Functions that are everywhere continuous and nowhere differentiable

1. Riemann's Function

According to publications of Du Bois Raymond [7] and Weierstrass [9], the function

$$R_a(x) = \sum_{n=1}^{\infty} \frac{\sin(n^a \pi x)}{n^a \pi}, \tag{1}$$

has been suggested by Riemann in 1861 (for $a = 2$) as an example for a function that is everywhere continuous but (almost) nowhere differentiable. Note that the sum is the Fourier series of an odd periodic function of period 2, thus it is sufficient to visualize its graph on the interval $[0, 1]$. Clearly, if $a > 1$ the series is uniformly convergent by the Weierstrass M-test, hence the series defines a function that is continuous everywhere. However, differentiating formally each term in the series (1) gives

$$R'_a(x) \sim \sum_{n=1}^{\infty} \cos(n^a \pi x), \tag{2}$$

and there exists no interval on which this series is uniformly convergent. Due to lack of uniform convergence, differentiability is delicate to study. Hardy [4] proved in 1916 that $R_a(x)$ is not differentiable at every irrational number using properties of certain Diophantine equations. It took over 45 years until the question of differentiability of (1) was fully solved including rational number as well: In the 1970's Gerver [2, 3] and Smith [8] proved that R_a is also not differentiable at rational numbers except for numbers of the form p/q with odd (and relatively prime) integers p and q. Certain interesting features of this function at rational points have been established as well.

2. Weiertrass' Function

Weiertrass' [9] celebrated function is defined by the series

$$W_{ab}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{3}$$

where here $0 < a < 1$ and b is an odd integer such that $ab > 1 + 3\pi/2$ (according to Weierstrass' original discussion). The minimum value of b which satisfies these constraints is $b = 7$. This construction, along with the proof that the function is nowhere differentiable, was first given by Weierstrass in a report in 1872 and published in 1875. The requirement that b be an odd integer makes the function periodic of period 2. The condition $a < 1$ allows to apply the Weierstrass M-test again (using a geometric series) and guarantees continuity. Term by term differentiation gives the series

$$W'_{ab}(x) \sim -\pi \sum_{n=0}^{\infty} (ab)^n \sin(b^n \pi x), \tag{4}$$

and already $ab > 1$ prevents uniform convergence of the series (4). Indeed, Hardy has shown that $W_{ab}(x)$ is nowhere differentiable if $ab > 1$.

Renewed interest in the the Weierstrass function arose in the 1970-1980's with the discovery of and interest in fractals. Fractals are geometrical objects that cannot be characterized as objects with an integer dimension (isolated points, 1d lines, 2d areas etc), Specifically the graph of the Weierstrass function does not appear like a smooth, 1d curve. When zooming progressively into a smooth curve one eventually gets closer and closer to a straight line, but not so with the graph of the Weierstrass function and other fractal curves.

There are various definitions of noninteger dimensions of sets. The most general definition is the Hausdorff dimension introduced by Feliz Hausdorff [5]. The Hausdorff dimension of the graph of the Weierstrass function is bounded above by 2 + ln(a)/ln(b), and it is generally believed to be exactly that value, but this had not been proven rigorously. Notice that $1 < D < 2$ if $ab > 1$.

The Weierstrass function has been taken up and generalized by Benoit Mandelbrot [6] in 1977. An extensive study of this function using asymptotic analysis, numerical computations as well as ideas from quantum mechanics was given by Berry and Lewis [1].

3. Piecewise linear versions of Weiertrass' Function

The term "Weierstrass function" is often used in real analysis to refer to any function with similar properties and construction to Weierstrass' original example. For example, the cosine function can be replaced in the infinite series by a piecewise linear "zigzag" function. An example of such a function is introduced in our text in Example 3 in Section 25 (p. 204) as follows: Let $g_0(x)$ be the "tent function" on $[0, 2]$,

$$g_0(x) = \left\{ \begin{array}{ccc} x & \text{if} & 0 \le x < 1 \\ 2 - x & \text{if} & 1 \le x < 2 \end{array} \right\} \tag{5}$$

and define the function $g(x)$ on $\mathbb{R}$ by extending $g(x)$ periodically,

$$g(x) = g_0(x - 2k) \;\text{ if }\; 2k \le x < 2(k+1) \;\; \forall\, k \in \mathbb{Z}. \tag{6}$$

(see Figure 25.1, p. 204, in the text). The Weierstrass-type function $f_a(x)$ associated with $g(x)$ ($g(x)$ takes the role of the cosine in the original Weierstrass function) is then defined as

$$f_a(x) = \sum_{n=0}^{\infty} a^n g(4^n x), \quad 0 < a < 1. \tag{7}$$

where again $0 < a < 1$. This function is continuous everywhere, however nowhere differentiable if $4a > 1$ (e.g. $a = \frac{3}{4}$). The proof of nondifferentiability is somewhat delicate as for the other cases. A discussion of nondifferentiability for another example is given in the text in Section 38 (last section).

4. Graphs of $\mathbf{R_a(x)}$ and $\mathbf{f_a(x)}$

Graphs of approximations of $R_a(x)$ and $f_a(x)$ are depicted in Figures 1 and 2.

To approximate the Riemann function, 600 cosine terms were used. For larger values of a the nondifferentiability is not so well apparent in finite sum approximations, but for smaller values it becomes clearly visible.

For the piecewise linear version of the Weierstrass function, the infinite sum in (7) was approximated by 10 terms; beyond that memory problems occurred.

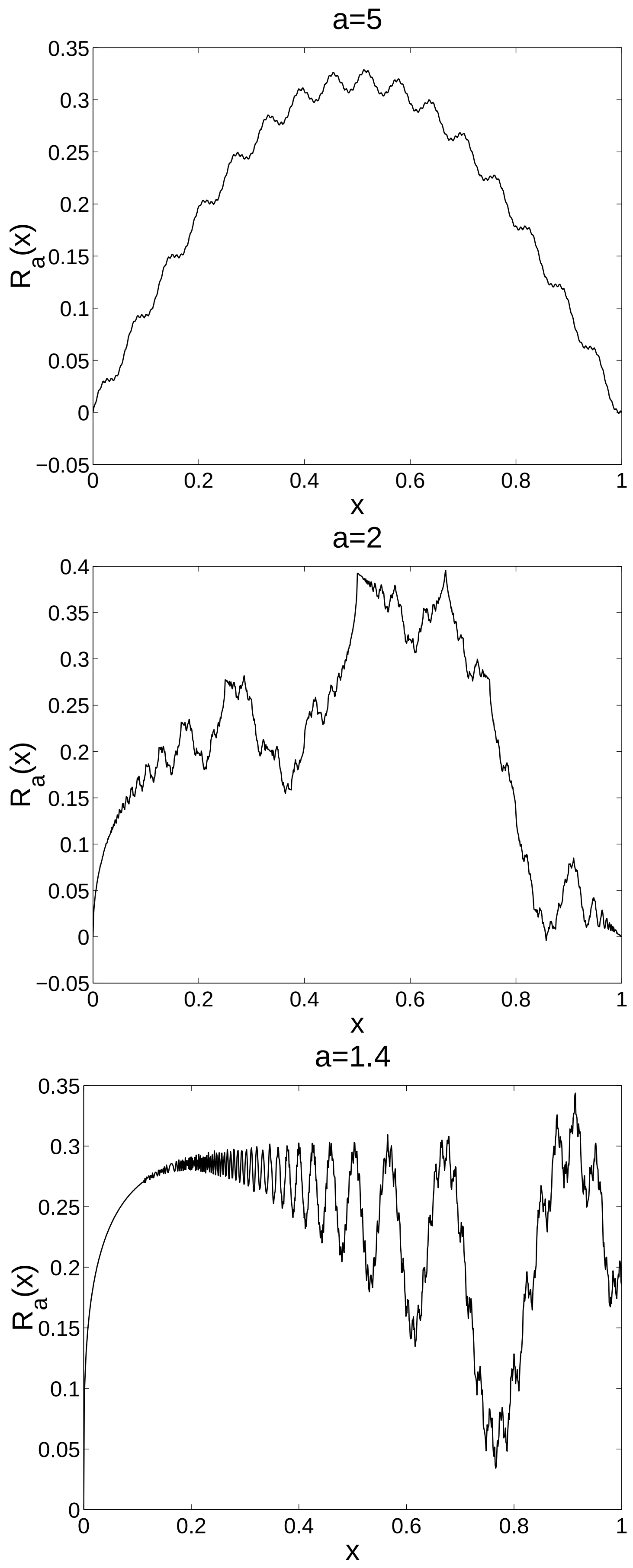

Figure 1: $R_a(x)$, equation (1), for three decreasing values of a.

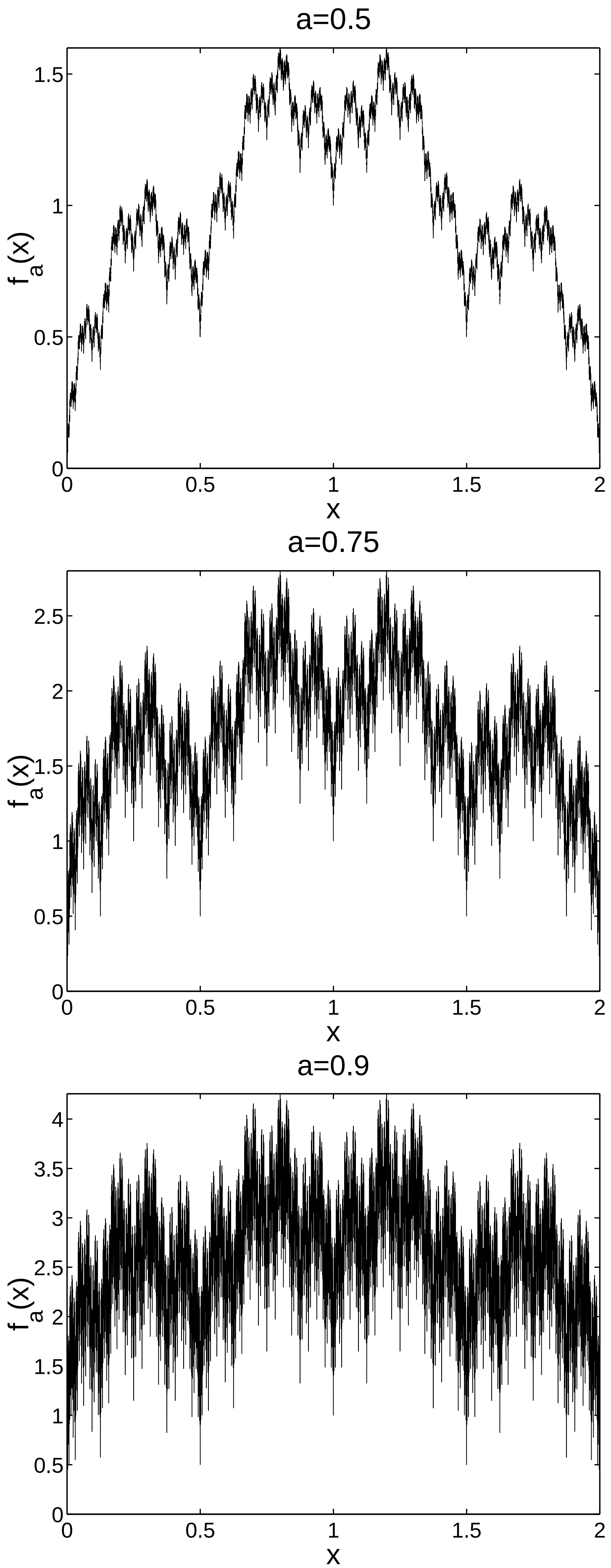

Figure 2: $f_a(x)$, equation (7), for three increasing values of a.

References

[1] Berry, M.V. and Lewis, Z,V. On the Weierstrass-Mandelbrot fractal function. Proceedings of the Royal Society of London, Series A, 370, pp. 459-484 (1980).

[2] Gerver, J. The Differentiability of the Riemann Function at Certain Rational Multiples of pi. Amer. J. Math. 92, pp. 33-55 (1970).

[3] Gerver, J. More on the Differentiability of the Riemann Function." Amer. J. Math. 93, pp. 33-41 (1971).

[4] Hardy, P. Weiertrass's non-differentiable function. Transactions of the American Mathematical Society 17, pp. 301-325 (1916).

[5] Hausdorff, F. Dimension und äusseres Mass. Mathematische Annalen 79, pp. 157-179 (1919).

[6] Mandelbrot, B.B. *Fractals: form, chance and dimension.* San Fracisco: Freeman 1977

[7] Du Bois Raymond, P. Versuch einer Klasifikation der Willkürlichen Funktionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen. Journal für Mathematik 79, p. 28 (1874).

[8] Smith, A. The Differentiability of Riemann's Functions. Proc. Amer. Math. Soc. 34, pp. 463-468 (1972).

[9] K. Weierstrass. Über kontinuierliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen. Gelesen Akad. Wiss. 18. Juli 1872, and J. für Mathematik **79**, 21-37 (1875).