasing* abbreviate *Strictly Increasing* and *Strictly Decreasing* respectively.

(2) Limits Involving Infinity

Suppose that $a \in \mathbb{R}$.

Limits

(i) $Lim_{x\to\infty} f(x) = a \quad \Leftrightarrow \quad \exists\, b \in \mathbb{R} \left[\ [b,\infty) \subseteq Domain\, f \ \right]$

$$\& \quad \forall \varepsilon > 0 \ \exists\, t \in \mathbb{R} \ \textit{with} \ t \geq b \quad \forall x \in (t,\infty) \quad \left[a - \varepsilon < f(x) < a + \varepsilon \right]$$

The definition of $Lim_{x\to -\infty} f(x) = a$ is by symmetry.

(ii) $Lim_{x\to a^+} f(x) = \infty \quad \Leftrightarrow \quad \exists\, b \in \mathbb{R} \left[a < b \wedge (a,b) \subseteq Domain\, f \right]$

$$\& \quad \forall n \in \mathbb{N} \ \exists\, t \in \mathbb{R} \ \textit{with} \ t \in (a,b) \quad \forall x \in (a,t) \quad \left[f(x) > n \right]$$

The definitions in the presence of minus signs are by symmetry.

Asymptotes

The *Open Planar Strip with Radius* $\varepsilon > 0$ *Centered at the Horizontal Line* $y = a$ is defined to be the cross product: $HorStrip_{\varepsilon}(a) = \mathbb{R} \times (a - \varepsilon, a + \varepsilon)$

Then, in the *Positive Direction of the Argument*, that is, as $x \to \infty$, the horizontal line $y = a$ is a *Horizontal Asymptote of* $Graph\ f$ provided that:

$$\exists\, b \in \mathbb{R}\ \left[\ [b, \infty) \subseteq Domain\ f\ \right]$$

$$\&\quad \forall \varepsilon > 0\ \ \exists\, t \in \mathbb{R}\ with\ t \geq b\ \ \left[Graph\ f \Big| (t, \infty)\ \subseteq\ HorStrip_{\varepsilon}(a)\right]$$

For the other kinds of asymptotes, the definitions are again by symmetry.

Asymptote Theorem

(i) Each of the limit conditions below is independently sufficient for the horizontal line $y = a$ to be a horizontal asymptote of $Graph\ f$:

$$Lim_{x \to \infty}\ f(x) = a \qquad\qquad Lim_{x \to -\infty}\ f(x) = a$$

(ii) Each of the limit conditions below is independently sufficient for the vertical line $x = a$ to be an upward vertical asymptote of $Graph\ f$:

$$Lim_{x \to a^+}\ f(x) = \infty \qquad\qquad Lim_{x \to a^-}\ f(x) = \infty$$

(iii) Each of the limit conditions below is independently sufficient for the vertical line $x = a$ to be an downward vertical asymptote of $Graph\ f$:

$$Lim_{x \to a^+}\ f(x) = -\infty \qquad\qquad Lim_{x \to a^-}\ f(x) = -\infty$$

2. Constructing the Graph from Sign Charts and Limits

(1) Sign Chart of a Function

Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$. Then, the *Sign Chart* of f is the function, taking values in a set of *Formal Symbols*, defined by:

$$Sign_f : \mathbb{R} \to \{-, +, \mathbf{0}, \infty\}$$

$Sign_f(x) = -$	$x \in f^{-1}[(-\infty, 0)]$	$f(x) < 0$
$Sign_f(x) = +$	$x \in f^{-1}[(0, \infty)]$	$f(x) > 0$
$Sign_f(x) = \mathbf{0}$	$x \in f^{-1}(0)$	$f(x) = 0$
$Sign_f(x) = \infty$	$x \in \mathbb{R} \setminus D$	$f(x)$ *Undefined*

(2) Graph Construction Theorem

The graph of a sufficiently smooth real–valued function f of one real variable can be constructed from the sign charts of f, f', and f'', and the limits of f involving infinity.