Fractals : Spectral properties
Statistical physics

Course 1

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PAZY
EXCELLING IN SCIENCE

קרן הלאומית למדע
ISRAEL SCIENCE FOUNDATION

6th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals, June 13-17, 2017

Benefitted from discussions and collaborations with:

Technion:

Evgeni Gurevich (KLA-Tencor)
Dor Gittelman
Eli Levy (+ Rafael)
Ariane Soret (ENS Cachan)
Or Raz (HUJI, Maths)
Omrie Ovdat
Yaroslav Don

Rafael:
Assaf Barak
Amnon Fisher

Elsewhere:

Gerald Dunne (UConn.)
Alexander Teplyaev (UConn.)
Jacqueline Bloch (LPN, Marcoussis)
Dimitri Tanese (LPN, Marcoussis)
Florent Baboux (LPN, Marcoussis)
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Plan of the 4 talks

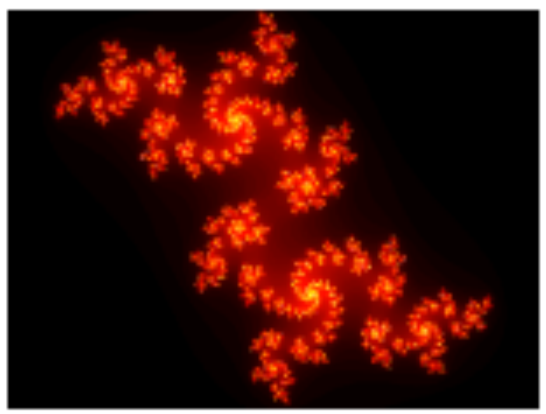
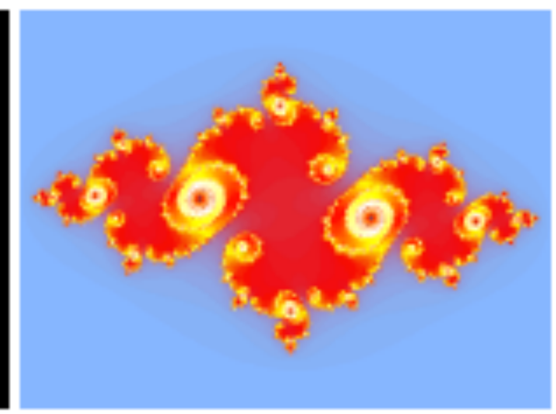
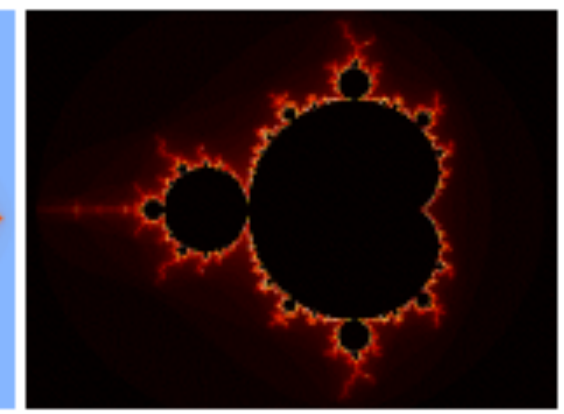
- Course 1 : Spectral properties of fractals - Application in statistical physics
- Talk : quantum phase transition - scale anomaly and fractals
- Course 2 : topology and fractals - measuring topological numbers with waves.
- Elaboration : Renormalisation group and Efimov physics

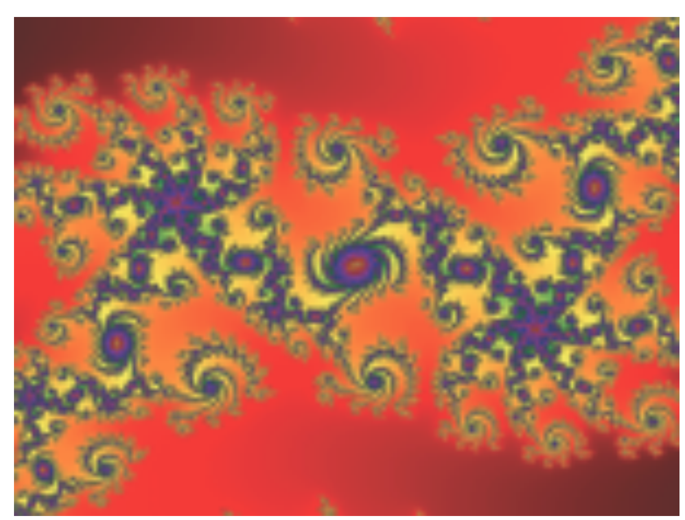
Program for today

- Introduction : spectral properties of self similar fractals.
- Heat kernel - Asymptotic behaviour - Weyl expansion - Spectral volume.
- Thermodynamics of the fractal blackbody.
- Summary - Phase transitions.

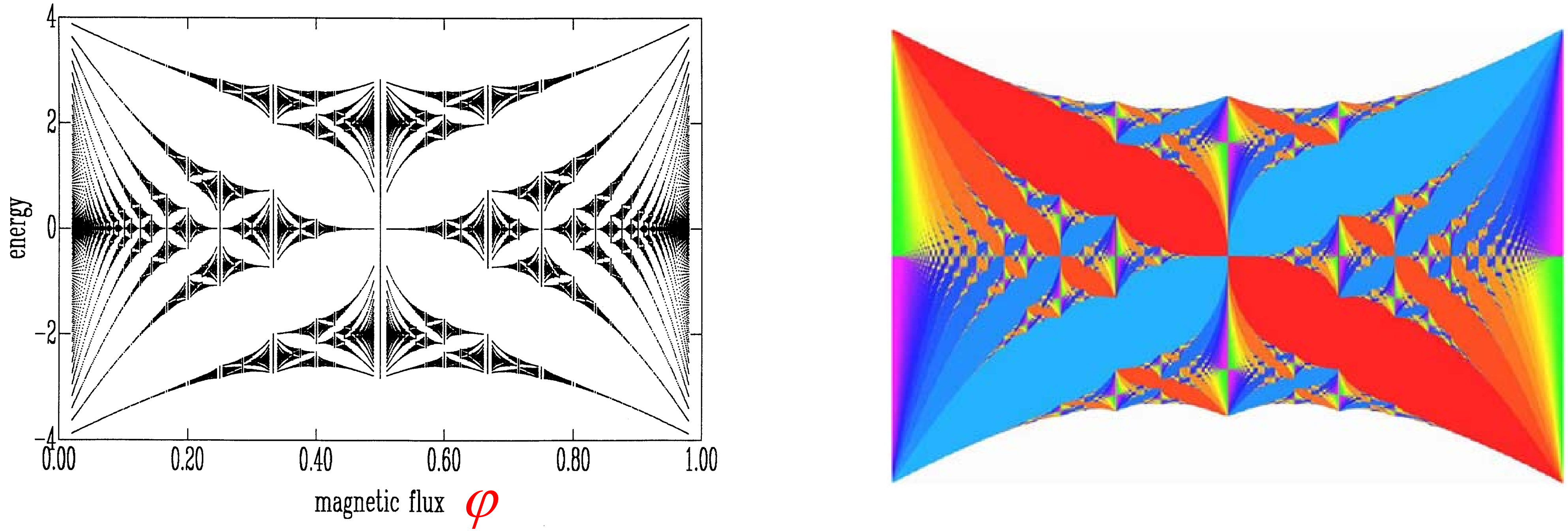
Introduction : spectral properties of self similar fractals.

- attractive objects - Bear exotic names

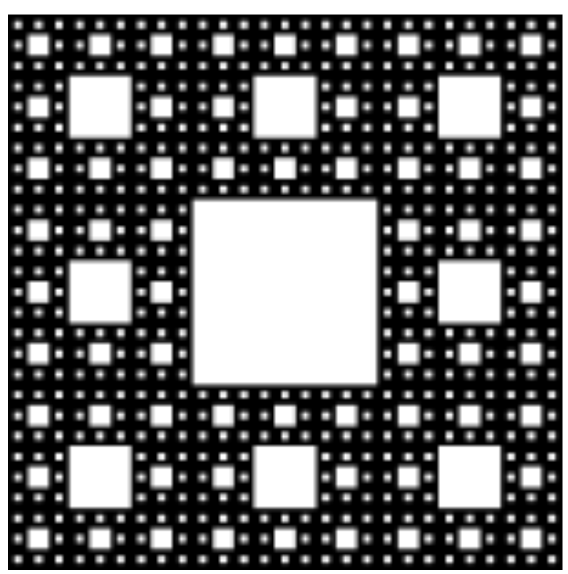






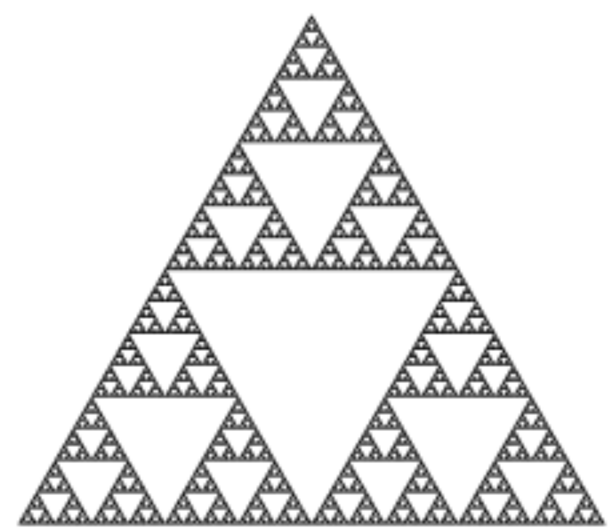
Julia sets



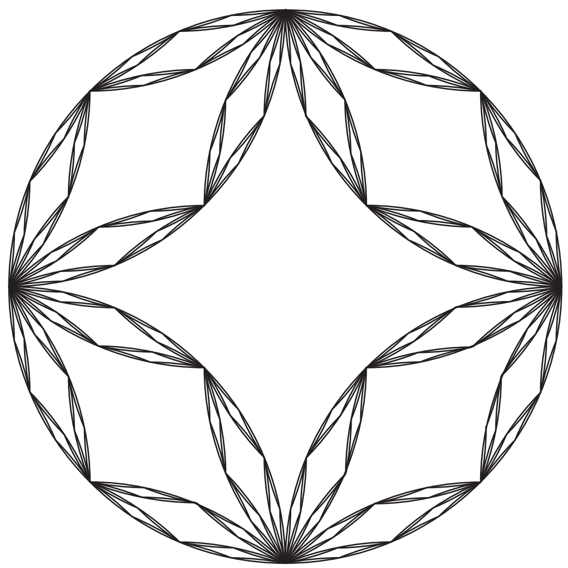
Hofstadter butterfly



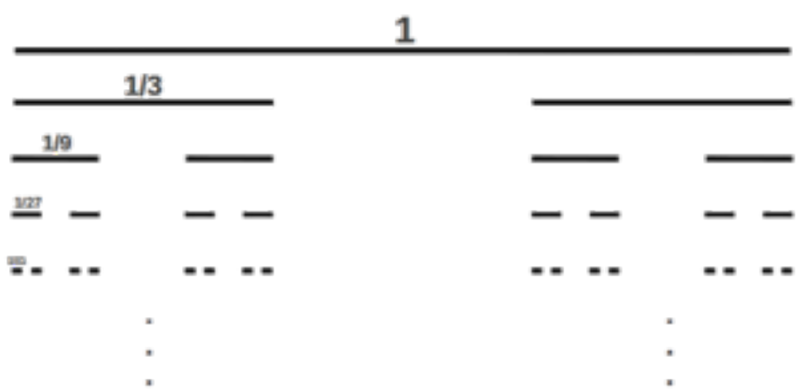
Sierpinski carpet



Sierpinski gasket



Diamond fractals



Triadic Cantor set

Convey the idea of highly symmetric objects yet with an unusual type of symmetry and a notion of extreme subdivision

Fractal : Iterative graph structure

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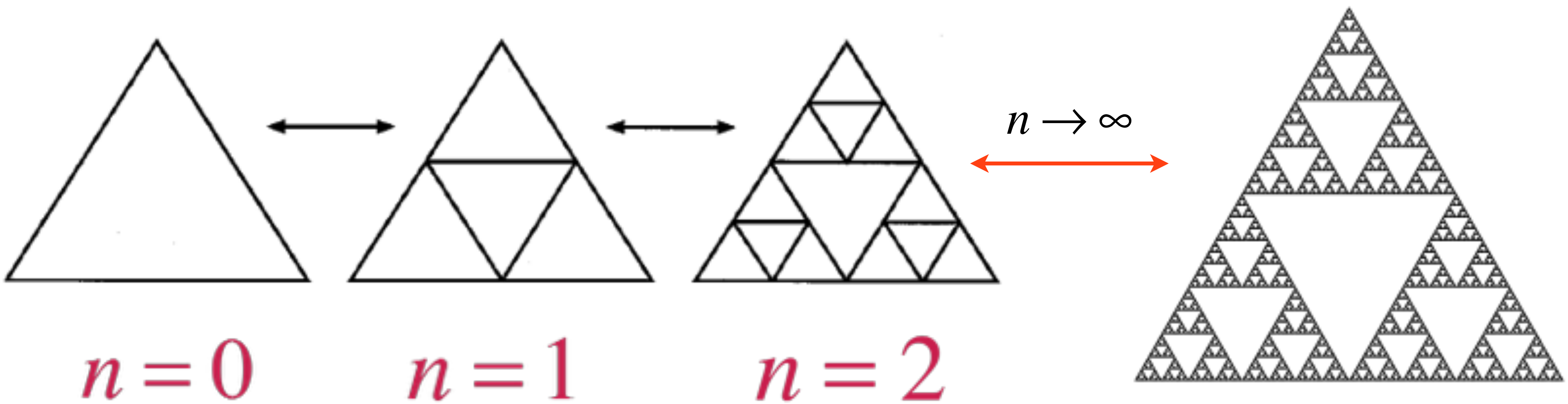

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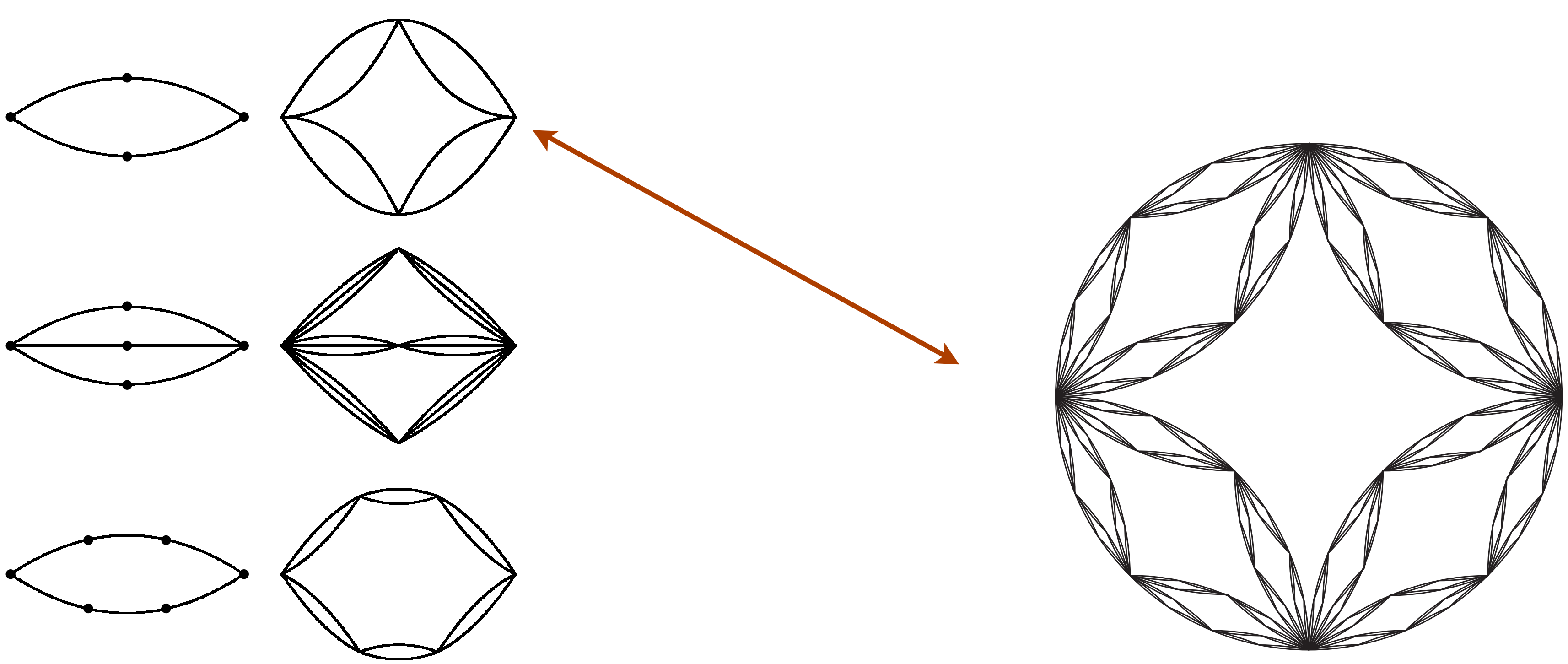
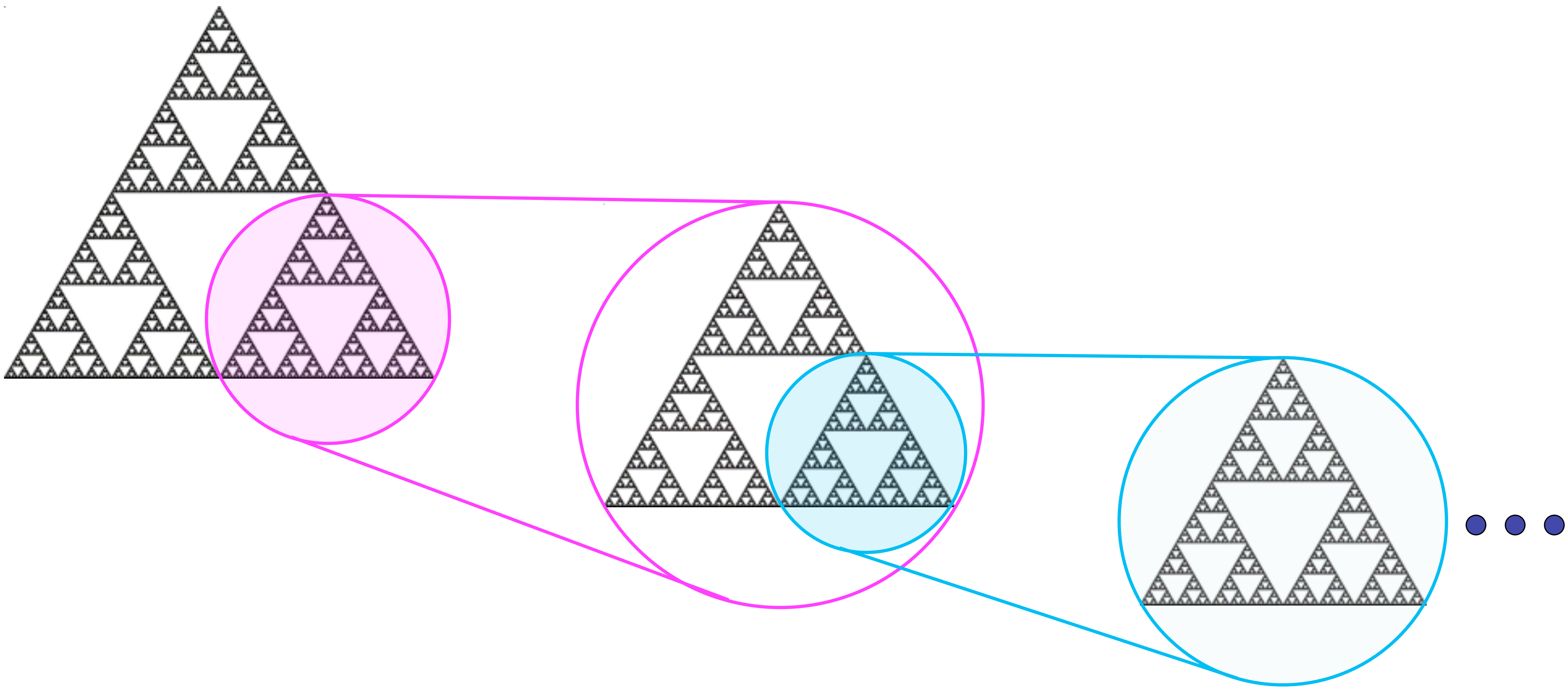


Diamond fractals

As opposed to Euclidean spaces characterised by translation symmetry, fractals possess a dilatation symmetry.

Fractals are self-similar objects

Fractal ↔ Self-similar



Discrete scaling symmetry

- But not all fractals are obvious, good faith geometrical objects.
    - Sometimes, the fractal structure is not geometrical but it is hidden at a more abstract level.

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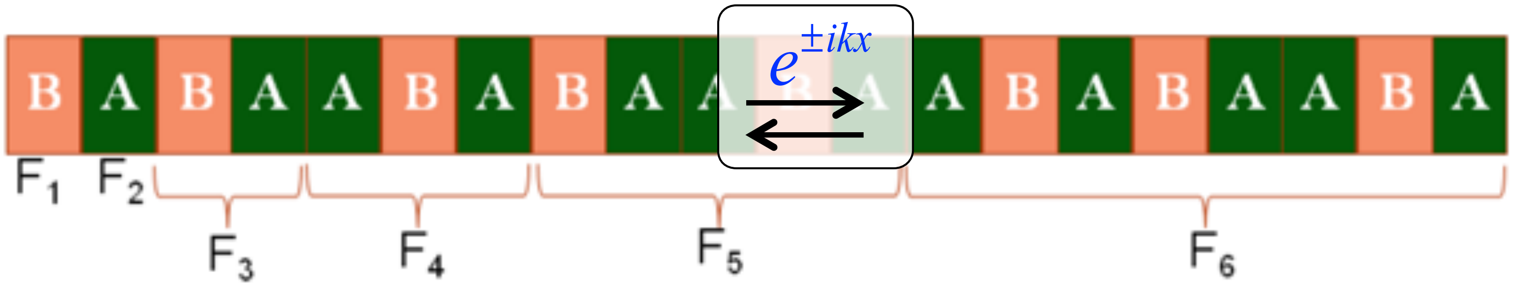
  - Sometimes, the fractal structure is not geometrical but it is hidden at a more abstract level.

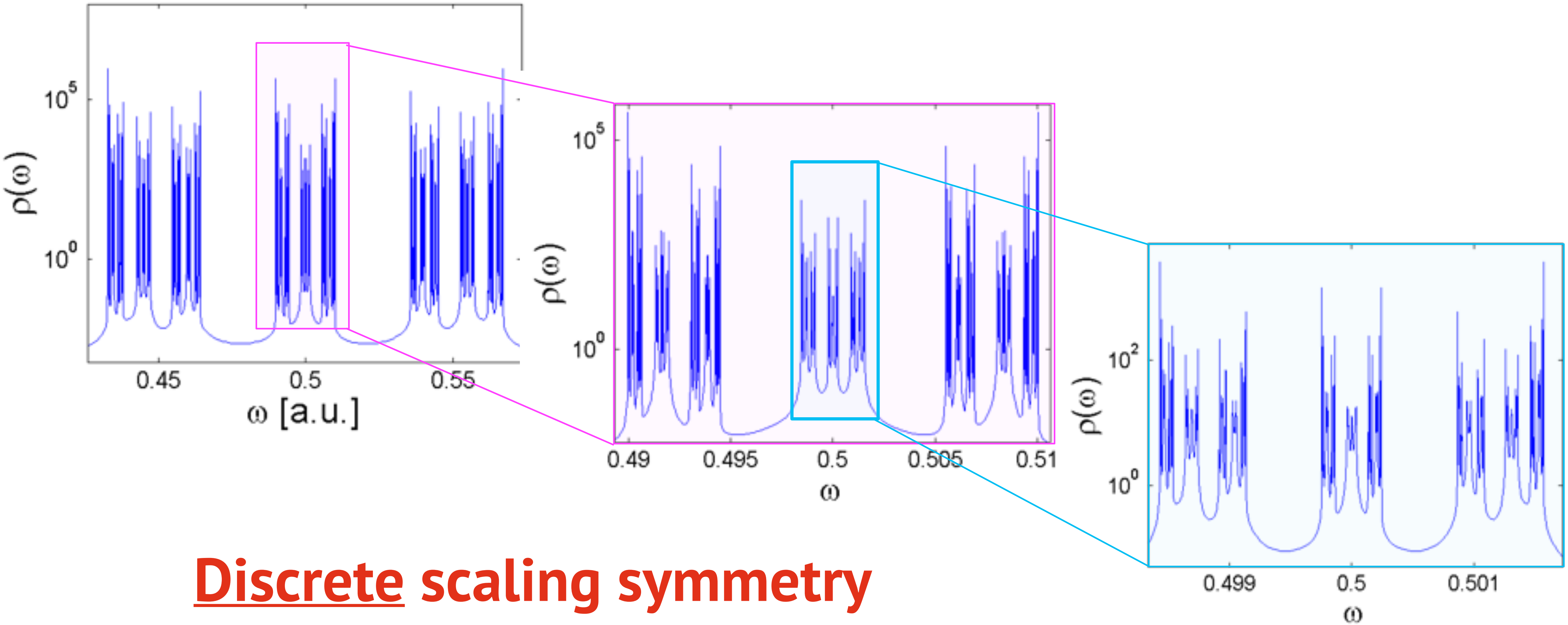
Exemple : Quasi-periodic chain of layers of 2 types A, B

Fibonacci sequence : $F_1 = B;\ F_2 = A;\ F_{j\geq 3} = \left[F_{j-2} F_{j-1} \right]$



Defines a cavity whose frequency spectrum is fractal.

Density of modes $\rho(\omega)$:



Discrete scaling symmetry

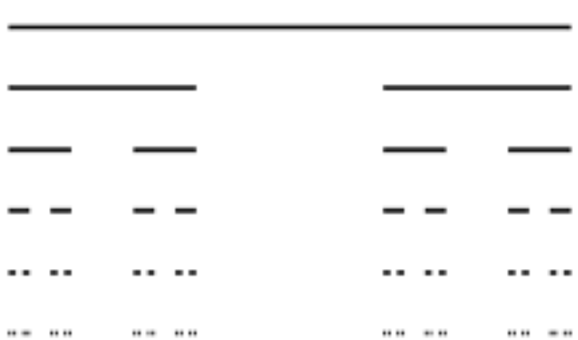
Minicourse 2 - Tomorrow

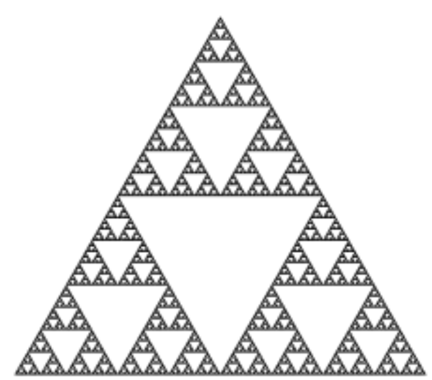
Operators and fields on fractal manifolds

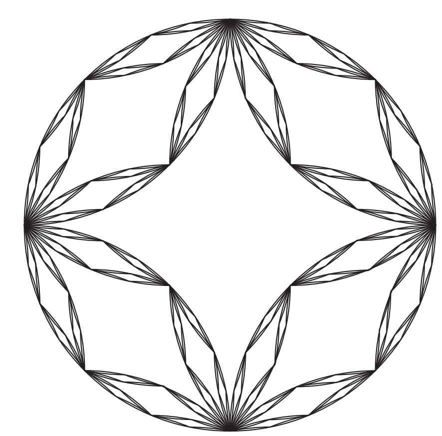
Operators are often expressed by local differential equations relating the space-time behaviour of a field

Ex. Wave equation $$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

Such local equations cannot be defined on a fractal







But operators are essential quantities for physics!

- Quantum transport in fractal structures :

  e.g., networks, waveguides, …

  electrons, photons
- Density of states
- Scattering matrix (transmission/reflection)

But operators are essential quantities for physics!

- Quantum fields on fractals, *e.g.*, fermions (spin 1/2), photons (spin 1) - canonical quantisation (Fourier modes) - path integral quantisation : path integrals, Brownian motion.

- “curved space QFT” or quantum gravity

- Scaling symmetry (renormalisation group) - critical behaviour.

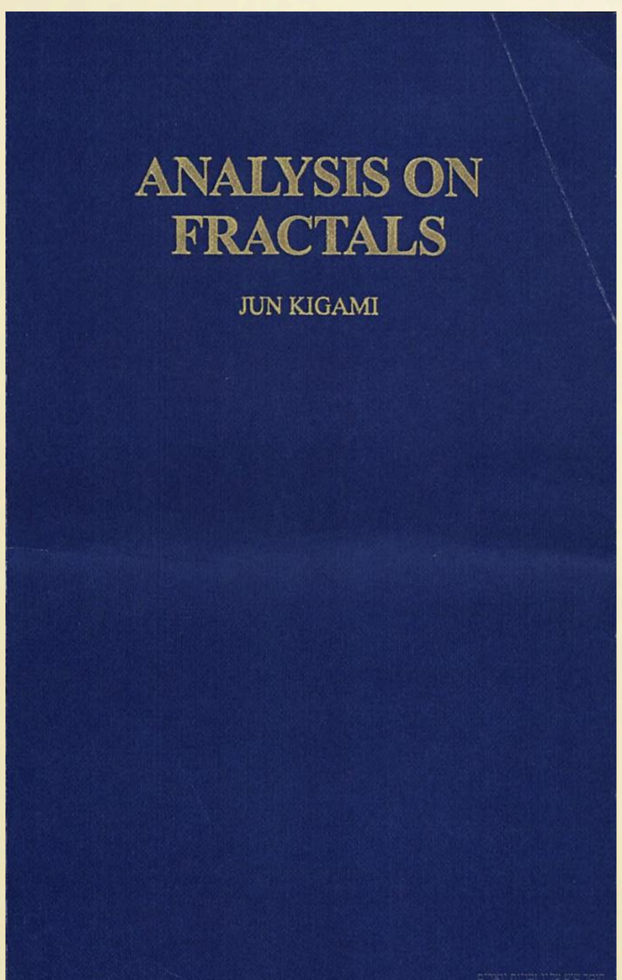
Michel Lapidus

Bob Strichartz

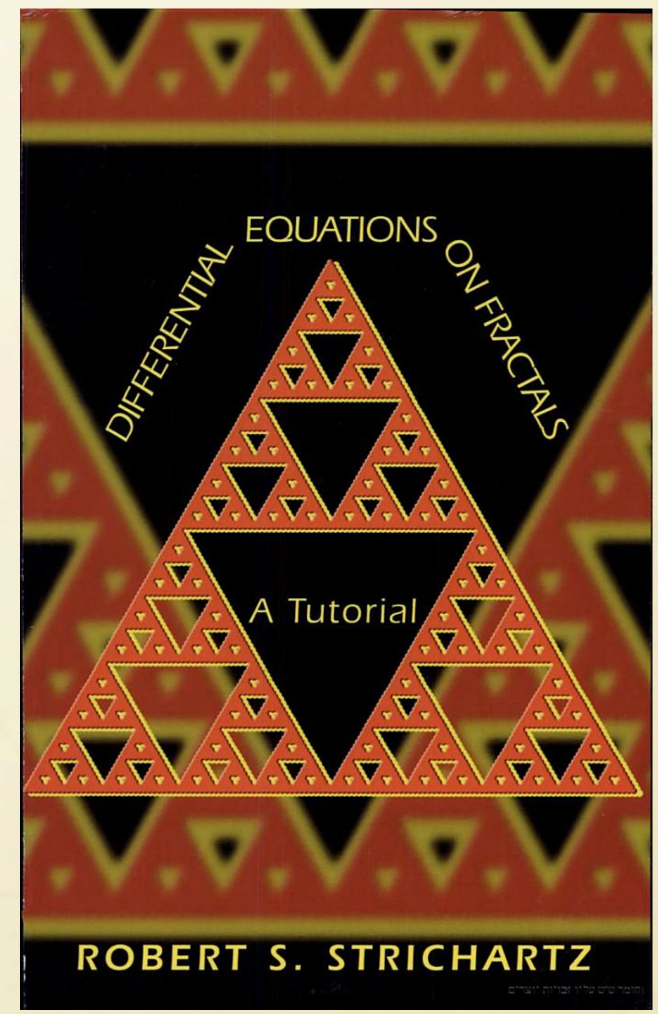
Recent new ideas

>2000

Maths.



Jun Kigami

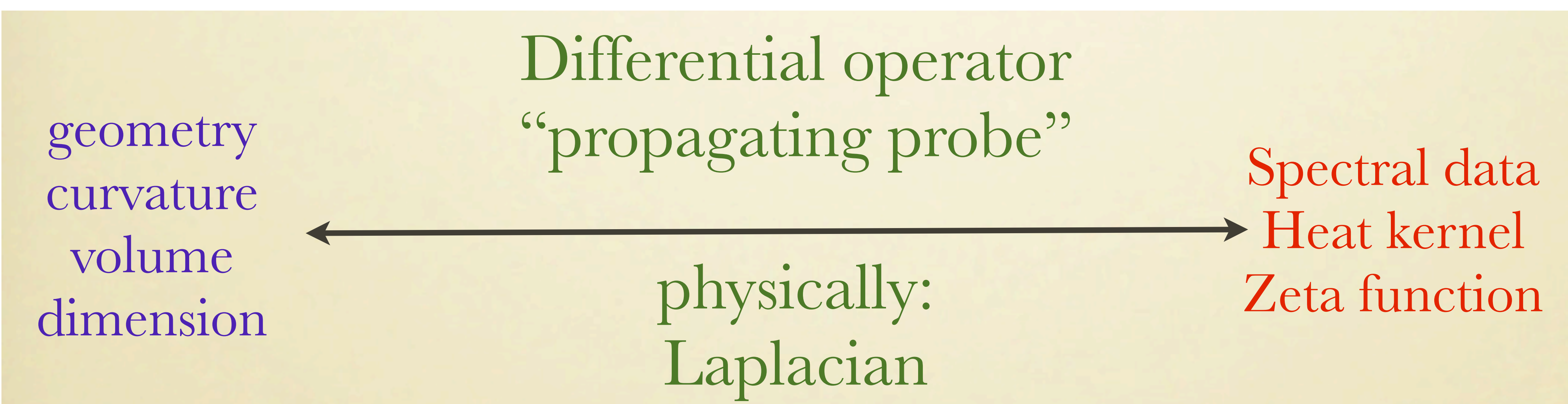


Intermezzo : heat and waves

From classical diffusion to wave propagation

Important relation between classical diffusion and wave propagation on a manifold.

Expresses the idea that it is possible to measure and characterise a manifold using waves (eigenvalue spectrum of the Laplace operator)



Use propagating waves/particles to probe :

- spectral information: density of states, transport, heat kernel, …
- geometric information: dimension, volume, boundaries, shape, …

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Mathematical physics

1910 Lorentz: why is the Jeans radiation law only dependent on the volume ?

1911 Weyl : relation between asymptotic eigenvalues and dimension/volume.

1966 Kac : can one hear the shape of a drum ?

Important examples

- Heat equation $$\frac{\partial u}{\partial t} = \Delta u$$
- Wave equation $$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

Schr. equation. $$i\frac{\partial u}{\partial t} = \Delta u$$

Important examples

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$$u(x,t) = \int d\mu(y) P_t(x,y) u(y,0)$$

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$$P_t(x,y) = \int_{x(0)=x,\, x(t)=y} \mathcal{D}x\, e^{-(i)\int_0^t \dot{x}^2 d\tau}$$

Brownian motion

$$P_t(x,y) \sim \frac{1}{t^{d/2}} \sum_n a_n(x,y) t^n$$

Heat kernel expansion

$$P_t(x,y) \sim \sum_{geodesics} (\#) e^{-(i) S_{classical}(x,y,t)}$$

Gutzwiller - instantons

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Spectral functions

$$P_t\left(x,y\right)=\left\langle y\right|e^{-\Delta t}\left|x\right\rangle=\sum_{\lambda}\psi_{\lambda}^{*}(y)\psi_{\lambda}(x)e^{-\lambda t}$$

Spectral functions

$$P_t(x,y) = \langle y | e^{-\Delta t} | x \rangle = \sum_\lambda \psi_\lambda^*(y)\psi_\lambda(x)e^{-\lambda t}$$

$$Z(t) = Tr\, e^{-\Delta t} = \int dx \langle x | e^{-\Delta t} | x \rangle = \sum_\lambda e^{-\lambda t}$$

Heat kernel

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Heat kernel

Return probability

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Heat kernel

$$\zeta_Z(s) \equiv \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} Z(t)$$

Mellin transform

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Weyl expansion

The heat kernel is related to the density of states of the Laplacian

There are "Laplace transform" of each other:

$$Z(t) = \int_0^\infty d\omega\, \rho(\omega)\, e^{-t\,\omega}$$

From the Weyl expansion, it is possible to obtain the density of states.

How does it work ?

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$

Diffusion (heat) equation in d=1

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Probability of diffusing from x to y in a time t.

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We can characterise the "spatial geometry" by watching how the heat flows.

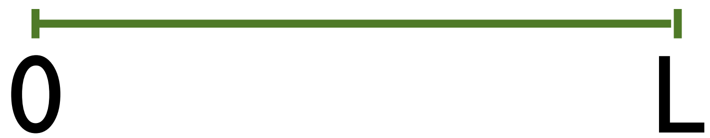
The heat kernel $Z_d(t)$ is

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi Dt)^{d/2}}$$

→ volume of the manifold

Boundary terms- Hearing the shape of a drum

Mark Kac (1966)



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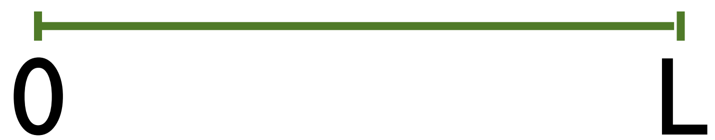

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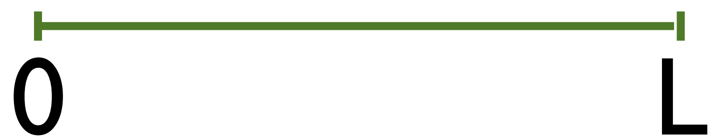

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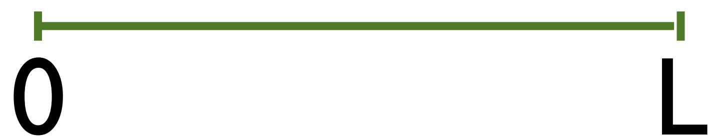

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Poisson formula

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Weyl expansion (1d)

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Weyl expansion (2d) :

$$Z_{d=2}(t) \sim \frac{Vol.}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}} + \frac{1}{6} + \ldots$$

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sensitive to boundary

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bulk

sensitive to boundary

integral of bound. curvature

ζ-function

$$\zeta_Z(s) = Tr\frac{1}{\Delta^s} = \sum_{\lambda}\frac{1}{\lambda^s}$$

$$Dirichlet:\ \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,\ldots$$

$$\zeta(s) = \sum_{n=0}^{\infty}\left(\frac{L^2}{n^2\pi^2}\right)^s = \frac{L^{2s}}{\pi^{2s}}\sum_{n=0}^{\infty}\frac{1}{n^{2s}} \equiv \frac{L^{2s}}{\pi^{2s}}\zeta_R(2s)$$

$\zeta_R(2s)$ has a simple pole at $s = \frac{1}{2}$ $\left(s = \frac{d}{2}\right)$ so that,

$$Z(t) = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} ds\, t^{-s}\,\Gamma(s)\zeta(s) \sim \frac{L}{2\pi}t^{-1/2}\Gamma(1/2) + \ldots$$

$$= \frac{L}{\sqrt{4\pi t}} + \ldots$$

How does it work on a fractal ?

How does it work on a fractal ?

Differently…

No simple access to the eigenvalue spectrum but we know how to calculate the heat kernel.

$$Z(t) = Tr\, e^{-\Delta t} = \int dx \langle x | e^{-\Delta t} | x \rangle = \sum_{\lambda} e^{-\lambda t}$$

and thus, the density of states,

$$Z(t) = \int_0^{\infty} d\omega\, \rho(\omega)\, e^{-t\omega}$$

More precisely,

$$Z(t) = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} + B \sum_{n=0}^{\infty} L_n^{d_h} \sum_{k=1}^{\infty} e^{-k^2\pi^2 t\, L_n^{d_w}}$$

$L_n = a^n$ is the total length upon iteration of the elementary step

$$\zeta(s) = \frac{\zeta_R(2s)}{\pi^{2s}} + \sum_{n=0}^{\infty} L_n^{d_h} \sum_{k=1}^{\infty} \left(\frac{1}{k^2\pi^2 L_n^{d_w}} \right)^s$$

$$= \frac{\zeta_R(2s)}{\pi^{2s}} \left(1 + \sum_{n=0}^{\infty} L_n^{d_h - d_w s} \right)$$

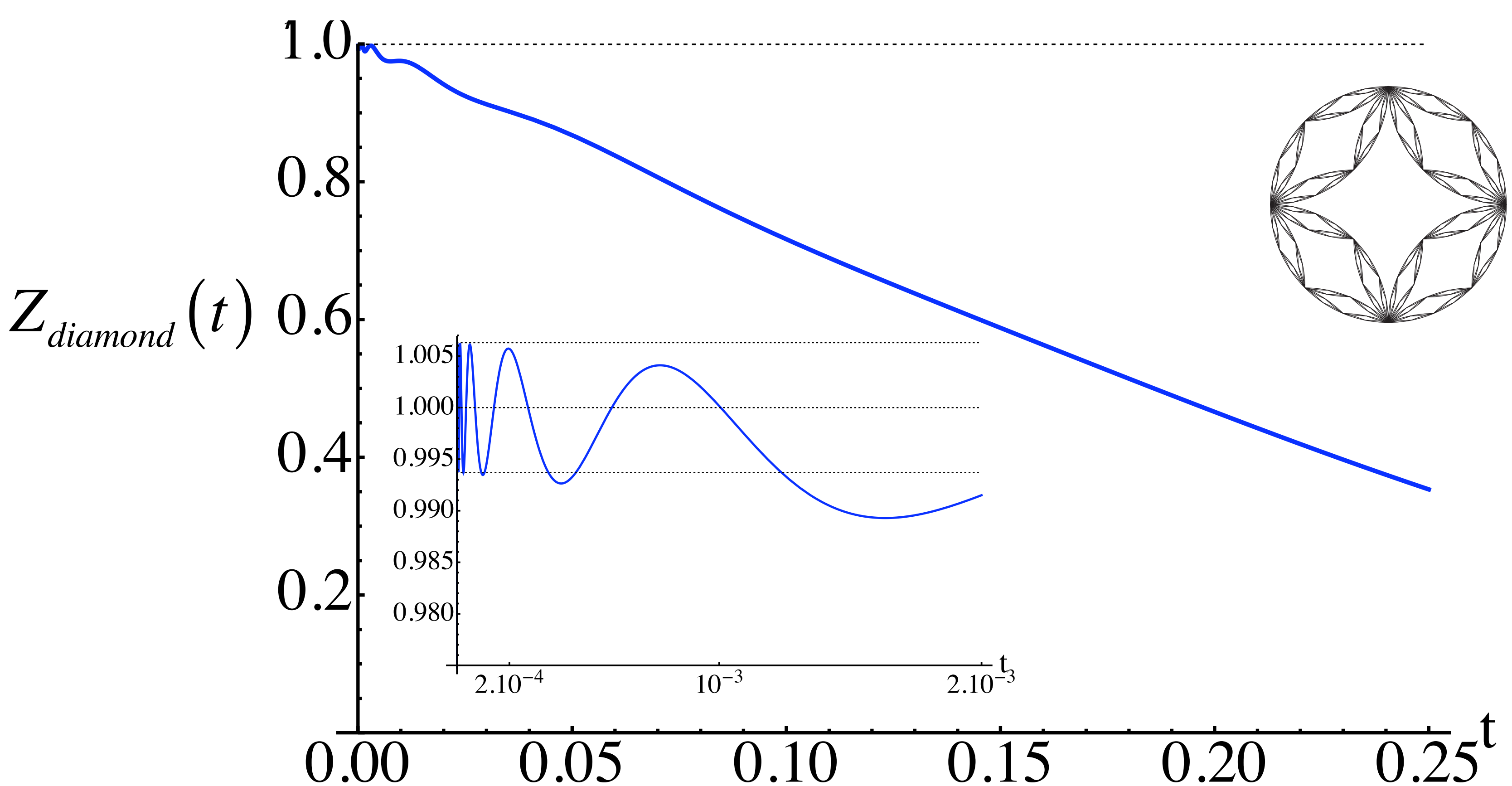
$$= \frac{\zeta_R(2s)}{\pi^{2s}} \left(\frac{2 - a^{d_h - d_w s}}{1 - a^{d_h - d_w s}} \right)$$

which has poles at $a^{d_h - d_w s} = 1$

$$a^{d_h - d_w s} = 1 \quad \Leftrightarrow \quad s_n = \frac{d_s}{2} + \frac{2i\pi n}{d_w \ln a}$$

Infinite number of complex poles : complex fractal dimensions.

They control the behaviour of the heat kernel which exhibits oscillations.



A new fractal dimension : **spectral dimension** d_s

Notion of spectral volume

From the previous expression we obtain $Z(t)$

Consider for simplicity $n = 1$, namely $s_1 = \frac{d_s}{2} + \frac{2i\pi}{d_w \ln a} \equiv \frac{d_s}{2} + i\delta$

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to compare with

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$

Spectral volume ?

Geometric volume described by the Hausdorff dimension is large (infinite)

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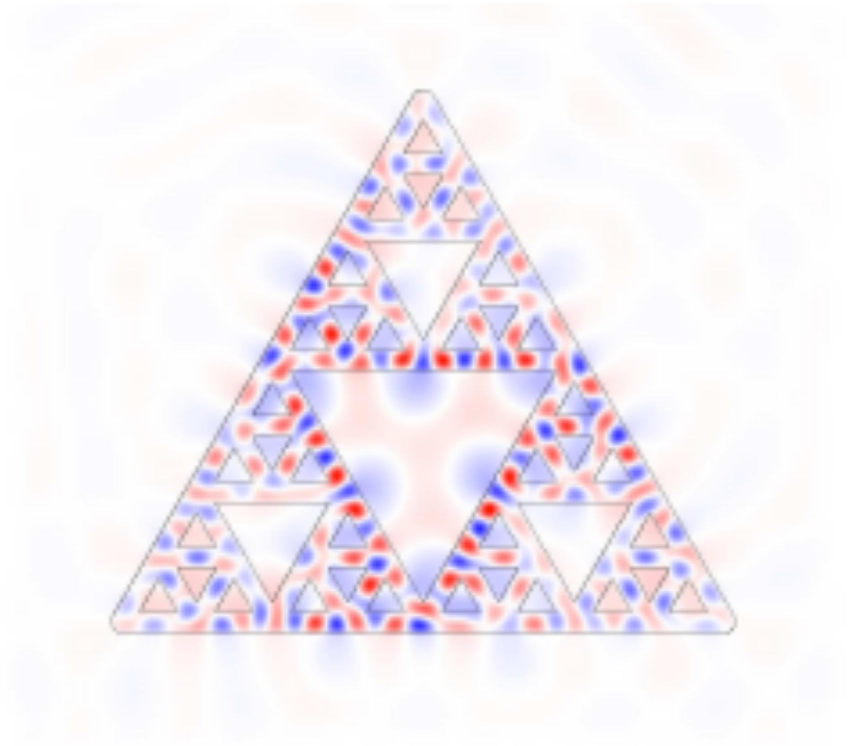
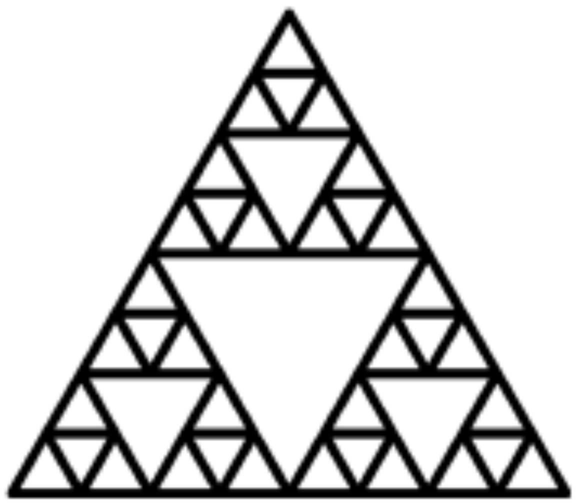
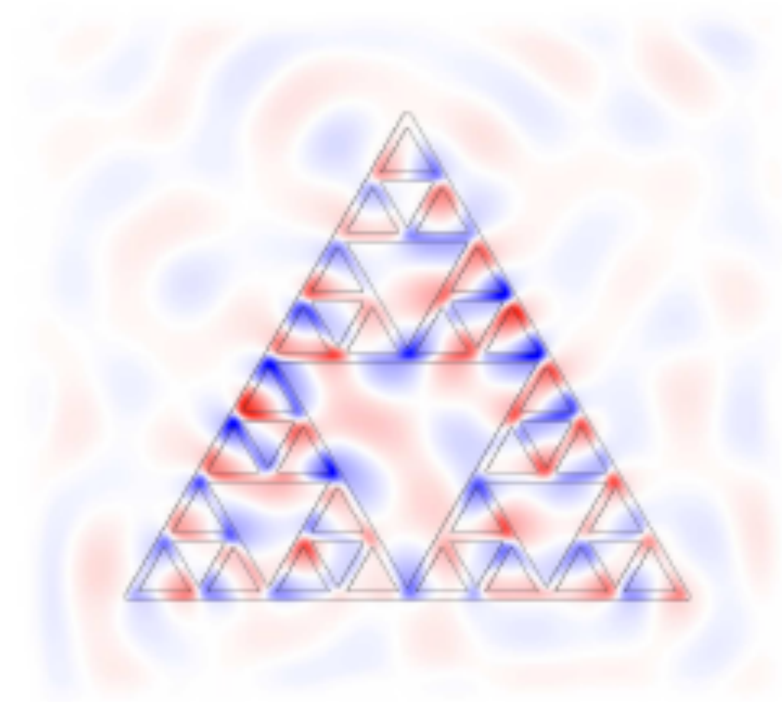


Spectral volume V_s is the finite volume occupied by the modes

Numerical solution of Maxwell eqs. on the Sierpinski gasket

Physical application :
Thermodynamics of photons on fractals

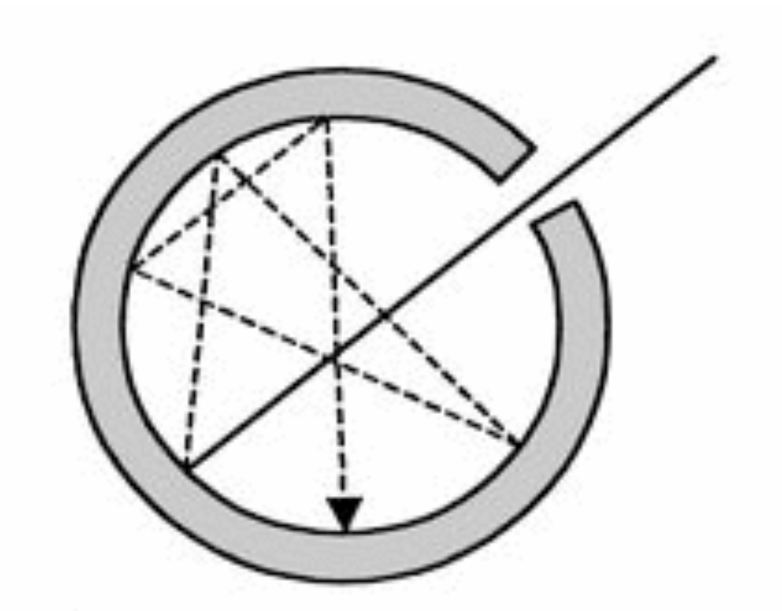




Electromagnetic field in a waveguide fractal structure.

How to measure the spectral volume ?

The radiating fractal blackbody



Equation of state at thermodynamic equilibrium relating pressure, volume and internal energy:

$$PV = U/d$$

In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$ ... It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones that lies in the interval ν to $\nu+d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume.

H. Lorentz, 1910

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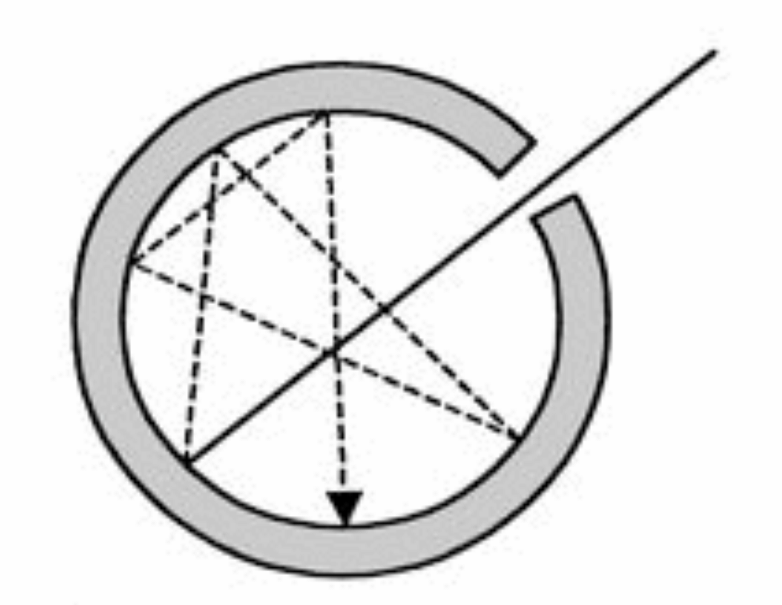

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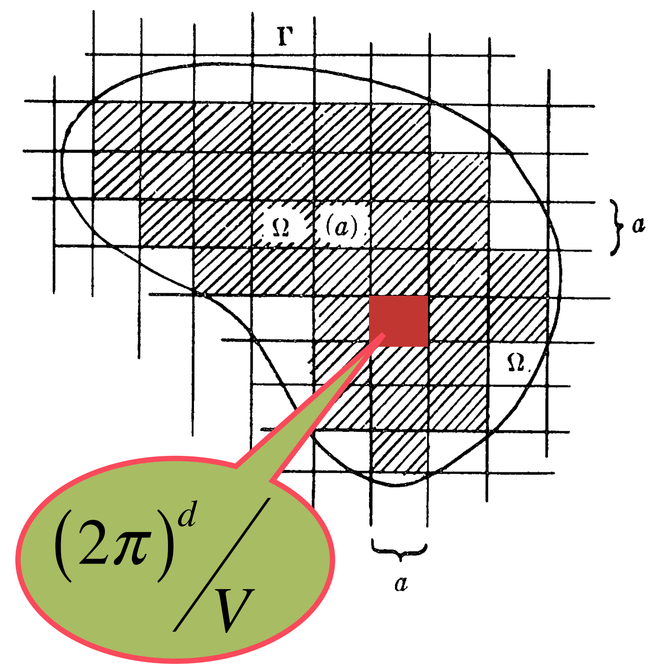
Spectral volume ?

Usual approach : count modes in momentum space

Calculate the partition (generating)

function $z(T,V)$ for a blackbody of

large volume V in dimension d



Mode decomposition of the field $\omega = c\left|\vec{k}\right| = c\,V^{-1/d}\,2\pi\left|\vec{n}\right|$

$$\ln z(T,V) = Q\left(L_\beta / V^{1/d}\right)$$

with $L_\beta \equiv \beta\hbar c$

(photon thermal wavelength)

$\beta = 1/k_B T$

Thermodynamics :

$$U = -\frac{\partial}{\partial \beta} \ln z(T, V) = -\left(\frac{dQ}{dx}\right) \hbar c V^{-1/d}$$

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Stefan-Boltzmann $U\propto V\,T^{d+1}$ is a consequence of $\left(\frac{\partial U}{\partial V}\right)_T=T\left(\frac{\partial P}{\partial T}\right)_V-P$

Adiabatic expansion $V\,T^{d}=Cte$

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Nevertheless,

$$PV_s = U/d_s$$

V_s is the “spectral volume”.

Re-phrase the thermodynamic problem in terms of heat kernel and zeta function.

Partition function of equilibrium quantum radiation

$$\ln z(T,V) = -\frac{1}{2}\ln Det_{M\times V}\left(\frac{\partial^2}{\partial\tau^2} + c^2\Delta\right)$$

Looks (almost) like a bona fide wave equation **but** proper time.

This expression does not rely on mode decomposition.

Rescale by $L_\beta \equiv \beta\hbar c$

Partition function of equilibrium quantum radiation

$$\ln z(T,V) = -\frac{1}{2}\ln Det_{M\times V}\left(\frac{\partial^2}{\partial u^2} + L_\beta^2 \Delta\right)$$

M: circle of radius $L_\beta \equiv \beta\hbar c$

Spatial manifold (fractal)

Thermal equilibrium of photons on a spatial manifold V at temperature T is described by the (scaled) wave equation on $M \times V$

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$$Z\left(L_\beta^2\tau\right)$$

Heat kernel

Large volume limit (a high temperature limit) $V \gg L_\beta^d \Leftrightarrow k_B T \gg \hbar c / V^{1/d}$

Weyl expansion:

$$Z\left(L_\beta^2\tau\right)\sim\frac{V}{\left(4\pi L_\beta^2\tau\right)^{d/2}}$$

$$\ln z(T,V) = \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau} f(\tau)\, Tr_V\, e^{-\tau L_\beta^2 \Delta}$$

+ Weyl expansion $\Longrightarrow$

$$\ln z(T,V) \sim \frac{V}{L_\beta^d}$$

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$$PV = U/d$$

Thermodynamics measures the spectral volume

On a fractal...

$$Z\left(L_\beta^2 \tau\right) \sim \frac{V_s}{\left(4\pi L_\beta^2 \tau\right)^{d_s/2}} f\left(\ln \tau\right)$$

On a fractal...

Spectral volume

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$$Z\left(L_\beta^2\tau\right)\sim\frac{V_s}{\left(4\pi L_\beta^2\tau\right)^{d_s/2}}f\left(\ln\tau\right)$$

Spectral dimension

On a fractal...

Spectral volume

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Spectral dimension

Thermodynamic equation of state for a fractal manifold

$$PV_s = U/d_s$$

Thermodynamics measures the spectral volume and the spectral dimension.

Summary

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- Thermodynamics is directly related to the heat kernel (partition function) - fractal blackbody - importance of the spectral volume.
- Phase transitions on fractals : scaling/hyperscaling relations are modified on fractals (dependence on distinct fractal dimensions).

- Non gaussian fixed points (limit cycles) - Harris criterion : fractal geometry is a specific type of disorder similar to quasicrystals.

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- Off-diagonal long range order - superfluidity (Mermin, Wagner, Coleman theorem) - Non diagonal Green's function.
- Applications to other problems : quantum phase transitions - quantum Einstein gravity, …

Thank you for your attention.