Chapter 4

Fractions and Mixed Numbers

4.1

Introduction to Fractions and Mixed Numbers

Parts of a Fraction

Whole numbers are used to count whole things. To refer to a part of a whole, **fractions** are used.

A fraction is a number of the form $\frac{a}{b}$,

where a and b are integers and b is not 0.

The parts of a fraction are

numerator → a

fraction bar ← (the bar in $\frac{a}{b}$)

denominator → b

$$\frac{a}{b}$$

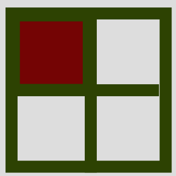
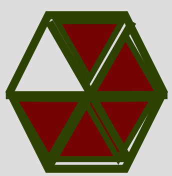
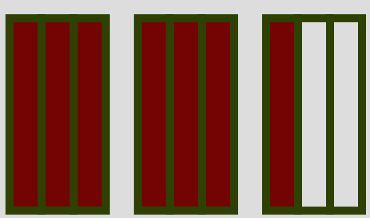
Helpful Hint

$\frac{4}{7}$ ← Remember that the bar in a fraction means division. Since division by 0 is undefined, a fraction with a denominator of 0 is undefined.

Visualizing Fractions

One way to visualize fractions is to picture them as shaded parts of a whole figure.

Visualizing Fractions

Picture	Fraction	Read as
	$\frac{1}{4}$ — 1 ← part shaded; 4 ← equal parts	one-fourth
	$\frac{5}{6}$ — 5 ← parts shaded; 6 ← equal parts	five-sixths
	$\frac{7}{3}$ — 7 ← parts shaded; 3 ← equal parts	seven-thirds

Types of Fractions

A **proper fraction** is a fraction whose numerator is less than its denominator.

Proper fractions have values that are less than 1. $\frac{1}{2}, \frac{3}{4}, \frac{2}{5}$

An **improper fraction** is a fraction whose numerator is greater than or equal to its denominator.

$\frac{8}{3}, \frac{5}{5}, \frac{4}{1}$

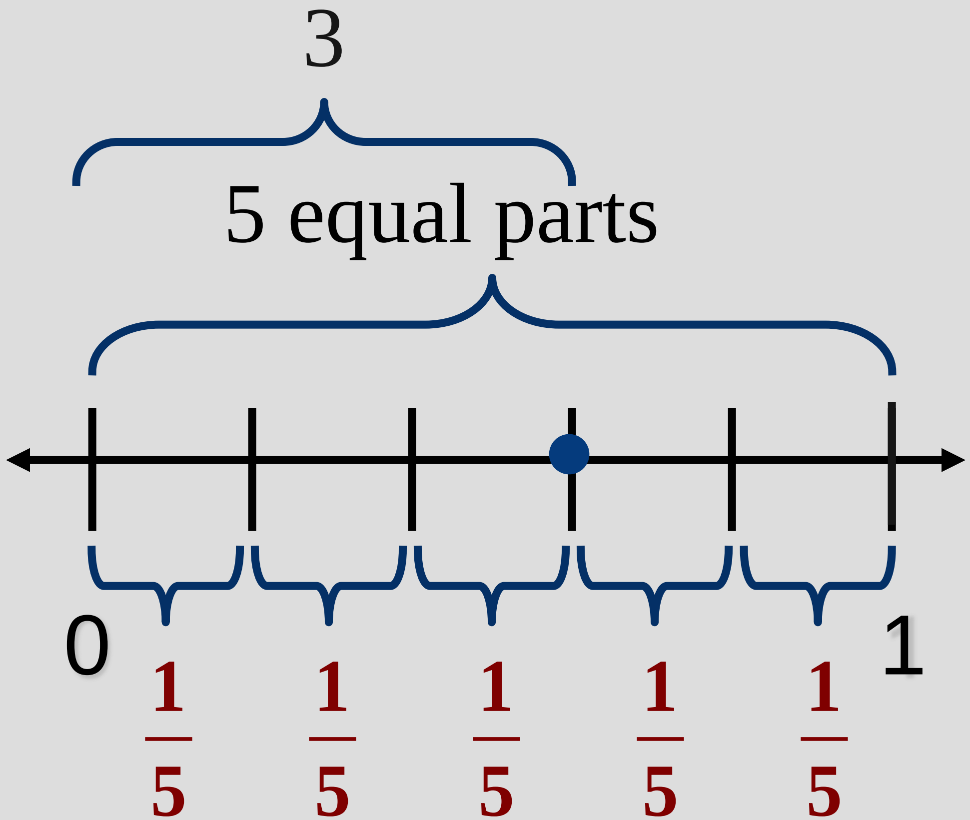
Improper fractions have values that are greater than or equal to 1.

A **mixed number** is a sum of a whole number and a proper fraction. $2\frac{2}{3}, 3\frac{1}{5}, 4\frac{2}{7}$

Fractions on Number Lines

Another way to visualize fractions is to graph them on a number line.

$$\frac{3}{5}$$



Fraction Properties of 1

If n is any integer other than 0, then

$$\frac{n}{n} = 1 \qquad \frac{5}{5} = 1$$

If n is any integer, then

$$\frac{n}{1} = n \qquad \frac{3}{1} = 3$$

Fraction Properties of 0

If *n* is any integer other than 0, then

$$\frac{0}{n} = 0 \qquad \frac{0}{5} = 0$$

If *n* is any integer, then

$$\frac{n}{0} = \text{undefined} \qquad \frac{3}{0} = \text{undefined}$$

Writing a Mixed Number as an Improper Fraction

Step 1: Multiply the denominator of the fraction by the whole number.

Step 2: Add the numerator of the fraction to the product from Step 1.

Step 3: Write the sum from Step 2 as the numerator of the improper fraction over the original denominator.

$$2\frac{3}{4} = \frac{2 \cdot 4 + 3}{4} = \frac{8 + 3}{4} = \frac{11}{4}$$

Writing an Improper Fraction as a Mixed Number or a Whole Number

Step 1: Divide the denominator into the numerator.

Step 2: The whole number part of the mixed number is the quotient. The fraction part of the mixed number is the remainder over the original denominator.

$$\text{quotient} \quad \frac{\text{remainder}}{\text{original denominator}}$$

4.2

Factors and Simplest Form

Prime and Composite Numbers

A **prime number** is a natural number greater than 1 whose only factors are 1 and itself. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, . . .

A **composite number** is a natural number greater than 1 that is not prime.

Helpful Hint

The natural number 1 is neither prime nor composite.

Prime Factorization

A **prime factorization** of a number expresses the number as a **product** of its **factors** and the **factors must be prime numbers**.

Helpful Hints

Remember a factor is any number that divides a number evenly (with a remainder of 0).

Prime Factorization

Every whole number greater than 1 has exactly one prime factorization.

$$12 = 2 \cdot 2 \cdot 3$$

2 and 3 are prime factors of 12 because they are prime numbers and they divide evenly into 12.

Divisibility Tests

A whole number is divisible by

2 if its last digit is 0, 2, 4, 6, or 8.

196 is divisible by 2

3 if the sum of its digits is divisible by 3.

117 is divisible by 3 since 1 + 1 + 7 = 9 is divisible by 3.

Divisibility Tests

A whole number is divisible by

5 if the ones digit is 0 or 5.

↓

2,345 is divisible by 5.

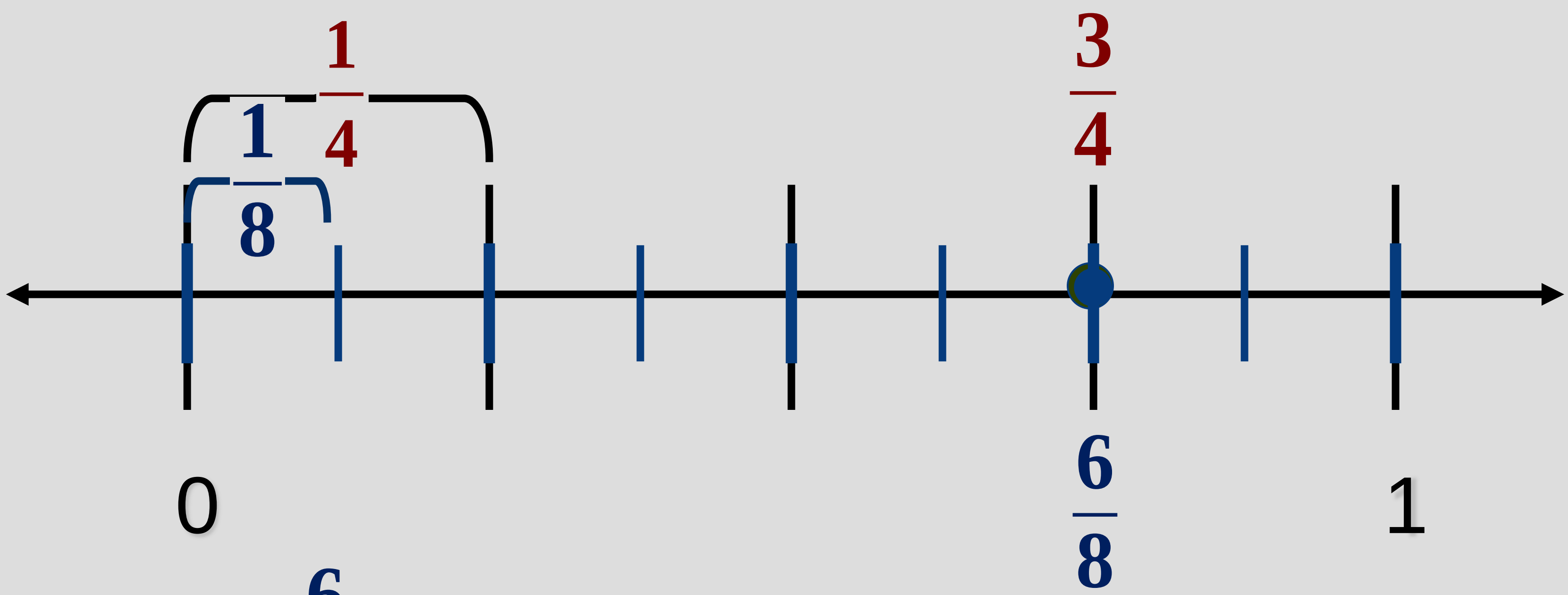
10 if its last digit is 0.

↓

8,470 is divisible by 10.

Equivalent Fractions

Graph $\frac{3}{4}$ on the number line.



Graph $\frac{6}{8}$ on the number line.

Equivalent Fractions

Fractions that represent the same portion of a whole or the same point on the number line are called **equivalent fractions**.

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

$$\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$$

Fundamental Property of Fractions

If *a*, *b*, and *c* are numbers, then

$$\frac{a}{b} = \frac{a \times c}{b \times c}$$

and also

$$\frac{a}{b} = \frac{a \div c}{b \div c}$$

as long as *b* and *c* are not 0. If the numerator and denominator are multiplied or divided by the same nonzero number, the result is an **equivalent fraction**.

Simplest Form

A fraction is in simplest form, or lowest terms, when the numerator and denominator have no common factors other than 1.

Using the fundamental principle of fractions, divide the numerator and denominator by the common factor of 7.

$$\frac{14}{21} = \frac{14 \div 7}{21 \div 7} = \frac{2}{3}$$

Using the prime factorization of the numerator and denominator, divide out common factors.

$$\frac{14}{21} = \frac{7 \cdot 2}{7 \cdot 3} = \frac{7 \cdot 2}{7 \cdot 3} = \frac{2}{3}$$

Writing a Fraction in Simplest Form

To write a fraction in simplest form, write the prime factorization of the numerator and the denominator and then divide both by all common factors.

The process of writing a fraction in simplest form is called **simplifying** the fraction.

Helpful Hints

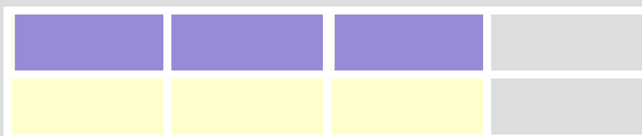
$$\frac{5}{10} = \frac{\cancel{5}}{\cancel{5} \cdot 2} = \frac{1}{2} \qquad \frac{15}{3} = \frac{\cancel{3} \cdot 5}{\cancel{3}} = \frac{5}{1} = 5$$

When all factors of the numerator or denominator are divided out, don't forget that 1 still remains in that numerator or denominator.

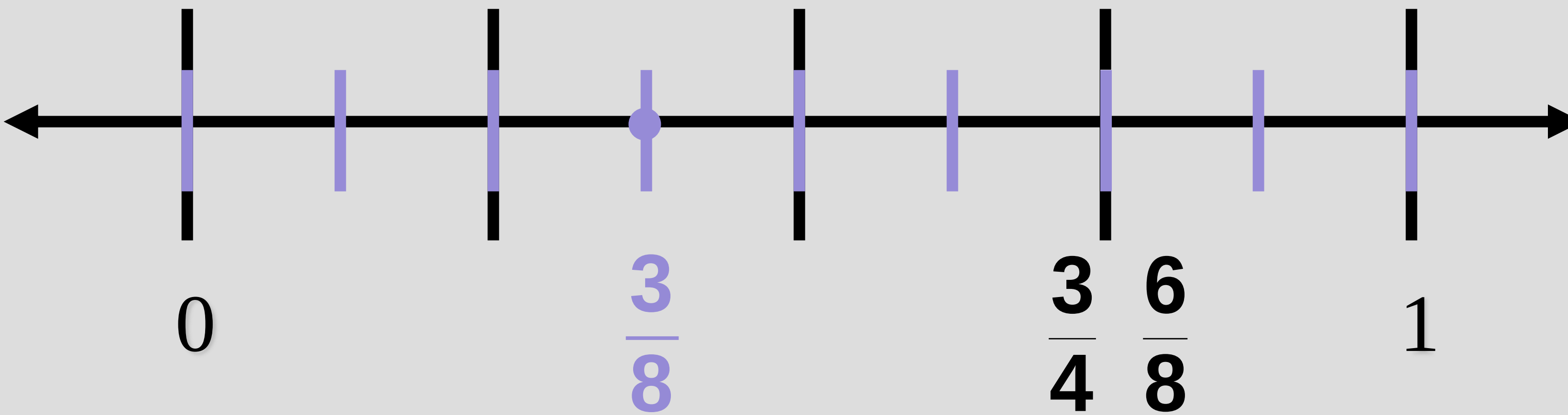
4.3

Multiplying and Dividing Fractions

Multiplying Fractions



$\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$



The word "of" means multiplication and "is" means equal to.

Multiplying Fractions

$$\frac{1}{2} \text{ of } \frac{3}{4} \text{ is } \frac{3}{8}$$

means

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

Multiplying Two Fractions

If *a*, *b*, *c*, and *d* are numbers and *b* and *d* are not 0, then

$$\frac{a}{b} \bullet \frac{c}{d} = \frac{a \bullet c}{b \bullet d}$$

In other words, to multiply two fractions, multiply the numerators and multiply the denominators.

Examples

$$\frac{3}{2}\cdot\frac{5}{7}=\frac{3\cdot 5}{2\cdot 7}=\frac{15}{14}$$

If the numerators have common factors with the denominators, divide out common factors before multiplying.

$$\frac{3}{4}\cdot\frac{2}{5}=\frac{3\cdot \cancel{2}}{\cancel{2}\cdot 2\cdot 5}=\frac{3}{10}$$

or

$$\frac{3}{\underset{2}{\cancel{4}}}\cdot\frac{\overset{1}{\cancel{2}}}{5}=\frac{3}{10}$$

Examples

$$\frac{3x}{4}\cdot\frac{8}{5x} = \frac{3\cdot \cancel{x}\cdot\cancel{4}\cdot 2}{\cancel{4}\cdot 5\cdot\cancel{x}} = \frac{6}{5}$$

or

$$\frac{3\cancel{x}}{\underset{1}{\cancel{4}}}\cdot\frac{\overset{2}{\cancel{8}}}{5\cancel{x}} = \frac{6}{5}$$

Helpful Hint

Recall that when the denominator of a fraction contains a variable, such as

$$\frac{8}{5x},$$

we assume that the variable is not 0.

Expressions with Fractional Bases

The base of an exponential expression can also be a fraction.

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \bullet \frac{2}{3} \bullet \frac{2}{3} = \frac{2 \bullet 2 \bullet 2}{3 \bullet 3 \bullet 3} = \frac{8}{27}$$

Reciprocal of a Fraction

Two numbers are **reciprocals** of each other if their product is 1. The reciprocal of the fraction

$$\frac{a}{b} \text{ is } \frac{b}{a}$$

because

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{ab}{ab} = 1$$

Dividing Two Fractions

If *b*, *c*, and *d* are not 0, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

In other words, to divide fractions, multiply the first fraction by the reciprocal of the second fraction.

$$\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \cdot \frac{7}{2} = \frac{21}{10}$$

Helpful Hint

Every number has a reciprocal except 0. The number 0 has no reciprocal. Why?

There is no number that when multiplied by 0 gives the result 1.

Helpful Hint

When dividing by a fraction, do not look for common factors to divide out until you rewrite the division as multiplication.

Do not try to divide out these two 2s.

$$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \bullet \frac{3}{2} = \frac{3}{4}$$

Fractional Replacement Values

If $x = \frac{5}{6}$ and $y = \frac{2}{5}$, evaluate $x \div y$.

Replace x with $\frac{5}{6}$ and y with $\frac{2}{5}$.

$$x \div y = \frac{5}{6} \div \frac{2}{5} = \frac{5}{6} \bullet \frac{5}{2} = \frac{25}{12}$$

4.4

Adding and Subtracting Like Fractions, Least Common Denominator, and Equivalent Fractions

Like and Unlike Fractions

Fractions that have the same or common denominator are called **like fractions**.

Fractions that have different denominators are called **unlike fractions**.

Like Fractions	**Unlike Fractions**
$\frac{2}{5}$ and $\frac{4}{5}$	$\frac{2}{3}$ and $\frac{3}{4}$
$\frac{5}{7}$ and $\frac{-3}{7}$	$\frac{5}{6}$ and $\frac{5}{12}$

Adding or Subtracting Like Fractions

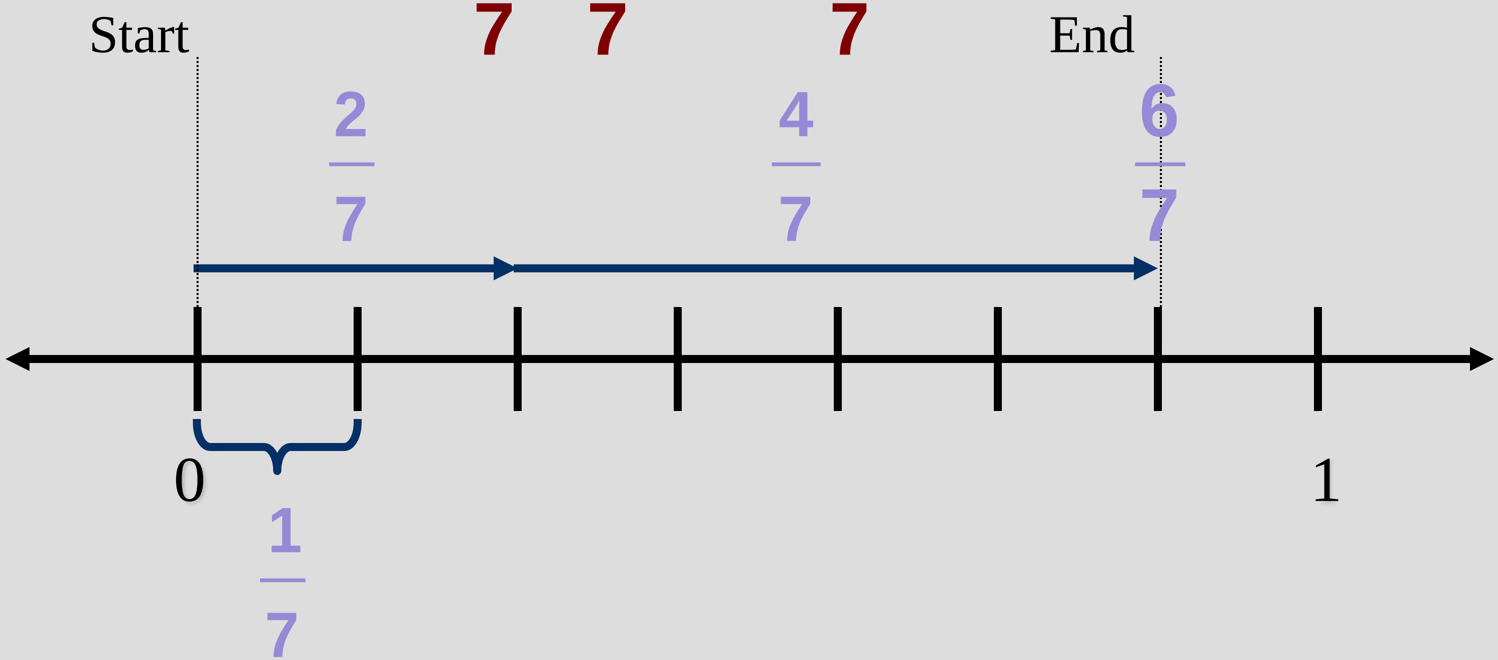
If *a*, *b*, and *c*, are numbers and *b* is not 0, then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \quad \text{also} \quad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

To add or subtract fractions with the same denominator, add or subtract their numerators and write the sum or difference over the common denominator.

Adding or Subtracting Like Fractions

$$\frac{2}{7} + \frac{4}{7} = \frac{6}{7}$$



To add like fractions, add the numerators and write the sum over the common denominator.

Helpful Hint

Do not forget to write the answer in **simplest form**. If it is not in simplest form, divide out all common factors larger than 1.

Equivalent Negative Fractions

$$\frac{-2}{3} = \frac{2}{-3} = -\frac{-2}{-3} = -\frac{2}{3}$$

Least Common Denominator

To add or subtract fractions that have unlike, or different, denominators, we write the fractions as equivalent fractions with a common denominator.

The smallest common denominator is called the **least common denominator** (LCD) or the least common multiple (LCM).

Least Common Multiple

The least common denominator (LCD) of a list of fractions is the smallest positive number divisible by all the denominators in the list. (The least common denominator is also the least common multiple (LCM) **of the denominators**.)

Least Common Denominator

To find the LCD of $\frac{5}{12}$ and $\frac{5}{18}$

First, write each denominator as a product of primes.

$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot 3 \cdot 3$

Then write each factor the greatest number of times it appears in any one prime factorization.

The greatest number of times that 2 appears is 2 times.
The greatest number of times that 3 appears is 2 times.

$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 = 36$

4.5

Adding and Subtracting Unlike Fractions

Adding or Subtracting Unlike Fractions

Step 1: Find the LCD of the denominators of the fractions.

Step 2: Write each fraction as an equivalent fraction whose denominator is the LCD.

Step 3: Add or subtract the like fractions.

Step 4: Write the sum or difference in simplest form.

Adding or Subtracting Unlike Fractions

Add: $\frac{1}{9}+\frac{7}{12}$

Step 1: Find the LCD of 9 and 12.

$9 = 3 \cdot 3$ and $12 = 2 \cdot 2 \cdot 3$

$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 = 36$

Step 2: Rewrite equivalent fractions with the LCD.

$$\frac{1}{9}=\frac{1 \cdot 4}{9 \cdot 4}=\frac{4}{36} \qquad \frac{7}{12}=\frac{7 \cdot 3}{12 \cdot 3}=\frac{21}{36}$$

Continued.

Adding or Subtracting Unlike Fractions

Continued:

Step 3: Add like fractions.

$$\frac{1\cdot 4}{9\cdot 4}+\frac{7\cdot 3}{12\cdot 3}=\frac{4}{36}+\frac{21}{36}=\frac{25}{36}$$

Step 4: Write the sum in simplest form.

$$\frac{25}{36}$$

Writing Fractions in Order

One important application of the least common denominator is to use the LCD to help order or compare fractions.

Insert < or > to form a true sentence. $\frac{3}{5}\ ?\ \frac{4}{7}$

The LCD for these fractions is 35.

Write each fraction as an equivalent fraction with a denominator of 35.

$$\frac{3}{5} = \frac{3\cdot 7}{5\cdot 7} = \frac{21}{35} \qquad \frac{4}{7} = \frac{4\cdot 5}{7\cdot 5} = \frac{20}{35}$$

Continued.

Writing Fractions in Order

Continued:

Compare the numerators of the equivalent fractions.

Since 21 > 20, then $\frac{21}{35} > \frac{20}{35}$

Thus, $\frac{3}{5} > \frac{4}{7}$

Evaluating Expressions

Evaluate $x - y$ if $x = \frac{2}{3}$ and $y = \frac{3}{4}$.

Replacing x with $\frac{2}{3}$ and y with $\frac{3}{4}$,

then, $x - y = \frac{2}{3} - \frac{3}{4}$

$$= \frac{2 \cdot 4}{3 \cdot 4} - \frac{3 \cdot 3}{4 \cdot 3} = \frac{8}{12} - \frac{9}{12} = -\frac{1}{12}$$

Solving Equations Containing Fractions

Solve: $x - \frac{1}{3} = \frac{5}{12}$

To get x by itself, add $\frac{1}{3}$ to both sides.

$$x - \frac{1}{3} + \frac{1}{3} = \frac{5}{12} + \frac{1}{3}$$

$$= \frac{5}{12} + \frac{1 \cdot 4}{3 \cdot 4}$$

Continued.

Solving Equations Containing Fractions

Continued:

$$x = \frac{5}{12} + \frac{4}{12}$$

$$= \frac{9}{12} = \frac{3}{4}$$

Write fraction in simplest form.

4.6

Complex Fractions and Review of Order of Operations

Complex Fraction

A fraction whose numerator or denominator or both numerator and denominator contain fractions is called a complex fraction.

$$\frac{\frac{2}{3}}{\frac{x}{4}} \qquad \frac{\frac{2}{3}+\frac{3}{5}}{\frac{y}{5}-\frac{1}{7}} \qquad \frac{\frac{2}{5}+4}{\frac{7}{8}}$$

Method 1: Simplifying Complex Fractions

This method makes use of the fact that a fraction bar means division.

$$\frac{\frac{2}{3}}{\frac{8}{9}} = \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \cdot \frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{8}}} = \frac{3}{4}$$

When dividing fractions, multiply by the reciprocal of the divisor.

Method 1: Simplifying Complex Fractions

Recall the order of operations. Since the fraction bar is a grouping symbol, simplify the numerator and denominator separately. Then divide.

$$\frac{\frac{1}{2}+\frac{1}{6}}{\frac{3}{4}-\frac{2}{3}} = \frac{\frac{1\cdot 3}{2\cdot 3}+\frac{1}{6}}{\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4}} = \frac{\frac{3}{6}+\frac{1}{6}}{\frac{9}{12}-\frac{8}{12}} = \frac{\frac{4}{6}}{\frac{1}{12}} = \frac{4}{\cancel{6}_1}\cdot\frac{\cancel{12}^2}{1} = 8$$

When dividing fractions, multiply by the reciprocal of the divisor.

Method 2: Simplifying Complex Fractions

This method is to multiply the numerator and the denominator of the complex fraction by the LCD of all the fractions in its numerator and its denominator. Since this LCD is divisible by all denominators, this has the effect of leaving sums and differences of terms in the numerator and the denominator and thus a simple fraction.

Let's use this method to simplify the complex fraction of the previous example.

Method 2: Simplifying Complex Fractions

$$\frac{\frac{1}{2}+\frac{1}{6}}{\frac{3}{4}-\frac{2}{3}} = \frac{12\left(\frac{1}{2}+\frac{1}{6}\right)}{12\left(\frac{3}{4}-\frac{2}{3}\right)} = \frac{12\left(\frac{1}{2}\right)+12\left(\frac{1}{6}\right)}{12\left(\frac{3}{4}\right)-12\left(\frac{2}{3}\right)} =$$

Step 1: The complex fraction contains fractions with denominators of 2, 6, 4, and 3. The LCD is 12. By the fundamental property of fractions, multiply the numerator and denominator of the complex fraction by 12.

Step 2: Apply the distributive property

Continued.

Method 2: Simplifying Complex Fractions

Continued:

$$\frac{\frac{1}{2}+\frac{1}{6}}{\frac{3}{4}-\frac{2}{3}} = \frac{12\left(\frac{1}{2}\right)+12\left(\frac{1}{6}\right)}{12\left(\frac{3}{4}\right)-12\left(\frac{2}{3}\right)} = \frac{6+2}{9-8} = \frac{8}{1} = 8$$

Step 3: Multiply.

Step 4: Simplify.

The result is the same no matter which method is used.

Reviewing the Order of Operations

1. Perform all operations within parentheses ( ), brackets [ ], or other grouping symbols such as fraction bars, starting with the innermost set.
2. Evaluate any expressions with exponents.
3. Multiply or divide in order from left to right.
4. Add or subtract in order from left to right.

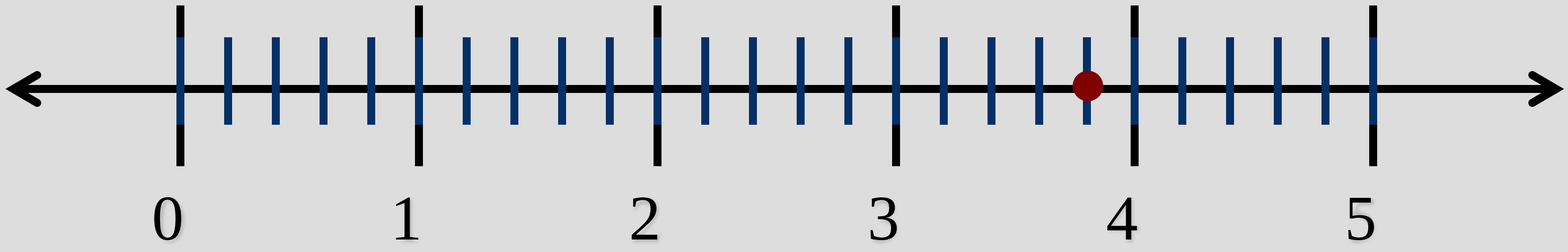
4.7

Operations on Mixed Numbers

Mixed Numbers

Recall that a **mixed number** is the sum of a whole number and a proper fraction.

$$3\frac{4}{5} = 3 + \frac{4}{5}$$



$$\frac{19}{5} = 3\frac{4}{5}$$

Multiplying or Dividing with Mixed Numbers

To multiply or divide with mixed numbers or whole numbers, first write each mixed number as an improper fraction.

Multiply: $3\frac{1}{5} \cdot 2\frac{1}{4}$

$$3\frac{1}{5} \cdot 2\frac{1}{4} = \frac{16}{5} \cdot \frac{9}{4}$$

Change mixed numbers to improper fractions.

$$\frac{4 \cdot 4}{5} \cdot \frac{9}{4} = \frac{36}{5} = 7\frac{1}{5}$$

Remove common factors and multiply.

Write the solution as a mixed number if possible.

Adding or Subtracting Mixed Numbers

We can add or subtract mixed numbers by first writing each mixed number as an improper fraction. But it is often easier to add or subtract the whole number parts and add or subtract the proper fraction parts vertically.

Adding or Subtracting Mixed Numbers

Add: $2\frac{5}{14} + 5\frac{6}{7}$

The LCD of 14 and 7 is 14.

$$\begin{array}{rcr} 2\frac{5}{14} & = & 2\frac{5}{14} \\ +\,5\frac{6}{7} & = & +\,5\frac{12}{14} \\ \hline & & 7\frac{17}{14} \end{array}$$

Write equivalent fractions with the LCD of 14.

Add the fractions, then add the whole numbers.

Notice that the fractional part is **improper**.

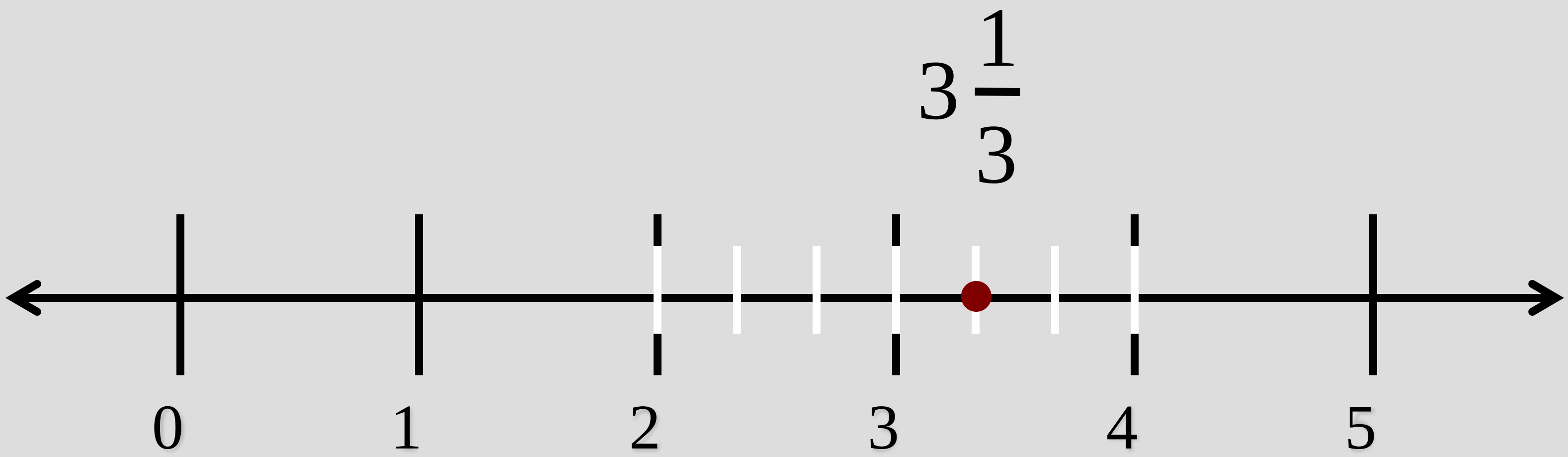
Since $\frac{17}{14}$ is $1\frac{3}{14}$, write the sum as

$$7\frac{17}{14} = 7 + 1\frac{3}{14} = 8\frac{3}{14}$$

Make sure the fractional part is always **proper**.

Adding or Subtracting Mixed Numbers

When subtracting mixed numbers, borrowing may be needed.



$$3\frac{1}{3} = 2 + 1\frac{1}{3} = 2 + 1 + \frac{1}{3} = 2 + \frac{3}{3} + \frac{1}{3} = 2\frac{4}{3}$$

Borrow 1 from 3.

Adding or Subtracting Mixed Numbers

Subtract: $5\frac{3}{14} - 3\frac{6}{7}$

The LCD of 14 and 7 is 14.

$$\begin{array}{r} 5\frac{3}{14} \\ -\ 3\frac{6}{7} \\ \hline \end{array} = \begin{array}{r} 5\frac{3}{14} \\ -\ 3\frac{12}{14} \\ \hline \end{array}$$

Write equivalent fractions with the LCD of 14.

To subtract the fractions, we have to **borrow**.

$$5\frac{3}{14} = 4 + 1\frac{3}{14} = 4 + \frac{17}{14} = 4\frac{17}{14}$$

$$\begin{array}{r} 5\frac{3}{14} \\ -\ 3\frac{6}{7} \\ \hline \end{array} = \begin{array}{r} 5\frac{3}{14} \\ -\ 3\frac{12}{14} \\ \hline \end{array} = \begin{array}{r} 4\frac{17}{14} \\ -\ 3\frac{12}{14} \\ \hline 1\frac{5}{14} \end{array}$$

Subtract the fractions, then subtract the whole numbers.

Notice that the fractional part is **proper**.

4.8

Solving Equations Containing Fractions

Addition Property of Equality

Let a, b, and c represent numbers.

If $a = b$, then

$$a + c = b + c$$

and

$$a - c = b - c$$

In other words, the same number may be added to or subtracted from both sides of an equation without changing the solution of the equation.

Multiplication Property of Equality

Let a, b, and c represent numbers and let $c \neq 0$.
If $a = b$, then

$$a \cdot c = b \cdot c \quad \text{and} \quad \frac{a}{c} = \frac{b}{c}$$

In other words, both sides of an equation may be multiplied or divided by the same nonzero number without changing the solution of the equation.

Solving an Equation in *x*

Step 1: If fractions are present, multiply both sides of the equation by the LCD of the fractions.

Step 2: If parentheses are present, use the distributive property.

Step 3: Combine any like terms on each side of the equation.

Solving an Equation in *x*

Step 4: Use the addition property of equality to rewrite the equation so that variable terms are on one side of the equation and constant terms are on the other side.

Step 5: Divide both sides of the equation by the numerical coefficient of x to solve.

Step 6: Check the answer in the **original equation**.

Solve for *x*

$$\frac{1}{7}x = \frac{5}{9}$$

$$7\left(\frac{1}{7}x\right) = \left(\frac{5}{9}\right)7$$ Multiply both sides by 7.

$$x = \frac{35}{9}$$ Simplify both sides.

Solve for *x*

$$\frac{3(y+3)}{5} = 2y+6$$

$$5\left[\frac{3(y+3)}{5}\right] = (2y+6)5$$ Multiply both sides by 5.

$$3y+9 = 10y+30$$ Simplify both sides.

$$9 = 7y+30$$ Add – 3*y* to both sides.

$$-21 = 7y$$ Add – 30 to both sides.

$$-3 = y$$ Divide both sides by 7.