Free-Surface Waves

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CAUSE OF OCEAN WAVES

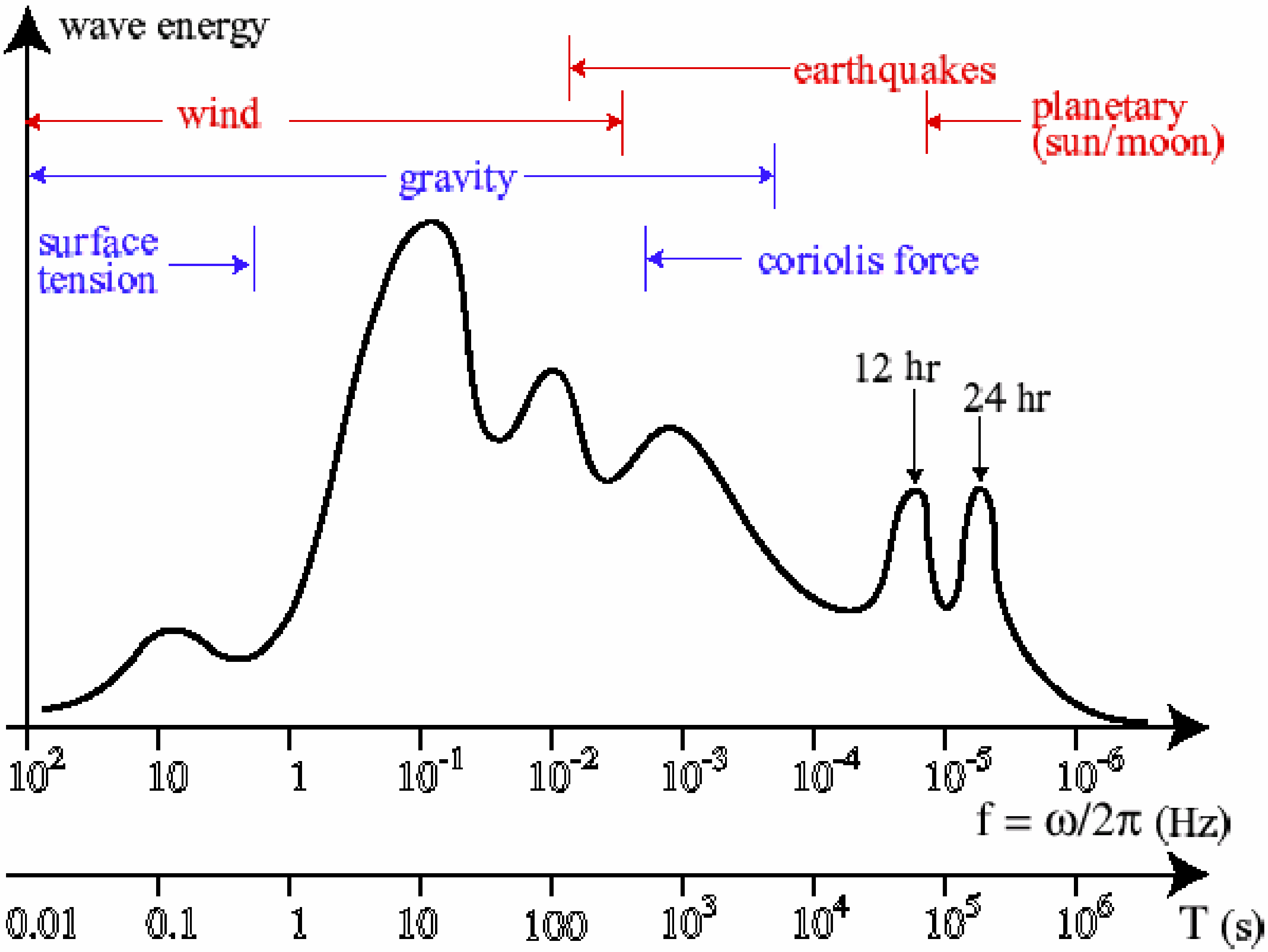
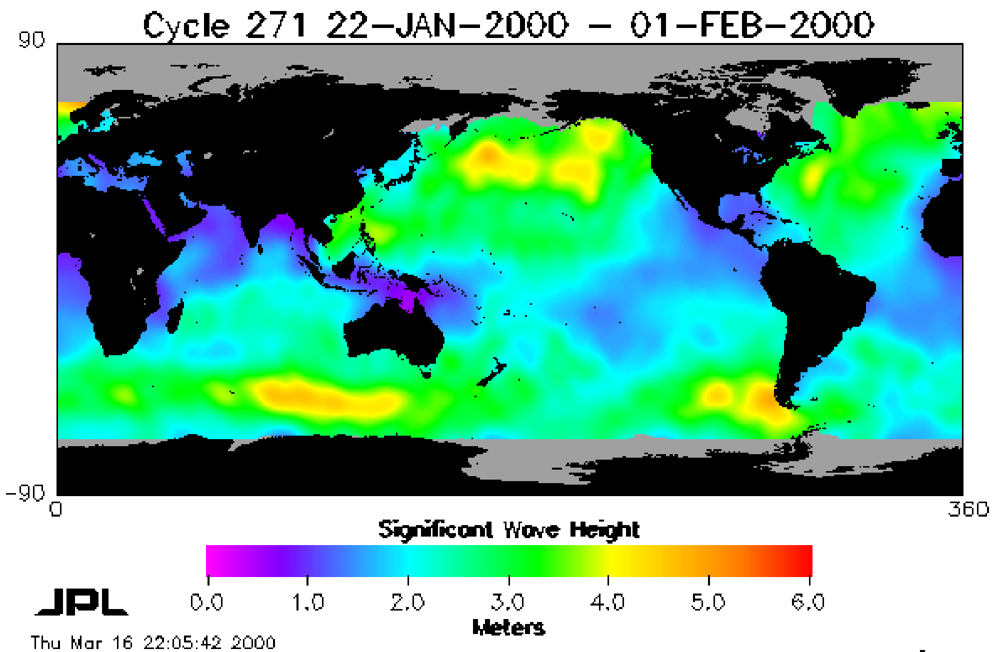


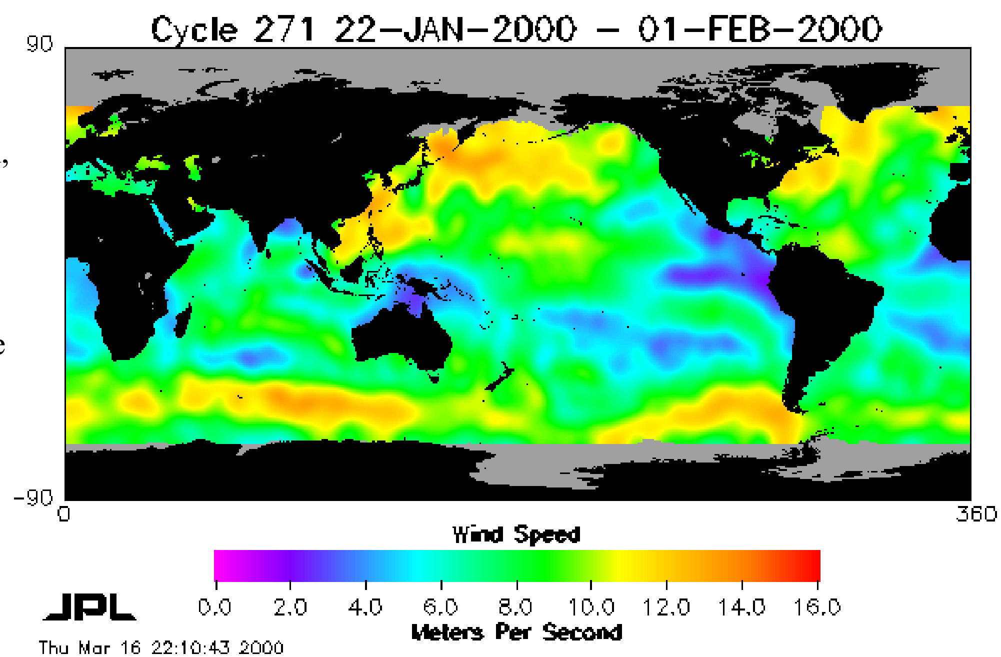
FIGURE 1. Wave energy spectra. Red text indicates wave generation mechanisms and blue text indicates damping/restoring forces.



The highest waves generally occur in the Southern Ocean, where waves over six meters in height (shown as red in images) are found. The strongest winds are also generally found in this region. The lowest waves (shown as purple in images) are found primarily in the tropical and subtropical oceans where the wind speed is also the lowest.

In general, there is a high degree of correlation between wind speed and wave height.

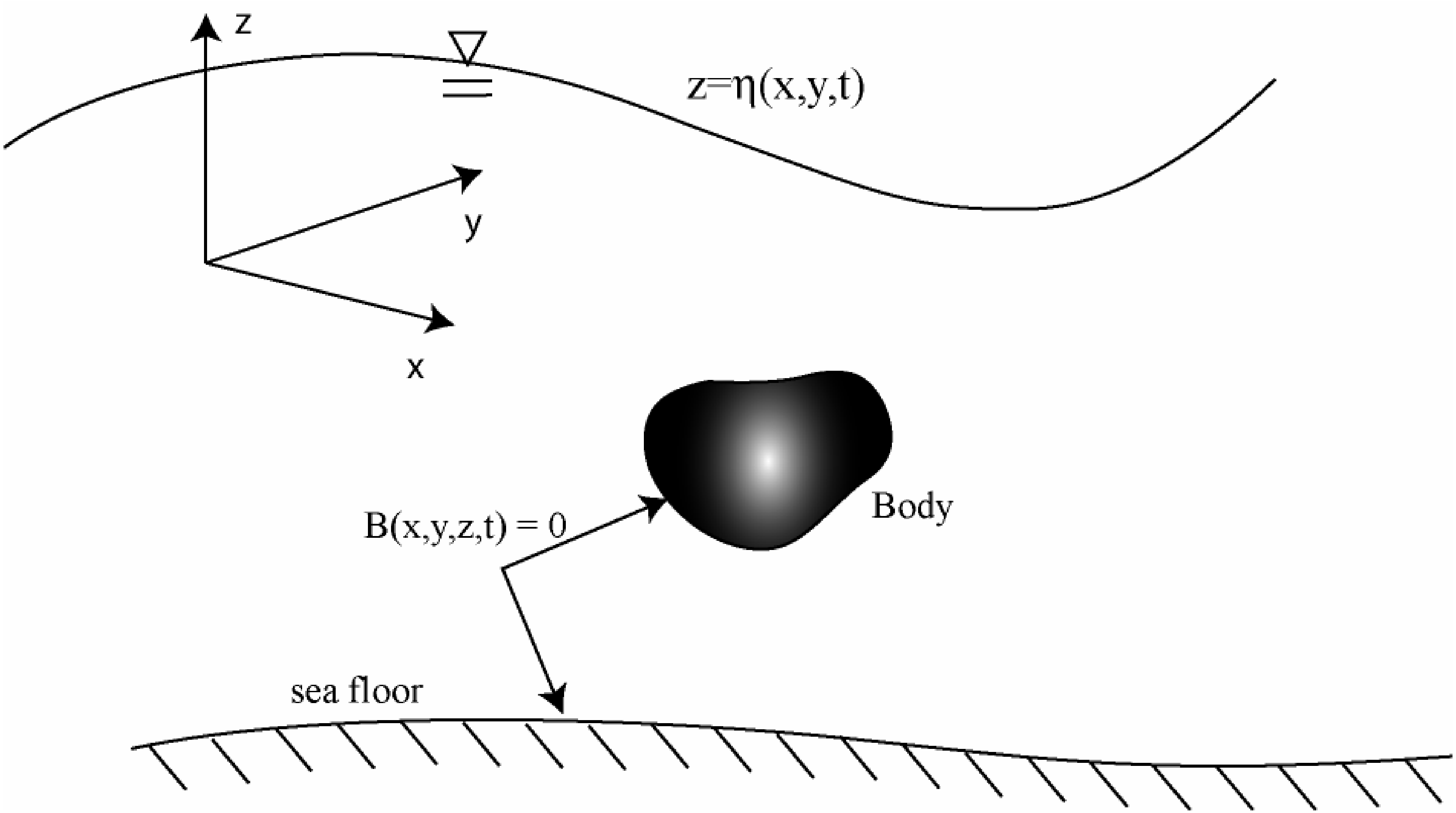
The highest winds generally occur in the Southern Ocean, where winds over 15 meters per second (represented by red in images) are found. The strongest waves are also generally found in this region. The lowest winds (indicated by the purple in the images) are found primarily in the tropical and subtropical oceans where the wave height is also the lowest.



World Meteorological Org. Sea State Codes

Sea State Code	Significant Wave Height		Description
	Range	Mean	
0	0 (meters)	0 (meters)	Calm (glassy)
1	0-0.1	0.05	Calm (rippled)
2	0.1-0.5	0.3	Smooth (mini-waves)
3	0.5-1.25	0.875	Slight
4	1.25-2.5	1.875	Moderate
5	2.5-4.0	3.25	Rough
6	4.0-6.0	5.0	Very Rough
7	6.0-9.0	7.5	High
8	9.0-14.0	11.5	Very High
9	> 14.0	> 14.0	Huge

General Wave Problem



Unknowns

- Velocity Field

$$\vec{V}(x, y, z, t) = \nabla \phi(x, y, z, t)$$

- Free Surface Elevation

$$z(x, y, t) = \eta(x, y, t)$$

- Pressure Field

$$p(x, y, z, t) = p_{dynamic} + p_{hydrostatic}$$

Governing Equations & Conditions

- Continuity (Conservation of Mass):

$$\boxed{\nabla^2\phi = 0 \text{ for } z < \eta}$$

- Bernoulli's Equation (given ϕ):

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p - p_a}{\rho} + gz = 0 \text{ for } z < \eta$$

- No disturbance exists far away:

$$\frac{\partial\phi}{\partial t}, \nabla\phi \rightarrow 0 \text{ and } p = p_a - \rho gz$$

Boundary Conditions

- In order to solve the boundary value problem for free surface waves we need to understand the boundary conditions on the free surface, any bodies under the waves, and on the sea floor:
  - Pressure is constant across the interface
  - Once a particle on the free surface, it remains there always.
  - No flow through an impervious boundary or body.

Pressure Across the FS Interface

$$p = p_{atm} \text{ on } z = \eta$$

Bernoulli's Eqn.
at the free surface

$$p + \rho\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}V^2 + gz\right\} = c(t)$$

$$p = -\rho\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}V^2 + gz\right\} + c(t) = p_{atm}$$

Since *c(t)* is arbitrary we can choose a suitable constant that fits our needs:

$$c(t) = p_{atm}$$

Thus our pressure boundary condition on $z = \eta$ becomes:

$$\boxed{\rho\{\frac{\partial\phi}{\partial t} + \frac{1}{2}V^2 + g\eta\} = 0}$$

Once a Particle on the FS…

The normal velocity of a particle on the FS follows the normal velocity of the surface itself:

$$z_p = \eta(x_p, t)$$ z-position of the particle

Look at small motion δz_p :

$$z_p + \delta z_p = \eta(x_p + \delta x_p, t + \delta t) = \eta(x_p, t) + \frac{\partial \eta}{\partial x}\delta x_p + \frac{\partial \eta}{\partial t}\delta t$$

On the surface, where $z_p = \eta,$ this reduces to:

$$\delta z_p = \frac{\partial \eta}{\partial t} u\ \delta t + \frac{\partial \eta}{\partial t}\delta t$$

$$\delta x_p = u\ \delta t$$

$$\delta z_p = w\ \delta t$$

$$\boxed{\therefore w = u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \quad \text{on} \quad z = \eta}$$

No flow through an impervious boundary

Velocity of the fluid normal to the body must be equal to the body velocity in that direction:

$$\text{On the Body: } B(x,y,z,t)=0$$

$$\vec{V}\cdot\hat{n}=\nabla\phi\cdot\hat{n}=\frac{\partial\phi}{\partial n}=\vec{U}(\vec{x},t)\cdot\hat{n}(\vec{x},t)=U_n \quad on \quad B=0$$

Alternately a particle *P* on *B* remains on *B* always; i.e. *B* is a material surface.

For example: if *P* is on *B* at some time $t = t_o$ such that $B(x,t_o) = 0$ and we were to follow *P* then *B* will be always zero:

$$\text{If } \ B(\vec{x},t_o)=0, \ \text{then } \ B(\vec{x},t)=0 \ \text{ for all } \ t,$$

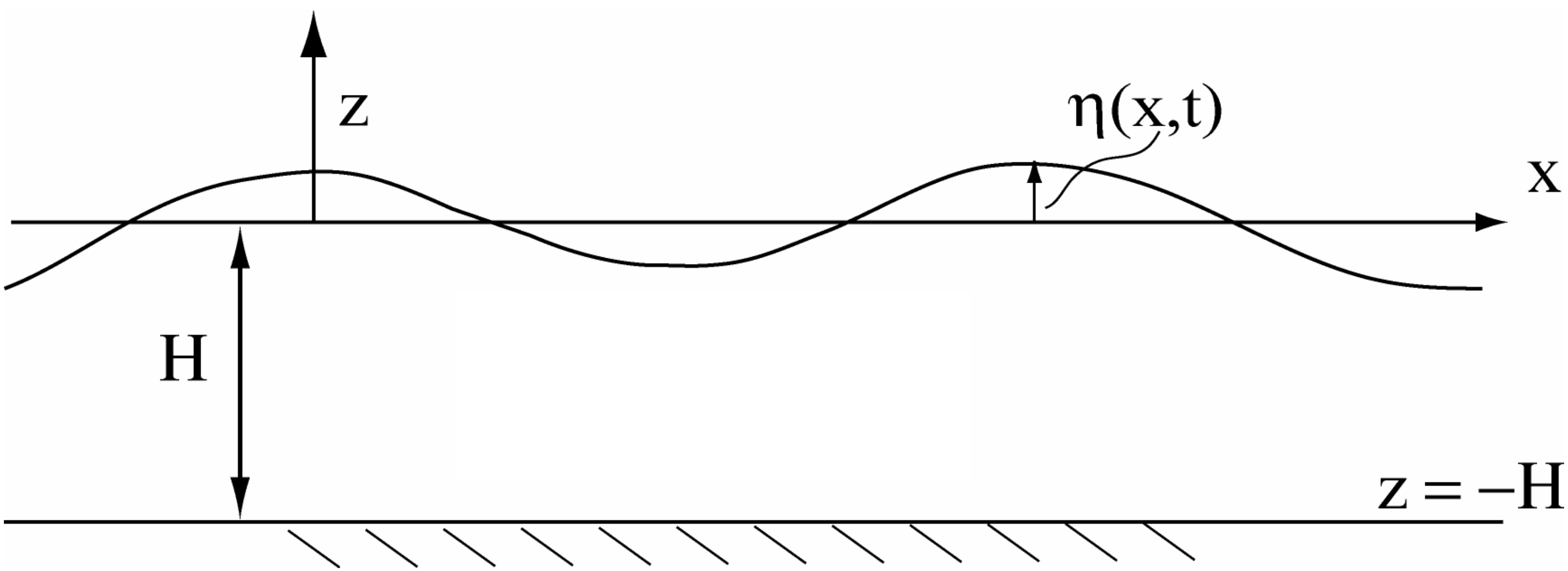
$$\therefore \frac{DB}{Dt}=\frac{\partial B}{\partial t}+(\nabla\phi\cdot\nabla)B=0 \quad on \quad B=0$$

Take for example a flat bottom at $z=-H$ then $\partial\phi/\partial z=0$

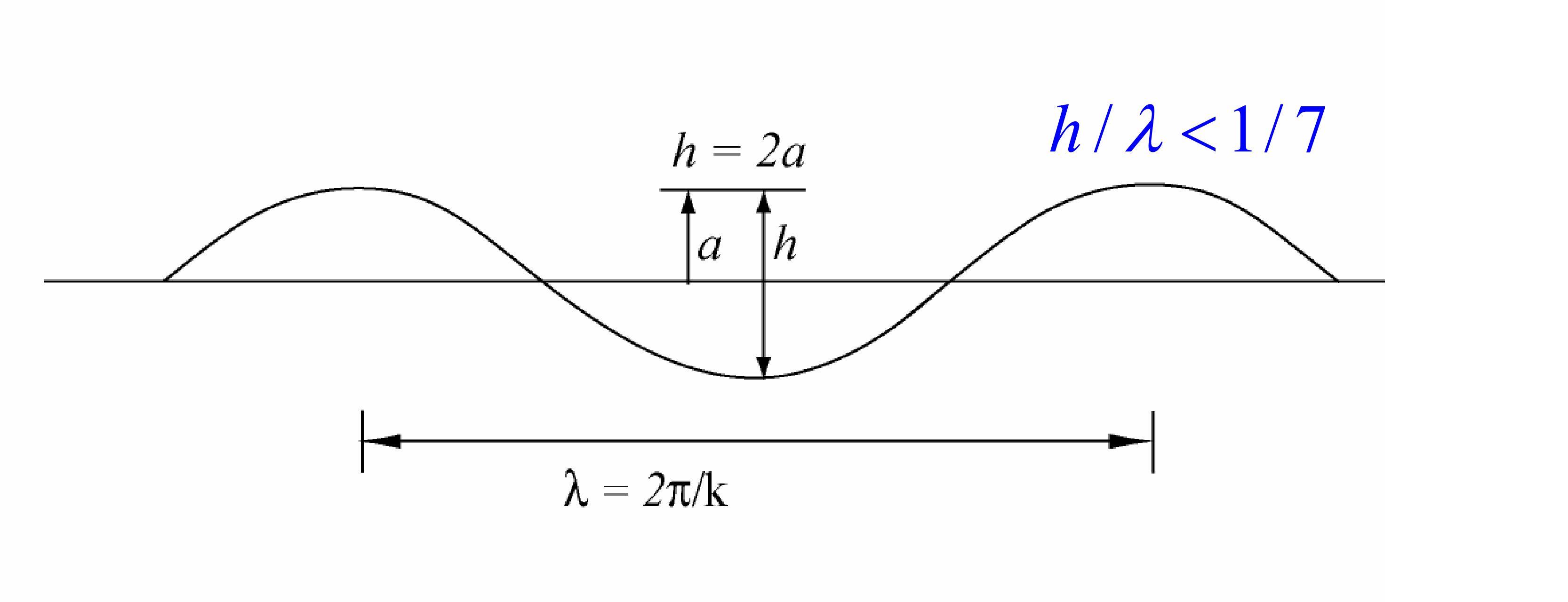
Linear Plane Progressive Waves

Linear free-surface gravity waves can be characterized by their amplitude, a, wavelength, $\lambda = 2\pi/k$, and frequency, ω.

$$\eta(x,t) = a\cos(kx - \omega t)$$



Linear Waves



- *a* is wave amplitude, $h = 2a$
- λ is wavelength, $\lambda = 2\pi/k$ where k is wave number
- Waves will start to be non-linear (and then break) when $h/\lambda > 1/7$

Linearization of Equations & Boundary Conditions

Non-dimensional variables :

$$\eta = a\eta^{*} \qquad u = a\omega u^{*} \qquad \omega t = t^{*}$$

$$\phi = a\omega\lambda\phi^{*} \qquad w = a\omega w^{*} \qquad x = \lambda x^{*}$$

$$dt = 1/\omega\, dt^{*} \qquad d\phi = a\omega\lambda\, d\phi^{*} \qquad dx = \lambda\, dx^{*}$$

FS Boundary Conditions

1. Dynamic FS BC: $\rho\{\frac{\partial\phi}{\partial t}+\frac{1}{2}V^2+g\eta\}=0$
(Pressure at the FS)

Compare $\frac{\partial\phi}{\partial t}$ and $V^2 \sim \left(\frac{\partial\phi}{\partial x}\right)^2$

$$\frac{(\frac{\partial\phi}{\partial x})^2}{\frac{\partial\phi}{\partial t}}=\frac{a^2\omega^2}{a\omega^2\lambda}\frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}}=\frac{a}{\lambda}\frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}}$$

If $h/\lambda << 1/7$ then $a/\lambda << 1/14$ since $h=2a$.

$$(\frac{\partial\phi}{\partial x})^2 << \frac{\partial\phi}{\partial t}$$

$\therefore\ \rho\{\frac{\partial\phi}{\partial t}+\frac{1}{2}V^2+g\eta\}=0$ becomes $\boxed{\frac{\partial\phi}{\partial t}+g\eta=0}$ on $z=\eta$

FS Boundary Conditions

2. Kinematic FS BC:
(Motion at the Surface) $w = u\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \eta}{\partial t}$

Non-dimensionalize:

$$\cancel{a\omega}\, w^{*} = \cancel{a\omega}\, u^{*}\,\frac{a}{\lambda}\,\frac{\partial \eta^{*}}{\partial x^{*}} + \cancel{a\omega}\,\frac{\partial \eta^{*}}{\partial t^{*}}$$

If $h/\lambda << 1/7$ $\Rightarrow$ $u\dfrac{\partial \eta}{\partial x} << \dfrac{\partial \eta}{\partial t}$ and $u\dfrac{\partial \eta}{\partial x} << w$

Therefore:

$w = u\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \eta}{\partial t}$ becomes $\boxed{\dfrac{\partial \phi}{\partial z} = \dfrac{\partial \eta}{\partial t}}$ on $z = \eta$

FSBC about $z=0$

Since wave elevation, η, is proportional to wave amplitude, *a*, and *a* is small compared to the wavelength, λ, we can simplify our boundary conditions one step further to show that they can be taken at $z = 0$ versus $z = \eta$.

First take the Taylor's series expansion of $\phi(x, z=\eta, t)$ about $z=0$:

$$\phi(x, z=\eta, t) = \phi(x, 0, t) + \frac{\partial \phi}{\partial z}\eta + \ldots$$

It can be shown that the second order term and all subsequent HOTs are very small and can be neglected. Thus we can substitute $\phi(x, z=0, t)$ for $\phi(x, z=\eta, t)$ everywhere:

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \Rightarrow \boxed{\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}} \Rightarrow \frac{\partial \eta}{\partial t} = -\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \Rightarrow \boxed{\frac{\partial \phi}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi}{\partial t^2} = 0 \text{ on } z = 0}$$

Linear Wave Boundary Value Problem

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \text{ for } z < 0$$

1. Sea Floor Boundary Condition (flat bottom)

$$\frac{\partial \phi}{\partial z} = 0 \text{ on } z = -H$$

2. Dynamic FSBC (Pressure at the FS)

$$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t} \text{ on } z = 0$$

3. Kinematic FSBC (Motion at the Surface)

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \text{ on } z = 0$$

Solution to Laplace's Equation

By Separation of Variables we can get a solution for linear FS waves:

$$\eta(x,t) = a\cos(kx - \omega t + \psi)$$

$$\phi(x,z,t) = -\frac{a\omega}{k} f(z)\sin(kx - \omega t + \psi)$$

$$u(x,z,t) = a\omega\, f(z)\cos(kx - \omega t + \psi)$$

$$w(x,z,t) = -a\omega f_1(z)\sin(kx - \omega t + \psi)$$

$$\omega^2 = gk\tanh(kH) \Rightarrow \textit{dispersion relation}$$

Where:

$$f(z) = \frac{\cosh[k(z+H)]}{\sinh(kH)} \qquad f_1(z) = \frac{\sinh[k(z+H)]}{\sinh(kH)}$$

Dispersion Relationship

The dispersion relationship uniquely relates the wave frequency and wave number given the depth of the water.

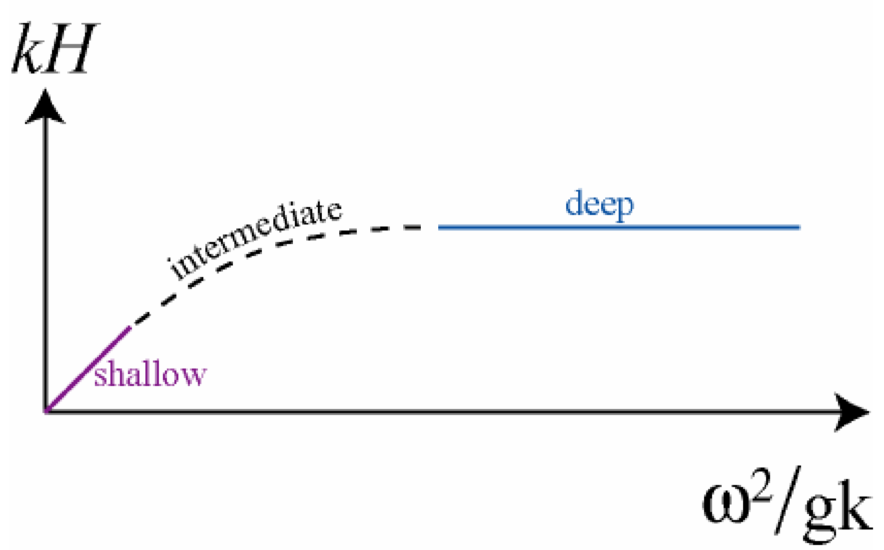
$$\omega^2 = gk\tanh(kH)$$

The solution f must satisfy all boundary conditions. Plugging f into the KBC yields the dispersion relationship

- <u>Approximations</u>
  - As $kH \to 0$ $\tanh(kH) \to kH$

    $\therefore \omega^2 \cong gk^2H$ (shallow)
  - As $kH \to \infty$ $\tanh(kH) \to 1$

    $\therefore \omega^2 \cong gk$ (deep)



Pressure Under Waves

Unsteady Bernoulli's Equation:

$$p = -\underbrace{\underbrace{\rho \frac{\partial \phi}{\partial t}}_{\substack{unsteady \\ fluctuation}} - \underbrace{\frac{1}{2}\rho V^2}_{\substack{2nd\ Order \\ term}}}_{DynamicPressure} - \underbrace{\rho g z}_{\substack{hydrostatic \\ pressure}}$$

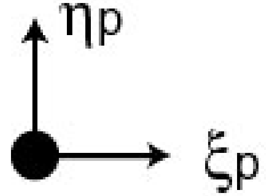
Since we are dealing with a linearized problem we can neglect the 2nd order term. Thus dynamic pressure is simply:

$$p_d(x,z,t) = -\rho \frac{\partial \phi}{\partial t}$$

$$= \frac{a\omega^2}{k} \rho \; f(z) \cos(\omega t - kx - \psi) = \rho \frac{\omega^2}{k} \; f(z) \, \eta(x,t)$$

$$= \rho g \; \eta(x,t) \frac{\cosh\left[k\left(z+H\right)\right]}{\cosh\left(kH\right)}$$

Motion of a Fluid Particle



$$u = \frac{d\xi p}{dt} \qquad w = \frac{d\eta p}{dt}$$

$$\xi p(x,z,t) = a\ f(z)\ \sin(\omega t - kx + \psi)$$

$$\eta p(x,z,t) = a\ f_1(z)\ \sin(\omega t - kx + \psi)$$

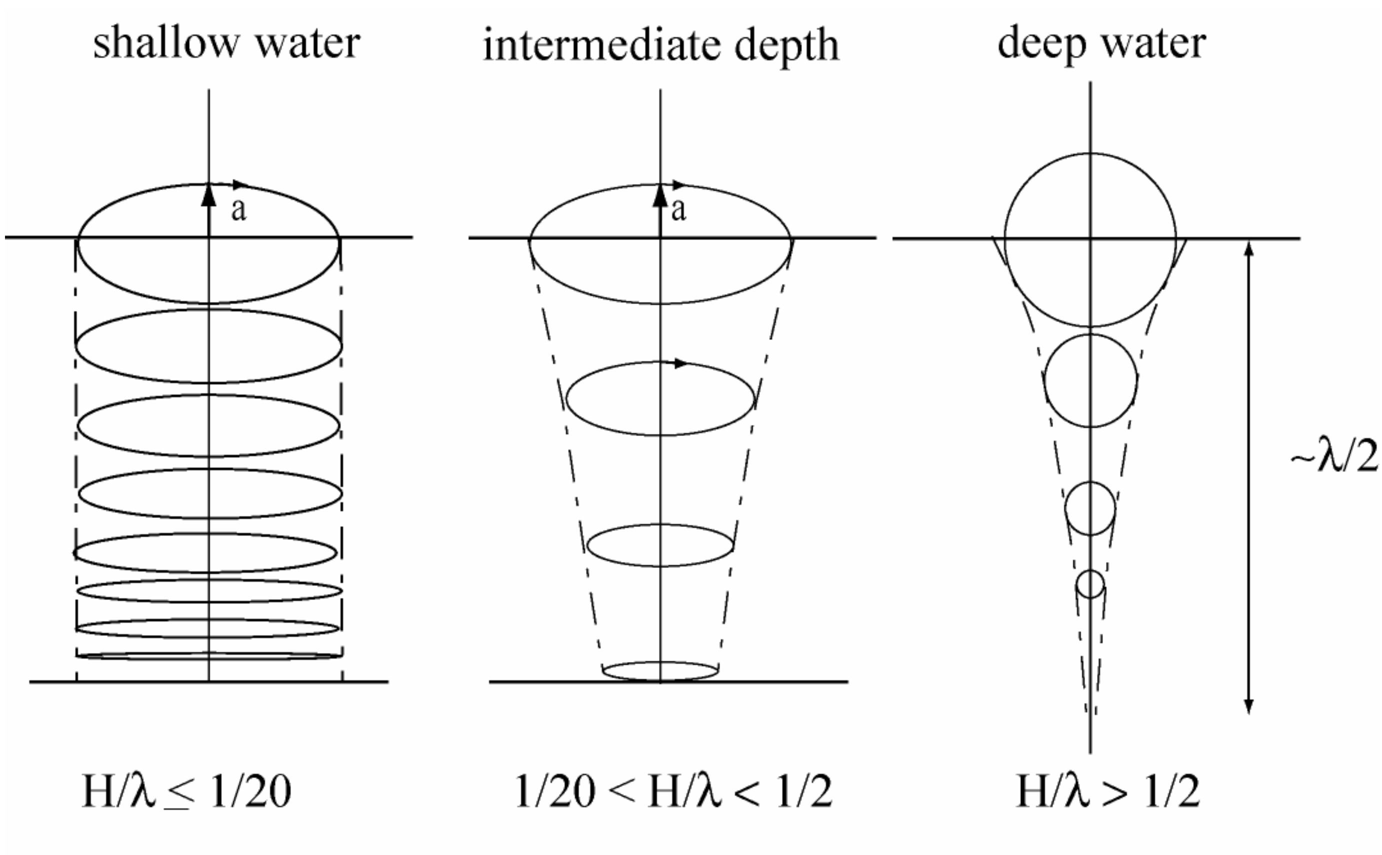
Elliptical Orbit Path

$$\left[\frac{\xi p}{f(z)}\right]^2 + \left[\frac{\eta p}{f_1(z)}\right]^2 = a^2$$

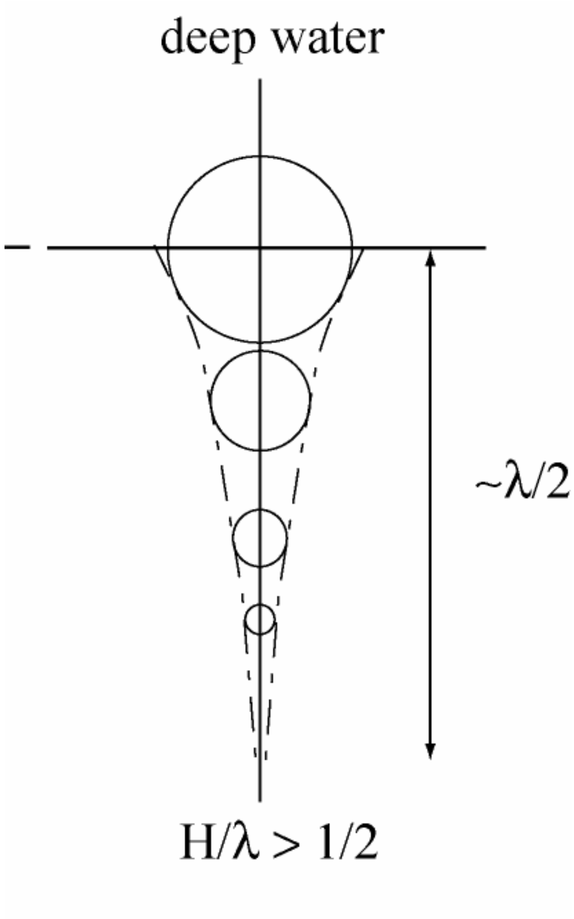
$$\left[\frac{\xi p}{\dfrac{a \cosh k(z+H)}{\sinh kH}}\right]^2 + \left[\frac{\eta p}{\dfrac{a \sinh k(z+H)}{\sinh kH}}\right]^2 = 1$$

Particle Orbits

Under the waves particles follow distinct orbits depending on whether the water is shallow, intermediate or deep. Water is considered deep when water depth is greater than one-half the wavelength of the wave.



Particle Orbits in Deep Water



$$H \to \infty \quad \left(kH \gg 1\right)$$

$$\omega^2 = gk \Leftrightarrow \text{dispersion relationship}$$

$$f(z) \cong f_1(z) \cong e^{kz}$$

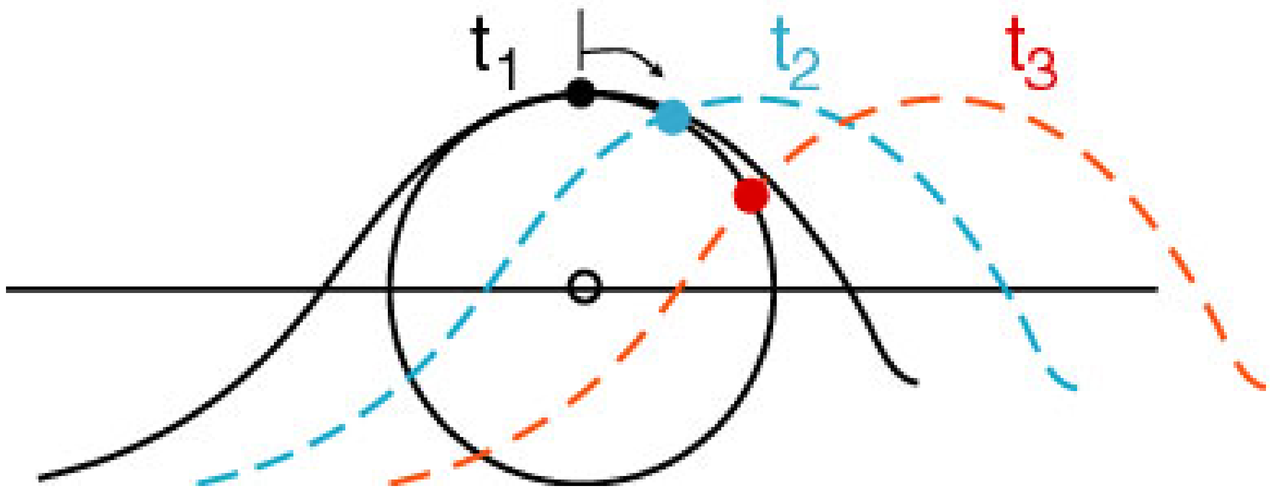
$$\xi_p{}^2 + \eta_p{}^2 = \left(a\, e^{kz}\right)^2$$

Circular orbits with
exponentially
decreasing radius

Particle motion extinct at z ≅ -λ/2

at the free surface…

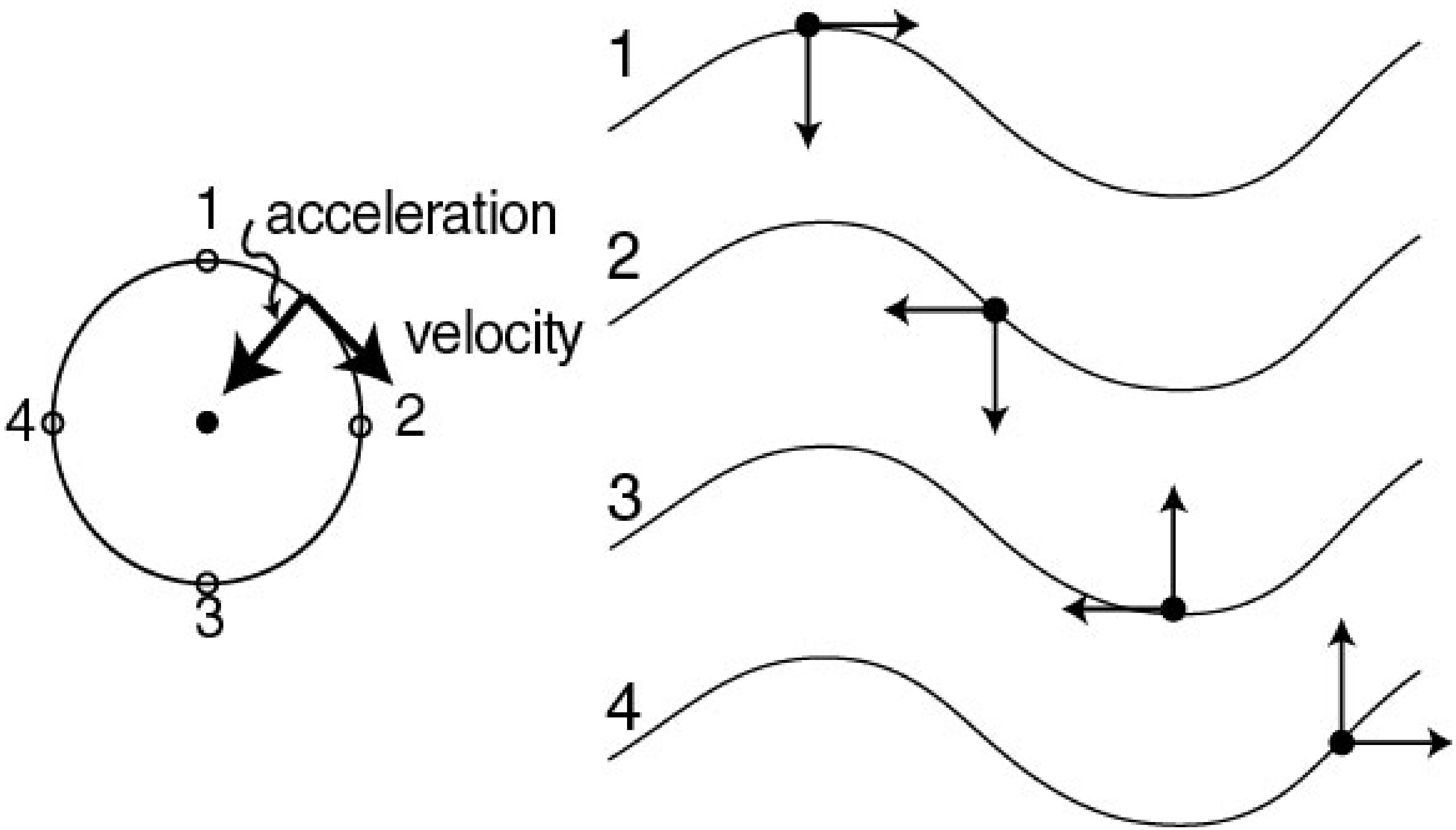
The intersection between the circle on which ξ_p *and* η_p *lie and the elevation profile* $\eta(x,t)$ *define the location of the particle.*



This applies at all depths, z:

$$\eta_p(x,z,t) = a\, e^{kz} \cos(\omega t - kx + \psi) = \eta(x,t)\, e^{kz}$$

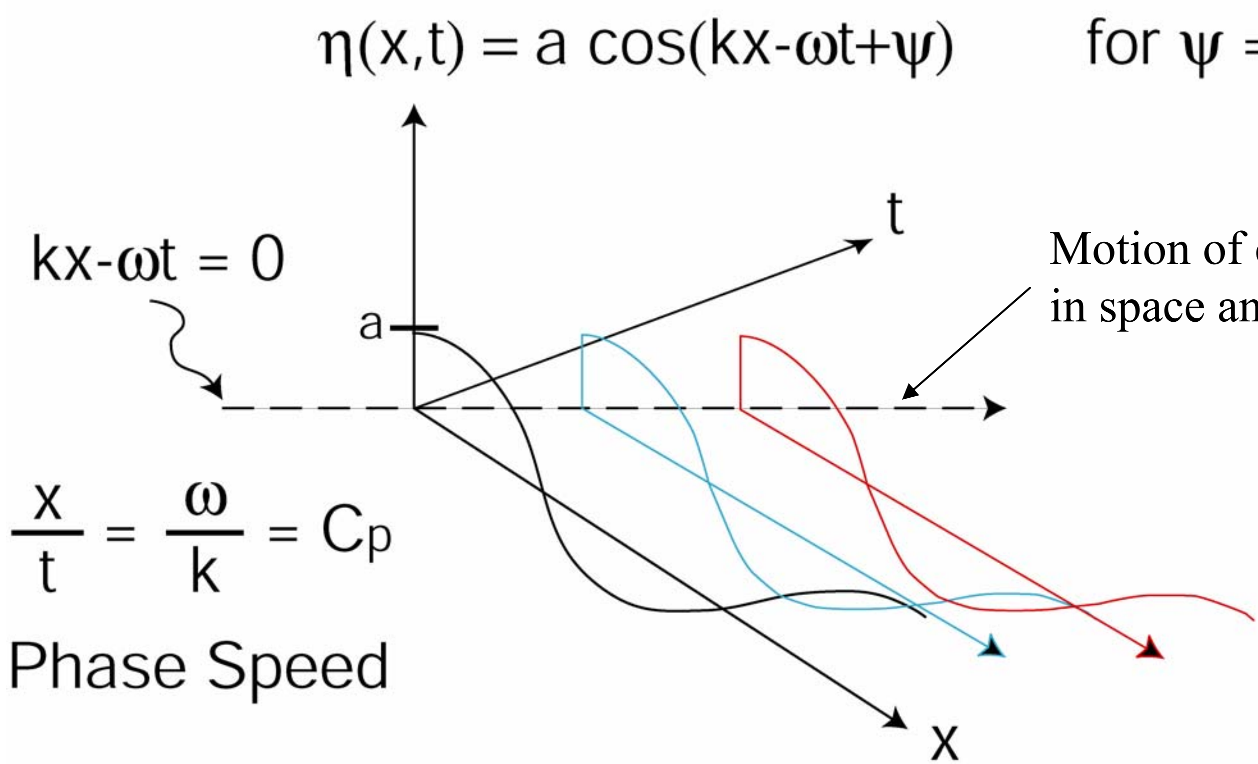
Particle acceleration and velocity



Phase Velocity

$$V_p = \frac{\lambda}{T} = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH}$$

Phase Velocity= speed of the wave crest



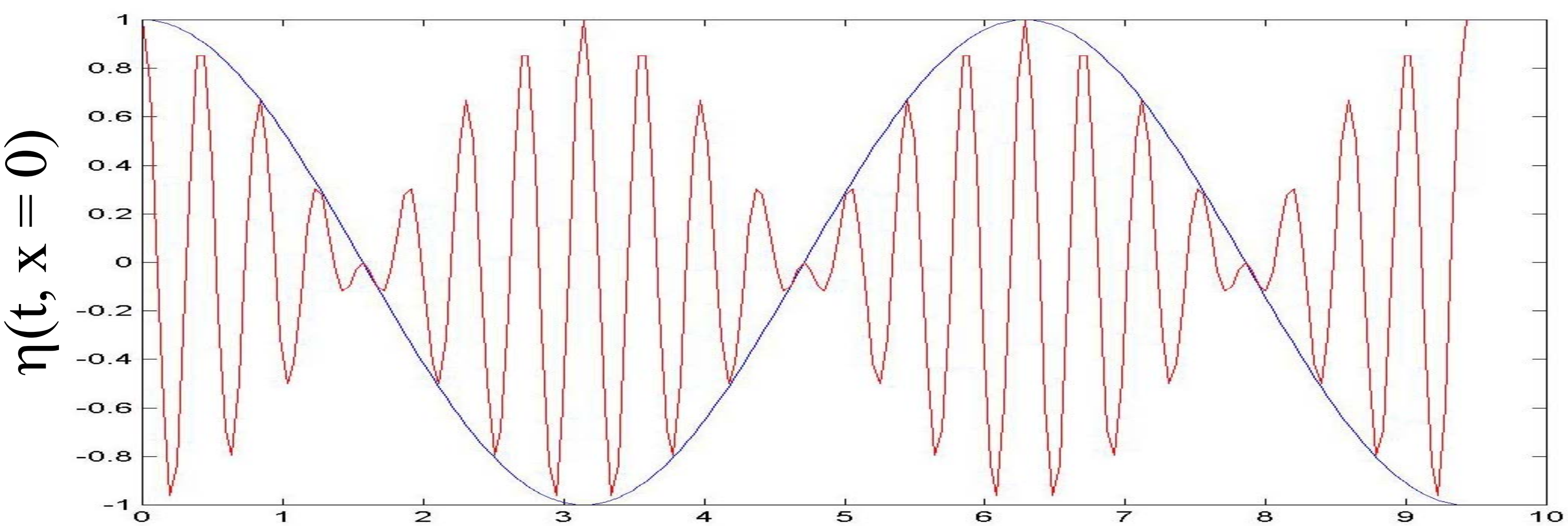
Motion of crest in space and time

Group Velocity

Simple Harmonic Wave: $\eta(x,t) = a \cos(\omega t - kx)$

$$\eta(x,t) = \lim_{\delta k, \delta\omega \to 0} \left\{ \tfrac{a}{2} \cos\left([\omega - \delta\omega]t - [k - \delta k]x \right) + \tfrac{a}{2} \cos\left([\omega + \delta\omega]t - [k + \delta k]x \right) \right\}$$

$$= \lim_{\delta k, \delta\omega \to 0} \left\{ a \cos(\omega t - kx) \cos(\delta\omega\, t - \delta k\ x) \right\}$$



Group Velocity

speed of the wave energy

$$V_g = \frac{\delta\omega}{\delta k} = \frac{d\omega}{dk} \qquad \omega^2 = gk\tanh(kH)$$

$$\frac{d}{dk}\{\omega^2\} = \frac{d}{dk}\{kg\tanh(kH)\}$$

$$2\omega\frac{d\omega}{dk} = g\tanh(kH) + \frac{kgH}{\cosh^2(kH)}$$

$$\frac{d\omega}{dk} = \frac{1}{2}\underbrace{\frac{g}{\omega}\tanh(kH)}_{\omega/k = V_p}\left\{1 + \frac{kH}{\sinh(kH)\cosh(kH)}\right\}$$

$$\therefore C_g = \frac{1}{2}C_p\left\{1 + \frac{kH}{\sinh kH \cosh kH}\right\}$$

Shallow water $H \to 0$

$$\omega = \sqrt{gH}\,k \qquad C_g = C_p$$

Deep water $H \to \infty$

$$\omega^2 = kg \qquad C_g = \tfrac{1}{2}C_p$$