From Characteristic Functions and Fourier Transforms to PDFs/CDFs and Option Prices

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Option Pricing

From a dynamic process to a terminal distribution

- If we specify a dynamic process for X and its initial conditions, we can compute the distribution of X over a certain time horizon, $[t, T]$.
- The reverse is not true.
  - Given a conditional distribution for X over $[t, T]$, there can be many processes that can generate this distribution.
  - Knowing the dynamic process is important for hedging practices — Dynamic hedging is more likely to work if the underlying process is continuous instead of discontinuous.
- The objective of this note: Given a dynamic process for X, derive the probability density (PDF) of X and price options based on integration of terminal payoffs over the probability densities (instead of based on dynamic hedging arguments and PDEs).

Formalizing the idea

- Assume X is a Markov process: Any information about X up to time t is summarized by X_t. Let $f(X_T|X_t)$ denote the conditional distribution of X_T conditional on time-t information (filtration $\mathcal{F}_t$) under the risk-neutral measure $\mathbb{Q}$.

- Let $\Pi(X_T)$ denote the payoff of a contingent claim (derivative), which we assume is a function of X_T. Then, its time-t value is

$$p_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\Pi(X_T)\right] = \int_X e^{-\int_t^T r_s ds}\Pi(X_T)f(X_T|X_t)dX_T$$

  Instead of solving PDEs, binomial trees, or simulating the process, we focus on doing the integration based on the risk-neutral densities.

- In most of the examples, I assume deterministic interest rates:

$$p_t = e^{-r\tau}\int_X \Pi(X_T)f(X_T|X_t)dX_T$$

  where r now is the time-t continuously compounded spot rate of maturity $T-t$.

The road map

- For most models (processes) discussed in this class on security prices S_t, we can derive the characteristic function or Fourier transform of the log security return ($\ln S_T/S_t$) (semi-)analytically.
- Given the characteristic function (CF), we just need one numerical integration to obtain the probability density function (PDF) or cumulative density function (CDF).
- Given the Fourier transforms (FT), we just need one numerical integration to obtain the value of vanilla options.
- The integration is one-dimensional in both cases no matter how many dimensions/factors the security price S_t is composed of.
- We can apply fast Fourier inversion (FFT or some other methods) to make the numerical integration (for both PDF and option prices) very fast.
- Solving PDEs becomes difficult as the dimension increases.
- Simulation works (slowly) for high-dimensional cases and is reserved for pricing exotics.

Characteristic function: definition

- In probability theory, the characteristic function (CF) of any random variable X completely defines its probability distribution. On the real line it is given by the following formula:

$$\phi_X(u) \equiv \mathbb{E}\left[e^{iuX}\right] = \int_{-\infty}^{\infty} e^{iux} f_X(x)dx = \int_{\Omega} e^{iux} dF_X(x), \quad u \in \mathbb{R}$$

  where u is a real number, i is the imaginary unit, and $\mathbb{E}$ denotes the expected value, $f_X(x)$ denotes the probability density function (PDF), and $F_X(x)$ denotes the cumulative density function (CDF).
- CF is well defined on the whole real line (u).
- For option pricing, we extend the definition to the complex plane, $u \in \mathcal{D} \subseteq \mathbb{C}$, where $\mathcal{D}$ denotes the subset of the complex plane on which the expectation is well defined. $\phi_X(u)$ under this extended definition is called the generalized Fourier transform.
  - The generalized Fourier transform includes as special cases the Laplace transform (when $im(u) > 0$) and cumulant generating function (when $im(u) < 0$) as special cases (when they are well defined).

Example: The Black-Scholes model

- Under BSM, the log security return follows,

$$s_t \equiv \ln S_t/S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

The return is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$.

- The PDF is $f_s(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\left(\mu-\frac{1}{2}\sigma^2\right)t)^2}{2\sigma^2 t}\right)$.
- The CF is:

$$\begin{aligned} \phi_s(u) &= \mathbb{E}\left[e^{ius_t}\right] = e^{iu\text{mean}-\frac{1}{2}u^2\text{variance}} = e^{\left(iu\mu t - iu\frac{1}{2}\sigma^2 t - \frac{1}{2}u^2\sigma^2 t\right)} \\ &= e^{\left(iu\mu t - \frac{1}{2}\sigma^2(iu+u^2)t\right)} \end{aligned}$$

- Under $\mathbb{Q}$, $\mu = r - q$.

The inversion: From CF to PDF and CDF

- There is a bijection between CDF and CFs: Two distinct probability distributions never share the same CF.
- Given a CF ϕ, it is possible to reconstruct the corresponding CDF:

$$F_X(y) - F_X(x) = \lim_{\tau\to\infty} \frac{1}{2\pi} \int_{-\tau}^{+\tau} \frac{e^{-iux} - e^{-iuy}}{iu} \phi_X(u) du$$

  In general this is an improper integral ...
- Another form of the inversion

$$F_X(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{iux}\phi_X(-u) - e^{-iux}\phi_X(u)}{iu} du.$$

- The inversion formula for PDF:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux}\phi_X(u) du = \frac{1}{\pi} \int_0^{\infty} e^{-iux}\phi_X(u) du.$$

- All the integrals here should be understood as a principal value if there is no separate convergence at the limits.

Proofs

- Preliminary results:

$$
\begin{array}{rl}
 & e^{iux} = \cos ux + i \sin ux, \\
 & \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{iu\zeta}}{iu} du = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin u\zeta}{u} du = sgn(\zeta), \\
 & \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin u\zeta}{u} du = sgn(\zeta), \\
\text{For fixed } x, & \int_{-\infty}^{\infty} sgn(y-x) dF(y) = -\int_{-\infty}^{x} dF(y) + \int_{x}^{\infty} dF(y) = 1 - 2F(x), \\
 & \phi(u) \& \phi(-u) \text{ are complex conjugates.}
\end{array}
$$

- CDF inversion:

$$
\begin{aligned}
I &= \int_0^\infty \frac{e^{iux}\phi_X(-u) - e^{-iux}\phi_X(u)}{iu} du \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{e^{iux}e^{-iuz} - e^{-iux}e^{iuz}}{iu} dF(z) du \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{2\sin u(x-z)}{u} dF(z) du \\
&= \int_{-\infty}^\infty \int_0^\infty \frac{2\sin u(x-z)}{u} du dF(z) = \int_{-\infty}^\infty \pi sgn(x-z) dF(z) \\
&= \pi(2F(x)-1)
\end{aligned}
$$

Hence, $F(x) = \frac{1}{2} + \frac{1}{2\pi} I$.

- PDF inversion:

$$
f(x) = F'(x) = \frac{1}{2\pi}\int_0^\infty \left(e^{iux}\phi(-u) + e^{-iux}\phi(u)\right) du = \frac{1}{\pi}\int_0^\infty e^{-iux}\phi(u) du
$$

Reference: Kendall's Advanced Theory of Statistics, Volume I, chapter 4

Fourier transforms and inversions of European options

- Take a European call option as an example. We perform the following rescaling and change of variables:

$$c(k) = e^{rt} c(K, t)/F_0 = \mathbb{E}_0^{\mathbb{Q}} \left[(e^{s_t} - e^k) 1_{s_t \geq k} \right],$$

  with $s_t = \ln F_t/F_0$ and $k = \ln K/F_0$.

  - $c(k)$: the option forward price in percentage of the underlying forward as a function of moneyness defined as the log strike over forward, k (at a fixed time to maturity).

- We can derive the Fourier transform of the call option in terms of the Fourier transform (CF) of the log return $\ln F_t/F_0$.

- Hence, if we know the CF of the return, we would know the transform of the option.

- Then, we can use numerical inversion to obtain option prices directly.

- There are many ways of doing this inversion.

I. The CDF analog

- Treat $c(k) = \mathbb{E}_0^{\mathbb{Q}}\left[\left(e^{s_t} - e^k\right) 1_{s_t \geq k}\right] = \int_{-\infty}^{\infty} \left(e^{s_t} - e^k\right) 1_{s_t \geq x} dF(s)$ as a *CDF*.
  - The option transform:
$$\chi_c^I(u) \equiv \int_{-\infty}^{\infty} e^{iuk} dc(k) = -\frac{\phi_s(u-i)}{iu+1}, \quad u \in \mathbb{R}.$$
    Thus, if we know the CF of the return, $\phi_s(u)$, we know the transform of the option, $\chi_c^I(u)$.
  - The inversion formula is analogous to the inversion of a CDF:
$$c(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{iux}\chi_c^I(-u) - e^{-iux}\chi_c^I(u)}{iu} du.$$
  - Use quadrature methods for the numerical integration.
    It can work well if done right.
  - The literature often writes: $c(x) = e^{-qt} Q_1(x) - e^{-rt} e^{-x} Q_2(x)$. Then, we must invert twice.
  - References: Duffie, Pan, Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump Diffusions, Econometrica, 68(6), 1343–1376.

    Singleton, 2001, Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function,"

    Journal of Econometrics, 102, 111-141.

Proofs: The option transform

$$\chi_c^I(u) \equiv \int_{-\infty}^{\infty} e^{iuk} dc(k) = e^{iuk} c(k)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} c(k)\, iue^{iuk} dk$$

Checking the boundary conditions, we have $c(\infty) = 0$ (when strike is infinity) and $c(-\infty) = (F_t - 0)/F_0 = 1$ when the strike is zero. Hence, $e^{iu\infty} c(\infty) = 0$ and we will carry the other non-convergent limit $e^{-iu\infty}$.

$$\begin{aligned}
\chi_c^I(u) &= -e^{-iu\infty} - \int_{-\infty}^{\infty} c(k)\, iue^{iuk} dk \\
&= -e^{-iu\infty} - iu \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(e^{s_t} - e^{k}\right) 1_{s_t \geq k} dF(s) \right] e^{iuk} dk \\
&= -e^{-iu\infty} - iu \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(e^{s_t} - e^{k}\right) 1_{s_t \geq k} e^{iuk} dk \right] dF(s) \\
&= -e^{-iu\infty} - iu \int_{-\infty}^{\infty} \left[\int_{-\infty}^{s_t} \left(e^{iuk + s_t} - e^{(iu+1)k}\right) dk \right] dF(s) \\
&= -e^{-iu\infty} - iu \int_{-\infty}^{\infty} \left[e^{s_t} \frac{e^{iuk}}{iu} - \frac{e^{(iu+1)k}}{iu+1} \Big|_{-\infty}^{s_t} \right] dF(s).
\end{aligned}$$

We need to check the boundary again. $\lim_{k \to -\infty} e^{(iu+1)k} = 0$ given the real component $e^{-\infty}$. The other boundary is non-convergent $e^{s_t} e^{-iu\infty}$, which we pull out and take the expectation to have

$$iu \int_{-\infty}^{\infty} \frac{e^{s_t} e^{-iu\infty}}{iu} dF(s) = e^{-iu\infty},$$

which cancels out the other nonconvergent term.

$$\chi_c^I(u) = -iu \int_{-\infty}^{\infty} \left[\frac{e^{(iu+1)s_t}}{iu} - \frac{e^{(iu+1)s_t}}{iu+1} \right] dF(s) = -\int_{-\infty}^{\infty} \frac{e^{(iu+1)s_t}}{iu+1} dF(s) = -\frac{\phi(u-i)}{iu+1}.$$

Proofs: The option transform inversion

The proof for the option transform inversion is similar to that for the CDF. In particular, our scaled option value $c(k)$ behaves just like a CDF: $c(\infty) = 0$ (when strike is infinity), and $c(-\infty) = 1$ (when strike is zero). Hence, the inversion formula is

$$
\begin{aligned}
I &\equiv \int_0^\infty \frac{e^{iux}\chi(-u) - e^{-iux}\chi(u)}{iu} du \\
&= \cdots \text{(as before)} = \int_{-\infty}^{\infty} \pi sgn(x - z)\, dF(z) = -\pi(1 - 2c(x)).
\end{aligned}
$$

Thus,

$$
c(x) = \frac{1}{2} + \frac{1}{2\pi} I.
$$

II. The PDF analog

- Treat $c(k)$ analogous to a *PDF*.
  - The option transform:
  $$\chi_c^{II}(z) \equiv \int_{-\infty}^{\infty} e^{izk} c(k) dk = \frac{\phi_s(z-i)}{(iz)(iz+1)}$$
  with $z = u - i\alpha$, $\alpha \in \mathcal{D} \subseteq \mathbb{R}^+$ for the option transform to be well defined.
    - The range of α depends on payoff structure and model.
    - The exact value choice of α is a numerical issue.
    - Carr and Madan (1999, Journal of Computational Finance) refer to α as the dampening coefficient.
    - Given the transform on return $\phi_s(u)$, we know the transform on call.
  - The inversion is analogous to that for a PDF:
  $$c(k) = \frac{1}{2\pi} \int_{-i\alpha-\infty}^{-i\alpha+\infty} e^{-izk} \chi_c^{II}(z) dz = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-iuk} \chi_c^{II}(u - i\alpha) du.$$
  - References: Carr&Wu, Time-Changed Levy Processes and Option Pricing, JFE, 2004, 17(1), 113–141.

Proofs

$$
\begin{aligned}
\chi^{II}(z) &\equiv \int_{-\infty}^{\infty} e^{izk} c(k)\, dk \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(e^{s_t} - e^{k} \right) 1_{s_t \geq k} dF(s) \right] e^{izk} dk \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(e^{s_t} - e^{k} \right) 1_{s_t \geq k} e^{izk} dk \right] dF(s) \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{s_t} \left(e^{izk + s_t} - e^{(iz+1)k} \right) dk \right] dF(s) \\
&= \int_{-\infty}^{\infty} \left[e^{s_t} \frac{e^{izk}}{iz} - \frac{e^{(iz+1)k}}{iz+1} \Big|_{-\infty}^{s_t} \right] dF(s)
\end{aligned}
$$

We need to consider the boundary conditions at $k = -\infty$. $\lim_{k \to -\infty} e^{(iz+1)k} = 0$ as long as the real component of iz is greater than -1. $\lim_{k \to -\infty} e^{izk} = 0$ as long as the real component of iz is greater than 0. Hence, taken together, we need the real component of iz to be greater than zero. If we write $z = u - i\alpha$, with both u and α real, we have $iz = iu + \alpha$. Hence, we need $\alpha > 0$ for the above boundary condition to converge. Given that $u_i > 0$, we have

$$
\begin{aligned}
\chi^{II}(z) &= \int_{-\infty}^{\infty} \left[\frac{e^{(iz+1)s_t}}{iz} - \frac{e^{(iz+1)s_t}}{iz+1} \right] dF(s) \\
&= \int_{-\infty}^{\infty} \frac{e^{(iz+1)s_t}}{iz(iz+1)} dF(s) = \frac{\phi(z-i)}{iz(iz+1)}.
\end{aligned}
$$

Fast Fourier Transform (FFT)

- FFT is an efficient algorithm for computing discrete Fourier coefficients.
- The discrete Fourier transform is a mapping of $\mathbf{f} = (f_0, \cdots, f_{N-1})^\top$ on the vector of Fourier coefficients $\mathbf{d} = (d_0, \cdots, d_{N-1})^\top$, such that

$$d_j = \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jm\frac{2\pi}{N} i}, \quad j = 0, 1, \cdots, N-1.$$

- FFT allows the efficient calculation of $\mathbf{d}$ if N is an even number, say $N = 2^n, n \in \mathbb{N}$. The algorithm reduces the number of multiplications in the required N summations from an order of 2^{2n} to that of $n2^{n-1}$, a very considerable reduction.
- By a suitable discretization, we can approximate the inversion of a PDF (also option price) in the above form to take advantage of the computational efficiency of FFT.

Return PDF inversion

Compare the PDF inversion with the FFT form:

$$f_X(x) = \frac{1}{\pi}\int_0^\infty e^{-iux}\phi_X(u)du. \quad d_j = \frac{1}{N}\sum_{m=0}^{N-1} f_m e^{-jm\frac{2\pi}{N}i}$$

- Discretize the integral using the trapezoid rule:
  $f_X(x) \approx \frac{1}{\pi}\sum_{m=0}^{N-1}\delta_m e^{-iu_m x}\phi_X(u_m)\Delta u$, $\delta_m = \frac{1}{2}$ when $m = 0$ and 1 otherwise.
  (trapezoid rule: $\int_a^b h(x)dx = \left(\frac{h(a)+h(b)}{2} + \sum_{k=1}^{N-1} h(a+k\Delta x)\right)\Delta x$.)
- Set $\eta = \Delta u$, $u_m = \eta m$.
- Set $x_j = -b + \lambda j$ with $\lambda = 2\pi/(\eta N)$ being the return grid and b being a parameter that controls the return range.
  - To center return around zero, set $b = \lambda N/2$.
- The PDF becomes

$$f_X(x_j) \approx \frac{1}{N}\sum_{m=0}^{N-1} f_m e^{-jm\frac{2\pi}{N}i}, \quad f_m = \delta_m \frac{N}{\pi} e^{iu_m b}\phi_X(u_m)\eta. \tag{1}$$

with $j = 0, 1, \cdots, N-1$. The summation has the FFT form and can hence

Call value inversion

Compare the call inversion (method II) with the FFT form:

$$c(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-iuk} \chi_c^{II}(u - i\alpha) du. \quad d_j = \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jm\frac{2\pi}{N} i}$$

- Discretize the integral using the trapezoid rule:
  $c(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{m=0}^{N} \delta_m e^{-iu_m k} \chi_c^{II}(u_m - i\alpha) \Delta u$
- Set $\eta = \Delta u$, $u_m = \eta m$.
- Set $k_j = -b + \lambda j$ with $\lambda = 2\pi/(\eta N)$ being the return grid and b being a parameter that controls the return range.
  - To center return around zero, set $b = \lambda N/2$.
- The call value becomes

$$c(k_j) \approx \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jm\frac{2\pi}{N} i}, \quad f_m = \delta_m \frac{N}{\pi} e^{-\alpha k_j + iu_m b} \chi_c^{II}(u_m - i\alpha)\eta.$$

with $j = 0, 1, \cdots, N-1$. The summation has the FFT form and can hence be computed efficiently.

FFT implementation

To implement the FFT, we need to fix the following parameters

- $N = 2^n$: The number of summation grids. Setting it to be the power of 2 can speed up the FFT calculation.
- $\eta = \Delta u$: The discrete summation grid width. The smaller the grid, the better the approximation.

However, given N, η also determines the strike grid $\lambda = 2\pi/(\eta N)$. The finer the summation grid η, the coarser the strike spacing returned from the FFT calculation. There is a trade off: If we want to have more option value calculated at a finer grid of strikes, we would need to use a coarser summation grid and hence less accuracy.

The lower and upper bound truncation $b = \lambda N/2$ is also determined by the summation grid choice.

FFT generates option values at N strikes simultaneously. However, if the strike grid is larger, many of the returned strikes are out of the interesting region.

III. Fractional FFT

- Fractional FFT (FRFT) separates the integration grid choice from the strike grids. With appropriate control, it can generate more accurate option values given the same amount of calculation.
- The method can efficiently compute,
$$d_j = \sum_{m=0}^{N-1} f_m e^{-jm\zeta i}, \qquad j = 0, 1, ..., N-1,$$
for any value of the parameter ζ.
- The standard FFT can be seen as a special case for $\zeta = 2\pi/N$. Therefore, we can use the FRFT method to compute,
$$c(k_j) \approx \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jm\eta\lambda i}, \quad f_m = \delta_m \frac{N}{\pi} e^{-\nu k_j + i u_m b} \chi_c^{II}(u_m)\eta.$$
without the trade-off between the summation grid η and the strike spacing λ.
    - We require $\eta\lambda = 2\pi/N$ under standard FFT.

Fractional FFT implementation

- Let $\mathbf{d} = D(\mathbf{f}, \zeta)$ denote the FRFT operation, with $D(\mathbf{f}) = D(\mathbf{f}, 2\pi/N)$ being the standard FFT as a special case.
- An N-point FRFT can be implemented by invoking three $2N$-point FFT procedures.
  - Define the following $2N$-point vectors:

$$\mathbf{y} = \left(\left(f_n e^{i\pi n^2 \zeta} \right)_{n=0}^{N-1}, (0)_{n=0}^{N-1} \right), \tag{2}$$

$$\mathbf{z} = \left(\left(e^{i\pi n^2 \zeta} \right)_{n=0}^{N-1}, \left(e^{i\pi (N-n)^2 \alpha} \right)_{n=0}^{N-1} \right). \tag{3}$$

  - The FRFT is given by,

$$D_k(\mathbf{h}, \zeta) = \left(e^{i\pi k^2 \zeta} \right)_{k=0}^{N-1} \odot D_k^{-1} \left(D_j(\mathbf{y}) \odot D_j(\mathbf{z}) \right), \tag{4}$$

  where $D_k^{-1}(\cdot)$ denotes the inverse FFT operation and $\odot$ denotes element-by-element vector multiplication.

Fractional FFT implementation

- Due to the multiple application of the FFT operations, an N-point FRFT procedure demands a similar number of elementary operations as a $4N$-point FFT procedure.
- Given the free choices on λ and η, FRFT can be applied more efficiently. Using a smaller N with FRFT can achieve the same option pricing accuracy as using a much larger N with FFT.
- The accuracy improvement is larger when we have a better understanding of the model and model parameters so that we can set the boundaries more tightly.
- Caveat: The more freedom also asks for more discretion and caution in applying this method to generate robust results in all situations. This concern becomes especially important for model estimation, during which the trial model parameters can vary greatly.
- Reference: Chourdakis, 2005, Option pricing using fractional FFT, JCF, 8(2).

IV. Fourier-cosine series expansions

Fang & Oosterlee, A novel pricing method for European options based on Fourier-cosine series expansions, 2008.

- Given a characteristic function $\phi(u)$, the density function can be numerically obtained via the Fourier-cosine series expansion,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\phi(u)du \approx \sum_{j=0}^{N-1} \delta_j \cos\left((x-a)u_j\right) V_j$$

  where $u_j = \frac{j\pi}{b-a}$, $V_j = \frac{2}{b-a}\text{Re}\left[\phi(u_j)e^{iu_j a}\right]$, and $[a, b]$ denotes a truncation of the return range. Choosing the range to be ± 10 standard deviation away from the mean seems to work well: $b, a = \mu \pm 10\sigma$.

- Applying the expansion to the option valuation, we have

$$C(K,t) \approx Ke^{-rt}\sum_{j=0}^{N-1} \delta_j \text{Re}\left[\phi_s\left(u_j\right) e^{-iu_j(k+a)} U_j\right]$$

  where $U_j = \frac{2}{b-a}\left(\chi_j(0,b) - \psi_j(0,b)\right)$ with

$$\begin{aligned}
\chi_j(c,d) &= \frac{1}{1+u_j^2}\left[\cos((d-a)u_j)e^d - \cos((c-a)u_j)e^c + u_j\sin((d-a)u_j)e^d - u_j\sin((c-a)u_j)e^c\right], \\
\psi_j(c,d) &= \begin{cases} \left[\sin((d-a)u_j) - \sin((c-a)u_j)\right]/u_j & j \neq 0 \\ (d-c) & j = 0 \end{cases}
\end{aligned}$$

- Works well. Some constraints on how $[a, b]$ are chosen.