5
Fourier and Laplace Transforms

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.", Nikolai Lobatchevsky (1792-1856)

5.1 Introduction

In this chapter we turn to the study of Fourier transforms, which provide integral representations of functions defined on the entire real line. Such functions can represent analog signals. Recall that analog signals are continuous signals which are sums over a continuous set of frequencies. Our starting point will be to rewrite Fourier trigonometric series as Fourier exponential series. The sums over discrete frequencies will lead to a sum (integral) over continuous frequencies. The resulting integrals will be complex integrals, which can be evaluated using contour methods. We will investigate the properties of these Fourier transforms and get prepared to ask how the analog signal representations are related to the Fourier series expansions over discrete frequencies which we had seen in Chapter 2. Fourier series represented functions which were defined over finite domains such as $x \in [0, L]$. Our explorations will lead us into a discussion of the sampling of signals in the next chapter.

We will also discuss a related integral transform, the Laplace transform. Laplace transforms are useful in solving initial value problems in differential equations and can be used to relate the input to the output of a linear system. Both transforms provide an introduction to a more general theory of transforms, which are used to transform specific problems to simpler ones.

In this chapter we will explore the use of integral transforms. Given a function $f(x)$, we define an integral transform to a new function $F(k)$ as

$$F(k) = \int_a^b f(x)K(x,k)\, dx.$$

Here $K(x,k)$ is called the kernel of the transform. We will concentrate specifically on Fourier transforms,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\, dx,$$

and Laplace transforms

$$F(s) = \int_0^{\infty} f(t)e^{-st}\, dt.$$

In Figure 5.1 we summarize the transform scheme for solving an initial value problem. One can solve the differential equation directly, evolving the initial condition $y(0)$ into the solution $y(t)$ at a later time.

However, the transform method can be used to solve the problem indirectly. Starting with the differential equation and an initial condition, one computes its Transform (T) using

$$Y(s) = \int_0^{\infty} y(t)e^{-st}\, dt.$$

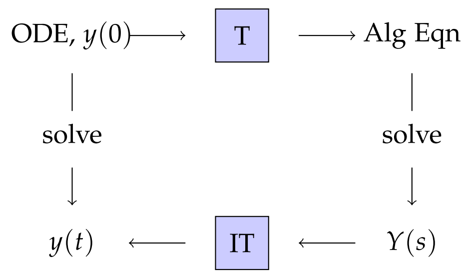

Figure 5.1: Schematic of using transforms to solve a linear ordinary differential equation.

Applying the transform to the differential equation, one obtains a simpler (algebraic) equation satisfied by $Y(s)$, which is simpler to solve than the original differential equation. Once $Y(s)$ has been found, then one applies the Inverse Transform (IT) to $Y(s)$ in order to get the desired solution, $y(t)$. We will see how all of this plays out by the end of the chapter.

We will begin by introducing the Fourier transform. First, we need to see how one can rewrite a trigonometric Fourier series as complex exponential series. Then we can extend the new representation of such series to analog signals, which typically have infinite periods. In later chapters we will highlight the connection between these analog signals and their associated digital signals.

5.2 Complex Exponential Fourier Series

Before deriving the Fourier transform, we will need to rewrite the trigonometric Fourier series representation as a complex exponential Fourier series. We first recall from Chapter 2 the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right), \tag{5.1}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.2}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms. This gives

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-inx}. \end{aligned} \tag{5.3}$$

The coefficients of the complex exponentials can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \tag{5.4}$$

This implies that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \tag{5.5}$$

So far, the representation is rewritten as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Re-indexing the first sum, by introducing $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Since k is a dummy index, we replace it with a new n as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=-1}^{-\infty} \bar{c}_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

We can now combine all the terms into a simple sum. We first define c_n for negative n's by

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Letting $c_0 = \frac{a_0}{2}$, we can write the complex exponential Fourier series representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{5.6}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots, \\ c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots, \\ c_0 &= \frac{a_0}{2}. \end{aligned} \tag{5.7}$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots,$

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n) \\ &= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx} + e^{-inx}}{2}\right) dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx} - e^{-inx}}{2i}\right) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx. \end{aligned} \tag{5.8}$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, we have obtained the complex exponential Fourier series coefficients for all n. Now we can define the complex exponential Fourier series for the function $f(x)$ defined on $[-\pi, \pi]$ as shown below.

Complex Exponential Series for $f(x)$ Defined on $[-\pi, \pi]$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{5.9}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \tag{5.10}$$

We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

This can be rewritten as an exponential Fourier series of the form

Complex Exponential Series for $f(x)$ Defined on $[-L, L]$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \tag{5.11}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx. \tag{5.12}$$

We can now use this complex exponential Fourier series for function defined on $[-L, L]$ to derive the Fourier transform by letting L get large. This will lead to a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series represent.

5.3 *Exponential Fourier Transform*

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions of finite period in terms of sums over a discrete set of frequencies. In particular, for functions defined on $x \in [-L, L]$, the period of the Fourier series representation is $2L$. We can write the arguments in the exponentials, $e^{-in\pi x/L}$, in terms of the angular frequency, $\omega_n = n\pi/L$, as $e^{-i\omega_n x}$. We note that the frequencies, ν_n, are then defined through $\omega_n = 2\pi\nu_n = \frac{n\pi}{L}$. Therefore, the complex exponential series is seen to be a sum over a discrete, or countable, set of frequencies.

We would now like to extend the finite interval to an infinite interval, $x \in (-\infty, \infty)$, and to extend the discrete set of (angular) frequencies to a continuous range of frequencies, $\omega \in (-\infty, \infty)$. One can do this rigorously. It amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed.

We first define $\Delta\omega = \frac{\pi}{L}$, so that $\omega_n = n\Delta\omega$. Inserting the Fourier coefficients (5.12) into Equation (5.11), we have

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\xi/L}\, d\xi \right) e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x}. \end{aligned} \tag{5.13}$$

Now, we let L get large, so that $\Delta\omega$ becomes small and ω_n approaches the angular frequency ω. Then,

$$\begin{aligned} f(x) &\sim \lim_{\Delta\omega \to 0, L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x} \Delta\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi \right) e^{-i\omega x}\, d\omega. \end{aligned} \tag{5.14}$$

Looking at this last result, we formally arrive at the definition of the Fourier transform. It is embodied in the inner integral and can be written as

Definitions of the Fourier transform and the inverse Fourier transform.

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{5.15}$$

This is a generalization of the Fourier coefficients (5.12).

Once we know the Fourier transform, $\hat{f}(\omega)$, we can reconstruct the original function, $f(x)$, using the inverse Fourier transform, which is given by the outer integration,

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{5.16}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Such functions are said to be L_1.

We combine these results below, defining the Fourier and inverse Fourier transforms and indicating that they are inverse operations of each other.

We will then prove the first of the equations, Equation (5.19). [The second equation, Equation (5.20), follows in a similar way.]

The **Fourier transform** and **inverse Fourier transform** are inverse operations. Defining the Fourier transform as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx. \tag{5.17}$$

and the inverse Fourier transform as

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}\, d\omega, \tag{5.18}$$

then

$$F^{-1}[F[f]] = f(x) \tag{5.19}$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega). \tag{5.20}$$

Proof. The proof is carried out by inserting the definition of the Fourier transform, Equation (5.17), into the inverse transform definition, Equation (5.18), and then interchanging the orders of integration. Thus, we have

$$\begin{aligned}
F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f]e^{-i\omega x}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}\, d\xi\right]e^{-i\omega x}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\, d\xi d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi. \qquad (5.21)
\end{aligned}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we first define

$$D_\Omega(x) = \int_{-\Omega}^{\Omega} e^{i\omega x}\, d\omega$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x).$$

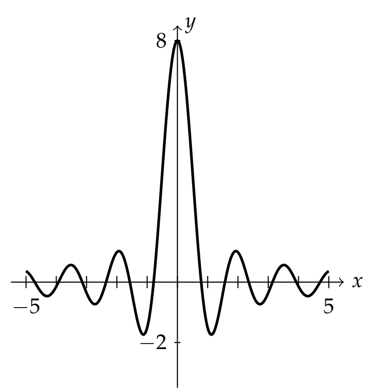

Figure 5.2: A plot of the function $D_\Omega(x)$ for $\Omega = 4$.

We can compute $D_\Omega(x)$. A simple evaluation yields

$$\begin{aligned}
D_\Omega(x) &= \int_{-\Omega}^{\Omega} e^{i\omega x}\, d\omega \\
&= \left.\frac{e^{i\omega x}}{ix}\right|_{-\Omega}^{\Omega} \\
&= \frac{e^{ix\Omega} - e^{-ix\Omega}}{2ix} \\
&= \frac{2\sin x\Omega}{x}. \qquad (5.22)
\end{aligned}$$

A plot of this function is given in Figure 5.2 for $\Omega = 4$. For large Ω, the peak grows and the values of $D_\Omega(x)$ for $x \neq 0$ tend to zero as shown in Figure 5.3. In fact, as x approaches 0, $D_\Omega(x)$ approaches 2Ω. For $x \neq 0$, the $D_\Omega(x)$ function tends to zero.

We further note that

$$\lim_{\Omega\to\infty} D_\Omega(x) = 0, \quad x \neq 0,$$

and $\lim_{\Omega\to\infty} D_\Omega(x)$ is infinite at $x = 0$. However, the area is constant for each Ω. In fact,

$$\int_{-\infty}^{\infty} D_\Omega(x)\, dx = 2\pi.$$

We can show this by recalling the computation in Example 4.42,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Then,

$$\begin{aligned}\int_{-\infty}^{\infty} D_\Omega(x)\, dx &= \int_{-\infty}^{\infty} \frac{2\sin x\Omega}{x}\, dx \\ &= \int_{-\infty}^{\infty} 2\frac{\sin y}{y}\, dy \\ &= 2\pi. \end{aligned} \tag{5.23}$$

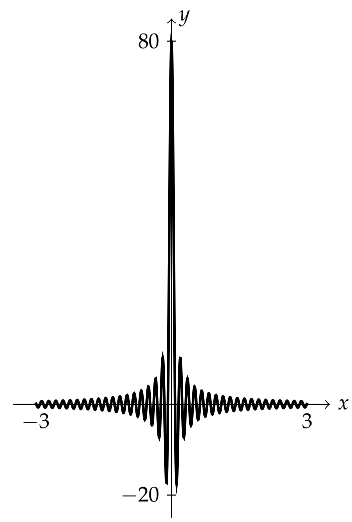

Figure 5.3: A plot of the function $D_\Omega(x)$ for $\Omega = 40$.

Another way to look at $D_\Omega(x)$ is to consider the sequence of functions $f_n(x) = \frac{\sin nx}{\pi x}$, $n = 1, 2, \ldots$. Thus we have shown that this sequence of functions satisfies the two properties,

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \neq 0,$$

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1.$$

This is a key representation of such generalized functions. The limiting value vanishes at all but one point, but the area is finite.

Such behavior can be seen for the limit of other sequences of functions. For example, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n}, \\ \frac{n}{2}, & |x| \leq \frac{1}{n}. \end{cases}$$

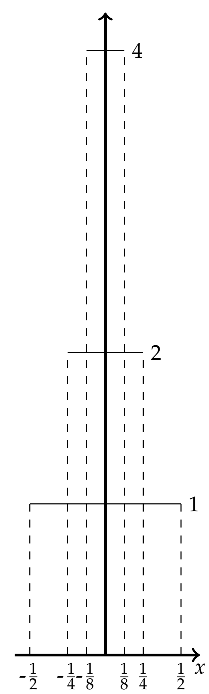

Figure 5.4: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

This is a sequence of functions as shown in Figure 5.4. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

The limit is not really a function. It is a generalized function. It is called the Dirac delta function, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

Before returning to the proof that the inverse Fourier transform of the Fourier transform is the identity, we state one more property of the Dirac delta function, which we will prove in the next section. Namely, we will show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega = \lim_{\Omega\to\infty} D_{\Omega}(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into Equation (5.21), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\, d\xi \\ &= f(x). \end{aligned} \tag{5.24}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f. □

5.4 *The Dirac Delta Function*

P. A. M. Dirac (1902 - 1984) introduced the δ function in his book, *The Principles of Quantum Mechanics*, 4th Ed., Oxford University Press, 1958, originally published in 1930, as part of his orthogonality statement for a basis of functions in a Hilbert space, $< \xi'|\xi'' >= c\delta(\xi' - \xi'')$ in the same way we introduced discrete orthogonality using the Kronecker delta.

In the last section we introduced the Dirac delta function, $\delta(x)$. As noted above, this is one example of what is known as a generalized function, or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Two properties were used in the last section. First, one has that the area under the delta function is one:

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Integration over more general intervals gives

$$\int_{a}^{b} \delta(x)\, dx = \begin{cases} 1, & 0 \in [a,b], \\ 0, & 0 \notin [a,b]. \end{cases} \tag{5.25}$$

The other property that was used was the sifting property:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx &= \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx \\ &= f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a). \qquad (5.26)\end{aligned}$$

Another property results from using a scaled argument, ax. In this case, we show that

$$\delta(ax) = |a|^{-1} \delta(x). \qquad (5.27)$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(ax)\, dx &= \lim_{L\to\infty} \int_{-L}^{L} \delta(ax)\, dx \\ &= \lim_{L\to\infty} \frac{1}{a} \int_{-aL}^{aL} \delta(y)\, dy. \qquad (5.28)\end{aligned}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{\infty}^{-\infty} \delta(y)\, dy = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\boxed{\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy. \qquad (5.29)}$$

Example 5.1. *Evaluate* $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx$. *This is a straight-forward integration:*

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{11}{4}.$$

The first step is to write $\delta(4(x-2)) = \frac{1}{4}\delta(x-2)$. *Then, the final evaluation is given by*

$$\frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{1}{4}(5(2)+1) = \frac{11}{4}.$$

Even more general than $\delta(ax)$ is the delta function $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated, depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|} \delta(x - x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|} \delta(x - x_1),$$

again keeping in mind that this only has meaning when placed under an integral.

Properties of the Dirac delta function:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \frac{\delta(x-x_j)}{|f'(x_j)|}\, dx.$$

(For n simple roots.)

These and other properties are often written outside the integral:

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

$$\delta(-x) = \delta(x).$$

$$\delta((x-a)(x-b)) = \frac{[\delta(x-a) + \delta(x-a)]}{|a-b|}.$$

$$\delta(f(x)) = \sum_j \frac{\delta(x-x_j)}{|f'(x_j)|},$$

for $f(x_j) = 0, f'(x_j) \neq 0$.

Example 5.2. *Evaluate* $\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx$.

This is not a simple $\delta(x-a)$. *So, we need to find the zeros of* $f(x) = 3x-2$. *There is only one,* $x = \frac{2}{3}$. *Also,* $|f'(x)| = 3$. *Therefore, we have*

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(x-\frac{2}{3})x^2\,dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way using the substitution $y = 3x-2$. *Then,* $dy = 3\,dx$ *and* $x = (y+2)/3$. *This gives*

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx = \frac{1}{3}\int_{-\infty}^{\infty} \delta(y)\left(\frac{y+2}{3}\right)^2 dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for $j = 1, 2, \ldots, n$, (i.e., when one has n simple zeros), then

$$\boxed{\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|}\delta(x-x_j).}$$

Example 5.3. *Evaluate* $\int_0^{2\pi} \cos x\,\delta(x^2-\pi^2)\,dx$.

In this case, the argument of the delta function has two simple roots. Namely, $f(x) = x^2-\pi^2 = 0$ *when* $x = \pm\pi$. *Furthermore,* $f'(x) = 2x$. *Therefore,* $|f'(\pm\pi)| = 2\pi$. *This gives*

$$\delta(x^2-\pi^2) = \frac{1}{2\pi}[\delta(x-\pi)+\delta(x+\pi)].$$

Inserting this expression into the integral and noting that $x = -\pi$ *is not in the integration interval, we have*

$$\begin{aligned} \int_0^{2\pi} \cos x\,\delta(x^2-\pi^2)\,dx &= \frac{1}{2\pi}\int_0^{2\pi} \cos x\,[\delta(x-\pi)+\delta(x+\pi)]\,dx \\ &= \frac{1}{2\pi}\cos\pi = -\frac{1}{2\pi}. \end{aligned} \tag{5.30}$$

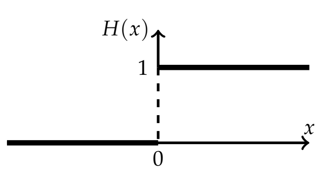

Figure 5.5: The Heaviside step function, $H(x)$.

Example 5.4. *Show* $H'(x) = \delta(x)$, *where the Heaviside function (or, step function) is defined as*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and is shown in Figure 5.5.

Looking at the plot, it is easy to see that $H'(x) = 0$ *for* $x \neq 0$. *In order to check that this gives the delta function, we need to compute the area integral. Therefore, we have*

$$\int_{-\infty}^{\infty} H'(x)\,dx = H(x)\Big|_{-\infty}^{\infty} = 1 - 0 = 1.$$

Thus, $H'(x)$ *satisfies the two properties of the Dirac delta function.*

5.5 Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications, functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt, \tag{5.31}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx. \tag{5.32}$$

ω is called the angular frequency and is related to the frequency ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the wavenumber. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

We explore a few basic properties of the Fourier transform and use them in examples in the next section.

1. **Linearity:** For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

   $$F[f+g] = F[f] + F[g]$$

   and

   $$F[af] = aF[f].$$

   These simply follow from the properties of integration and establish the linearity of the Fourier transform.

2. **Transform of a Derivative:** $F\left[\dfrac{df}{dx}\right] = -ik\hat{f}(k)$

   Here we compute the Fourier transform (5.17) of the derivative by inserting the derivative in the Fourier integral and using integration by parts:

   $$\begin{aligned} F\left[\frac{df}{dx}\right] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx \\ &= \lim_{L\to\infty} \left[f(x)e^{ikx}\right]_{-L}^{L} - ik\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx. \end{aligned} \tag{5.33}$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. This last integral is recognized as the Fourier transform of f, proving the given property.

3. **Higher Order Derivatives:** $F\left[\frac{d^n f}{dx^n}\right] = (-ik)^n \hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx} f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2 \hat{f}(k). \end{aligned} \tag{5.34}$$

   This result will be true if

$$\lim_{x\to\pm\infty} f(x) = 0 \text{ and } \lim_{x\to\pm\infty} f'(x) = 0.$$

   The generalization to the transform of the nth derivative easily follows.

4. **Multiplication by x:** $F[xf(x)] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and the ability to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)\,dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \tag{5.35}$$

   This result can be generalized to $F[x^n f(x)]$ as an exercise.

5. **Shifting Properties:** For constant a, we have the following shifting properties:

These are the first and second shifting properties, or First and Second Shift Theorems.

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{5.36}$$

$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{5.37}$$

   Here we have denoted the Fourier transform pairs using a double arrow as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proved by inserting the desired forms into the definition of the Fourier transform (5.17), or inverse Fourier transform (5.18). The first shift property (5.36) is shown by the following argument. We evaluate the Fourier transform:

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

   Now perform the substitution $y = x - a$. Then,

$$\begin{aligned} F[f(x-a)] &= \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy \\ &= e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy \\ &= e^{ika}\hat{f}(k). \end{aligned} \tag{5.38}$$

The second shift property (5.37) follows in a similar way.

6. **Convolution of Functions:** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dx. \tag{5.39}$$

Then, the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions:

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{5.40}$$

We will return to the proof of this property in Section 5.6.

5.5.1 *Fourier Transform Examples*

In this section we will compute the Fourier transforms of several functions.

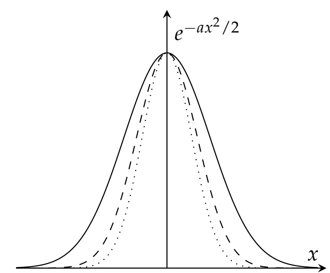

Figure 5.6: Plots of the Gaussian function $f(x) = e^{-ax^2/2}$ for $a = 1, 2, 3$.

Example 5.5. *Find the Fourier transform of a Gaussian, $f(x) = e^{-ax^2/2}$.*

This function, shown in Figure 5.6, is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability, and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation, we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{5.41}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\,dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned}
-\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\
&= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\
&= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}.
\end{aligned} \tag{5.42}$$

We now put this expression into the integral and make the substitutions $y = x - \frac{ik}{a}$ and $\beta = \frac{a}{2}$.

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2}\,dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2}\,dy.
\end{aligned} \tag{5.43}$$

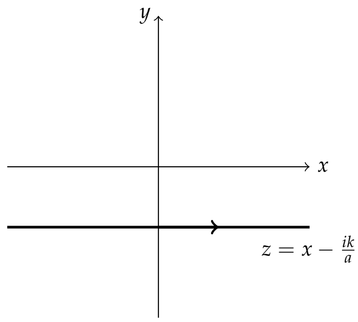

Figure 5.7: Simple horizontal contour.

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane as shown in Figure 5.7.

In this case, we can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we now safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Define I by[1]

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Note that we needed to change the integration variable so that we can write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2}\, r dr d\theta \\ &= 2\pi \int_0^{\infty} e^{-\beta r^2}\, r dr \\ &= -\frac{\pi}{\beta}\left[e^{-\beta r^2}\right]_0^{\infty} = \frac{\pi}{\beta}. \end{aligned} \tag{5.44}$$

The final result is obtained by taking the square root, yielding

$$I = \sqrt{\frac{\pi}{\beta}}.$$

[1] Here we show

$$\int_{-\infty}^{\infty} e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

Note that we solved the $\beta = 1$ case in Example 3.14, so a simple variable transformation $z = \sqrt{\beta} y$ is all that is needed to get the answer. However, it cannot hurt to see this classic derivation again.

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{5.45}$$

The Fourier transform of a Gaussian is a Gaussian.

Therefore, we have shown that the Fourier transform of a Gaussian is a Gaussian.

Example 5.6. *Find the Fourier transform of the box, or gate, function,*

$$f(x) = \begin{cases} b, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

This function is called the box function, or gate function. It is shown in Figure 5.8. The Fourier transform of the box function is relatively easy to compute. It is given by

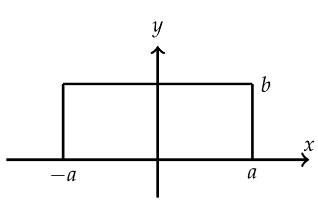

Figure 5.8: A plot of the box function in Example 5.6.

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx$$

$$\begin{aligned} &= \int_{-a}^{a} b e^{ikx}\, dx \\ &= \frac{b}{ik} e^{ikx}\Big|_{-a}^{a} \\ &= \frac{2b}{k}\sin ka. \end{aligned} \tag{5.46}$$

We can rewrite this as

$$\hat{f}(k) = 2ab\frac{\sin ka}{ka} \equiv 2ab \operatorname{sinc} ka.$$

Here we introduced the sinc function,

$$\operatorname{sinc} x = \frac{\sin x}{x}.$$

A plot of this function is shown in Figure 5.9. This function appears often in signal analysis and it plays a role in the study of diffraction.

We will now consider special limiting values for the box function and its transform. This will lead us to the Uncertainty Principle for signals, connecting the relationship between the localization properties of a signal and its transform.

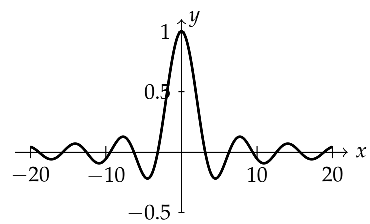

Figure 5.9: A plot of the Fourier transform of the box function in Example 5.6. This is the general shape of the sinc function.

1. *$a \to \infty$ and b fixed.*

   In this case, as a gets large, the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 5.9 with Figure 5.2. In fact, $\hat{f}(k) = bD_a(k)$. [Recall the definition of $D_\Omega(x)$ in Equation (5.22).] So, in the limit, we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit implies the fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes wider, the Fourier transform becomes more localized. In fact, we have arrived at the important result that

$$\boxed{\int_{-\infty}^{\infty} e^{ikx}\, dx = 2\pi\delta(k).} \tag{5.47}$$

$$\int_{-\infty}^{\infty} e^{ikx}\, dx = 2\pi\delta(k).$$

2. *$b \to \infty$, $a \to 0$, and $2ab = 1$.*

   In this case, the box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

$$\int_{-\infty}^{\infty} \delta(x) e^{ikx}\, dx = 1. \tag{5.48}$$

   In this case, we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform, $\hat{f}(k)$, is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.

3. *The Uncertainty Principle:* $\Delta x \Delta k = 4\pi$.

The widths of the box function and its Fourier transform are related, as we have seen in the last two limiting cases. It is natural to define the width, Δx, of the box function as

$$\Delta x = 2a.$$

The width of the Fourier transform is a little trickier. This function actually extends along the entire k-axis. However, as $\hat{f}(k)$ became more localized, the central peak in Figure 5.9 became narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe as shown in Figure 5.10. This gives

$$\Delta k = \frac{2\pi}{a}.$$

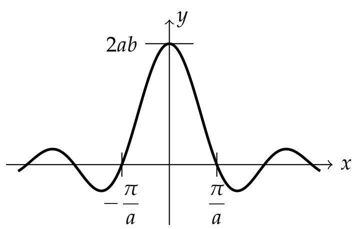

Figure 5.10: The width of the function $2ab\frac{\sin ka}{ka}$ is defined as the distance between the smallest magnitude zeros.

Combining these two relations, we find that

$$\Delta x \Delta k = 4\pi.$$

Thus, the more localized a signal, the less localized its transform and vice versa. This notion is referred to as the Uncertainty Principle. For general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq c > 0.$$

More formally, the Uncertainty Principle for signals is about the relation between duration and bandwidth, which are defined by $\Delta t = \frac{\|tf\|_2}{\|f\|_2}$ and $\Delta\omega = \frac{\|\omega\hat{f}\|_2}{\|\hat{f}\|_2}$, respectively, where $\|f\|_2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt$ and $\|\hat{f}\|_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$. Under appropriate conditions, one can prove that $\Delta t \Delta\omega \geq \frac{1}{2}$. Equality holds for Gaussian signals. Werner Heisenberg (1901 - 1976) introduced the Uncertainty Principle into quantum physics in 1926, relating uncertainties in the position (Δx) and momentum (Δp_x) of particles. In this case, $\Delta x \Delta p_x \geq \frac{1}{2}\hbar$. Here, the uncertainties are defined as the positive square roots of the quantum mechanical variances of the position and momentum.

We now turn to other examples of Fourier transforms.

Example 5.7. *Find the Fourier transform of* $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $a > 0$.

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\ &= \int_{0}^{\infty} e^{ikx-ax}\, dx \\ &= \frac{1}{a-ik}. \end{aligned} \tag{5.49}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 5.8. *Find the inverse Fourier transform of* $\hat{f}(k) = \frac{1}{a-ik}$.

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\, dk.$$

This integral can be evaluated using contour integral methods. We evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz}\, dz,$$

using Jordan's Lemma from Section 4.4.8. According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$, as shown in Figure 5.11.

The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk \\ &= \pm\frac{1}{2\pi}\oint_C \frac{e^{-ixz}}{a-iz}\,dz \\ &= \begin{cases} 0, & x<0 \\ -\frac{1}{2\pi}2\pi i \;Res\;[z=-ia], & x>0 \end{cases} \\ &= \begin{cases} 0, & x<0 \\ e^{-ax}, & x>0 \end{cases}. \end{aligned} \tag{5.50}$$

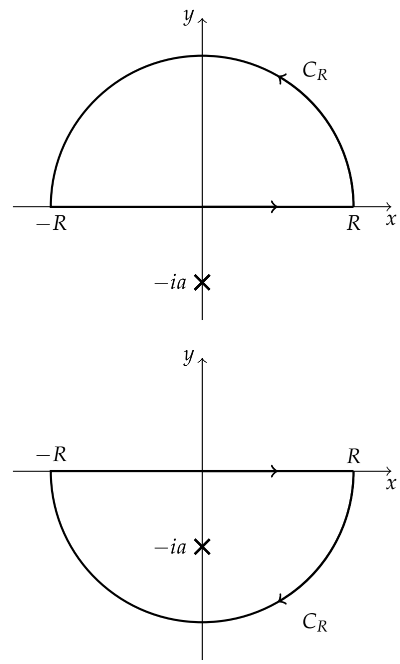

Figure 5.11: Contours for inverting $\hat{f}(k) = \frac{1}{a-ik}$.

Note that without paying careful attention to Jordan's Lemma, one might not retrieve the function from the last example.

Example 5.9. *Find the inverse Fourier transform of $\hat{f}(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$.*

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega-\omega_0)$. This is a shifted function. From the shift theorems in Equations (5.36) and (5.37) we have the Fourier transform pair

$$e^{i\omega_0 t} f(t) \leftrightarrow \hat{f}(\omega-\omega_0).$$

Recalling from Example 5.6 that

$$\int_{-\infty}^{\infty} e^{i\omega t}\,dt = 2\pi\delta(\omega),$$

we have from the shift property that

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}e^{-i\omega_0 t}.$$

The second term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0] = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 5.10. *Find the Fourier transform of the finite wave train.*

$$f(t) = \begin{cases} \cos\omega_0 t, & |t| \le a, \\ 0, & |t| > a. \end{cases}$$

For the last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. In Figure 5.12 we show a plot of this function.

A straight-forward computation gives

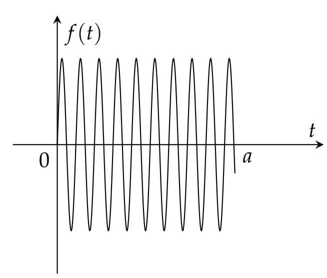

Figure 5.12: A plot of the finite wave train.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$$

$$\begin{aligned}
&= \int_{-a}^{a} [\cos\omega_0 t + i\sin\omega_0 t] e^{i\omega t}\, dt \\
&= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\, dt + i\int_{-a}^{a} \sin\omega_0 t \sin\omega t\, dt \\
&= \frac{1}{2}\int_{-a}^{a} \left[\cos((\omega+\omega_0)t) + \cos((\omega-\omega_0)t)\right]\, dt \\
&= \frac{\sin((\omega+\omega_0)a)}{\omega+\omega_0} + \frac{\sin((\omega-\omega_0)a)}{\omega-\omega_0}. \qquad (5.51)
\end{aligned}$$

5.6 The Convolution Operation

In the list of properties of the Fourier transform, we defined the convolution of two functions, $f(x)$ and $g(x)$, to be the integral

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt. \qquad (5.52)$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is *faltung*, which means "folding" and in old texts this is referred to as the Faltung Theorem. In this section we will look into the convolution operation and its Fourier transform.

Before we get too involved with the convolution operation, it should be noted that there are really two things you need to take away from this discussion. The rest is detail. First, the convolution of two functions is a new functions as defined by Equation (5.52) when dealing with the Fourier transform. The second and most relevant is that the Fourier transform of the convolution of two functions is the product of the transforms of each function. The rest is all about the use and consequences of these two statements. In this section we will show how the convolution works and how it is useful.

The convolution is commutative.

First, we note that the convolution is commutative: $f*g = g*f$. This is easily shown by replacing $x-t$ with a new variable, $y = x-t$ and $dy = -dt$.

$$\begin{aligned}
(g*f)(x) &= \int_{-\infty}^{\infty} g(t)f(x-t)\, dt \\
&= -\int_{\infty}^{-\infty} g(x-y)f(y)\, dy \\
&= \int_{-\infty}^{\infty} f(y)g(x-y)\, dy \\
&= (f*g)(x). \qquad (5.53)
\end{aligned}$$

The best way to understand the folding of the functions in the convolution is to take two functions and convolve them. The next example gives a graphical rendition followed by a direct computation of the convolution. The reader is encouraged to carry out these analyses for other functions.

Example 5.11. *Graphical convolution of the box function and a triangle function.*

In order to understand the convolution operation, we need to apply it to specific *functions. We will first do this graphically for the box function*

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \textit{otherwise}, \end{cases}$$

as shown in Figure 5.13.

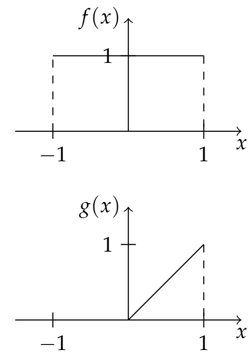

Figure 5.13: A plot of the box function $f(x)$ and the triangle function $g(x)$.

Next, we determine the contributions to the integrand. We consider the shifted and reflected function $g(t-x)$ in Equation (5.52) for various values of t. For $t=0$, we have $g(x-0) = g(-x)$. This function is a reflection of the triangle function, $g(x)$, as shown in Figure 5.14.

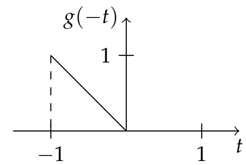

Figure 5.14: A plot of the reflected triangle function, $g(-t)$.

We then translate the triangle function performing horizontal shifts by t. In Figure 5.15 we show such a shifted and reflected $g(x)$ for $t=2$, or $g(2-x)$.

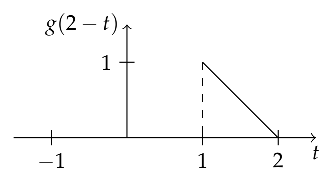

Figure 5.15: A plot of the reflected triangle function shifted by two units, $g(2-t)$.

In Figure 5.15 we show several plots of other shifts, $g(x-t)$, superimposed on $f(x)$.

The integrand is the product of $f(t)$ and $g(x-t)$ and the integral of the product $f(t)g(x-t)$ is given by the sum of the shaded areas for each value of x.

In the first plot of Figure 5.16, the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in the other plots in Figure 5.16. The value of the convolution at x is shown by the area under the product of the two functions for each value of x.

Plots of the areas of the convolution of the box and triangle functions for several values of x are given in Figure 5.15. We see that the value of the convolution integral builds up and then quickly drops to zero as a function of x. In Figure 5.17 the values of these areas is shown as a function of x.

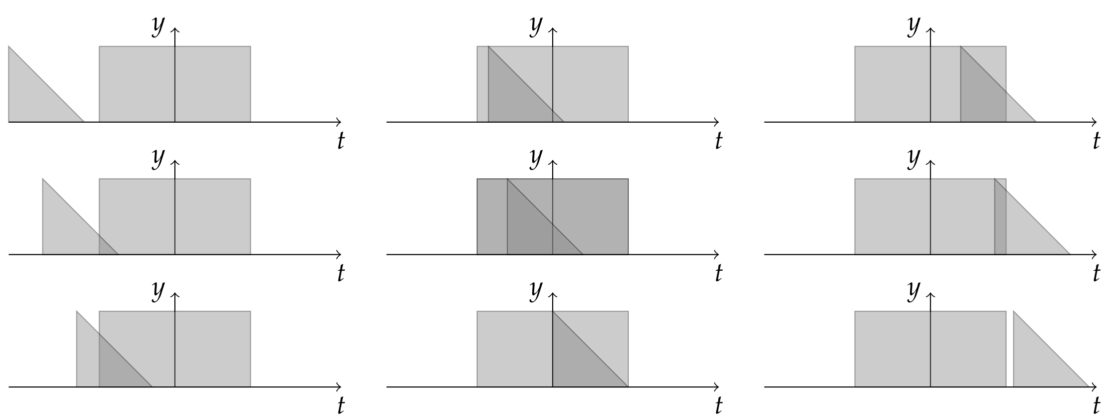

Figure 5.16: A plot of the box and triangle functions with the overlap indicated by the shaded area.

The plot of the convolution in Figure 5.17 is not easily determined using the graphical method. However, we can directly compute the convolution as shown in the next example.

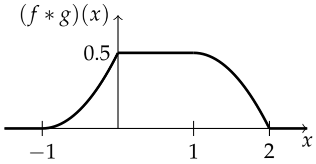

Figure 5.17: A plot of the convolution of the box and triangle functions.

Example 5.12. *Analytically find the convolution of the box function and the triangle function.*

The nonvanishing contributions to the convolution integral are when both $f(t)$ and $g(x-t)$ do not vanish. $f(t)$ is nonzero for $|t| \leq 1$, or $-1 \leq t \leq 1$. $g(x-t)$ is nonzero for $0 \leq x-t \leq 1$, or $x-1 \leq t \leq x$. These two regions are shown in Figure 5.18. On this region, $f(t)g(x-t) = x-t$.

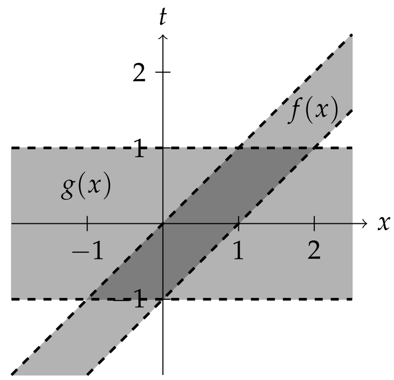

Figure 5.18: Intersection of the support of $g(x)$ and $f(x)$.

Isolating the intersection in Figure 5.19, we see in Figure 5.19 that there are three regions as shown by different shadings. These regions lead to a piecewise defined function with three different branches of nonzero values for $-1 < x < 0$, $0 < x < 1$, and $1 < x < 2$.

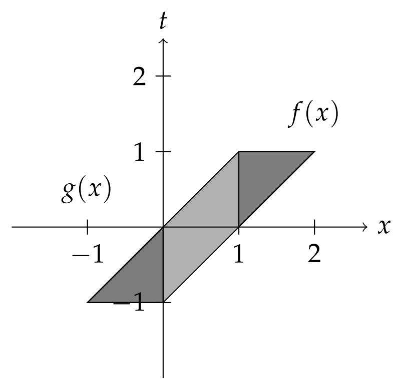

Figure 5.19: Intersection of the support of $g(x)$ and $f(x)$ showing the integration regions.

The values of the convolution can be determined through careful integration. The resulting integrals are given as

$$
\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \\
&= \left\{ \begin{array}{ll} \int_{-1}^{x}(x-t)\,dt, & -1 < x < 0, \\ \int_{x-1}^{x}(x-t)\,dt, & 0 < x < 1, \\ \int_{x-1}^{1}(x-t)\,dt, & 1 < x < 2 \end{array} \right. \\
&= \left\{ \begin{array}{ll} \frac{1}{2}(x+1)^2, & -1 < x < 0, \\ \frac{1}{2}, & 0 < x < 1, \\ \frac{1}{2}\left[1-(x-1)^2\right] & 1 < x < 2. \end{array} \right.
\end{aligned} \tag{5.54}
$$

A plot of this function is shown in Figure 5.17.

5.6.1 Convolution Theorem for Fourier Transforms

IN THIS SECTION WE COMPUTE the Fourier transform of the convolution integral and show that the Fourier transform of the convolution is the product of the transforms of each function,

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{5.55}$$

First, we use the definitions of the Fourier transform and the convolution to write the transform as

$$\begin{aligned} F[f * g] &= \int_{-\infty}^{\infty} (f * g)(x) e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) g(x-t)\, dt \right) e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt. \end{aligned} \tag{5.56}$$

We now substitute $y = x - t$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f * g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{ik(y+t)}\, dy \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right) f(t) e^{ikt}\, dt \\ &= \left(\int_{-\infty}^{\infty} f(t) e^{ikt}\, dt \right) \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right). \end{aligned} \tag{5.57}$$

We see that the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f * g] = \hat{f}(k)\hat{g}(k).$$

Example 5.13. *Compute the convolution of the box function of height one and width two with itself.*

Let $\hat{f}(k)$ be the Fourier transform of $f(x)$. Then, the Convolution Theorem says that $F[f * f](k) = \hat{f}^2(k)$, or

$$(f * f)(x) = F^{-1}[\hat{f}^2(k)].$$

For the box function, we have already found that

$$\hat{f}(k) = \frac{2}{k} \sin k.$$

So, we need to compute

$$\begin{aligned} (f * f)(x) &= F^{-1}[\frac{4}{k^2} \sin^2 k] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{4}{k^2} \sin^2 k \right) e^{-ikx}\, dk. \end{aligned} \tag{5.58}$$

One way to compute this integral is to extend the computation into the complex k-plane. We first need to rewrite the integrand. Thus,

$$\begin{aligned}
(f*f)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{k^2}\sin^2 k e^{-ikx}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}[1-\cos 2k]e^{-ikx}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}\left[1-\frac{1}{2}(e^{ik}+e^{-ik})\right]e^{-ikx}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2}\left[e^{-ikx}-\frac{1}{2}(e^{-i(1-k)}+e^{-i(1+k)})\right]dk. \quad (5.59)
\end{aligned}$$

We can compute the above integrals if we know how to compute the integral

$$I(y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-iky}}{k^2}\,dk.$$

Then, the result can be found in terms of $I(y)$ as

$$(f*f)(x)=I(x)-\frac{1}{2}[I(1-k)+I(1+k)].$$

We consider the integral

$$\oint_C \frac{e^{-iyz}}{\pi z^2}\,dz$$

over the contour in Figure 5.20.

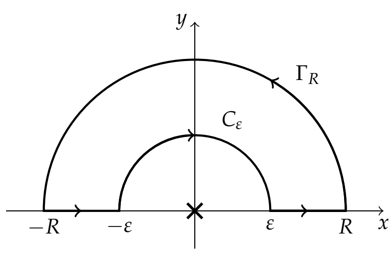

Figure 5.20: Contour for computing $P\int_{-\infty}^{\infty}\frac{e^{-iyz}}{\pi z^2}\,dz$.

We can see that there is a double pole at $z=0$. The pole is on the real axis. So, we will need to cut out the pole as we seek the value of the principal value integral.

Recall from Chapter 4 that

$$\oint_{C_R}\frac{e^{-iyz}}{\pi z^2}\,dz=\int_{\Gamma_R}\frac{e^{-iyz}}{\pi z^2}\,dz+\int_{-R}^{-\epsilon}\frac{e^{-iyz}}{\pi z^2}\,dz+\int_{C_\epsilon}\frac{e^{-iyz}}{\pi z^2}\,dz+\int_{\epsilon}^{R}\frac{e^{-iyz}}{\pi z^2}\,dz.$$

The integral $\oint_{C_R}\frac{e^{-iyz}}{\pi z^2}\,dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon\to 0$ and $R\to\infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma provided $y<0$. That leaves the integral over the small semicircle.

As before, we can show that

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,dz=-\pi i\,\mathrm{Res}[f(z);z=0].$$

Therefore, we find

$$I(y)=P\int_{-\infty}^{\infty}\frac{e^{-iyz}}{\pi z^2}\,dz=\pi i\,\mathrm{Res}\left[\frac{e^{-iyz}}{\pi z^2};z=0\right].$$

A simple computation of the residue gives $I(y)=-y$, for $y<0$.

When $y>0$, we need to close the contour in the lower half plane in order to apply Jordan's Lemma. Carrying out the computation, one finds $I(y)=y$, for $y>0$. Thus,

$$I(y)=\begin{cases}-y, & y>0,\\ y, & y<0,\end{cases} \quad (5.60)$$

We are now ready to finish the computation of the convolution. We have to combine the integrals $I(y)$, $I(y+1)$, *and* $I(y-1)$, *since* $(f*f)(x) = I(x) - \frac{1}{2}[I(1-k) + I(1+k)]$. *This gives different results in four intervals:*

$$\begin{aligned}(f*f)(x) &= x - \frac{1}{2}[(x-2)+(x+2)] = 0, \quad x<-2,\\ &= x - \frac{1}{2}[(x-2)-(x+2)] = 2+x \quad -2<x<0,\\ &= -x - \frac{1}{2}[(x-2)-(x+2)] = 2-x, \quad 0<x<2,\\ &= -x - \frac{1}{2}[-(x-2)-(x+2)] = 0, \quad x>2. \end{aligned} \tag{5.61}$$

A plot of this solution is the triangle function:

$$(f*f)(x) = \begin{cases} 0, & x<-2 \\ 2+x, & -2<x<0 \\ 2-x, & 0<x<2 \\ 0, & x>2, \end{cases} \tag{5.62}$$

which was shown in the last example.

Example 5.14. *Find the convolution of the box function of height one and width two with itself using a direct computation of the convolution integral.*

The nonvanishing contributions to the convolution integral are when both $f(t)$ *and* $f(x-t)$ *do not vanish.* $f(t)$ *is nonzero for* $|t| \le 1$, *or* $-1 \le t \le 1$. $f(x-t)$ *is nonzero for* $|x-t| \le 1$, *or* $x-1 \le t \le x+1$. *These two regions are shown in Figure 5.22. On this region,* $f(t)g(x-t) = 1$.

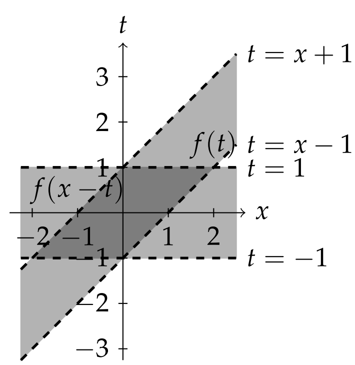

Figure 5.21: Plot of the regions of support for $f(t)$ and $f(x-t)$..

Thus, the nonzero contributions to the convolution are

$$(f*f)(x) = \begin{cases} \int_{-1}^{x+1} dt, & 0 \le x \le 2, \\ \int_{x-1}^{1} dt, & -2 \le x \le 0, \end{cases} = \begin{cases} 2+x, & 0 \le x \le 2, \\ 2-x, & -2 \le x \le 0. \end{cases}$$

Once again, we arrive at the triangle function.

In the last section we showed the graphical convolution. For completeness, we do the same for this example. In Figure 5.22 we show the results. We see that the convolution of two box functions is a triangle function.

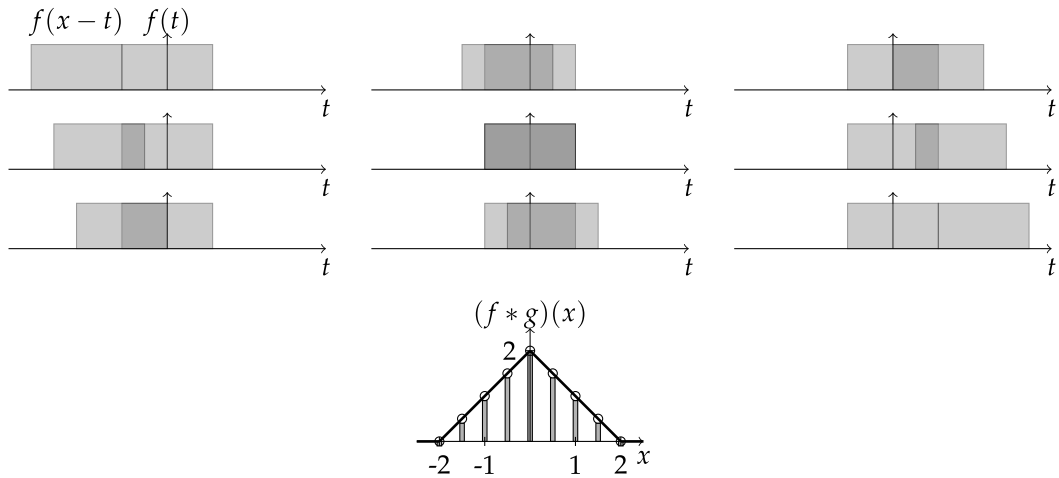

Figure 5.22: A plot of the convolution of a box function with itself. The areas of the overlaps of as $f(x-t)$ is translated across $f(t)$ are shown as well. The result is the triangular function.

Example 5.15. *Show the graphical convolution of the box function of height one and width two with itself.*

Let's consider a slightly more complicated example, the convolution of two Gaussian functions.

Example 5.16. *Convolution of two Gaussian functions* $f(x) = e^{-ax^2}$.

In this example we will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ *and* $g(x) = e^{-bx^2}$. *A direct evaluation of the integral would be to compute*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} e^{-at^2-b(x-t)^2}\,dt.$$

This integral can be rewritten as

$$(f*g)(x) = e^{-bx^2}\int_{-\infty}^{\infty} e^{-(a+b)t^2+2bxt}\,dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution and leave the evaluation of this integral to Problem 12.

Recalling the Fourier transform of a Gaussian from Example 5.5, we have

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}e^{-k^2/4a} \tag{5.63}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{\pi}{b}}e^{-k^2/4b}.$$

Denoting the convolution function by $h(x) = (f*g)(x)$, *the Convolution Theorem gives*

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{\pi}{\sqrt{ab}}e^{-k^2/4a}e^{-k^2/4b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{\pi}{\sqrt{ab}} e^{-\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{\pi}{\sqrt{ab}} e^{-\frac{a+b}{4ab}k^2}. \tag{5.64}$$

In order to complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (5.64). We can do this by looking at Equation (5.63). We have first that

$$F^{-1}\left[\sqrt{\frac{\pi}{a}} e^{-k^2/4a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/4a}] = \sqrt{\frac{a}{\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{4a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/4\alpha}.$$

This is in the form needed to invert Equation (5.64). Thus, for $\alpha = \frac{a+b}{4ab}$, we find

$$(f * g)(x) = h(x) = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

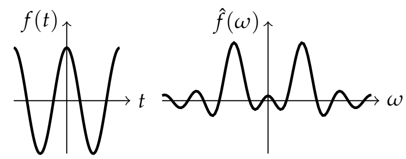

Figure 5.23: Schematic plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

5.6.2 Application to Signal Analysis

There are many applications of the convolution operation. One of these areas is the study of analog signals. An analog signal is a continuous signal and may contain either a finite or continuous set of frequencies. Fourier transforms can be used to represent such signals as a sum over the frequency content of these signals. In this section we will describe how convolutions can be used in studying signal analysis.

The first application is filtering. For a given signal, there might be some noise in the signal, or some undesirable high frequencies. For example, a device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal such as the example provided in Figure 5.23. Recall that the Fourier transform gives the frequency content of the signal.

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all the high (angular) frequencies. For example, for some cutoff frequency ω_0, frequencies $|\omega| > \omega_0$ will be removed. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could use the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \le \omega_0, \\ 0, & |\omega| > \omega_0 \end{cases} \tag{5.65}$$

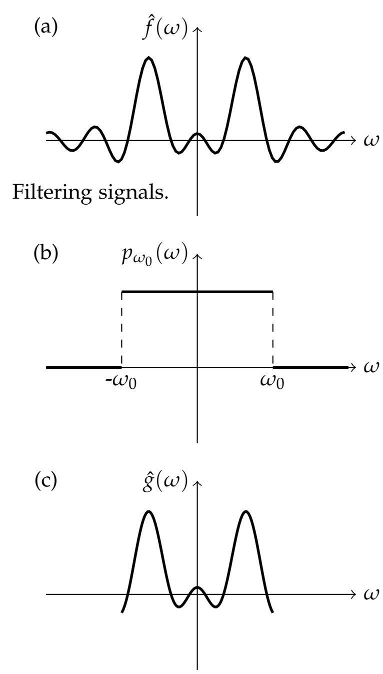

Figure 5.24: (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b) shows how the filters cuts out high frequencies, $|\omega| > \omega_0$.

as shown in Figure 5.24.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{5.66}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device, which in turn produces the output, $g(t)$. The function $h(t)$ is called the impulse response. This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\,d\tau = h(t).$$

Windowing signals.

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead, data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

Example 5.17. Finite Wave Train, Revisited.

We return to the finite wave train in Example 5.10 given by

$$h(t) = \begin{cases} \cos\omega_0 t, & |t| \le a, \\ 0, & |t| > a. \end{cases}$$

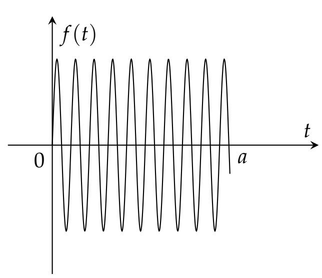

Figure 5.25: A plot of the finite wave train.

We can view this as a windowed version of $f(t) = \cos\omega_0 t$ obtained by multiplying $f(t)$ by the gate function

$$g_a(t) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases} \tag{5.67}$$

This is shown in Figure 5.25. Then, the Fourier transform is given as a convolution,

The convolution in spectral space is defined with an extra factor of $1/2\pi$ so as to preserve the idea that the inverse Fourier transform of a convolution is the product of the corresponding signals.

$$\begin{aligned} \hat{h}(\omega) &= (\hat{f} * \hat{g}_a)(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\,d\nu. \end{aligned} \tag{5.68}$$

Note that the convolution in frequency space requires the extra factor of $1/(2\pi)$.

We need the Fourier transforms of f and g_a in order to finish the computation. The Fourier transform of the box function was found in Example 5.6 as

$$\hat{g}_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

The Fourier transform of the cosine function, $f(t) = \cos\omega_0 t$, is

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{i\omega t}\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right)\, dt \\
&= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]. \qquad (5.69)
\end{aligned}$$

Note that we had earlier computed the inverse Fourier transform of this function in Example 5.9.

Inserting these results in the convolution integral, we have

$$\begin{aligned}
\hat{h}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\, d\nu \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\omega-\nu+\omega_0) + \delta(\omega-\nu-\omega_0)\right] \frac{2}{\nu}\sin \nu a\, d\nu \\
&= \frac{\sin(\omega+\omega_0)a}{\omega+\omega_0} + \frac{\sin(\omega-\omega_0)a}{\omega-\omega_0}. \qquad (5.70)
\end{aligned}$$

This is the same result we had obtained in Example 5.10.

5.6.3 *Parseval's Equality*

The integral/sum of the (modulus) square of a function is the integral/sum of the (modulus) square of the transform.

As another example of the convolution theorem, we derive Parseval's Equality (named after Marc-Antoine Parseval (1755 - 1836)):

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega. \qquad (5.71)$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's Equality, is simply a statement that the energy is invariant under the Fourier transform. Parseval's Equality is a special case of Plancherel's Formula (named after Michel Plancherel, 1885 - 1967).

Let's rewrite the Convolution Theorem in its inverse form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f * g)(t). \qquad (5.72)$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty} f(t-u)g(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}\, d\omega.$$

Setting $t = 0$,

$$\int_{-\infty}^{\infty} f(-u)g(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)\, d\omega. \qquad (5.73)$$

Now, let $g(t) = \overline{f(-t)}$, or $f(-t) = \overline{g(t)}$. We note that the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned} \hat{g}(\omega) &= \int_{-\infty}^{\infty} \overline{f(-t)} e^{i\omega t}\, dt \\ &= -\int_{\infty}^{-\infty} \overline{f(\tau)} e^{-i\omega\tau}\, d\tau \\ &= \overline{\int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau}\, d\tau} = \overline{\hat{f}(\omega)}. \end{aligned} \tag{5.74}$$

So, inserting this result into Equation (5.73), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega,$$

which yields Parseval's Equality in the form in Equation (5.71) after substituting $t = -u$ on the left.

As noted above, the forms in Equations (5.71) and (5.73) are often referred to as the Plancherel Formula or Parseval Formula. A more commonly defined Parseval equation is that given for Fourier series. For example, for a function $f(x)$ defined on $[-\pi, \pi]$, which has a Fourier series representation, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2\, dx.$$

In general, there is a Parseval identity for functions that can be expanded in a complete sets of orthonormal functions, $\{\phi_n(x)\}$, $n = 1, 2, \ldots$, which is given by

$$\sum_{n=1}^{\infty} < f, \phi_n >^2 = \|f\|^2.$$

Here, $\|f\|^2 =< f, f >$. The Fourier series example is just a special case of this formula.

5.7 *The Laplace Transform*

The Laplace transform is named after Pierre-Simon de Laplace (1749 - 1827). Laplace made major contributions, especially to celestial mechanics, tidal analysis, and probability.

Up to this point we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving initial value problems for ordinary differential equations.

Integral transform on $[a, b]$ with respect to the integral kernel, $K(x, k)$.

The Fourier and Laplace transforms are examples of a broader class of transforms known as integral transforms. For a function $f(x)$ defined on an interval (a, b), we define the integral transform

$$F(k) = \int_a^b K(x, k) f(x)\, dx,$$

where $K(x,k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x,k) = e^{ikx}$. In Table 5.1 we show several types of integral transforms.

Laplace Transform	$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$
Fourier Transform	$F(k) = \int_{-\infty}^\infty e^{ikx} f(x)\, dx$
Fourier Cosine Transform	$F(k) = \int_0^\infty \cos(kx) f(x)\, dx$
Fourier Sine Transform	$F(k) = \int_0^\infty \sin(kx) f(x)\, dx$
Mellin Transform	$F(k) = \int_0^\infty x^{k-1} f(x)\, dx$
Hankel Transform	$F(k) = \int_0^\infty x J_n(kx) f(x)\, dx$

Table 5.1: A Table of Common Integral Transforms.

It should be noted that these integral transforms inherit the linearity of integration. Namely, let $h(x) = \alpha f(x) + \beta g(x)$, where α and β are constants. Then,

$$\begin{aligned} H(k) &= \int_a^b K(x,k) h(x)\, dx, \\ &= \int_a^b K(x,k)(\alpha f(x) + \beta g(x))\, dx, \\ &= \alpha \int_a^b K(x,k) f(x)\, dx + \beta \int_a^b K(x,k) g(x)\, dx, \\ &= \alpha F(x) + \beta G(x). \end{aligned} \tag{5.75}$$

Therefore, we have shown linearity of the integral transforms. We have seen the linearity property used for Fourier transforms and we will use linearity in the study of Laplace transforms.

The Laplace transform of f, $F = \mathcal{L}[f]$.

We now turn to Laplace transforms. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t) e^{-st}\, dt, \quad s > 0. \tag{5.76}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t) e^{-st} = 0$$

to guarantee convergence.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 5.26 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

$$\begin{tikzcd}
x(t) \arrow[r, dashed] \arrow[d, "\text{Laplace Transform}"'] & \boxed{h(t)} \arrow[r, dashed] \arrow[d] & y(t) = h(t) * x(t) \\
X(s) \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = H(s)X(s) \arrow[u, "\text{Inverse Laplace Transform}"']
\end{tikzcd}$$

Figure 5.26: A schematic depicting the use of Laplace transforms in systems theory.

5.7.1 Properties and Examples of Laplace Transforms

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 5.2. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 5.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these transforms can be used to sum infinite series and to solve initial value problems for ordinary differential equations.

Table 5.2: Table of Selected Laplace Transform Pairs.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$	$\delta(t-a)$	$e^{-as}, \quad a \geq 0, s > 0$

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 5.18. *Show that* $\mathcal{L}[1] = \frac{1}{s}$.

For this example, we insert $f(t) = 1$ *into the definition of the Laplace transform:*

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a *and then letting* $a \to \infty$. *We will not always write this limit, but it will be understood that this is how one computes such improper integrals.*

Proceeding with the computation, we have

$$\begin{aligned} \mathcal{L}[1] &= \int_0^\infty e^{-st}\,dt \\ &= \lim_{a\to\infty}\int_0^a e^{-st}\,dt \\ &= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-st}\right)_0^a \\ &= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-sa}+\frac{1}{s}\right)=\frac{1}{s}. \end{aligned} \tag{5.77}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant c, using the linearity of the transform, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 5.19. *Show that* $\mathcal{L}[e^{at}] = \frac{1}{s-a}$*, for* $s > a$.

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$\begin{aligned} \mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-st}\,dt \\ &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty \\ &= \lim_{t\to\infty}\frac{1}{a-s}e^{(a-s)t}-\frac{1}{a-s}=\frac{1}{s-a}. \end{aligned} \tag{5.78}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$*. This is only true if* $a - s < 0$*, or* $s > a$*. [Actually,* a *could be complex. In this case we would only need* s *to be greater than the real part of* a*,* $s > Re(a)$*.]*

Example 5.20. *Show that* $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$ *and* $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st}\cos at\,dt.$$

Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i\sin at$*. Making use of the linearity of the Laplace transform, we have*

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions!

The transform is simply computed as

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat}e^{-st}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case, $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts, we find the sought-after transforms,

$$\begin{aligned} \mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}, \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}. \end{aligned} \tag{5.79}$$

Example 5.21. *Show that* $\mathcal{L}[t] = \frac{1}{s^2}$.

For this example we evaluate

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

This integral can be evaluated using the method of integration by parts:

$$\begin{aligned} \int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}. \end{aligned} \tag{5.80}$$

Example 5.22. *Show that* $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ *for nonnegative integer* n.

We have seen the $n = 0$ *and* $n = 1$ *cases:* $\mathcal{L}[1] = \frac{1}{s}$ *and* $\mathcal{L}[t] = \frac{1}{s^2}$. *We now generalize these results to nonnegative integer powers,* $n > 1$, *of* t. *We consider the integral*

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

Following the previous example, we again integrate by parts:[2]

$$\begin{aligned} \int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt. \end{aligned} \tag{5.81}$$

[2] This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\,dt = \int_0^\infty t^n e^{-st}\,dt.$$

Since

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s},$$

$$\int_0^\infty t^n e^{-st}\,dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. *So, we can write the result as*

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

We compute $\int_0^\infty t^n e^{-st}\,dt$ by turning it into an initial value problem for a first-order difference equation and finding the solution using an iterative method.

This is an example of a recursive definition of a sequence. In this case, we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I_0 = \mathcal{L}[1] = \frac{1}{s}$, *we have the following:*

$$I_n = \frac{n}{s} I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{5.82}$$

This is also what is called a difference equation. It is a first-order difference equation with an "initial condition," I_0. *The next step is to solve this difference equation.*

Finding the solution of this first-order difference equation is easy to do using simple iteration. Note that replacing n *with* $n-1$, *we have*

$$I_{n-1} = \frac{n-1}{s} I_{n-2}.$$

Repeating the process, we find

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s} I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3}. \end{aligned} \tag{5.83}$$

We can repeat this process until we get to I_0, *which we know. We have to carefully count the number of iterations. We do this by iterating* k *times and then figure out how many steps will get us to the known initial value. A list of iterates is easily written out:*

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \ldots \\ &= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}. \end{aligned} \tag{5.84}$$

Since we know $I_0 = \frac{1}{s}$, *we choose to stop at* $k = n$ *obtaining*

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.

Such iterative techniques are useful in obtaining a variety of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$.

As a final note, one can extend this result to cases when n is not an integer. To do this, we use the Gamma function, which was discussed in Section 3.5. Recall that the Gamma function is the generalization of the factorial function and is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt. \tag{5.85}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x - 1$ an integer and $s = 1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform in Table 5.3. We have already discussed the first property, which is a consequence of the linearity of integral transforms. We will prove the other properties in this and the following sections.

Table 5.3: Table of selected Laplace transform properties.

Laplace Transform Properties
$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$
$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\, du] = F(s)G(s)$

Example 5.23. *Show that* $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$.

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st}\, dt.$$

We can move the derivative off f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have

$$\begin{aligned} \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt} e^{-st}\, dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt \\ &= -f(0) + sF(s). \end{aligned} \tag{5.86}$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

Example 6: *Show that* $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$.

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, *we have*

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s) - g(0) = sG(s) - f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

So,

$$\begin{aligned}\mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0)\\ &= s\left[sF(s) - f(0)\right] - f'(0)\\ &= s^2F(s) - sf(0) - f'(0). \qquad (5.87)\end{aligned}$$

We will return to the other properties in Table 5.3 after looking at a few applications.

5.8 Applications of Laplace Transforms

Although the Laplace transform is a very useful transform, it is often encountered only as a method for solving initial value problems in introductory differential equations. In this section we will show how to solve simple differential equations. Along the way we will introduce step and impulse functions and show how the Convolution Theorem for Laplace transforms plays a role in finding solutions. However, we will first explore an unrelated application of Laplace transforms. We will see that the Laplace transform is useful in finding sums of infinite series.

5.8.1 Series Summation Using Laplace Transforms

We saw in Chapter 2 that Fourier series can be used to sum series. For example, in Problem 2.13, one proves that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this section we will show how Laplace transforms can be used to sum series.[3] There is an interesting history of using integral transforms to sum series. For example, Richard Feynman[4] (1918 - 1988) described how one can use the Convolution Theorem for Laplace transforms to sum series with denominators that involved products. We will describe this and simpler sums in this section.

[3] Albert D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, 1968.

[4] R. P. Feynman, 1949, *Phys. Rev.* **76**, p. 769

We begin by considering the Laplace transform of a known function,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Inserting this expression into the sum $\sum_n F(n)$ and interchanging the sum and integral, we find

$$\begin{aligned}\sum_{n=0}^{\infty} F(n) &= \sum_{n=0}^{\infty} \int_0^{\infty} f(t) e^{-nt}\, dt \\ &= \int_0^{\infty} f(t) \sum_{n=0}^{\infty} \left(e^{-t}\right)^n\, dt \\ &= \int_0^{\infty} f(t) \frac{1}{1-e^{-t}}\, dt. \end{aligned} \tag{5.88}$$

The last step was obtained using the sum of a geometric series. The key is being able to carry out the final integral as we show in the next example.

Example 5.24. *Evaluate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.*

Since, $\mathcal{L}[1] = 1/s$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \sum_{n=1}^{\infty} \int_0^{\infty} (-1)^{n+1} e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}}\, dt \\ &= \int_1^2 \frac{du}{u} = \ln 2. \end{aligned} \tag{5.89}$$

Example 5.25. *Evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.*

This is a special case of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{5.90}$$

The Riemann zeta function[5] *is important in the study of prime numbers and more recently has seen applications in the study of dynamical systems. The series in this example is $\zeta(2)$. We have already seen in Problem 2.13 that*

$$\zeta(2) = \frac{\pi^2}{6}.$$

Using Laplace transforms, we can provide an integral representation of $\zeta(2)$.

The first step is to find the correct Laplace transform pair. The sum involves the function $F(n) = 1/n^2$. So, we look for a function $f(t)$ whose Laplace transform is $F(s) = 1/s^2$. We know by now that the inverse Laplace transform of $F(s) = 1/s^2$ is $f(t) = t$. As before, we replace each term in the series by a Laplace transform, exchange the summation and integration, and sum the resulting geometric series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{t}{e^t - 1}\, dt. \end{aligned} \tag{5.91}$$

So, we have that

$$\int_0^{\infty} \frac{t}{e^t-1}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

[5] A translation of Riemann, Bernhard (1859), "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" is in H. M. Edwards (1974). *Riemann's Zeta Function.* Academic Press. Riemann had shown that the Riemann zeta function can be obtained through contour integral representation, $2\sin(\pi s)\Gamma\zeta(s) = i \oint_C \frac{(-x)^{s-1}}{e^x-1}\, dx$, for a specific contour C.

Integrals of this type occur often in statistical mechanics in the form of Bose-Einstein integrals. These are of the form

$$G_n(z) = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx.$$

Note that $G_n(1) = \Gamma(n)\zeta(n)$.

In general, the Riemann zeta function must be tabulated through other means. In some special cases, one can use closed form expressions. For example,

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n,$$

where the B_n's are the Bernoulli numbers. Bernoulli numbers are defined through the Maclaurin series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The first few Riemann zeta functions are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

We can extend this method of using Laplace transforms to summing series whose terms take special general forms. For example, from Feynman's 1949 paper, we note that

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a} \int_0^\infty e^{-s(a+bn)}\, ds.$$

This identity can be shown easily by first noting

$$\int_0^\infty e^{-s(a+bn)}\, ds = \left[\frac{-e^{-s(a+bn)}}{a+bn} \right]_0^\infty = \frac{1}{a+bn}.$$

Now, differentiate the result with respect to a and the result follows.

The latter identity can be generalized further as

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial a^k} \int_0^\infty e^{-s(a+bn)}\, ds.$$

In Feynman's 1949 paper, he develops methods for handling several other general sums using the Convolution Theorem. Wheelon gives more examples of these. We will just provide one such result and an example. First, we note that

$$\frac{1}{ab} = \int_0^1 \frac{du}{[a(1-u)+bu]^2}.$$

However,

$$\frac{1}{[a(1-u)+bu]^2} = \int_0^\infty t e^{-t[a(1-u)+bu]}\, dt.$$

So, we have

$$\frac{1}{ab} = \int_0^1 du \int_0^\infty t e^{-t[a(1-u)+bu]}\, dt.$$

We see in the next example how this representation can be useful.

Example 5.26. *Evaluate* $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$.

We sum this series by first letting $a = 2n+1$ *and* $b = 2n+2$ *in the formula for* $1/ab$. *Collecting the n-dependent terms, we can sum the series leaving a double integral computation in ut-space. The details are as follows:*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} &= \sum_{n=0}^{\infty} \int_0^1 \frac{du}{[(2n+1)(1-u)+(2n+2)u]^2} \\
&= \sum_{n=0}^{\infty} \int_0^1 du \int_0^{\infty} t e^{-t(2n+1+u)}\, dt \\
&= \int_0^1 du \int_0^{\infty} t e^{-t(1+u)} \sum_{n=0}^{\infty} e^{-2nt}\, dt \\
&= \int_0^{\infty} \frac{te^{-t}}{1-e^{-2t}} \int_0^1 e^{-tu}\, du\, dt \\
&= \int_0^{\infty} \frac{te^{-t}}{1-e^{-2t}} \frac{1-e^{-t}}{t}\, dt \\
&= \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}}\, dt \\
&= -\ln(1+e^{-t})\Big|_0^{\infty} = \ln 2. \qquad (5.92)
\end{aligned}$$

5.8.2 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then, one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 5.27.

$$\begin{tikzcd}
\mathcal{L}[y] = g \arrow[rr, "\text{Laplace Transform}"] \arrow[d, dashed, "\begin{array}{c}\text{ODE}\\ \text{for } y(t)\end{array}"'] & & F(Y) = G \arrow[d, "\begin{array}{c}\text{Algebraic}\\ \text{Equation}\\ Y(s)\end{array}"] \\
y(t) & & Y(s) \arrow[ll, "\text{Inverse Laplace Transform}"]
\end{tikzcd}$$

Figure 5.27: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform gives the solution to the initial value problem.

Example 5.27. *Solve the initial value problem* $y' + 3y = e^{2t}$, $y(0) = 1$.

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the righthand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

Combining these two results, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. We will use the tables of Laplace transform pairs. Later we will show that there are other methods for carrying out the Laplace transform inversion.

The inverse transform of the first term is e^{-3t}. However, we have not seen anything that looks like the second form in the table of transforms that we have compiled, but we can rewrite the second term using a partial fraction decomposition. Let's recall how to do this.

The goal is to find constants A and B such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}. \tag{5.93}$$

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterward. So, adding the two terms, we have

This is an example of carrying out a partial fraction decomposition.

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3) + B(s-2).$$

There are several ways to proceed at this point.

a. *Method 1.*

   We can rewrite the equation by gathering terms with common powers of s, we have

   $$(A+B)s + 3A - 2B = 1.$$

   The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

   $$\begin{aligned} A+B &= 0, \\ 3A-2B &= 1. \end{aligned} \tag{5.94}$$

   The first equation gives $A = -B$, so the second equation becomes $-5B = 1$. The solution is then $A = -B = \frac{1}{5}$.

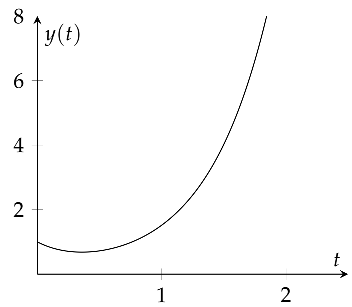

Figure 5.28: A plot of the solution to Example 5.27.

b. *Method 2.*

Since the equation $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$ *is true for all* s*, we can pick specific values. For* $s = 2$*, we find* $1 = 5A$*, or* $A = \frac{1}{5}$*. For* $s = -3$*, we find* $1 = -5B$*, or* $B = -\frac{1}{5}$*. Thus, we obtain the same result as Method 1, but much quicker.*

c. *Method 3.*

We could just inspect the original partial fraction problem. Since the numerator has no s *terms, we might guess the form*

$$\frac{1}{(s-2)(s+3)} = \frac{1}{s-2} - \frac{1}{s+3}.$$

But, recombining the terms on the right hand side, we see that

$$\frac{1}{s-2} - \frac{1}{s+3} = \frac{5}{(s-2)(s+3)}.$$

Since we were off by 5, we divide the partial fractions by 5 to obtain

$$\frac{1}{(s-2)(s+3)} = \frac{1}{5}\left[\frac{1}{s-2} - \frac{1}{s+3}\right],$$

which once again gives the desired form.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

We can now see that the function with this Laplace transform is given by

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right)\right] = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

We can verify that we have solved the initial value problem.

$$y' + 3y = \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t} + 3\left(\frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}\right) = e^{2t}$$

and $y(0) = \frac{1}{5} + \frac{4}{5} = 1$.

Example 5.28. *Solve the initial value problem* $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{5.95}$$

Solving for Y*, we have*

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

The first term is now recognizable as the transform of $\cos 2t$. *The second term is not the transform of* $\sin 2t$. *It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:*

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

The solution is then found as

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

The reader can verify that this is the solution of the initial value problem.

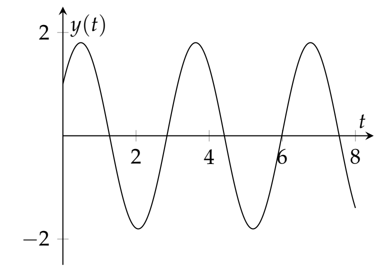

Figure 5.29: A plot of the solution to Example 5.28.

5.8.3 Step and Impulse Functions

Often, the initial value problems that one faces in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems, and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{5.96}$$

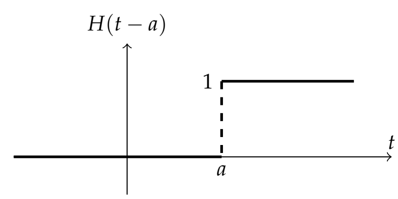

Figure 5.30: A shifted Heaviside function, $H(t-a)$.

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 5.30. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \left.\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{5.97}$$

Just like the Fourier transform, the Laplace transform has two Shift Theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The First and Second Shift Properties/Theorems are given by

The Shift Theorems.

$$\boxed{\begin{aligned} \mathcal{L}[e^{at}f(t)] &= F(s-a), & (5.98)\\ \mathcal{L}[f(t-a)H(t-a)] &= e^{-as}F(s). & (5.99)\end{aligned}}$$

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned} \mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \end{aligned} \tag{5.100}$$

Example 5.29. *Compute the Laplace transform of* $e^{-at}\sin\omega t$.

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The First Shift Theorem tells us that we first need the transform of the sine function. So, for $f(t) = \sin\omega t$, *we have*

$$F(s) = \frac{\omega}{s^2+\omega^2}.$$

Using this transform, we can obtain the solution to this problem as

$$\mathcal{L}[e^{-at}\sin\omega t] = F(s+a) = \frac{\omega}{(s+a)^2+\omega^2}.$$

More interesting examples can be found using piecewise defined functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$, both terms are zero. In the interval $[0, a]$, the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0, a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. The graph of $H(t) - H(t-a)$ is shown in Figure 5.31.

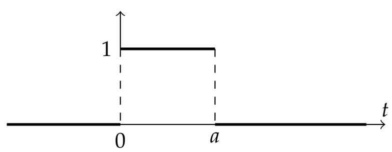

Figure 5.31: The box function, $H(t) - H(t-a)$.

We now consider the piecewise defined function:

$$g(t) = \begin{cases} f(t), & 0 \le t \le a, \\ 0, & t < 0, t > a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$g(t) = f(t)[H(t) - H(t-a)].$$

We depict this in Figure 5.32.

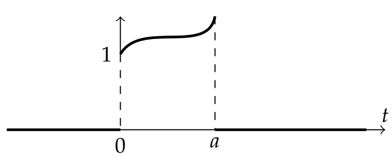

Figure 5.32: Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

Even more complicated functions can be written in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a) - H(t-b)]$ for $b > a$. This is similar to a box function. It is nonzero between a and b and has height $f(t)$.

We show as an example the square wave function in Figure 5.33. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)],$$

for $a > 0$.

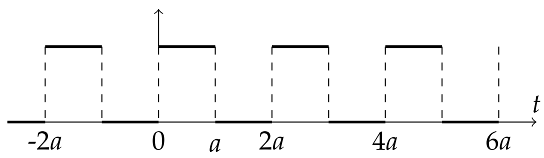

Figure 5.33: A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)]$.

Example 5.30. *Find the Laplace Transform of a square wave "turned on" at $t = 0$.*

.

We let

$$f(t) = \sum_{n=0}^{\infty}[H(t-2na) - H(t-(2n+1)a)], \quad a > 0.$$

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty}\left[\mathcal{L}[H(t-2na)] - \mathcal{L}[H(t-(2n+1)a)]\right] \\
&= \sum_{n=0}^{\infty}\left[\frac{e^{-2nas}}{s} - \frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s}\sum_{n=0}^{\infty}\left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s}\left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (5.101)
\end{aligned}$$

Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form since $e^{-2as} < 1$.

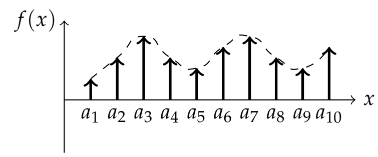

Figure 5.34: Plot representing impulse forces of height $f(a_i)$. The sum $\sum_{i=1}^{n} f(a_i)\delta(x-a_i)$ describes a general impulse function.

Other interesting examples are provided by the delta function. The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function as shown in Figure 5.34. The sum of impulses located at points a_i, $i = 1, \dots, n$, with strengths $f(a_i)$ would be given by

$$f(x) = \sum_{i=1}^{n} f(a_i)\delta(x-a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,d\xi.$$

This is simply an application of the sifting property of the delta function. We will investigate a case when one would use a single impulse. While a mass on a spring is undergoing simple harmonic motion, we hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$.

$\mathcal{L}[\delta(t-a)] = e^{-as}$.

One would then need the Laplace transform of the delta function to solve the associated initial value problem. Inserting the delta function into the

Laplace transform, we find that for $a > 0$,

$$\begin{aligned}\mathcal{L}[\delta(t-a)] &= \int_0^\infty \delta(t-a)e^{-st}\,dt \\ &= \int_{-\infty}^\infty \delta(t-a)e^{-st}\,dt \\ &= e^{-as}. \qquad (5.102)\end{aligned}$$

Example 5.31. *Solve the initial value problem $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.*

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.

First, we transform the differential equation:

$$s^2Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. Thus, we need the Second Shift Theorem since the solution is of the form $Y(s) = e^{-2s}F(s)$ for

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

We need to find the corresponding $f(t)$ of the Laplace transform pair. The denominator in $F(s)$ suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi}\frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1}\sin 2\pi t$.

We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$, or

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-2s}F(s)\right] \\ &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \qquad (5.103)\end{aligned}$$

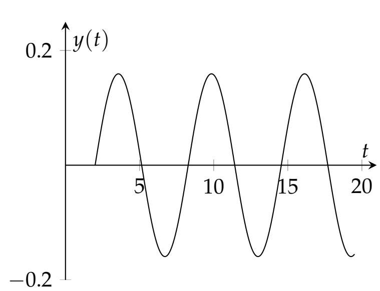

Figure 5.35: A plot of the solution to Example 5.31 in which a spring at rest experiences an impulse force at $t = 2$.

This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 5.35

Example 5.32. *Solve the initial value problem*

$$y'' + y = f(t), \quad y(0) = 0, y'(0) = 0,$$

where

$$f(t) = \begin{cases} cos \pi t, & 0 \leq t \leq 2, \\ 0, & otherwise. \end{cases}$$

We need the Laplace transform of $f(t)$. This function can be written in terms of a Heaviside function, $f(t) = \cos \pi t H(t-2)$. In order to apply the Second Shift Theorem, we need a shifted version of the cosine function. We find the shifted version by noting that $\cos \pi(t-2) = \cos \pi t$. Thus, we have

$$\begin{aligned} f(t) &= \cos \pi t \left[H(t) - H(t-2)\right] \\ &= \cos \pi t - \cos \pi(t-2) H(t-2), \quad t \geq 0. \end{aligned} \tag{5.104}$$

The Laplace transform of this driving term is

$$F(s) = (1 - e^{-2s}) \mathcal{L}[\cos \pi t] = (1 - e^{-2s}) \frac{s}{s^2 + \pi^2}.$$

Now we can proceed to solve the initial value problem. The Laplace transform of the initial value problem yields

$$(s^2 + 1) Y(s) = (1 - e^{-2s}) \frac{s}{s^2 + \pi^2}.$$

Therefore,

$$Y(s) = (1 - e^{-2s}) \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

We can retrieve the solution to the initial value problem using the Second Shift Theorem. The solution is of the form $Y(s) = (1 - e^{-2s}) G(s)$ for

$$G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

Then, the final solution takes the form

$$y(t) = g(t) - g(t-2) H(t-2).$$

We only need to find $g(t)$ in order to finish the problem. This is easily done using the partial fraction decomposition

$$G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)} = \frac{1}{\pi^2 - 1} \left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + \pi^2} \right].$$

Then,

$$g(t) = \mathcal{L}^{-1} \left[\frac{s}{(s^2 + \pi^2)(s^2 + 1)} \right] = \frac{1}{\pi^2 - 1} \left(\cos t - \cos \pi t \right).$$

The final solution is then given by

$$y(t) = \frac{1}{\pi^2 - 1} \left[\cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t) \right].$$

A plot of this solution is shown in Figure 5.36

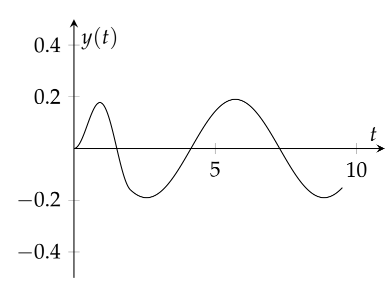

Figure 5.36: A plot of the solution to Example 5.32 in which a spring at rest experiences an piecewise defined force.

5.9 *The Convolution Theorem*

FINALLY, WE CONSIDER THE CONVOLUTION of two functions. Often, we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say we have obtained $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. In this case, we could find a partial fraction decomposition. But, there are other ways to find the inverse transform, especially if we cannot perform a partial fraction decomposition. We could use the Convolution Theorem for Laplace transforms or we could compute the inverse transform directly. We will look into these methods in the next two sections.We begin with defining the convolution.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The convolution $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{5.105}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

The convolution is commutative.

1. The convolution is commutative: $f * g = g * f$

   Proof. The key is to make a substitution $y = t - u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned} (g * f)(t) &= \int_0^t g(u)f(t-u)\,du \\ &= -\int_t^0 g(t-y)f(y)\,dy \\ &= \int_0^t f(y)g(t-y)\,dy \\ &= (f * g)(t). \end{aligned} \tag{5.106}$$

□

The Convolution Theorem for Laplace transforms.

2. The Convolution Theorem: The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s).$$

   Proof. Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Second, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

Carrying out the computation, we have

$$\begin{aligned}
\mathcal{L}[f*g] &= \int_0^\infty \left(\int_0^t f(u)g(t-u)\,du \right) e^{-st}\,dt \\
&= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)\,du \right) e^{-st}\,dt \\
&= \int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du \qquad (5.107)
\end{aligned}$$

Now, make the substitution $\tau = t-u$. We note that

$$int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du = \int_0^\infty f(u) \left(\int_{-u}^\infty g(\tau)e^{-s(\tau+u)}\,d\tau \right) du$$

However, since $g(\tau)$ is a causal function, we have that it vanishes for $\tau < 0$ and we can change the integration interval to $[0,\infty)$. So, after a little rearranging, we can proceed to the result.

$$\begin{aligned}
\mathcal{L}[f*g] &= \int_0^\infty f(u) \left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\,d\tau \right) du \\
&= \int_0^\infty f(u)e^{-su} \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau \right) du \\
&= \left(\int_0^\infty f(u)e^{-su}\,du \right) \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau \right) \\
&= F(s)G(s). \qquad (5.108)
\end{aligned}$$

□

We make use of the Convolution Theorem to do the following examples.

Example 5.33. *Find* $y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right]$.

We note that this is a product of two functions:

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors: $f(t) = e^t$ *and* $g(t) = e^{2t}$.

Using the Convolution Theorem, we find $y(t) = (f*g)(t)$. *We compute the convolution:*

$$\begin{aligned}
y(t) &= \int_0^t f(u)g(t-u)\,du \\
&= \int_0^t e^u e^{2(t-u)}\,du \\
&= e^{2t}\int_0^t e^{-u}\,du \\
&= e^{2t}[-e^t+1] = e^{2t}-e^t. \qquad (5.109)
\end{aligned}$$

One can also confirm this by carrying out a partial fraction decomposition.

Example 5.34. *Consider the initial value problem, $y'' + 9y = 2\sin 3t$, $y(0) = 1$, $y'(0) = 0$.*

The Laplace transform of this problem is given by

$$(s^2+9)Y - s = \frac{6}{s^2+9}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}.$$

The inverse Laplace transform of the second term is easily found as $\cos(3t)$; however, the first term is more complicated.

We can use the Convolution Theorem to find the Laplace transform of the first term. We note that

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}$$

is a product of two Laplace transforms (up to the constant factor). Thus,

$$\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] = \frac{2}{3}(f * g)(t),$$

where $f(t) = g(t) = sin3t$. Evaluating this convolution product, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] &= \frac{2}{3}(f * g)(t) \\
&= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)\,du \\
&= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]\,du \\
&= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\
&= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \qquad (5.110)
\end{aligned}$$

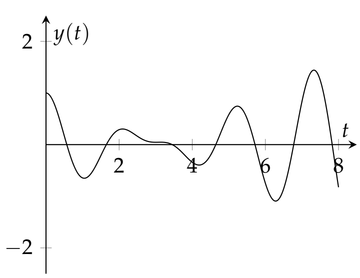

Figure 5.37: Plot of the solution to Example 5.34 showing a resonance.

Combining this with the inverse transform of the second term of $Y(s)$, the solution to the initial value problem is

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 5.37. This is known as a resonance.

Example 5.35. *Find $\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}]$ using partial fraction decomposition.*

If we look at Table 5.2, we see that the Laplace transform pairs with the denominator $(s^2+\omega^2)^2$ are

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2},$$

and

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

So, we might consider rewriting a partial fraction decomposition as

$$\frac{6}{(s^2+9)^2} = \frac{A6s}{(s^2+9)^2} + \frac{B(s^2-9)}{(s^2+9)^2} + \frac{Cs+D}{s^2+9}.$$

Combining the terms on the right over a common denominator, we find

$$6 = 6As + B(s^2-9) + (Cs+D)(s^2+9).$$

Collecting like powers of s, we have

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

Therefore, $C = 0$, $A = 0$, $D + B = 0$, *and* $D - B = \frac{2}{3}$. *Solving the last two equations, we find* $D = -B = \frac{1}{3}$.

Using these results, we find

$$\frac{6}{(s^2+9)^2} = -\frac{1}{3}\frac{(s^2-9)}{(s^2+9)^2} + \frac{1}{3}\frac{1}{s^2+9}.$$

This is the result we had obtained in the last example using the Convolution Theorem.

5.10 The Inverse Laplace Transform

UP TO THIS POINT WE HAVE SEEN that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used in a differential equations course. One can do the same for Fourier transforms. However, in the case of Fourier transforms, we introduced an inverse transform in the form of an integral. Does such an inverse integral transform exist for the Laplace transform? Yes, it does! In this section we will derive the inverse Laplace transform integral and show how it is used.

We begin by considering a causal function $f(t)$, which vanishes for $t < 0$, and define the function $g(t) = f(t)e^{-ct}$ with $c > 0$. For $g(t)$ absolutely integrable,

$$\int_{-\infty}^{\infty} |g(t)|\, dt = \int_0^{\infty} |f(t)|e^{-ct}\, dt < \infty,$$

A function $f(t)$ is said to be of exponential order if $\int_0^\infty |f(t)|e^{-ct}\,dt < \infty$

we can write the Fourier transform,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \int_0^{\infty} f(t)e^{i\omega t - ct}dt$$

and the inverse Fourier transform,

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t}\, d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\tau)e^{(i\omega - c)\tau}d\tau e^{-(i\omega-c)t}\, d\omega.$$

Letting $s = c - i\omega$ (so $d\omega = ids$), we have

$$f(t) = \frac{i}{2\pi} \int_{c+i\infty}^{c-i\infty} \int_0^\infty f(\tau) e^{-s\tau} d\tau e^{st}\, ds.$$

Note that the inside integral is simply $F(s)$. So, we have

$$\boxed{f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st}\, ds. \qquad (5.111)}$$

The integral in the last equation is the inverse Laplace transform, called the Bromwich Integral and is named after Thomas John I'Anson Bromwich (1875 - 1929) . This inverse transform is not usually covered in differential equations courses because the integration takes place in the complex plane. This integral is evaluated along a path in the complex plane called the Bromwich contour. The typical way to compute this integral is to first choose c so that all poles are to the left of the contour. This guarantees that $f(t)$ is of exponential type. The contour is a closed semicircle enclosing all the poles. One then relies on a generalization of Jordan's Lemma to the second and third quadrants.[6]

[6] Closing the contour to the left of the contour can be reasoned in a manner similar to what we saw in Jordan's Lemma. Write the exponential as $e^{st} = e^{(s_R + is_I)t} = e^{s_R t} e^{is_I t}$. The second factor is an oscillating factor and the growth in the exponential can only come from the first factor. In order for the exponential to decay as the radius of the semicircle grows, $s_R t < 0$. Since $t > 0$, we need $s < 0$ which is done by closing the contour to the left. If $t < 0$, then the contour to the right would enclose no singularities and preserve the causality of $f(t)$.

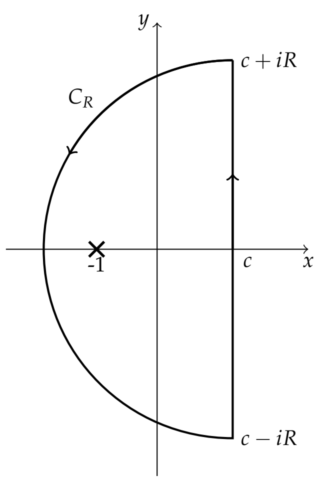

Figure 5.38: The contour used for applying the Bromwich integral to the Laplace transform $F(s) = \frac{1}{s(s+1)}$.

Example 5.36. *Find the inverse Laplace transform of $F(s) = \frac{1}{s(s+1)}$.*

The integral we have to compute is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+1)}\, ds.$$

This integral has poles at $s = 0$ and $s = -1$. The contour we will use is shown in Figure 5.38. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as $2\pi i$ times the sum of the residues. The residues in this case are

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = 0\right] = \lim_{z\to 0} \frac{e^{zt}}{(z+1)} = 1$$

and

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = -1\right] = \lim_{z\to -1} \frac{e^{zt}}{z} = -e^{-t}.$$

Therefore, we have

$$f(t) = 2\pi i \left[\frac{1}{2\pi i}(1) + \frac{1}{2\pi i}(-e^{-t})\right] = 1 - e^{-t}.$$

We can verify this result using the Convolution Theorem or using a partial fraction decomposition. The latter method is simplest. We note that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives e^{-t}. So,

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}.$$

Thus, we have verified the result from doing contour integration.

Example 5.37. *Find the inverse Laplace transform of* $F(s) = \frac{1}{s(1+e^s)}$.

In this case, we need to compute

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(1+e^s)}\, ds.$$

This integral has poles at complex values of s such that $1 + e^s = 0$, *or* $e^s = -1$. *Letting* $s = x + iy$, *we see that*

$$e^s = e^{x+iy} = e^x(\cos y + i \sin y) = -1.$$

We see $x = 0$ *and y satisfies* $\cos y = -1$ *and* $\sin y = 0$. *Therefore,* $y = n\pi$ *for n an odd integer. Therefore, the integrand has an infinite number of simple poles at* $s = n\pi i$, $n = \pm 1, \pm 3, \ldots$. *It also has a simple pole at* $s = 0$.

In Figure 5.39, we indicate the poles. We need to compute the residues at each pole. At $s = n\pi i$, *we have*

$$\begin{aligned} Res\left[\frac{e^{st}}{s(1+e^s)}; s = n\pi i\right] &= \lim_{s\to n\pi i}(s - n\pi i)\frac{e^{st}}{s(1+e^s)} \\ &= \lim_{s\to n\pi i}\frac{e^{st}}{se^s} \\ &= -\frac{e^{n\pi i t}}{n\pi i}, \quad n \text{ odd}. \end{aligned} \tag{5.112}$$

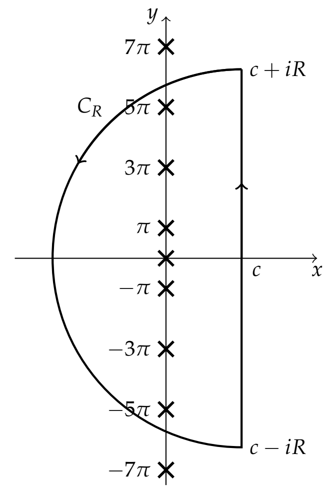

Figure 5.39: The contour used for applying the Bromwich integral to the Laplace transform $F(s) = \frac{1}{1+e^s}$.

At $s = 0$, *the residue is*

$$Res\left[\frac{e^{st}}{s(1+e^s)}; s = 0\right] = \lim_{s\to 0}\frac{e^{st}}{1+e^s} = \frac{1}{2}.$$

Summing the residues and noting the exponentials for $\pm n$ *can be combined to form sine functions, we arrive at the inverse transform.*

$$\begin{aligned} f(t) &= \frac{1}{2} - \sum_{n \text{ odd}} \frac{e^{n\pi i t}}{n\pi i} \\ &= \frac{1}{2} - 2\sum_{k=1}^{\infty}\frac{\sin{(2k-1)\pi t}}{(2k-1)\pi}. \end{aligned} \tag{5.113}$$

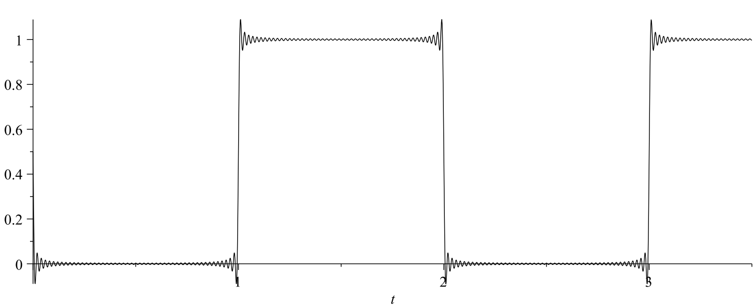

Figure 5.40: Plot of the square wave result as the inverse Laplace transform of $F(s) = \frac{1}{s(1+e^s)}$ with 50 terms.

The series in this example might look familiar. It is a Fourier sine series with odd harmonics whose amplitudes decay like $1/n$. *It is a vertically shifted square* *wave. In fact, we had computed the Laplace transform of a general square wave in Example 5.30.*

In that example, we found

$$\begin{aligned}\mathcal{L}\left[\sum_{n=0}^{\infty}[H(t-2na)-H(t-(2n+1)a)]\right] &= \frac{1-e^{-as}}{s(1-e^{-2as})} \\ &= \frac{1}{s(1+e^{-as})}. \qquad (5.114)\end{aligned}$$

In this example, one can show that

$$f(t) = \sum_{n=0}^{\infty}[H(t-2n+1)-H(t-2n)].$$

The reader should verify that this result is indeed the square wave shown in Figure 5.40.

5.11 *Transforms and Partial Differential Equations*

As another application of the transforms, we will see that we can use transforms to solve some linear partial differential equations. We will first solve the one-dimensional heat equation and the two-dimensional Laplace equations using Fourier transforms. The transforms of the partial differential equations lead to ordinary differential equations which are easier to solve. The final solutions are then obtained using inverse transforms.

We could go further by applying a Fourier transform in space and a Laplace transform in time to convert the heat equation into an algebraic equation. We will also show that we can use a finite sine transform to solve nonhomogeneous problems on finite intervals. Along the way we will identify several Green's functions.

5.11.1 *Fourier Transform and the Heat Equation*

We will first consider the solution of the heat equation on an infinite interval using Fourier transforms. The basic scheme was discussed earlier and is outlined in Figure 5.41.

Consider the heat equation on the infinite line:

$$\begin{aligned} u_t = \alpha u_{xx}, &\quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), &\quad -\infty < x < \infty. \end{aligned} \qquad (5.115)$$

We can Fourier transform the heat equation using the Fourier transform of $u(x,t)$,

$$\mathcal{F}[u(x,t)] = \hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{ikx}\,dx.$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,0) \arrow[r, "\textit{Fourier Transform}"] \arrow[d, dashed, "u_t = \alpha u_{xx}"] & \hat{u}(k,0) \arrow[d, "\hat{u}_t = -\alpha k^2 \hat{u}"] \\
u(x,t) & \hat{u}(k,t) \arrow[l, "\textit{Inverse Fourier Transform}"]
\end{tikzcd}$$

Figure 5.41: Using Fourier transforms to solve a linear partial differential equation.

We need to transform the derivatives in the equation. First we note that

$$\begin{aligned}
\mathcal{F}[u_t] &= \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t} e^{ikx}\, dx \\
&= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) e^{ikx}\, dx \\
&= \frac{\partial \hat{u}(k,t)}{\partial t}. \qquad (5.116)
\end{aligned}$$

Assuming that $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$, we also have that

$$\begin{aligned}
\mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{ikx}\, dx \\
&= -k^2 \hat{u}(k,t). \qquad (5.117)
\end{aligned}$$

Therefore, the heat equation becomes

The transformed heat equation.

$$\frac{\partial \hat{u}(k,t)}{\partial t} = -\alpha k^2 \hat{u}(k,t).$$

This is a first-order differential equation which is readily solved as

$$\hat{u}(k,t) = A(k) e^{-\alpha k^2 t},$$

where $A(k)$ is an arbitrary function of k. The inverse Fourier transform is

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{-ikx}\, dk. \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{A}(k) e^{-\alpha k^2 t} e^{-ikx}\, dk. \qquad (5.118)
\end{aligned}$$

We can determine $A(k)$ using the initial condition. Note that

$$\mathcal{F}[u(x,0)] = \hat{u}(k,0) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx.$$

But we also have from the solution that

$$u(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{A}(k) e^{-ikx}\, dk.$$

Comparing these two expressions for $\hat{u}(k,0)$, we see that

$$A(k) = \mathcal{F}[f(x)].$$

We note that $\hat{u}(k,t)$ is given by the product of two Fourier transforms, $\hat{u}(k,t) = A(k)e^{-\alpha k^2 t}$. So, by the Convolution Theorem, we expect that $u(x,t)$ is the convolution of the inverse transforms:

$$u(x,t) = (f * g)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi,t)g(x-\xi,t)\,d\xi,$$

where

$$g(x,t) = \mathcal{F}^{-1}[e^{-\alpha k^2 t}].$$

In order to determine $g(x,t)$, we need only recall Example 5.5. In that example, we saw that the Fourier transform of a Gaussian is a Gaussian. Namely, we found that

$$\mathcal{F}[e^{-ax^2/2}] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a},$$

or,

$$\mathcal{F}^{-1}[\sqrt{\frac{2\pi}{a}}e^{-k^2/2a}] = e^{-ax^2/2}.$$

Applying this to the current problem, we have

$$g(x) = \mathcal{F}^{-1}[e^{-\alpha k^2 t}] = \sqrt{\frac{\pi}{\alpha t}}e^{-x^2/4t}.$$

Finally, we can write the solution to the problem:

$$u(x,t) = (f * g)(x,t) = \int_{-\infty}^{\infty} f(\xi,t)\frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi\alpha t}}\,d\xi,$$

The function in the integrand,

$$K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi\alpha t}},$$

$K(x,t)$ is called the heat kernel.

is called the heat kernel and acts as an initial value Green's function. The solution takes the form

$$u(x,t) = \int_{-\infty} \infty f(\xi,t)K(x,\xi;t)\,d\xi.$$

5.11.2 *Laplace's Equation on the Half Plane*

We consider a steady-state solution in two dimensions. In particular, we look for the steady-state solution, $u(x,y)$, satisfying the two-dimensional Laplace equation on a semi-infinite slab with given boundary conditions as shown in Figure 5.42. The boundary value problem is given as

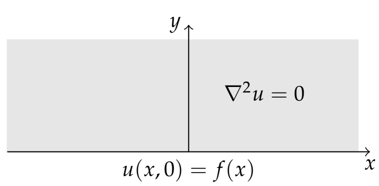

Figure 5.42: This is the domain for a semi-infinite slab with boundary value $u(x,0) = f(x)$ and governed by Laplace's equation.

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, \quad y > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ \lim_{y\to\infty} u(x,y) = 0, &\quad \lim_{|x|\to\infty} u(x,y) = 0. \end{aligned} \tag{5.119}$$

This problem can be solved using a Fourier transform of $u(x, y)$ with respect to x. The transform scheme for doing this is shown in Figure 5.43. We begin by defining the Fourier transform

$$\hat{u}(k, y) = \mathcal{F}[u] = \int_{-\infty}^{\infty} u(x, y) e^{ikx}\, dx.$$

We can transform Laplace's equation. We first note from the properties of Fourier transforms that

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -k^2 \hat{u}(k, y),$$

if $\lim_{|x|\to\infty} u(x, y) = 0$ and $\lim_{|x|\to\infty} u_x(x, y) = 0$. Also,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial^2 \hat{u}(k, y)}{\partial y^2}.$$

Thus, the transform of Laplace's equation gives $\hat{u}_{yy} = k^2 \hat{u}$.

$$\begin{tikzcd}
u(x,0) \arrow[r, "\textit{Fourier Transform}"] \arrow[d, dashed, "u_{xx} + u_{yy} = 0"'] & \hat{u}(k,0) \arrow[d, "\hat{u}_{yy} = k^2 \hat{u}"] \\
u(x,y) & \hat{u}(k,y) \arrow[l, "\textit{Inverse Fourier Transform}"]
\end{tikzcd}$$

Figure 5.43: The transform scheme used to convert Laplace's equation to an ordinary differential equation, which is easier to solve.

This is a simple ordinary differential equation. We can solve this equation using the boundary conditions. The general solution is

The transformed Laplace equation.

$$\hat{u}(k, y) = a(k) e^{ky} + b(k) e^{-ky}.$$

Since $\lim_{y\to\infty} u(x, y) = 0$ and k can be positive or negative, we have that $\hat{u}(k, y) = a(k) e^{-|k|y}$. The coefficient $a(k)$ can be determined using the remaining boundary condition, $u(x, 0) = f(x)$. We find that $a(k) = \hat{f}(k)$ since

$$a(k) = \hat{u}(k, 0) = \int_{-\infty}^{\infty} u(x, 0) e^{ikx}\, dx = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx = \hat{f}(k).$$

We have found that $\hat{u}(k, y) = \hat{f}(k) e^{-|k|y}$. We can obtain the solution using the inverse Fourier transform,

$$u(x, t) = \mathcal{F}^{-1}[\hat{f}(k) e^{-|k|y}].$$

We note that this is a product of Fourier transforms and use the Convolution Theorem for Fourier transforms. Namely, we have that $a(k) = \mathcal{F}[f]$ and $e^{-|k|y} = \mathcal{F}[g]$ for $g(x, y) = \frac{2y}{x^2+y^2}$. This last result is essentially proven in Problem 6.

Then, the Convolution Theorem gives the solution

$$
\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\frac{2y}{(x-\xi)^2+y^2}\,d\xi.
\end{aligned} \tag{5.120}
$$

We note that this solution is in the form

$$
u(x,y) = \int_{-\infty}^{\infty} f(\xi)G(x,\xi;y)\,d\xi,
$$

where

$$
G(x,\xi;y) = \frac{2y}{\pi((x-\xi)^2+y^2)}
$$

is the Green's function for this problem.

The Green's function for the Laplace equation.

5.11.3 Heat Equation on Infinite Interval, Revisited

WE NEXT CONSIDER THE INITIAL VALUE PROBLEM for the heat equation on an infinite interval,

$$
\begin{aligned}
u_t = u_{xx}, &\quad -\infty < x < \infty, \quad t > 0, \\
u(x,0) = f(x), &\quad -\infty < x < \infty.
\end{aligned} \tag{5.121}
$$

We can apply both a Fourier and a Laplace transform to convert this to an algebraic problem. The general solution will then be written in terms of an initial value Green's function as

$$
u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi)f(\xi)\,d\xi.
$$

For the time dependence, we can use the Laplace transform; and, for the spatial dependence, we use the Fourier transform. These combined transforms lead us to define

$$
\hat{u}(k,s) = \mathcal{F}[\mathcal{L}[u]] = \int_{-\infty}^{\infty}\int_{0}^{\infty} u(x,t)e^{-st}e^{ikx}\,dtdx.
$$

Applying this to the terms in the heat equation, we have

$$
\begin{aligned}
\mathcal{F}[\mathcal{L}[u_t]] &= s\hat{u}(k,s) - \mathcal{F}[u(x,0)] \\
&= s\hat{u}(k,s) - \hat{f}(k) \\
\mathcal{F}[\mathcal{L}[u_{xx}]] &= -k^2\hat{u}(k,s).
\end{aligned} \tag{5.122}
$$

Here we have assumed that

$$
\lim_{t\to\infty} u(x,t)e^{-st} = 0, \quad \lim_{|x|\to\infty} u(x,t) = 0, \quad \lim_{|x|\to\infty} u_x(x,t) = 0.
$$

Therefore, the heat equation can be turned into an algebraic equation for the transformed solution:

$$
(s+k^2)\hat{u}(k,s) = \hat{f}(k),
$$

tion.

or

$$\hat{u}(k,s) = \frac{\hat{f}(k)}{s+k^2}.$$

The solution to the heat equation is obtained using the inverse transforms for both the Fourier and Laplace transforms. Thus, we have

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\mathcal{L}^{-1}[\hat{u}]] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}(k)}{s+k^2}e^{st}\,ds\right)e^{-ikx}\,dk. \qquad (5.123) \end{aligned}$$

Since the inside integral has a simple pole at $s=-k^2$, we can compute the Bromwich Integral by choosing $c > -k^2$. Thus,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}(k)}{s+k^2}e^{st}\,ds = \text{Res}\left[\frac{\hat{f}(k)}{s+k^2}e^{st}; s=-k^2\right] = e^{-k^2 t}\hat{f}(k).$$

Inserting this result into the solution, we have

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\mathcal{L}^{-1}[\hat{u}]] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{-k^2 t}e^{-ikx}\,dk. \qquad (5.124) \end{aligned}$$

This solution is of the form

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}\hat{g}]$$

for $\hat{g}(k) = e^{-k^2 t}$. So, by the Convolution Theorem for Fourier transforms,the solution is a convolution:

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

All we need is the inverse transform of $\hat{g}(k)$.

We note that $\hat{g}(k) = e^{-k^2 t}$ is a Gaussian. Since the Fourier transform of a Gaussian is a Gaussian, we need only recall Example 5.5:

$$\mathcal{F}[e^{-ax^2/2}] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}.$$

Setting $a = 1/2t$, this becomes

$$\mathcal{F}[e^{-x^2/4t}] = \sqrt{4\pi t}e^{-k^2 t}.$$

So,

$$g(x) = \mathcal{F}^{-1}[e^{-k^2 t}] = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

Inserting $g(x)$ into the solution, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/4t}\,d\xi \\ &= \int_{-\infty}^{\infty} G(x,t;\xi)f(\xi)\,d\xi. \qquad (5.125) \end{aligned}$$

Here we have identified the initial value Green's function

The initial value Green's function for the heat equation.

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi t}}e^{-(x-\xi)^2/4t}.$$

5.11.4 Nonhomogeneous Heat Equation

We now consider the nonhomogeneous heat equation with homogeneous boundary conditions defined on a finite interval.

$$\begin{aligned} u_t - k u_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{5.126}$$

When $h(x,t) \equiv 0$, the general solution of the heat equation satisfying the above boundary conditions, $u(0,t) = 0$, $u(L,t) = 0$, for $t > 0$, can be written as a Fourier Sine Series:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, when $h(x,t) \neq 0$, we might assume that the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$$

where the b_n's are the Finite Fourier Sine Transform of the desired solution,

$$b_n(t) = \mathcal{F}_s[u] = \frac{2}{L} \int_0^L u(x,t) \sin \frac{n\pi x}{L}\, dx$$

Note that the Finite Fourier Sine Transform is essentially the Fourier Sine Series which we encountered in Section 2.4.

$$\begin{tikzcd}[column sep=huge, row sep=large]
u(x,0) \arrow[r, "\text{Finite Fourier Sine Transform}"] \arrow[d, dashed, "u_t - u_{xx} = h(x,t)"'] & A(k,0) \arrow[d, "\frac{db_n}{dt} + \omega_n^2 b_b = B_n(t)"'] \\
u(x,t) & A(k,t) \arrow[l, "\text{Inverse Finite Fourier Sine Transform}"]
\end{tikzcd}$$

Figure 5.44: Using finite Fourier transforms to solve the heat equation by solving an ODE instead of a PDE.

The idea behind using the Finite Fourier Sine Transform is to solve the given heat equation by transforming the heat equation to a simpler equation for the transform, $b_n(t)$, solve for $bn(t)$, and then do an inverse transform, that is, insert the $b_n(t)$'s back into the series representation. This is depicted in Figure 5.44. Note that we had explored a similar diagram earlier when discussing the use of transforms to solve differential equations.

First, we need to transform the partial differential equation. The finite transform of the derivative terms are given by

$$\mathcal{F}_s[u_t] \quad = \quad \frac{2}{L} \int_0^L \frac{\partial u}{\partial t}(x,t) \sin \frac{n\pi x}{L}\, dx$$

$$\begin{aligned}
&= \frac{d}{dt}\left(\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}\,dx\right) \\
&= \frac{db_n}{dt}. \qquad (5.127)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_s[u_{xx}] &= \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\sin\frac{n\pi x}{L}\,dx \\
&= \left[u_x \sin\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)\frac{2}{L}\int_0^L \frac{\partial u}{\partial x}(x,t)\cos\frac{n\pi x}{L}\,dx \\
&= -\left[\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)^2\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{n\pi}{L}[u(0,t) - u(L,0)\cos n\pi] - \left(\frac{n\pi}{L}\right)^2 b_n^2 \\
&= -\omega_n^2 b_n^2, \qquad (5.128)
\end{aligned}$$

where $\omega_n = \frac{n\pi}{L}$.

Furthermore, we define

$$H_n(t) = \mathcal{F}_s[h] = \frac{2}{L}\int_0^L h(x,t)\sin\frac{n\pi x}{L}\,dx.$$

Then, the heat equation is transformed to

$$\frac{db_n}{dt} + \omega_n^2 b_n = H_n(t), \quad n = 1,2,3,\ldots.$$

This is a simple linear first-order differential equation. We can supplement this equation with the initial condition

$$b_n(0) = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx.$$

The differential equation for b_n is easily solved using the integrating factor, $\mu(t) = e^{\omega_n^2 t}$. Thus,

$$\frac{d}{dt}\left(e^{\omega_n^2 t} b_n(t)\right) = H_n(t)e^{\omega_n^2 t}$$

and the solution is

$$b_n(t) = b_n(0)e^{-\omega_n^2 t} + \int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau.$$

The final step is to insert these coefficients (Finite Fourier Sine Transform) into the series expansion (inverse finite Fourier sine transform) for $u(x,t)$. The result is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\left[\int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L}.$$

This solution can be written in a more compact form in order to identify the Green's function. We insert the expressions for $b_n(0)$ and $H_n(t)$ in terms of the initial profile and source term and interchange sums and integrals.

This leads to

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L u(\xi,0)\sin\frac{n\pi\xi}{L}\,d\xi\right)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} \\
&\quad+\sum_{n=1}^{\infty}\left[\int_0^t\left(\frac{2}{L}\int_0^L h(\xi,\tau)\sin\frac{n\pi\xi}{L}\,d\xi\right)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L} \\
&= \int_0^L u(\xi,0)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2 t}\right]d\xi \\
&\quad+\int_0^t\int_0^L h(\xi,\tau)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}\right] \\
&= \int_0^L u(\xi,0)G(x,\xi;t,0)d\xi+\int_0^t\int_0^L h(\xi,\tau)G(x,\xi;t,\tau)\,d\xi d\tau.
\end{aligned} \tag{5.129}$$

Here we have defined the Green's function

$$G(x,\xi;t,\tau)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}.$$

We note that $G(x,\xi;t,0)$ gives the initial value Green's function.

Evaluating the Green's function at $t=\tau$, we have

$$G(x,\xi;t,t)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

This is actually a series representation of the Dirac delta function. The Fourier Sine Transform of the delta function is

$$\mathcal{F}_s[\delta(x-\xi)]=\frac{2}{L}\int_0^L\delta(x-\xi)\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\sin\frac{n\pi\xi}{L}.$$

Then, the representation becomes

$$\delta(x-\xi)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}=G(x,\xi;t,\tau).$$

Also, we note that

$$\begin{aligned}
\frac{\partial G}{\partial t} &= -\omega_n^2 G \\
\frac{\partial^2 G}{\partial x^2} &= -\left(\frac{n\pi}{L}\right)^2 G.
\end{aligned}$$

Therefore, $G_t=G_{xx}$, at least for $\tau\neq t$ and $\xi\neq x$.

We can modify this problem by adding nonhomogeneous boundary conditions.

$$\begin{aligned}
u_t-ku_{xx}=h(x,t),&\quad 0\le x\le L,\quad t>0,\\
u(0,t)=A,\quad u(L,t)=B,&\quad t>0,\\
u(x,0)=f(x),&\quad 0\le x\le L.
\end{aligned} \tag{5.130}$$

One way to treat these conditions is to assume $u(x,t)=w(x)+v(x,t)$ where $v_t-kv_{xx}=h(x,t)$ and $w_{xx}=0$. Then, $u(x,t)=w(x)+v(x,t)$ satisfies the original nonhomogeneous heat equation.

If $v(x,t)$ satisfies $v(0,t) = v(L,t) = 0$ and $w(x)$ satisfies $w(0) = A$ and $w(L) = B$, then $u(0,t) = w(0) + v(0,t) = A$ $u(L,t) = w(L) + v(L,t) = B$

Finally, we note that

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x).$$

Therefore, $u(x,t) = w(x) + v(x,t)$ satisfies the original problem if

$$\begin{aligned} v_t - kv_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - w(x), \quad 0 \le x \le L, \end{aligned} \tag{5.131}$$

and

$$\begin{aligned} w_{xx} &= 0, \quad 0 \le x \le L, \\ w(0) &= A, \quad w(L) = B. \end{aligned} \tag{5.132}$$

We can solve the last problem to obtain $w(x) = A + \frac{B-A}{L}x$. The solution to the problem for $v(x,t)$ is simply the problem we solved already in terms of Green's functions with the new initial condition, $f(x) = A - \frac{B-A}{L}x$.

Problems

1. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.

b. Show that the area under each function is one.

2. Verify that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, defined by $f_n(x) = \frac{n}{2}e^{-n|x|}$, approaches a delta function.

3. Evaluate the following integrals:

a. $\int_0^{\pi} \sin x\delta\left(x - \frac{\pi}{2}\right)\, dx$.

b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\, dx$.

c. $\int_0^{\pi} x^2\delta\left(x + \frac{\pi}{2}\right)\, dx$.

d. $\int_0^{\infty} e^{-2x}\delta(x^2 - 5x + 6)\, dx$. [See Problem 4.]

e. $\int_{-\infty}^{\infty}(x^2 - 2x + 3)\delta(x^2 - 9)\, dx$. [See Problem 4.]

4. For the case that a function has multiple roots, $f(x_i) = 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2 - 5x - 6)(3x^2 - 7x + 2)\, dx$.

5. Find a Fourier series representation of the Dirac delta function, $\delta(x)$, on $[-L, L]$.

6. For $a > 0$, find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-a|x|}$.

7. Use the result from Problem 6 plus properties of the Fourier transform to find the Fourier transform, of $f(x) = x^2 e^{-a|x|}$ for $a > 0$.

8. Find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-2x^2+x}$.

9. Prove the Second Shift Property in the form

$$F\left[e^{i\beta x} f(x)\right] = \hat{f}(k + \beta).$$

10. A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t} e^{i\omega_0 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

.

a. Find $\hat{f}(\omega)$ and

b. the frequency distribution $|\hat{f}(\omega)|^2$.

c. Sketch the frequency distribution.

11. Show that the convolution operation is associative: $(f * (g * h))(t) = ((f * g) * h)(t)$.

12. In this problem, you will directly compute the convolution of two Gaussian functions in two steps.

a. Use completing the square to evaluate

$$\int_{-\infty}^{\infty} e^{-\alpha t^2 + \beta t}\, dt.$$

b. Use the result from part a. to directly compute the convolution in Example 5.16:

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2bxt}\, dt.$$

13. You will compute the (Fourier) convolution of two box functions of the same width. Recall, the box function is given by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

Consider $(f_a * f_a)(x)$ for different intervals of x. A few preliminary sketches will help. In Figure 5.45, the factors in the convolution integrand are show for one value of x. The integrand is the product of the first two functions. The convolution at x is the area of the overlap in the third figure. Think about how these pictures change as you vary x. Plot the resulting areas as a function of x. This is the graph of the desired convolution.

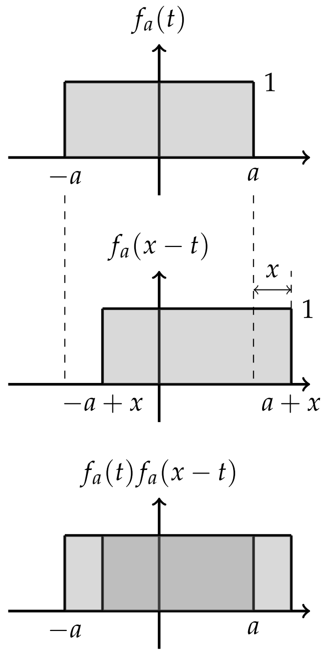

Figure 5.45: Sketch used to compute the convolution of the box function with itself. In the top figure is the box function. The middle figure shows the box shifted by x. The bottom figure indicates the overlap of the functions.

14. Define the integrals $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$. Noting that $I_0 = \sqrt{\pi}$,

a. Find a recursive relation between I_n and I_{n-1}.

b. Use this relation to determine I_1, I_2, and I_3.

c. Find an expression in terms of n for I_n.

15. Find the Laplace transform of the following functions:

a. $f(t) = 9t^2 - 7$.

b. $f(t) = e^{5t-3}$.

c. $f(t) = \cos 7t$.

d. $f(t) = e^{4t} \sin 2t$.

e. $f(t) = e^{2t}(t + \cosh t)$.

f. $f(t) = t^2 H(t-1)$.

g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi. \end{cases}$

h. $f(t) = \int_0^t (t-u)^2 \sin u\, du$.

i. $f(t) = (t+5)^2 + te^{2t} \cos 3t$ and write the answer in the simplest form.

16. Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

a. $F(s) = \dfrac{18}{s^3} + \dfrac{7}{s}$.

b. $F(s) = \dfrac{1}{s-5} - \dfrac{2}{s^2+4}$.

c. $F(s) = \dfrac{s+1}{s^2+1}$.

d. $F(s) = \dfrac{3}{s^2+2s+2}$.

e. $F(s) = \dfrac{1}{(s-1)^2}$.

f. $F(s) = \dfrac{e^{-3s}}{s^2-1}$.

g. $F(s) = \dfrac{1}{s^2+4s-5}$.

h. $F(s) = \dfrac{s+3}{s^2+8s+17}$.

17. Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

a. $f(t) = t^2$, $g(t) = t^3$.

b. $f(t) = t^2$, $g(t) = \cos 2t$.

c. $f(t) = 3t^2 - 2t + 1$, $g(t) = e^{-3t}$.

d. $f(t) = \delta\left(t - \frac{\pi}{4}\right)$, $g(t) = \sin 5t$.

18. For the following problems, draw the given function and find the Laplace transform in closed form.

a. $f(t) = 1 + \sum_{n=1}^{\infty}(-1)^n H(t-n)$.

b. $f(t) = \sum_{n=0}^{\infty}[H(t-2n+1) - H(t-2n)]$.

c. $f(t) = \sum_{n=0}^{\infty}(t-2n)[H(t-2n) - H(t-2n-1)] + (2n+2-t)[H(t-2n-1) - H(t-2n-2)]$.

19. Use the Convolution Theorem to compute the inverse transform of the following:

a. $F(s) = \dfrac{2}{s^2(s^2+1)}$.

b. $F(s) = \dfrac{e^{-3s}}{s^2}$.

c. $F(s) = \dfrac{1}{s(s^2+2s+5)}$.

20. Find the inverse Laplace transform in two different ways: (i) Use tables. (ii) Use the Bromwich Integral.

a. $F(s) = \dfrac{1}{s^3(s+4)^2}$.

b. $F(s) = \dfrac{1}{s^2-4s-5}$.

c. $F(s) = \dfrac{s+3}{s^2+8s+17}$.

d. $F(s) = \dfrac{s+1}{(s-2)^2(s+4)}$.

e. $F(s) = \dfrac{s^2 + 8s - 3}{(s^2 + 2s + 1)(s^2 + 1)}$.

21. Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.

b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.

c. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.

d. $y'' + 6y' + 18y = 2H(\pi - t)$, $y(0) = 0$, $y'(0) = 0$.

22. Use Laplace transforms to convert the following system of differential equations into an algebraic system and find the solution of the differential equations.

$$\begin{aligned} x'' &= 3x - 6y, \quad x(0) = 1, \quad x'(0) = 0, \\ y'' &= x + y, \quad y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

23. Use Laplace transforms to convert the following nonhomogeneous systems of differential equations into an algebraic system and find the solutions of the differential equations.

a.

$$\begin{aligned} x' &= 2x + 3y + 2\sin 2t, \quad x(0) = 1, \\ y' &= -3x + 2y, \quad y(0) = 0. \end{aligned}$$

b.

$$\begin{aligned} x' &= -4x - y + e^{-t}, \quad x(0) = 2, \\ y' &= x - 2y + 2e^{-3t}, \quad y(0) = -1. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y + 2\cos t, \quad x(0) = 3, \\ y' &= x + y - 3\sin t, \quad y(0) = 2. \end{aligned}$$

24. Use Laplace transforms to sum the following series:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2n}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+3)}$.

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 - a^2}$.

e. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 - a^2}$.

f. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} e^{-an}$.

25. Use Laplace transforms to prove

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{b-a} \int_0^1 \frac{u^a - u^b}{1-u}\, du.$$

Use this result to evaluate the following sums:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$.

26. Do the following:

a. Find the first four nonvanishing terms of the Maclaurin series expansion of $f(x) = \dfrac{x}{e^x - 1}$.

b. Use the result in part a. to determine the first four nonvanishing Bernoulli numbers, B_n.

c. Use these results to compute $\zeta(2n)$ for $n = 1, 2, 3, 4$.

27. Given the following Laplace transforms, $F(s)$, find the function $f(t)$. Note that in each case there are an infinite number of poles, resulting in an infinite series representation.

a. $F(s) = \dfrac{1}{s^2(1+e^{-s})}$.

b. $F(s) = \dfrac{1}{s \sinh s}$.

c. $F(s) = \dfrac{\sinh s}{s^2 \cosh s}$.

d. $F(s) = \dfrac{\sinh(\beta\sqrt{s}x)}{s \sinh(\beta\sqrt{s}L)}$.

28. Consider the initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= 2u_{xx}, && 0 < t, \quad 0 \le x \le 1, \\ u(x,0) &= x(1-x), && 0 < x < 1, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) &= 0, && t > 0. \end{aligned}$$

Use the finite transform method to solve this problem. Namely, assume that the solution takes the form $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$ and obtain an ordinary differential equation for b_n and solve for the b_n's for each n.