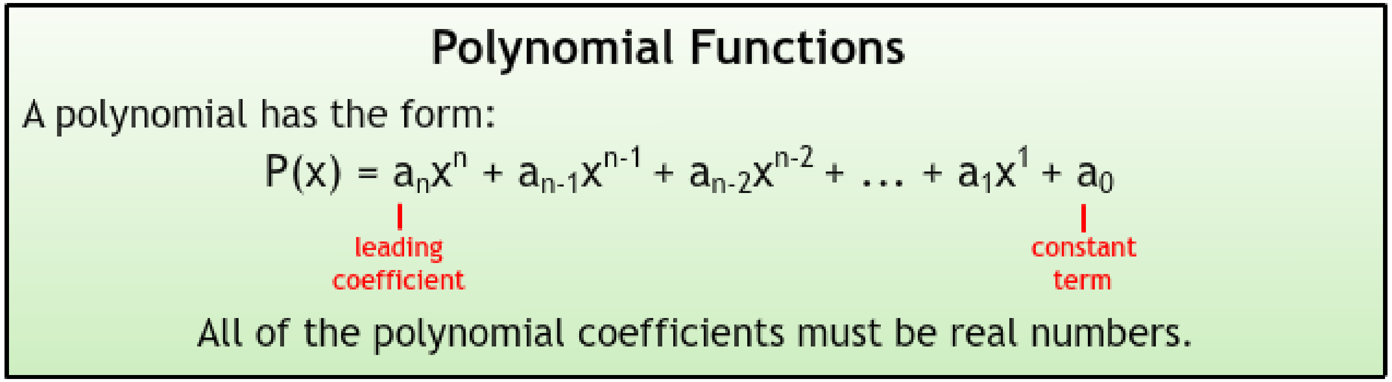
Functions and Relations

RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

Functions and Relations

RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).



Polynomial Functions

A polynomial has the form:

$$P(x) = \underset{\text{leading coefficient}}{a_n} x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + \underset{\text{constant term}}{a_0}$$

All of the polynomial coefficients must be real numbers.

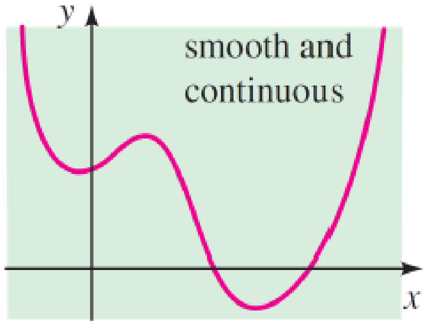
Note:

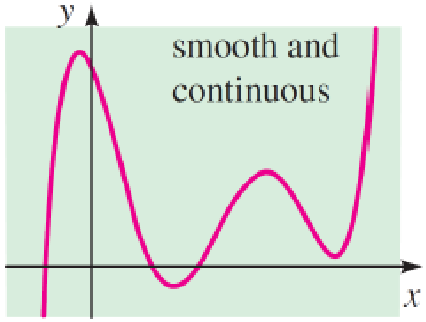
- n is a whole number
- x is a variable
- the coefficients a_n to a_0 are real numbers

Each exponent of a polynomial must be a whole number defined as {0, 1, 2, 3, ...}.

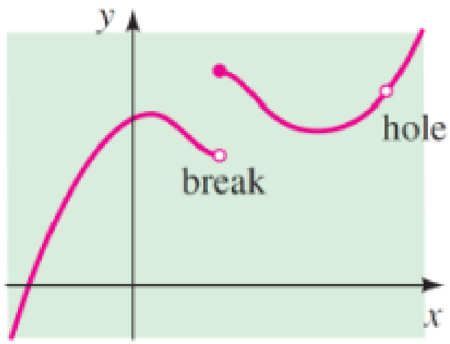
The Shape of a Polynomial

The graph of a polynomial function is continuous. This means that you can draw the complete graph without lifting your pencil from the paper.

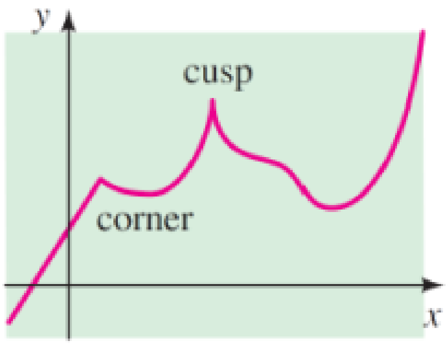




A polynomial has no breaks or sharp corners.



Not the graph of a polynomial function



Not the graph of a polynomial function

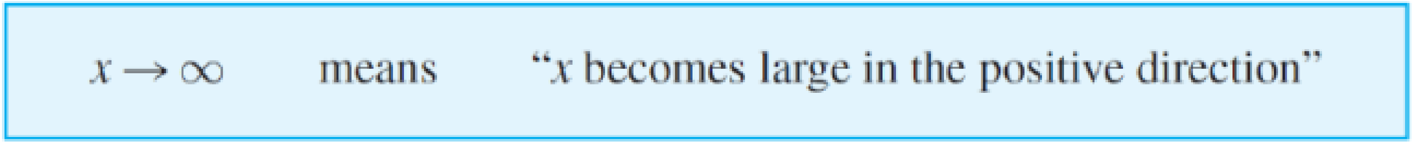
Polynomial Functions

No absolute value shapes

No reciprocal shapes

End Behavior of Polynomials

The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we can use the following notation:



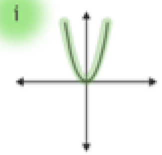
$x \to \infty$ means "x becomes large in the positive direction"

$x \to -\infty$ means "x becomes large in the negative direction"

End Behavior of Polynomials

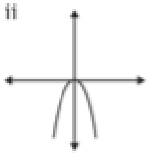
Even-degree polynomials with a **positive leading coefficient** have a trendline that matches an upright parabola.
End behaviour: The graph starts in the upper-left quadrant and ends in the upper-right quadrant.

i



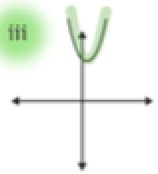
$f(x) = x^2$
quadratic

ii



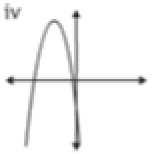
$f(x) = -x^2$
quadratic

iii



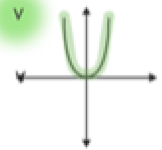
$f(x) = x^2 - x + 6$
quadratic

iv



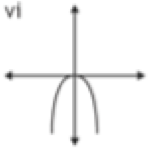
$f(x) = -x^2 - 8x - 7$
quadratic

v



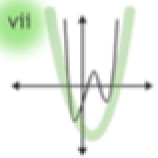
$f(x) = x^4$
quartic

vi



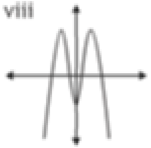
$f(x) = -x^4$
quartic

vii



$f(x) = x^4 - 4x^3 + x^2 + 7x - 3$
quartic

viii

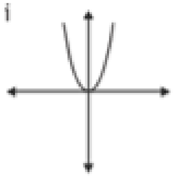


$f(x) = -x^4 + 7x^2 - 5$
quartic

Even-degree polynomials with a negative leading coefficient have a trendline that matches an upside-down parabola.

End behaviour: The graph starts in the lower-left quadrant and ends in the lower-right quadrant.

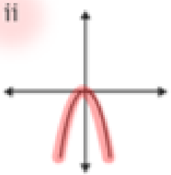
i



$f(x) = x^2$

quadratic

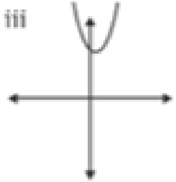
ii



$f(x) = -x^2$

quadratic

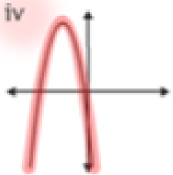
iii



$f(x) = x^2 - x + 6$

quadratic

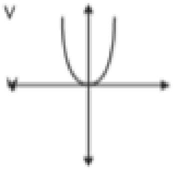
iv



$f(x) = -x^2 - 8x - 7$

quadratic

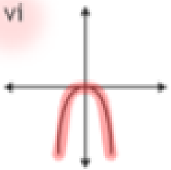
v



$f(x) = x^4$

quartic

vi



$f(x) = -x^4$

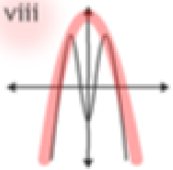
quartic

vii

$f(x) = x^4 - 4x^3 + x^2 + 7x - 3$

quartic

viii



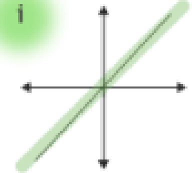
$f(x) = -x^4 + 7x^2 - 5$

quartic

Odd-degree polynomials with a positive leading coefficient have a trendline matching the line $y = x$.

End behaviour: The graph starts in the lower-left quadrant and ends in the upper-right quadrant.

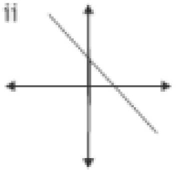
i



$f(x) = x$

linear

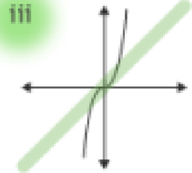
ii



$f(x) = -x + 4$

linear

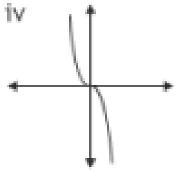
iii



$f(x) = x^3$

cubic

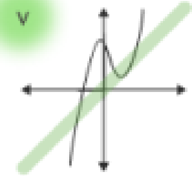
iv



$f(x) = -x^3$

cubic

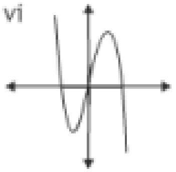
v



$f(x) = x^3 - 2x^2 - 2x + 6$

cubic

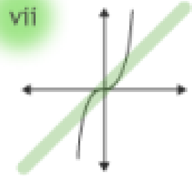
vi



$f(x) = -x^3 + 7x$

cubic

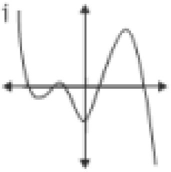
vii



$f(x) = x^5$

quintic

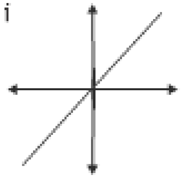
viii

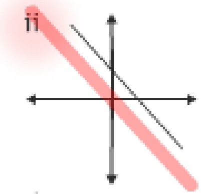


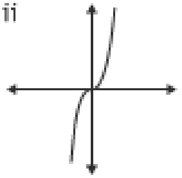
$f(x) = -x^5 - 4x^4 + 40x^3 + 160x^2 - 144x - 576$

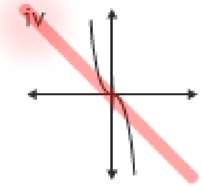
quintic

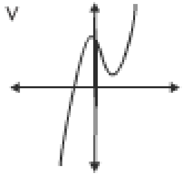
Odd-degree polynomials with a negative leading coefficient have a trendline matching the line $y = -x$.
End behaviour: The graph starts in the upper-left quadrant and ends in the lower-right quadrant.

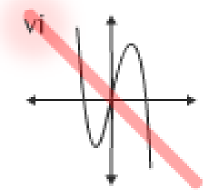
i

$f(x) = x$
linear

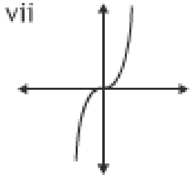
ii

$f(x) = -x + 4$
linear

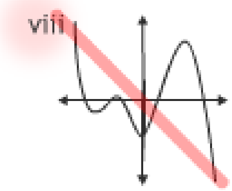
iii

$f(x) = x^3$
cubic

iv

$f(x) = -x^3$
cubic

v

$f(x) = x^3 - 2x^2 - 2x + 6$
cubic

vi

$f(x) = -x^3 + 7x$
cubic

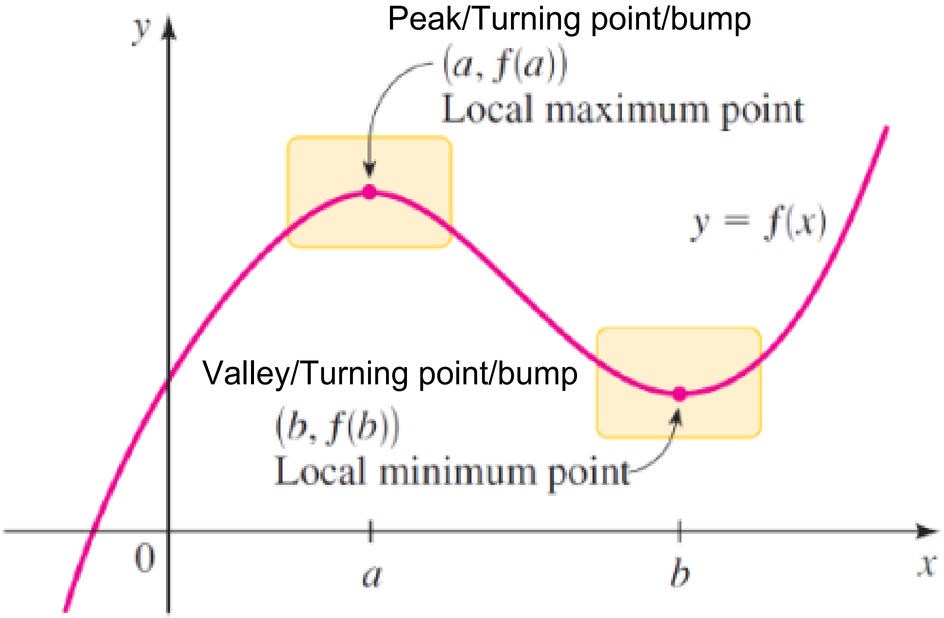
vii

$f(x) = x^5$
quintic

viii

$f(x) = -x^5 - 4x^4 + 40x^3 + 160x^2 - 144x - 576$
quintic

Peaks and Valleys

(Turning Points/BUMPS/Extreme Points)

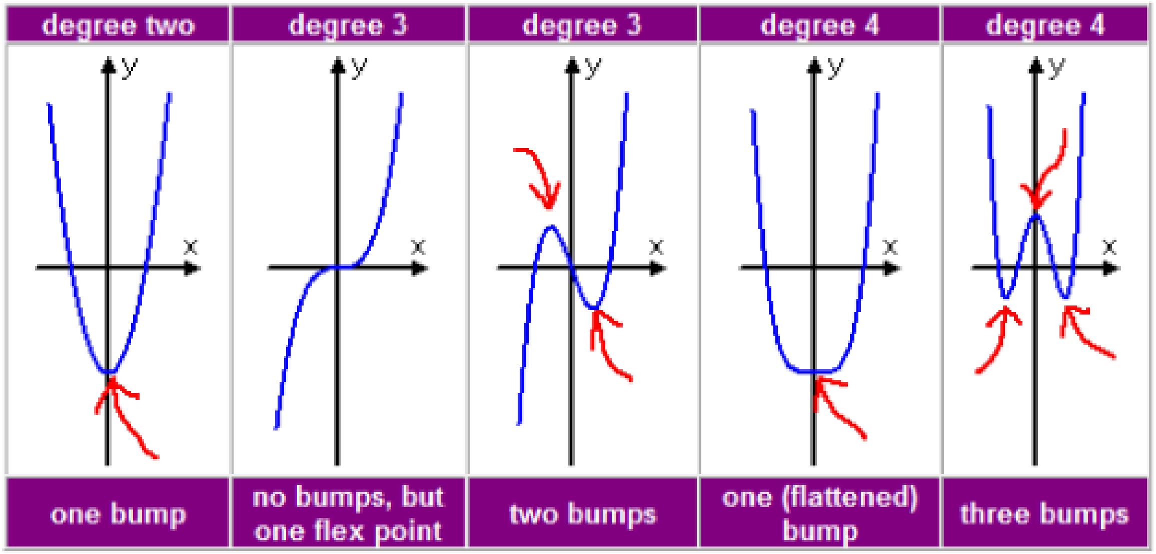
For degree n, a polynomial graph will have, at most, $n-1$ bumps.

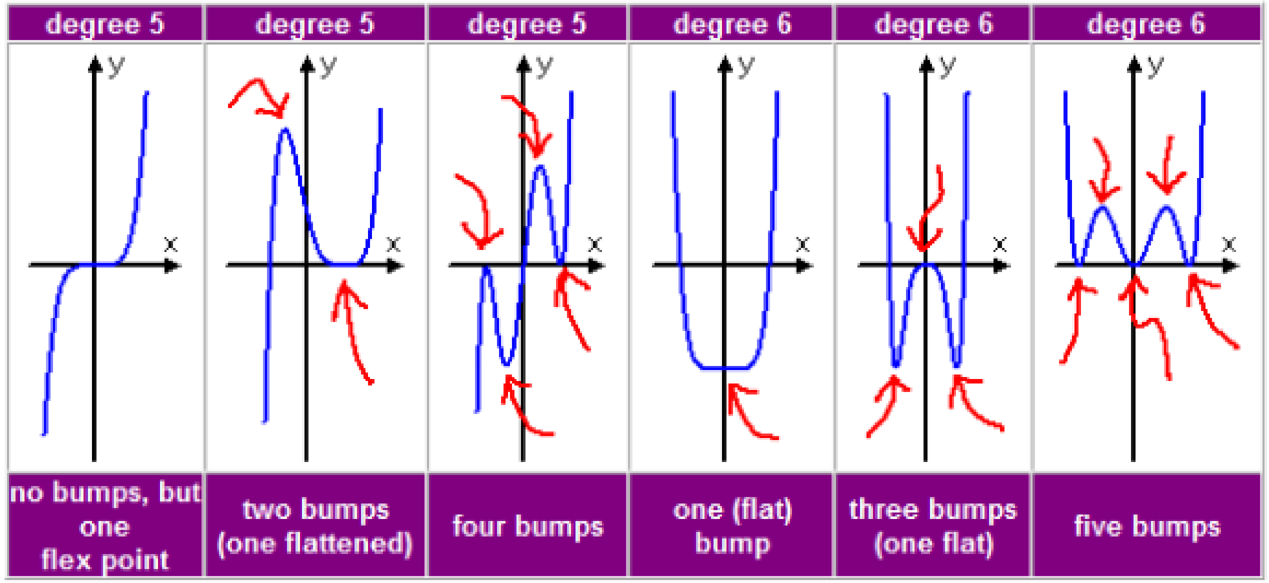


Peaks and Valleys

(Turning Points/BUMPS/Extreme Points)

For degree n, a polynomial graph will have, at most, $n-1$ peaks and valleys.

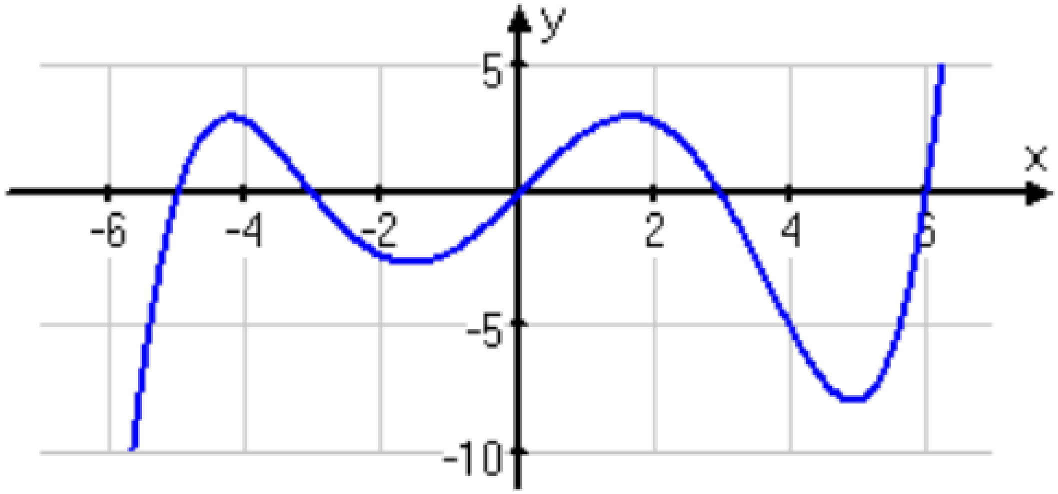




degree 5	degree 5	degree 5	degree 6	degree 6	degree 6
no bumps, but one flex point	two bumps (one flattened)	four bumps	one (flat) bump	three bumps (one flat)	five bumps

Sample Problem

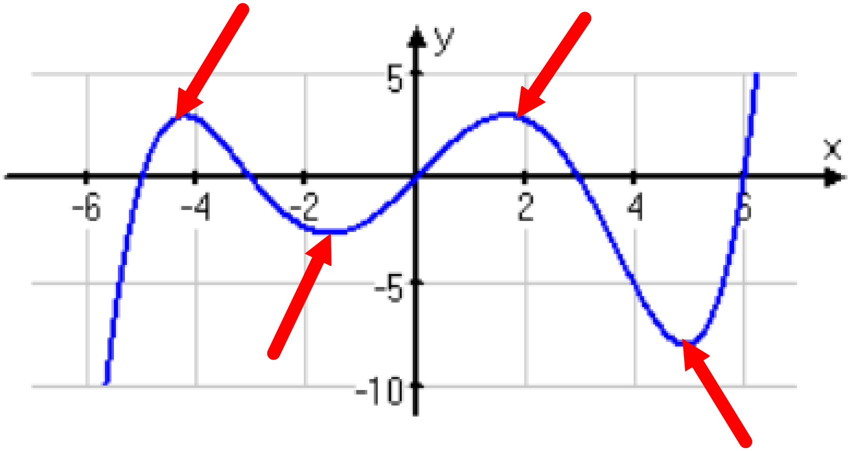
What is the minimum possible degree of the polynomial graphed below?



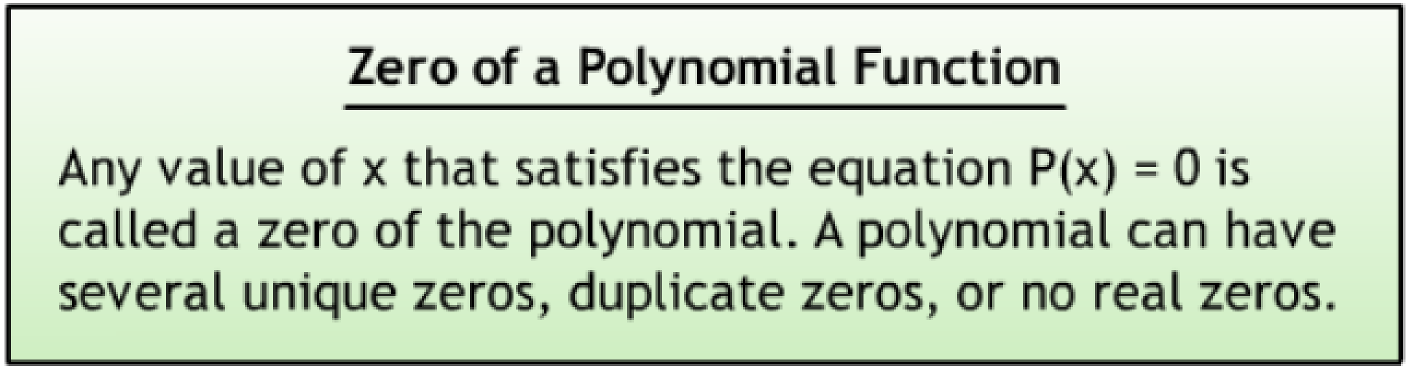
Answer Key

Since there are four bumps on the graph, and since the end-behavior says that this is an odd-degree polynomial, then the degree of the polynomial is 5, 7, or 9...

Therefore, the minimum possible degree is 5.



Zeros, Roots and X-Intercepts



Zero of a Polynomial Function

Any value of x that satisfies the equation P(x) = 0 is called a zero of the polynomial. A polynomial can have several unique zeros, duplicate zeros, or no real zeros.

Sample Problem

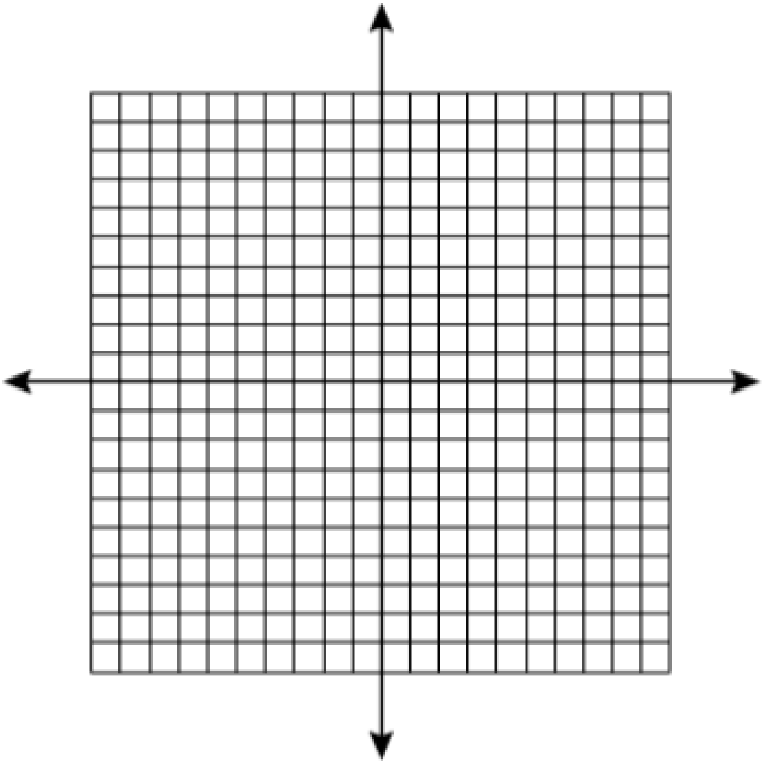
a) Determine if each given value is a zero of $P(x) = x^2 - 4x - 5$.

i) $x = -1$

ii) $x = 3$

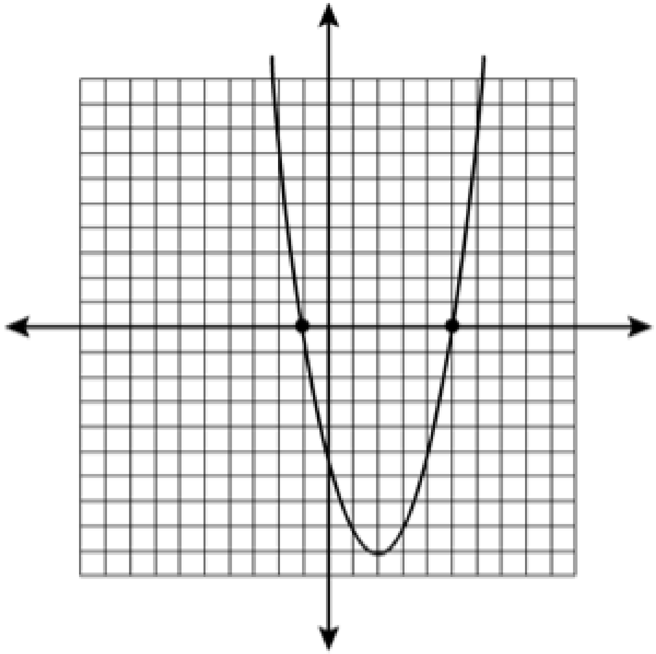
b) Find the zeros of $P(x) = x^2 - 4x - 5$ by solving for the roots of the related equation, $P(x) = 0$.

c) Use a graphing calculator to graph $P(x) = x^2 - 4x - 5$.

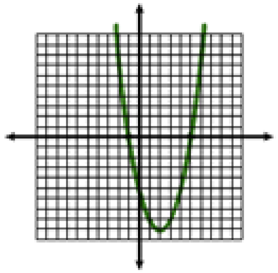


Answer Key

$$P(x) = x^2 - 4x - 5$$



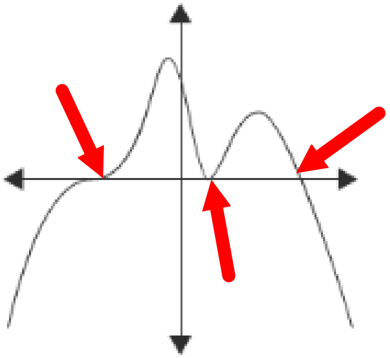
The zeros, roots, and x-intercepts of a polynomial all have the same numeric values. However, the term we use to describe them is context-sensitive.

Term Usage	Example
We use the term ***zero*** to describe a property of a function.	The polynomial $P(x) = x^2 - 4x - 5$ has ***zeros*** of -1 and 5.
We use the term ***root*** to describe a property of an equation.	The equation $x^2 - 4x - 5 = 0$ has ***roots*** of -1 and 5.
We use the term ***x-intercept*** to describe a property of a graph.	 The graph of $P(x) = x^2 - 4x - 5$ has ***x-intercepts*** of -1 and 5.

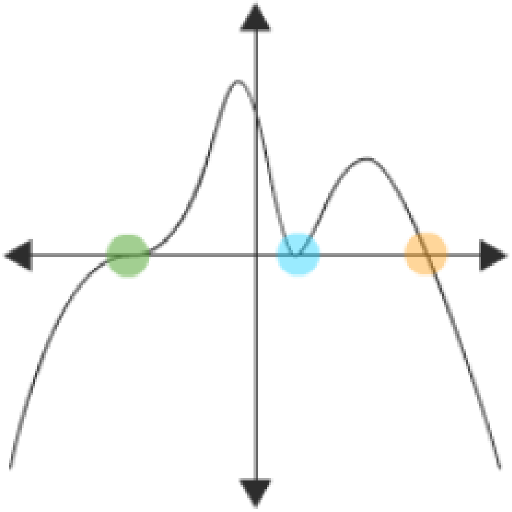
Multiplicity of a Zero

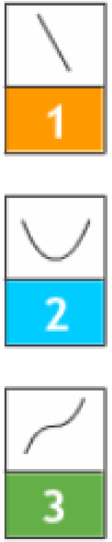
When we solve an equation of the form P(x) = 0, some of the roots may be duplicated.
The multiplicity of a root is how many times the root appears as a solution.

The multiplicity of a root gives an indication as to how the graph will behave near the x-intercept corresponding to the root.



We will be working with polynomial functions that have a multiplicity of 1, 2 or 3.





A root with a multiplicity of 1 will pass straight through the x-axis.

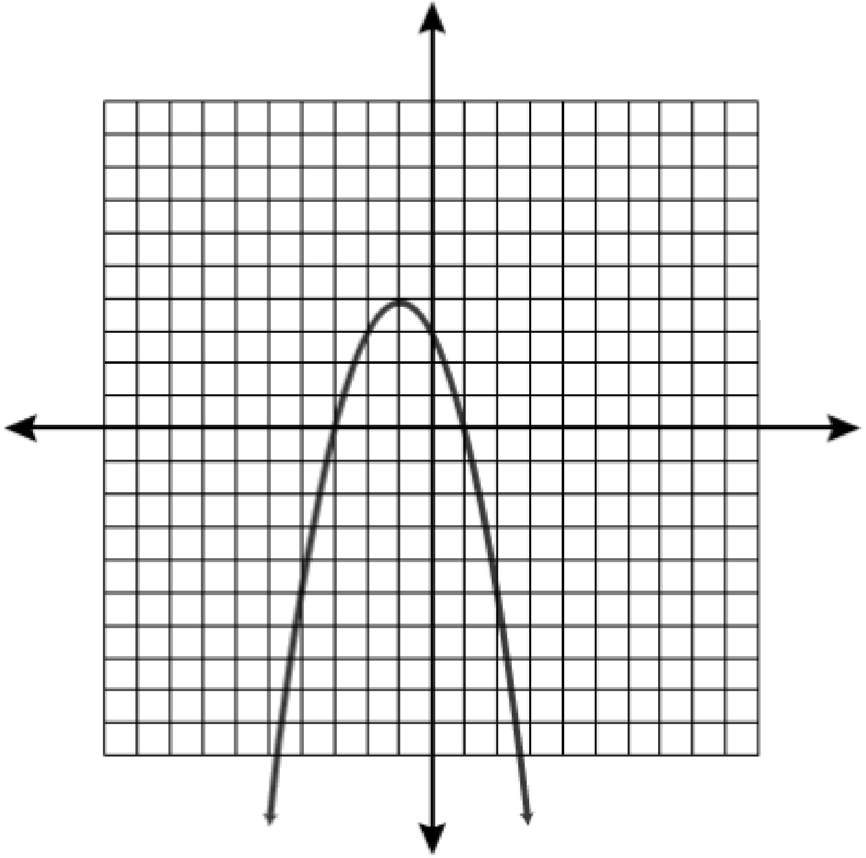
A root with a multiplicity of 2 will touch the x-axis, but not cross it.

A root with a multiplicity of 3 will have a cubic shape at the x-axis.

Check Your Understanding

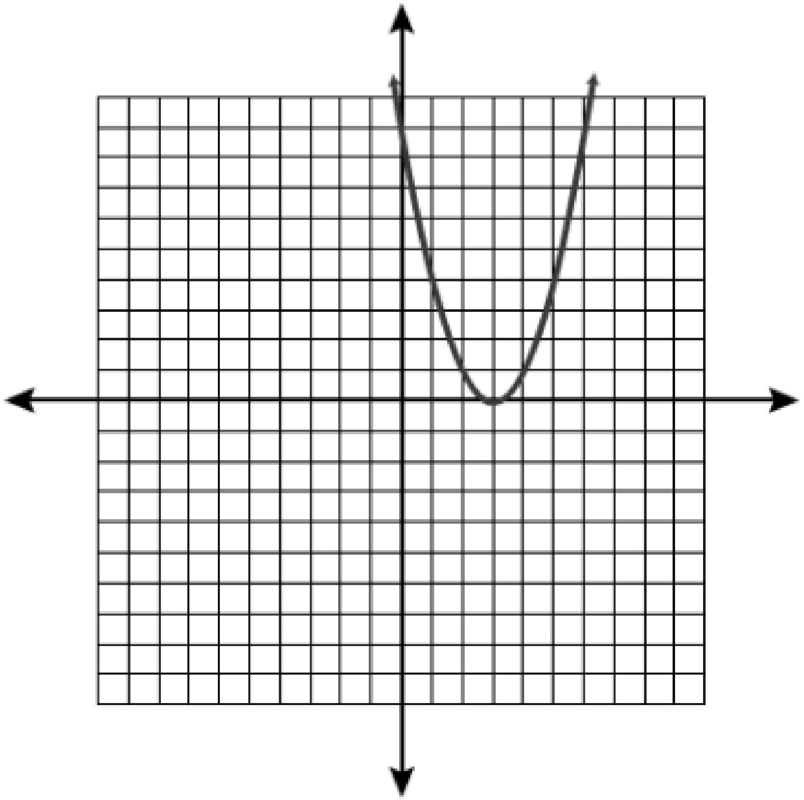
For the following graphs determine the zeros and state each zero's multiplicity.

a) P(x) = -(x + 3)(x - 1)



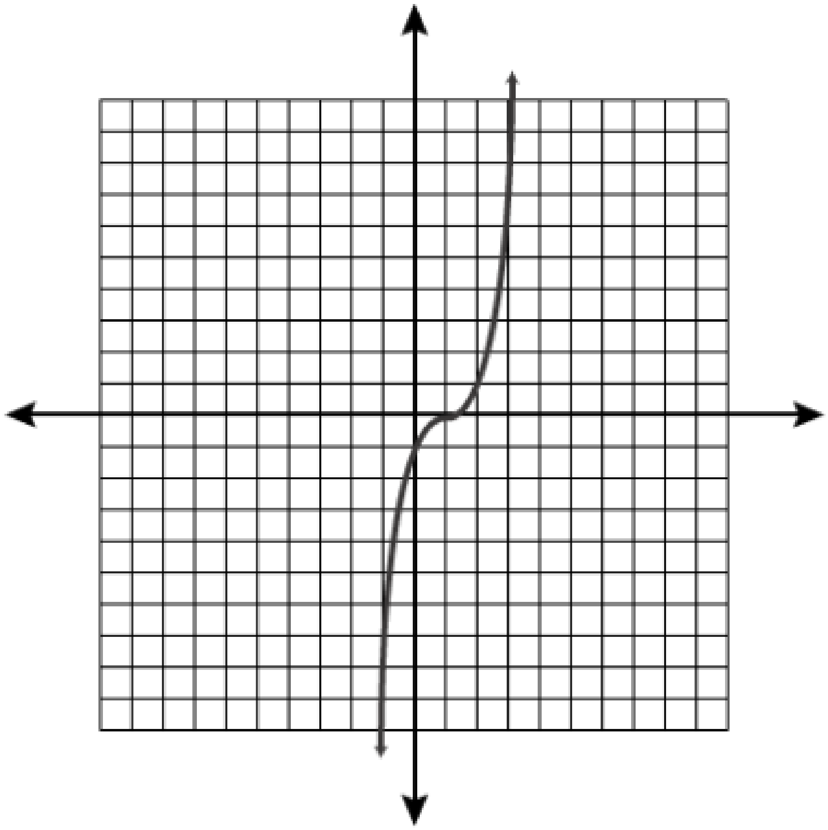
Zeros	Multiplicity

b) $P(x) = (x - 3)^2$



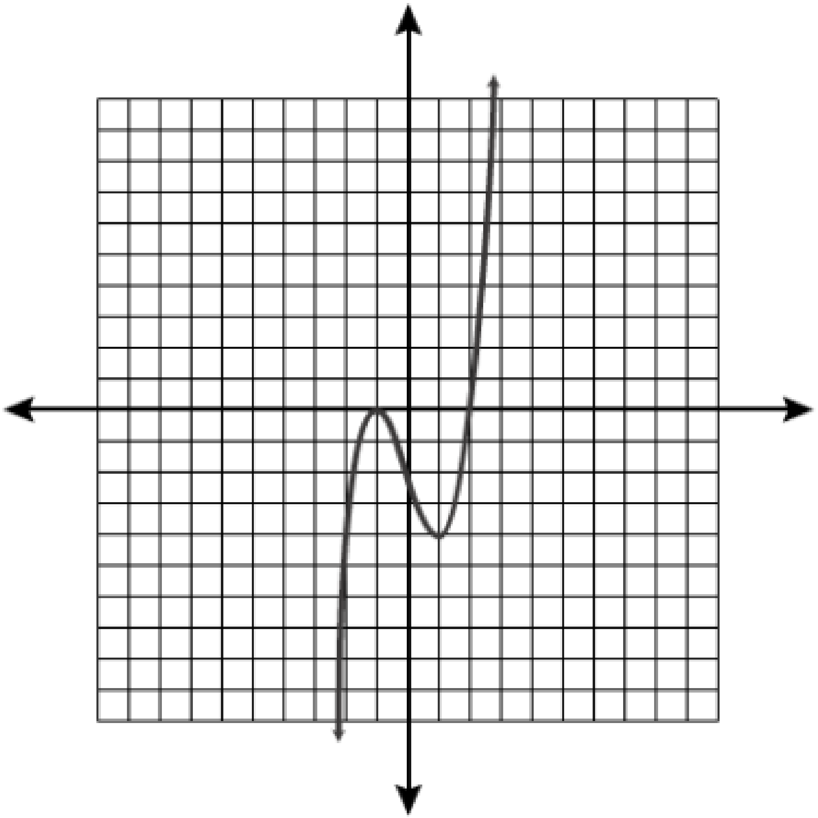
Zeros	Multiplicity

c) $P(x) = (x - 1)^3$



Zeros	Multiplicity

d) $P(x) = (x + 1)^2(x - 2)$



Zeros	Multiplicity

Sample Problems

For each polynomial function:

i) Find the zeros and their multiplicities
ii) Find the y-intercept
iii) Describe the end behaviour
iv) What other points are required to draw the graph accurately?

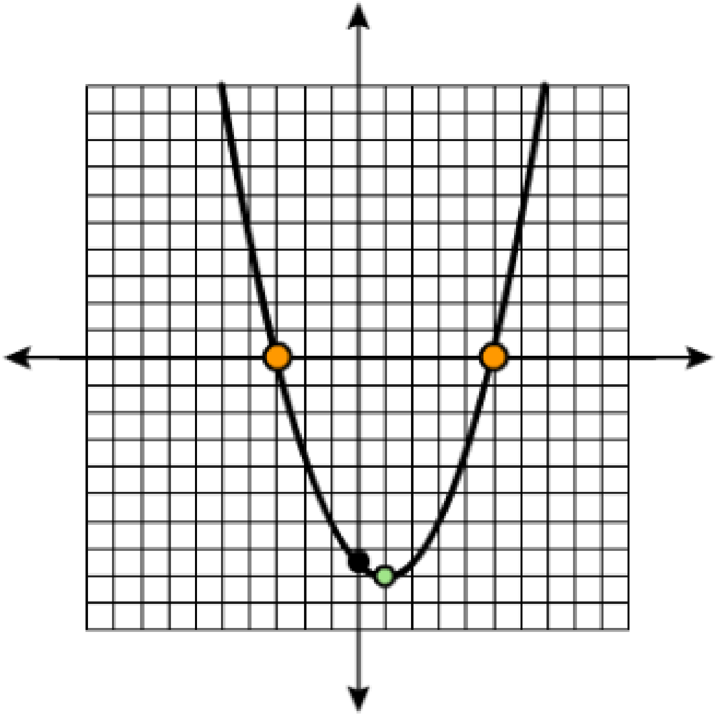
Use this information to sketch the graph.

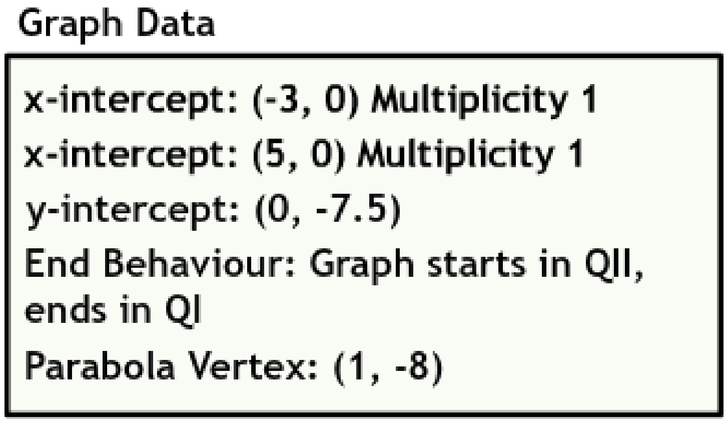
a) $P(x) = \frac{1}{2}(x - 5)(x + 3)$

b) $P(x) = -x^2(x + 1)$

Answer Key

a) $P(x) = \frac{1}{2}(x - 5)(x + 3)$





Graph Data

x-intercept: (-3, 0) Multiplicity 1

x-intercept: (5, 0) Multiplicity 1

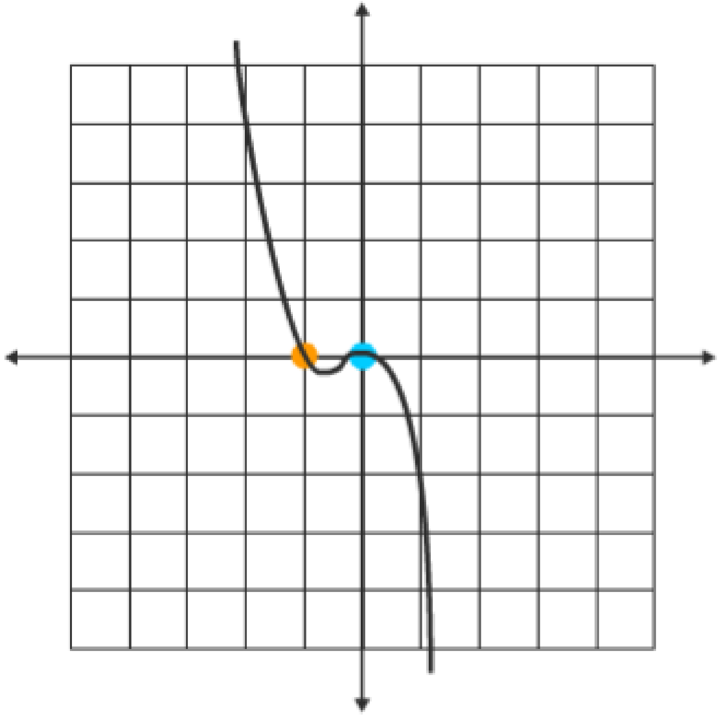
y-intercept: (0, -7.5)

End Behaviour: Graph starts in QII, ends in QI

Parabola Vertex: (1, -8)

Answer Key

b) $P(x) = -x^2(x + 1)$



Graph Data

x-intercept: (-1, 0), Multiplicity 1

x-intercept: (0, 0), Multiplicity 2

y-intercept: (0, 0)

End Behaviour: The graph starts in QII and ends in QIV.

Other: (-2, 4), (-0.67, -0.15), and (1, -2)

RF4 - Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5)

Sample Problem

Consider the polynomial function $f(x) = (x + 3)(x + 1)^2(x - 2)^3$.

Complete the chart and sketch the graph of $f(x)$.

Degree		
Leading Coefficient		
End Behavior	$x \to -\infty$	
	$x \to \infty$	
Zeroes	*Multiplicity*	*Description*
x-intercepts		
y-intercept		
Other Points		
Interval(s) where *f*(*x*) is positive		
Interval(s) where *f*(*x*) is negative		

FOOPLOT

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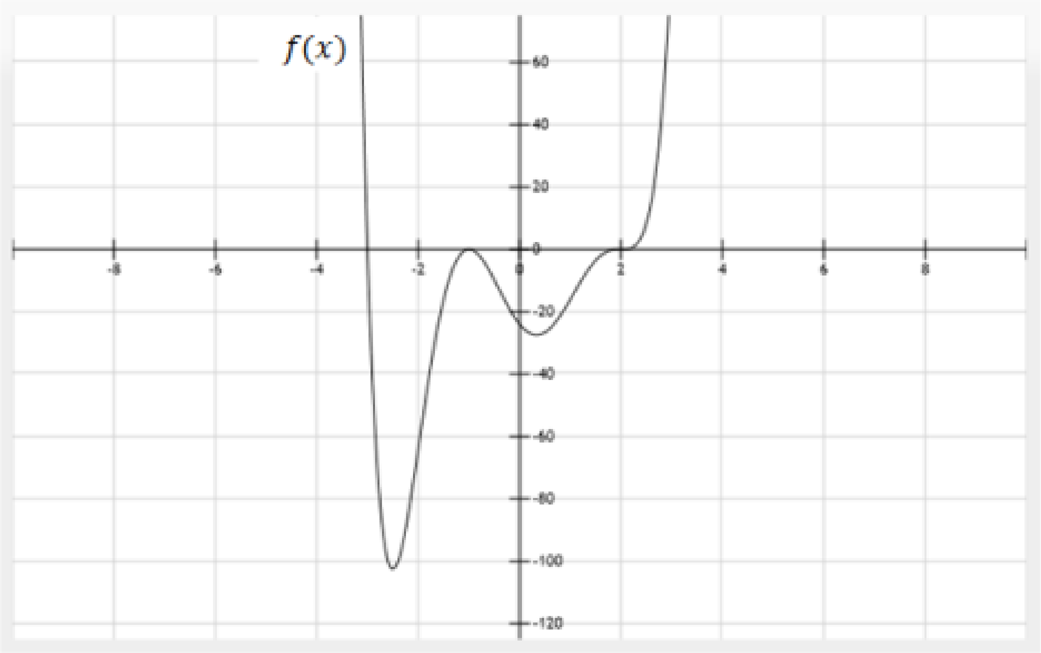


Check Your Understanding

For each polynomial function:

i) Find the zeros and their multiplicities.
ii) Find the y-intercept.
iii) Describe the end behaviour.
iv) What other points are required to draw the graph accurately?

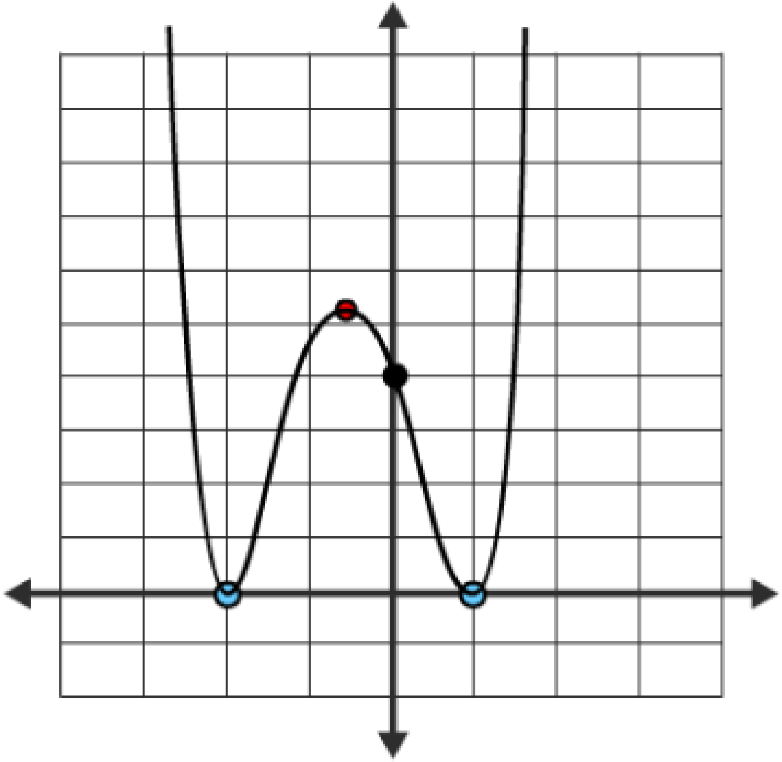
Use this information to sketch the graph.

a) $P(x) = (x - 1)^2(x + 2)^2$

b) $P(x) = x(x + 1)^3(x - 2)^2$

Answer Key

a) $P(x) = (x - 1)^2(x + 2)^2$



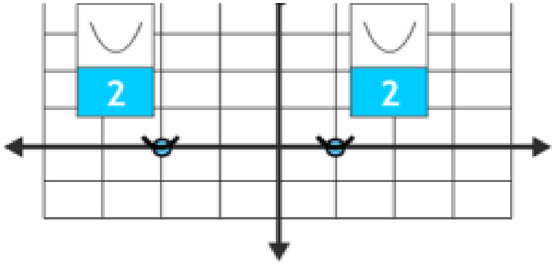
Graph Data

x-intercept: (-2, 0) Multiplicity 2
x-intercept: (1, 0) Multiplicity 2

y-intercept: (0, 4)

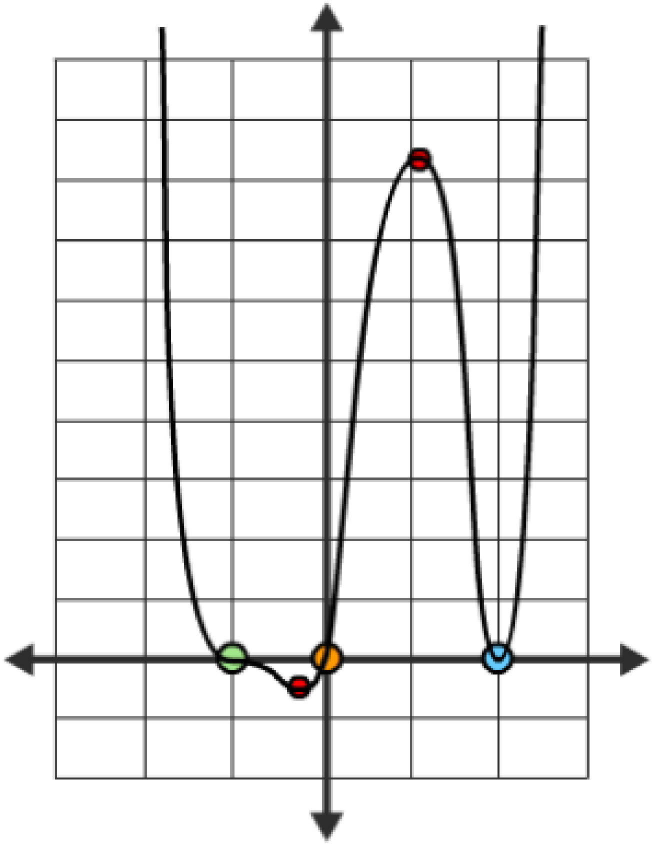
End Behaviour: Starts in QII, ends in QI

Other: (-3, 16), (-0.5, 5.0625), (2, 16)



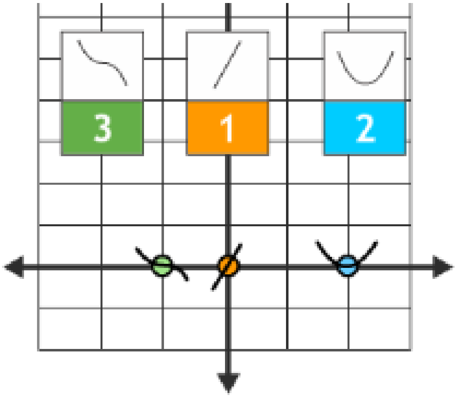
Answer Key

b) $P(x) = x(x + 1)^3(x - 2)^2$

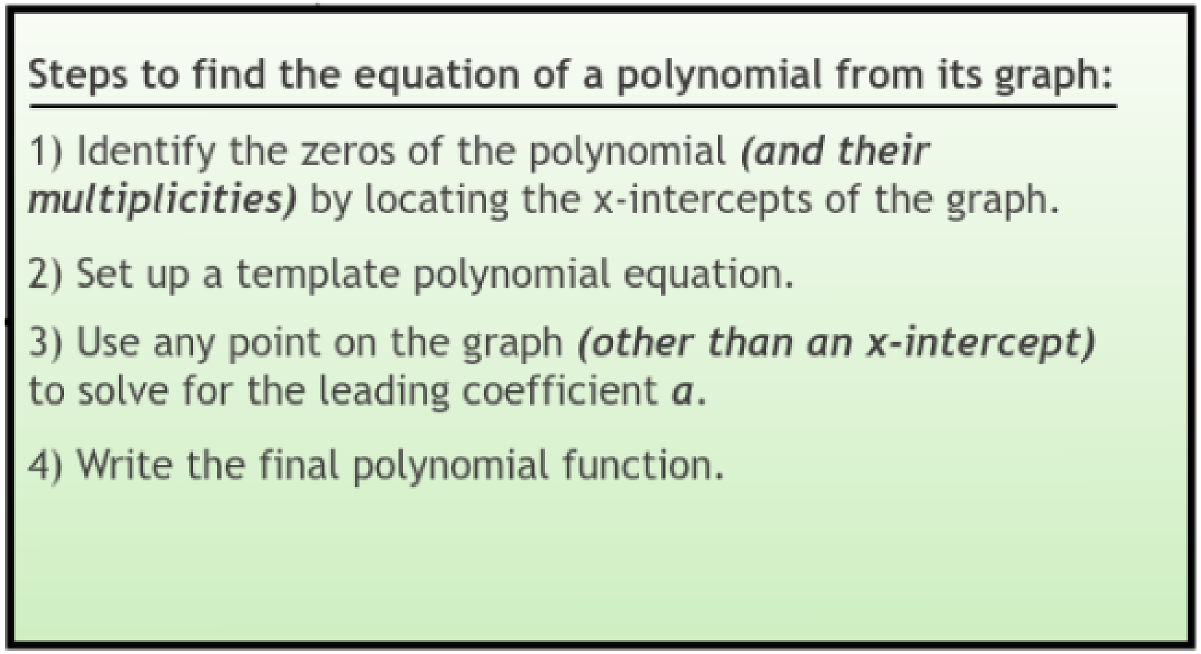


Graph Data

x-intercept: (-1, 0) Multiplicity 3
x-intercept: (0, 0) Multiplicity 1
x-intercept: (2, 0) Multiplicity 2
y-intercept: (0, 0)
End behaviour: Starts in QI, ends in QII
Other: (-2, 32), (-0.3, -0.5), (1.1, 8.3), (3, 192)



Writing an Equation from a Graph



Steps to find the equation of a polynomial from its graph:

1) Identify the zeros of the polynomial ***(and their multiplicities)*** by locating the x-intercepts of the graph.

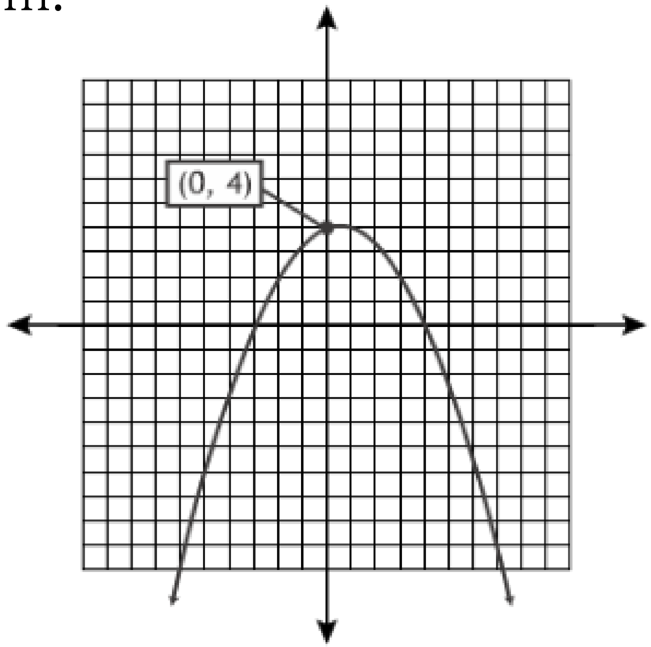
2) Set up a template polynomial equation.

3) Use any point on the graph ***(other than an x-intercept)*** to solve for the leading coefficient ***a***.

4) Write the final polynomial function.

Sample Problem

Determine the polynomial function corresponding to the graph. You may leave your answer in factored form.



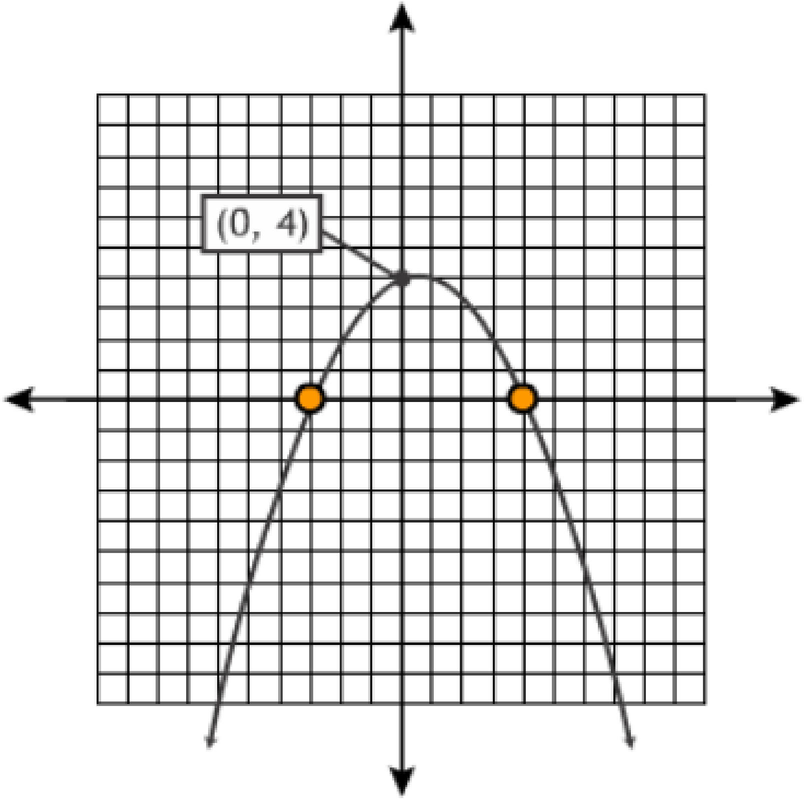
1) Identify the zeros of the polynomial and their multiplicities.

2) Set up a template polynomial equation.

3) Use a point on the graph to solve for the leading coefficient *a*.

4) Write the final polynomial function.

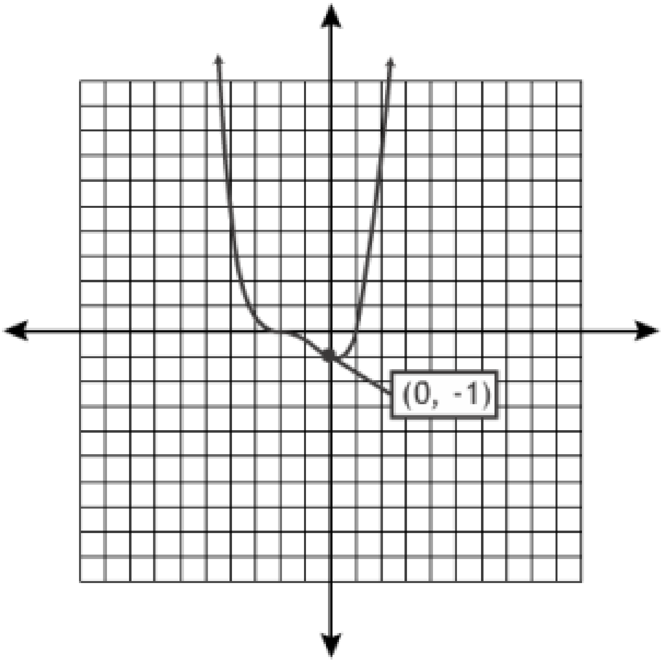
Answer Key



$$P(x) = -\frac{1}{3}(x + 3)(x - 4)$$ ✓

Sample Problem

Determine the polynomial function corresponding to the graph. You may leave your answer in factored form.



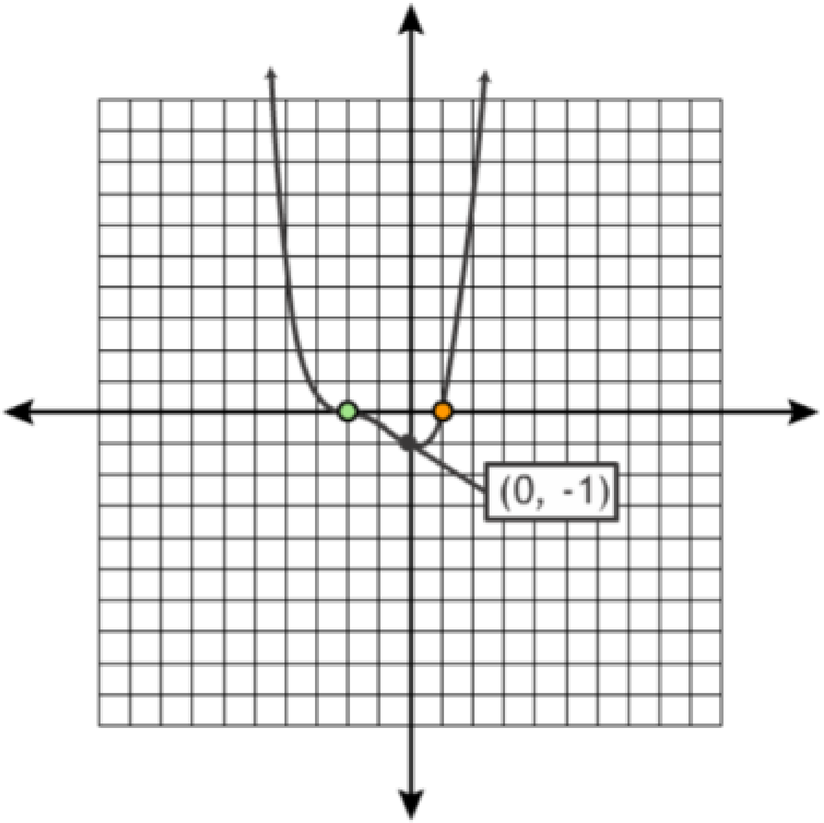
1) Identify the zeros of the polynomial and their multiplicities.

2) Set up a template polynomial equation.

3) Use a point on the graph to solve for the leading coefficient ***a*.**

4) Write the final polynomial function.

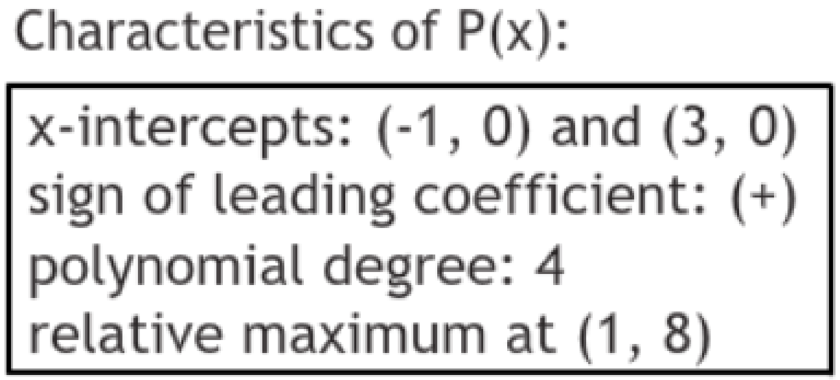
Answer Key



$P(x) = \frac{1}{8}(x + 2)^3(x - 1)$ ✓

Sample Problem

Given the characteristics of a polynomial,
draw the graph and derive the actual function.



Characteristics of P(x):

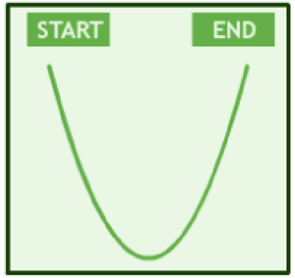
x-intercepts: (-1, 0) and (3, 0)
sign of leading coefficient: (+)
polynomial degree: 4
relative maximum at (1, 8)

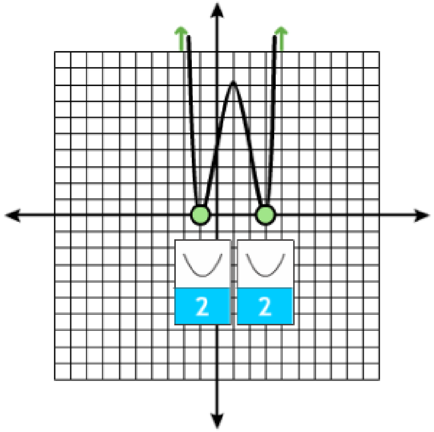
Answer Key

Characteristics of P(x):

x-intercepts: (-1, 0) and (3, 0)
sign of leading coefficient: (+)
polynomial degree: 4
relative maximum at (1, 8)

The trendline of a fourth-degree polynomial with a positive leading coefficient matches an upright parabola.





Find the Polynomial

$P(x) = a(x + 1)^2(x - 3)^2$

$8 = a(1 + 1)^2(1 - 3)^2$

$8 = a(2)^2(-2)^2$

$8 = 16a$

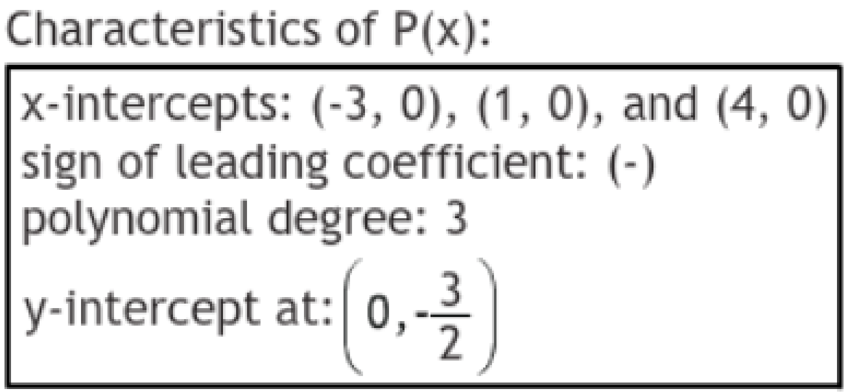
$a = \frac{8}{16}$

$a = \frac{1}{2}$

$P(x) = \frac{1}{2}(x + 1)^2(x - 3)^2$ ✓

Sample Problem

Given the characteristics of a polynomial, draw the graph and derive the actual function.



Characteristics of P(x):

x-intercepts: (-3, 0), (1, 0), and (4, 0)
sign of leading coefficient: (-)
polynomial degree: 3
y-intercept at: $\left(0,-\frac{3}{2}\right)$

Answer Key

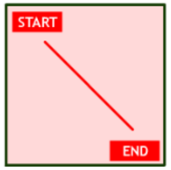
Characteristics of P(x):

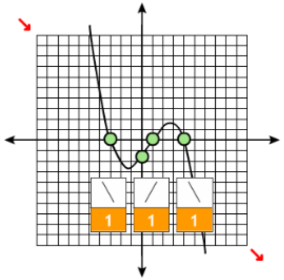
x-intercepts: (-3, 0), (1, 0), and (4, 0)

sign of leading coefficient: (-)
polynomial degree: 3

y-intercept at: $\left(0, -\frac{3}{2}\right)$

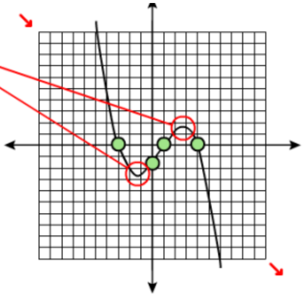
The trendline of a third-degree polynomial with a negative leading coefficient matches the line y = -x.





Note: The relative maximum and minimum are not required to find the function, so we can guess their position.

(We can always adjust the graph after finding the function)



Find the Polynomial

P(x) = a(x + 3)(x - 1)(x - 4)

$$-\frac{3}{2} = a(0+3)(0-1)(0-4)$$

$$-\frac{3}{2} = a(3)(-1)(-4)$$

$$-\frac{3}{2} = 12a$$

$$3 = -24a$$

$$a = -\frac{3}{24}$$

$$a = -\frac{1}{8}$$

$P(x) = -\frac{1}{8}(x + 3)(x - 1)(x - 4)$ ✓