FOURIER SERIES, HAAR WAVELETS AND FAST FOURIER TRANSFORM

VESA KAARNIOJA, JESSE RAILO AND SAMULI SILTANEN

Abstract. This handout is for the course *Applications of matrix computations* at the University of Helsinki in Spring 2018. We recall basic algebra of complex numbers, define the Fourier series, the Haar wavelets and discrete Fourier transform, and describe the famous Fast Fourier Transform (FFT) algorithm. The discrete convolution is also considered.

Contents

1. Revision on complex numbers

A complex number is a pair $x + iy := (x, y) \in \mathbb{C}$ of $x, y \in \mathbb{R}$. Let $z = x + iy, w = a + ib \in \mathbb{C}$ be two complex numbers. We define the *sum* as

$$z + w := (x + a) + i(y + b)$$

- **Version 1.** *Suggestions and corrections could be send to jesse.railo@helsinki.fi.*
- Things labaled with * symbol are supposed to be extra material and might be more advanced.
- There are some exercises in the text that are supposed to be relatively easy, even trivial, and support understanding of the main concepts. These are not part of the official course work.

and the *product* as

$$zw := (xa - yb) + i(ya + xb).$$

These satisfy all the same algebraic rules as the real numbers. Recall and verify that $i^2 = -1$. However, $\mathbb{C}$ is not an ordered field, i.e. there is not a natural way to say "$z < w$" for two complex numbers $z, w \in \mathbb{C}$ such that the algebraic rules behave nicely with respect to an order $<$ in $\mathbb{C}$.

We define the complex exponential function

$$\exp(x + i\phi) := e^x(\cos(\phi) + i\sin(\phi))$$

for all complex numbers with $x, \phi \in \mathbb{R}$. Notice that here $e^x, \cos(x)$ and $\sin(x)$ are real functions. The *Euler formula*

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

holds for all $\phi \in \mathbb{R}$. One easily notice that $E : \mathbb{R} \to \mathbb{C}, E(\phi) := e^{i\phi}$, has 2π-period since sine and cosine functions have. Using the Euler formula one can notice that every complex number $z = x + iy \in \mathbb{C}$ can be represented in the *polar coordinates* as $z = rE(\phi)$ where $r = |z| := \sqrt{zz^*} = \sqrt{x^2 + y^2}$, $z^* := x - iy$ is the *complex conjugate* (transpose), and $\phi \in \mathbb{R}$. This representation is unique up to the period 2π in the variable ϕ. One usually then requires that the polar coordinates are chosen such a way that $\phi \in [0, 2\pi)$ or $(0, 2\pi]$.

One also notices that the function

$$e : \mathbb{R} \to \mathbb{C}, e(t) := E(2\pi t) = e^{2\pi i t},$$

has period 1. We will use this function e later for the discrete Fourier transform. One also sees that $|e(t)| = 1$ for every $t \in \mathbb{R}$.

Exercise 1.1. a) Given a complex number (r, ϕ) in the polar coordinates, find the Cartesian coordinates (x, y).

b) Given a complex number in the Cartesian coordinates find (x, y), find the polar coordinates (r, ϕ).

Exercise 1.2. Verify the formulas

$$\cos(x) = \frac{E(x) + E(-x)}{2}$$

and

$$\sin(x) = \frac{E(x) - E(-x)}{2i}$$

for every $x \in \mathbb{R}$ using the Euler formula.

Exercise 1.3 (*). Using the real Taylor series[1] for $e^x, \cos(x)$ and $\sin(x)$, verify the Euler formula by plugging in the correct complex variables and assuming that this is well justified.

Exercise 1.4 (*). Write $e^{ix} = r(x)(\cos(\phi(x)) + i\sin(\phi(x)))$ $(*)$ for some real functions $r(x)$ and $\phi(x)$ using the fact that every complex number can be written in the polar coordinates. Further, assume that ϕ and r are differentiable at the point x and that the rule $\frac{d}{dx}e^{ix} = ie^{ix}$ is valid. Show now that the Euler formula is correct by differentiating $(*)$ with respect to x and considering the real and imaginary parts separately. (You need to argue that $r(x) = 1$ and $\phi(x) = x$. Why?)

2. Fourier series

Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic (in other words, satisfies $f(x) = f(x + \nu 2\pi)$ for any $\nu \in \mathbb{Z}$) and can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right), \tag{2.1}$$

where $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ are real-valued coefficients.

Computationally it is very useful to consider approximations of functions and signals by a truncated Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right). \tag{2.2}$$

Then the practical question is: given f, how to determine the coefficients $a_0, a_1, a_2, \dots, a_N$ and $b_1, b_2, \dots, b_N$? Let us derive formulas for them.

The constant coefficient a_0 is found as follows. Integrate both sides of (2.1) from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} f(x)dx &= a_0 \int_0^{2\pi} dx + \\ &\quad + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)dx + \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(nx)dx, \end{aligned} \tag{2.3}$$

where we assumed that the orders of infinite summing and integration can be interchanged. Now it is easy to check that $\int_0^{2\pi} \cos(nx)dx = 0$

[1]See `https://en.wikipedia.org/wiki/Taylor_series#List_of_Maclaurin_series_of_some_common_functions`.

and $\int_0^{2\pi} \sin(nx)dx = 0$ for every $n \in \mathbb{Z}_+$, and trivially $\int_0^{2\pi} dx = 2\pi$. Therefore,

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \tag{2.4}$$

which can be interpreted as the average value of the function f over the interval $[0, 2\pi]$.

Exercise 2.1. Show that $\int_0^{2\pi} \cos(nx)dx = 0$ and $\int_0^{2\pi} \sin(nx)dx = 0$ holds for every $n \in \mathbb{Z}$, $n \neq 0$. Evaluate the integrals also when $n = 0$.

Further, fix any integer $m \geq 1$ and multiply both sides of (2.1) by $\cos(mx)$. Integration from 0 to 2π gives

$$\begin{aligned} \int_0^{2\pi} f(x)\cos(mx)dx &= a_0 \int_0^{2\pi} \cos(mx)dx + \\ &\quad + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)\cos(mx)dx + \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(nx)\cos(mx)dx. \end{aligned} \tag{2.5}$$

We already know that $\int_0^{2\pi} \cos(mx)dx = 0$, so the term containing a_0 in the right hand side of (2.5) vanishes. Clever use of trigonometric identities allows one to see that

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1, \tag{2.6}$$

and that

$$\int_0^{2\pi} \cos(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1 \text{ with } n \neq m. \tag{2.7}$$

Exercise 2.2. Show that the identities (2.6) and (2.7) are valid.

So actually the only nonzero term in the right hand side of (2.5) is the one containing the coefficient a_m.

Exercise 2.3. Verify this identity:

$$\int_0^{2\pi} \cos(nx)\cos(nx)dx = \pi. \tag{2.8}$$

Therefore, substituting (2.8) into (2.5) gives

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos(nx)dx. \tag{2.9}$$

A similar derivation shows that

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\sin(nx)dx. \tag{2.10}$$

One might be tempted to ask: what kind of functions allow a representation of the form (2.1)? Or: in what sense does the right-hand sum converge in (2.2) as $N \to \infty$? Also: under what assumptions can the order of infinite summing and integration can be interchanged in the derivations of (2.3) and (2.5)? These are deep and interesting mathematical questions which will not be further discussed in this short note.

2.1. **Fourier series: complex formulation.** The unit circle (the boundary of the unit disk) can be parametrized as

$$\{(\cos\theta, \sin\theta) \,|\, 0 \le \theta < 2\pi\}.$$

We will use the *Fourier basis functions*

$$\varphi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad n \in \mathbb{Z}. \tag{2.11}$$

We can approximate 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ following the lead of the great applied mathematician Joseph Fourier (1768–1830). Define cosine series coefficients using the L^2 inner product

$$\widehat{f}_n := \langle f, \varphi_n \rangle := \int_0^{2\pi} f(\theta)\, \overline{\varphi_n(\theta)}\, d\theta, \qquad n \in \mathbb{Z}.$$

Then, for nice enough functions f, we have

$$f(\theta) \approx \sum_{n=-N}^{N} \widehat{f}_n \, \varphi_n(\theta)$$

with the approximation getting better when N grows.

Exercise 2.4. Note that the functions $\{\varphi_n\}_{n\in\mathbb{Z}}$ are L^2 orthonormal:

$$\langle \varphi_n, \varphi_m \rangle = \delta_{nm} \quad \forall n, m \in \mathbb{Z}$$

where

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

3. *Haar wavelets

For a wonderful introduction to wavelets, please see the classic book *Ten lectures on wavelets* by Ingrid Daubechies.

3.1. ***Theoretical approach as an orthonormal basis of $L^2([0,1])$.** Consider real-valued functions defined on the interval $[0,1]$. There are two especially important functions, namely the *scaling function* $\varphi(x)$ and the *mother wavelet* $\psi(x)$ related to the Haar wavelet basis, defined as follows:

$$\varphi(x) \equiv 1, \qquad \psi(x) = \begin{cases} 1, & \text{for } 0 \le x < 1/2, \\ -1 & \text{for } 1/2 \le x \le 1. \end{cases}$$

Also, let us define *wavelets* as scaled and translated versions of the mother wavelet:

$$\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k) \qquad \text{for } j,k \in \mathbb{N} \text{ and } k \le 2^j - 1,$$

where $x \in [0,1]$. Let

$$H_{(0,1)} := \{\, (j,k) \;:\; j,k \in \mathbb{N}, k \le 2^j - 1 \,\}.$$

Let $f, g : [0,1] \to \mathbb{R}$. Define the $L^2(0,1)$ inner product between f and g by

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\,dx. \tag{3.1}$$

(Note that the complex conjugate over g in (3.1) is not relevant here as g is real-valued. We just have it there for mathematical completeness.)

Exercise 3.1. Please convince yourself about the fact that wavelets are orthogonormal:

$$\langle \psi_{jk}, \psi_{j'k'} \rangle = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

(Start by understanding why $\langle \psi, \varphi \rangle = 0$ and $\langle \psi, \psi \rangle = 1$, then look at smaller scales corresponding to $j > 0$. Basically it is the same phenomenon always.)

The *Haar wavelet series* of $f \in L^2(0,1)$ is defined as

$$\mathcal{W}(f)(x) := \sum_{(j,k) \in H_{(0,1)}} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x)$$

for $x \in [0,1]$. The collection of coefficients

$$(w_{jk})_{(j,k) \in H_{(0,1)}} := \langle f, \psi_{i,j} \rangle_{(j,k) \in H_{(0,1)}}$$

can be called the *Wavelet transform* of f.

Note that such a series decomposition can be always done for any orthonormal family $\Phi := \{\phi_k \;:\; k \in \mathbb{Z}\} \subset L^2(0,1)$. This is always a *projection* of $f \in L^2(0,1)$ into the (metric) completion of the linear span of Φ, or equivalently the closure of the linear span of Φ in $L^2(0,1)$.

Wheather $f = \mathcal{W}_\Phi(f)$ holds or not depends on the question: Is Φ a Hilbert basis[2] of $L^2(0,1)$?

Theorem 3.2. *The Haar wavelet system is an orthonormal basis of $L^2(0,1)$. Therefore*

$$f(x) = \mathcal{W}(f)(x) \quad \forall x \in [0,1]$$

for any $f \in L^2(0,1)$.

Proof. We do not consider proof of this fact in this course since it requires better knowledge of functional analysis which is an advanced topic. (One should show that every function in $L^2(0,1)$ can be approximated by the functions in the linear span of $\{\, \phi_{ij} \,:\, i,j \in \mathbb{Z} \,\}$. One already knows that the family is orthonormal by Exercise 3.1.) $\square$

The success of Fourier series is based on this very same phenomenon: the functions $\{\, e^{2\pi i n x} \,:\, n \in \mathbb{Z} \,\}$ form an orthonormal basis of $L^2(0,1)$ in the complex case, and $\{\, 1, \cos(2\pi n x), \sin(2\pi n x) \,:\, n \in \mathbb{Z}_+ \,\}$ in the real case.

Exercise 3.3. Can you say what is the orthonormal Fourier basis of $L^2(0,2\pi)$? (Hint: Look at Section 2.)

4. Discrete Fourier transform

In practice one always uses discrete data sets. We next describe how to perform Fourier analysis on such sets: Let $N \in \mathbb{Z}_+$ be positive integer and define the set $A_N := \{0, \dots, N-1\}$. Consider a complex function $f : A_N \to \mathbb{C}$, $f := (z_0, \dots, z_{N-1})$, where $z_j \in \mathbb{C}$ for every $j = 0, \dots, N-1$. Notice that such a function is naturally presented as a vector in $\mathbb{C}^N$. For example, $f(2) = z_2$, and $f(x) = z_x$ if $x = 0, \dots, N-1$.

The *discrete Fourier transform (DFT)* $\mathcal{F}(f) : A_N \to \mathbb{C}$ is defined via the formula

$$\mathcal{F}(f)(\xi) := \frac{1}{N} \sum_{x \in A_N} f(x) e\left(-\frac{x\xi}{N}\right)$$

for every $\xi \in A_N$. The (discrete) Fourier transform is often denoted by $\hat{f} := \mathcal{F}(f)$.

If we write $f = (z_0, \dots, z_{N-1})$ and $\mathcal{F}(f) = (Z_0, \dots, Z_{N-1})$, then we can write more naturally (from the computational point of view) that

$$Z_k = \sum_{j=0}^{N-1} z_j e\left(-\frac{jk}{N}\right) = \sum_{j=0}^{N-1} z_j e^{-2\pi i jk/N}$$

[2]See `https://en.wikipedia.org/wiki/Hilbert_space#Orthonormal_bases`.

for all $k = 0, \ldots, N-1$. Let us call this as the *computational representation*.

Exercise 4.1. Verify that $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ is a linear mapping.

We next seek for a formula of the inverse Fourier transform and the matrix of $\mathcal{F}$. These are essentially helpful in many applications.

Theorem 4.2 (Discrete Fourier inverse formula)**.** *Let $f : A_N \to \mathbb{C}$. Then*

$$f(x) = \sum_{\xi \in A_N} \hat{f}(\xi) e\left(\frac{x\xi}{N}\right)$$

for every $x \in A_n$. Therefore,

$$z_j = \sum_{k=0}^{N-1} Z_k e^{2\pi i jk/N}$$

for every $j = 0, \ldots, N-1$ in the computational representation.

Proof. The proof of this formula is a short calculation. We have that

$$\begin{aligned}
\sum_{\xi \in A_N} \hat{f}(\xi) e\left(\frac{x\xi}{N}\right) &= \sum_{\xi \in A_N} \left[\frac{1}{N} \sum_{y \in A_N} f(y) e\left(-\frac{y\xi}{N}\right)\right] e\left(\frac{x\xi}{N}\right) \\
&= \sum_{y \in A_N} f(y) \frac{1}{N} \sum_{\xi \in A_N} e\left(\frac{(x-y)\xi}{N}\right).
\end{aligned}$$

We have that

$$\sum_{\xi \in A_N} e\left(\frac{(x-y)\xi}{N}\right) = \begin{cases} 0 & \text{if } x \neq y \\ N & \text{if } x = y \end{cases}$$

using the exercise below. Now we plug these two formulas together to obtain the result. The computational representation is nothing else than rewriting the formula using the vector notation. □

We denote the *discrete inverse Fourier transform* by $\mathcal{F}^{-1}$, i.e.

$$\mathcal{F}^{-1}(f) := \sum_{\xi \in A_N} f(\xi) e\left(\frac{x\xi}{N}\right).$$

It is again a linear map by results of elementary linear algebra. (You may want to recall how it is proved that if $A : V \to W$ is an invertible linear map, then $A^{-1} : W \to V$ is an invertible linear map as well.)

Exercise 4.3. Verify that

$$\sum_{\xi \in A_N} e\left(\frac{x\xi}{N}\right) = \begin{cases} 0 & \text{if } x \neq 0 \\ N & \text{if } x = 0. \end{cases}$$

Basic formulas for the geometric series might be useful.

Exercise 4.4. Write down $\mathcal{F}^{-1}f(\xi)$ using the computational representation (vector notation).

Let us then describe $\mathcal{F}$ as an invertible matrix in $\mathbb{C}^{N\times N}$ as it is a linear automorphism (invertible linear map from the space onto itself) of $\mathbb{C}^N$. It is moreover an *isometry* (up to an normalizing factor), i.e. $\|\mathcal{F}f - \mathcal{F}g\| = C\,\|f-g\|$, when $\mathbb{C}^N$ is equipped with the standard 2-norm, i.e. $f = (f_0, \ldots, f_{N-1}) \in \mathbb{C}^N$ has the norm

$$\|f\| = \sqrt{|f_0|^2 + \cdots + |f_{N-1}|^2}.$$

We do not consider this fact further here (this is a finite version of the *Plancherel theorem*).

One has that

$$\mathcal{F}_{kl} = \frac{1}{N}(e(-kl/N)) = \frac{1}{N}((e(-k/N))^l), \quad k, l = 0, \ldots, N-1.$$

This is called the *DFT matrix*. This is a *Vandermonde matrix*[3] which provides another proof for invertibility using the determinant formula for Vandermonde matrices and the elementary fact that a linear map is invertible if and only if $\det \neq 0$.

Exercise 4.5. Write down $\mathcal{F}$ using the matrix notation:

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_{00} & \mathcal{F}_{01} & \cdots & \mathcal{F}_{0(N-1)} \\ \mathcal{F}_{10} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{F}_{(N-1)0} & \cdots & \cdots & \mathcal{F}_{(N-1)(N-1)} \end{pmatrix}$$

when $N = 4$. Write also the numbers $e(-kl/N)$ using the form $e^{2\pi i ?}$.

Exercise 4.6 (*). Write down the matrix of $\mathcal{F}^{-1}$ directly from the formula that defines the discrete inverse Fourier transform. (We should be thankful for this since it avoids a need to invert the matrix of $\mathcal{F}$ using the general formula for calculating an inverse matrix.) Then verify that

$$\mathcal{F}^{-1} = N\mathcal{F}^*$$

where * denotes the conjugate transpose. (If one defines $U = \sqrt{N}\mathcal{F}$, then $U^{-1} = U^*$ and U would be an unitary operator[4]. Sometimes this might be useful in practical computations, especially if dimensions are high to avoid unnecessary multiplications. It is perhaps an eternal

[3]See `https://en.wikipedia.org/wiki/Vandermonde_matrix`.

[4]See `https://en.wikipedia.org/wiki/Unitary_matrix`.

argue how to define the constant in all different versions of Fourier transforms.)

4.1. ***Discrete convolutions.** One can also define the convolution of two functions $f, g : A_N \to \mathbb{C}$, i.e. $f, g \in \mathbb{C}^N$, as a function $A_N \to \mathbb{C}$ via the formula

$$(f*g)(x) := \frac{1}{N}\left(\sum_{y\in A_N, y\le x} f(y)g(x-y) + \sum_{y\in A_N, y>x} f(y)g(x-y+N)\right).$$

One can also represent the convolution simply as

$$(f*g)(x) = \frac{1}{N}\sum_{y\in\mathbb{Z}_N} f(y)g(x-y)$$

where $A_N = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ is thought as a finite group of N elements. However, the first definition is the one that is often used to calculate convolutions numerically in practice whereas the second reveals its relation to group and number theories. If you are not familiar with groups yet, then you do not have to worry the second formulation.

Exercise 4.7. Write down the convolution operation using the computational representation (vector notation). Define the operator $K_f : \mathbb{C}^N \to \mathbb{C}^N, K_f(g) := f*g$. Show that K_f is a linear operator and find its matrix.

Exercise 4.8. Verify that $\mathcal{F}(f*g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. This means also that the (Fast) Fourier transform can be used to calculate a convolution using the formula

$$f*g = \mathcal{F}^{-1}(\hat{f}\hat{g}).$$

Show that $f*g = g*f$. (You can see this directly from the definition, but it is now highly recommended to use its relation to the Fourier transform to argue this.) Does this also imply that $K_f = K_g$?

5. Fast Fourier transform (FFT)

Next we describe the Fast Fourier transform (FFT) which is widely used algorithm in practice due importance of Fourier transforms in science and engineering. It is called *fast* since it is computationally faster than the direct use of the formulas given in the earlier sections.

Let now $N = 2^\alpha$ for some $\alpha \in \mathbb{Z}_+$. We call products and sums of complex numbers *elementary operations*. We will show that the DFT of $f : A_N \to \mathbb{C}$ can be calculated using only $O(N\log N)$ elementary operations. This *big-O notation* $O(N\log N)$ means that the number of required calculations is at most a constant $C > 0$ times $N\log N$. It is frequently used notation in analysis of algorithms and applied mathematics. We presume that the all values of $e(x/N)$ are already calculated, e.g. in a database where it can be read.

Define $f_0(x) = f(2x)$ and $f_1(x) = f(2x+1)$ for all $x = 0, \ldots, N/2-1$. Notice that this is well defined since $N/2 = 2^\alpha/2 = 2^{\alpha-1}$ is an integer and $\alpha \in \mathbb{Z}_+$. One has also that $f_0 : A_{N/2} \to \mathbb{C}$ and $f_1 : A_{N/2} \to \mathbb{C}$. We will prove the following recursion formula in the end of this section.

Theorem 5.1 (Radix-2 FFT)**.** *Let $N = 2^\alpha$ for some $\alpha \in \mathbb{Z}_+$ and $f : A_N \to \mathbb{C}$. Then*

$$\hat{f}(\xi) = \frac{1}{2}\left(\hat{f}_0(\xi) + \hat{f}_1(\xi) e\left(\frac{-\xi}{N}\right)\right).$$

Notice that $\hat{f}_0$ and $\hat{f}_1$ are discrete Fourier transforms of lower dimension than $\hat{f}$. If we already know $\hat{f}_0(\xi)$ and $\hat{f}_1(\xi)$, then we know that $\hat{f}(\xi)$ can be calculated from those using two products and one sum. In this case only 3 new elementary operations are required. If we know $\hat{f}_0$ and $\hat{f}_1$ for all values of ξ, then $\hat{f}$ can be calculated using $3N$ operations. Theorem 5.1 leads to a recursive algorithm, denoted by FFT_α, that calculates the DFT.

Theorem 5.2. *The FFT algorithm $\hat{f} = FFT_\alpha(f)$ has the computational complexity $O(N \log N)$ for any input $f \in \mathbb{C}^N$.*

Proof. If $\alpha = 1$, then $\hat{f}(0) = \frac{f(0)+f(1)}{2}$ and $\hat{f}(1) = \frac{f(0)-f(1)}{2}$. Hence FFT_1 uses 4 elementary operations.

Let $|FFT_\alpha|$ denote the maximum number of elementary operations needed for any input of FFT_α (this is the *computational complexity* of the algorithm) and $\#(FFT_\alpha(f))$ the number of elementary operations needed for the input f. Let $\alpha > 1$. Now

$$\#(FFT_\alpha(f)) \leq 2\,|FFT_{\alpha-1}| + 3N$$

using the recursion formula for the FFT. We can iterate k times

$$\begin{aligned} 2\,|FFT_{\alpha-1}| + 3N &\leq 2(|FFT_{\alpha-2}| + 3N/2) + 3N \\ &= 2^2\,|FFT_{\alpha-2}| + 2 \cdot (3N) \\ &\leq 2^k\,|FFT_1| + 3kN \end{aligned}$$

until $\alpha - k = 1$. We have that $k = \alpha - 1$. Since this estimate holds for any input f, we have that

$$|FFT_\alpha| \leq 2^{\alpha-1}\,|FFT_1| + 3(\alpha-1)N = 2^{\alpha-1} \cdot 4 + (\alpha-1)3N$$

where we used the fact that $|FFT_1| = 4$.

Finally, we use that $\alpha = \log(N)/\log(2)$ and perform an elementary estimation that

$$2^{\alpha-1}\cdot 4 + (\alpha-1)3N \leq 2^{\alpha+1} + 3\alpha N = 2N + 3\alpha N \leq CN\log(N)$$

for a sufficiently large $C > 0$. This completes the proof. $\square$

Exercise 5.3. Find a suitable constant $C > 0$ in the previous proof.

Exercise 5.4. Given N (not necessarily $N = 2^\alpha$), what is $O(DFT)$ using directly the algorithm based on the definition of the DFT?

Exercise 5.5 (*). Try to figure out an analog of the FFT for $N = 3^\alpha$ (the recursion formula). What is the computational complexity of such algorithm? (This would be a *radix-3 FFT*.)

Proof of Theorem 5.1. This a simple calcutation using definitions.

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{N}\sum_{x\in A_N} f(x)e\left(\frac{-x\xi}{N}\right)\\
&= \frac{1}{2}\left(\frac{1}{N/2}\sum_{x\in A_{N/2}} f(2x)e\left(\frac{-x\xi}{N/2}\right) + \frac{1}{N/2}\sum_{x\in A_{N/2}} f(2x+1)e\left(\frac{-(2x+1)\xi}{N}\right)\right)\\
&= \frac{1}{2}\left(\frac{1}{N/2}\sum_{x\in A_{N/2}} f_0(x)e\left(\frac{-x\xi}{N/2}\right) + \frac{1}{N/2}\sum_{x\in A_{N/2}} f_1(x)e\left(\frac{-x\xi}{N/2}\right)e\left(\frac{-\xi}{N}\right)\right)\\
&= \frac{1}{2}\left(\hat{f}_0(\xi) + \hat{f}_1(\xi)e\left(\frac{-\xi}{N}\right)\right).
\end{aligned}$$

$\square$

One can apply the FFT for example to a fast multiplication of polynomials. This recursive algorithm is also very important in many practical algorithms. Some practical examples are considered at the live coding lectures and official course exercises. There are various ways to really program an FFT algorithm, for example: *Cooley-Tukey FFT algorithm*[5], *Prime-factor FFT algorithm*, *Bruun's FFT algorithm*, *Rader's FFT algorithm*, *Bluestein's FFT algorithm*, and *Hexagonal Fast Fourier Transform*.[6]

[5]The radix-2 FFT desribed in this note is the mathematical basis for this algorithm.

[6]See `https://en.wikipedia.org/wiki/Fast_Fourier_transform`. The Prime-factor FFT algorithm can be used for any $N \in \mathbb{Z}_+$. In this case one divides the problem into smaller problems using the prime factorization of N similarly to the radix-2 FFT (where 2 is the only prime factor of $N = 2^\alpha$ with multiplicity α).

Exercise 5.6. There are several applications of the FFT. Learn one cool application of the FFT from the internet!