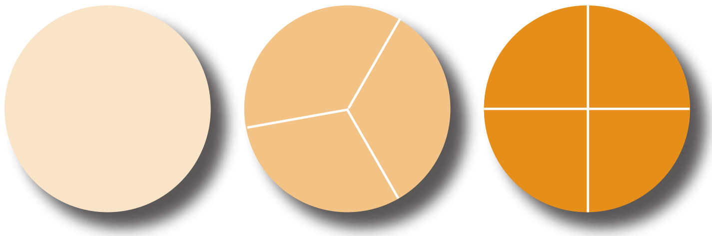
Fractions

pikelets and lamingtons



Fractions: pikelets and lamingtons

Cover artwork by Stephen Axelsen

ISBN 0 7313 8278 1
SCIS number 1146670

Contents

Introduction 5
Why do some students find working with fractions difficult?
Models used with fractions
Building on sharing
Parts and wholes
Recording thinking with diagrams

Fraction units 9

Fold, open and draw 11
Introducing sharing diagrams

Sharing pikelets 13
Developing a part-whole model of fractions using sharing diagrams

Half the pikelets 20
Linking part-whole models of fractions (discrete and continuous)

A dozen pikelets 24
Linking part–whole models of fractions (discrete and continuous)

A piece of cake 25
Forming an image of thirds

How many pikelets? 27
Part–whole models beyond one (discrete and continuous)

A birthday secret 29
Recreating the whole from a part

A pikelet recipe 32
Using sharing diagrams to operate on continuous models of fractions

Lamington bars 37
Forming equivalent fractions

Mrs Packer's visitors 39
Comparing fractions

Related fractions 1 41
One-half, one-quarter and one-eighth

Related fractions 2 43
One-third, one-sixth, one-ninth and one-twelfth

Building the fraction bridge: 1 48
Constructing and comparing unit fractions

Building the fraction bridge: 2 50
Constructing and comparing unit fractions

Crossing the wall 52
Linking and using equivalent fractions

What can we learn from the research? 59
The instructional sequence

References 63

Acknowledgements

No mathematics curriculum publication is ever developed without the contributions of many people. Penny Lane, St George District Mathematics Consultant, assisted with the early discussions on the ways of representing fractions and developed the activities *A piece of cake*, *How many pikelets*? and *A birthday secret*. Brett Butterfield assisted with the trialling of *Related fractions* and *Building the fraction bridge*. Chris Francis developed the writing tasks associated with *Related Fractions 2* and provided many refinements to the development and trialling of the tasks.

The activity, *Mrs Packer's visitors*, was inspired by *Chocolate Cake* from *Fractions in Action* by The Task Centre Collective Pty Ltd (1999), which in turn was an adaptation of *Share it out* from *Using the CSF- NPDP Mathematics Assessment Activity* (1995).

Elaine Watkins and Lee Brown assisted with the trials and collection of work samples. In particular, the teachers and students of Roselea Primary School, Dennistone East Primary School, Beaumont Road Primary School, Kellyville Primary School and West Pymble Primary School all contributed to the development of this resource.

Peter Gould built on the work of many colleagues in developing the teaching activities, the teaching sequence and the related summary of research.

Introduction

The history of teaching fractions is long and colourful. In 1958 Hartung wrote, "The fraction concept is complex and cannot be grasped all at once. It must be acquired through a long process of sequential development." This sequential development of the fraction concept needs to be well understood if we are to develop widespread access to learning fractions with understanding.

The recent history of fraction use has been echoed in this area of syllabus development. The day-to-day manipulation of common fractions became less frequent as our money system and our measurement system went decimal. Added to this, access to cheap and efficient calculating devices reduced the need to labour over tedious fraction algorithms. There was an almost audible collective sigh of relief when operating with fractions became yet another thing that calculators could do for you.

We know that many students experience difficulties in working with fractions. Beyond the algorithmic manipulation of fractions lie the related difficulties of the underpinning concept.

Why do some students find working with fractions difficult?

Imagine a student encountering the symbols we use to record fractions. She is told that $\frac{3}{4}$ is the same as three out of four. Explaining what we mean by the numerator and the denominator of a fraction might expand this "definition". The student then demonstrates her understanding of fraction notation by stating that three people out of four people is the same as $\frac{3}{4}$, two people out of five people is the same as $\frac{2}{5}$ and five people out of nine people is the same as $\frac{5}{9}$. All appears well until your precocious student surprises you by writing $\frac{3}{4} + \frac{2}{5} = \frac{5}{9}$. Now you have a lot of explaining to do!

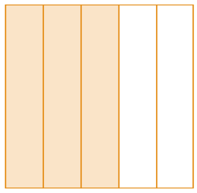
Emphasising numeric rules too soon without underlying meaning discourages students from attempting to see rational numbers as something sensible.

The rapid transition from modelling fractions to recording fractions in symbolic form, numerator over denominator, can contribute to many students' confusion. The result of this rapid transition to recording fractions is that many students see fractions as two whole numbers—three-quarters is the whole number three written over the whole number four. In a National Assessment of Educational Progress, more than half of U.S. eighth graders appeared to believe that fractions were a form of recording whole numbers and chose 19 or 21 as the best estimate of $\frac{12}{13} + \frac{7}{8}$.

Recording fractions in symbolic form needs to build on an underpinning conceptual framework. This framework highlights the role of equal parts and collections of parts that form new units. Emphasising numeric rules too soon without underlying meaning discourages students from attempting to see rational numbers as something sensible.

Models used with fractions

As well as writing fractions as symbols, we are all familiar with using an area model to describe fractions. This is sometimes described as using a continuous model of fractions

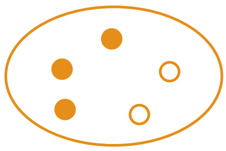


Divide the whole into 5 equal parts and shade 3 of them.

A common difficulty that arises with the area model is that some students focus on the number of parts, rather than the equality of those parts.

A common difficulty that arises with the area model is that some students focus on the number of parts, rather than the equality of those parts.

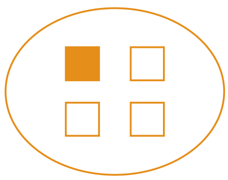
Sometimes, instead of a continuous model of fractions, we use a discrete model of fractions.



$\frac{2}{5}$ of the dots are white.

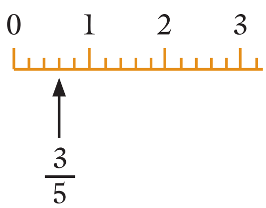
This is called a discrete model because the "parts" are separate things. Within the discrete model it is often harder to see the "whole". The emphasis on the parts appears stronger than that on the whole. The difficulties many students experience in working with a discrete part-whole model of fractions are well documented. In an assessment of 11 year-old students in England reported in 1980 (Assessment of Performance Unit) only 64% were successful on the following task.

> *Students were presented with 4 square tiles, 3 yellow and 1 red and asked, "What fraction of these squares are red?"*



Many of those who were incorrect gave the answer one-third.

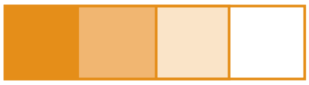
We can also represent fractions in a linear model as a location on the number line.



Although it is reasonable to expect that the number line model of fractions would be of the same order of difficulty as the area model or the discrete model, it turns out to be more difficult. Various reasons have been put forward as to why this might be so. Part of the difficulty may be due to the problem associated with allocating a marker to zero. This is often a problem for students who believe that measuring is nothing more than counting markers. Added to this, the number line requires recognising that three-fifths *is a number rather than a comparison of two numbers.*

The number line interpretation of fractions needs to be linked to the idea of measurement. That is, the number line representation builds on finding a fraction part of a unit (say, a paper streamer) by folding. The location of a fraction is then the same as an accumulation of distance.

The number line interpretation of fractions needs to be linked to the idea of measurement.



$\frac{3}{4}$

Learning fractions should never start from the symbols. Fraction symbols look like two whole numbers. The fraction concept should be based on the process of equal sharing.

Building on sharing

Fractions arise from the process of equal partitioning or sharing.

> *Megan has 15 pikelets to share equally among 3 people. How many pikelets does each person get?*

This type of question is known as partitive division. Partitive division problems give the total number of objects and the number of groups to be formed; the number of objects in each group is unknown. Changing the numbers slightly in the above question will increase the difficulty significantly as well as providing a link between division and fractions.

> *Megan has 8 pikelets to share equally among 3 people. How many pikelets does each person get?*

If materials or diagrams are used in the solution of this question it is easy to see how both wholes and parts play a role in the solution.

The formation of three equal parts from a continuous area model of circles is not easy through folding or cutting[4] even though it ties strongly to the idea of angles. Partitioning circles using the idea of sharing pikelets provides a context for fractions as well as a link between fractions and division. It is also possible this way to link fractions to time and angle measure.

[4] It is easier to find one-third of a rectangular strip of paper through folding. Finding one-third of a circle requires locating the centre.

Fractions can also arise from using a unit to measure. This is similar to dividing the total length by a unit of a given size.

This link between fractions and measurement with a focus on the whole unit was used in a special Russian curriculum. For example, a piece of string is measured by a small piece of tape and found to be equal to five copies of the tape. Rational numbers arise quite naturally when the quantity is not measured by the unit an exact number of times. The remainder can be measured by subdividing the unit to create a fraction.

Parts and wholes

A focus on re-dividing the unit is important in the development of the idea of equivalent fractions. Imagine a block of chocolate. We can equally share a block of chocolate between three people just as we can share the block between five people. Learning experiences must be used that emphasise the importance of the unit and its subdivision into equal parts. These experiences need to focus on the development of conceptual knowledge prior to formal work with symbols and algorithms. This does not restrict the use of recording fractions as a form of shorthand notation.

Traditional questions comparing unit fractions, such as “Which is bigger, one-third or one-quarter?” become much simpler through a focus on division or equal sharing. Sharing one pikelet between three people means that each will receive more than sharing one pikelet between four people. The relationship between the number of equal parts and the size of the parts needs to be established.

Once the relationship between the number of equal parts and the size of the parts has been established, it can be used to answer questions such as “which is bigger, two-thirds or three-quarters?”

Recording thinking with diagrams

Sharing diagrams provide a good method of representing and calculating with fractions. Not only are they more closely linked to the nature of fractions arising from division than the traditional symbolic notation, they frequently provide access to the images students hold of fractions.

Sharing diagrams are offered to a student as a tool to represent and support his or her thinking. Representational tools are forms of symbolising that support thinking. Students’ diagrams should represent fraction problems in the way that they think about the problems. The value of sharing diagrams is in their congruence with the way that problems are interpreted. Standard fraction symbols are dissimilar from both the problem and the thinking involved in solving the problem.

Fractions

pikelets and lamingtons

Fraction units

Fold, open and draw ES1 S1 S2

Introducing sharing diagrams

Overview

In this multi-stage activity, students fold shapes into equal parts and are introduced to sharing diagrams by drawing what they have formed. The activity aims to promote part-whole understanding and to assist students perform the process of forming equal parts.

Outcomes

Describes halves, encountered in everyday contexts, as two equal parts of an object (NES1.4).

Describes and models halves and quarters, of objects and collections, occurring in everyday situations (NS1.4).

Models, compares and represents commonly used fractions … (NS2.4)

Development of activity

1. *If we wanted to share a lamington bar fairly between Chris and Elaine, how could we do it?*

   Draw a rectangle on the board to represent the lamington bar. Invite students to the board to draw a line to show where you would cut the lamington bar to make it fair. Do not erase each "cut". *Why do you think that the cut is correct or incorrect?* **Adjustments are important in developing the idea of equal parts**.

ES1

2. Introduce the idea of folding to make equal parts. Hold up a brown rectangular piece of paper. *If this were the lamington bar, who can show me where you would cut it … and, prove to everyone that it is fair?* Distribute brown paper rectangles, at least three per student. Allow students time to engage with the problem. Folding the rectangle to form half is preferable to cutting it, as both the parts and the whole remain present. *Remember that you have to explain to everyone why it is fair.*

   Allow students to use their paper rectangles to justify why their divisions are fair. Try to have the idea of folding to show equal parts come from your students' justifications. *How do you know that it is fair? Can you draw how you would share the lamington bar?*

S1

3. *When Chris and Elaine looked at their share of the lamington bar, they both said that it was too much to eat. They decided to have some now but to leave the same amount for later. Can you use the paper lamington bars or your drawing to show how much Chris and Elaine would eat and how much they would leave for later? Draw your answer.*

S2

4. *If Chris and Elaine wanted to share the lamington bar equally with Fiona, can you use another piece of paper to show how this could be done? There are now three people and they each want the same sized piece.* Allow students time to engage with the problem. Folding the rectangle to form three equal pieces is difficult because of the need to make multiple adjustments. *Remember that you have to explain to everyone why it is fair. Can you draw how you would share the lamington bar?*

ES1

S2

5. *If Chris and Elaine had to equally share a round lamington cake, could you show them where to cut it?* Distribute brown circular pieces of paper. Fold a circle in half, open and draw your answer.

6. *To cut the cake into six equal pieces they must cut each half into three equal pieces. Use the circle of paper you have folded in half. Fold this half of a circle in thirds, open and draw your answer.*

Comments

- Folding a paper rectangle to create halves, is better than colouring in a half or cutting to form a half. Folding has the advantage of forming *equal halves* by modelling the process of aligning and matching to create equal parts.

- Both rectangular and circular models are used to emphasise the process of aligning the equal pieces over the individual shape. The final activity is quite difficult because in sequencing halving and folding to form thirds, we have developed the foundation of multiplying fractions or fractions as operators.

Indicators of conceptual understanding

- Students form equal parts by aligning and matching, describing how any "whole" can be divided into halves through folding.

- The link between the parts and the whole is clear in students' recordings.

Sharing pikelets S1

Developing a part-whole model of fractions using sharing diagrams

Overview

In this activity, students explore dividing *wholes* into equal parts and are introduced to sharing diagrams. The activity aims to promote part–whole conceptual understanding and to assist students perform simple fraction mental computations through visualisation of a whole divided into equal parts.

Outcomes

Describes and models halves and quarters, of objects and collections, occurring in everyday situations (NS1.4).

Development of activity

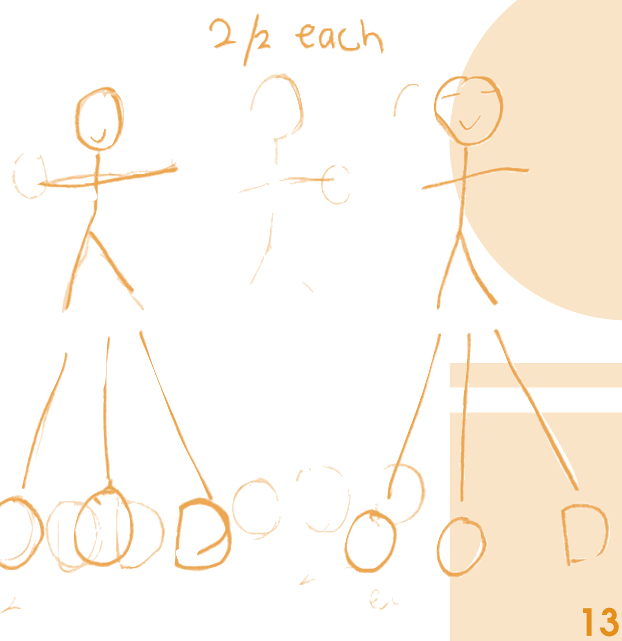
1. Introduce the problem of sharing pikelets. *Who can tell me what pikelets are? If we wanted to share 4 pikelets between 2 people, how could we do it?*

   Use four equal-sized circles (these could be pikelets or Brenex circles to represent the pikelets) and two students to model the process.

2. *If we wanted to share 3 pikelets between 2 people, how could we do it?* Allow students time to engage with the problem. Use three equal-sized circles (Brenex circles or other) to represent the pikelets. Have one student show, using the circles, how many pikelets each person would get. Folding the circle to form half is preferable to cutting the circle, as both the parts and the whole remain present.

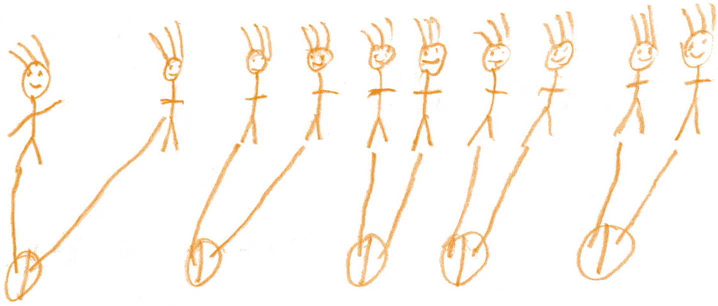
   Draw 2 stick figures and 3 circles on the board. Ask one student to add lines to your diagram on the board to show how he or she would share the pikelets.

3. *What would we do if we had 5 pikelets to share between 2 people? Can you draw your answer?*



Work sample 1

4. *What would we do if we had 5 pikelets to share among 10 people? Can you draw your answer?*



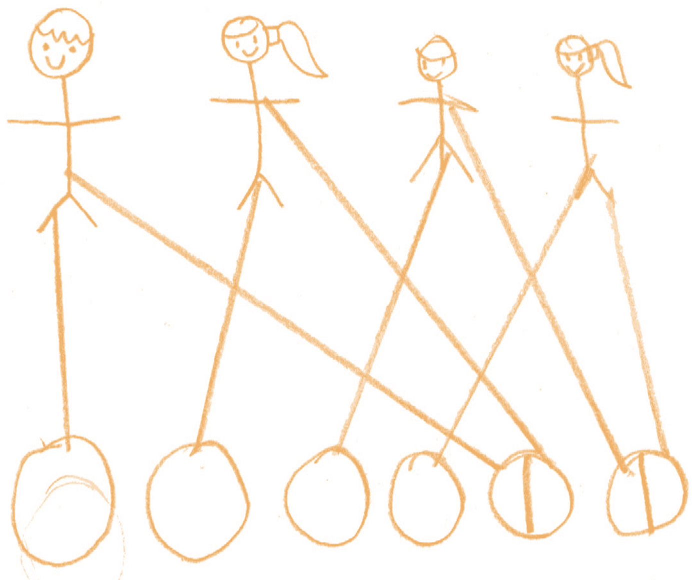
Work sample 2

Work sample 3

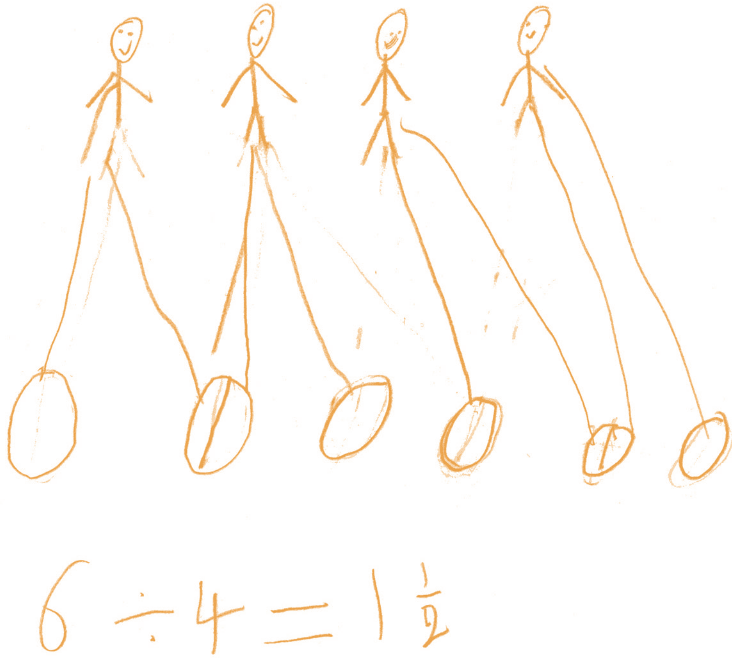
Comments

Both of the above work samples emphasise pairing to form five groups of two halves.

5. *Who can draw what would happen if we had 6 pikelets to share among 4 people?*



Work sample 4

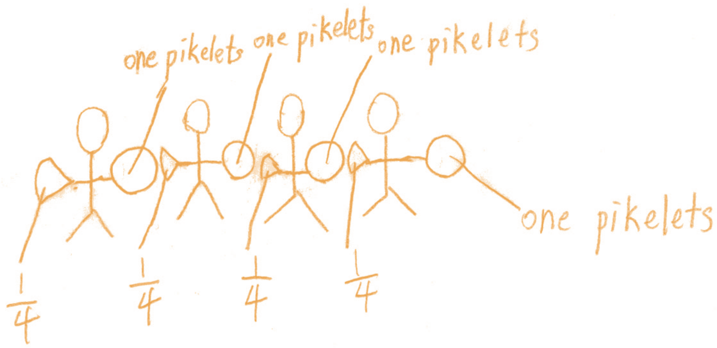


Work sample 5

Comments

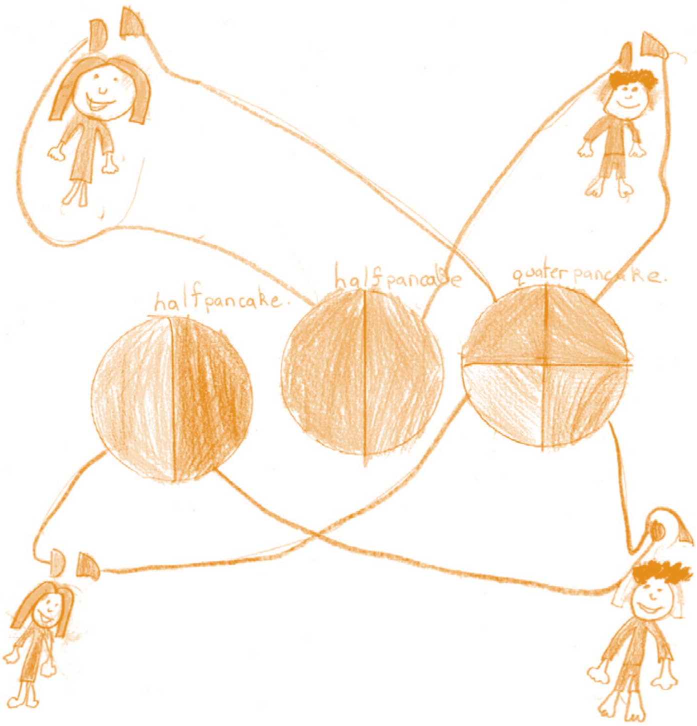
In work sample 5 the student has spontaneously recorded the division in symbolic form without the need to use remainder notation or to deal with equivalent fractions ($\frac{2}{4} = \frac{1}{2}$).

6. *What would happen if we had 5 pikelets to share among 4 people? Can you draw your answer?*



Work sample 6

7. *What would happen if we had 3 pikelets to share among 4 people? Can you draw your answer?*

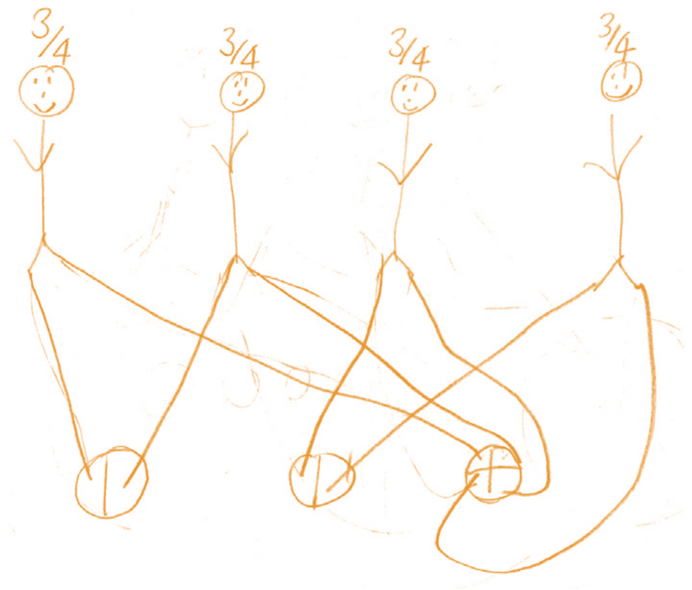


Work sample 7

Comments

In the above work sample, the student has used colour coding to emphasise ownership of the pikelet parts. The image of halves and quarters is clearly represented and a good approximation to the spelling of quarter has been made.

Work sample 8

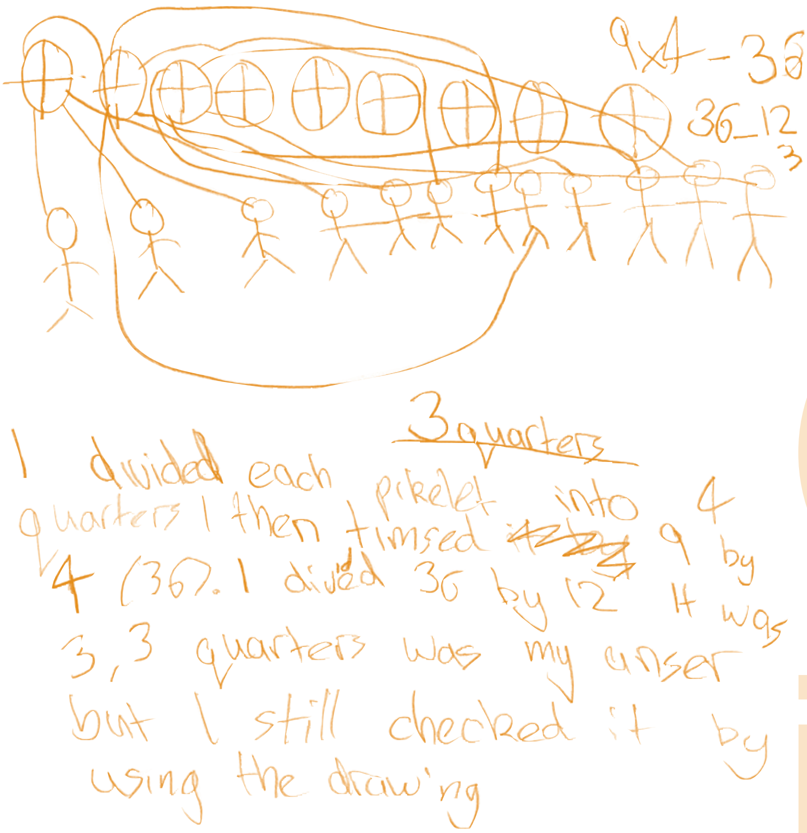


Work sample 9

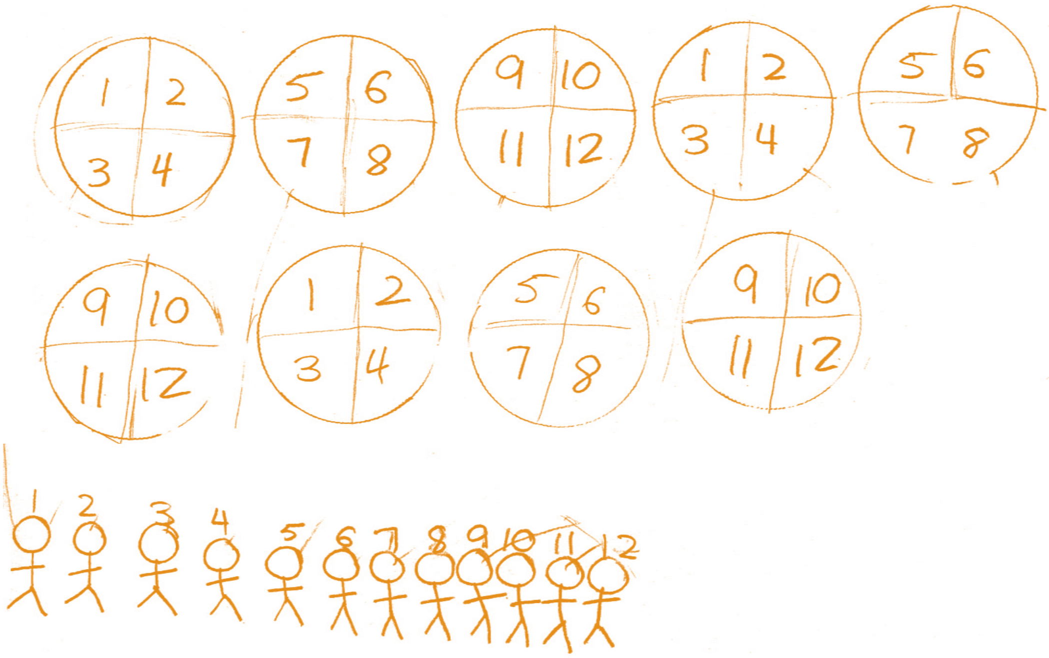
Comments

The above work sample shows the result of sharing as one-half plus one-quarter equalling three-quarters.

8. *What would happen if we had 9 pikelets to share among 12 people? Can you draw your answer?*

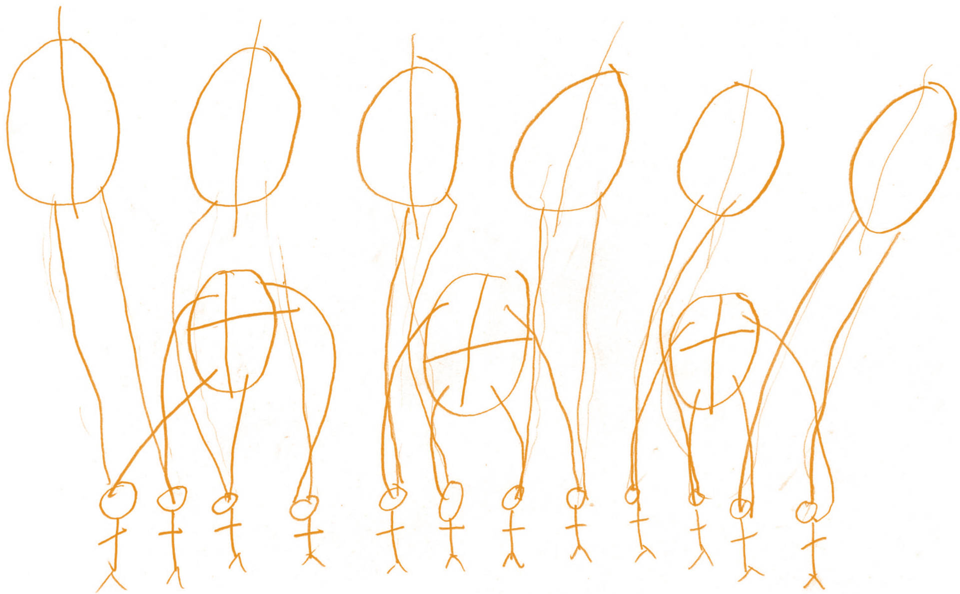


Work sample 10



They each get 3.

Work sample 11



each get three quarters

each get one half and 1 quarter, 1 half from the top row and one quarter from the bottom row.

Work sample 12

Comments

Work sample 10 shows the way fractions can be described as having a multiplicative structure. Converting each fraction into quarters multiplies the number of pieces to be shared by four. Work sample 11 uses a similar process but the student has given each person a number to allocate the pieces and then forgotten to name the pieces.

General comments

- The fraction names halves and quarter should be used as appropriate but the fraction notation should not be introduced unless a student offers it as another way of recording. Brenex paper circles act as a means of representing the fraction parts before introducing the sharing diagram. This has the advantage of forming *equal halves* by folding and models the process of creating equal parts.
- The students became very excited when they realised that they could share the pikelets by cutting some in half.
- This sequence of questions has been designed to use only halves or quarters in the partitions of the pikelets.

Indicators of conceptual understanding

- Students readily discuss their visual images of fractions.
- Students subdivide a “whole” into halves or quarters to create a requisite number of equal parts.
- The link between division as sharing and fractions is clear in students’ recordings.

Half the pikelets S1

Linking part-whole models of fractions (discrete and continuous)

Overview

In this activity, students focus on recognising the whole and a part of the whole when the whole is made of discrete parts. The activity aims to link part–whole understanding of a continuous model with a discrete model of part–whole.

Outcomes

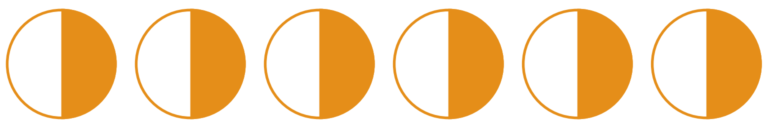
Describes and models halves and quarters of objects and collections, occurring in everyday situations (NS1.4).

Development of activity

1. *I have 6 pikelets and I want to put jam on half of them. How could I do that?*

   Allow time to engage with the problem. Draw 6 circles on the board. Ask one student to show on the board, how he or she would put jam on half of the pikelets.

   Chris decided to put jam on half of the 6 pikelets like this.

   Circles of two-coloured paper could be used to represent the pikelets. Draw an outline on the board of each pikelet by tracing around the circle of coloured paper. When folded the colour coming to the fore would represent the jam on the outline of the original circle. The circles of paper could be attached to a whieboard with a reuseable adhesive.

2. *If I have 3 pikelets and I want to put jam on half of them, could I use the method Chris used? Draw your answer.*

3. *If I have 5 pikelets and I want to put jam on half of them, show two ways that I could do this. Draw your answer and show why the two ways are the same.*

Work sample 1

Work sample 2

Comments

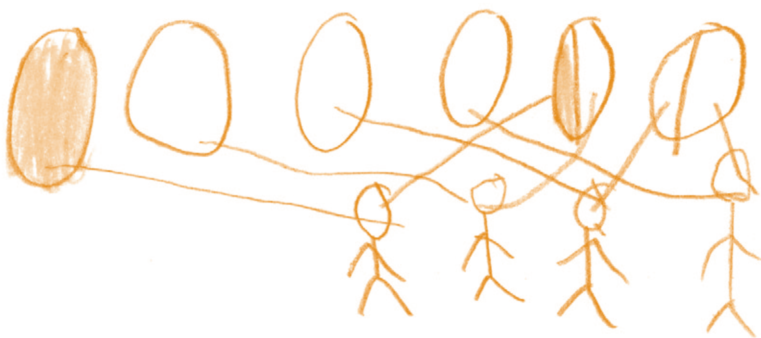
Many students focused on the orienation of the halves. Students considered vertical halves to be different from horizontal halves although they were understood to have the same area. Explaining why the two ways are the same also caused difficulties for many students.

4. Remove the outline and coloured paper representing the jam topping from the whiteboard, leaving five pikelets with half of each pikelet identified. *If this is Chris's way of putting jam on half of the pikelets, who can move the jam halves to show another way of putting jam on half of five pikelets*? Allow time to discuss the way the five halves can also be seen as two whole pikelets and one-half.

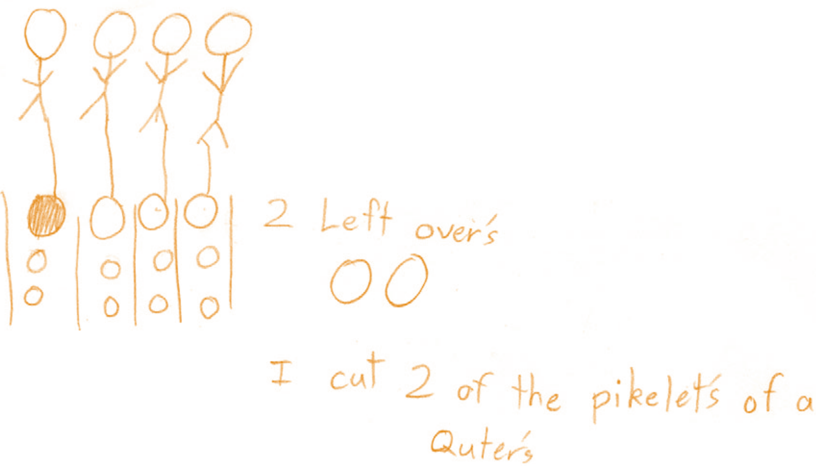
5. *If I have 6 pikelets and I want to put jam on a quarter of them, draw a diagram to show how Chris would do this.*

Work sample 3

6. *Draw what would happen if we had 6 pikelets to share among 4 people and one person wanted jam on his pikelets.*



Work sample 4



Work sample 5

Comments

This task is quite difficult because students need to not only share six pikelets among four people but also to identify and indicate one share in a different way. In addressing multiple units it exceeds the expectations of the Stage 1 outcome. In Work sample 5 shares the six pikelets by allocating one pikelet to each person, leave two pikelets. The two pikelets are then cut into quarters and redistributed, only the quarter pikelets appear as smaller circles.

General comments

- The fraction names halves and quarter should be used as appropriate but the fraction notation should not be introduced unless a student offers it as another way of recording.
- The purpose of this activity is to link the idea that half of the total is the same as the total taken as halves (and similarly for quarters). In this sense the activity is similar to the activity, *How many pikelets*?

Indicators of conceptual understanding

- Students readily discuss their images of fractions when comparing two different ways of forming half of six.
- The link between division as sharing and fractions is clear in students' recordings.
- Students can reassemble halves and quarters into whole.

A dozen pikelets S1

Linking part–whole models of fractions (discrete and continuous)

Overview	Outcomes
In this activity, students focus on recognising the whole and sub-units made of discrete parts.	Describes and models halves and quarters, of objects and collections, occurring in everyday situations (NS1.4).

Development of activity

1. *I have 12 pikelets on the table. 3 pikelets are plain, 3 pikelets have strawberry jam, 3 pikelets have honey and the rest have butter. How many have butter? How many different plates are needed if they each have the same number and type of pikelets? What fraction of the pikelets has butter?*

   Allow time to engage with the problem. Draw 12 circles on the board or use coloured disks of paper to represent each different type of topping.

2. *If the 3 pikelets with strawberry jam are eaten first, what fraction remains? How do you know?*

3. *I now have 12 plain pikelets and a supply of plates. Draw how you could share the pikelets between three people, Alice, Brian and Carole. What fraction of the pikelets does Alice get?*

4. *If three more people arrive, David, Eva and Frank, draw the new plates of pikelets. What fraction of the pikelets does Alice get now?*

Comments

- The informal knowledge of fractions students bring with them is often based on counting. The focus on the sub-units (plates) is important in moving to the idea that the whole can be regrouped.

Indicators of conceptual understanding

- Students readily discuss their visual images of fractions.
- Students discuss the appropriateness of various visual images of fractions, describing how any “whole” can be divided into halves.
- The link between division as sharing and fractions is clear in students’ recordings.

A piece of cake

S1 S2 S3

Forming an image of thirds

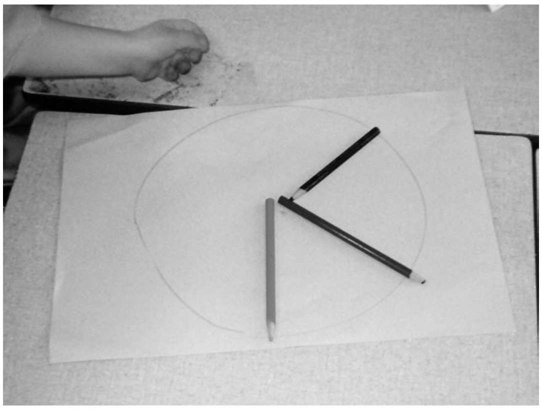
Overview	Outcomes	Materials
In this activity, students focus on dividing a circle into three equal pieces.	Models, compares and represents commonly used fractions … (NS2.4)	Paper and pencils

Development of activity

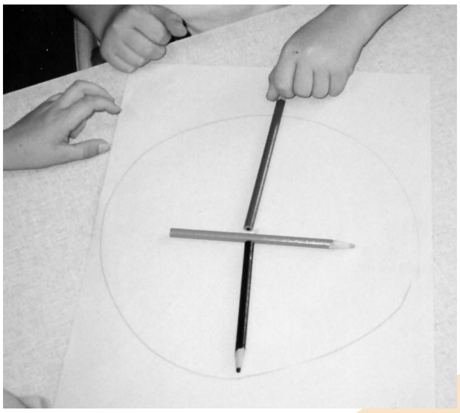
1. Ask the students to draw a circle on a sheet of paper and to imagine that it is the top view of a round lamington cake. *I want you to work out where we would cut the cake to have three equal slices with none left over. Use pencils or popsticks to work out where the cuts would go before you draw them.*

2. Observe the sequence of approximations that students use in developing the idea of dividing a circle into three equal parts.

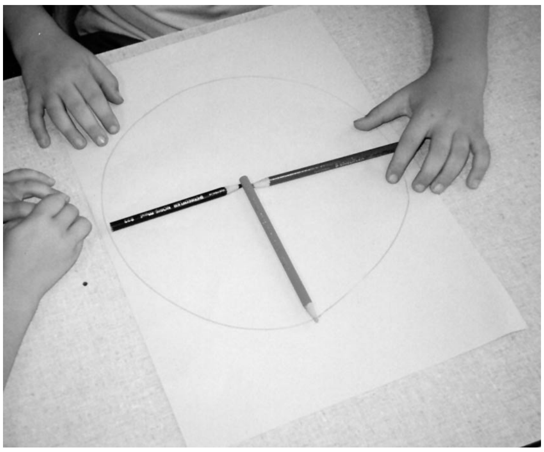
(a)



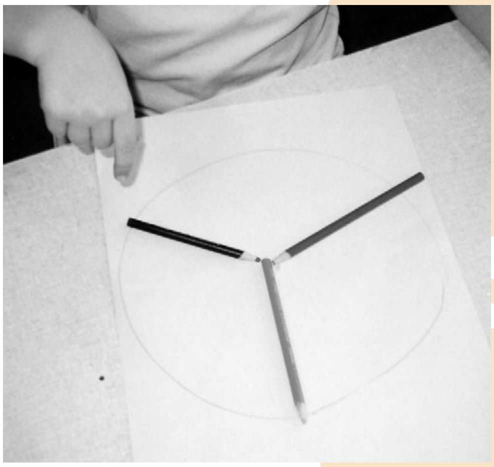
(b)



(c)



(d)



3. Have a number of students display their answers. Ask the students, *Do you think that everyone's answers will be the same?*

Comments

- This activity can be used with Stage 1 students as it is important that students recognise that not all fractions are halves and quarters.
- The sequence of images above shows the progress of three Year 1 boys in partitioning the circle into three equal parts.
- This activity helps to develop the image of a circle divided into thirds that students access in Stage 3.

Indicators of conceptual understanding

- Students make adjustments to the portions to divide the whole into thirds.
- The link between division as sharing and fractions is clear in students' recordings.

How many pikelets? S1 S2 S3

Part–whole models beyond one (discrete and continuous)

Overview	Outcomes	Materials
In this activity, students focus on forming wholes from fractional parts.	Describes and models halves and quarters, of objects and collections, occurring in everyday situations (NS1.4). Models, compares and represents commonly used fractions … (NS2.4) Manipulates, sorts, represents, describes and explores various two-dimensional shapes (SGS1.2).	24 quarter circles, all the same size (18 thirds of circles, all the same size)

Development of activity

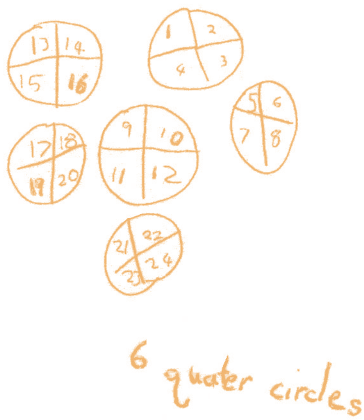
S1

1. With the students sitting in a circle on the floor, place the quarter circles in a pile in the middle and ask, *What are these?*

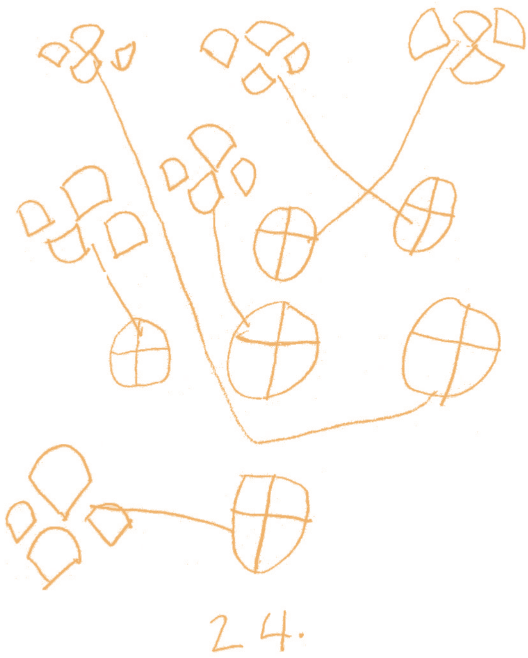
2. *Can we use these shapes to make circles? How many circles do you think that we can make with these shapes?* As the quarter circles are in a pile, this task requires students to make estimates of the answer. Ask some students to explain how they worked out their estimates.

S2

3. Count the quarter circles, then put them away and ask the students to work out how many circles they could make with 24 quarter circles. Have the students record how they arrived at the answer.



Work sample 1



Work sample 2

S3

4. Hold up one-third of a circle and ask, *If this is a piece of a pikelet what would we call it?* Likely answers include "big quarters". *How could we check to see if our name is correct?* When students have determined that the pikelet pieces are thirds, repeat the process using the 18 thirds of circles, all the same size.

Comments

- Many students believe that a fraction can never be bigger than one. This building up of units from fractions is useful because of its relationship with sharing activities.

Indicators of conceptual understanding

- Students readily discuss their visual images of fractions.
- Addition or multiplication of fraction parts is clear in students' recordings.

A birthday secret S1 S2

Recreating the whole from a part

Overview

In this activity, students focus on reconstructing a circle from a single piece of the circle. They also have an opportunity to form a unit-of-units (equal cake slices).

Outcomes

Models, compares and represents commonly used fractions … (NS2.4)

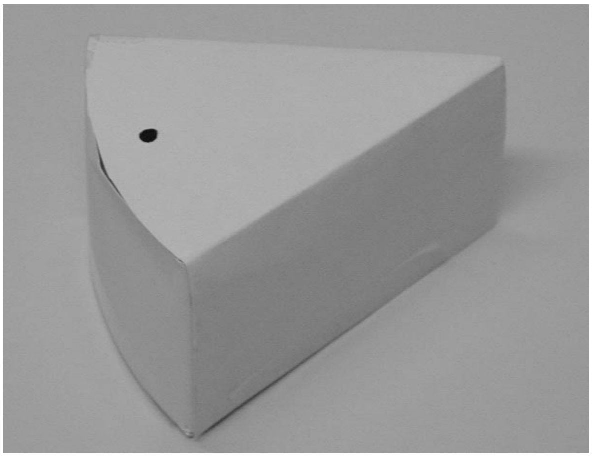
Uses a range of mental strategies and concrete materials for multiplication and division. (NS1.3)

Materials

A model of a slice of birthday cake, cardboard sectors with dots indicating where the candles would be placed, paper and pencils.

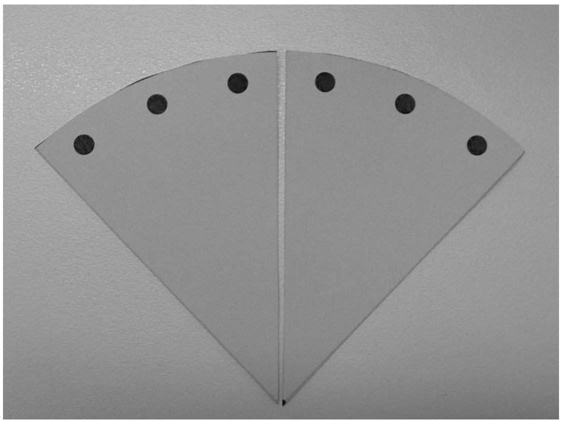
Development of activity

1. Show the 3-dimensional model of a slice of birthday cake. Explain that the mark on the cake is where a candle was and that the candles were equally spaced around the cake. *How old was the person having the birthday? How could you work it out?* Allow students time to think about the problem.



2. Give pairs of students cardboard sectors representing slices of cake and ask them to work out the age of the person having the birthday. From the cardboard model of a cake you can create pieces with 2 candles, 3 candles, 4 candles, 6 candles or 8 candles depending upon which multiples you wish to work with, or how many times you require students to repeat the unit. Have paper and pencils available as students often draw around the slice a sufficient number of times to create the whole cake.

3. Provide opportunities for students to report on their solution methods. *How many people could have a piece of cake the same size as the one you have?*



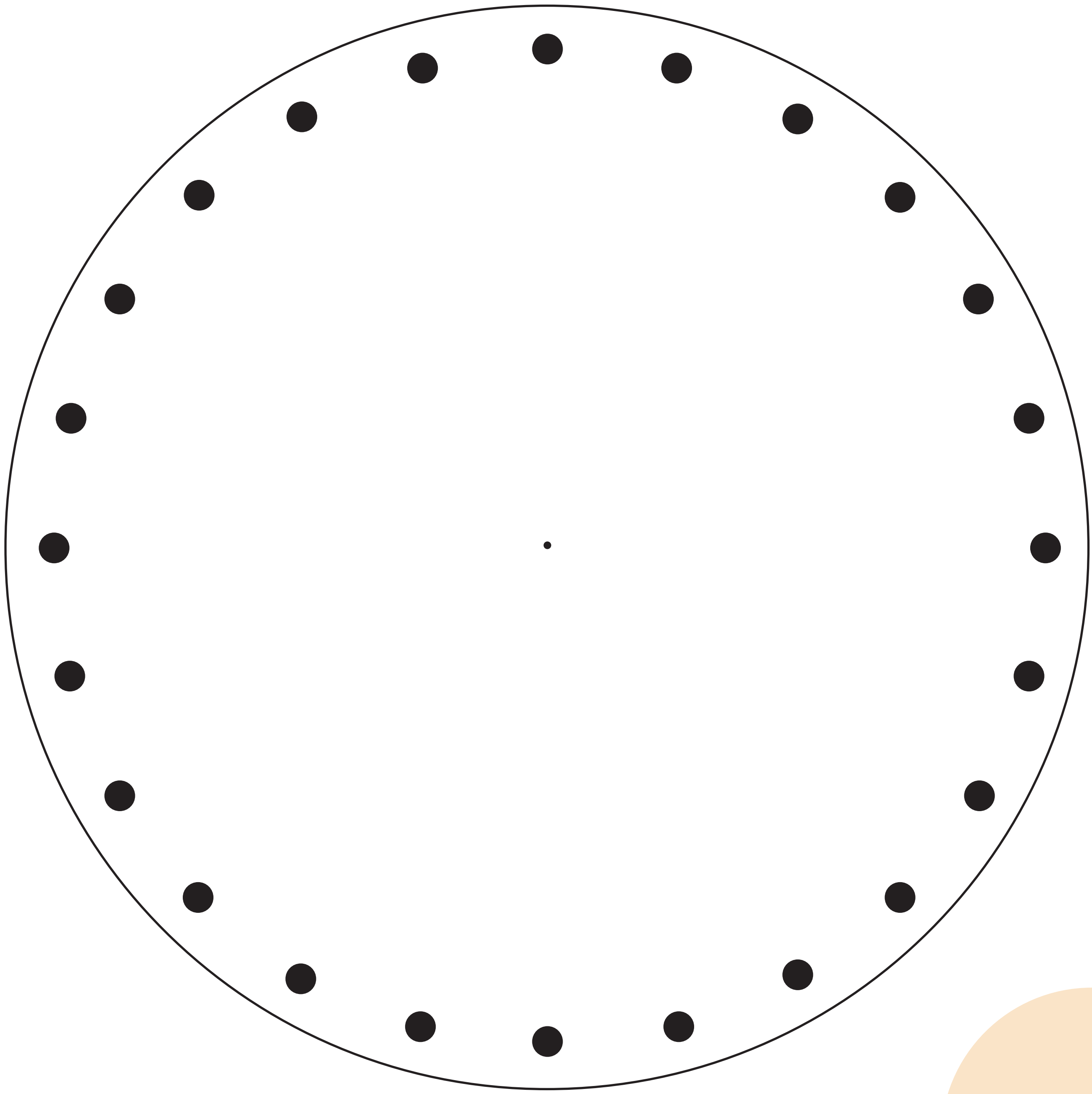
This person is 24. We did 4 slices and that's 12 candles because it's 3, 6, 9, 12, and it's half the cake so we doubled it.

Comments

- This activity recreates the whole given a part. It requires students to construct a unit-of-units and to use skip counting or related methods to determine the total number of candles on the cake.
- Opportunities exist to link the activity to angles as well as a variety of fraction units. The predominant feature of the activity is linking the continuous and discrete forms of fraction units.

Indicators of conceptual understanding

- Students abbreviate the solution process through recognition of the value of repeated units.
- The link between multiplication as coordinating units-of-units and fractions is clear in students' recordings.



A birthday secret template

A pikelet recipe S2 S3

Using sharing diagrams to operate on continuous models of fractions

Overview

In this activity, students explore dividing wholes into equal parts and use sharing diagrams to divide by fractions. The activity aims to promote part–whole conceptual understanding and to assist students perform fraction computations based on using a sound understanding of the fraction concept.

Outcomes

Compares, orders and calculates with decimals, simple fractions and simple percentages. (NS3.4)

Materials

A pouring jug full of water (food colouring or cordial, optional), 4 cylindrical clear plastic tumblers, thin strips of masking tape or similar.

Development of activity

1. Place 4 identical empty cylindrical clear plastic tumblers near each other on a table. [You can have your students fill the tumblers to the desired amount if you have a convenient "wet-area"]. *I want to pour half a glass of drink. Who can show me where about on the glass I would need to fill it to?* Provide the student with a thin piece of masking tape to record his or her answer. A marking pen can be used to identify the exact level. *Who thinks that this is the place we should fill the tumbler to get half a glass?* Allow an opportunity for class discussion and if the student wishes, he or she can move the tape. *How can we know if we are right?*

2. Put out another transparent tumbler with vertical sides. *Can you show me where I would have to fill this glass to get one-quarter of a glass?* Attach a small piece of thin black tape at the indicated location. *Does this look correct?* Adjust as directed.

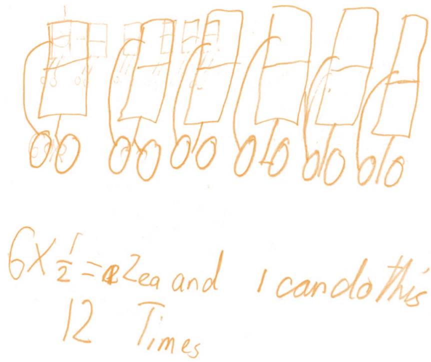
   Draw a sketch of the tumbler on the board. Ask one student to add a line to your diagram on the board to show one-quarter of a glass.

3. Put out three empty transparent tumblers with vertical sides and one tumbler full of water. *By pouring, and using any of these other glasses, show me exactly a third of a glass of water? What fraction remains in the glass?*

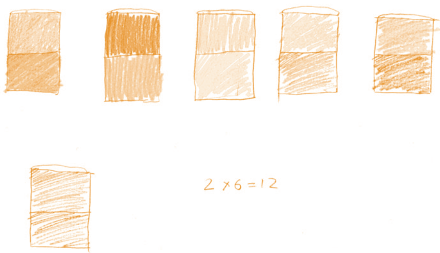
   Draw a sketch of the three tumblers on the board. Ask one student to add a line to your diagram on the board to show one-third of a glass.

   Who can show me two-thirds of a glass by drawing a line on the glass I have drawn on the board?

4. *I have 6 cups of milk. A recipe needs $\frac{1}{2}$ of a cup of milk. How many times can I make the recipe before I run out of milk? Can you draw your answer?*

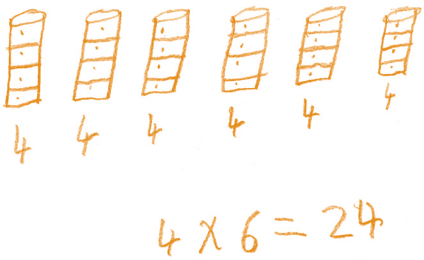


Work sample 1



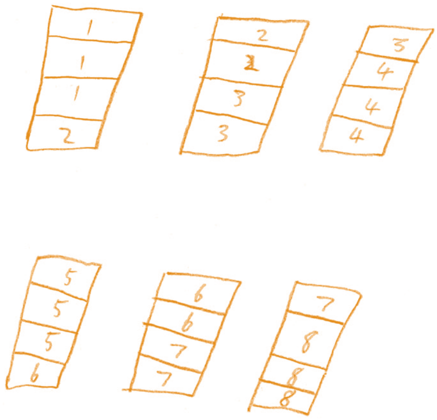
Work sample 2

5. *I have 6 cups of milk. A recipe needs one-quarter ($\frac{1}{4}$) of a cup of milk. How many times can I make the recipe before I run out of milk? Can you draw your answer?*

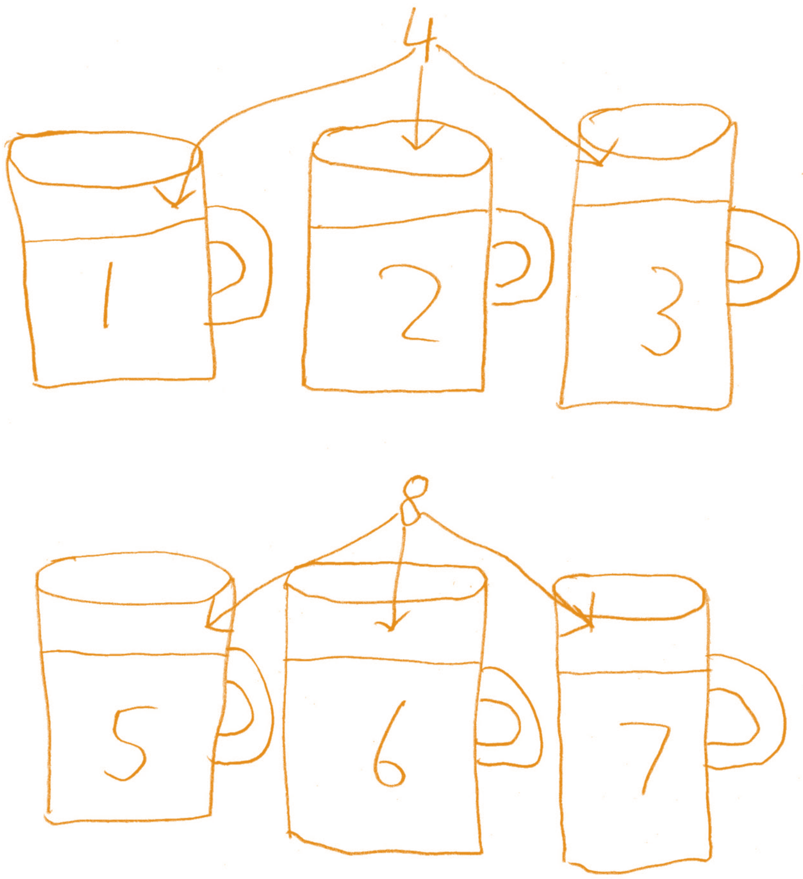


Work sample 3

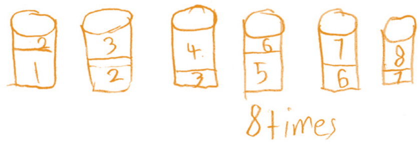
6. *Draw what would happen if I have 6 cups of milk and a recipe needs three-quarters ($\frac{3}{4}$) of a cup of milk. How many times can I make the recipe before I run out of milk?*



Work sample 4



Work sample 5

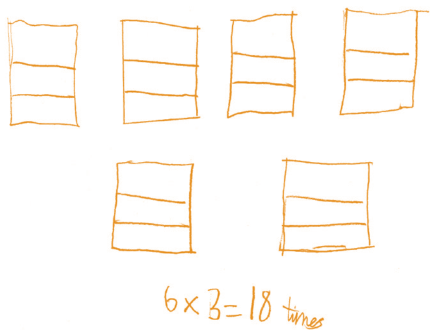


Work sample 6

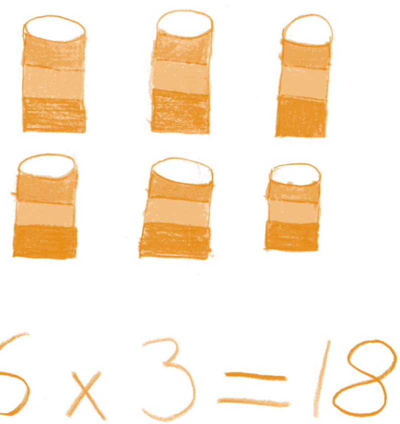
Comments

The above work samples show three different approaches in determining how many three-quarter cups are in six cups. Work sample 4 subdivides the cups into quarters and allocates quarters in groups of three. Work sample 5 focuses on three-quarters of a cup as a unit and accumulates the remaining quarters. Work sample 6 progressively accumulates each three-quarter cup, aggregating quarters and halves as needed.

7. *Who can draw what would happen if I have 6 cups of milk and a recipe needs one-third* ($\frac{1}{3}$) *of a cup of milk? How many times can I make the recipe before I run out of milk?*



Work sample 7

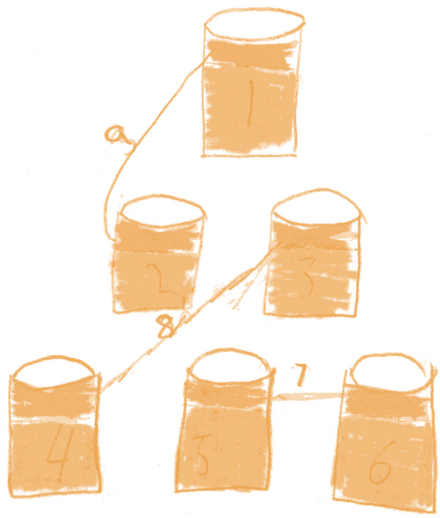


Work sample 8

Comments

In responding to this task, some students subdivide the cups into thirds and then count each third while others treat the three thirds in a cup as a unit and multiply by six.

8. *I have 6 cups of milk. A recipe needs two-thirds ($\frac{2}{3}$) of a cup of milk. How many times can I make the recipe before I run out of milk? Can you draw your answer?*



Work sample 9

Comments

Most students responded to this task in a similar way to finding out how many three-quarters in six cups. Some students subdivide the cups into thirds and then count two-thirds of a cup while others treat the two thirds in a cup as a unit and aggregate the remaining thirds.

Comments on activity

- Students appreciate this context for division by fractions. The activity also provides opportunities to deal with the "fraction units" in ways that require regrouping fraction parts.
- Students often use a consistent recording strategy across all of the tasks. If a student used colour-coding or counting individual units, he or she used the same strategy throughout.

Indicators of conceptual understanding

- Students readily illustrate their images of fractions and their processes for accumulating units.
- Students discuss appropriateness of various visual images of fractions, describing how any "whole" can be divided into fraction amounts.

Lamington bars S2

Forming equivalent fractions

Overview

In this activity, students encounter partitioning a rectangle in two directions. The activity aims to promote part–whole conceptual understanding leading to simple fraction multiplication.

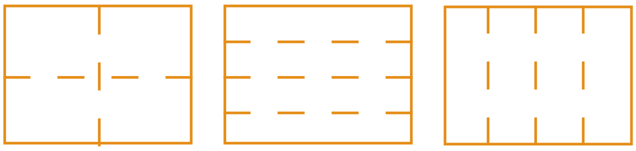
Outcomes

Models, compares and represents commonly used fractions and decimals, adds and subtracts decimals to two decimal places, and interprets everyday percentages. (NS2.4)

Development of activity

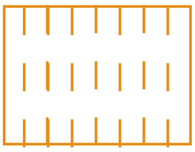
1. *Lamingtons are pieces of sponge cake covered in chocolate icing and dipped in shredded coconut. Mrs Packer makes excellent lamingtons and she likes to put a layer of whipped cream in the middle of her lamingtons. Mrs Packer starts with a large rectangular sponge cake.*

2. Distribute rectangular sheets of brown paper. *Show by folding the piece of paper how Mrs Packer could make four lamington bars.*



   Check to see which way the paper has been divided. If your students use different methods to form quarters ask them if each person would still get the same cut of cake. If all students create quarters by folding in the same direction take your piece of paper and fold it a different way to the direction the class has chosen. Compare the different ways of forming quarters shown above. Ask your students to show how the pieces of cake are equal.

3. *I am going to make eight smaller lamington bars.* Fold the rectangle into eighths as below.



   If I wanted to eat this much (show three-quarters of the horizontally divided rectangle) *how many of the smaller lamington bars would this be equal to? Remember that you have to explain your answer.*

Comments on activity

- An equal area comparison of fractions is basic to this activity. It is important that students acknowledge that they are starting from the same area regardless of the way it is partitioned. This may take some time to establish.

Indicators of conceptual understanding

- Students readily discuss their visual images of fractions.

- Students discuss appropriateness of various visual images of fractions, describing how any “whole” can be divided into equivalent fraction amounts.

Mrs Packer's visitors S2

Comparing fractions

Overview

In this activity, students encounter partitioning a rectangle into different amounts and comparing the resulting fractions.

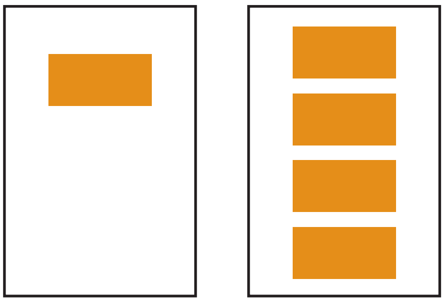
Outcomes

Models, compares and represents commonly used fractions and decimals, adds and subtracts decimals to two decimal places, and interprets everyday percentages. (NS2.4)

Development of activity

1. *Mrs Packer was expecting guests. She made five lamington bars and put them on two tables ready for the guests. As each guest arrived Mrs Packer asked the guest to choose a table. Once seated, the guests cannot change tables but must equally share the lamington bars with all the guests at the table.*

2. *Mrs Packer has placed one lamington bar on one table and four lamington bars on the other table.*



   Place one rectangular sheet of brown paper on one table and four rectangles of brown paper on another table.

3. *Mrs Packer is expecting eight guests. I want eight of you to play the part of the guests. The aim is to get as much of the lamington bars as you can but you cannot change tables after you sit down and everyone must wait until the last person sits down to share the lamington bars at their table.* Send eight students out of the class and give each one a number to represent the order in which they should return. As each student comes in and sits down, ask the class to record how much each person at that table will receive. *Remember that as you sit down you will have to explain why you chose the table you sat at.*

4. *Show by folding the piece of paper how much each person on your table receives. What would be the best solution? Record your answer.*

5. Repeat the activity with two lamington bars on one able and three on the other.

Comments on activity

- An equal area comparison of fractions is basic to this activity.
- The first person goes to the second table and so does the second person because if no one else sat there they would have two lamington bars each. The third person also sits at the second table, as each person on this table will receive more than one lamington bar. Where the fourth person sits is interesting because it makes no difference to his or her share but it does to the other people.

Indicators of conceptual understanding

- Students readily discuss their visual images of fractions.
- Students discuss the relationship between the number of pieces and the relative size of the pieces.

Related fractions 1 S2

One-half, one-quarter and one-eighth

Overview

In this activity, students explore the relationships between the unit fractions $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ through dividing a continuous unit. They then express the equivalence between various units, as well as the relationship between the unit fraction and the whole. The activity aims to promote an understanding of the relationship between unit fractions with related denominators.

Outcomes

Models, compares and represents commonly used fractions and decimals, adds and subtracts decimals to two decimal places, and interprets everyday percentages. (NS2.4)

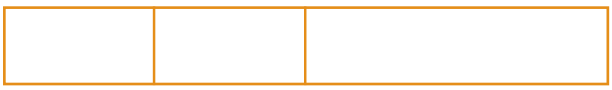
Materials

a paper streamer, scissors

Development of activity

1. Write the fractions one-half ($\frac{1}{2}$), one-quarter ($\frac{1}{4}$) and one-eighth ($\frac{1}{8}$) on the board. Hold up a paper streamer approximately 90 cm long. *Using this paper streamer, how could you make one of these fractions?* Allow the students some time to think about the question. *Which of these fractions will be the easiest to make? Why?* Focus the questions on: *How do you know that you have one-half (or one-quarter or one-eighth)?*

2. Fold the paper streamer in half and then fold one half in half. Unfold the streamer and display it to the class.



   Point to each part in turn and ask, *What fraction of the streamer is this part? How do you know?*

3. *If I fold one-quarter in half, what will I have?* Fold the quarter in half and, as before, point to each part in turn and ask, *What fraction of the streamer is this part? How do you know?* Emphasise reversibility: *If I fold the quarter in half I get two-eighths and two-eighths is the same as one quarter.*

4. *Which is the biggest part? Which is the smallest part? Can anyone see two fractions that would be the same as another fraction?*

5. *Show me two-eighths. Show me two-quarters. Show me two-halves.*

6. *Draw the streamer and show how halves, quarters and eighths are related to each other.*

Comments on activity

- The repetition of the two-partition or halving is quite straight forward. The focus of the activity is on what fractions are formed by the repeated halving and how the fractions relate.

Indicators of conceptual understanding

- Students readily describe a process of forming equal parts.

- Students discuss the relationship between creating a two-partition and the resulting unit fractions.

Related fractions 2 (S3)

One-third, one-sixth, one-ninth and one-twelfth

Overview

In this activity, students explore the relationships between the unit fractions $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$ and $\frac{1}{12}$ through dividing a continuous unit. They then express the equivalence between various units, as well as the relationship between the unit fraction and the whole.

Outcomes

Compares, orders and calculates with decimals, simple fractions and simple percentages. (NS3.4)

Materials

wool, streamers or strips of paper

Development of activity

1. Write the fractions $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$ and $\frac{1}{12}$ on the board. Hold up a piece of wool approximately 90 cm long. *Using this piece of wool, how could you make one of these fractions?* Allow the students some time to think about the question. *Which of these fractions will be the easiest to make? Why?* Select two students to demonstrate how to make one-third. Give the piece of wool to the two students and send them to a quiet corner to work on their demonstration.

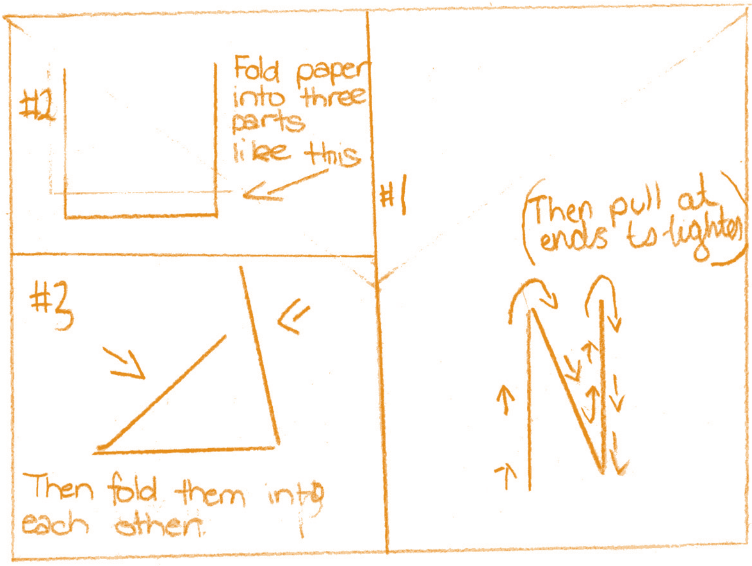
2. Distribute streamers or strips of paper or light card to each student. Explain that as well as creating each of the fractions written on the board, the task is to write a procedure using appropriate diagrams to allow other students to follow the methods developed.

3. Have the two students demonstrate how they made one-third of the length of wool and justify why the answer is one-third. *Now that you have made one-third, which of the fractions on the board would be the easiest to make next? Why?*

4. Provide sufficient opportunities in the class discussion to clarify the result of repeated partitioning, say, halving one-third or finding one-third of one-third. Allocate the task of writing the procedures for finding $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$ and $\frac{1}{12}$ of a strip of paper.

5. Have students share their procedures and ask students to explain what is the same and what is different about the procedures.

How would you fold a strip of paper to show one-third?



#1 Form a loose "N" with the strip of paper, Shuffle paper around to make all pieces equal. Pull pieces tight and fold on the creases.
#2 Fold paper into a square shape with top missing. Then crease on edges
#3 Fold paper into a burrito style way. Crease and you'll have 1/3.

Work sample 1

Comments

In responding to this task, this student describes three procedures for making one-third. Note that the third method draws heavily on the contextual knowledge of making a burrito.

How would you fold a strip of paper to show one-sixth?

- Fold a third and then fold that third in half.
- Here is a diagram how to fold a third:

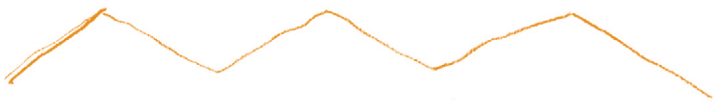
- Now that you have folded a third, fold the third in half

- Unfold your piece of paper and you should have six spaces

Work sample 2

- Fold ends to opposite sides. Over lap them so it has 3 equal squares if you open it.

- Fold it in ½ and open it to see 6 sections.

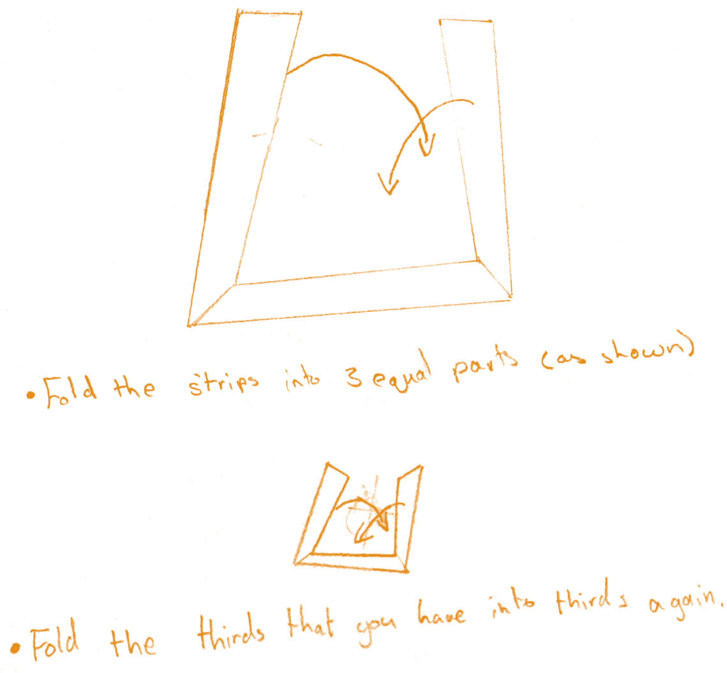


Work sample 3

Comments

Work samples for these procedures also provide information on understanding of terms used in two-dimensional space. Note the reference to "squares" in work sample 3.

How would you fold a strip of paper to show $\frac{1}{9}$?

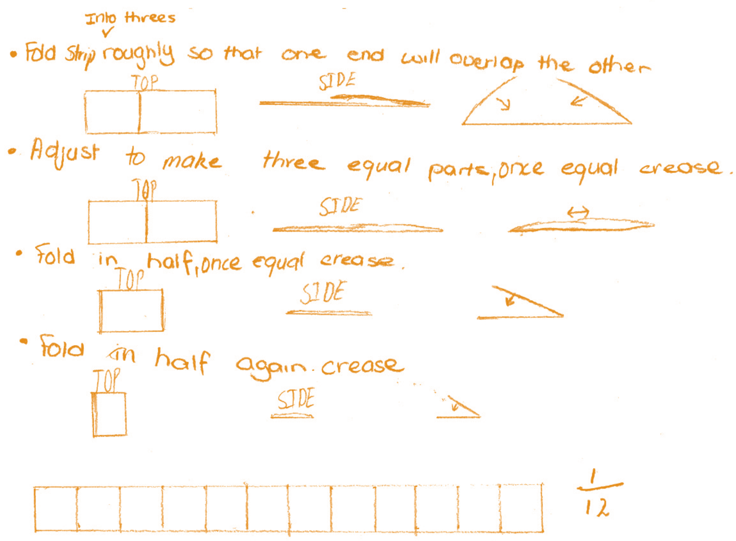


Work sample 4

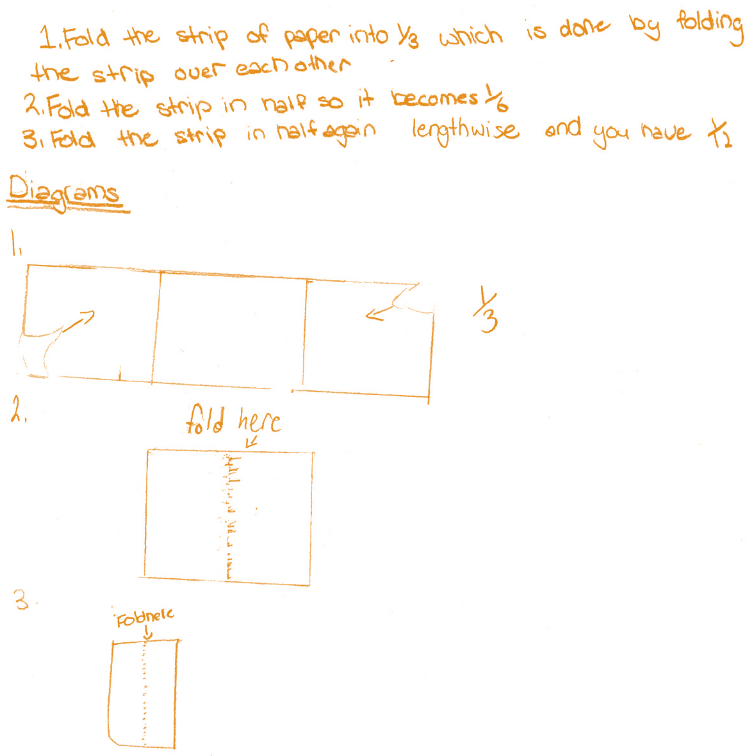
Comments

Repeating the process to find one-third of one-third is not as easy as moving from one-third to one-sixth. The procedure of halving dominates many students fraction understanding. As one student said, “You can’t make one-ninth because you can’t halve anything”.

How would you fold a strip of paper to show 1/12?



Work sample 5



Work sample 6

Comments on activity

- The activity promotes an understanding of the relationship between the process of partitioning and the resulting unit fractions. This assists with an appreciation of the process of multiplying fractions.

- You could have students swap procedures to see if some else could follow the method from the description.

Indicators of conceptual understanding

- Students procedures and related diagrams accurately describe a process for creating a three-partition.

- Students discuss the relationship between creating a three-partition and the resulting unit fractions.

- Repeated partitioning is used as a form of multiplication of fractions.

Building the fraction bridge: 1 S3

Constructing and comparing unit fractions

Overview

In this activity, students create unit fraction partitions of a given rectangle and relate the subdivided units. That is, if I have made one-third how can I make one-sixth?

Outcomes

Compares, orders and calculates with decimals, simple fractions and simple percentages. (NS3.4)

Development of activity

1. Display a picture of a suspension bridge. *Who has seen a bridge like this one?*

A suspension bridge

Today we are going to begin to make our own bridge, called a fraction bridge.

2. Write the following unit fractions on the board: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$ and $\frac{1}{12}$. Have students work in pairs and give each pair three paper streamers or strips of light card, each 60 cm long and of equal widths.

   Hold up a strip of paper. *Using this strip of paper, how could you make any one of these fractions?* Allow time for discussion and listen for examples that use repeated halving.

3. *I don't want to waste streamers and I want to see how many of these fractions we can make out of each steamer. In your pairs, I want you to discuss which three fractions you could make from one streamer and how you would do it. There is no need to make the fractions yet.* Allow time for discussion before selecting two teams to report back. Circulate to hear the discussions and to select one of the teams that suggests a relationship between the three fractions, such as halving (say making $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$).

4. Have the selected teams report on which fractions they could make and how they would make them. Ask the students questions such as: *How did you know that those three fractions would work? Did anyone try three fractions that were too big?* Focus the questions on *How do you know that it is one-eighth (or one-quarter or one-half)?* Encourage the idea of reversibility: *If I fold the quarter in half I get two-eighths and two-eighths is the same as one quarter.*

5. Cross off the three fractions from the list and ask: *Can anyone think of another three fractions from the list that could be done in a similar way?* Ask your students to explain how they would make the three fractions suggested. If students suggest $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{9}$, listen carefully to the explanation of how this would be constructed and, in particular, the order in which the fractions are calculated. If the explanation is incorrect, select students to explain what is wrong with the method.

6. Create one-third of a streamer and emphasise the third of a unit by folding. *If this is one-third of a streamer, how could I make one-ninth of a streamer?* Ask the class to create a third of a streamer. *Now I want you to discuss this in your teams and be prepared to show the rest of the class how to make one-ninth. I will also ask you, "How do you know that you have made one-ninth?"*

7. *If this streamer is 60 cm long, how long would one-tenth of the streamer be?* Measure this length along the streamer and fold it to show that it is one-tenth of a streamer. *If this is one-tenth of a streamer, how could I make one-fifth of a streamer?* Allow time for students to think about the question and have one student demonstrate his or her answer with the streamer. *How do you know that you have made one-fifth of the streamer?*

8. Assign the task of making $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ from one streamer, and have students record the fractions on the streamer parts.

9. Assign the task of making $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, and $\frac{1}{12}$ with a second streamer and have students record the fractions on the streamer parts.

10. Assign the task of making $\frac{1}{5}$ and $\frac{1}{10}$ with the third streamer and have students record the fractions on the streamer parts.

11. Have your students check their results with other teams and resolve any disputes over the size of the fraction parts.

12. Save the streamer parts for the next lesson.

13. *Next lesson we will make* $\frac{1}{7}$ *and* $\frac{1}{11}$. *I want you to think of a way that we could make* $\frac{1}{7}$ *or* $\frac{1}{11}$ *of a 60 cm streamer.*

Comments on activity

- The focus of this activity is not additive measuring but rather the multiplicative structure of fractions. The multiplicative structure of fractions means that when you halve a unit you double the number of pieces and when you find a third of a unit you create three times as many units.

- Discussion and demonstration needs to follow students construction of the fractions, emphasising the link between the fractional part and the whole unit as well as the relationship between the various fractions.

Building the fraction bridge: 2 S3

Constructing and comparing unit fractions

Overview

In this activity, students create unit fraction partitions of a given rectangle and relate the sub-divided units.

Outcomes

Compares, orders and calculates with decimals, simple fractions and simple percentages. (NS3.4)

Development of activity

1. Make multiple copies of the remaining fractions parts on streamers. That is, mark $\frac{1}{7}$ of a streamer and $\frac{1}{11}$ of a different streamer. *I have marked where I think $\frac{1}{7}$ of this streamer is. How could I work out if I am correct?*

2. Invite one student to demonstrate his or her method on the streamer. *Do you think that $\frac{1}{7}$ of a streamer would be bigger or smaller than $\frac{1}{6}$ of a streamer? Can you explain your answer?* Responses should include the idea that if you divide the same length among 7 people each would get less than if you shared the same length among 6 people. Follow up with the following question: *Can you explain which is bigger, $\frac{1}{11}$ or $\frac{1}{7}$?*

3. Present the second streamer with the length corresponding to $\frac{1}{11}$ of the streamer marked. Invite a student to see if he or she can work out what fraction of the streamer has been marked.

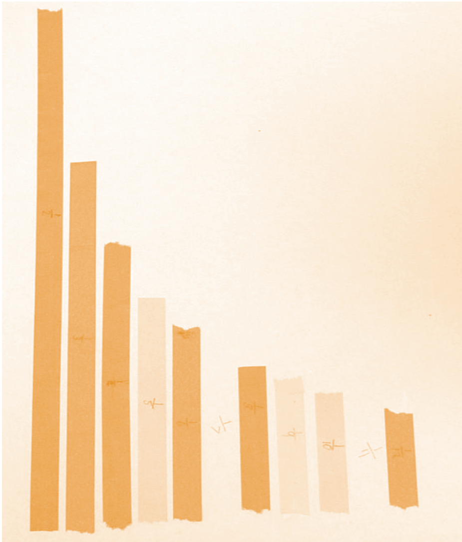
4. Cut up the streamer marked in sevenths and distribute the parts to act as templates. *Using some of your remaining streamers from yesterday, make $\frac{1}{7}$ of a streamer and write the fraction name on the streamer.*

5. Repeat the process for $\frac{1}{11}$ of the streamer.

6. *You should now have streamers that are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$ and $\frac{1}{12}$ of a unit long. Can you find one streamer that is the same length as two other streamers? Could we write this as a sum?* [$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$].

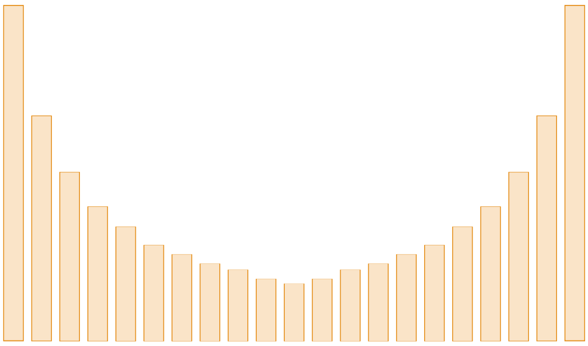
7. *Can you find three streamers that could make up the original streamer?* Hold up the original streamer. *Could we write this as a sum?* [$\frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$].

8. *I want you to put the eleven streamers in a row from biggest to smallest.* After students have done this you might play the "missing part" game. Put your fraction streamers in a row but leave one aside. Students have to guess which fraction you have left out by asking appropriate questions on size such as "Is it bigger than $\frac{1}{7}$?"



A partially completed fraction bridge

9. Have students construct the outline of a suspension bridge by combining two sets of streamers from largest to smallest and then from smallest to largest. One of the strips representing $\frac{1}{12}$ could be used as a "spacer" instead of having two-twelfths in the bridge.



A fraction bridge

Does the bridge have a line of symmetry? Where would it be? What happens to the size of the fractions as we get close to the centre of the bridge? Why do you think this happens?

Comments on activity

- The comparison of unit fractions in this activity emphasises the role that the denominator plays in determining the size of the fraction. It also highlights the relationship between the number of fractions in the unit and the size of each fraction.

- It is a common misconception, arising from an additive interpretation of the fraction symbolism, that $\frac{1}{11}$ is thought to be larger than $\frac{1}{5}$.

Crossing the wall (S3)

Linking and using equivalent fractions

Overview

In this activity, students focus on re-dividing the same unit. The activity aims to emphasise equivalent subdivisions of a common unit by aligning parts of a common whole.

Outcomes

Compares, orders and calculates with decimals, simple fractions and simple percentages. (NS3.4)

Materials

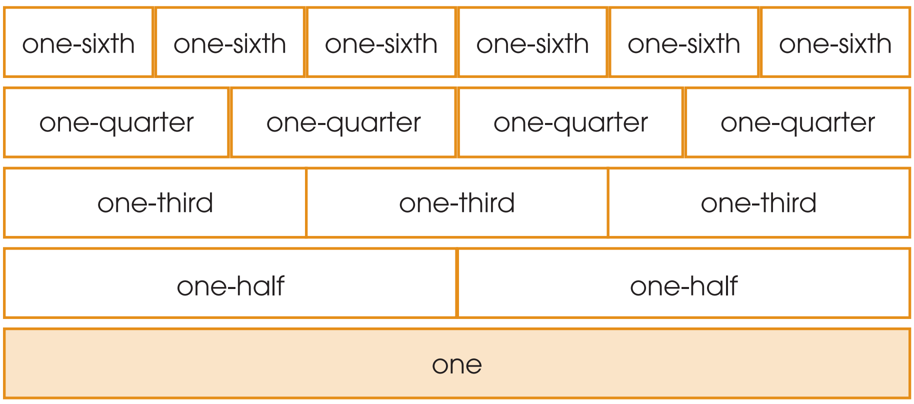
Sheets of one-centimetre grid paper, overhead transparency (optional)

Development of activity

1. *Ben and Mason work at the quarry. Their boss has asked them to cut the stone to make a part of a garden wall 1200 millimetres long. The sandstone blocks will be stacked to make the wall. When they have cut and moved the first block they realise that they don't have the strength to move another 1200 millimetre block. Ben says that they could make two blocks for the next layer of the wall. How long would each block of sandstone be? When they have moved the two blocks for the next layer and put them in place, Mason says that they should make three blocks this time. How long would each block of sandstone be? For the next layer of the wall they use four blocks and finally six blocks for the last layer.*

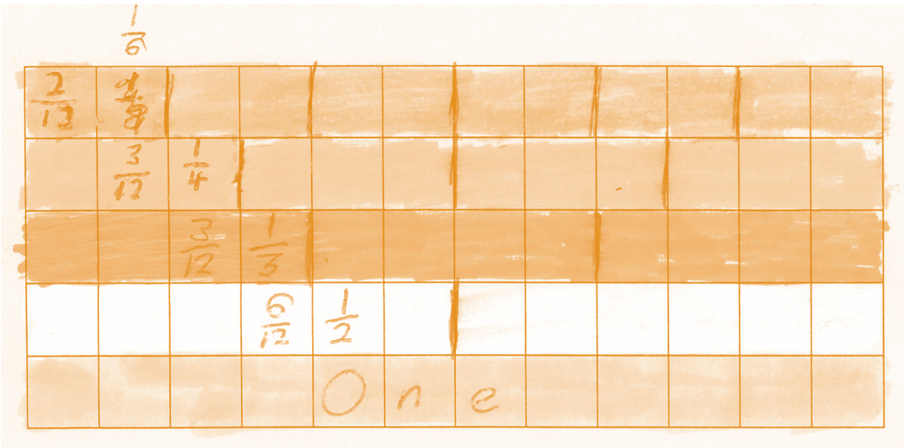
2. Record the number of blocks on the board: 1, 2, 3, 4 and 6. Distribute sheets of one-centimetre grid paper.

3. *Draw a diagram of the way the wall would look when it was finished.*



4. Have your students share their diagrams. *If we call the first block one, write the fraction name of each of the other blocks on your diagram. Can anyone find where the cracks in the wall line up? Why do these cracks line up?*

5. *Can you use your diagram of the sandstone wall to work out what one-half take away one-sixth is? Explain your answer.* Using an overhead transparency of the sandstone wall can help in sharing explanations. Check that the explanations emphasise that the end of one-half lines up with the end of the third one-sixth. Next come back one-sixth and then emphasise that the second one-sixth aligns with one-third.



5. 1/3 or 4 squares, one half minus one 3rd (on 6th) and = 1/3
6. 1/2 or 6 squares or 1 half. I added 1/3 plus one 6th = 1 half, 1/2
7. 1/2 or 6 squares or 1 half. I added 5 6th (5/6) and minused 1 3rd (1/3) = 1/2
8. 1/2 or 1 half I added 2 thirds (2/3) then minused 1 6th (1/6) = 1/3 or 2/8
9. 1/6. I took away one 6th from one quarter
10. 14 squares or 1 and one 6th. 1 half plus two thirds = 14 squares

Work sample 1

Comments

This work sample shows confidence in moving between "squares", where 12 squares represent one whole unit, and the standard fraction notation. The context is used in calculating the answers although the answers are not always correctly read. Note the use of ordinals (e.g. 3rd, 6th) instead of fractions.

6. *What is one-third plus one-sixth? Explain your answer.*

5. $\frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

6. $\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{3}$

7. $\frac{5}{6} - \frac{1}{3} = \frac{5}{6} - \frac{2}{6} = \frac{3}{6} = \frac{1}{3}$

8. $\frac{2}{3} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

9. $\frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}$

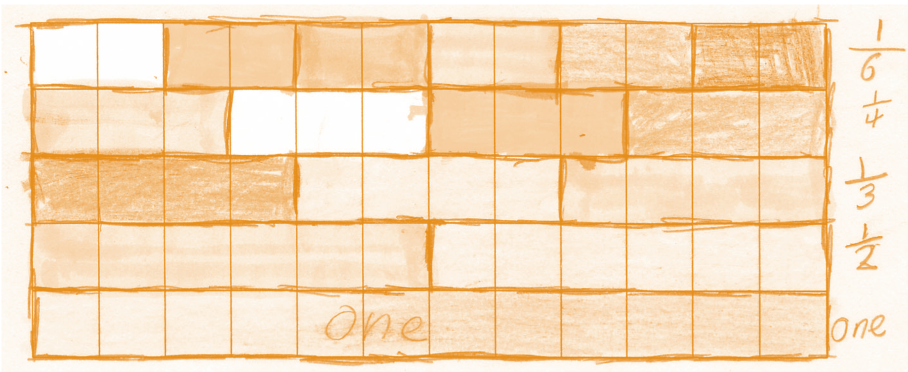
10. $\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6} = 1\frac{1}{6}$

Work sample 2

Comments

In this work sample, the wall has been correctly subdivided and labelled but the student has chosen to use the standard algorithms and to ignore the context. The lowest common denominators are always correctly selected and used. However, three-sixths is simplified in two questions to be equal to one-third!

7. *What is five-sixths take away one-third? Explain your answer.*



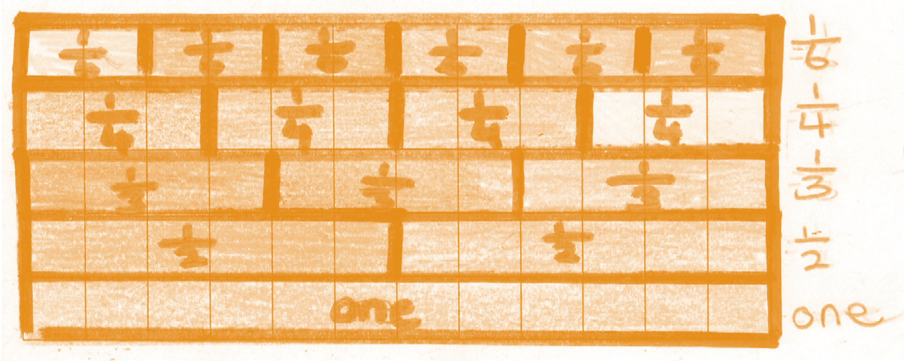
5. 4 squares, I took away 1 half and one sixth
6. one half, I added one third and one sixth
7 one half, I took away 1 third from 5 sixths,
8 one half, I took away 1 sixth from 2 thirds,
9 1 squares I took one sixth away from 1 quarter
10. 14 squares, the whole wall is 12 squares long and if you add 1 half and 2 thirds is 2 more squares than the wall so 12 plus 2 is 14.

Work sample 3

Comments

This response shows a partial understanding of the use of fractions. Converting the individual squares back into parts of the whole (re-unitising) when the result is other than one-half is not apparent.

8. *What is two-thirds minus one-sixth? Explain your answer.*



5. $\frac{1}{2} - \frac{1}{6} = \frac{1\times3}{2\times3} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

6. $\frac{1}{3} + \frac{1}{6} = \frac{1\times2}{3\times2} + \frac{1}{6} = \frac{3}{6} = \frac{1}{3}$

7. $\frac{5}{6} - \frac{1}{3} = \frac{5}{6} - \frac{1\times2}{3\times2} = \frac{3}{6} = \frac{1}{2}$

8. $\frac{2}{3} - \frac{1}{6} = \frac{2\times2}{3\times2} - \frac{1}{6} = \frac{3}{6} = \frac{1}{3}$

9. $\frac{1}{4} - \frac{1}{6} = \frac{1\times6}{4\times6} - \frac{1\times4}{6\times4} = \frac{2}{24} = \frac{1}{12}$

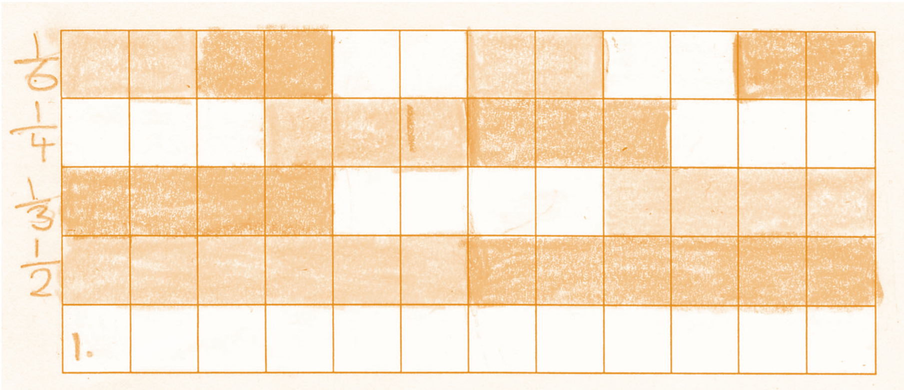
10. $\frac{1}{2} + \frac{2}{3} = \frac{1\times3}{2\times3} + \frac{2\times2}{3\times2} = \frac{7}{6} = 1\frac{1}{6}$

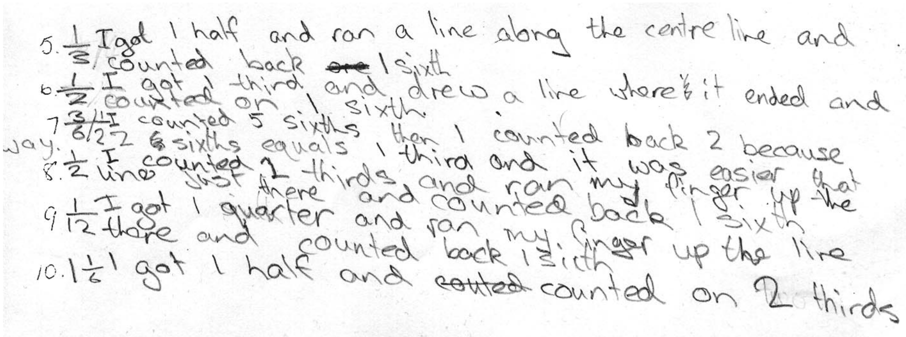
Work sample 4

Comments

In this work sample, the student has also chosen to use the standard algorithms and to ignore the context. Again, as with work sample 2, three-sixths is incorrectly simplified in two questions to be equal to one-third.

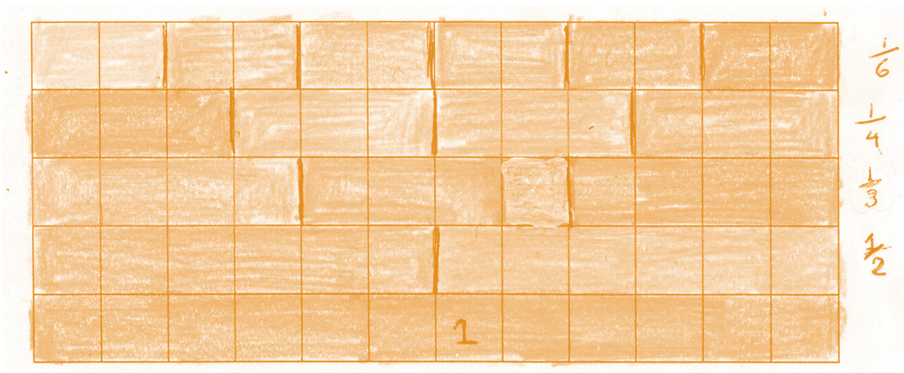
9. *What do you think one-quarter take away one-sixth would be equal to? Why do you think this and how could you check?*

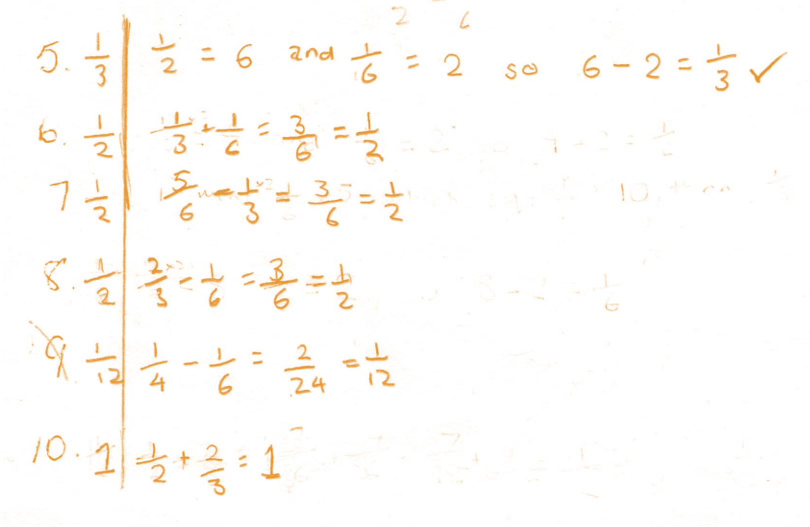



5. 1/3 I got 1 half and ran a line along the centre line and counted back ~~one~~ 1 sixth
6. 1/2 I got 1 third and drew a line where it ended and counted on 1 sixth.
7. 3/6 1/2 I counted 5 sixths then I counted back 2 because 2 sixths equals 1 third and it was easier that way.
8. 1/2 I counted 1 thirds and ran my finger up the line just there and counted back 1 sixth
9. 1/12 I got 1 quarter and ran my finger up the line there and counted back 1 sixth
10. $1\frac{1}{6}$ I got 1 half and ~~couted~~ counted on 2 thirds

Work sample 5

10. *What do you think one-half plus two-thirds would be equal to? Why do you think this and how could you check?*

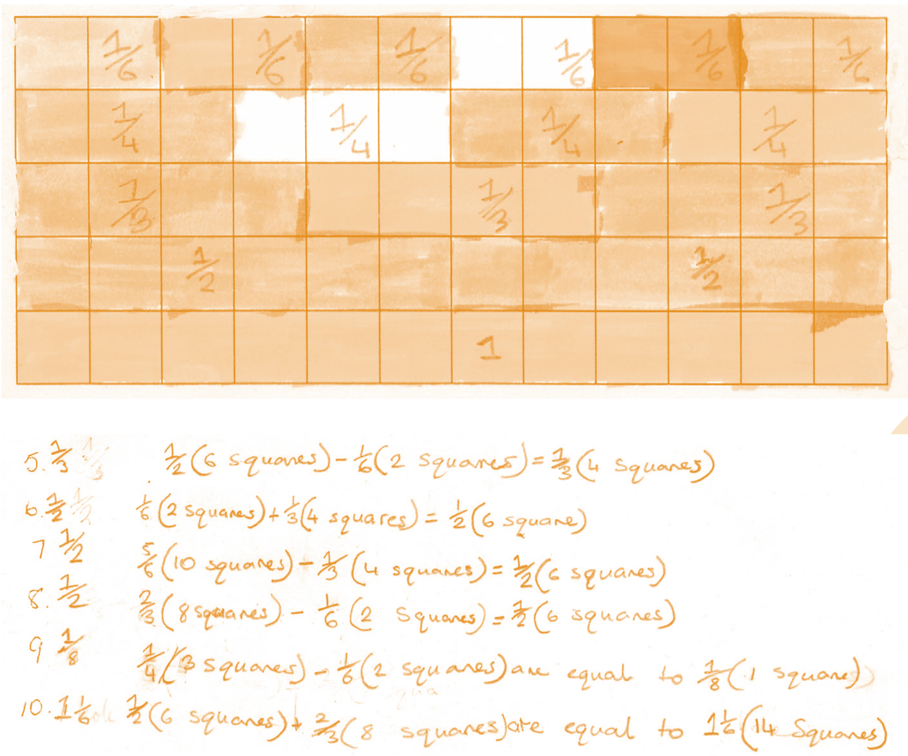




Work sample 6

Comments

What isn't clear from this work sample is the degree to which the context was used to calculate the answers. Judging from the working out that has been erased, the questions appear to have been worked out in a similar fashion to the first question, using squares. Given this approach, the last question appears to suffer from a transcription error.



Work sample 7

Comments

This work sample provides an indication that this student can work with units of units and reform units as needed. Twelve squares is one unit and two squares is one-sixth of a unit. Equivalent fractions are quickly ascertained. Note that the student appears to incorrectly follow an arithmetic pattern to suggest that one square represents one-eighth.

Comments on activity

- It is worthwhile noting that many Stage 3 students experienced difficulty in adding one-half plus two-thirds. This could have occurred because the response went beyond the limits of the model.
- The work samples also provide an indication of what students take to be valid explanations. In some cases computations take the place of explanations and in others the explanation is a kind of procedural recount.

Indicators of conceptual understanding

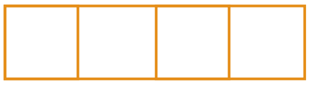
- The relationship between the equivalence of fraction parts is used as needed.
- Explanations draw upon the equivalence of fractions.

What can we learn from the research?

The process of equal sharing or division underpins the fraction concept.

What is clear from the research (Pitkethly & Hunting, 1996) is that the fraction concept develops from partitioning discrete and continuous quantities leading to identification of the unit and then building up amounts using the unit to form units-of-units. That is, the process of equal sharing or division underpins the fraction concept.

The development of fractions hinges on the role played by units of different types. Compare the process of determining a quarter of a continuous quantity, such as a strip of paper, with finding a quarter of 12 discrete items. What unit does the student use in each case?



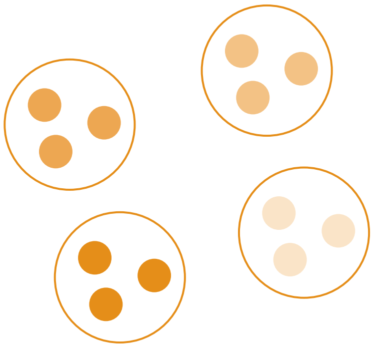
Finding a quarter by folding in half and folding in half again is the repetition of a process of halving or splitting (Confrey 1995). This process or scheme of splitting does not by itself imply a robust concept of a half. A student can apply this process and still hold that the following represent different amounts of the square[1].

The basic process of halving is similar to one-to-one dealing. The act of halving by subdividing continuous quantities is a powerful scheme which Pothier and Sawada (1983) called algorithmic halving. As a foundation form of partitioning, and a successful method, it can inhibit partitioning into odd numbers such as thirds. Subdividing continuous quantities needs to include thirds or fifths to develop the scheme of adjusting to form equal parts.

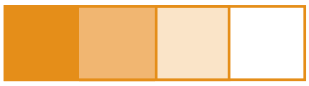
Unitising means thinking about a quantity by using a convenient unit of measure. For example, the given unit in a fraction problem may be 24 drinks, but a student may think about that quantity as 2 (12-packs), 4 (6-packs) or 1 (24-pack). That is, new units are formed in the student's mind to address the problem.

[1] This could arise from focusing on the different perimeters of the parts.

Steffe (1988; Steffe & Cobb, 1988) describes four different types of units: counting units, composite units, measurement units and units-of-units. Composite units are a collection of units sharing a common attribute. For example, if a plate of 12 pikelets had 3 plain pikelets, 3 with strawberry jam, 3 with honey and 3 with butter, each topping could be considered as a unit and the plate of pikelets could be considered as one unit composed of four sub units.

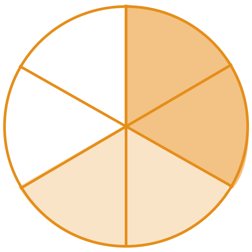


Three-quarters of a strip of paper could be considered as one unit composed of three units of one-quarter each.

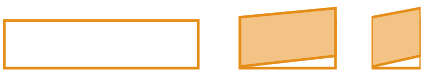


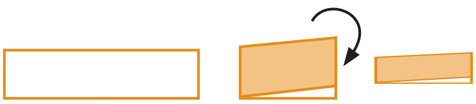
$\frac{3}{4}$

Units-of-units are composite units that are re-conceptualised into a new, encompassing unit. For example, the plate of 12 pikelets, or four units of three, could be re-conceptualised as one unit of 12, or one dozen. Similarly, four-sixths of a pikelet could be re-conceptualised as two units of two-sixths or as one unit of two-thirds.

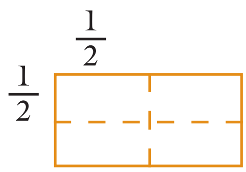


Behr, et al (1983) have indicated that the idea of part-whole, which makes use of the ability to partition either a continuous quantity or a set of discrete objects into equal sized sub-parts, is fundamental to the fraction concept. The procedures used to form equal sized sub-parts are not necessarily equivalent. If we take a rectangular piece of paper, there are two distinct ways of forming quarters:

(a) 

(b) 

The second method of folding, models area multiplication of fractions, whereas the first models repeated halving.



The process of halving is applied in two directions at right-angles. The quarters produced by the two methods are not identical.

Finally, symbolic operations should only become the focus of instruction once students have developed coherent and stable meanings for fractions that may be expressed symbolically (Thompson & Saldanha, 2003).

The instructional sequence

Students' informal notions of partitioning, sharing and measuring provide a starting point for developing the fraction concept.

Students' informal notions of partitioning, sharing and measuring provide a starting point for developing the fraction concept. Drawing on the research on developing the fraction concept[2] and the identified problem of representing fractions as $\frac{a}{b}$, the following sequence is used to develop the fraction concept.

- Subdividing continuous quantities into halves and quarters using the names halves and quarters and identifying the sub-units.
- Sharing of numbers of continuous models. First using the halving strategy (e.g. Share 6 pikelets between 2 people, share 3 pikelets between 2 people, share 3 pikelets between 4 people)[3] then coordinating the partitioning with the number of sharers.
- Recordings using sharing diagrams for the continuous model (typically circles or rectangles) showing the partitioning.
- Traditional recording of fractions used as "environmental print" linked to sharing diagrams. If a student choses to write one-quarter as $\frac{1}{4}$ it is accepted as a common shorthand form.
- Comparison of units by re-dividing a continuous quantity, such as a paper streamer, leading to representing equivalent fractions (e.g. fraction wall).
- Identifying fractions as numbers and locations on a number line (comparison of location, Which is larger, $\frac{1}{3}$ or $\frac{1}{2}$?)

[2] Behr, Lesh, Post and Silver, 1983; Confrey, 1994, 1995; Empson, 1999; Kieren, 1992; Mack, 1990, 1995; Pothier and Sawada, 1983; Streefland, 1991, 1993; Thompson & Saldanha, 2003.
[3] Kerslake (1986) stated: *It seems that the perception of a fraction as part of a whole shape, usually a circle or a square, is so strongly held by children that they find it impossible to adapt this model even to include the notion of three circles to be shared into four equal parts.* (p. 120)

It is important to build meaning for fractions arising from the process of division. In this way halves, quarters, thirds and fifths arise from problems of sharing rather than being defined as abstract numerical quantities.

- Addition and subtraction of fractions using representations of equivalent fractions as subdivisions of the same unit of length ($\frac{2}{3} + \frac{1}{4}$). For example, addition or subtraction using the "fraction wall".
- Symbolic operations with fractions based on students' coherent and stable meanings for fractions that may be expressed symbolically.

It is important to build meaning for fractions arising from the process of division. In this way halves, quarters, thirds and fifths arise from problems of sharing rather than being defined as abstract numerical quantities.

References

Behr, M. J., Harel, G., Post, T. & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 121–176). Albany, NY: SUNY Press.

Behr, M. J., Lesh, R., Post, T. & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). New York: Academic Press.

Confrey, J. (1994). Splitting, similarity, and the rate of change: New approaches to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 293–332). Albany: State University of New York Press.

Confrey, J. (1995). Splitting Reexamined: Results from a Three-Year Longitudinal Study of Children in Grades Three to Five. In *Proceedings of the 17th Annual Meeting, International Group for the Psychology of Mathematics Education*, NA, Vol. 1, edited by Douglas T. Owens, Michelle K. Reed and Gayle M. Millsaps, pp. 421–26. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, 17, 283–342.

Gravemeijer, K. P. E., Cobb, P., Bowers, J. & Whitenack, J. (2000). Symbolizing, modelling, and instructional design. In P. Cobb & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225–274). Hillsdale, NJ: Erlbaum.

Hartung, M. L. (1958). Fractions and Related Symbolism in Elementary School Instruction. *Elementary School Journal*, 58, pp. 377–384.

Hunting, R. P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, 14, 182–197.

Kerslake, D. (1986). *Fractions: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project*, Nfer-Nelson, Windsor, Berks.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge; Implications for curriculum and instruction. In G. Leinhardt & R. T. Putman (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323–371). Hillsdale, NJ: Erlbaum.

Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding: Essential Content knowledge and Instructional Strategies for Teachers.* Mahwah, N. J.: Lawrence Erlbaum Associates.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16–32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, 26, 422–441.

Mack, N. K. (2001). Building on Informal Knowledge Through Instruction in a Complex Content Domain: Partitioning, Units, and Understanding Multiplication of Fractions. *Journal for Research in Mathematics Education*, 32 (3), pp. 267–295.

Pitkethly, A. & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, 30, pp. 5–38.

Pothier, Y. & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 14 (5), pp. 307–317.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119–140). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Dordrecht, The Netherlands: Kluwer.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289–325). Hillsdale, NJ: Erlbaum.

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and Multiplicative Reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.