Appendix B

Fourier Transforms & Generalized Functions

B.1 Introduction to Fourier Transforms

The original application of the techniques of Fourier analysis was in Fourier' s studies of heat flow, Thorie Analytique de la Chaleur (The Analytical Theory of Heat), published in 1822. Fourier unwittingly revolutionized both mathematics and physics. Although similar trigonometric series were previously used by Euler, d'Alembert, Daniel Bernoulli and Gauss, Fourier was the first to recognize that such trigonometric series could represent arbitrary functions, even those with discontinuities. It required many years to clarify this insight, and it has led to important theories of convergence, function space, and harmonic analysis.

The classic text for Fourier transforms for physicists is Titchmarsh, [Titchmarsh 1948]. Here we review the aspects of the theory that is relevant to Electromagnetism. Although Fourier's original interest was in the analysis of heat flow, the simplicity of these techniques is generally applicable to any field theory. In particular, the expansion of functions in a series of special functions such as the sin and cos functions is at its center a result of the underlying symmetry of the space. For these functions form a complete set of irreducible representations of the symmetry group of translations and inversions, the $\sin \frac{2\pi n x}{\lambda}, n = 1, 2, \cdots$, series for the transformation $x' = x + \lambda$ with λ discrete and odd under $x' = -x$ and the $\cos \frac{2\pi n x}{\lambda}$, $n = 0, 1, 2, \cdots$ series for the transformation $x' = x + \lambda$ and even under $x' = -x$. For the case of continuous translation symmetry in which the translation parameter can take any value, $x' = x + a$, the set $\sin kx$ forms a complete set of irreducible representations of the odd functions and $\cos kx$ for the even functions. The label of the representation, k, is called the wavenumber and the wavelength is defined as $\lambda \equiv \frac{2\pi}{k}$. For the general case without the even and odd requirement, the representation is simply $eikx$ for all k, $-\infty < k < \infty$. The intimate relationship between the Fourier transform and generalized functions has improved the understanding of both, a route that we follow.

The analysis of electromagnetic phenomena in wavenumber space, the space of the Fourier transform, is especially fruitful. Since local field theories describe interactions through field values and derivatives at the field point, the wave-number description eliminates the derivatives for algebraic relations: a significant gain in analytic simplicity. It is also the case that the kinematics of the vector fields are easier to implement in wave-number space. These advantages, of course, are offset by the difficulty of developing an intuitive interpretation of the behavior of vector fields in wave-number space; the Fourier transform is a mapping that is intrinsically non-local and maps points in configuration space to regions in wave-number space.

After reviewing Fourier transforms in one dimension in Section B.2, we will study the relationship of these transforms to the notion of generalized function, Section B.3. We conclude with the application of these techniques in higher dimensional spaces, Section B.4

B.2 One Dimensional Fourier Transforms

The idea of Fourier transforms is a natural extension of the idea of Fourier series[1]. A function, $F(x)$, with periodicity, λ, in the sense $F(x+\lambda) = F(x)$ is represented by the series

$$F(x) = \frac{1}{\sqrt{2\lambda}} a_0 + \sum_{n=1}^{\infty} \left(a_n \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi n x}{\lambda} + b_n \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi n x}{\lambda} \right) \tag{B.1}$$

where

$$a_m = \sqrt{\frac{2}{\lambda}} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} F(t) \cos \frac{2\pi m t}{\lambda} dt \tag{B.2}$$

[1]There are many conventions on the normalizations of the Fourier transforms. Here, I am using a convention that is the one most commonly followed by physicists. It is a slight variation of the one of Titchmarsh, [Titchmarsh 1948], which is advantageous for the development of formal manipulations and proofs. Mathematicians commonly use another convention such as the one in Lighthill, [Lighthill 1958]. The primary difference is in the location of the 2π conversion from cycles to radians.

and

$$b_m = \sqrt{\frac{2}{\lambda}} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} F(t) \sin \frac{2\pi m t}{\lambda} dt. \tag{B.3}$$

Note that, if $F(x)$ is an even function, the b_m's are all zero and, thus, for even functions, the Fourier series and the Fourier cosine series are the same. Similarly, for odd functions, the Fourier sine series and the Fourier series coincide.

Inserting Equations B.2 and B.3 into Equation B.1,

$$F(x) = \frac{1}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} F(t) dt + \frac{2}{\lambda} \sum_{n=1}^{\infty} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} F(t) \cos \frac{2\pi n \left(x - t\right)}{\lambda} dt. \tag{B.4}$$

If $F(t)$ has compact support, and putting $k = \frac{2\pi n}{\lambda}$ and identifying $\delta k = \frac{2\pi}{\lambda}$ and letting $\lambda \to \infty$, Equation B.4 becomes

$$F(x) = \frac{1}{\pi} \int_0^{\infty} dk \int_{-\infty}^{\infty} F(t) \cos\left(k\left(x - t\right)\right) dt. \tag{B.5}$$

This is Fourier's integral formula.

It is worthwhile to interrupt this development at this point to note that in the more modern language of generalized functions, Section B.3, we would identify $\frac{1}{\pi} \int_0^{\infty} dk \cos\left(k\left(x - t\right)\right)$ as one of the many possible manifestations of the delta function, $\delta(x - t)$. We will have several more examples of this kind in our development of Fourier transforms, a rich source of generalized functions, see Section B.3.

Using the Fourier integral formula, Equation B.5, an expansion similar to the Fourier series expansion, Equation B.1, and the separation of even and odd functions with the resultant Fourier sine and cos series and resulting Fourier sine and cosine integrals is possible.

$$F(x) = \int_0^{\infty} \left\{a(k) \cos kx + b(k) \sin kx\right\} dk \tag{B.6}$$

where

$$a(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} F_e(t) \cos kt \, dt \tag{B.7}$$

and

$$b(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} F_o(t) \sin kt \, dt \tag{B.8}$$

and $F_e(t)$ and $F_o(t)$ are the even and odd parts of $F(t)$ in the sense

$$\begin{aligned} F_e(t) &\equiv \frac{1}{2}\{F(t)+F(-t)\} \\ F_o(t) &\equiv \frac{1}{2}\{F(t)-F(-t)\} \end{aligned} \tag{B.9}$$

and $F(t) = F_e(t) + F_o(t)$. Thus, for even and odd functions, we have the Fourier cosine formula and Fourier sine formula

$$F_e(x) = \frac{2}{\pi}\int_0^\infty \cos kx\, dk \int_0^\infty F_e(t)\cos kt\, dt \tag{B.10}$$

and

$$F_o(x) = \frac{2}{\pi}\int_0^\infty \sin kx\, dk \int_0^\infty F_o(t)\sin kt\, dt. \tag{B.11}$$

The form most useful to us is found by expanding the cos in Equation B.5 to yield

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty dk \int_{-\infty}^\infty dt F(t) e^{ik(t-x)} \tag{B.12}$$

Cauchy was the first to realize that these Fourier integral formulas lead to a reciprocal relation between pairs of functions. From the exponential form of the Fourier integral formula, Equation B.12, we obtain the Fourier transformation relations

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(t)e^{-ikt}dt \tag{B.13}$$

and

$$F(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathcal{F}(k)e^{ikt}dk. \tag{B.14}$$

The Cauchy reciprocity is that, if $\mathcal{F}(k)$ is the Fourier transform of $F(t)$, then $F(t)$ is the Fourier transform of $\mathcal{F}(-k)$ which is $\mathcal{F}^*(k)$ if $F(t)$ is real.

The sine and cosine transformations follow similarly as

$$\begin{aligned} \mathcal{F}_c(k) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kt\, F(t)dt \\ F(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kt\, \mathcal{F}_c(k)dk \end{aligned} \tag{B.15}$$

and

$$\begin{aligned} \mathcal{F}_s(k) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin kt\, F(t)dt \\ F(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin kt\, \mathcal{F}_s(k)dk. \end{aligned} \tag{B.16}$$

Clearly, $\mathcal{F}_c(k)$ is an even function of k for any $F(t)$ for which it is defined, i. e. $L(0, \infty)$, and from the latter of the Equations B.15, $F(t)$, now defined on the domain $-\infty$ to ∞, is an even function of t. The Cauchy reciprocity, in this case is that if $\mathcal{F}_c(k)$ is the Fourier cosine transformation of $F(t)$, then the even function $F_e(t)$ which coincides with $F(t)$ in the domain 0 to ∞, is the Fourier transformation of $\mathcal{F}_c(k)$. If $F(t)$ is real, $\mathcal{F}_c(k)$ is real.

In order to further develop the ideas and limitations of these transforms, it is worthwhile to work through a few examples to show these important properties.

Consider the function e^{-x}. It should be clear that any attempt of use the Fourier transform, Equations B.13 and B.14, is questionable in this case since the integrals involved may not be defined. In fact the derivation of Equation B.5, Fourier's integral formula, our basic starting point, required that $F(x)$ have compact support. A more general requirement that should allow all the integrals to be defined would be $\int_{-\infty}^{\infty} |F(t)|dt < \infty$ or we can say $F(t)$ must be in $L(-\infty.\infty)$ where L represents the Lebesgue measure. Later in this section, we will discuss other conditions that are possible. For now at least, it should be obvious that our example, e^{-x} in the domain $-\infty$ to ∞ is not eligible for Fourier transformation. In this case though, we can implement a transformation based on the domain 0 to ∞ as required for the cosine and sine transformations. Using Equations B.15, and directly integrating, the Fourier cosine transformation of e^{-x} is $\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}$. There is the even function, $e^{-|x|}$, that coincides with e^{-x} in the domain 0 to ∞ and is $L(-\infty, \infty)$. Thus the Fourier transformation of the even function $e^{-|x|}$ is the same as the Fourier cosine transformation, $\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}$. Note that the even function $\cosh x$ constructed from e^{-x} by Equations B.9 is not $L(-\infty, \infty)$ and is not the appropriate Fourier partner to $\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}$. Note also the the Fourier cosine transformation is real by definition and thus the Fourier transformation of $e^{-|x|}$ is real and even in k. Similarly by direct integration, the Fourier sine transformation to e^{-x} is $\sqrt{\frac{2}{\pi}}\frac{k}{1+k^2}$. It is real and odd in k. Thus the Fourier transform of the odd function

$$F(x) = \begin{cases} e^{-x} : & 0 < x < \infty \\ 0 : & x = 0 \\ -e^{x} : & -\infty < x < 0 \end{cases}$$

is $\frac{1}{\sqrt{2\pi}}\frac{ik}{1+k^2}$. It is imaginary and odd in k. Like the Fourier series, Fourier transforms can handle discontinuities. In order to remain integrable, the functions must have bounded variation though.

Some of the results of the previous example can be readily generalized. For real $F(t)$, since both the Fourier sine and cosine transformations are real and respectively odd in k and even in k, the Fourier transform of an even function is even and real and the Fourier transform of an odd function is odd and imaginary. In general for real $F(t)$,

$$\mathcal{F}(k) = \mathcal{F}^*(-k). \tag{B.17}$$

Admitting the case of complex $F(t)$, we can derive an important set of relations called the Parseval identities. First, define

$$\mathcal{F}_*(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^*(t) e^{-ikt} dt,$$

the Fourier transform of the complex conjugate of $F(t)$. Then Equation B.17 generalizes to $\mathcal{F}_*(k) = \mathcal{F}^*(-k)$. Given two complex functions $F(t)$ and $G(t)$ with corresponding Fourier transforms $\mathcal{F}$ and $\mathcal{G}$,

$$\begin{aligned} \int_{-\infty}^{\infty} F(t) G^*(t) dt &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \left\{ \int_{-\infty}^{\infty} \mathcal{F}(k) e^{ikt} dk \right\} G^*(t) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \mathcal{F}(k) \left\{ \int_{-\infty}^{\infty} G^*(t) e^{ikt} dt \right\} \\ &= \int_{-\infty}^{\infty} dk \mathcal{F}(k) \mathcal{G}_*(-k) \\ &= \int_{-\infty}^{\infty} dk \mathcal{F}(k) \mathcal{G}^*(k). \end{aligned} \tag{B.18}$$

For the special case of $G(t) = F(t)$, we have the important result

$$\int_{-\infty}^{\infty} |F(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}(k)|^2 dk. \tag{B.19}$$

These relations are known as the Parseval Identities.

Similarly, consider the inverse Fourier transform, Equation B.14, of the product of two Fourier transforms, $\mathcal{F}(k)$ and $\mathcal{G}(k)$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \mathcal{F}(k) \mathcal{G}(k) e^{ikt} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \mathcal{F}(k) e^{ikt} \int_{-\infty}^{\infty} dx G(x) e^{-ikx} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx G(x) \int_{-\infty}^{\infty} \mathcal{F}(k) e^{-ik(t-x)} dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx G(x) F(t-x). \end{aligned} \tag{B.20}$$

Thus, the inverse Fourier transform of a product of two Fourier transforms is the convolution of the original functions. Of course, these relations hold in the sense of the Cauchy reciprocity mentioned after Equation B.14 suitably generalized to complex functions when appropriate.

Suppose our problem is one which is not centered on the origin but instead about some point a. For example for the case that we have been analyzing, $F(x)$ is $e^{-|x-a|}$ is the translate of $e^{-|x|}$. From Equation B.13, it follows that the Fourier transform, $\mathcal{F}_t(k)$, of the translated function, $F_t(x) = F(x+a)$ is

$$\mathcal{F}_t\left\{F(x+a)\right\}(k) = e^{ika}\mathcal{F}\left\{F(x)\right\}(k). \tag{B.21}$$

Thus the Fourier transform of $e^{-|x-a|}$ is $\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}e^{-ika}$.

Another important issue for physical applications is that the prior discussions are based on mathematical examples. In mathematics, the variables are all dimensionless, whereas in physical applications the variables generally have dimensional content; x or t are often lengths or times. In this regard, a formula such as e^{-x} will appear in a physics context as $e^{-\frac{x}{a}}$, $a > 0$, the magnitude of the effect falls by $\frac{1}{e}$th when x advances by an amount a. The dimensional parameter a plays two roles. It has the same dimensions as x and thus provides dimensional consistency. It also provides a scale for the phenomena. The problem is that the Fourier transforms do not appear reciprocal when the variable has dimension. From Equation B.13, k must have the inverse dimension of t and the Fourier transform must have different dimensions from $F(t)$. $\mathcal{F}(k)$ has the dimension of $F(t)$ divided by the dimension of t. Consider the Fourier transform of a scaled variable,

$$\begin{aligned}\mathcal{F}_s(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\frac{t}{a})e^{-ikt}dt \\ &= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t')e^{-iakt'}dt' \\ &= a\mathcal{F}(ak),\end{aligned} \tag{B.22}$$

where $\mathcal{F}_s(k)$ is the Fourier transform for the scaled function. For our example, $e^{-\frac{|x|}{a}}$, the Fourier transform is $\sqrt{\frac{2}{\pi}}\frac{a}{1+a^2k^2}$.

An observation that follows from this scaling rule is an approximate inverse relationship between the regions of support in configuration space and the regions of support in Fourier space. This is best exemplified by the special case of the Gaussian function. The Fourier transform of a Gaussian is a Gaussian. Starting from $F_1(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$, we have $\mathcal{F}_1(k) =$

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}e^{-ikt}dt = \frac{1}{2\pi}e^{-\frac{k^2}{2}}\int_{-\infty}^{\infty}e^{-\frac{(t+ik)^2}{2}}dt = \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}$ or

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} \longleftrightarrow \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}} \tag{B.23}$$

form a Fourier transform pair. Extending this result by scaling the coordinate, $t \longrightarrow \frac{t}{\sigma}$, by the standard deviation and using the result of Equation B.22, we have for the normed Gaussian

$$\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{\sigma^2 k^2}{2}} \tag{B.24}$$

This result can easily be extended to the set of Hermite functions[2]. The generator of the Hermite functions is

$$e^{\left(-\frac{x^2}{2}+2xt-t^2\right)} = \sum_{n=0}^{\infty} e^{\left(-\frac{x^2}{2}\right)} H_n(x)\frac{t^n}{n!}. \tag{B.25}$$

Using the results of Equation B.24, the Fourier transform of the generator on the variable x,

$$\begin{aligned}\mathcal{F}\left\{e^{\left(-\frac{x^2}{2}+2xt-t^2\right)}\right\}(k) &= e^{\left(-\frac{k^2}{2}-2ikt+t^2\right)} \\ &= \sum_{n=0}^{\infty} e^{\left(-\frac{k^2}{2}\right)} H_n(k)\frac{(-it)^n}{n!}.\end{aligned}$$

Performing the Fourier transform of the terms of the generator expansion in Equation B.25,

$$\mathcal{F}\left\{\sum_{n=0}^{\infty} e^{\left(-\frac{x^2}{2}\right)} H_n(x)\frac{t^n}{n!}\right\} = \sum_{n=0}^{\infty}\mathcal{F}\left\{e^{\left(-\frac{x^2}{2}\right)} H_n(x)\right\}\frac{t^n}{n!}, \tag{B.26}$$

and comparing coeficients, we have

$$\mathcal{F}\left\{e^{\left(-\frac{x^2}{2}\right)} H_n(x)\right\} = (-i)^n e^{\left(-\frac{k^2}{2}\right)} H_n(k). \tag{B.27}$$

The previous examples of scaling, Equation B.22, and translation, Equation B.21, can be combined to yield the general result that if $\mathcal{F}(k)$ is the

[2] The Hermite functions are the normalized products of the Hermite polynomials and a Gaussian, $\psi_n(x) = \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}e^{-\frac{x^2}{2}}H_n(x)$.

Fourier transform of $F(x)$, then the Fourier transform of the rescaled[3] translated $F(x)$,

$$F_{st}(x) \equiv F(ax+b),$$

with a and b constant, is

$$\mathcal{F}_{st}(k) = \frac{e^{i\frac{kb}{a}}}{a}\mathcal{F}\left(\frac{k}{a}\right). \tag{B.28}$$

This kind of relationship between translation and scaling and the Fourier transform can be interpreted through the concepts associated with symmetries and symmetry group representations. In a one dimensional world, the action of the rescaling and translation $x \to x' = ax + b$ is the most general linear transformations that can be implemented. This is a two parameter Lie group. The sub groups of translations, $a = 1$, and scaling, $b = 0$, are both abelian and have simple representation structure. The Fourier transforms form an irreducible

A common and useful trick for the evaluation of a Fourier transform is to allow k to become a complex variable and consider the integration in Equation B.14 to be an integral along the real axis, see Figure B.1.

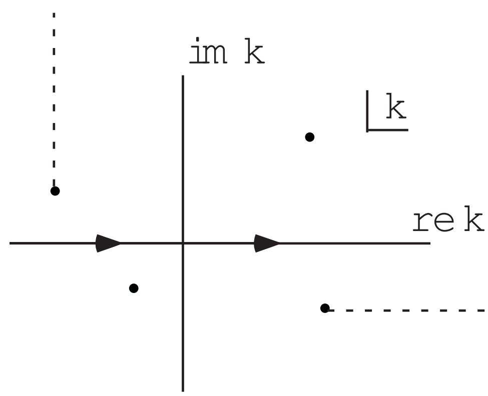

Figure B.1: **Complex k Plane** Converting the real axis integral into a line integral in the complex k plane. The dots represent singularities in $\mathcal{F}(k)$. The isolated points are poles or higher power singularities and the dashed lines ending on a point are branch cuts.

Because of the exponential in Equation B.14 and the controlled nature of the $\mathcal{F}(k)$ at large k, The contour can be closed either above or below

[3]Note that the rescale variable here is the inverse of the one used above, Equation B.22

and depending on the sign of t often evaluated or at least estimated. For example for the case shown in Figure B.1 the contour can be shifted to the one shown in Figure B.2. If $t > 0$, the integral over the large arc is zero and the integral is reduced to one along branch cuts and poles. The poles can be evaluated by the Cauchy Residue Theorem. If $t < 0$, a similar translation of the contour reduces the integral to one involving the branch cuts and isolated singularities in the lower half plane.

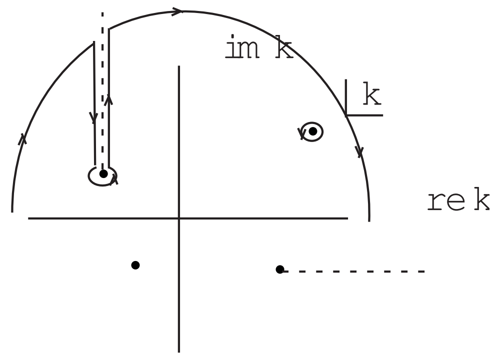

Figure B.2: **Complex Integral Shifted** Converting the real axis integral into a line integral in the complex k plane. For the case of the singularity structure in Figure B.1, the contour integral here is equal to the real integral shown there.

Some examples will clarify the use of these techniques. Consider $F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+k^2}} e^{ikt}$. The singularities are a square root branch cut at $k = \pm ia$ and we can run the branch cut along the imaginary axis with the cuts directed away from the origin. For the case of $t > 0$, the original contour can be pushed upward as shown in Figure B.3. Because of the exponential, the integral along the large arc can be neglected as the arc radius goes to infinity. The problem reduces to the integral around the singularity and the integrals along the sides of the branch cut. The integral around the

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B.3 Generalized Functions

B.3.1 Introduction

The first generalized function was the Dirac delta function, $\delta(x)$. Introduced by him to allow for the analysis of some aspects of quantum mechanics. It

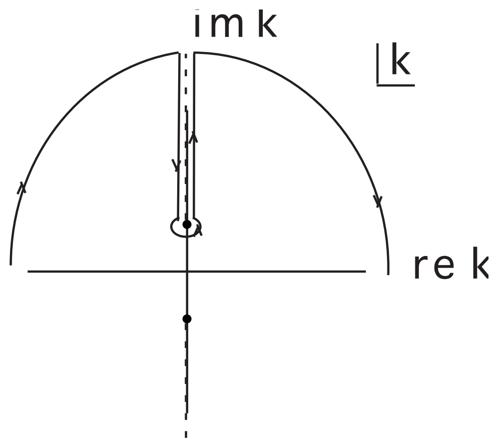

Figure B.3: **Cut Example** The contour for the integral of $F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+k^2}} e^{ikt}$ when $t > 0$.

is not a function in the ordinary sense because it is zero everywhere except at the argument value zero where it is infinitely large is such a way that

$$\int_{x_1}^{x_2} \delta(x)F(x)dx = \begin{cases} F(0) & : x_1 < 0 < x_2 \\ -F(0) & : x_2 < 0 < x_1 \\ 0 & : \text{ otherwise} \end{cases} \tag{B.29}$$

for any reasonable function $F(x)$.

A more general definition of generalized functions would be a limit of functions that is itself not a function but which still has well prescribed analytic properties. In this sense, Dirac was not the first to use these forms. As discussed in Section B.2, our analysis of the Fourier transform which follows that of Fourier used the step in the analysis that identified an integral identity that utilized what today would be called a Dirac delta function. Cauchy in his development of the Fourier transform also used limit forms to carry out important steps in the derivation. These were all specific applications of limit forms. A rigorous theory of generalized functions derived from limit forms has been developed by Laurent Schwartz and given the general name of distribution theory. A very good, short, and rigorous introduction is given by Lighthill [Lighthill 1958]. The development of the theory of distributions provided a very nice extension of and has broad implications for Fourier transforms.

B.3.2 Dirac Delta Function

The Dirac delta function is the generalized function given by Equation B.29. Some of the commonly used limit forms are

$$\delta(x) = \lim_{\epsilon \longrightarrow 0} \begin{cases} \frac{1}{\epsilon\sqrt{\pi}} e^{(-\frac{x^2}{\epsilon^2})} & : \text{ Gaussian} \\ \frac{\frac{\epsilon}{\pi}}{x^2+\epsilon^2} & : \text{Lorentzian} \\ \frac{\sin\frac{x}{\epsilon}}{\pi x} & : \text{ Dirichlet} \end{cases} \tag{B.30}$$

Each of these forms have the obvious property of vanishing for all x other than $x = 0$ where it is undefined and, in an integral with a reasonable function, yields Equation B.29. The many limit forms that produce the same generalized function are said to be equivalent.

Several simple properties follow from these forms for the delta function, when it is used inside an integral. If $f(x)$ has multiple zeros,

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}, \tag{B.31}$$

where the x_n are the zeros of $f(x)$ and $f'(x) = \frac{df}{dx}|_{x_n}$. A special case of this result is when $f(x) = ax$.

B.3.3 Other Generalized Functions

Probably the simplest generalized function is the function **1** defined as

$$\int_{x_1}^{x_2} \mathbf{1} F(x)dx \equiv \int_{x_1}^{x_2} F(x)dx. \tag{B.32}$$

A limit form that produces this function is

$$\mathbf{1} = \lim_{\alpha \longrightarrow 0} e^{-\alpha x^2}. \tag{B.33}$$

This especially simple example provides an insight into the connections among the generalized functions. In this case, the derivative of this limit form produces a new limit form, $\lim_{\alpha\rightarrow 0}(-\alpha 2x)e^{-\alpha x^2}$ which produces the generalized function **0** or

$$\int_{x_1}^{x_2} \mathbf{0} F(x)dx \equiv 0. \tag{B.34}$$

This introduces the idea of the derivative of a generalized function; the derivative of the limit form is the limit form for a new generalized function. In other words, $\frac{d\mathbf{1}}{dx} = \mathbf{1}' = \mathbf{0}$.

It is straight forward using the Gaussian limit form, Equation B.30, and the properties of the Hermite polynomials and the fact that $F(x)$ is a good function to show

$$\lim_{\epsilon\to 0}\int_{x_1}^{x_2}\left(\frac{d}{dx}\right)^n \frac{1}{\epsilon\sqrt{\pi}}\exp\left(-\frac{x^2}{\epsilon^2}\right)F(x)dx$$
$$=\int_{x_1}^{x_2}\left\{\left(\frac{d}{dx}\right)^n \delta(x)\right\}F(x)dx$$
$$=(-1)^n\int_{x_1}^{x_2}\delta(x)\left\{\left(\frac{d}{dx}\right)^n F(x)\right\}dx=(-1)^n\frac{d^n}{dx^n}F(0). \quad \text{(B.35)}$$

A particularly interesting and useful form for the Dirac delta function that was already alluded to in Section B.2 follows from the Dirchelet limit form, Equation B.30. Changing the Dirchelet form trivially to $\lim_{K\to\infty}\frac{\sin Kx}{\pi x}=\delta(x)$, we can identify

$$2\pi\delta(x)=\lim_{K\to\infty}\left\{2\pi\frac{\sin Kx}{\pi x}=\int_{-K}^{K}dke^{ikx}\right\} \quad \text{(B.36)}$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dke^{ikx}=\delta(x). \quad \text{(B.37)}$$

Of course, this is the essence of Fourier's proof of the validity of Fourier's integral in the exponential form, Equation B.12. This representation of the Dirac delta function is clearly important to the interplay of generalized functions and Fourier transforms. In this form, the derivatives of the Dirac delta function take on a specially important form:

$$\frac{d^n}{dx^n}\delta(x)=\frac{(i)^n}{2\pi}\int_{-\infty}^{\infty}dkk^n e^{ikx} \quad \text{(B.38)}$$

On the face of it, this integral expression is meaningless. It must be remembered that generalized functions are used only in the context of integration with reasonable functions.

Another important related expression for the Dirac delta function follows from the properties of the Fourier series, Equations B.1-B.3. The functions $\cos\frac{2\pi nx}{\lambda}$ and $\sin\frac{2\pi nx}{\lambda}$ are an example of a complete orthonormal set of functions on the interval $-\frac{\lambda}{2}<x<\frac{\lambda}{2}$. Obviously, this given this function defined on the interval, there is the periodic function $F(t)$ that has period λ and equals the given function in the interval. Thus it satisfies Equations B.1-B.3 and, more significantly, even functions in the interval are expressed in the cosine series and odd functions in the sine series. This is a specific example of a complete orthonormal set of functions $\{\phi_n\}$. Given some interval, $x_1 < x < x_2$, any reasonable function $f(x)$ defined in the interval can be expressed as $f(x) = \sum_n c_n \phi_n(x)$ if the set satisfies the conditions of orthonormality

$$\int_{x_1}^{x_2} \phi_n^*(x)\phi_{n'}(x)dx = \delta_{nn'} \tag{B.39}$$

and completeness

$$\sum_n \phi_n^*(x')\phi_n(x) = \delta(x'-x). \tag{B.40}$$

For example, for odd functions in the interval $-\frac{\lambda}{2} < x < \frac{\lambda}{2}$, the set of functions $\sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nx}{\lambda}$ form a complete orthonormal basis and thus

$$\begin{aligned} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nx}{\lambda} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi mx}{\lambda} dx &= \delta_{n,m} \\ \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nx}{\lambda} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nx'}{\lambda} &= \delta(x-x'). \end{aligned} \tag{B.41}$$

This last result is most easily seen from the Fourier series identity, Equation B.4, modified for odd functions,

$$\begin{aligned} F_o(x) &= \frac{2}{\lambda} \sum_{n=1}^{\infty} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} F_o(t) \cos \frac{2\pi n\,(x-t)}{\lambda} dt. \\ &= \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} dt F_o(t) \left\{ \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nx}{\lambda} \sqrt{\frac{2}{\lambda}} \sin \frac{2\pi nt}{\lambda} \right\}. \end{aligned} \tag{B.42}$$

The cosine series is slightly more complicated since the expansion $f(x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi nx}{\lambda}$ does not cover functions with a non-zero average in the interval. For this case, completeness is expressed as

$$\delta(x-t) = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi nx}{\lambda} \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi nx}{\lambda} \tag{B.43}$$

and the orthonormality is

$$\int_{-\infty}^{\infty} \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi nx}{\lambda} \sqrt{\frac{2}{\lambda}} \cos \frac{2\pi mx}{\lambda} dx = \delta_{n,m}, \tag{B.44}$$

for $n, m = 1, 2, 3, \ldots$.

A very important generalized function is the Heaviside or step function, $\theta(x)$ which is defined as

$$\int_{x_1}^{x_2} \theta(x)F(x)dx = \begin{cases} 0 & : x_1 < 0 \quad x_2 < 0 \\ \int_0^{x_2} F(x)dx & : x_1 < 0 \quad x_2 > 0 \\ -\int_0^{x_1} F(x)dx & : x_1 > 0 \quad x_2 < 0 \\ \int_{x_1}^{x_2} F(x)dx & : x_1 > 0 \quad x_2 > 0 \end{cases} \tag{B.45}$$

for any reasonable $F(x)$. This generalized function is usually expressed as

$$\theta(x) = \begin{cases} 0 & : x < 0 \\ \frac{1}{2} & : x = 0 \\ 1 & : x > 0 \end{cases} . \tag{B.46}$$

A limit form for this generalized function is

$$\theta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \left(\frac{\pi}{2} + \tan^{-1} \left(\frac{x}{\epsilon} \right) \right) . \tag{B.47}$$

The derivative of this limit form produces the Lorentzean limit form for the Dirac delta function. In other words, $\theta'(x) = \delta(x)$. Another interesting application of the Heaviside function is as a limit form for the Dirac delta function, $\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \theta \left(\frac{\epsilon}{2} - |x| \right)$.

There are many general results for generalized functions that can be derived simply from the properties of the limit forms. For a generalized function, $G(x)$ with Fourier transform $\mathcal{G}(k)$,

$$\begin{aligned} \int_{-\infty}^{\infty} G'(x)F(x)dx &= -\int_{-\infty}^{\infty} G(x)F'(x)dx \\ \int_{-\infty}^{\infty} G(ax+b)F(x)dx &= \frac{1}{|a|} \int_{-\infty}^{\infty} G(x)F(\frac{x-b}{a})dx \\ \int_{-\infty}^{\infty} \{\phi(x)G(x)\} F(x)dx &= \int_{-\infty}^{\infty} G(x) \{\phi(x)F(x)\} dx \\ \int_{-\infty}^{\infty} \mathcal{G}(k)\mathcal{F}(k)dk &= \int_{-\infty}^{\infty} G(x)F(-x)dx \end{aligned} \tag{B.48}$$

where $\phi(x)$ is a fairly good function and $F(x)$ is a good function and its Fourier transform is $\mathcal{F}(k)$.

Given two generalized functions $G(x)$ and $H(x)$

$$\frac{d}{dx} \{G(x) + H(x)\} = G'(x) + H'(x)$$

$$\begin{aligned}\frac{d}{dx}\{\phi(x)G(x)\} &= \phi'(x)G(x)+\phi(x)G'(x)\\ \frac{d}{dx}G(ax+b) &= aG'(ax+b)\\ \phi(ax+b)G(ax+b) &= H(ax+b) \text{ if } \phi(x)G(x)=H(x)\end{aligned} \tag{B.49}$$

where again $\phi(x)$ is a fairly good function. It is important to realize that there is no definition for the product of two generalized functions.

B.4 Three Space

It is simply a fact that the space that we have is not one dimensional. Applications to physics requires that we understand the nature of Fourier transforms and generalized functions in at least three dimension. Of course, since the three space is an $(\mathbb{R}^1)^3$, it should simply require the multiplying of each of the elements. Thus using the Fourier transform form for the Dirac delta function, Equation B.37, we have the simple construction

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk_x e^{ik_x x}\int_{-\infty}^{\infty} dk_y e^{ik_y y}\int_{-\infty}^{\infty} dk_z e^{ik_z z} = \delta^3(\vec{x}), \tag{B.50}$$

or in a simpler notation $\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3\vec{k}e^{i\vec{k}\cdot\vec{x}} = \delta^3(\vec{x})$. Of course this equation is meaningful only in the context of integration with a good function; $\int_{-\infty}^{\infty}\delta^3(\vec{x})F(\vec{x})d^3\vec{x} = F(\vec{0})$. This approach to higher dimensions merely treats each direction independently. This carries with it then the usual intuition from the one dimensional cases. For example, we understand that the amount of support in configuration space and the amount of support in Fourier space are inversely related, see Equation B.24.

The complication of the use of these formulas to many applications is that the logical coordinate basis may not be the simple direct product basis used in Equation B.50. Important examples are spherical polar and cylindrical coordinate systems.

The basis for the problem is the difference between iterative integration in a multidimensional space and integration on the manifold. As an example consider the three dimensional case in cartesian and spherical polar coordinates. Equation B.50 is a Fourier transform in all three spatial coordinates. If we realize the relationship between

B.4.1 Diversion on the Transverse Propagator

The transverse propagator is $\delta_{ij} - \frac{k_i k_j}{k^2}$ Some important three dimensional examples.

$$\frac{1}{4\pi r} \quad \longleftrightarrow \quad \frac{1}{(\sqrt{2\pi})^3}\frac{1}{k^2} \tag{B.51}$$

$$\frac{\vec{r}}{4\pi r^3} \quad \longleftrightarrow \quad \frac{1}{(\sqrt{2\pi})^3}\frac{-i\vec{k}}{k^2} \tag{B.52}$$

$$\delta(\vec{r} - \vec{r}_a) \quad \longleftrightarrow \quad \frac{1}{(\sqrt{2\pi})^3} e^{-i\vec{k}\cdot\vec{r_a}} \tag{B.53}$$

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