Functions

After completing this section, students should be able to:

- Decide whether a relationship between input and output values is a function or not, based on an equation, a graph, or a table of values.
- Find the corresponding output value for a given input value for a function given in equation, graphical, or tabular form.
- Find the corresponding input value(s) for a given output value for a function given in equation, graphical, or tabular form.
- Find the domain and range of a function or relation based on a graph or table of values.
- Find the domains of functions given in equation form involving square roots and denominators. .

Definition. A function is correspondence between input numbers (x-values) and output numbers (y-value) that sends each input number (x-value) to exactly one output number (y-value).

Sometimes, a function is described with an equation.

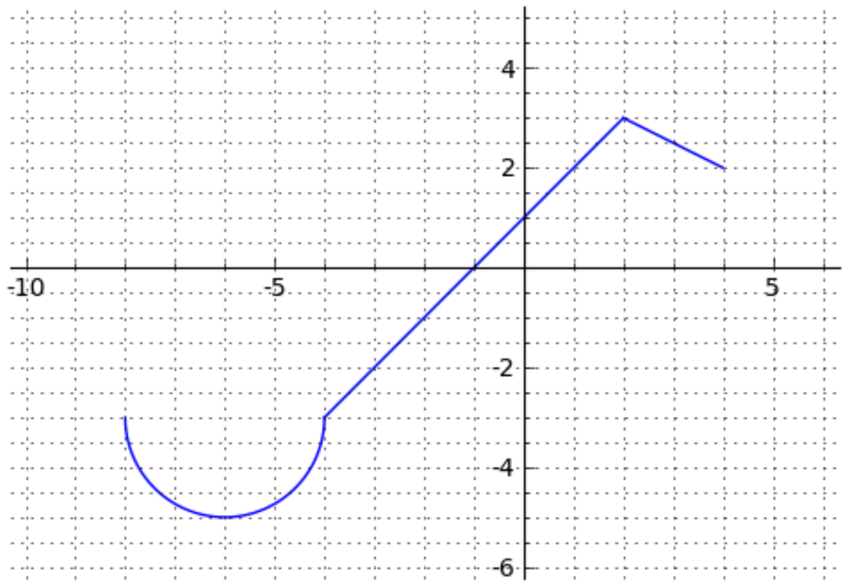
Example. $y = x^2 + 1$, which can also be written as $f(x) = x^2 + 1$

What is $f(2)$? $f(5)$?

What is $f(a+3)$?

Sometimes, a function is described with a graph.

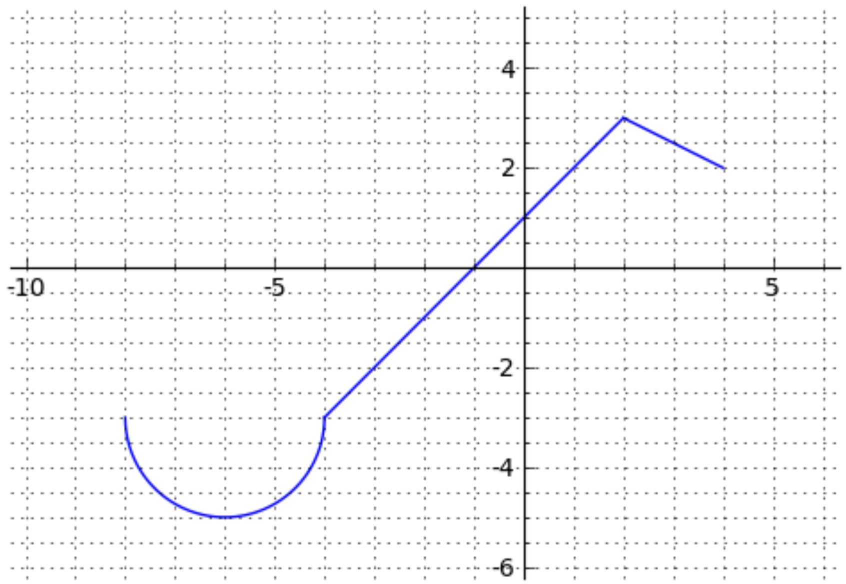
Example. The graph of $y = g(x)$ is shown below



What is $g(2)$? $g(5)$?

Definition. The *domain* of a function is all possible x-values. The *range* is the y-values.

Example. What is the domain and range of the function $g(x)$ graphed below?



Example. What are the domains of these functions?

A. $g(x) = \dfrac{x}{x^2 - 4x + 3}$

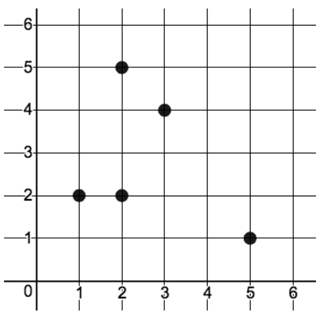
B. $f(x) = \sqrt{3 - 2x}$

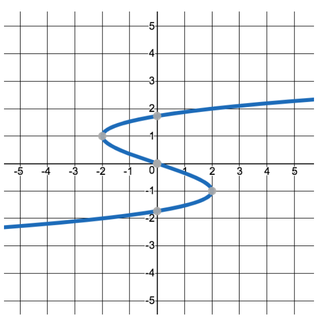
C. $h(x) = \dfrac{\sqrt{3-2x}}{x^2 - 4x + 3}$

END OF VIDEO

Definition. A relation is ...

Example. Which of these relations represent functions?

1. $\{(4,4),(6,4),(-4,6),(7,4)\}$
2. 

3. $y = \dfrac{1}{2}x^2$
4. $y^2 = 3x$
5. 

Example. Find the domain of $h(x) = \dfrac{2}{x-9} + \sqrt{6x+5}$

Example. Find the domain of $g(x) = \dfrac{\sqrt{2-x}}{x+3}$

A. $(-\infty, 2)$

B. $(-\infty, 2]$

C. $(-3, 2]$

D. $(-\infty, -3) \cup (-3, 2)$

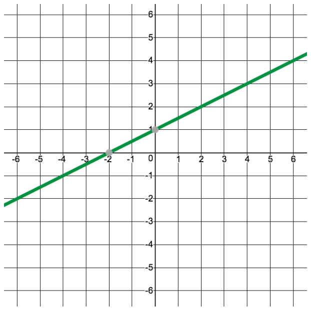
E. $(-\infty, -3) \cup (-3, 2]$

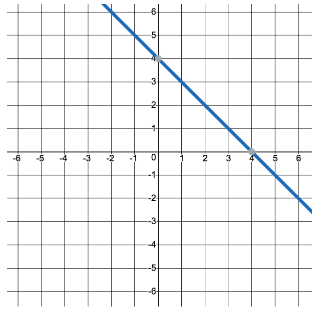
Increasing and Decreasing Functions, Maximums and Minimums

After completing this sections, students should be able to:

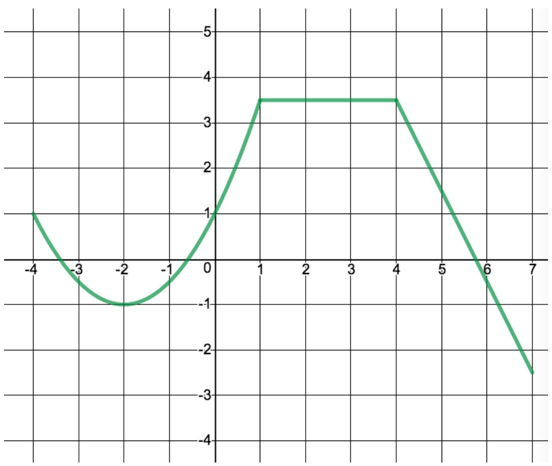
- Identify the intervals on which a function is increasing and decreasing based on a graph.
- Define absolute maximum and minimum values and points and local minimum values and points.
- Identify absolute and local max and min values and points based on a graph.

Example. Which function is increasing? Which is decreasing?





Example. On what intervals is the function graphed below increasing? Decreasing?



Definition. A function $f(x)$ has an **absolute maximum** at $x = c$ if

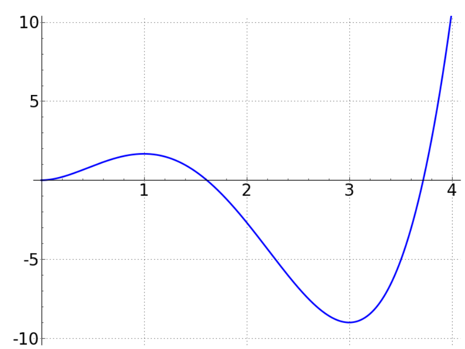
The y-value $f(c)$ is called the ______________________________

and the point $(c, f(c))$ is called ______________________________

Definition. A function $f(x)$ has an **absolute minimum** at $x = c$ if

The y-value $f(c)$ is called the ______________________________

and the point $(c, f(c))$ is called ______________________________



Definition. Absolute maximum and minimum values can also be called

Definition. A function $f(x)$ has an **local maximum** at $x = c$ if

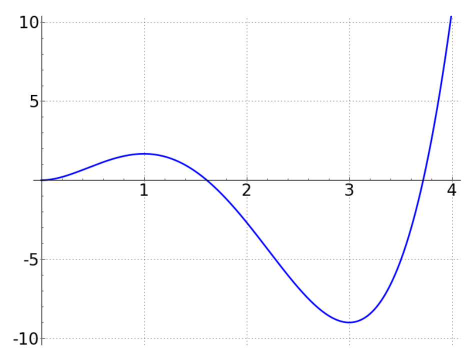
The y-value $f(c)$ is called the ______________________________

and the point $(c, f(c))$ is called ______________________________

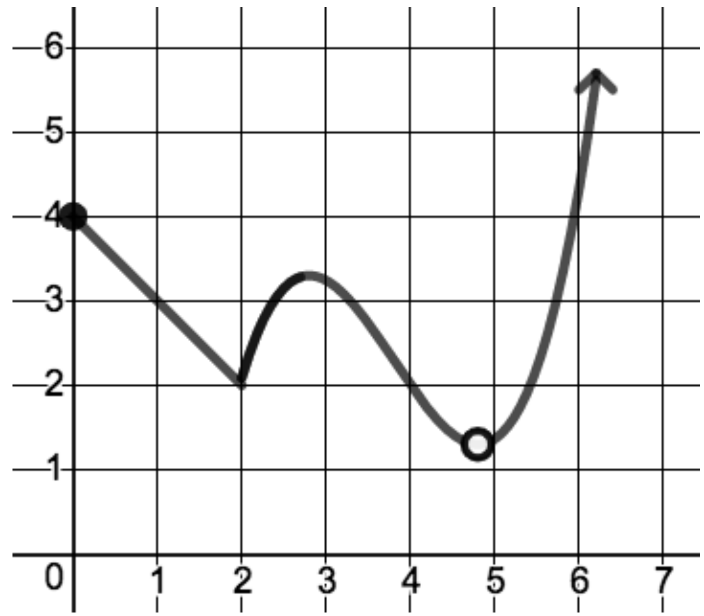
Definition. A function $f(x)$ has an **local minimum** at $x = c$ if

The y-value $f(c)$ is called the ______________________________

and the point $(c, f(c))$ is called ______________________________



Definition. Local maximum and minimum values can also be called



Example. .

1. Mark all local maximum and minimum points.
2. Mark all absolut maximum and and minimum points.
3. What are the local maximum and minimum values of the function?
4. What are the absolute maximum and minimum values of the function?

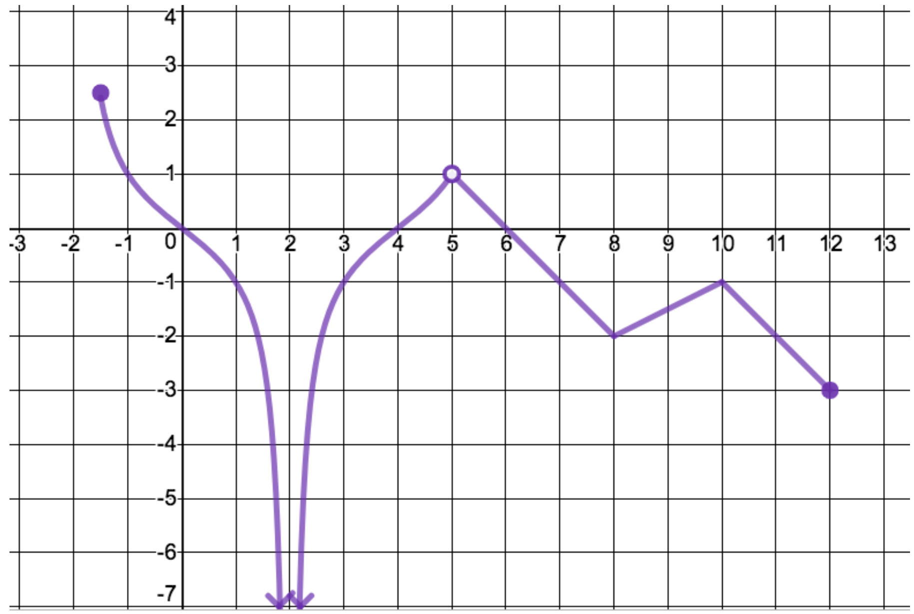
END OF VIDEOS

Question. What is the difference between a maximum *point* and a maximum *value*?

Question. What is difference between an *absolute* maximum value and a *local* maximum value?

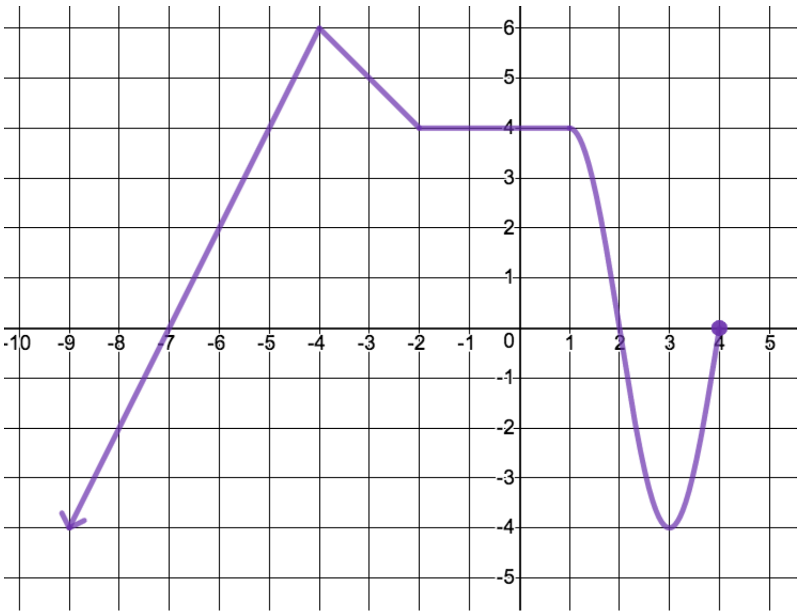
Question. Is it possible to have more than one absolute maximum *value*? More than one absolute maximum *point*?

Example. Find the local maximum points, the local minimum points, the absolute maximum points, and the absolute minimum points.



What are the absolute maximum and minimum values?

Example. On what intervals is the function graphed below increasing? Decreasing?



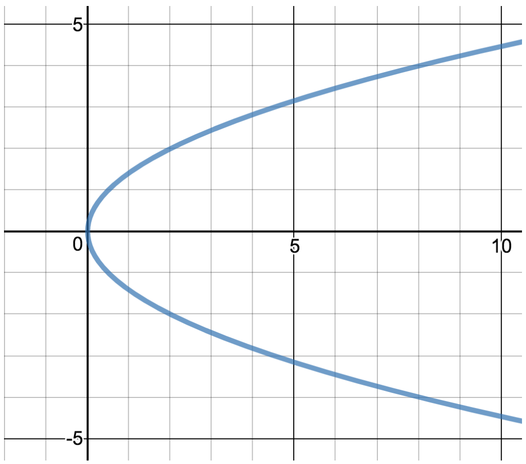
What are its absolute max and min points? Absolute max and min values?

Symmetry and Even and Odd Functions

After completing this sections, students should be able to:

- Identify whether a graph is symmetric with respect to the x-axis, symmetric with respect to the y-axis, symmetric with respect to the origin, or none of these.
- Determine whether a function is even or odd or neither, based in its graph or its equation.
- Explain the relationship between even and odd functions and the symmetry of their graphs.

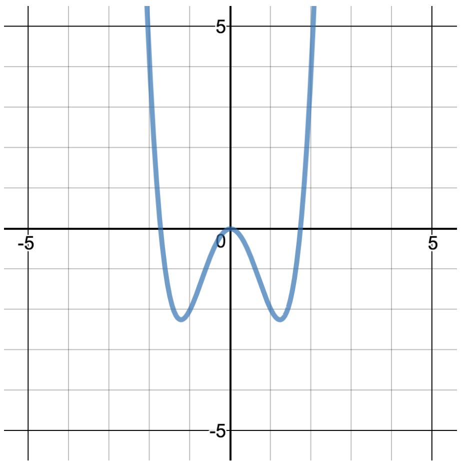
Definition. A graph is *symmetric with respect to the x-axis* if ...



Whenever a point (x, y) is on the graph, the point

is also on the graph.

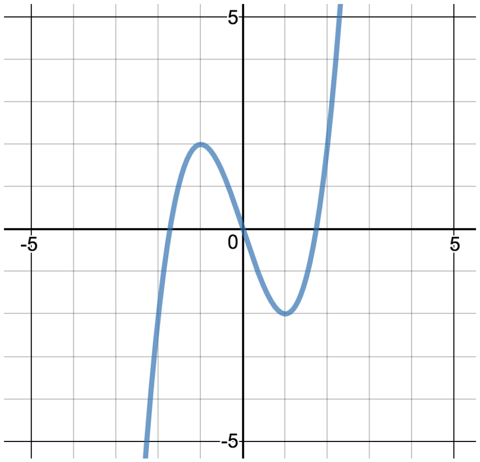
Definition. A graph is *symmetric with respect to the y-axis* if ...



Whenever a point (x, y) is on the graph, the point

is also on the graph.

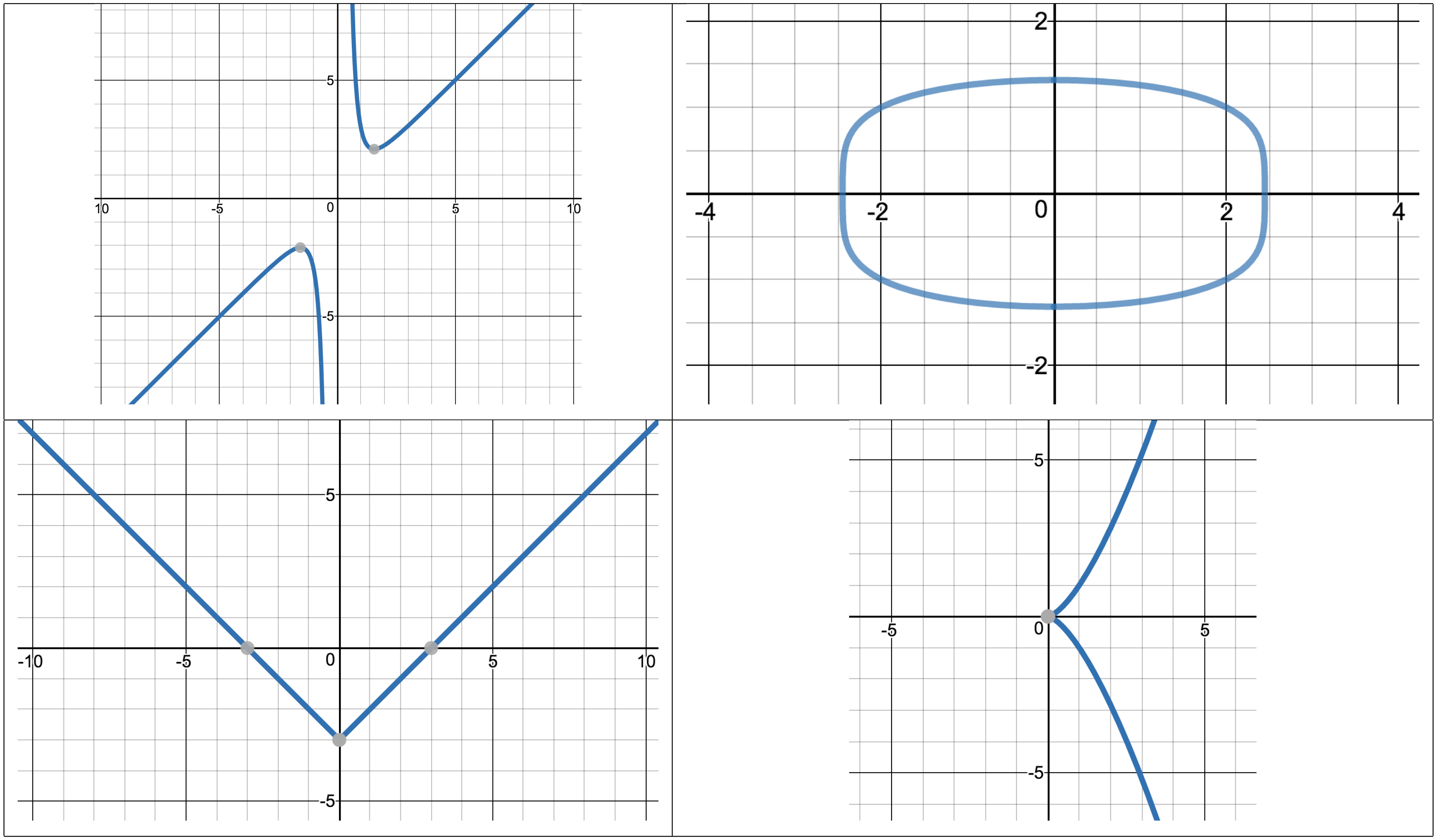
Definition. A graph is *symmetric with respect to the origin* if ...



Whenever a point (x, y) is on the graph, the point

is also on the graph.

Example. Which graphs are symmetric with respect to the x-axis, the y-axis, the origin, or neither?



Definition. A function $f(x)$ is *even* if ...

Example. $f(x) = x^2 + 3$ is even because ...

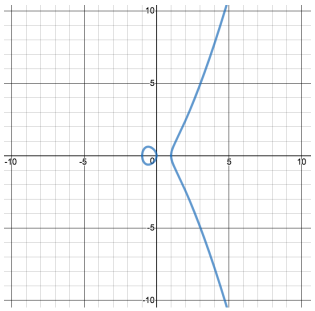
Definition. A function $f(x)$ is *odd* if ...

Example. $f(x) = 5x - \frac{1}{x}$ is odd because ...

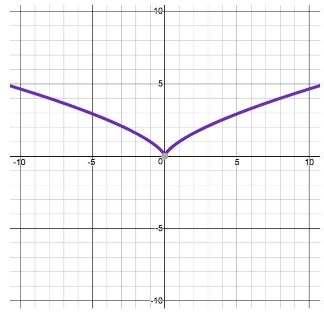
Question. There is no word like even or odd for when a function's graph is symmetric with respect to the x-axis. Why not?

END OF VIDEO

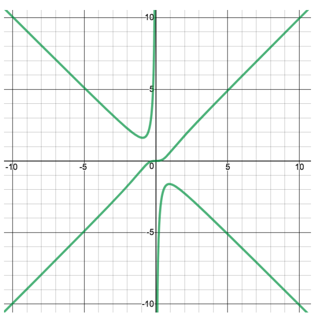
Example. Are these graphs symmetric with respect to the x-axis, the y-axis, the origin, or neither?



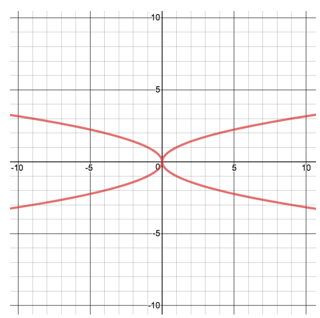
$y^2 = x^3 - x$



$y^3 = x^2$



$y = x^3 - y^2 x$



$y^2 = |x|$

Example. Do these equations have graphs that are symmetric with respect to the x-axis, the y-axis, the origin, or neither?

$y = \dfrac{2}{x^3} + x$ $\qquad$ $x^2 + 2y^4 = 6$ $\qquad$ $y = |x| + x$ $\qquad$ $y = |x| + x^2$

Which equations represent even functions? Odd functions?

Example. Determine whether the functions are even, odd, or neither.

1. $f(x) = 4x^3 + 2x$

2. $g(x) = 5x^4 - 3x^2 + 1$

3. $h(x) = 2x^3 + 7x^2$

Transforming Functions

After completing this section, students should be able to

- Identify the motions corresponding to adding or multiplying numbers or introducing a negative sign on the inside or the outside of a function.
- Draw the transformed graph, given an original graph of $y = f(x)$ and an equation like $y = -3f(x+2)$, using a point by point analysis or a wholistic approach.
- Identify the equation for transformed graphs of toolkit functions like $y = |x|$ and $y = x^2$
- Identify a point on a transformed graph, given a point on the original graph and the equation of the transformed graph.

Review of Function Notation

Example. Rewrite the following, if $g(x) = \sqrt{x}$.

a) $g(x) - 2 =$

b) $g(x - 2) =$

c) $g(3x) =$

d) $3g(x) =$

e) $g(-x) =$

Example. Rewrite the following in terms of $g(x)$, if $g(x) = \sqrt{x}$.

f) $\sqrt{x} + 17 =$

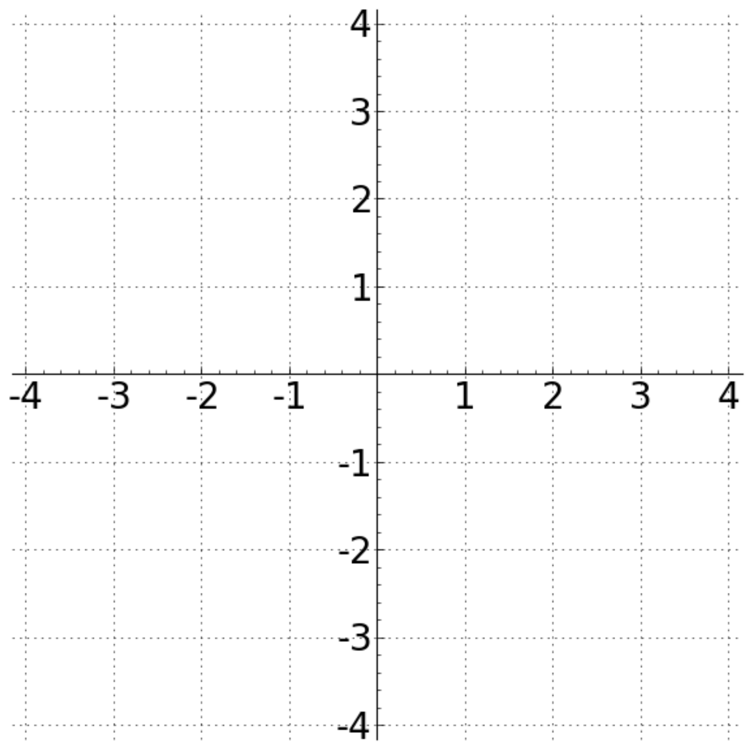
g) $\sqrt{x + 12} =$

h) $-36 \cdot \sqrt{x} =$

i) $\sqrt{\frac{1}{4}x} =$

Example. Graph

- $y = \sqrt{x}$
- $y = \sqrt{x} - 2$
- $y = \sqrt{x-2}$



Rules for transformations:

- Numbers on the *outside* of the function affect the y-values and result in vertical motions. These motions are in the direction you expect.

- Numbers on the *inside* of the function affect the x-values and result in horizontal motions. **These motions go in the opposite direction from what you expect.**

- Adding results in a shift (translations)

- Multiplying results in a stretch or shrink

- A negative sign results in a reflection

Example. Consider $g(x) = \sqrt{x}$. How do the graphs of the following functions compare to the graph of $y = \sqrt{x}$?

a) $y = \sqrt{x} - 4$

b) $y = \sqrt{x + 12}$

c) $y = -3 \cdot \sqrt{x}$

d) $y = \sqrt{\frac{1}{4}x}$

END OF VIDEO

Rules of Function Transformations (see graph animations involving $y = \sin(x)$)

- A number added on the OUTSIDE of a function ...

- A number added on the INSIDE of a function ....

- A number multiplied on the OUTSIDE of a function ....

- A number mulitplied on the INSIDE of a function ....

- A negative sign on the OUTSIDE of a function ....

- A negative sign on the INSIDE of a function ....

Example. Consider $h(x) = x^3$. How do the graphs of the following functions compare to the graph of $y = x^3$?

a) $y = -(x+1)^3$

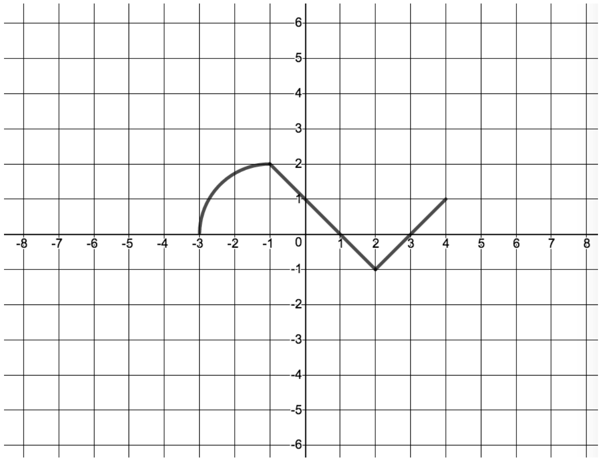
b) $y = (2x)^3$

c) $y = \dfrac{x^3}{2} - 5$

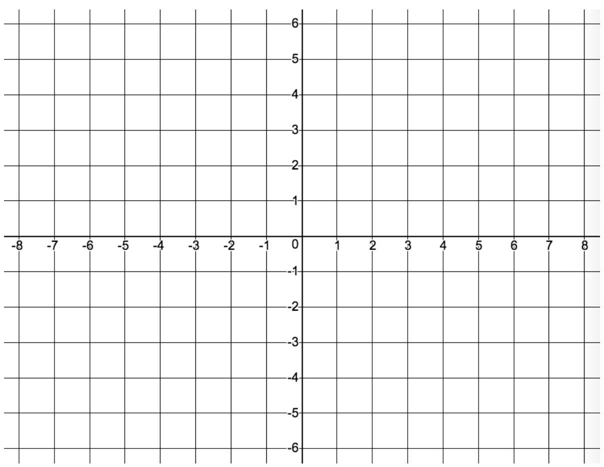
d) $y = (-x)^3 + 3$

Note. There are two approaches to graphing transformed functions:

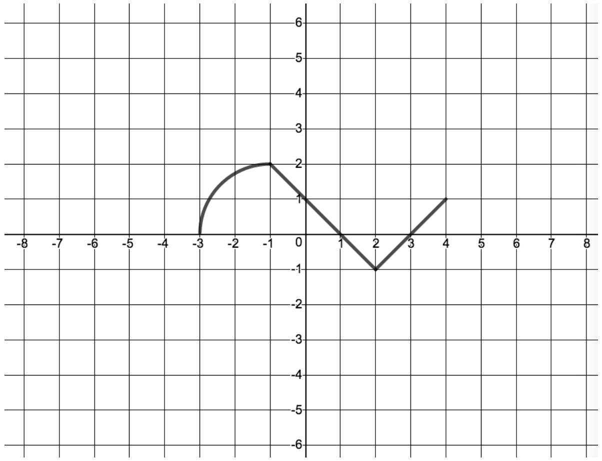
Example. The graph of a certain function $y = f(x)$ is shown below.



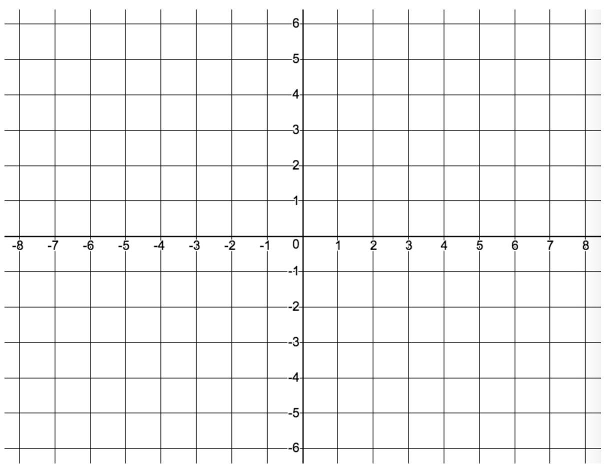
Use transformations to draw the graph of the function $y = -f(2x) + 4$. Label points on your final graph.



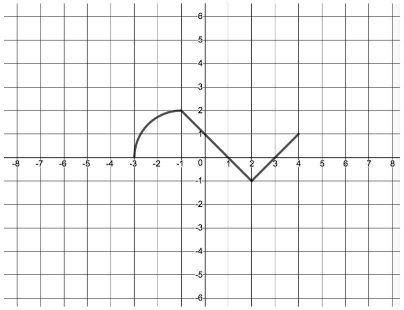
Example. The graph of a certain function $y = f(x)$ is shown below.



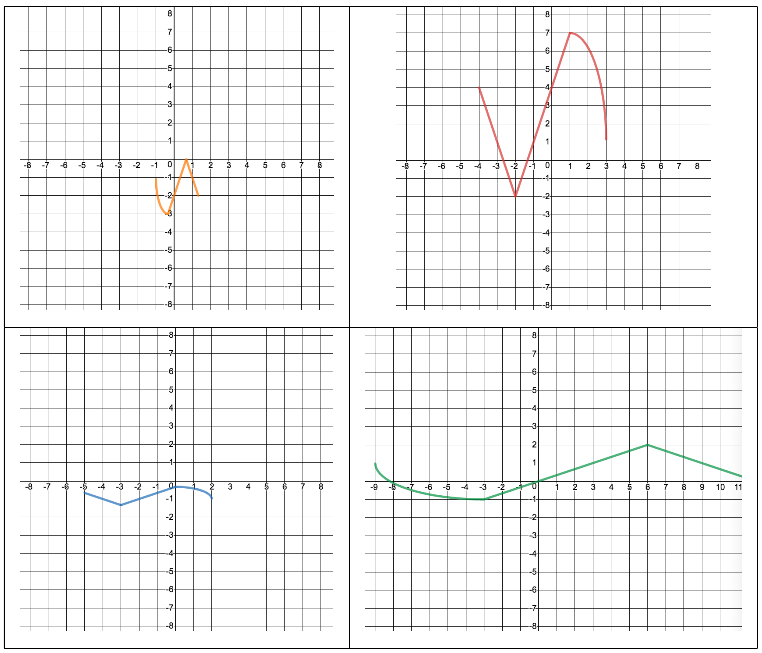
Use transformations to draw the graph of the function $y = 3f(-x) - 1$. Label points on your final graph.



Example. Given the original graph if $y = f(x)$



Find the graph of $y = 3f(-x) + 1$, and write down the equations of the other graphs.



Extra Example. Suppose the graph of $y = f(x)$ contains the point $(3, -1)$. Identify a point that must be on the graph of $y = 2f(x-1)$.

A. $(2, -1)$

B. $(2, -1)$

C. $(4, -1)$

D. $(4, -2)$

Piecewise Functions

After completing this sections, students should be able to:

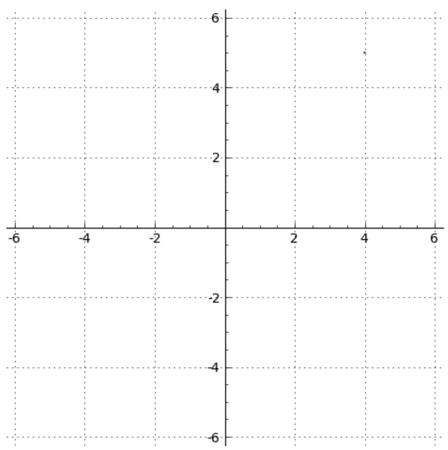
- Evaluate piecewise functions at a given x-value.
- Graph piecewise functions.

Example. The function f is defined as follows:

$$f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ -2x + 3 & \text{if } x \geq 1 \end{cases}$$

1. What is $f(-2)$? What is $f(1)$ What is $f(3)$?

2. Graph $y = f(x)$.



3. Is $f(x)$ continuous?

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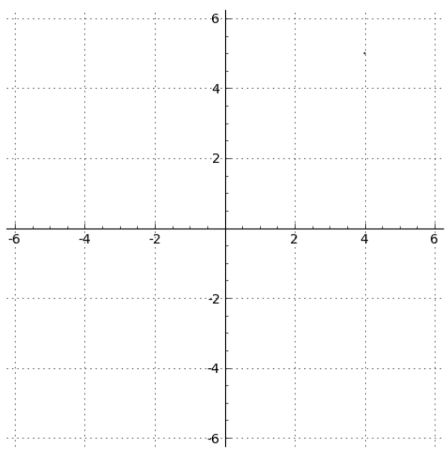
Example. The function $g(x)$ is defined by:

$$g(x) = \begin{cases} (x+2)^2 & \text{if } -4 \leq x \leq -2 \\ \frac{1}{2}x + 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } 2 \leq x \leq 4 \end{cases}$$

1. What is $g(-2)$? What is $g(1)$ What is $g(3.5)$?

2. Graph $y = g(x)$.



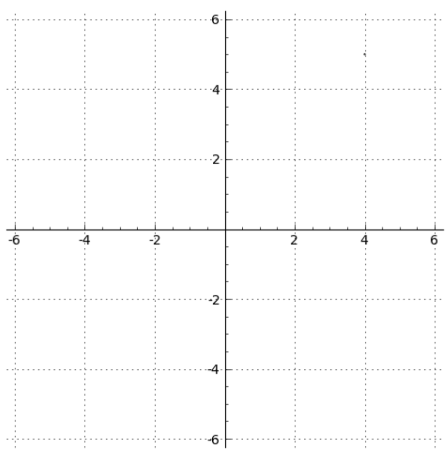
3. What are the domain and range of $g(x)$?

Extra Example. The function $h(x)$ is defined by:

$$h(x) = \begin{cases} 2|x| & \text{if } -2 \leq x < 1 \\ -\sqrt{x} - 1 & \text{if } 1 \leq x < 4 \\ 1 - x & \text{if } 4 \leq x \leq 6 \end{cases}$$

1. What is $h(1)$? What is $h(5)$?

2. Graph $y = h(x)$.



3. What are the domain and range of $h(x)$?

Inverse functions

After completing this section, students should be able to:

- Based on the graph of a function, determine if the function has an inverse that is a function.
- Draw the graph of an inverse function, given the graph of the original.
- Use a table of values for a function to write a table of values for its inverse.
- Determine if two given functions are inverses of each other by computing their compositions.
- Use a formula for a function to find a formula for its inverse.
- Find the range of the inverse function from the domain of the original function.
- Find the domain of the inverse function from the range of the original function.

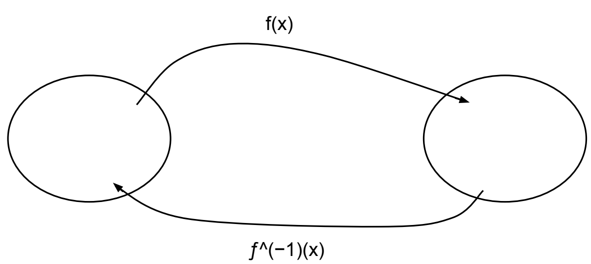
Example. Suppose $f(x)$ is the function defined by the chart below:

x	2	3	4	5
$f(x)$	3	5	6	1

In other words,

- $f(2) = 3$
- $f(3) = 5$
- $f(4) = 6$
- $f(5) = 1$

Definition. The **inverse function** for f, written $f^{-1}(x)$, undoes what f does.

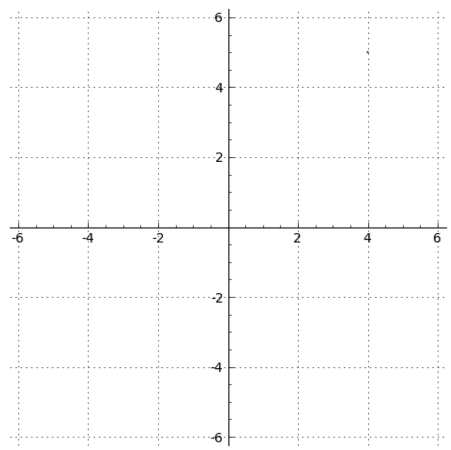


- $f^{-1}(\ 3\ ) = 2$
- $f^{-1}(\quad) =$
- $f^{-1}(\quad) =$
- $f^{-1}(\quad) =$

x	3			
$f^{-1}(x)$	2			

Key Fact 1. Inverse functions reverse the roles of y and x.

Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same axes below. What do you notice about the points on the graph of $y = f(x)$ and the points on the graph of $y = f^{-1}$?



Key Fact 2. The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line ________ .

In our same example, compute:

$f^{-1} \circ f(2) =$	$f \circ f^{-1}(3) =$
$f^{-1} \circ f(3) =$	$f \circ f^{-1}(5) =$
$f^{-1} \circ f(4) =$	$f \circ f^{-1}(6) =$
$f^{-1} \circ f(5) =$	$f \circ f^{-1}(1) =$

Key Fact 3. $f^{-1} \circ f(x) =$ ________ and $f \circ f^{-1}(x) =$ ________ . This is the mathematical way of saying that f and f^{-1} undo each other.

Example. $f(x) = x^3$. Guess what the inverse of f should be. Remember, f^{-1} undoes the work that f does.

Example. Find the inverse of the function:

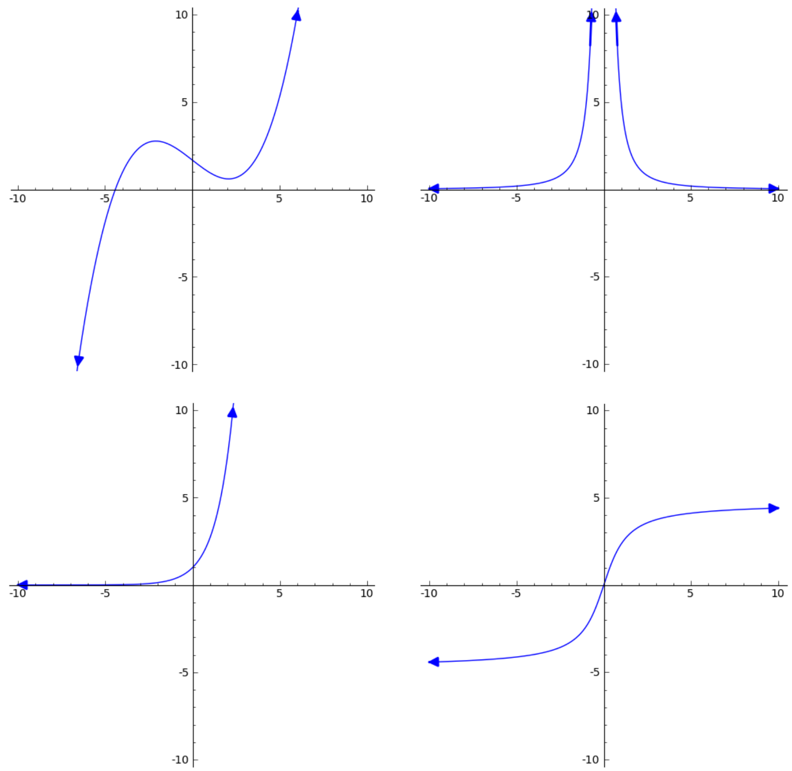
$$f(x) = \frac{5 - x}{3x}$$

Note. $f^{-1}(x)$ means the inverse function for $f(x)$. Note that $f^{-1}(x) \neq \frac{1}{f(x)}$.

Question. Do all functions have inverse functions? That is, for any function that you might encounter, is there always a **function** that is its inverse?

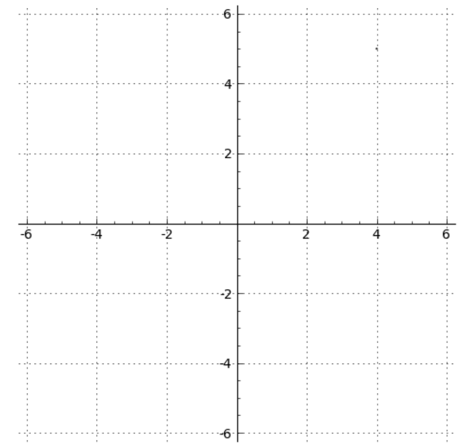
Try to find an example of a function that does **not** have an inverse **function**.

Key Fact 4. A function f has an inverse function if and only if the graph of f satisfies the **horizontal line test** (i.e. every horizontal line intersects the graph of $y = f(x)$ in at most one point.)



Definition. A function is **one-to-one** if it passes the horizontal line test. Equivalently, a function is one-to-one if for any two different x-values x_1 and x_2, $f(x_1)$ and $f(x_2)$ are different numbers. Sometimes, this is said: f is one-to-one if, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Example. (Tricky) Find $p^{-1}(x)$, where $p(x) = \sqrt{x-2}$ drawn above. Graph $p^{-1}(x)$ on the same axes as $p(x)$.



For the function $p(x) = \sqrt{x-2}$, what is:

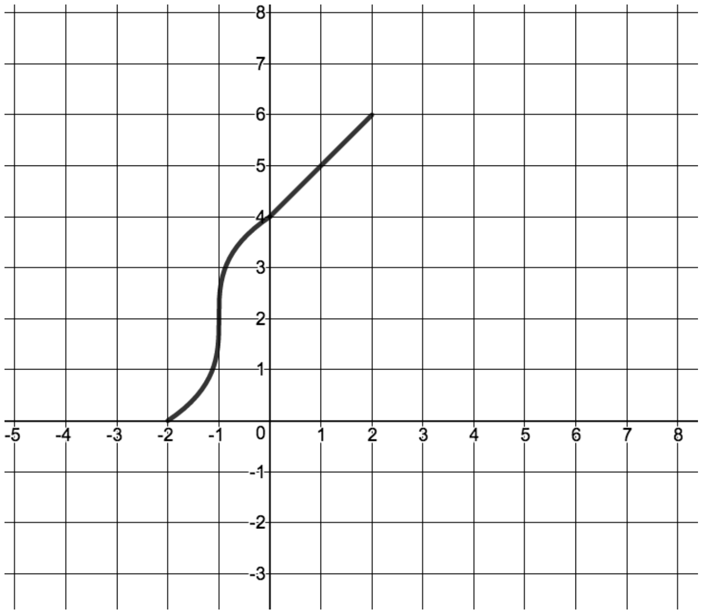
- the domain of p?
- the range of p?
- the domain of p^{-1}?
- the range of p^{-1} ?

Key Fact 5. For any invertible function f, the domain of $f^{-1}(x)$ is ________________ and the range of $f^{-1}(x)$ is ________________ .

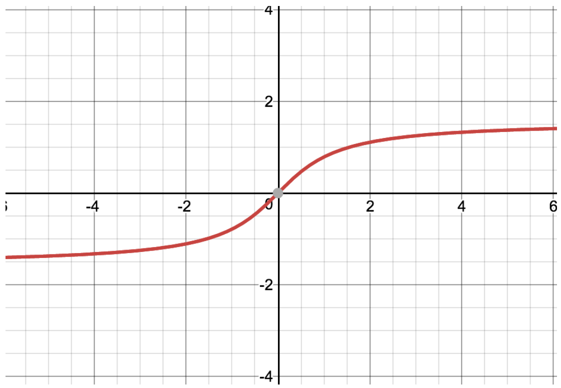
END OF VIDEO

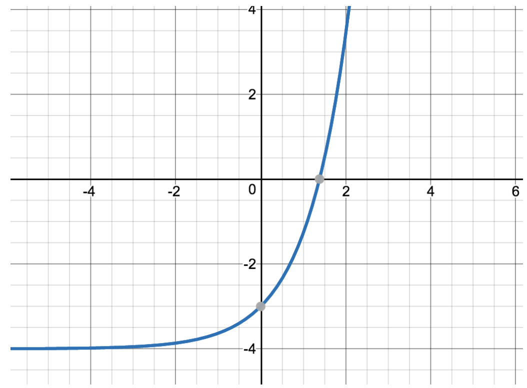
What are some facts about inverse functions?

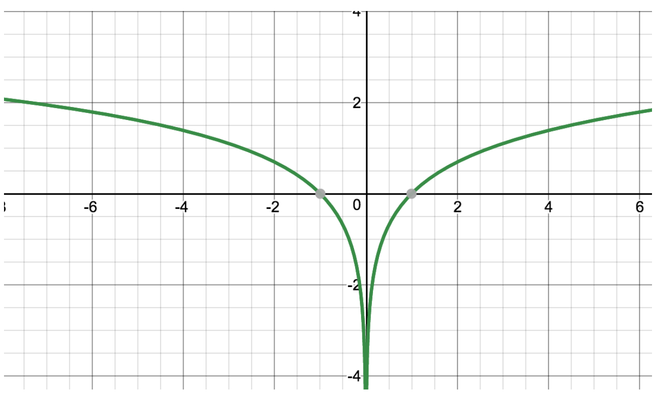
Example. The graph of $f(x)$ is show below. Find the graph of $f^{-1}(x)$.

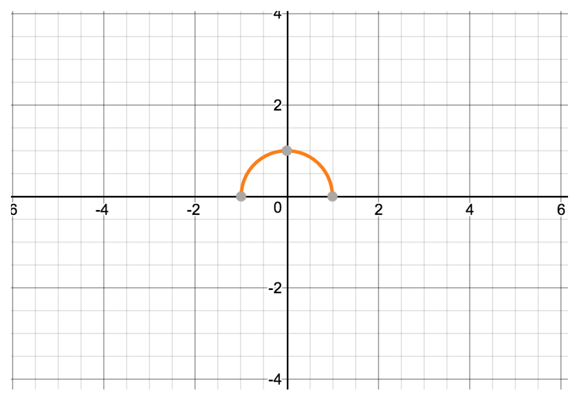


Example. For each function graph, determine whether it has an inverse function.

A. 

B. 

C. 

D. 

Example. $h(x) = x^3 - 7$. Find $h^{-1}(x)$.

A) $p(x) = \dfrac{1}{x^3 - 7}$

B) $q(x) = \sqrt[3]{x} + 7$

C) $r(x) = \sqrt[3]{x + 7}$

D) $v(x) = \sqrt[3]{x - 7}$

Check your answer by ...

Example. Find the inverse of the function:

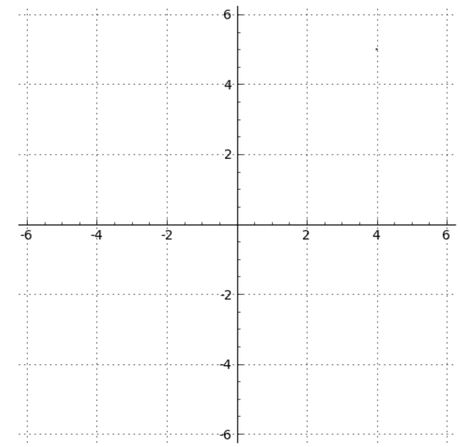
$$f(x) = \frac{3x + 1}{x - 6}$$

Extra Example. Find the inverse of the function: $$f(x) = \frac{7 - x}{2x + 3}$$
Find the domain and range of $f(x)$ and $f^{-1}(x)$.

Example. Consider the function $f(x) = x^2 + 4$.

1. $f(x)$ does not have an inverse that is a function. Why not?
2. Restrict the domain of $f(x)$ so that it has an inverse that is a function. Call the restricted $f(x)$ by the name $\hat{f}(x)$.
3. Find $\hat{f}^{-1}(x)$.
4. What are the domain and range of $\hat{f}(x)$ and $\hat{f}^{-1}(x)$
5. Draw $\hat{f}(x)$ and $\hat{f}^{-1}(x)$ on the same axes.

Extra Example. Find $f^{-1}(x)$, where $f(x) = \sqrt{x+1}$. Graph $f^{-1}(x)$ on the same axes as $f(x)$.



For the function $f(x) = \sqrt{x+1}$, what is:

- the domain of f?
- the range of f?
- the domain of f^{-1}?
- the range of f^{-1} ?

Extra Example. According to math lore, if you are age x, the oldest person that it is okay for you to date is given by the formula $d(x) = 2x - 14$. Plug in your own age for x and see how old a person you can date.

Suppose you want to date a younger person instead of an older person. Invert the formula to find out how young a person someone of a given age can date.

Plug in your own age for x into $d^{-1}(x)$ and see how young a person you can date.