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Full fuzzy linear systems of the form Ax+b=Cx+d

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Abstract: This paper mainly intends to discuss the solution of the full fuzzy linear systems (FFLS) Ax+b=Cx+d, where A and C are fuzzy matrices, b and d are fuzzy vectors. Ming Ma et al. introduced a new fuzzy arithmetic based on parametric form of fuzzy numbers, which we apply it for our purpose.

Keywords: Fuzzy number, Full fuzzy linear systems, Fuzzy arithmetic, Parametric epresentation

1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [16], Dubois and Prade [7]. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [5] to control haotic systems [8,11], fuzzy metric spaces [14], fuzzy differential equations [3], fuzzy linear systems [1,2].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Friedman et al. [9] introduced a general model for solving a fuzzy $n\times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n\times n$ linear system by a crisp $2n\times 2n$ linear system and studied duality in fuzzy linear systems Ax=Bx+y where A , B are real $n\times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers and the constant y is vector consisting of n fuzzy numbers, in [10]. In [1,2] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Wang et al. [15] presented an iterative algorithm for solving dual linear system of the form x=Ax+u, where A is real $n\times n$ matrix, the unknown vector x and the constant u are all vectors consisting of fuzzy numbers and abbasbandy [4] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form Ax+f=Bx+c, where A, B are real $m\times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers and the onstant f, c are vectors consisting of m fuzzy numbers. Recently, Muzziloi et al. [13] considered fully fuzzy linear systems of the form

$A_1x+b_1=A_2x+b_2$ square matrices of fuzzy coefficients and b_1,b_2 fuzzy number vectors and Dehghan et al. [6] considered fully fuzzy linear systems of the form Ax=b where A and b are a fuzzy matrices and a fuzzy vector, respectively and them discussed the iterative solution of fully fuzzy linear systems.

In this paper, we are finding the solution of a fully fuzzy linear system of the form Ax+b=Cx+d based on a new arithmetic calculation [12], with A, C square matrices of fuzzy coefficients and b, d fuzzy number vectors and the unknown vector x is vector consisting of n fuzzy numbers. In Section 2, we recall some fundamental results on fuzzy numbers. The proposed model for solving the system Ax+b=Cx+d are discussed in Section3. Numerical examples are given in Section 4 followed by a discussion and concluding in Section 5.

2. Preliminaries

The parametric form of an arbitrary fuzzy number is given in [12] as follows. A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0\le r\le 1$ which satisfy the following requirements:

1. $\underline{u}$ is a bounded left continuous non-decreasing function over [0,1],
2. $\overline{u}$ is a bounded left continuous non-increasing function over [0,1],
3. $\underline{u}(r)\le\overline{u}(r), 0\le r\le 1$.

For arbitrary fuzzy numbers $x=(\underline{x}(r),\overline{x}(r)),\ y=(\underline{y}(r),\overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [15]

(a) $x=y$ if and only if $\underline{x}(r)=\underline{y}(r)$ and $\overline{x}(r)=\overline{y}(r)$,

(b) $x+y=(\underline{x}(r)+\underline{y}(r),\overline{x}(r)+\overline{y}(r))$,

(c) $kx=\begin{cases}(k\underline{x},k\overline{x}), & k\ge 0,\\ (k\overline{x},k\underline{x}), & k<0.\end{cases}$

The collection of all the fuzzy numbers with addition and scalar multiplication as defined by above equations is denoted by E which is a complete metric space with Hausdorff distance.

A popular fuzzy number is the triangular fuzzy number u=(a,b,c) where the membership function is

$$\mu_u(x)=\begin{cases}\frac{x-a}{b-a} & a\le x\le b,\\ \frac{c-x}{c-b} & b\le x\le c,\\ 0 & otherwise.\end{cases}$$

Also, the parametric form of the triangular fuzzy number u=(a,b,c) is

$$\underline{u}(r)=(b-a)r+a,\quad \overline{u}(r)=c-(c-b)r.$$

Definition 1. We introduce a lattice L as $L=\{h|h:[0,1]\to[0,\infty)\}$ is non-decreasing and left continuous .

The order in L is the natural order defined by $h\le g$ if and only if $h(r)\le g(r)$ for all $r\in[0,1]$. It is easy to show that

$$[h\vee g](r)=\max\{h(r),g(r)\},\qquad (1)$$
$$[h\wedge g](r)=\min\{h(r),g(r)\},$$

where $h\vee g$ and $h\wedge g$ are supremum and infimum of h and g.

Definition 2. [12] For arbitrary fuzzy number, the number $u=(\underline{u},\overline{u})$ the number $u_0=\frac{1}{2}(\underline{u}(1),\overline{u}(1))$ is said to be a location index number of u, and two non-decreasing left continuous functions

$$u_*=u_0-\underline{u},\quad u^*=\overline{u}-u_0,\qquad (2)$$

are called the left fuzziness index function and the right fuzziness index function, respectively.

According to Definition 2, every fuzzy number can be represented by (u_0,u_*,u^*). It is obvious that a fuzzy number u is symmetric if and only if $u_*=u^*$.

Definition 3. [12] For arbitrary fuzzy number (u_0,u_*,u^*) and (v_0,v_*,v^*) the four arithmetic operations are defined by

$$u\odot v=(u_0\odot v_0,u_*\vee v_*,u^*\vee v^*),$$

where $u\odot v$ is either of u+v, u-v, u.v and u/v. Here and after this, we operate all fuzzy arithmetic calculation based on this definition.

Theorem 1. For all fuzzy numbers u,v and w, we have

1. $u+v=v+u$,
2. $(u+v)+w=u+(v+w)$,

3. $u.v = v.u$,

4. $(u.v).w = u.(v.w)$,

5. $u.(v+w) = u.v + u.w$,

6. there exists $1 \in E$ such that $u.1 = u$ for all $u \in E$,

Proof. It is trivial by Definition 3.

Definition 4. A matrix $A=(a_{ij})$ is called a fuzzy matrix, if each element of A is a fuzzy number, we represent $A=(a_{ij})$ that

$$a_{ij} = ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*)$$

where $(a_{ij})_0$ is location index of a_{ij} and $(a_{ij})_*, (a_{ij})^*$ are left fuzziness index and right fuzziness index.

Definition 5. A vector $b=(b_i)$ is called a fuzzy vector, if each element of b is a fuzzy number, with new notation

$b = ((b_i)_0, (b_i)_*, (b_i)^*)$

where $(b_i)_0$ is location index of b_i, $(b_i)_*$ and $(b_i)^*$ are left fuzziness index and right fuzziness index.

3. The solution of Ax+b=Cx+d

Consider the $n \times n$ general dual full fuzzy linear system of equations:

$$\begin{cases} a_{11}\underline{x}_1 + \cdots + a_{1n}\underline{x}_n + b_1 = \\ c_{11}\underline{x}_1 + \cdots + c_{1n}\underline{x}_n + d_1, \\ a_{21}\underline{x}_1 + \cdots + a_{2n}\underline{x}_n + b_2 = \\ c_{21}\underline{x}_1 + \cdots + c_{2n}\underline{x}_n + d_2, \\ \vdots \\ a_{n1}\underline{x}_1 + \cdots + a_{nn}\underline{x}_n + b_n = \\ c_{n1}\underline{x}_1 + \cdots + c_{nn}\underline{x}_n + d_n, \end{cases} \tag{3}$$

and hence the matrix form of above equation is

$$Ax + b = Cx + d,$$

where the coefficient matrices

$A = (a_{ij}) = ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*)$

and

$$C = (c_{ij}) = ((c_{ij})_0, (c_{ij})_*, (c_{ij})^*),$$

are $n \times n$ fuzzy matrices,

$$b = (b_i) = ((b_i)_0, (b_i)_*, (b_i)^*),$$

$$d = (d_i) = ((d_i)_0, (d_i)_*, (d_i)^*)$$

and

$$x = (x_i) = ((x_i)_0, (x_i)_*, (x_i)^*),$$

$$1 \le i \le n$$

fuzzy vectors.

Let x_i be a solution of (3), that is

$$\sum_{j=1}^n a_{ij}x_j + b_i = \sum_{j=1}^n c_{ij}x_j + d_i, \quad i = 1, 2, \ldots, n, \tag{4}$$

therefore, by definitions 2 and 3 for $i = 1, \ldots, n$,

$$\sum_{j=1}^n ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*) .((x_j)_0, (x_j)_*, (x_j)^*) + ((b_i)_0, (b_i)_*, (b_i)^*)$$

$$= \sum_{j=1}^n ((c_{ij})_0, (c_{ij})_*, (c_{ij})^*) .((x_j)_0, (x_j)_*, (x_j)^*) + ((d_i)_0, (d_i)_*, (d_i)^*).$$

This implies for $i = 1, \ldots, n$

$$\sum_{j=1}^n ((a_{ij})_0(x_j)_0, \max\{(a_{ij})_*, (x_j)_*\}, \max\{(a_{ij})^*, (x_j)^*\}) + ((b_i)_0, (b_i)_*, (b_i)^*) =$$

$$\sum_{j=1}^n ((c_{ij})_0(x_j)_0, \max\{(c_{ij})_*, (x_j)_*\}, \max\{(c_{ij})^*, (x_j)^*\}) + ((d_i)_0, (d_i)_*, (d_i)^*).$$

Let

$$\begin{aligned} \max\{(a_{ij})_*, (x_j)_*\} &= (m_{ij})_*, \\ \max\{(a_{ij})^*, (x_j)^*\} &= (m_{ij})^*, \\ \max\{(c_{ij})_*, (x_j)_*\} &= (n_{ij})_*, \\ \max\{(c_{ij})^*, (x_j)^*\} &= (n_{ij})^*, \end{aligned} \tag{5}$$

then we have for $i = 1, \ldots, n$

$$\sum_{j=1}^n ((a_{ij})_0(x_j)_0 + (b_i)_0,$$

$$\max\{(m_{ij})_*, (b_i)_*\},$$

$$\max\{(m_{ij})^*,(b_i)^*\}) = \sum_{j=1}^{n}((c_{ij})_0(x_j)_0+(d_i)_0,$$

$$\max\{(n_{ij})_*,(d_i)_*\},$$

$$\max\{(n_{ij})^*,(d_i)^*\}).$$

Theorem 2. If for $(x_j)_0$ for $j=1,\ldots,n$ are the solution of the crisp linear system

$$\sum_{j=1}^{n}(a_{ij})_0(x_j)_0+(b_i)_0 = \sum_{j=1}^{n}(c_{ij})_0(x_j)_0+(d_i)_0,$$

$$i=1,2,\ldots,n,$$

and $(x_j)_*,(x_j)^*$ for $j=1,\ldots,n$ are obtained by

$$(x_j)_* = \max_{1\le i\le n}\{(a_{ij})_*,(b_i)_*,(c_{ij})_*,(d_i)_*\},$$

$$(x_j)^* = \max_{1\le i\le n}\{(a_{ij})^*,(b_i)^*,(c_{ij})^*,(d_i)^*\}.$$

Then the fuzzy vector $x=(x_j)=(\underline{x}_j,\overline{x}_j)$ obtained by

$$\underline{x_j}=(x_j)_0+(x_j)_*,\quad \overline{x_j}=(x_j)_0+(x_j)^*,$$

for $j=1,\ldots,n$ is a solution of (3).

Proof. The attention of the results, it is clear. $\square$

4 Numerical examples and applications

Example 1. Consider the 2×2 fully fuzzylinear system

$$\begin{cases}(1,2,3)x_1+(4,6,9)x_2+(1,3,4)\\ =(0,1,3)x_1+(5,6,8)x_2+(0,1,7),\\ (5,6,8)x_1+(3,5,6)x_2+(0,7,8)\\ =(1,4,5)x_1+(2,3,4)x_2+(8,9,12).\end{cases}$$

By simple calculations of the new arithmetic, we have following system for finding location index number of x_1 and x_2:

$$\begin{cases}2(x_1)_0+6(x_2)_0+3\\ =(x_1)_0+6(x_2)_0+1,\\ 6(x_1)_0+5(x_2)_0+7\\ =4(x_1)_0+3(x_2)_0+9,\end{cases}$$

therefore, we have

$$(x_1)_0=-2,\qquad (x_2)_0=3.$$

We obtain left fuzziness index and right fuzziness index function x_1 and x_2 by Theorem 2:

$$(x_1)_*=1-r,\ (x_2)_*=2-2r,$$

$$(x_1)^*=1-r,\ (x_2)^*=1-r.$$

The parametric form of x_1 and x_2 are the following form:

$$\underline{x}_1(r)=r-3,\ \underline{x}_2(r)=2r+1,$$

$$\overline{x}_1(r)=-1-r,\ \overline{x}_2(r)=4-r.$$

Example 2. For production of a high uality chemical compound, we need about 0.4 ((0.3,0.4,0.5)) kg poly ethylene high density (PEHD) and about 0.3 ((0.1,0.3,0.45)) kg poly ethylene low density(PELD) and from poly propylen (PP), we need exactly 0.267 kg which its price is about 3 ((1.217,3,4.775)) dollar. Now from same chemical compound with lower quality with same cost so the roducts would have higher expansion. We need from the PEHD about 0.5 ((0.3,0.5,0.8)) kg and from PELD about 0.4 ((0.2,0.4,0.5)) kg and from PP exactly 0.1 kg which its price is about 3 ((1,3,3.5)) dollar and for production of second high quality chemical compound, we need about 0.2 ((0.15,0.2,0.3)) kg PEHD and about 0.7 ((0.6,0.7,0.95)) kg PELD and from poly estyrene (PE), we need exactly 0.1 kg which its price is about 5 ((2.625,5,6.75)) dollar. Now from same chemical mpound with lower quality with same cost so the products would have higher expansion. We need from the PEHD about 0.3 ((0.2,0.3,0.5)) kg and from PELD about 0.3 ((0.15,0.3,0.4)) kg and from PE exactly 0.3 kg which its price is about 5 ((4,5,7)) dollar. For obtaining this two chemical compound with different qualities how much would be about the price of PEHD and PELD ?

Let x_1 and x_2 show the price of PEHD and PELD. Together, these equations form a fully fuzzy linear system

$$\begin{cases} (0.3,0.4,0.5)x_1+(0.1,0.3,0.45)x_2 \\ +(0.325,0.8,1.275)= \\ (0.3,0.5,0.8)x_1+(0.2,0.4,0.5)x_2 \\ +(0.1,0.3,0.35), \\ (0.15,0.2,0.3)x_1+(0.6,0.7,0.95)x_2 \\ +(0.2625,0.5,0.67)= \\ (0.2,0.3,0.5)x_1+(0.15,0.3,0.4)x_2 \\ +(1.2,1.5,2.1). \end{cases}$$

By simple calculations of the new arithmetic, we have following system for finding location index number of x_1 and x_2:

$$\begin{cases} 0.4(x_1)_0+0.3(x_2)_0+0.8 \\ =0.5(x_1)_0+0.4(x_2)_0+0.3, \\ 0.2(x_1)_0+0.7(x_2)_0+0.5 \\ =0.3(x_1)_0+0.3(x_2)_0+1.5, \end{cases}$$

therefore, we have

$$(x_1)_0=2, \quad (x_2)_0=3.$$

We obtain left fuzziness index and right fuzziness index function x_1 and x_2 by Theorem 2:

$$(x_1)_*=0.5-0.5r, \quad (x_2)_*=0.75-0.75r,$$
$$(x_1)^*=0.5-0.5r, \quad (x_2)^*=0.5-0.5r.$$

The parametric form of x_1 and x_2 are the following form:

$$\underline{x}_1(r)=1.5+0.5r, \ \underline{x}_2(r)=2.25+0.75r,$$
$$\overline{x}_1(r)=2.5-0.5r, \overline{x}_2(r)=3.5-0.5r.$$

5 Summary and conclusions

In this paper, we propose a general model for solving a system of n fuzzy linear equations with n variables. The original system with coefficient matrices A and C are fuzzy matrices, b and d are fuzzy vectors. For finding the solution of general dual fully fuzzy linear system, we used a new arithmetic. Also, a condition for the existence of a fuzzy solution to the general dual fully fuzzy linear system, is presented. Initially researchers in the [1,2,4,9,10] assumed the coefficient matrices were crisp and in the [6] researchers assumed the solution is positive fuzzy vector. This is too restrictive for applications, but in this paper we don't have these restrictions.

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