FOUNDATIONS OF STATISTICAL LEARNING THEORY, I. THE LINEAR MODEL FOR SIMPLE LEARNING

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1. Introduction.

By simple learning we refer to the changes in behavioral probabilities which occur as a function of trials in the following type of situation. Each trial of a series begins with presentation of the same stimulating situation (e.g., a ready signal, conditioned stimulus, or the like). The organism responds to the stimulus with one or another of a set of alternative actions (responses). Then the trial terminates with occurrence of some one of a set of outcomes (e.g., reward, non-reward, unconditioned stimulus and response, knowledge of results). In contemporary learning theories (see, e.g., [4], [14]), it is assumed that if other parameters of the situation are fixed, the course of learning is determined by the trial outcomes. These outcomes can be classified, relative to any given response, according to their effects on response probability. We shall speak of events which increase probability of a given response as reinforcing events for that response and of events which leave response probabilities unchanged as instances of non-reinforcement.

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In the following sections we shall present the primitive notions, axiomatic structure, and important general theorems of a model which formalizes and generalizes the models for simple learning that have developed within contemporary statistical learning theory. Several considerations suggest the desirability of examining rigorously the mathematical foundations of these models. Almost without exception, contemporary theories that deal quantitatively with simple learning have in their development tended to follow the lead of experimental explorations. Thus they have for the most part proceeded from simplified special cases to more general ones and the common mathematical structure of the models has not received detailed analysis in its own right. Our formulation will draw upon the concepts and methods developed by Bush and Mosteller [3] and Estes and Burke [6], and we anticipate that it may clarify the set-theoretical foundations of the entire group of models associated with these investigators and their associates. Since all of the models are in one way or another probabilistic, we shall address ourselves first of all to the problem of characterizing exactly and explicitly the sample spaces of the stochastic processes involved and the methods of defining events and probabilities. In the matter of deriving theorems, we shall rely primarily upon set-theoretical and probabilistic techniques which have not heretofore appeared in the literature of learning theory.

This report will be limited to linear models, i.e., those which assume that response probabilities undergo linear transformations, exactly one such transformation representing the effects of any given type of reinforcing event. The level of generality at which we have aimed is that of a model just broad enough to include as special cases all of the particular linear models that have been applied to learning experiments. We do not, for example, consider any of the obvious generalizations that could be achieved by including free parameters which have no definite interpretations in existing learning theories. It will be seen that even without the luxury of extra parameters, the linear model presented here, involving as it does a stochastic process with an infinite number of states, poses mathematical problems of a higher order of difficulty than do the stimulus sampling models which we shall treat in a later report.

2. Primitive and Defined Notions.

The main point of this section is to discuss the three primitive notions on which our analysis of simple probability learning is based and to define a number of further notions in terms of these three.

However, some mathematical concepts are required for this discussion, and to these we first turn. To begin with, we use the familiar notation of elementary set theory: $A \subseteq X$ means that A is a subset of X; 0 designates the empty set (as well as the number zero); $A \cap B$ designates the intersection of sets A and B, that is, the set of all elements common to A and B; $A \cup B$ designates the union of sets A and B, that is, the set of all elements which belong to at least one of two sets; we also make use of the corresponding notation for the intersection and union of families of sets; $x \in A$ means that x is a member of the set A, and $x \notin A$ means that x is not a member of A. We use the notation:

$$\{x: \varphi(x)\}$$

to designate the set of all elements x satisfying the property φ. For example,

$$\{x: x \text{ is an integer } \& \ 0 \leq x < 6\}$$

is simply the set $\{0,1,2,3,4,5\}$. We use the notation $< x_1, x_2, \ldots, x_n >$ to designate the ordered n-tuple whose first member is x_1, second member x_2, etc. Similarly, the notation $< x_1, x_2, \ldots, x_n, \ldots >$ designates the infinite sequence whose first term is x_1, second term x_2, etc.

A family $\mathcal{F}$ of subsets of a set X is a field if and only if for every A and B in $\mathcal{F}$

(i) $A \cup B \in \mathcal{F}$

(ii) $\tilde{A} \in \mathcal{F}$,

where $\tilde{A}$ is the complement of A relative to X, that is

$$\tilde{A} = \{x\colon x \in X \ \&\ x \notin A\} .$$

A field $\mathcal{F}$ of subsets of X is a Borel field if for any sequence $\langle A_1, A_2, \ldots, A_n, \ldots \rangle$ of elements of $\mathcal{F}$,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F} .$$

For explicit subsequent reference we formalize in a numbered definition the standard notion of a probability measure on a field. The axioms are those of Kolmogorov [10]. The last axiom, asserting that the probability measure is continuous from above at zero, is easier to verify in constructing measures than the equivalent axiom that the measure is countably additive.

Definition 2.1. *Let X be a set and $\mathcal{F}$ a field of subsets of X. Then a real-valued function P is a probability measure on $\mathcal{F}$ if, and only if, the following axioms are satisfied:*

Axiom P1. *For every A in $\mathcal{F}$* , $P(A) \geq 0$;

Axiom P2. $P(X) = 1$;

Axiom P3. *For every A and B in $\mathcal{F}$ if $A \cap B = 0$ then*

$$P(A \cup B) = P(A) + P(B);$$

Axiom P4. If $< A_1, A_2, \ldots, A_n, \ldots >$ is a decreasing sequence of elements of $\mathcal{F}$, i.e., $A_{n+1} \subseteq A_n$ for every n, and

$$\bigcap_{n=1}^{\infty} A_n = 0,$$

then

$$\lim_{n \to \infty} P(A_n) = 0.$$

Some further mathematical notions we want to introduce are most easily explained by considering the first of our primitive notions, the sample space. Let r be a positive integer--the intuitive interpretation of r is that it is the number of possible responses on any trial--, and let

$$Z(r) = \{< x,y >: x,y \text{ are integers } \& \ 1 \leq x \leq r \ \& \ 0 \leq y \leq r \}$$

that is, $Z(r)$ is the set of all ordered couples of the indicated integers. The intuitive significance of $Z(r)$ is that on a given trial what actually happens can be represented by one of the ordered couples in $Z(r)$. For example, $< 3,4 >$ would represent the occurrence of the third response followed by the fourth reinforcing event on a given trial. The special case when 0 is the second member of a couple should be mentioned: $< 1,0 >$ would represent occurrence of the first response followed by no reinforcing event.

The set $Z(r)$ of ordered couples is adequate only for a single trial. To obtain the basic sample space for a sequence of trials we consider the denumerable Cartesian product of $Z(r)$ with itself, that is the set X defined as:

$$X = \bigtimes_{n=1}^{\infty} X_n,$$

where for every n, $X_n = Z(r)$. Thus in an experiment with two possible responses, a typical element of X would be a sequence

$$\langle\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,0 \rangle, \langle 2,1 \rangle, \ldots \rangle.$$

Here on the first trial the first response and the first reinforcing event occurred; on the second trial the first response and the second reinforcing event occurred, etc. We call the sample space X the r-*response space*.

If $x \in X$, i.e., if x is a sequence which is a member of the sample space X, then x_n is the n^{th} term of this sequence. Moreover, if $x_n = \langle j,k \rangle$, then $x_{A,n}$ designates j and $x_{E,n}$ designates k. The letters 'A' and 'E' have been used to pick out the first and second members respectively of the ordered couple in deference to the use in the psychological literature of 'A' to designate responses and 'E' to designate reinforcing events. The rather cumbersome notation $x_{A,n}$ is not used extensively in the remainder of the paper, but it or its equivalent is needed to define the basic notions we need.

Certain subsets of X are of particular interest to us, namely, the *cylinder sets*. Intuitively a cylinder set of X is a subset restricted on a finite number of "dimensions", that is, restricted for a finite number of the X_n's. For instance, the set of all sequences having response 1 on the first trial is a cylinder set. Formally this set is defined as

$$\{x: x \in X \ \&\ x_{A,1} = 1\} .$$

As a second example of a cylinder set, the set of all sequences having response 2 on the third trial and reinforcing event 1 on the fifth trial would be defined as

$$\{x: x \in X \ \&\ x_{A,3} = 2 \ \&\ x_{E,5} = 1\} .$$

The formal definition of cylinder sets of X generalizes on these examples. Moreover, we formally define the notion of an n-*cylinder set*, which is a cylinder set restricted on no more than the first n dimensions. This notion of an n-cylinder set is important in properly formulating various facts about conditional probabilities, as we shall see in subsequent sections.

Definition 2.2. C *is an* n-*cylinder set of* X *if, and only if, there is a finite set* I *of positive integers and a function* T *defined on* I *such that*

(i) $\max I = n$

(ii) *For each* $i \in I$, $T_i \subseteq X_i$

(iii) $C = \{x: x \in X \ \&\ x_i \in T_i$ *for* $i \in I\}$.

Moreover, C *is a cylinder set if for some* n, C *is an* n-*cylinder set*.

In the first example above of a cylinder set

$$I = \{1\}$$

and

$$T_1 = \{< 1,0 >, < 1,1 >,\ldots,< 1,r >\} .$$

In the second example,

$$I = \{3,5\}$$

$$T_3 = \{< 2,0 >, < 2,1 >,\ldots,< 2,r >\}$$

$$T_5 = \{< 1,1 >, < 2,1 >,\ldots,< r,1 >\} .$$

In terms of the notion of n-cylinder set, the first example is a 1-cylinder set and the second a 5-cylinder set. Note that if a set is an n-cylinder set and $m > n$ then it is also an m-cylinder set. In defining various special cylinder sets in the sequel, we use the kind of notation used in the two examples rather than the less intuitive notation which requires listing the various sets T_i. However, it will always be obvious that it is a trivial matter to re-cast our definitions in a form explicitly agreeing with the requirements of Definition 2.2.

An experimentally minded psychologist reflecting on these cylinder sets, might soon raise the following objection: it is not clear that every conceivable experiment can be described by means of cylinder sets. In the kind of experiments relevant here the experimenter has control over the pattern of reinforcing events, and his rule for generating the sequence of reinforcing events may be defined for all n. Consider, for instance, a two response experiment in which the first reinforcing event occurs on all odd trials and the second reinforcing event on all even trials, that is, the experimenter's rule is simply to alternate the two reinforcing events. The set of sequences which would represent possible outcomes of this experiment is not a cylinder set, and certainly we want our theory to be adequate to handle such an experiment.*/ By a slight extension this may be accomplished. We first observe that the family of all cylinder sets of X forms a field, for clearly the union of two cylinder sets is a cylinder set, and the complement of a cylinder set is a cylinder set. Now it is well known that given any field there is a unique smallest Borel field containing the given field. It is the smallest Borel field containing the field of cylinder sets of X with which we want to work. We designate this Borel field by $\mathcal{B}(X)$. The set of sequences representing possible outcomes of the experiment just described is clearly a member of this Borel field, for consider the sequence of cylinder sets $< C_1, C_2, \ldots, C_n, \ldots >$ where

$$C_n = \{x: x \in X \ \&\ x_{E,n} = 1 \text{ if } n \text{ is odd } \&\ x_{E,n} = 2 \text{ if } n \text{ is even}\}.$$

Each C_n is a cylinder set, hence the union $\bigcup_{n=1}^{\infty} C_n$ is in $\mathcal{B}(X)$.

We have been able to think of no empirically realizable experiment which

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*/ Although any actual experiment terminates in a finite number of trials, we will want to deal with concepts such as asymptotic response probability which are defined relative to infinite sequences of trials.

would require a set not in $\mathcal{B}(X)$ to represent it.*/

Since none of the existing literature on statistical learning theory seems to be explicitly concerned with cylinder sets, it is natural to wonder why they have been introduced in this paper. Fortunately or unfortunately we were forced to introduce them once it became apparent that the theory should be so set up that the initial probabilities of response of the subject and the experimenter-controlled conditional probabilities of reinforcement determine a unique model of simple probability learning. If the basic probability measure of the theory is defined on some broader family of subsets of X than $\mathcal{B}(X)$ it is in general not possible to prove it is uniquely determined by the initial probabilities of response and the conditional probabilities of reinforcement. Details are to be found in the proof of Theorem 4.7.

These remarks about probability lead us directly to our second primitive notion: a probability measure P on $\mathcal{B}(X)$. It is particularly important to note that all probability notions in which we are interested can be defined in terms of the basic probability measure P. Philosophers of science (see, e.g., [8]) have recently emphasized the importance of theoretical concepts which are tied to experimental facts only in terms of certain defined notions. In our opinion the basic probability measure P affords a clearcut example of such a concept in psychology. The measure P cannot be observed directly, but both the probability of a response and conditional probabilities of

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*/ In fact there is no obvious construction of a subset of X not in $\mathcal{B}(X)$ which does not require the axiom of choice.

reinforcing events are defined in terms of it.

Before defining any special probabilities, we need first to define the special events to which they are attached. Here as usual in probability theory events are certain subsets of X, our basic sample space. We begin with the event of response j on trial n.*/

Definition 2.3.

$$A_{j,n} = \{x: x \in X \ \& \ x_{A,n} = j\} .$$

Similarly, we define the event of reinforcing event k on trial n.

Definition 2.4.

$$E_{k,n} = \{x: x \in X \ \& \ x_{E,n} = k\} .$$

The probability of response j on trial n we define in more or less familiar notation.

Definition 2.5.

$$p_{j,n} = P(A_{j,n}) .$$

Since our axioms for learning are concerned with particular sequences of reinforcing events, we want to define the equivalence class of sequences which have the same outcomes through the n^{th} trial. We use a notation of square brackets common in mathematics.

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*/ It is understood throughout this paper that the range of the variable 'j' is $1,\ldots,r$ and the range of 'k' is $0,1,\ldots,r$.

Definition 2.6.

$$[x]_n = \{y\colon y \in X \ \& \ \text{for} \ m \leq n,\ y_m = x_m\}.$$

Some obvious relations are:

$$[x]_n \subseteq [x]_{n-1};$$

and there is a j and a k such that

$$[x]_n = [x]_{n-1} \cap A_{j,n} \cap E_{k,n} .$$

Furthermore, in terms of the notation: x_A and x_E, $[x_A]_n$ is simply the set of sequences y in X identical in the first n responses with x, and $[x_E]_n$ is the set of sequences y in X identical in the first n reinforcing events with x.

We next define the probability of response j on trial n given the first $n-1$ responses and the first $n-1$ reinforcing events.

Definition 2.7.

$$p_{xj,n} = P(A_{j,n} | [x]_{n-1}).$$

Here and subsequently we use one of the standard notations for conditional probability, and the elementary theory of conditional probabilities is assumed throughout this paper. However, to avoid a lot of unimportant technicalities, contrary to the usual practice we define conditional probabilities when the given event has a probability of zero. Namely, if $P(B) = 0$

(1) $$P(A|B) = \frac{0}{0} .$$

We take $\frac{0}{0}$ as a mathematical entity whose only known useful property is that

(2) $$\frac{0}{0} \cdot 0 = 0.$$

Adopting (1) and (2) greatly simplifies later work with summations of products. From a working standpoint a convention close to ours is usually adopted by most writers.

Some further probability notions are needed later, but we reserve their statement until after consideration of our axioms for simple learning.

We do need to characterize our third and final primitive notion θ. The mathematical entity θ is a real number between 0 and 1 with the intended interpretation that it is the learning parameter. In statistical learning theories ([4], [6]), it is assumed that the value of θ in any experiment is determined both by characteristics of the organism, e.g., sensory acuity, and by characteristics of environmental stimulus sources. In deriving experimental implications of the model developed in this paper, we assume always that θ is a fixed constant throughout an experiment; consequently, the model should be applied only to situations for which this assumption appears reasonable. Consequences of systematic variation in value of the parameter θ over a series of learning trials have been discussed by Restle [13] and LaBerge [11].

3. Axioms.

Our three axioms for simple probability learning are embodied in the following definition.

Definition 3.1. An ordered triple $\chi = \langle X, P, \theta \rangle$ is a (single parameter) linear model for simple learning if, and only if, there is an r such that X is the r-response space, P is a probability measure on $\mathscr{B}(X)$, θ is in the open interval $(0,1)$ and the following three axioms are satisfied for every n, every x in X with $P([x]_n) > 0$ and every j and k:

Axiom 1. If $x \in E_{k,n}$ & $j = k$ & $k \neq 0$ then

$$p_{xj,n+1} = (1-\theta)p_{xj,n} + \theta.$$

Axiom 2. If $x \in E_{k,n}$ & $j \neq k$ & $k \neq 0$ then

$$p_{xj,n+1} = (1-\theta)p_{xj,n} .$$

Axiom 3. If $x \in E_{0,n}$ then

$$p_{xj,n+1} = p_{xj,n} .$$

The three axioms express assumptions concerning the effects of reinforcement and non-reinforcement. The first two say, in effect, that when a reinforcing event occurs, the response class corresponding to it increases in probability and all others decrease. This is the same assumption utilized in a number of contemporary stochastic and statistical models for learning ([4], [6], [13]). The difference equations expressing Axioms 1 and 2 are derivable, in quite different ways, as special cases of learning functions which occur in Estes and Burke's set-theoretical model [6] and in Bush and Mosteller's linear operator model [3]; a general model elaborating the implications of this pair of equations for reinforcement theory has been reported by one of the writers [4]. The third axiom expresses the assumption that response probabilities are unchanged on nonreinforced trials.

In empirical applications of the model defined by 3.1, the term $p_{xj,n}$ is to be interpreted as the probability of response A_j for a particular subject on trial n. In principle the values of $p_{xj,n}$ can be predicted for all sequences and all n, given $p_{j,1}$, r and θ (see Theorem 4.1 below). In practice, however, it is impracticable to evaluate trial by trial probabilities for individual subjects, so in experimental tests of the model we usually deal only with the average value of $p_{xj,n}$ over all sequences terminating on a given trial, i.e., with $p_{j,n}$. The latter can be predicted for all n, given the values of $p_{j,1}$, r, and θ, and sufficient information concerning probabilities or reinforcement and nonreinforcement (see Theorem 4.7 below).

It may appear at first reading because we define only one reinforcing event E_j corresponding to each response A_j, together with the event E_0, we will be greatly restricted, as compared say to Bush and Mosteller [3], in the variety of empirical situations we can interpret by means of the model. However, this restriction is only apparent; actually the reinforcing effects of innumerable different experimental outcomes of a learning trial can be represented in terms of the model. Firstly, one should note that all experimentally distinguishable outcomes which are assumed to produce the same effects on response probabilities will be represented by the same event E_j in the model. Secondly, if a trial on which a given response A_j occurs may terminate in several different ways, each of which has a different effect on response probability, the different outcomes may be represented by different probabilistic combinations of E_j, $E_k (k \neq j)$, and E_0. To illustrate this last consideration by means of a simple example, consider a two-choice situation, e.g., a T-maze, in which A_1 and A_2 responses are followed by different magnitudes of reward. Applying Bush and Mosteller's model to this hypothetical situation, one might define a pair of operators Q_1 and Q_2, to represent the effect of the two outcomes. Then

$$Q_1 p = \alpha_1 p + a_1$$

would be the new probability of A_1 after an A_1 occurrence and

$$Q_2 p = \alpha_2 p + a_2$$

would be the new probability of an A_1 after an A_2 occurrence. These operators lead to the recursive expression

$$p_{n+1} = (a_1 - a_2 + \alpha_2) p_n + a_2 + (\alpha_1 - \alpha_2) V_{2,n}$$

where $V_{2,n}$ is the second raw moment of the A_1 response probabilities on trial n. Applying our linear model to the same situation, we might assume that the reward given on A_1 trials produces events E_1, E_2, and E_0 with probabilities c_{11}, c_{12}, and c_{10} respectively, while the reward given on A_2 trials produces E_1, E_2, and E_0 events with probabilities c_{21}, c_{22}, and c_{20} respectively. Using results of a later section (5.2) one can show that with these interpretations, the model yields the following recursive expression for probability of an A_1 response

$$p_{n+1} = (1-\theta + \theta c_{11} - \theta c_{12} + \theta c_{20})p_n + \theta c_{21} + \theta(c_{10} - c_{20})V_{2,n}$$

where p_n and $V_{2,n}$ have the same interpretations as in the Bush and Mosteller model. (In the notation of Section 5, $V_{2,n} = \alpha_{2,1,n}$.) It appears to be the case that for any experiment which has an interpretation in the Bush and Mosteller model, one can find an interpretation in the present linear model which yields the same recursive expression for $p_{j,n}$. (Although not in general the same expressions for higher order moments of the response probabilities). We should add that the example just discussed gives a possible interpretation of the differential reward experiment, not necessarily the interpretation that would be dictated by an analysis of the problem in terms of any particular learning theory.

4. General Theorems.

We now turn to some general theorems which do not lead immediately to experimental tests of the theory, but rather have the function of clarifying the structure of linear models for simple learning. We begin with the theorem, already alluded to, which says that if $p_{j,1}$, r and θ are given, then $p_{xj,n}$ is determined for all sequences x and all trials n. In formulating the theorem we make this idea precise by considering two models of the theory for which $p_{j,1}$, r and θ are the same (we get identity of r by having the same r-response space X in both models).

Theorem 4.1. Let $\chi = < X,P,\theta >$ and $\chi' = < X,P',\theta >$ be two linear models for simple learning such that $p_{j,1} = p'_{j,1}$. Then if $P([x]_{n-1}) > 0$ and $P'([x]_{n-1}) > 0$, we have:

$$p_{xj,n} = p'_{xj,n} \ .$$

Proof: Suppose the theorem is false. Let n be the smallest integer such that (for some j and x)

$$(1) \qquad p_{xj,n} \neq p'_{xj,n} .$$

(By hypothesis of the theorem, $n > 1$.) Now if

$$(2) \qquad P([x]_{n-1}) > 0$$

and

$$(3) \qquad P'([x]_{n-1}) > 0,$$

then by our hypothesis on n we have:

$$(4) \qquad p_{xj,n-1} = p'_{xj,n-1} .$$

There are now three cases to consider: $x \in E_{j,n}$, $x \in E_{k,n}$ with $k \neq j$ and $k \neq 0$, and $x \in E_{0,n}$. Since the proof is similar for all three cases, each requiring application of the appropriate one of the three axioms, we consider only the first case:

$$(5) \qquad x \in E_{j,n} .$$

From (2), (3), (5) and Axiom 1 we infer immediately:

$$(6) \qquad \begin{cases} p_{xj,n} = (1-\theta)p_{xj,n-1} + \theta \\ p_{xj,n} = (1-\theta)p'_{xj,n-1} + \theta . \end{cases}$$

From (4) and (6) we conclude:

$$p_{xj,n} = p'_{xj,n} ,$$

which contradicts (1) and establishes our supposition as false. Q.E.D.

The next theorem establishes the fundamental result that given the initial probabilities of response of the subject, and the conditional probabilities of reinforcement, then a unique model of simple learning is determined. Moreover, no restrictions on these probabilities are required to establish the theorem. The significant intuitive content of this last assertion is that the experimenter may conditionalize the probabilities of reinforcement upon preceding events of the sample space in whatever manner he pleases.

Some preliminary definitions and lemmas are needed. The third definition introduces the notion of an <u>experimenter's partition</u> of X. The intuitive idea is that the conditional probabilities of reinforcing events on trial n depend on any partition of the equivalence classes $[x]_{n-1}$ and responses on the n^{th} trial.*/ The most general cases as yet studied experimentally are those for which the conditional probability of a reinforcing event depends on the response ν trials earlier. Such cases are treated in some detail in Section 5. It is important to emphasize that the results in the present section are in no way restricted to conditionalization on a <u>single</u> previous response; the probability pattern of reinforcement may depend on any selected sequence of prior responses and reinforcements.

<u>Definition</u> 4.2.

$$\Xi(n) = \{\xi : \text{there is an } x \text{ in } X \text{ and a } j \text{ such that } \xi = [x]_{n-1} \cap A_{j,n}\}$$

$\Xi(n)$ is the finest experimenter's partition of X which we can use on the n^{th} trial. It is immediately obvious that

<u>Lemma</u> 4.3. <u>For every</u> n, $\Xi(n)$ <u>is a partition of</u> X.

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*/ A partition of a non-empty set X is a family of pairwise disjoint, non-empty subsets of X whose union is equal to X.

We now use $\Xi(n)$ to define the general notion of an experimenter's partition $H(n)$, but for this definition we explicitly need the notion of one partition of a set being finer than another. (The definition is so phrased that any partition is finer than itself.)

Definition 4.4. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are partitions of* X, *then* $\mathcal{A}$ *is finer than* $\mathcal{B}$ *if, and only if, for every set* A *in* $\mathcal{A}$ *there is a set* B *in* $\mathcal{B}$ *such that* $A \subseteq B$.

We then have:

Definition 4.5. $H(n)$ *is an experimenter's partition of* X *(at trial* n) *if, and only if,* $H(n)$ *is a partition of* X *and* $\Xi(n)$ *is finer than* $H(n)$.

Finally, we need a lemma which provides a recursive equation for $P([x]_n)$ in terms of a given experimenter's partition on trial n. Notice that (iv) of the hypothesis of the lemma is a condition controlled by the experimenter, not by the subject.

Lemma 4.6. *Let* $H(n)$ *be an experimenter's partition of* X. *Let*

(i) $\eta \in H(n)$

(ii) $[x]_n \subseteq A_{j,n} \cap E_{k,n} \cap \eta$

(iii) $P(A_{j,n} \cap [x]_{n-1}) > 0$

(iv) $P(E_{k,n} \mid A_{j,n} \cap [x]_{n-1}) = P(E_{k,n} \mid \eta)$.

Then

$$P([x]_n) = P(E_{k,n}|\eta)p_{xj,n}\ P([x]_{n-1}).$$

Proof: By (ii) of the hypothesis

$$P([x]_n) = P(E_{k,n} \cap A_{j,n} \cap [x]_{n-1}),$$

whence,

$$P([x]_n) = P(E_{k,n}| A_{j,n} \cap [x]_{n-1})P(A_{j,n}|[x]_{n-1})P([x]_{n-1}).$$

Applying (iii) and (iv) to the first term on the right and Definition 2.7 to the second, we obtain the desired result. Q.E.D.

We are now prepared to state and prove the uniqueness theorem. Regarding the notation of the theorem it may be helpful to keep in mind that $q_{j,1}$ is the a priori probability of making response j on the first trial, and $\gamma_{\eta k,n}$ is the conditional probability of reinforcing event k on trial n given the event η of an experimenter's partition $H(n)$. It should be obvious why we use the notation $q_{j,1}$ rather than $p_{j,1}$ (and at the beginning of the proof $q_{xj,n}$ rather than $p_{xj,n}$); namely, the function p is defined in terms of the measure P whose unique existence we are establishing.

Theorem 4.7. Let X be an r-response space and let θ be a real number in the open interval $(0,1)$, and let the numbers $q_{j,1}$ be such that

$$q_{j,1} \geq 0$$

$$\sum_{j=1}^{r} q_{j,1} = 1.$$

For every n let $H(n)$ be an experimenter's partition of X, and let γ be a function defined for every n and k and every $\eta \in H(n)$ such that

$$\gamma_{\eta k,n} \geq 0$$

$$\sum_{k=0}^{r} \gamma_{\eta k,n}' = 1.$$

Then there exists a unique probability measure P on $\mathscr{B}(X)$ such that

(i) $< X,P,\theta >$ is a linear model of simple learning,

(ii) $q_{j,1} = p_{j,1}$

(iii) $\gamma_{\eta k,n} = P(E_{k,n}|\eta)$

(iv) If $\eta \in H(n)$ and W is an $n-1$ cylinder set such that $W \subseteq \eta$ and $P(W) > 0$ then

$$P(E_{k,n}|W) = P(E_{k,n}|\eta).$$

Proof: We first define recursively a function q intuitively corresponding to p, i.e., $q_{xj,n} = p_{xj,n}$.

(1) $$q_{xj,1} = q_{j,1}$$

(2) $$q_{xj,n} = (1-\theta)q_{xj,n-1} + \theta\, \delta(j, \mathcal{E}(x,n-1)) + \theta q_{xj,n-1}\, \delta(0, \mathcal{E}(x,n-1)),$$

where δ is the usual Kronecker delta function:

$$\delta(j,k) = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases},$$

and

(3) $$\mathcal{E}(x,n) = k \quad \text{if, and only if, } [x]_n \subseteq E_{k,n}.$$

(In effect, (2) combines all three axioms of Definition 3.1 into one to provide this recursive definition.)

For subsequent use we prove by induction that

(4) $$\sum_j q_{xj,n} = 1.$$

For $n = 1$, the proof follows at once from (1) and the hypothesis of the theorem that

$$\sum_j q_{xj,1} = 1.$$

Suppose now that

$$\sum_j q_{xj,n-1} = 1.$$

There are two cases to consider. If $x \in E_{k,n}$ for some $k \neq 0$ then from (2) and (3) we have at once:

$$\begin{aligned} \sum_j q_{xj,n} &= \sum_j (1-\theta) q_{xj,n-1} + \theta \\ &= (1-\theta) \sum_j q_{xj,n-1} + \theta \\ &= (1-\theta) + \theta \\ &= 1. \end{aligned}$$

If $x \in E_{0,n}$, then

$$\sum_j q_{xj,n} = \sum_j [(1-\theta)q_{xj,n-1} + \theta q_{xj,n-1}]$$

$$= \sum_j q_{xj,n-1}$$

$$= 1.$$

Following Lemma 4.6 we now recursively define $P([x]_n)$ in terms of q and the function γ introduced in the hypothesis of the theorem.

$$(5) \qquad \begin{cases} P([x]_1) = q_{j,1}\gamma_{\eta_1 \mathcal{E}(x,1),1} \\ \\ P([x]_n) = \gamma_{\eta\, \mathcal{E}(x,n),n}\; q_{xj',n-1}\; P([x]_{n-1}), \end{cases}$$

where

$$[x]_1 \subseteq A_{j,1}$$

$$[x]_1 \subseteq \eta_1 \in H(1)$$

$$[x]_n \subseteq A_{j',n}$$

$$[x]_n \subseteq \eta \in H(n).$$

We first need to show that the function P may be extended in a well-defined manner to any cylinder set C.*/ To this end we prove by

*/ In connection with the remarks in Section 2 on cylinder sets it may be pointed out that it is impossible to extend P in a well-defined manner to every subset of X.

induction that if

$$C = \bigcup_{i=1}^{m_1} [x_i]_{n_1} = \bigcup_{i=1}^{m_2} [y_i]_{n_2}$$

then

$$\sum_{i=1}^{m_1} P([x_i]_{n_1}) = \sum_{i=1}^{m_2} P([y_i]_{n_2}). \tag{6}$$

When $n_1 = n_2$ the proof is trivial. Without loss of generality we may assume than $n_1 < n_2$; i.e., there is a positive integer t such that $n_1 + t = n_2$. We proceed by induction on t. But first we observe that the family of sets $[x_i]_{n_1}$ constitutes a partition of C, as does the family of sets $[y_i]_{n_1+t}$, and the latter is a refinement of the former. Whence for each set $[x_i]_{n_1}$ there is a subset I of the first m_2 positive integers such that

$$[x_i]_{n_1} = \bigcup_{h \in I} [y_h]_{n_1+t}. \tag{7}$$

And on the basis of (7) to establish (6) it is obviously sufficient to show that

$$P([x_i]_{n_1} = \sum_{h \in I} P([y_h]_{n_1+t} .$$

Now if $t = 1$ then

$$[x_i]_{n_1} = [x_i]_{n_1} \cap \bigcup_j A_{j,n_1+1} \cap \bigcup_k E_{k,n_1+1}$$

$$= \bigcup_j \bigcup_k ([x_i]_{n_1} \cap A_{j,n_1+1} \cap E_{k,n+1})$$

$$= \bigcup_{h \in I} [y_h]_{n_1+1}.$$

Since for $h \in I$, $[y_h]_{n_1} = [x_i]_{n_1}$, we infer from the above and (5) that

$$\sum_{h \in I} P([y_h]_{n_1+1}) = \sum_j \sum_k \gamma_{\eta k, n_1+1}\, q_{xj,n_1}\, P([x_i]_{n_1})$$

$$= \sum_j q_{xj,n_1}\, P([x_i]_{n_1}) \qquad \text{by hypothesis on } \gamma$$

$$= P([x_i]_{n_1}) \qquad \text{by (4).}$$

Suppose now that (6) holds for t. Then there are sets I_1 and I_2 of positive integers such that

$$[x_i]_{n_1} = \bigcup_{h \in I_1} [y_h]_{n_1+t} \cap \bigcup_j A_{j,n_1+t+1} \cap \bigcup_k E_{k,n_1+t+1}$$

$$= \bigcup_{g \in I_2} [z_g]_{n_1+t+1}.$$

Since for each $g \in I_2$ there is an h in I_1 such that

$$[z_g]_{n_1+t} = [y_h]_{n_1+t},$$

similarly to the case for $t = 1$ we infer that

$$\sum_{g \in I_2} P([z_g]_{n_1+t+1}) = \sum_j \sum_k \sum_{h \in I_1} \gamma_{\eta k, n_1+t+1}\, q_{xj, n_1+t}\, P([y_h]_{n_1+t})$$

$$= \sum_{h \in I_1} P([y_h]_{n_1+t})$$

$$= P([x_i]_{n_1}),$$

by our inductive hypothesis, which completes the proof of (6) and justifies the extension of P to any cylinder set: if

$$C = \bigcup_{i=1}^{m} [x_i]_n$$

then

$$P(C) = \sum_{i=1}^{m} P([x_i]_n).\text{*/} \tag{8}$$

We now want to show that P is a probability measure (in the sense of Definition 2.1) on the field of cylinder sets of X. Since the functions q and γ are non-negative it follows at once from

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*/ In using throughout this paper the notation

$$C = \bigcup_{i=1}^{m} [x_i]_n$$

we always assume the sets $[x_i]_n$ are distinct; otherwise the extension of P would be incorrect.

(5) and (8) that Axiom P1 is satisfied, i.e., for every cylinder set C, $P(C) \geq 0$.

Now it is easy to select a subset Y of X such that

$$X = \bigcup_{x \in Y} [x]_1 ,$$

whence by virtue of (5) and (8)

$$P(X) = \sum_{x \in Y} P([x]_1) = \sum_j \sum_k q_{j,1} \gamma_{\eta k,1}$$

$$= \sum_j q_{j,1} \sum_k \gamma_{\eta k,1}$$

$$= 1 \cdot 1$$

$$= 1,$$

which establishes Axiom P2.

To verify Axiom P3, let C_1 and C_2 be two cylinder sets such that $C_1 \cap C_2 = 0$. Without loss of generality we may assume they are both non-empty n-cylinder sets, and we may represent them each by

$$C_1 = \bigcup_{i=1}^{m_1} [x_i]_n$$

$$C_2 = \bigcup_{h=m_1+1}^{m_2} [x_h]_n,$$

and by hypothesis, for each $i=1,\ldots,m_1$ and $h=m_1+1,\ldots,m_2$

$$[x_i]_n \cap [x_h]_n = 0.$$

Whence

$$P(C_1 \cup C_2) = P\left(\bigcup_{i=1}^{m_2} [x_i]_n\right)$$

$$= \sum_{i=1}^{m_2} P([x_i]_n)$$

$$= \sum_{i=1}^{m_1} P([x_i]_n) + \sum_{h=m_1+1}^{m_2} P([x_h]_n)$$

$$= P(C_1) + P(C_2).$$

Now for Axiom P4. Let $< C_1, C_2, \ldots, C_n, \ldots >$ be a decreasing sequence of cylinder sets, that is,

(9) $$C_{n+1} \subseteq C_n$$

and

(10) $$\bigcap_{n=1}^{\infty} C_n = 0.$$

Suppose now that

(11) $$\lim_{n \to \infty} P(C_n) \neq 0 .$$

(This limit must exist since the sequence is bounded and monotone decreasing. The monotonicity follows from (9) and Axioms P1 and P3.)

In fact, let

$$\lim_{n \to \infty} P(C_n) = s > 0.$$

Hence for every n

$$P(C_n) \geq x,$$

and it follows at once from Axioms P1-P3 that

(12) $$C_n \neq 0.$$

We now use a topological argument to show that

$$\bigcap_{n=1}^{\infty} C_n \neq 0,$$

contrary to (10). The idea is simple; details will be omitted to avoid too serious a diversion. We know from Section 2 that X is the countably infinite product of the finite set $Z(r)$. Hence every cylinder set of X is compact in the product topology of the discrete topology on $Z(r)$; in particular for every n, C_n is compact. Also by virtue of (12) every C_n is non-empty. But it is a well-known theorem of topology that a decreasing sequence of non-empty compact sets has a non-empty intersection, which contradicts (5). Thus our supposition (11) is false and P satisfies Axiom P4.

Finally, the unique extension of P to the Borel field $\mathcal{B}(X)$ follows from the standard theorem on this extension (see [10], p. 17). The verification that the measure P defined by (5), (8) and the extension just mentioned has properties (ii)-(iv) of the theorem is straightforward and will be omitted. Q.E.D.

We now want to show that the probability of a given response on trial n depends only on the probability of the response on the first trial and the sequence of preceding reinforcing events, and is independent of the sequence of preceding responses.

Theorem 4.8. *If* W_n *is an* n-*cylinder set such that* $W \subseteq [x_E]_n$ *and* $P(W_n) > 0$, *then*

$$P(A_{j,n+1} | W_n) = P(A_{j,n+1} | [x]_n).$$

Proof: The proof is by induction on n. For $n=1$ the proof follows immediately from the axioms. Suppose now that the theorem holds for n. There are three cases to consider: $x \in E_{j,n+1}$; $x \in E_{0,n+1}$; $x \in E_{k,n+1}$ with $k \neq j$ and $k \neq 0$. Since the proof for all three cases is the same we consider only the last one. Now W_{n+1} may be of the form

$$W_n \cap E_{k,n+1} \tag{1}$$

or

$$W_n \cap E_{k,n+1} \cap A_{j',n+1}. \tag{2}$$

The proof is similar for (1) and (2), so we shall consider only (2). We then have the following equalities:

$$P(A_{j,n+2}|W_n \cap E_{k,n+1} \cap A_{j',n+1})$$

$$= \sum_{[x']_n \subseteq W_n} P(A_{j,n+2}|E_{k,n+1} \cap A_{j',n+1} \cap [x']_n) \frac{P(E_{k,n+1} \cap A_{j',n+1} \cap [x']_n)}{P(E_{k,n+1} \cap A_{j',n+1} \cap W_n)}$$

by elementary theorems about conditional probability

$$= \sum_{[x']_n \subseteq W_n} (1-\theta)P(A_{j,n+1}|[x']_n) \frac{P(E_{k,n+1} \cap A_{j',n+1} \cap [x']_n)}{P(E_{k,n+1} \cap A_{j',n+1} \cap W_n)}$$

by Axiom 2

$$= (1-\theta)P(A_{j,n+1}|W_n) \sum_{[x']_n \subseteq W_n} \frac{P(E_{k,n+1} \cap A_{j',n+1} \cap [x']_n)}{P(E_{k,n+1} \cap A_{j',n+1} \cap W_n)}$$

by inductive hypothesis

$$= (1-\theta)P(A_{j,n+1}|W_n)$$

by summing over W_n

$$= (1-\theta)P(A_{j,n+1}|[x]_n)$$

by inductive hypothesis

$$= P(A_{j,n+2}|[x]_{n+1})$$

by Axiom 2 and hypothesis that $x \in E_{k,n+1}$. Q.E.D.

It follows immediately from this theorem that the three axioms (of Definition 3.1) hold for such cylinder sets W_n. Since this result is sometimes useful, we state it as a corollary.

Corollary 4.9. If W_n is an n-cylinder set such that $W_n \subseteq [x_E]_n$ and $P(W_n) > 0$, then:

(i) if $W_n = W_{n-1} \cap E_{j,n}$

$$P(A_{j,n+1} | W_n) = (1-\theta)P(A_{j,n} | W_{n-1}) + \theta$$

(ii) if $W_n = W_{n-1} \cap E_{k,n}$ with $k \neq j$ and $k \neq 0$,

$$P(A_{j,n+1} | W_n) = (1-\theta)P(A_{j,n} | W_{n-1})$$

(iii) if $W_n = W_{n-1} \cap E_{0,n}$

$$P(A_{j,n+1} | W_n) = P(A_{j,n} | W_{n-1}).$$

The next two theorems show that the linear model possesses the property which Luce [12] has designated "independence of irrelevant alternatives." That is, given that the response made on trial n of an experiment belongs to a subset T of the set of possible responses, the relative probabilities of any two responses in the subset are the same as their relative probabilities in the complete set. The first of the two theorems guarantees this property for the conditional probabilities $p_{xj,n}$ and the second for the unconditional probabilities $p_{j,n}$.

Definition 4.10. If $T \subseteq N(r)$, where $N(r)$ is the set consisting of the first r positive integers then

$$P_T(A_{j,n} | [x]_{n-1}) = P(A_{j,n} | [x]_{n-1} \cap \bigcup_{j' \in T} A_{j',n}).$$

<u>Theorem</u> 4.11. <u>If</u> $j \in T$ <u>and</u> $\sum_{j' \in T} P(A_{j',n} | [x]_{n-1}) > 0$, <u>then</u>

$$P_T(A_{j,n} | [x]_{n-1}) = \frac{P(A_{j,n} | [x]_{n-1})}{\sum_{j' \in T} P(A_{j',n} | [x]_{n-1})} .$$

<u>Proof</u>: By definition of conditional probability,

$$P(A_{j,n} | [x]_{n-1} \cap \bigcup_{j' \in T} A_{j',n}) = \frac{P(A_{j,n} \cap [x]_{n-1} \cap \bigcup_{j' \in T} A_{j',n})}{P([x]_{n-1} \cap \bigcup_{j' \in T} A_{j',n})}$$

$$= \frac{P(A_{j,n} \cap \bigcup_{j' \in T} A_{j',n} | [x]_{n-1}) P([x]_{n-1})}{P(\bigcup_{j' \in T} A_{j',n} | [x]_{n-1}) P([x]_{n-1})} .$$

Distributing the set unions in the numerator and denominator, this last expression reduces to

$$\frac{P(A_{j,n} | [x]_{n-1})}{\sum_{j' \in T} P(A_{j',n} | [x]_{n-1})} .$$

<u>Definition</u> 4.12. <u>If</u> $T \subseteq N(r)$ <u>then</u>

$$P_T(A_{j,n}) = P(A_{j,n} | \bigcup_{j' \in T} A_{j',n}).$$

<u>Theorem</u> 4.13. <u>If</u> $j \in T$ <u>and</u> $\sum_{j' \in T} P(A_{j',n}) > 0$, <u>then</u>

$$P_T(A_{j,n}) = \frac{P(A_{j,n})}{\sum_{j' \in T} P(A_{j',n})} .$$

The proof is analogous to that of 4.11.

5. Recursive Expression for Moments.

In this section we derive a recursive expression for the moments of $p_{xj,n}$ (i.e., $P(A_{j,n}|[x]_{n-1})$). The results are then used in the analysis of various special cases which have been studied experimentally.

We first need to introduce a notation for the partition of the sample space X into sets agreeing on the first n trials, n responses or n reinforcing events.

Definition 5.1.

$$X(n) = \{[x]_n : x \in X\}$$

$$X_A(n) = \{[x_A]_n : x \in X\}$$

$$X_E(n) = \{[x_E]_n : x \in X\} .$$

We formally define the moments of $p_{xj,n}$ as follows:

Definition 5.2.

$$\alpha_{v,j,n} = \sum_{[x]_{n-1} \in X(n-1)} P^v(A_{j,n}|[x]_{n-1})P([x]_{n-1}).$$

The first moment is simply $p_{j,n}$, that is,

$$\alpha_{1,j,n} = p_{j,n} = P(A_{j,n}).$$

Naturally the variable v ranges over positive integers. The general recursive expression we first establish is computationally unworkable, but it specializes in different directions to something manageable. Note that in this general expression the evaluation of $\alpha_{v,j,n+1}$ depends upon summing over the partition $\chi(n)$. Generally speaking, any expression which depends in this way on the partition $\chi(n)$ cannot feasibly be computed for any n beyond the first few positive integers.

Theorem 5.3.

$$\alpha_{v,j,n+1} = (1-\theta)^v \alpha_{v,j,n} + \sum_{[x]_{n-1} \in \chi(n-1)} \Bigg[\sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i p_{xj,n}^{v-i} \cdot P(E_{j,n} \cap [x]_{n-1}) + (1-(1-\theta)^v) p_{xj,n}^v P(E_{0,n} \cap [x]_{n-1}) \Bigg] .$$

Proof: By Definition 5.2

$$(1) \qquad \alpha_{v,j,n+1} = \sum_{[x]_n \in \chi(n)} P^v(A_{j,n+1} | [x]_n) P([x]_n) .$$

Since for every x in X and for every n there is exactly one j' and one k such that

$$[x]_n \subseteq A_{j',n} \cap E_{k,n},$$

we may re-write the right-hand side of (1) to obtain:

$$(2) \qquad \alpha_{v,j,n+1} = \sum_{\chi(n-1)} \sum_{j'} \sum_{k} P^v(A_{j,n+1} | [x]_{n-1} \cap A_{j',n} \cap E_{k,n}) \cdot P([x]_{n-1} \cap A_{j',n} \cap E_{k,n}) .$$

(For brevity in (2) and hence forth we write: $\sum_{\chi(n-1)}$ rather than:

$\sum_{[x]_{n-1}\in X(n-1)}$.) Applying to (2) the axioms of Definition 3.1 we infer that

$$(3) \qquad \alpha_{v,j,n+1} = \sum_{X(n-1)} \sum_{j'} \Big[[(1-\theta)P(A_{j,n}|[x]_{n-1})+\theta]^v P([x]_{n-1}\cap A_{j',n}\cap E_{j,n}) + \sum_{\substack{k\neq j\\ k\neq 0}} [(1-\theta)P(A_{j,n}|[x]_{n-1})]^v P([x]_{n-1}\cap A_{j',n}\cap E_{k,n}) + P^v(A_{j,n}|[x]_{n-1})P([x]_{n-1}\cap A_{j',n}\cap E_{0,n}) \Big] .$$

Since the various terms of the right-hand side of (3) which are raised to the power v are independent of the summation over j' we may eliminate this summation, using the fact that for any $Y \subseteq X$

$$\sum_{j'} P(Y\cap A_{j',n}) = P(Y\cap \bigcup_{j'} A_{j',n}) = P(Y).$$

Moreover, we may use the following binomial expansion for the first term on the right of (2), writing here and hereafter $p_{xj,n}$ for $P(A_{j,n}|[x]_{n-1})$:

$$[(1-\theta)p_{xj,n} + \theta]^v = (1-\theta)^v p_{xj,n}^v + \sum_{i=1}^{v} \binom{v}{i}(1-\theta)^{v-i}\,\theta^i\, p_{xj,n}^{v-i} .$$

Finally, adding and subtracting the term

$$(1-\theta)^v p_{xj,n}^v P([x]_{n-1}\cap E_{0,n}),$$

we infer from (3) as the result of these three changes

$$(4)\quad \alpha_{v,j,n+1} = \sum_{\chi(n-1)} \Big[(1-\theta)^v \, p^v_{xj,n} \, P([x]_{n-1} \cap E_{j,n}) +$$

$$\sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i p^{v-i}_{xj,n} \, P([x]_{n-1} \cap E_{j,n}) +$$

$$\sum_{\substack{k \neq j \\ k \neq 0}} (1-\theta)^v \, p^v_{xj,n} P([x]_{n-1} \cap E_{k,n}) + (1-\theta)^v \, p^v_{xj,n} P([x]_{n-1} \cap E_{0,n})$$

$$+ \, (1-(1-\theta)^v) p^v_{xj,n} \, P([x]_{n-1} \cap E_{0,n}) \Big] .$$

Combining now the first, third and fourth terms on the right of (4) and summing k out, we get:

$$(5)\quad \alpha_{v,j,n+1} = \sum_{\chi(n-1)} \Big[(1-\theta)^v \, p^v_{xj,n} \, P([x]_{n-1}) +$$

$$\sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \, \theta^i \, p^{v-i}_{xj,n} \, P([x]_{n-1} \cap E_{j,n}) +$$

$$(1 - (1-\theta)^v) p^v_{xj,n} \, P([x]_{n-1} \cap E_{0,n}) \Big] .$$

Since the first term on the right of (5) is simply $(1-\theta)^v \, \alpha_{v,j,n}$, we have our desired result. Q.E.D.

It is natural to ask what is the most general special case for which we can derive a recursive expression for the moments that does not depend on the partition $\chi(n-1)$. The following corollary is addressed to this question. We restrict the number of responses to two;

we make the conditional probability of reinforcement depend only on the immediately preceding response (condition (ii) below), and independent of the trial number ((iii) below); and we make the probability of E_0 non-contingent, that is, independent of previous responses. Note that (ii) requires the notion of an $(n-1)$-cylinder set.

<u>Corollary</u> 5.4. <u>If</u>

(i) $r = 2$

(ii) <u>if</u> W <u>is an</u> $(n-1)$-<u>cylinder set and</u> $P(W \cap A_{j,n}) > 0$ <u>then</u>

$$P(E_{k,n} | W \cap A_{j,n}) = P(E_{k,n} | A_{j,n})$$

(iii) $P(E_{1,n} | A_{1,n}) = \pi_{11}$

$P(E_{1,n} | A_{2,n}) = \pi_{21}$

$P(E_{0,n} | A_{1,n}) = P(E_{0,n} | A_{2,n}) = \pi_0,$

<u>then</u>

$$\alpha_{v,1,n+1} = (\pi_0 + (1-\pi_0)(1-\theta)^v)\alpha_{v,1,n} + \sum_{i=1}^{v} \binom{v}{i}(1-\theta)^{v-i}\, \theta^i(\pi_{21}\, \alpha_{v-i,1,n} + (\pi_{11} - \pi_{21})\alpha_{v-i+1,1,n} .$$

<u>Proof</u>: To simplify the recursive expression for $\alpha_{v,1,n+1}$ given by the theorem, we first note that on the specialized hypothesis of the corollary

$$P(E_{0,n} \cap [x]_{n-1}) = P(E_{0,n} | [x]_{n-1})P([x]_{n-1}) = \pi_0 P([x]_{n-1}),$$

and

$$P(E_{1,n} \cap [x]_{n-1}) = P(E_{1,n} \cap \bigcup_{j'} A_{j,n} \cap [x]_{n-1})$$

$$= P(E_{1,n} | A_{1,n} \cap [x]_{n-1}) P(A_{1,n} | [x]_{n-1}) P([x]_{n-1})$$

$$+ P(E_{1,n} | A_{2,n} \cap [x]_{n-1}) P(A_{2,n} | [x]_{n-1}) P([x]_{n-1}).$$

Using (ii) and (iii) of the hypothesis and the fact that

$$P(A_{2,n} | [x]_{n-1}) = 1 - P(A_{1,n} | [x]_{n-1}),$$

we get:

$$P(E_{1,n} \cap [x]_{n-1}) = [\pi_{11} p_{x1,n} + \pi_{21}(1-p_{x1,n})] P([x]_{n-1}).$$

Substituting these two results in 5.3 we have:

$$\alpha_{v,1,n+1} = (1-\theta)^v \alpha_{v,1,n} + \sum_{\chi(n-1)} \left[\sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i p_{x1,n}^{v-i} \cdot \right.$$

$$(\pi_{11} p_{x1,n} + \pi_{21}(1-p_{x1,n})) P([x]_{n-1}) + (1-(1-\theta)^v) p_{x1,n}^v \pi_0 P([x]_{n-1})$$

$$= (1-\theta)^v \alpha_{v,1,n} + \sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i \left[\sum_{\chi(n-1)} (\pi_{11} p_{x1,n}^{v-i+1} + \right.$$

$$\left. \pi_{21} p_{x1,n}^{v-i} - \pi_{21} p_{x1,n}^{v-i+1}) P([x]_{n-1}) \right] + (1-(1-\theta)^v) \pi_0 \sum_{\chi(n-1)} p_{x1,n}^v P([x]_{n-1}).$$

Applying Definition 5.2 to the right-hand side we infer:

$$\alpha_{v,1,n+1} = (1-\theta)^v \alpha_{v,1,n} + \sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i [\pi_{11}\alpha_{v-i+1,1,n} +$$

$$\pi_{21}\alpha_{v-i,1,n} - \pi_{21}\alpha_{v-i+1,1,n} + (1-(1-\theta)^v)\pi_0\alpha_{v,1,n}.$$

Simple rearrangement of terms of the right-hand side immediately yields the desired conclusion. Q.E.D.

As a second corollary we want to derive a general recursive expression for $\alpha_{1,j,n} = p_{j,n}$ which does not depend on the partition $\chi(n-1)$. It is possible to obtain a recursive expression in terms of an arbitrary experimenter's partition. However, practically no asymptotic results can be derived at this level of generality, so that it is convenient to specialize to conditionalization of the probability of a reinforcing event to a single previous response. To this end, we introduce the notation:

<u>Definition</u> 5.5. <u>If</u> $n > \nu$ <u>then</u>

$$c_{jk}(n,\nu) = P(E_{k,n} | A_{j,n-\nu}).$$

Thus $c_{jk}(n,\nu)$ is the probability that reinforcing event k will occur on trial n given that response j occurred ν trials earlier. It is understood throughout that 'ν' is a variable ranging over non-negative integers. The probabilities $c_{jk}(n,\nu)$ are ordinarily completely determined by the experimenter. If $\nu > 0$ we say that there is a <u>lag</u> ν in the probability pattern of reinforcing events. The corollary covers the case where the reinforcing events $1,\ldots,r$ are contingent with lag, and the occurrence of no reinforcing event on trial n, that is, $E_{0,n}$, has a fixed probability independent of previous responses or reinforcing events.

An obvious but cumbersome definition formalizing use of the conditional probabilities $c_{jk}(n,v)$ is needed. Since we consider only the case with non-contingent E_0 trials, this special condition is included in the definition.

Definition 5.6. A linear model $\chi = < X,P,\theta >$ satisfies the simple contingent condition with lag and with non-contingent E_0 trials if, and only if, (i) for every n there is a unique ν such that for all j, k and W if W is an $(n-1)$-cylinder set and $P(W \cap A_{j,n-\nu}) > 0$ then

$$P(E_{k,n}|W \cap A_{j,n-\nu}) = c_{jk}(n,\nu),$$

(ii) there is a number c_0 such that for all j, n, and ν with $n > \nu$,

$$c_{j0}(n,\nu) = c_0.$$

The phrase in (i) of the definiens "for every n there is a unique ν..." indicates that we are permitting ν to depend on n, that is, the amount of lag may vary from trial to trial. The lag could be 1 on odd-numbered trials and 2 on even-numbered trials for instance.

Corollary 5.7. If the simple contingent condition with lag and with non-contingent E_0 trials is satisfied, then

$$p_{j,n+1} = (1+\theta c_0-\theta)p_{j,n} + \theta \sum_{j'} c_{j'j}(n,\nu)p_{j',n-\nu},$$

<u>where</u> ν <u>depends</u> <u>on</u> n.

<u>Proof</u>: We have from Theorem 5.3 that

$$(1) \qquad p_{j,n+1} = \alpha_{1,j,n+1} = (1-\theta)p_{j,n} + \sum_{\chi(n-1)} \theta\, P(E_{j,n} \cap [x]_{n-1})$$

$$+ (1-(1-\theta)) \sum_{\chi(n-1)} p_{xj,n}\, P(E_{0,n} \cap [x]_{n-1}).$$

Applying (ii) of Definition 5.6, summing over the partition $\chi(n-1)$ (since

$$\sum_{\chi(n-1)} P(E_{j,n} \cap [x]_{n-1}) = P(E_{j,n} \cap \bigcup_{\chi(n-1)} [x]_{n-1}) = P(E_{j,n}))$$

and rearranging terms, we have:

$$(2) \qquad p_{j,n+1} = (1+\theta c_0-\theta)p_{j,n} + \theta\, P(E_{j,n}).$$

We obtain the desired result from (2) by using (i) of 5.6 and observing that

$$P(E_{j,n}) = P(E_{j,n} \cap \bigcup_{j'} A_{j',n-\nu})$$

$$= \sum_{j'} P(E_{j,n} | A_{j',n-\nu}) P(A_{j',n-\nu})$$

$$= \sum_{j'} c_{j'j}(n,\nu) p_{j',n-\nu} . \qquad \text{Q.E.D.}$$

In the special case for which $c_{j'j}(n,\nu)$ is independent of n, we can derive from the recursive expression of Corollary 5.7 various experimentally useful relations involving the Cesàro means (arithmetic means over trials) of the response probabilities. These means are defined as follows:

Definition 5.8.

$$\overline{p}_j(N) = \frac{1}{N}\sum_{n=1}^{N} p_{j,n}.$$

We restrict ourselves here to one asymptotic theorem, for which ν is fixed as well as independent of n. To make clear what we mean by ν being constant or fixed, we may describe an experiment in which it is not: on each trial the experimenter uses lag 1 with probability $\frac{1}{2}$ and lag 2 with probability $\frac{1}{2}$. In this case the lag would be independent of n but not fixed.

Theorem 5.9. Let the simple contingent condition with lag and with non-contingent E_0 trials be satisfied, and let the lag ν be fixed and independent of n (with lag zero on the first ν trials). Then in order for the limit of $\overline{p}_j(N)$, for $j=1,\ldots,r$, to exist as $N \to \infty$ and to be independent of the initial probabilities $p_{j,1}$, it is necessary and sufficient that unity be a simple root of the matrix

$$C = \begin{pmatrix} c_0 + c_{11} & c_{21} & \cdots & c_{r1} \\ c_{12} & c_0 + c_{22} & \cdots & c_{r2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1r} & c_{2r} & & c_0 + c_{rr} \end{pmatrix}$$

<u>Moreover, if the limit of</u> $\overline{p}_j(N)$ <u>does exist and is independent of</u> $p_{j,1}$

$$\lim_{N \to \infty} \overline{p}_j(N) = \lambda_j$$

<u>where the</u> λ_j <u>are the unique solutions of the</u> r+1 <u>equations</u>

$$(1-c_0)\lambda_j = \sum_{j'} c_{j'j}(\nu)\lambda_{j'} \qquad \text{for} \qquad j=1,\ldots,r$$

$$\sum \lambda_j = 1.$$

<u>Proof</u>: If we sum both sides of 5.7 from 1 to N, and then divide by N, we obtain:

$$\overline{p}_j(N) + \frac{p_{j,N+1} - p_{j,1}}{N} = (1+\theta c_0-\theta)\overline{p}_j(N) + \frac{\theta}{N}\sum_{j'} c_{j'j}(0)\sum_{n=1}^{\nu} p_{j,n}$$

$$+ \frac{\theta}{N}\sum_{j'} c_{j'j}(\nu)\sum_{n=1}^{N-\nu} p_{j,n}$$

$$= (1+\theta c_0-\theta)\overline{p}_j(N) + \frac{\theta}{N}\sum_{j'} c_{j'j}(0)\overline{p}_j(\nu)$$

$$+ \theta\sum_{j'} c_{j'j}(\nu)\overline{p}_j(N) - \frac{\theta}{N}\sum_{j'} c_{j'j}(\nu)\sum_{n=N-\nu+1}^{N} p_{j,n}.$$

Clearly as $N \to \infty$

$$\lim \frac{p_{j,N+1} - p_{j,1}}{N} = 0$$

$$\lim \frac{\theta}{N} \sum_{j'} c_{j'j}(0)\overline{p}_j(\nu) = 0$$

$$\lim \frac{\theta}{N} \sum_{j'} c_{j'j}(\nu) \sum_{n=N-\nu+1}^{N} p_{j,n} = 0.$$

Consequently as $N \to \infty$, we have r linear homogeneous equations, writing $\overline{p}_j$ for $\lim_{N \to \infty} p_j(N)$:

$$\overline{p}_j = (1-\theta c_0-\theta)\overline{p}_j + \theta \sum_{j'} c_{j'j}(\nu)\overline{p}_{j'}$$

which reduce to:

$$(1) \qquad (1-c_0)\overline{p}_j = \sum_{j'} c_{j'j}(\nu)\overline{p}_{j'} \qquad \text{for} \qquad j=1,\ldots,r.$$

The r homogeneous linear equations (1) have a non-trivial solution if and only if the rank of the matrix $C-I$ (where C is defined as in the statement of the theorem and I is the identity matrix) is less than r. That the rank of $C-I$ is less than r is easily seen by adding every row to the first, thereby obtaining a row of zeros.

Clearly the limit of $\overline{p}_j(N)$ exists and is independent of $p_{j,1}$ if, and only if, there is a unique non-trivial solution of the $r+1$ equations (1) and $\sum \overline{p}_j = 1$. To complete the proof of the theorem we thus need to show that in order for there to be a unique non-trivial solution it is necessary and sufficient that the matrix C have unity as a simple root. Now it is well known ([7a], p. 111) from the literature of Markov processes that the $r+1$ equations

$$(2) \qquad \bar{p}_j = \sum_{i=1}^{r} \bar{p}_i \, \bar{c}_{ij}$$

$$\sum_j \bar{p}_j = 1,$$

where $(\bar{c}_{ij})$ is the matrix C, have a unique solution if and only if unity is a simple root of the matrix C. Moreover, the r equations (1) are equivalent to the r equations (2) since

$$\bar{c}_{jj} = c_{jj}(\nu) + c_0.$$

Thus the proof of our theorem is complete.

6. Some Random Variables.

In this section we define some useful random variables and assert some obvious general theorems about them. Certain theorems of experimental interest are proved in Section 8 for special conditions of reinforcement.

Since this report is partially methodological we give a rather explicit treatment of several elementary questions. To begin with, we recall some familiar facts about random variables. A random variable is a (measurable) function defined on the sample space.*/ The common notation for the probability that a random variable U has a certain value u is ordinarily devoted by:

$$P(U = u), \tag{1}$$

and this probability is equal to the probability

$$P(\{x: x \in X \ \&\ U(x) = u\}).$$

For the expected value of a random variable U, we use the standard notation $E(U)$.

A random variable is discrete if the set of its possible values is countable, i.e., either finite or denumerable. If U is a discrete random variable then its discrete density is the function q such that for any possible value u of U

$$q(u) = P(U = u) .$$

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*/ Problems of measurability will not arise here.

All the random variables considered here are discrete. We shall not introduce an explicit notation for their discrete density functions, but it is well to remember that the quantities computed, e.g., expected values and variances, are defined with respect to these density functions.

Corresponding to events $A_{j,n}$ and $E_{k,n}$ we now introduce random variables denoted by corresponding letters.

<u>Definition</u> 6.1. <u>If</u> $x \in X$

$$\underline{A}_{j,n}(x) = \begin{cases} 1 & \underline{\text{if}} \quad x \in A_{j,n} \\ 0 & \underline{\text{otherwise}} \end{cases}$$

<u>and</u>

$$\underline{E}_{k,n}(x) = \begin{cases} 1 & \underline{\text{if}} \quad x \in E_{k,n} \\ 0 & \underline{\text{otherwise}} \end{cases} .$$

We shall also be interested in the sum of random variables $\underline{A}_{j,n}$.

<u>Definition</u> 6.2. If $x \in X$

$$\underline{S}_{j,n,N}(x) = \sum_{m=n+1}^{n+N} \underline{A}_{j,m}(x).$$

Some obvious general results concerning these random variables are formulated in the following theorem. The main object of the theorem is to relate expected values and variances to probabilities of events in the sample space.

Theorem 6.3.

(i) $P(\underline{A}_{j,n} = 1) = P(A_{j,n})$

(ii) $E(\underline{A}_{j,n}) = P(A_{j,n})$

(iii) $\text{Var}\ (\underline{A}_{j,n}) = P(A_{j,n})(1-P(A_{j,n}))$

(iv) $E(\underline{A}_{j,n}\ \underline{A}_{j,n+1}\cdots\underline{A}_{j,n+r}) = P(A_{j,n}\cap A_{j,n+1}\cap\cdots\cap A_{j,n+r})$

(v) $E(\underline{S}_{j,n,N}) = \sum_{m=n+1}^{n+N} P(A_{j,m})$

(vi) $\text{Var}\ (\underline{S}_{j,n,N}) = \sum_{m=n+1}^{n+N} P(A_{j,m})(1-P(A_{j,m})) + 2 \sum_{n<i<k\leq n+N} \text{cov}(\underline{A}_{j,i}\underline{A}_{j,k})$.

Proof: A few remarks will suffice. (i) and (ii) are obvious. (iii) follows from the equalities:

$$\begin{aligned}\text{var}(\underline{A}_{j,n}) &= (1-P(A_{j,n}))^2\ P(A_{j,n}) + (0-P(A_{j,n}))^2\ (1-P(A_{j,n}))\\ &= (1-P(A_{j,n}))[P(A_{j,n}) - P^2(A_{j,n}) + P^2(A_{j,n})]\\ &= [1-P(A_{j,n})]P(A_{j,n}).\end{aligned}$$

(iv) follows at once from the fact that the product of the various possible values of the random variable is zero except when they all have the value one. (v) may be inferred from (ii) and the familiar fact that the expected value of a sum of random variables is equal to the sum of their expected values. Finally, (vi) is a consequence of (iii) and the familiar relationship between the variance of a random variable which is a linear function of certain random variables and the variance and covariances of these latter random variables (cf. [7]). Q.E.D.

7. Some General Limit Theorems.

It is of intrinsic interest to know how responses and reinforcing events are related under very general conditions of reinforcement. The following four theorems describe in a precise way the kind of asymptotic matching relationships which obtain between them.

The first theorem asserts that for a given outcome x of an experiment the limiting ratio of the sum (with respect to n) of $p_{xj,n}$ to the sum of E_j reinforcements is 1, provided that there are no E_0 trials and that E_j trials do not stop occurring after some n. (It does not matter how rare the density of these E_j occurrences is, so long as there is an infinite number of them.)

Theorem 7.1. If $x \notin E_{0,n}$ for every n and if $\sum_{n=1}^{N} \underline{E}_{j,n}(x)$ diverges as $N \to \infty$, then

$$\lim_{N \to \infty} \left[\frac{\sum_{n=1}^{N} p_{xj,n}}{\sum_{n=1}^{N} \underline{E}_{j,n}(x)} \right]$$

exists and is equal to 1.

Proof: Since $x \notin E_{0,n}$ by hypothesis, we have immediately from our axioms:

$$p_{x,n+1} = (1-\theta)p_{x,n} + \theta\underline{E}_n(x) \tag{1}$$

(we have dropped the subscript j throughout the proof.) Summing both sides of (1) from 1 to N, we get:

$$(2)\quad \sum_{n=1}^{N} p_{x,n} + p_{x,N+1} - p_{x,1} = \sum_{n=1}^{N} p_{x,n} - \theta \sum_{n=1}^{N} p_{x,n} + \theta \sum_{n=1}^{N} \underline{E}_n(x).$$

We may derive immediately from (2):

$$(3)\qquad \frac{\sum_{n=1}^{N} p_{x,n}}{\sum_{n=1}^{N} E_n(x)} = 1 + \frac{p_{x,1} - p_{x,N+1}}{\theta \sum_{n=1}^{N} \underline{E}_n(x)} ,$$

provided only that we choose N large enough to have $E_n(x) = 1$ for at least one $n \leq N$. Since $\sum_{n=1}^{N} \underline{E}_n(x)$ diverges as $N \to \infty$ the second term on the right-hand side of (3) has zero as a limit as $N \to \infty$ and the theorem is proved.

The second theorem asserts a corresponding result for the ratio of the sum of the mean probabilities $P(A_{j,n})$ of responses and the sum of the mean probabilities $P(E_{j,n})$ of reinforcements. It should be noted that if the quantity

$$(1)\qquad \frac{1}{N} \sum_{n=1}^{N} P(E_{j,n})$$

is bounded away from zero as $N \to \infty$, then the theorem holds as well for the Cesaro mean probabilities of responses and reinforcements; it is not necessary that the limit of (1) exist.

Theorem 7.2. If $P(\bigcup_{n=1}^{\infty} E_{0,n}) = 0$ and $\sum_{n=1}^{N} P(E_{j,n})$ diverges as $N \to \infty$, then

$$\lim_{n \to \infty} \left[\frac{\sum_{n=1}^{N} P(A_{j,n})}{\sum_{n=1}^{N} P(E_{j,n})} \right] = 1.$$

Proof: By Theorem 5.3 and the hypothesis that the probability of E_0 trials is zero,

$$(1) \qquad p_{j,n+1} = (1-\theta)p_{j,n} + \theta P(E_{j,n}).$$

The same sequence of operations is applied to (1) as to the corresponding equation of the previous proof. Q.E.D.

We now turn to the consideration of the same kind of theorems when the conditions of reinforcement do not assign zero probabilities to E_0 trials. For this purpose we introduce some notation whose intuitive meaning is clear. If x is a possible experimental outcome, i.e., a sequence which is a member of X, then x^* is the subsequence of x which results from deleting all E_0 trials and no others. Clearly x^* is a well defined sequence only if the number of non-E_0 trials in x is infinite; this condition is equivalent to the divergence of the sum $\sum_{n=1}^{N} \sum_{k=1}^{r} \underline{E}_{k,n}(x)$ as $N \to \infty$. Formally we thus have:

Definition 7.3. If $x, y \in X$ and the sum $\sum_{n=1}^{N} \sum_{k=1}^{r} \underline{E}_{k,n}(x)$ diverges *as* $N \to \infty$ *then* $x^* = y$ *if, and only if, there exists an increasing sequence* φ *of positive integers such that for every* n

(i) $y_n = x_{\varphi(n)}$

(ii) $\underline{E}_{0,n}(y) = 0$

(iii) *if* $\underline{E}_{0,n}(x) = 0$ *then* n *is in the range of* φ.

Condition (iii) of the definiens simply insures that the subsequence y generated by φ from x does not omit any non-E_0 trials.

Analogous to the first theorem we then have a corresponding limit theorem for the subsequence x^*.

Theorem 7.4. *If* $\sum_{n=1}^{N} \underline{E}_{j,n}(x)$ *diverges as* $N \to \infty$, *then*

$$\lim_{N \to \infty} \left[\frac{\sum_{n=1}^{N} p_{x^*j,n}}{\sum_{n=1}^{N} \underline{E}_{j,n}(x^*)} \right] = 1.$$

Proof: In order to apply the argument used in the proof of Theorem 7.1, we need to prove that

(1) $$p_{x^*,n+1} = (1-\theta)p_{x^*,n} + \theta\underline{E}_n(x^*).$$

But as in the case of 7.1, (1) follows at once from our basic axioms and the fact that for all n, $x^* \notin E_{0,n}$. Q.E.D.

The analogue of Theorem 7.2 requires in a rather interesting way direct consideration of countable intersections and unions of cylinder sets. The theorem concerns the mean probabilities $P(A^*_{j,n})$ and $P(E^*_{j,n})$, where $A^*_{j,n}$ and $E^*_{j,n}$ are events yet to be defined in an exact manner but whose intuitive significance should be clear. It will be convenient to define at this point the set F of sequences with only a finite number of non-E_0 trials. F is the countable union of a countable intersection of cylinder sets. We first define the cylinder sets $F_n(k)$.

Definition 7.6.

$$F_n(k) = \{x\colon x \in X \;\&\; \underline{E}_{0,n}(x) = 0 \text{ and for } n < m \leq n+k,\ \underline{E}_{0,m}(x) = 1\}.$$

Evidently $F_n(k)$ is the set of all sequences whose n^{th} trial is a non-E_0 trial and whose next k trials are E_0 trials. F_n is defined simply as the countable intersection of sets $F_n(k)$.

Definition 7.7.

$$F_n = \bigcap_{k=1}^{\infty} F_n(k).$$

Clearly F_n is the set of all sequences whose last non-E_0 trial was trial n. Finally, then, we define F as the countable union of sets F_n, and obviously F is the set of sequences which have a finite number of non-E_0 trials. For completeness and explicitness we need the set F_0

to define F. As expected F_0 is the set of all sequences which have nothing but E_0 trials; it is, like any F_n for $n \geq 1$, a countable intersection of cylinder sets.

Definition 7.8.

$$F_0 = \{x\colon \text{for every } n, \underline{E}_{0,n}(x) = 1\}.$$

Definition 7.9.

$$F = \bigcup_{n=0}^{\infty} F_n.$$

We may now use F to define $A^*_{j,n}$ and $E^*_{k,n}$, as well as $[x]^*_n$.

Definition 7.10.

$$A^*_{j,n} = \{x\colon x \in X \ \&\ x \notin F \ \&\ \underline{A}_{j,n}(x^*) = 1\}.$$

Thus $A^*_{j,n}$ is the set of all sequences which have the j^{th} response on the n^{th} non-E_0 trial.

Definition 7.11.

$$E^*_{k,n} = \{x\colon x \in X \ \&\ x \notin F \ \&\ \underline{E}_{k,n}(x^*) = 1\}.$$

Clearly $E^*_{k,n}$ is the set of all sequences which have the k^{th} reinforcing event on the n^{th} non-E_0 trial.

Definition 9.12. If $x \notin F$ then

$$[x]_n^* = \{y: y \in X \ \& \ y \notin F \ \& \ \text{for every} \ m \leq n \ y_m^* = x_m^*\}.$$

Thus $[x]_n^*$ is the equivalence class of sequences which have the same first n non-E_0 trials that x does. Finally, we define $\chi^*(n)$ in the expected manner.

Definition 7.13.

$$\chi^*(n) = \{[x]_n^*: x \in X \ \& \ x \notin F\}.$$

It is obvious that for each n, $\chi^*(n)$ is a partition of $X \sim F$, where X is the sample space. Clearly sequences in F which do have n non-E_0 trials could have been included in $[x]_n^*$ for some x, and thus in the partition $\chi^*(n)$, but it is technically simpler to exclude them.

The simplest restrictive hypothesis under which the final theorem of the four may be proved is that the probability of the set F of sequences with only a finite number of non-E_0 trials is zero.

Theorem 7.14. If $P(F) = 0$ and $\sum_{n=1}^{N} P(E_{j,n}^*)$ diverges as $N \to \infty$ then

$$\lim_{N \to \infty} \left[\frac{\sum_{n=1}^{N} P(A_{j,n}^*)}{\sum_{n=1}^{N} P(E_{j,n}^*)} \right] = 1.$$

Proof: To apply directly the method of proof of 7.1 and the other two theorems (7.2 and 7.4), we need to show that

$$(1) \qquad P(A^*_{j,n+1}) = (1-\Theta)P(A^*_{j,n}) + \Theta P(E^*_{j,n}).$$

We may regard our theorem as complete when we have established (1).

In terms of the notation defined above, the whole sample space X may be represented by:

$$(2) \qquad X = \bigcup_{X^*(n)} [x]^*_n \cup F.$$

On the basis of (2) we have:

$$P(A^*_{j,n+1}) = P(A^*_{j,n+1} \cap (\bigcup_{X^*(n)} [x]^*_n \cup F))$$

$$= \sum_{X^*(n)} P(A^*_{j,n+1} \cap [x]^*_n) + P(A^*_{j,n+1} \cap F).$$

By virtue of the hypothesis that $P(F) = 0$, the second term on the right is zero, and we get:

$$(3) \qquad P(A^*_{j,n+1}) = \sum_{X^*(n)} P(A^*_{j,n+1} \cap [x]^*_n).$$

Now as in the case of the proof of 7.4 it is obvious that

$$(4) \qquad P(A^*_{j,n+1} | [x]^*_n) = (1-\Theta)P(A^*_{j,n} | [x]^*_{n-1}) + \Theta \underline{E}_n(x^*).$$

We use (4) to operate on (3):

$$P(A^*_{j,n+1}) = \sum_{\chi^*(n-1)} \sum_{j'} \sum_{k\neq 0} P(A^*_{j,n+1} | A^*_{j',n} \cap E^*_{k,n} \cap [x]^*_{n-1}) \cdot$$

$$P(A^*_{j',n} \cap E^*_{k,n} \cap [x]^*_{n-1})$$

$$= \sum_{\chi^*(n-1)} [(1-\theta)P(A^*_{j,n} | [x]^*_{n-1})P([x]^*_{n-1}) +$$

$$\theta P(E^*_{j,n} \cap [x]^*_{n-1})P([x]^*_{n-1})]$$

$$= (1-\theta)P(A^*_{j,n}) + \theta P(E^*_{j,n}),$$

which establishes (1). Q.E.D.

It will be useful to summarize the experimental implications of the four theorems proved in this section. By virtue of 7.1 we can predict that for any experiment in which E_0 trials do not occur, the proportion of A_j occurrences will approach the proportion of E_j occurrences for each individual subject as the number of trials becomes large. Similarly, using 7.4 we can predict the same result for any experiment in which E_0 trials do occur, provided that we delete the E_0 trials from the subject's protocol before computing the proportion of responses and reinforcing events. Corresponding predictions concerning mean proportions of A_j responses and E_j reinforcing events for populations of subjects follow from 7.2 and 7.14. Experimental applications of 7.1 and 7.2 are illustrated in [5].

8. The Non-Contingent Case.

Of the various special cases that may be derived from the general linear model, the constant event-probability, non-contingent case is the simplest from a mathematical viewpoint and also has been the most studied experimentally. We can conveniently illustrate numerous derivational techniques and at the same time assemble a variety of experimentally useful results by giving a detailed analysis of this case, which we now define in the obvious manner.

Definition 8.1. *A linear model* $\chi = \langle X, P, \theta \rangle$ *satisfies the simple, non-contingent condition if, and only if, for every* k *there is a* π_k *such that for every* n *and every* η *in an experimenter's partition* $H(n)$ *if* $P(\eta) > 0$ *then*

$$P(E_{k,n} | \eta) = \pi_k$$

and

$$P(E_{0,n}) = 0.$$

In order to simplify our notation, we shall adopt the following conventions throughout this section. We shall be concerned with probabilities and various statistical quantities associated with some arbitrarily selected response A_j (and its reinforcing event E_j). When only A_j is explicitly involved in a derivation, we shall drop the subscript j, yielding A_n for the event "A_j occurs on trial n," $P(A_n)$ or p_n for "probability of A_j on trial n," and so on. Also, in the interest of brevity, we shall omit from the numerous theorems to follow the standard hypothesis "*if* $\chi = < X,P,\theta >$ *is a simple, non-contingent, linear model, then*-".

Moments of the Response Probabilities

A recursive expression for the raw moments $\alpha_{v,n}$ of the response probabilities is obtainable simply by specialization of Theorem 5.3.

Theorem 8.2. *If* v *is a positive integer, then*

$$\alpha_{v,n+1} = (1-\theta)^v \alpha_{v,n} + \pi \sum_{i=1}^{v} \binom{v}{i} (1-\theta)^{v-i} \theta^i \alpha_{v-i,n}.$$

Since the first and second raw moments will frequently be required in later derivations, we include explicit expressions for them here. For the first moment, we have:

Corollary 8.3.

$$\alpha_{1,n+1} = (1-\theta)\alpha_{1,n} + \theta\pi,$$

or equivalently

$$p_{n+1} = (1-\theta)p_n + \theta\pi.$$

It can readily be shown by induction that this difference equation has the solution:

Corollary 8.4.

$$\alpha_{1,n} = p_n = \pi-(\pi-p_1)(1-\theta)^{n-1}.$$

For the second moment we have:

<u>Corollary</u> 8.5.

$$\alpha_{2,n+1} = (1-\theta)^2\, \alpha_{2,n} + \pi[2(1-\theta)\theta\alpha_{1,n} + \theta^2],$$

and it can be shown by induction that this difference equation has the solution:

<u>Corollary</u> 8.6.

$$\alpha_{2,n} = \frac{\pi(2\pi-2\theta\pi+\theta)}{2-\theta} - 2\pi(\pi-p_1)(1-\theta)^{n-1}$$

$$- [\frac{\pi(1-\pi)\theta}{2-\theta} - (\pi-p_1)^2](1-\theta)^{2(n-1)}.$$

Asymptotic moments appear so ubiquitously in experimental applications of the model that it is desirable to introduce special notation for them.

<u>Definition</u> 8.7. <u>If</u> $\lim_{n\to\infty} \alpha_{v,n}$ <u>exists</u>, <u>then</u>

$$\alpha_v = \lim_{n\to\infty} \alpha_{v,n}.$$

It is an obvious consequence of Corollaries 8.4 and 8.6 that for $\theta > 0$,

<u>Corollary</u> 8.8.

(i) $\alpha_1 = \pi$,

<u>and</u>

(ii) $\alpha_2 = \pi \dfrac{[2(1-\theta)\pi+\theta]}{2-\theta}$.

Taking these results together with the recursive relation of

Theorem 8.2, we see immediately that asymptotic moments of all orders exist for this case of the linear model.

Concerning such properties as dispersion, skewness, etc, central moments are usually more instructive than raw moments. For the distribution of response probabilities, central moments are defined in the standard manner:

Definition 8.9. *If* v *is a non-negative integer, then*

$$\mu_{v,n} = \sum_{\chi(n-1)} p_{x,n}-p_n)^v \, P([x]_{n-1}).$$

To obtain a useable recursive expression for the v^{th} central moment, we proceed as follows:

$$\mu_{v,n+1} = \sum_{\chi(n)} (p_{x,n+1}-p_{n+1})^v \, P([x]_n)$$

$$= \sum_{\chi(n-1)} \pi[(1-\theta)p_{x,n} + \theta-(1-\theta)p_n-\theta\pi]^v +$$

$$(1-\pi)[(1-\theta)p_{x,n} - (1-\theta)p_n - \theta\pi]^v \quad ,$$

where we have substituted for $p_{x,n+1}$ in terms of the axioms and for p_{n+1} in terms of Corollary 8.3. Expanding the bracketed terms by the binomial theorem and simplifying, we arrive at

Theorem 8.10. *For* $v \geq 2$ (*of course,* $\mu_{0,n} = 1$ *and* $\mu_{1,n} = 0$ *for all* n),

$$\mu_{v,n+1} = (1-\theta)^v \mu_{v,n} + \pi(1-\pi)\Big\{\binom{v}{2}(1-\theta)^{v-2}\theta^2 \mu_{v-2,n} +$$

$$\binom{v}{3}(1-\theta)^{v-3}\theta^3 \mu_{v-3,n}[(1-\pi)^2 - \pi^2] + \binom{v}{4}(1-\theta)^{v-4}\theta^4[(1-\pi)^3 + \pi^3] +$$

$$\dots + \binom{v}{v}\theta^v[(1-\pi)^{v-1} - (-\pi)^{v-1}]\Big\}.$$

Specialization of Theorem 8.10 yields relatively simple expressions for the first few central moments.

Corollary 8.11.

$$\mu_{2,n+1} = (1-\theta)^2 \mu_{2,n} + \pi(1-\pi)\theta^2.$$

[Or, in conventional notation,

$$\sigma^2_{n+1} = (1-\theta)^2 \sigma^2_n + \pi(1-\pi)\theta^2].$$

Asymptotically,

$$\mu_2 = \frac{\pi(1-\pi)\theta}{2-\theta}.$$

Corollary 8.12.

$$\mu_{3,n+1} = (1-\theta)^3 \mu_{3,n} + \pi(1-\pi)\theta^3(1-2\pi),$$

and asymptotically,

$$\mu_3 = \frac{\pi(1-\pi)\theta^3(1-2\pi)}{1-(1-\theta)^3}.$$

Corollary 8.13.

$$\mu_{4,n+1} = (1-\theta)^4 \mu_{4,n} + \pi(1-\pi)\{6(1-\theta)^2 \theta^2 \mu_{2,n} + \theta^4[(1-\pi)^3 + \pi^3]\},$$

and asymptotically,

$$\mu_4 = \frac{\pi(1-\pi)\theta^4}{1-(1-\theta)^4}\left\{1-3\pi(1-\pi)[\frac{1-3(1-\theta)^2}{1-(1-\theta)^2}]\right\}.$$

The properties of two limiting special cases are immediately apparent. If $\theta = 0$, all of the central moments vanish; i.e., for every n, all density of the distribution of response probabilities is concentrated at the point $p_{x,n} = p_1$. If $\theta = 1$, the central moments reduce to those of a binomial distribution; i.e., for all $n \geqq 2$,

$$\mu_{v,n} = \pi(1-\pi)[(1-\pi)^{v-1} - (-\pi)^{v-1}].$$

This last relation can be shown to be the solution of a recursive expression given by Kendall ([9], p. 118) for central moments of the binomial distribution. For θ between zero and unity, the distribution of asymptotic response probabilities has a smaller dispersion and less skewness than a point binomial distribution having the same mean.

Covariance in the Response Sequence

We now derive the doublet, triplet and quadruplet probability, $P(A_{n+1} \cap A_n)$, $P(A_{n+2} \cap A_{n+1} \cap A_n)$, and $P(A_{n+3} \cap A_{n+2} \cap A_{n+1} \cap A_n)$, which may be used to check the experimental adequacy of the model with respect to sequential effects. It is to be emphasized, for those who like random variables formulations, that in view of Theorem 6.3 the following theorem could just as well be expressed in terms of expectations of random variables, namely,

$$E(\underline{A}_{n+1}\ \underline{A}_n)$$

$$E(\underline{A}_{n+2}\ \underline{A}_{n+1}\ \underline{A}_n)$$

$$E(\underline{A}_{n+3}\ \underline{A}_{n+2}\ \underline{A}_{n+1}\ \underline{A}_n).$$

Our method of derivation may be compared with that which is used when the random variable approach is followed and expectations are first taken with respect to a fixed sequence of responses and reinforcements, for our method appears somewhat more direct and simple from a mathematical viewpoint.

Theorem 8.14.

(i) $P(A_{n+1} \cap A_n) = (1-\theta)\,\alpha_{2,n} + \theta\,\pi\,\alpha_{1,n}$,

(ii) $P(A_{n+2} \cap A_{n+1} \cap A_n) = \theta\,\pi\,P(A_{n+1} \cap A_n) + (1-\theta)^3\,\alpha_{1,n} +$

$2(1-\theta)^2\theta\,\pi\,\alpha_{2,n} + (1-\theta)\theta^2\,\pi\,\alpha_{1,n}$,

(iii) $P(A_{n+3} \cap A_{n+2} \cap A_{n+1} \cap A_n) = \theta\pi\,P(A_{n+2} \cap A_{n+1} \cap A_n) +$

$(1-\theta)^6\,\alpha_{4,n} + (1-\theta)^4\,\theta(5-3\theta)\pi\,\alpha_{3,n} + (1-\theta)^3\,\theta^2\,\pi[3(1-\theta)+4\pi]\alpha_{2,n} +$

$(1-\theta)^2\,\theta^3\,\pi\,[(1-\theta) + 2\pi]\alpha_{1,n}$.

Proof:

$$P(A_{j,n+1} \cap A_{j,n}) = \sum_{\chi(n-1)} \sum_{k} P(A_{j,n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \sum_{\chi(n-1)} [P(A_{j,n+1} | A_{j,n} \cap E_{j,n} \cap [x]_{n-1})\pi_j$$

$$+ \sum_{k \neq j} P(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1})(1-\pi_j)] \cdot$$

$$P(A_{j,n} | [x]_{n-1}) P([x]_{n-1})$$

$$= \sum_{\chi(n-1)} [(1-\theta)P(A_{j,n} | [x]_{n-1}) + \theta\pi_j] P(A_{j,n} | [x]_{n-1}) P([x]_{n-1})$$

$$= (1-\theta)\alpha_{2,j,n} + \theta\pi_j \, \alpha_{1,j,n}.$$

As for the second part,

$$P(A_{j,n+2} \cap A_{j,n+1} \cap A_{j,n}) = \sum_{\chi(n-1)} \sum_{k} \sum_{k'} P(A_{j,n+2} \cap A_{j,n+1} \cap E_{k',n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \sum_{\chi(n-1)} \sum_{k} [(1-\theta)P(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1}) + \theta\pi_j] P(A_{j,n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \theta\pi_j P(A_{j,n+1} \cap A_{j,n}) + (1-\theta) \sum_{\chi(n-1)} \Big\{ [1-\theta)P(A_{j,n} | [x]_{n-1}) + \theta]^2 \, \pi_j +$$

$$(1-\theta)^2 P^2(A_{j,n} | [x]_{n-1})(1-\pi_j) \Big\} P(A_{j,n} | [x]_{n-1}) P([x]_{n-1})$$

$$= \Theta\pi_j P(A_{j,n+1} \cap A_{j,n}) + \sum_{\chi(n-1)} [(1-\theta)^3 P^3(A_{j,n} | [x]_{n-1}) +$$

$$+ 2(1-\theta)^2 \theta\pi_j P^2(A_{j,n} | [x]_{n-1}) + (1-\theta)\theta^2 \pi_j P(A_{j,n} | [x]_{n-1})] P([x]_{n-1}).$$

And the desired result follows immediately from the right-hand side.

Finally, we consider the quadruplet term.

$$P(A_{j,n+3} \cap A_{j,n+2} \cap A_{j,n+1} \cap A_{j,n})$$

$$= \sum_{\chi(n-1)} \sum_{k} \sum_{k'} \sum_{k''} P(A_{j,n+3} \cap A_{j,n+2} \cap E_{k'',n+2} \cap A_{j,n+1} \cap E_{k',n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \theta\pi_j P(A_{j,n+2} \cap A_{j,n+1} \cap A_{j,n})$$

$$+ (1-\theta) \sum_{\chi(n-1)} \sum_{k} \sum_{k'} P^2(A_{j,n+2} | A_{j,n+1} \cap E_{k',n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1}) \cdot$$

$$P(E_{k',n+1} | A_{j,n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1}) \, P(A_{j,n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \theta\pi_j P(A_{j,n+2} \cap A_{j,n+1} \cap A_{j,n}) + (1-\theta) \sum_{\chi(n-1)} \sum_{k} \Big\{ [(1-\theta) P(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1}) + \theta]^2 \pi_j$$

$$+ (1-\theta)^2 P^2(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1})(1-\pi_j) \Big\} P(A_{j,n+1} \cap A_{j,n} \cap E_{k,n} \cap [x]_{n-1})$$

$$= \theta\pi_j P(A_{j,n+2} \cap A_{j,n+1} \cap A_{j,n}) + (1-\theta)\theta^2 \pi_j P(A_{j,n+1} \cap A_{j,n}) +$$

$$(1-\theta) \sum_{\chi(n-1)} \sum_{k} [(1-\theta)^2 P^3(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1}) +$$

$$2(1-\theta)\theta \pi_j P^2(A_{j,n+1} | A_{j,n} \cap E_{k,n} \cap [x]_{n-1})] \cdot$$

$$P(E_{k,n} | A_{j,n} \cap [x]_{n-1}) P(A_{j,n} \cap [x]_{n-1}).$$

To facilitate analysis of this last expression, let

$$y = P(A_{j,n} | [x]_{n-1}).$$

Thus by applying the axioms we now have the term summed over the partition $\chi(n-1)$ equal to:

$$(1-\theta) \sum_{\chi(n-1)} [\{(1-\theta)^2[1-\theta)y + \theta]^3 + 2(1-\theta)\theta\pi_j[(1-\theta)y + \theta]^2\}\pi_j +$$

$$\{(1-\theta)^2(1-\theta)^3y^3 + 2(1-\theta)\theta\pi_j(1-\theta)^2 y^2\}(1-\pi_j)]yP([x]_{n-1})$$

$$= (1-\theta)^6 \alpha_{4,j,n} + (1-\theta)^2[3(1-\theta)^3\theta\pi_j + 2(1-\theta)^2 \theta\pi_j]\alpha_{3,j,n}$$

$$+ (1-\theta)^2[3(1-\theta)^2\theta^2\pi_j + 2\theta\pi_j^2(2(1-\theta)\theta]\alpha_{2,j,n}$$

$$+ (1-\theta)^2[(1-\theta)\theta^3\pi_j + 2\theta\pi_j^2 \theta^2]\alpha_{1,j,n}.$$

$$= (1-\theta)^6 \alpha_{4,j,n} + (1-\theta)^4\theta(5-3\theta)\pi_j\alpha_{3,j,n} + (1-\theta)^3 \theta^2\pi_j[3(1-\theta)+4\pi_j]\alpha_{2,j,n}$$

$$+ (1-\theta)^2 \theta^3\pi_j[(1-\theta) + 2\pi_j]\alpha_{1,j,n},$$

which combines with preceding results to yield the desired expression. Q.E.D.

The same methods used in the proof of the above theorem may be applied to obtain terms like

$$P(A_{j,n+1} \cap A_{j',n})$$

where $j \neq j'$. On the other hand, the cases studied in the theorem are cases for which the results also hold for the simple contingent model

$$P(E_{k,n}|A_{j,n}) = \pi_{kj}.$$

To extend the results to this model, merely replace π_j by π_{jj}. Naturally the raw moments $\alpha_{v,j,n}$ are different, but the same expressions in terms of the moments hold. Unfortunately, for obvious reasons, this ready generalization does not hold for doublets $P(A_{j,n+1} \cap A_{j',n})$ (or for triplets or quadruplets).

The results of Theorems 8.2 and 8.14 may be combined to yield asymptotic results in terms solely of θ and π_j for the doublets, triplets and quadruplets of 8.14. We shall not formulate these results as a theorem because of their lengthy form. Even the doublet term is

$$\lim_{n \to \infty} P(A_{j,n+1} \cap A_{j,n}) = \frac{(1-\theta)\pi_j[2(1-\theta)\pi_j + \theta]}{2-\theta} + \theta\pi_j^2.$$

The covariance of the random variables $\underline{A}_n$ and $\underline{A}_{n+m}$ is of interest in its own right and also as a term entering into the derivation of an expression for the sum $S_{n,N}$. As a preliminary to calculation of the covariance, we require the following theorem concerning probability of the compound event $A_n \cap A_{n+m}$:

Theorem 8.15.

$$P(A_{n+m} \cap A_n) = \pi P(A_n) - [\pi P(A_n) - P(A_{n+1} \cap A_n)](1-\theta)^{m-1}.$$

Proof: The proof proceds by induction on m. For m=1, we have an identity. Suppose now the theorem holds for m. Utilizing the usual technique in order to apply the axioms (via 4.9) we have:

$$P(A_{j,n+m+1} \cap A_{j,n}) = \sum_{\chi(n+m-1)} \sum_{k} P(A_{j,n+m+1} \cap E_{k,n+m} \cap A_{j,n} \cap [x]_{n+m-1})$$

$$= \sum_{\chi(n+m-1)} [(1-\theta)P(A_{j,n+m} | [x]_{n+m-1} \cap A_{j,n})$$

$$+ \theta\pi_j]P([x]_{n+m-1} \cap A_{j,n})$$

$$= (1-\theta)P(A_{j,n+m} \cap A_{j,n}) + \theta\pi_j)P(A_{j,n}).$$

Applying now our inductive hypothesis to the last expression we obtain

$$P(A_{j,n+m+1} \cap A_{j,n}) = (1-\theta) \left\{ \pi_j P(A_{j,n}) - [\pi_j P(A_{j,n}) - P(A_{j,n+1} \cap A_{j,n})](1-\theta)^{m-1} \right\}$$

$$+ \theta\pi_j P(A_{j,n})$$

$$= \pi_j P(A_{j,n}) - [\pi_j P(A_{j,n}) - P(A_{j,n+1} \cap A_{j,n})](1-\theta)^m.$$

Q.E.D.

Using the results of Section 6, this theorem may be rewritten in terms of expectations of random variables.

Corollary 8.16.

$$E(\underline{A}_{n+m}\, \underline{A}_n) = \pi E(\underline{A}_n) - [\pi E(\underline{A}_n) - E(\underline{A}_{n+1}\, \underline{A}_n)](1-\theta)^{m-1}.$$

Applying the usual definition of a covariance and making the appropriate substitutions from 8.16, 6.3 and 8.4, we have, after simplification

Theorem 8.17.

$$\mathrm{Cov}(\underline{A}_{n+m}\, \underline{A}_n) = \pi(1-\pi)\, \frac{\theta(1-\theta)^m}{2-\theta}\, [1-(1-\theta)^{2(n-1)}].$$

As one would expect the covariance tends to zero as $\theta \to 0$, $\theta \to 1$, or $m \to \infty$. An outcome one might well have failed to anticipate is that for all $\underline{n}$, the covariance is independent of p_1, the initial probability of an A_j response, but depends in a very simple way on the variance of the response probabilities. (Referring back to 8.11, we see that $\mathrm{Cov}(\underline{A}_{n+m}\, \underline{A}_n) = (1-\theta)^m \sigma_n^2$.) Using 8.17 together with 8.11, we can now write an asymptotic expression for the serial correlation coefficient with lag m, conventionally defined

$$r_m = E\left\{ \frac{\mathrm{Cov}(\underline{A}_{n+m}\, \underline{A}_n)}{(\mu_{2,n+m}\, \mu_{2,n})^{1/2}} \right\},$$

where the expectation is taken over n. Making the appropriate substitutions and taking the limit as $n \to \infty$, we obtain

Theorem 8.18. In the limit as $n \to \infty$,

$$r_m = (1-\theta)^m.$$

Proof: We let $n \to \infty$ in 8.17 and 8.11. Then

$$\lim \mu_{2,n+m} = \lim \mu_{2,n} = \frac{\pi(1-\pi)\theta}{2-\theta}$$

and we have

$$r_m = E\left\{ \frac{\pi(1-\pi)\theta(1-\theta)^m}{2-\theta} \middle/ \frac{\pi(1-\pi)\theta}{2-\theta} \right\} = (1-\theta)^m.$$

This statistic may prove useful for estimation of the parameter θ from asymptotic data.

Also, we are now in a position to give a general formula for the variance of $S_{n,N}$, the variance of A_j response frequency in the block of N trials following any arbitrary trial n.

Theorem 8.19.

$$\mathrm{Var}(S_{n,N}) = N\pi(1-\pi) - (1-2\pi)(\pi-p_1)(1-\theta)^n \frac{[1-(1-\theta)^N]}{\theta}$$

$$- (\pi-p_1)^2(1-\theta)^{2n} \frac{[1-(1-\theta)^{2N}]}{\theta(2-\theta)} + \frac{2(1-\theta)\pi(1-\pi)}{2-\theta} [N-\frac{1}{\theta}+\frac{(1-\theta)^N}{\theta}]$$

$$- \frac{2\pi(1-\pi)}{\theta(2-\theta)^2} (1-\theta)^{2n+1} [1-(1-\theta)^{N-1}][1-(1-\theta)^N].$$

Proof: Since $S_{n,N}$ is the sum of N random variables $\underline{A}_{n+m}(m=1,2,\ldots,N)$, we can use the well known expression for the variance of a sum of random variables:

$$\mathrm{Var}(S_{n,N}) = \sum_{m=1}^{N} \mathrm{var}(\underline{A}_{n+m}) + 2 \sum_{1 \leq j < k \leq N} \mathrm{Cov}(\underline{A}_{n+j}\ \underline{A}_{n+k}).$$

From 6.3 we have

$$\mathrm{Var}(\underline{A}_{n+m}) = P(A_{n+m}) - P^2(A_{n+m});$$

substituting for $P(A_{n+m})$ from 8.4 and summing over $\underline{m}$, we obtain

$$\sum_{m=1}^{N} \mathrm{Var}(\underline{A}_{n+m}) = N\pi(1-\pi) - (1-2\pi)(\pi-p_1)(1-\theta)^n \frac{[1-(1-\theta)^N}{\theta}$$

$$- (\pi-p_1)^2(1-\theta) \frac{2n[1-(1-\theta)^{2N}]}{1-(1-\theta)^2}.$$

Secondly, we substitute for $\mathrm{Cov}(\underline{A}_{n+j}\ \underline{A}_{n+k})$ from 8.17 and sum over j and k, obtaining

$$\sum_{1 \leq j < k \leq N} \mathrm{Cov}(\underline{A}_{n+j}\ \underline{A}_{n+k}) = \frac{\pi(1-\pi)(1-\theta)}{2-\theta} \left\{ N - \frac{1}{\theta} + \frac{(1-\theta)^N}{\theta} \right.$$

$$\left. - \frac{(1-\theta)^{2n}}{\theta(2-\theta)} [1-(1-\theta)^{N-1}][1-(1-\theta)^N] \right\}.$$

Entering these two sums in the general formula for $\mathrm{Var}(S_{n,N})$, we arrive at Theorem 8.19. Q.E.D.

A number of special cases of 8.19 are of experimental interest. To obtain an expression for the variance of A_j frequency in N trials at the limit of learning, we take the limit of $Var(S_{n,N})$ as $n \to \infty$.

Corollary 8.20.

$$Var(S_{\infty,N}) = \lim_{n \to \infty} Var(S_{n,N}) = N\pi(1-\pi) + \frac{2(1-\theta)\pi(1-\pi)}{2-\theta}[N - \frac{1}{\theta} + \frac{(1-\theta)^N}{\theta}]$$

$$= \frac{\pi(1-\pi)}{(2-\theta)\theta}\{N\theta(4-3\theta) - 2(1-\theta)[1-(1-\theta)^N]\}.$$

If $\theta = 1$ or $\theta = 0$ in 8.20, $Var(S_{\infty,N})$ reduces to $N\pi(1-\pi)$, the variance of a sum of N independent random variables.

Letting $n = 0$ in 8.19, we obtain an expression for variance of A_j frequency over the first N trials of a series:

Corollary 8.21.

$$Var(S_{0,N}) = N\pi(1-\pi) + \frac{2(1-\theta)}{2-\theta}\pi(1-\pi)[N - \frac{1}{\theta} + \frac{(1-\theta)^N}{\theta}]$$

$$- (1-2\pi)(\pi-p_1)\frac{[1-(1-\theta)^N]}{\theta} - (\pi-p_1)^2\frac{[1-(1-\theta)^{2N}]}{\theta(2-\theta)}$$

$$- \frac{2\pi(1-\pi)}{\theta(2-\theta)^2}(1-\theta)[1-(1-\theta)^{N-1}][1-(1-\theta)^N].$$

Finally, dividing the right side of either 8.20 or 8.21 by N^2, we find that for large N, the variance of the proportion of A_j responses in

N trials is approximately equal to $\frac{\pi(1-\pi)}{N} \frac{(4-3\theta)}{2-\theta}$. The approximation fails at $\theta = 0$, but it should be relatively good over the range of θ values commonly observed experimentally.

Covariance of Responses and Reinforcing Events

In the simple non-contingent case, the probability of a response given that it was reinforced on the preceding trial takes the following very simple form:

Theorem 8.22. If $\pi > 0$, then

$$P(A_{n+1} | E_n) = (1-\theta)P(A_n) + \theta.$$

Proof:

$$P(A_{n+1} \cap E_n) = \sum_{x(n-1)} P(A_{n+1} \cap E_n \cap [x]_{n-1})$$

$$= \sum_{x(n-1)} P(A_{n+1} | E_n \cap [x]_{n-1}) \pi P([x]_{n-1})$$

$$= \sum_{x(n-1)} [(1-\theta)P(A_n | [x]_{n-1}) + \theta] \pi P([x]_{n-1})$$

$$= [(1-\theta)P(A_n) + \theta]\pi.$$

And obviously

$$P(A_{n+1} | E_n) = \frac{1}{\pi} P(A_{n+1} \cap E_n)$$

$$= (1-\theta)P(A_n) + \theta. \qquad \text{Q.E.D.}$$

More generally, the probability of a response given that it was reinforced on the ν^{th} preceding trial can be expressed as follows:

Theorem 8.23. If ν is a positive integer and $\pi > 0$ then,

$$P(A_{n+\nu}|E_n) = (1-\theta)^{\nu} P(A_n) + \theta(1-\theta)^{\nu-1} + [1-(1-\theta)^{\nu-1}]\pi.$$

Proof: The proof is inductive. We know from 8.22 that the theorem holds for $\nu = 1$. Preparatory to introducing our inductive hypothesis, we apply the learning axioms (via 4.9) as follows:

$$P(A_{n+\nu+1}|E_n) = \frac{P(A_{n+\nu+1} \cap E_n)}{\pi}$$

$$= \frac{1}{\pi}\,[\sum_{[x]_{n+\nu-1} \subseteq E_n} \sum_k P(A_{n+\nu+1} \cap E_{k,n+\nu} \cap [x]_{n+\nu-1})]$$

$$= \frac{1}{\pi}\,[\sum_{E_n} \sum_k P(A_{n+\nu+1}|E_{k,n+\nu} \cap [x]_{n+\nu-1}) P(E_{k,n+\nu}|[x]_{n+\nu-1})$$

$$\cdot P([x]_{n+\nu-1})]$$

$$= \frac{1}{\pi}\,[\sum_{E_n} [(1-\theta)P(A_{n+\nu}|[x]_{n+\nu-1}) + \theta\pi]\; P([x]_{n+\nu-1})$$

$$= \frac{1}{\pi}\,[(1-\theta)P(A_{n+\nu}|E_n) + \theta\pi]\cdot\pi$$

$$= (1-\theta)P(A_{n+\nu}|E_n) + \theta\pi.$$

Applying now the inductive hypothesis to the right-hand side of the last line, we infer:

$$P(A_{n+\nu+1}|E_n) = (1-\theta)\{(1-\theta)^{\nu} P(A_n) + \theta(1-\theta)^{\nu-1} + [1-(1-\theta)^{\nu-1}]\pi\} + \theta\pi$$

$$= (1-\theta)^{\nu+1} P(A_n) + \theta(1-\theta)^{\nu} + [1-(1-\theta)^{\nu}]\pi. \qquad \text{Q.E.D.}$$

Similarly, we have for the probability of a response given that some alternative response was reinforced on the ν^{th} preceding trial:

Theorem 8.24. *If* ν *is a positive integer and* $\pi_k > 0$ *and* $j \neq k$

$$P(A_{j,n+\nu}|E_{k,n}) = (1-\theta)^{\nu} P(A_{j,n}) + [1-(1-\theta)^{\nu-1}]\pi_j.$$

The proof is analogous to that of 8.23.

We conclude this section with two theorems concerning the repetition of "correct" and "incorrect" responses. The first of these specifies probability of a response given that it *occurred and was reinforced* on the preceding trial:

Theorem 8.25. *Provided both* $\pi > 0$ *and* $\alpha_{1,n} > 0$,

$$P(A_{n+1}|A_n \cap E_n) = (1-\theta)\frac{\alpha_{2,n}}{\alpha_{1,n}} + \theta .$$

Proof:

$$P(A_{n+1} \cap A_n \cap E_n) = \sum_{\chi(n-1)} P(A_{n+1} \cap A_n \cap E_n \cap [x]_{n-1})$$

$$= \sum_{\chi(n-1)} P(A_{n+1} | A_n \cap E_n \cap [x]_{n-1}) \pi P(A_n | [x]_{n-1}) P([x]_{n-1})$$

$$= \sum_{\chi(n-1)} [(1-\theta)P(A_n | [x]_{n-1}) + \theta]\pi \, P(A_n | [x]_{n-1}) P([x]_{n-1})$$

$$= \pi[(1-\theta)\alpha_{2,n} + \theta \, \alpha_{1,n}].$$

By virtue of the fact that

$$P(A_{n+1} | A_n \cap E_n) = P(A_{n+1} \cap A_n \cap E_n)/P(A_n \cap E_n)$$

and

$$P(A_n \cap E_n) = P(E_n | A_n) P(A_n) = \pi \, \alpha_{1,n},$$

we infer from the last line of the above identities:

$$P(A_{n+1} | A_n \cap E_n) = \frac{\pi[(1-\theta)\alpha_{2,n} + \theta\alpha_{1,n}]}{\pi \alpha_{1,n}}$$

$$= (1-\theta) \frac{\alpha_{2,n}}{\alpha_{1,n}} + \theta. \qquad \text{Q.E.D.}$$

The second of these theorems specifies probability of a response given that it occurred but was not reinforced on the preceding trial:

Theorem 8.26. Provided that $P(A_{j,n} \cap E_{k,n}) > 0$ and $j \neq k$,

$$P(A_{j,n+1} | A_{j,n} \cap E_{k,n}) = (1-\theta) \frac{\alpha_{2,j,n}}{\alpha_{1,j,n}} .$$

The proof is analogous to that of 8.25.

It may be noted that the difference between the two conditional probabilities of 8.25 and 8.26 equals θ; consequently the difference between the two corresponding conditional relative frequencies provides an additional estimator for this parameter.

9. Applications to Two-Person Game Situations.

By an obvious generalization of the primitive notions and axioms of Sections 2 and 3 we may apply the linear model to two-person game situations (for experimental results, see [1], [2], [5]). As might be expected, we restrict ourselves to games in normal form with a finite number of strategies available to each player. Each play of the game constitutes a trial in the learning sense. The choice of a strategy for each trial corresponds to the choice of a response. To avoid the problem of measuring utility we assume a unit reward which is assigned on an all-or-none basis. Rules of the game are taken to be such that on each trial there is exactly one correct choice leading to the unit reward for each player. (For simplicity we are omitting non-reinforced trials.) However, it should be noted that from a game standpoint, this unit reward is not the payoff on a trial, but rather the payoff is the expected value of the reward. This expected value depends on the reinforcement scheme selected by the experimenter; this scheme may be represented by a payoff matrix (a_{ij}, b_{ij}) where $i=1,\ldots,r_1$ and $j=1,\ldots,r_2$ with r_1 the number of responses available to the first player, player A, and r_2 the number available to the second player, player B. Thus the entries in the payoff matrix are pairs of numbers (a_{ij}, b_{ij}). The number a_{ij} is the probability that player A wins when A makes response i and B response j. Correspondingly, b_{ij} is the probability that player B wins when A makes response i and B response j.

Just as for the one-person linear model, the axioms are independent of the selection of any particular probabilistic scheme of reinforcement, and they apply to more complicated two-person situations than the game paradigm just described. Moreover, although the axioms are stated for two subjects, it is apparent that all notions generalize immediately to n-person situations. The restriction to $n=2$ has been imposed here because all the interesting deductive consequences of the axioms which we have yet considered are for two-person situations.

We turn now to a rapid capitulation of formal developments corresponding to those in Sections 2 and 3. The result of any trial may be represented by an ordered quadruple $< j_1, k_1, j_2, k_2 >$, where j_1 is the response of A, j_2 the response of B, k_1 the reinforcing event for A, and k_2 the reinforcing event for B, with $j_1, k_1 = 1, \ldots, r_1$ and $j_2, k_2 = 1, \ldots, r_2$. Our first primitive notion is then the sample space X which is the set of all sequences of such ordered quadruples. We call X the $< r_1, r_2 >$ - <u>response space</u>. As before, our second primitive notion is a countably additive probability measure P on the Borel field $\mathcal{B}(X)$ of cylinder sets. Our third and fourth primitive notions are the two learning parameters θ_A and θ_B for players A and B respectively.

A notation which is essentially needed only for the definitions immediately to follow is that if $x \in X$ and $x_n = < j_1, k_1, j_2, k_2 >$ is the n^{th} term of the sequence x, then

$$x^{(1)}_{A,n} = j_1$$

$$x^{(1)}_{E,n} = k_1$$

$$x^{(2)}_{A,n} = j_2$$

$$x^{(2)}_{E,n} = k_2.$$

(This dual use of the subscript A for responses in the next few lines should not be confusing; subsequently A refers once again to player A.) The event consisting of *response* j by subject i on *trial* n is defined as might be expected.

Definition 9.1.

$$A^{(i)}_{j,n} = \{x:\ x \in X \ \&\ x^{(i)}_{A,n} = j\}.$$

Similarly, we define the event of *reinforcing event* k *by subject* i *on trial* n.

Definition 9.2.

$$E^{(i)}_{k,n} = \ x:\{x \in X \ \&\ x^{(i)}_{E,n} = k\}.$$

Our next definition is for the *probability of response* j *by subject* i *on trial* n.

Definition 9.3.

$$p^{(i)}_{j,n} = P(A^{(i)}_{j,n}).$$

Since we shall be concerned with sequences which are identical, or identical for a given subject, through the n^{th} trial, we want to define the appropriate equivalence classes.

Definition 9.4.

$$[x]_n = \{y: y \in X \ \&\ \text{for}\ m \leq n,\ y_m = x_m\}$$

$$[x]_n^{(i)} = \{y: y \in X \ \&\ \text{for}\ m \leq n,\ y_m^{(i)} = x_m^{(i)}\}.$$

We next define the probability of response j by subject i on trial n given the first $n-1$ responses and the first $n-1$ reinforcing events for this subject.

Definition 9.5.

$$p_{xj,n}^{(i)} = P(A_{j,n}^{(i)} | [x]_{n-1}^{(i)}).$$

We are now in a position to state our axioms. It is understood that $\theta^{(i)} = \theta_A$ if $i=1$, and $\theta^{(i)} = \theta_B$ if $i=2$.

Definition 9.6. An ordered quadruple $\chi = \langle X, P, \theta_A, \theta_B \rangle$ is a two-person linear model if, and only if, there are integers r_1 and r_2 such that X is the $\langle r_1, r_2 \rangle$ - response space; P is a probability measure on $\mathcal{B}(X)$; θ_A, $\theta_B \in (0,1)$; and the following two axioms are satisfied for every positive integer n, for $i=1,2$, for every x in X with $P([x]_n^{(i)}) > 0$, and for every j and k:

Axiom 1. If $x \in E_{k,n}^{(i)}$ and $j = k$ then

$$p^{(i)}_{xj,n+1} = (1-\theta^{(i)})p^{(i)}_{xj,n} + \theta^{(i)}.$$

Axiom 2. If $x \in E^{(i)}_{k,n}$ and $j \neq k$ then

$$p^{(i)}_{xj,n+1} = (1-\theta^{(i)})p^{(i)}_{xj,n}.$$

In subsequent developments the superscript i will be mainly replaced by other devices. In any case it must not be thought of as an exponent.

We may now formally define the special reinforcement scheme yielding the payoff matrix (a_{ij}, b_{ij}).

Definition 9.7.

$$a_{ij} = P(E^{(1)}_{i,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n})$$

$$b_{ij} = P(E^{(2)}_{j,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n}).$$

In addition to 9.7 we also need:

Definition 9.8. A two-person linear model $\chi = \langle X, P, \theta_A, \theta_B \rangle$ has a simple payoff matrix if, and only if, for every n and for every set W which is an $n-1$ cylinder set if $P(A^{(1)}_{i,n} \cap A^{(2)}_{j,n} \cap W) > 0$ then

$$P(E^{(1)}_{i,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n} \cap W) = P(E^{(1)}_{i,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n})$$

and

$$P(E^{(2)}_{j,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n} \cap W) = P(E^{(2)}_{j,n} | A^{(1)}_{i,n} \cap A^{(2)}_{j,n}).$$

In the remainder of this section we consider only two-person linear models which have simple payoff matrices and exactly two responses for each player, i.e., $r_1 = r_2 = 2$.

Finally, one last definition to eliminate continual use of the superscript i. We denote player A's probability by α and player B's by β, whereas we denote the joint probability of an $A^{(1)}_1$ and $A^{(2)}_1$ response by γ.

Definition 9.9.

$$\alpha_n = p^{(1)}_{1,n}$$

$$\beta_n = p^{(2)}_{1,n}$$

$$\gamma_n = P(A^{(1)}_{1,n} \cap A^{(2)}_{1,n})$$

$$\bar{\alpha}_N = \frac{1}{N} \sum_{n=1}^{N} \alpha_n$$

$$\bar{\beta}_N = \frac{1}{N} \sum_{n=1}^{N} \beta_n.$$

Coming now to our first theorem of this section we derive recursive expressions for α_n and β_n. It is an unfortunate fact, and enormously complicating to the mathematics of two-person situations, that both α_{n+1} and β_{n+1} depend on the joint probability term γ_n. But this awkward mathematical consequence of the theory is psychologically sound: if the two subjects are interacting at all it would be surprising indeed if their probabilities of responses were statistically independent.

Theorem 9.10.

$$\alpha_{n+1} = (1-\theta_A(2-a_{12}-a_{22}))\alpha_n + \theta_A(a_{22}-a_{21})\beta_n + \theta_A(a_{11}+a_{21}-a_{12}-a_{22})\gamma_n + \theta_A(1-a_{22})$$

$$\beta_{n+1} = (1-\theta_B(2-b_{21}-b_{22}))\beta_n + \theta_B(b_{22}-b_{12})\alpha_n + \theta_B(b_{11}+b_{12}-b_{21}-b_{22})\gamma_n + \theta_B(1-b_{22}).$$

Proof: It will suffice to derive the difference equation for α_{n+1} since the derivation for β_{n+1} is identical. To begin with we observe that

$$(1) \qquad P(E_{1,n}^{(1)}) = P(E_{1,n}^{(1)} \cap \bigcup_j A_{j,n}^{(1)} \cap \bigcup_{j'} A_{j',n}^{(2)})$$

$$= \sum_j \sum_{j'} P(E_{1,n}^{(1)} | A_{j,n}^{(1)} \cap A_{j',n}^{(2)}) P(A_{j,n}^{(1)} \cap A_{j',n}^{(2)})$$

$$= a_{11}P(A_{1,n}^{(1)} \cap A_{1,n}^{(2)}) + a_{12}P(A_{1,n}^{(1)} \cap A_{2,n}^{(2)}) + (1-a_{21})P(A_{2,n}^{(1)} \cap A_{1,n}^{(2)})$$

$$+ (1-a_{22})P(A_{2,n}^{(1)} \cap A_{2,n}^{(2)}).$$

We next observe that

$$P(A_{1,n}^{(1)} \cap A_{2,n}^{(2)}) = P(A_{2,n}^{(2)} | A_{1,n}^{(1)})P(A_{1,n}^{(1)}) \tag{2}$$

$$= (1-P(A_{1,n}^{(2)} | A_{1,n}^{(1)}))P(A_{1,n}^{(1)})$$

$$= P(A_{1,n}^{(1)}) - P(A_{1,n}^{(1)} \cap A_{1,n}^{(2)}).$$

Similarly,

$$P(A_{2,n}^{(1)} \cap A_{1,n}^{(2)}) = P(A_{1,n}^{(2)}) - P(A_{1,n}^{(1)} \cap A_{1,n}^{(2)}) \tag{3}$$

and

$$P(A_{2,n}^{(1)} \cap A_{2,n}^{(2)}) = P(A_{2,n}^{(1)} | A_{2,n}^{(2)})P(A_{2,n}^{(2)}) \tag{4}$$

$$= (1-P(A_{1,n}^{(1)} | A_{2,n}^{(2)}))P(A_{2,n}^{(2)})$$

$$= 1-P(A_{1,n}^{(2)}) - P(A_{1,n}^{(1)} \cap A_{2,n}^{(2)})$$

$$= 1-P(A_{1,n}^{(1)}) - P(A_{1,n}^{(2)}) + P(A_{1,n}^{(1)} \cap A_{1,n}^{(2)}).$$

From (1)-(4) we conclude:

$$(5) \qquad P(E^{(1)}_{1,n}) = a_{11}\gamma_n + a_{12}(\alpha_n-\gamma_n) + (1-a_{21})(\beta_n-\gamma_n) + (1-a_{22})(1-\alpha_n-\beta_n+\gamma_n)$$

$$= -(1-a_{12}-a_{22})\alpha_n + (a_{22}-a_{21})\beta_n + (a_{11}+a_{21}-a_{12}-a_{22})\gamma_n + (1-a_{22}).$$

Now by a proof similar to that of Theorem 5.3 we may show that

$$(6) \qquad \alpha_{n+1} = (1-\theta_A)\alpha_n + \theta_A P(E^{(1)}_{1,n}).$$

Substituting (5) into (6) we have:

$$\alpha_{n+1} = (1-\theta_A)\alpha_n - \theta_A(1-a_{12}-a_{22})\alpha_n + \theta_A(a_{22}-a_{21})\beta_n + \theta_A(a_{11}+a_{21}-a_{12}-a_{22})\gamma_n + \theta_A(1-a_{22})$$

$$= (1-\theta_A(2-a_{12}-a_{22}))\alpha_n + \theta_A(a_{22}-a_{21})\beta_n + \theta_A(a_{11}+a_{21}-a_{12}-a_{22})\gamma_n + \theta_A(1-a_{22}). \qquad \text{Q.E.D.}$$

The investigation of the asymptotes of α_n and β_n, or even the asymptotes of the mean probabilities (over trials) $\bar{\alpha}_N$ and $\bar{\beta}_N$ is difficult and complicated. The reason is not hard to find. The pair of simultaneous recursive equations of the preceding theorem contain three, rather than two, expressions depending on n, namely, α_n, β_n and γ_n. And a recursive expression for γ_n in terms of these three quantities cannot be derived. Fortunately, without pursuing these questions we may prove directly that an asymptotic linear sum of $\overline{\alpha}_N$ and $\overline{\beta}_N$ exists and is independent of θ_A and θ_B. Moreover this linear relationship requires no restrictive hypotheses and may be compared in a straightforward manner with experimental data.

Theorem 9.11.

$$\lim_{N \to \infty} [(ag-ce)\overline{\alpha}_N + (bg-cf)\overline{\beta}_N] = ch-dg,$$

where

$$a = 2-a_{12}-a_{22}$$

$$b = a_{22}-a_{21}$$

$$c = a_{11}+a_{21}-a_{12}-a_{22}$$

$$d = 1-a_{22}$$

$$e = b_{22}-b_{12}$$

$$f = 2-b_{21}-b_{22}$$

$$g = b_{11}+b_{12}-b_{21}-b_{22}$$

$$h = 1-b_{22}.$$

Proof: From Theorem 9.10 and the definitions of the numbers a to h above, we have:

(1) $$\frac{\alpha_{n+1} - \alpha_n}{\Theta_A} = a\alpha_n + b\beta_n + c\gamma_n + d$$

and

(2) $$\frac{\beta_{n+1} - \beta_n}{\Theta_B} = e\alpha_n + f\beta_n + g\gamma_n + h.$$

Summing both (1) and (2) from 1 to N and dividing by N, we obtain:

(3) $$\frac{\alpha_{N+1} - \alpha_1}{N\Theta_A} = a\bar{\alpha}_N + b\bar{\beta}_N + c\bar{\gamma}_N + d$$

and

(4) $$\frac{\beta_{N+1} - \beta_1}{N\Theta_B} = e\bar{\alpha}_N + f\bar{\beta}_N + g\bar{\gamma}_N + h.$$

Multiplying (3) by g and (4) by c, and then subtracting (4) from (3), rearranging terms slightly, and letting $N \to \infty$ we obtain the desired result, since the left-hand side of (3) and of (4) goes to zero. Q.E.D.

By imposing various restrictions on the experimental parameters a_{ij} and b_{ij} a variety of results can be obtained. We restrict ourselves here to the consideration of one such case: choice of the parameters so that the coefficients of γ_n in the two recursive equations of Theorem 9.10 vanish. Moreover, we compute only the asymptotes of α_n and β_n, although it is not difficult to find explicit expressions for arbitrary n. The method of proof proceeds via consideration of the mean probabilities $\bar{\alpha}_N$ and $\bar{\beta}_N$. Direct solution of the difference equations is possible, but more tedious.

Theorem 9.12. *Let numbers* a *to* h *be defined as in* 9.11, *and let*

$$c = g = 0$$

$$af - be \neq 0$$

then

$$\lim_{n \to \infty} \alpha_n = \frac{bh-df}{af-be}$$

and

$$\lim_{n \to \infty} \beta_n = \frac{ah-de}{af-be} .$$

Proof: From Theorem 9.10, the definition of *a* to *h*, and the hypothesis that $c = g = 0$, we have:

(1) $$\frac{\alpha_{n+1} - \alpha_n}{\theta_A} = a\alpha_n + b\beta_n + d$$

and

$$(2) \qquad \frac{\beta_{n+1} - \beta_n}{\theta_A} = e\alpha_n + f\beta_n + h.$$

Summing both (1) and (2) from 1 to N and dividing by N we obtain:

$$(3) \qquad \frac{\alpha_{N+1} - \alpha_1}{N\theta_A} = a\overline{\alpha}_N + b\overline{\beta}_N + d$$

and

$$(4) \qquad \frac{\beta_{N+1} - \beta_1}{N\theta_B} = e\overline{\alpha}_N + f\overline{\beta}_N + h.$$

Multiplying (3) by f and (4) by b, subtracting then (4) from (3), and letting $N \to \infty$, we get:

$$\lim_{N \to \infty} [af\overline{\alpha}_N + df - be\overline{\alpha}_N - bh] = 0,$$

whence

$$(5) \qquad \lim_{N \to \infty} \overline{\alpha}_N = \frac{bh-df}{af-be} .$$

Since (1) and (2) are simultaneous linear difference equations with constant coefficients, we know that the asymptotes of α_n and β_n exist. Hence by the well known theorem that if a sequence $< y_1, y_2, \ldots, y_n, \ldots >$ of numbers converges to a finite limit y then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} y_n = y,$$

we conclude from (5) that

$$\lim_{n \to \infty} \alpha_n = \frac{bh-df}{af-be} . \tag{6}$$

The argument establishing the limit of β_n is similar in character. Q.E.D.

Some experimental cases falling within the province of this theorem have been studied by Atkinson and Suppes [2].

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