# Fundamentals of Electromagnetics with MATLAB®

## Second Edition

*To our wives: Vicki, Rossi, and Vickie*

# Fundamentals of Electromagnetics with MATLAB®
## Second Edition

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# Publisher's Note to the Second Printing

The diligent efforts of many dedicated individuals to help eliminate errata and improve fine points of this text bear special commendation. In barely six months from the first printing, the authors, SciTech advisors, adopting instructors, their students, and very talented teaching assistants went through every page, figure, example, equation and problem to bring subsequent students and users this remarkably clean printing. Among the improvements that will benefits students and readers are the following:

- Chapter One – *MATLAB, Vectors, and Phasors* was rewritten to provide an even better review of vector analysis and a more detailed introduction to phasors and phasor notation.
- Sixteen new problems were added to Chapter Two – *Electrostatic Fields*
- All problems were checked for clarity and 100% accurate answers in Appendix G
- Highly detailed Solutions were derived for instructors in editable Word files
- Short sections were added on topics of EMF, power flow and energy deposition, and complex vectors

We continue to encourage submission of errata and suggestions for improvements.

We welcome **Dr. Jonathan Bagby** of Florida Atlantic University to the author team. Though Dr. Bagby is primarily concerned with the Optional Topics on the Student CD that will evolve into the *Intermediate Electromagnetics with MATLAB* text, his strong background in Mathematics was ideal in strengthening Chapter 1 and overseeing numerous improvements in Chapter 7's *Transmission Lines* notation.

**Prof. Sven Bilen** and his colleague **Prof. Svetla Jivkova** of Pennsylvania State University used their teaching experience from the text and invited student feedback to submit numerous corrections and suggested improvements to factual and conceptual aspects of the book. In addition, their student **Mickey Rhoades** had a keen eye for errors that went undetected by others. Taken all together, a profound debt of gratitude is owed to the PSU spring 2007 EM course team. Similarly, **Prof. Doran Baker** and **Prof. Donald Cripps** at Utah State gathered student feedback and submitted a summary for our reprint consideration.

The accuracy checks of problems and their selected answers in Appendix G, plus the improvements to the detailed steps in the Solutions Manual for instructors, are the work of three talented PhD honors students: **David Padgett** of North Carolina State University, **Zhihong Hu** of the University of Iowa, and **Avinash Uppuluri** of

Utah State University. The feedback of a remarkable undergraduate student who provided a voice to the eyes and mind of the typical neophyte reader must also be acknowledged: **David Ristov** of South Dakota State University.

We like to call our textbook and its supplements "organic" because they are continually growing and being nurtured by those who are passionate about the subject and teaching undergraduates. Using book reprintings and the electronic opportunities of CDs and the internet, our authors and contributors are dedicated to fixing flaws and adding helpful resources whenever they present themselves. We gratefully thank all who share our dedication to the subject, the process, and the delight in learning.

Dudley R. Kay – President and Founder
SciTech Publishing, Inc.
Raleigh, NC
July, 2007

# Contents

# Preface

## Overview

Professors ask, "Why another textbook (edition)?" while students ask, "Why do I need to study electromagnetics?" The concise answers are that today's instructor needs more flexible options in topic selection, and students will better understand a difficult subject in their world of microelectronics and wireless if offered the opportunity to apply their considerable computer skills to problems and applications. We see many good textbooks but none with the built-in flexibility for instructors and computer-augmented orientation that we have found successful with our own students.

Virtually every four-year electrical and computer engineering program requires a course in electromagnetic fields and waves encompassing Maxwell's equations. Understanding and appreciating the laws of Nature that govern the speed of even the smallest computer chip or largest power line is fundamental for every electrical and computer engineer. Practicing engineers review these principles constantly, many regretting either their inattention as undergrads or the condensed, rushed nature of the single course. What used to be two or more terms of required study has been whittled down to one very intense term, with variations of emphasis and order. Recently, there has been a resurgence of the two-term course, or at least an elective second term, that is gathering momentum as a desirable, career-enhancing option in a wireless world. Students today have grown up with computers; they employ sophisticated simulation and calculation programs quite literally as child's play. When one considers the difficult challenges of this field of study, the variation among schools and individual instructors in course structure and emphasis, and the diverse backgrounds and abilities of students, you have the reason for another textbook in electromagnetics: learning by doing on the computer, using the premier software tool available in electrical engineering education today: MATLAB.

## Textbook and Supplements on CD

Actually, this is much more than a mere textbook. The book itself offers a structural framework of principles, key equations, illustrations, and problems. With that crucial supporting structure, each instructor, student, or reader can turn to the supplemental files provided with this book or available online to customize and decorate each topic room. The entire learning package is "organic" as we the authors, contributing EM instructors, and SciTech Publishing strive to bring you an array of supporting material through the CD, the Internet, and files stored on your computer. It is very important, therefore, that you **register your book** and bookmark the URL that will always be available as a starting and reference point for ever-changing supplementary materials: **www.scitechpub.com/lonngren2e.htm**.

## Approach Using MATLAB®[1]

Our underlying philosophy is that you can learn and apply this subject's difficult principles much more easily, and possibly even enjoyably, using MATLAB. Numerical computations are readily solved using MATLAB. Also, abstract theory of unobservable waves can be strikingly visualized using MATLAB. Perhaps you are either familiar with MATLAB through personal use or through a previous course and can immediately apply it to your study of electromagnetics. However, if you are unfamiliar with MATLAB, you can learn to use it on your own very quickly. A *MATLAB Tutorial* is supplied on the book's enclosed student CD. The extensive Lesson 0 is all you really need to establish a solid starting point and build on it. If you do not have the CD or a computer handy, Chapter One provides a brief overview of MATLAB operations and a review of vector analysis. For more information and instruction on using MATLAB, SciTech Publishing provides a list of MATLAB books and CDs in Appendix F, available at special discount prices to registered users of this book. If your book came without the CD (a used book purchase, perhaps), or even if you want to be sure of obtaining the latest files, you can purchase an electronic license for a year's access to all files at the URL shown above.

Within the book, MATLAB is used numerous ways. You will always be able to see where MATLAB is either applied or has the potential to be applied by the universal icon furnished by MATLAB's parent company MathWorks. Each time this icon appears, you will know that either MATLAB's M-files of program code are supplied on your student CD or else your instructor or TA has them, most typically on Problem solutions. Use and distribution of these solution M-files are at the discretion of individual instructors and are not, therefore, furnished to students. Non-student readers can contact the publisher for selected solutions and M-files if registered.

[1] MATLAB is a registered trademark of The MathWorks, Inc. For MATLAB product information and cool user code contributions, go to www.mathworks.com, write The MathWorks, Inc., 3 Apple Hill Dr., Natick, MA 01760-2098 or call (508) 647-7101.

- *Examples* – Worked-out examples run throughout the text to show how a proof can be derived or a problem solved in steps. Each is clearly marked with a heading and also appears in a tinted blue box. When the MATLAB icon appears, it means the worked solution also has an equivalent M-file. Here is how Example sections appear:

**EXAMPLE 7.3**

The voltage wave that propagates along a transmission line is detected at the indicated points. From this data, write an expression for the wave. Note that there is a propagation of the sinusoidal signal to increasing values of the coordinate $z$.

- *Figures* – Numerous figures within the text were generated using MATLAB. Not only can you obtain and manipulate the program code with its corresponding M-file, but you can also view this figure in full color from the CD. Here is how a MATLAB-generated figure will appear:

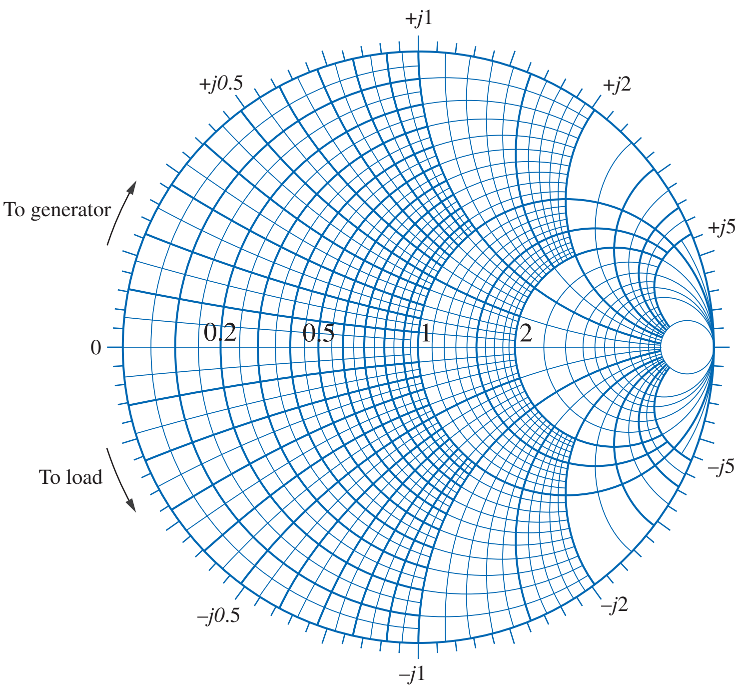



**FIGURE 7–9**

A Smith chart created with MATLAB.

- *Problems* – Each chapter contains numerous problems of varying complexity. Problem numbers correspond to the text sections so you can review any problems that prove difficult to handle at first pass. When an icon appears, you will know this problem can be solved using your MATLAB skills. Discuss with your instructor if the M-files will be made available for checking your work. Answers to selected problems are provided in Appendix G

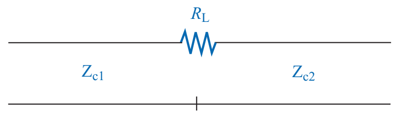
**7.6.1** Using a Smith chart, find the impedance $Z_{in}$ of a 50-Ω coaxial cable that is terminated in a load $Z_L = (25 + j25)\ \Omega$. The coaxial cable has a length of $3\lambda/8$.

**7.6.2** Using a Smith chart, find the admittance $Y_{in}$ of a 50-Ω coaxial cable that is terminated in a load $Z_L = (25 + j25)\ \Omega$. The coaxial cable has a length of $\lambda/8$.

**7.6.3** Using a Smith chart, find the distance from a load impedance $Z_L = (25 + j25)\ \Omega$ that is connected to a 50-Ω coaxial cable where the normalized input

**7.7.3.** Sketch the current profile at $z = \mathscr{L}/2$ as a function of time $0 < t < 4(\mathscr{L}/v)$ for the transmission line stated in Problem 7.7.2.

**7.7.4.** Two transmission lines are joined with a resistor $R_L$.




- *Animations* – It is possible to portray electromagnetic principles in animation, also sometimes called "movies." Authors, instructors, and even students have contributed a number of such animations showing principles at work. However, the real fun may be in manipulating the variables to produce different results. Thus the M-files become a starting point for that and also an instructive demo for leading you to your own creations. Animations at the time of this book's printing are indicated by a "flying disk" in the margin, close to the most pertinent text discussion of the underlying principle. You can create and submit animations to the publisher for posting and credit to you, using the submission form on the website. Also, check back for new animations.

ANIMATIONS

## Student CD

The Student CD enclosed with your textbook is a powerful resource. Not only does it contain files that are immediately and directly related to your course study, but it also offers a wealth of supplementary and advanced material for a 2nd term of study or your personal explorations. By registering your book, any new material produced for future CDs will be offered to you as web downloads. Here is what the Student CD contains:

## MATLAB Tutorial

Readers will come to this book with widely varying exposures to MATLAB. A self-paced tutorial has been included on the CD. Divided into lessons, MATLAB operations and tools are introduced within the context of Electromagnetics extensive notation, subject areas, examples, and problems. That is, the MATLAB tutorial gets you started with basics first and then develops text topics incrementally. You will eventually learn to perform relatively complex operations, problem-solving, and visualizations. Your instructor may later choose to assign projects involving multi-step problem-solving in MATLAB. Independent readers seeking these Projects should contact SciTech after registering the book. Also, an array of helpful books and tutorials about MATLAB for engineers is kept up to date on the SciTech website, always at discount prices.

## Optional Topics

Some schools require a second term of Electromagnetics and most at least offer a second term as an elective Advanced Electromagnetics course. Most of today's textbooks contain somewhat more material than can be covered in one term but not enough material for a full and flexible second term. Therefore, a second textbook is often required for the follow-up course, one that is costly, probably does not match the notation of the first textbook, and may even contradict the first book in places because of the difference in notation. These inconveniences are overcome with the CD's extended topics. *Optional Topics* are provided in PDF files that match the two-color design of the text, integrate MATLAB throughout, and contain the same array of problems. The book and CD, therefore, satisfy the needs of most two-term courses and many elective second-term courses. Additional optional topics are being added continually, as they are suggested and contributed by instructors with course-specific needs, such as biomedical engineering, wireless communications, materials science, military applications, and so forth. Contact SciTech Publishing if you wish to suggest or contribute a new topic.

## Applications

While brief references are made throughout the text to real-world applications of electromagnetic principles, we have chosen not to interrupt text flow with lengthy application discussions. Instead, applications are done proper justice in three–five page descriptions with graphics, in most cases, on the CD in PDF format. These applications point the way toward the utility of later courses in Microwave and RF, Wireless Communications, Antennas, High-speed Electronics, and many other career and research interests. They answer the age-old student question: "Why do I have to know this stuff?" Instructors, their TAs, and students are encouraged to submit additional applications for inclusion in the web-based Shareware Community.

## M-Files

The Student CD contains M-files for selected examples, figures, and animations. Any additional MATLAB items will usually be accompanied by M-files, so check the website and register your book for notifications of new items. New submissions and suggested improvements to existing M code will always be gratefully received and acknowledged.

## Note to Students: Equation Importance and Notational Schemes May Vary

You will encounter scads of equations in your study of electromagnetics. They are not all of equal importance. We have tried to make them clear by setting off most of them from the text and numbering them for reference. Note that we have taken the additional step of drawing a box around the most important equations, so look for those when you review.

"Mathematical notation" is how physical quantities, unit dimensions, and concepts are put into mathematical expression. While some notation is simple and common to the physical sciences, others can be arbitrary and a matter of choice to the author. The only firm rule is that the author, or other user of notation (such as an instructor writing on the board, in PowerPoint, or on a test), be clear about what the symbols represent and be consistent in their usage. Students may sometimes be unduly concerned about their textbook's notation if it is different from a previous text or what their instructor uses. Adapting to varying notational schemes is simply part of the learning process that "comes with the territory" in physics and engineering. In choosing the notation for this text we called upon our Editorial Advisory Board to determine preferences and precedents in the most widely referenced electromagnetics books. Our notational scheme is shown on page xix, and we define our symbols when first used in the text. We are acutely aware of the confusion and worry that poor notation causes students. However, be forewarned that your instructor or TA may choose to use their own notational schemes. There is no 'right" or "wrong" method. If you find any apparent inconsistencies or absence of clear definition within the textbook, please report them to us.

## Instructor Resources

Apart from the text and Student CD, numerous resources are available to instructors. Our Shareware Community of invited contributions is intended to grow the nature and number of teaching and evaluation tools. Instructors using this book as the required text are entitled to the following materials:

- *Solutions* to all chapter exercises: step-by-step Word files and MATLAB M-files for MATLAB-solvable problems designed by the MATLAB icon

- *Exam Sets* comprised of three exams in each set, including answers and solutions. Additional exams are solicited to add to the Shareware database

- *PowerPoint Slides* of all figures in the text, organized by chapter, including the full color version of MATLAB-generated figures. Additional supplementary figures may be added over time as part of the Shareware Program.

- *Projects* – the "starter set" of complex, multi-step problems involving use of MATLAB, including recommended student evaluation scoring sheets that break down the credit to be given for every step and aspect. Projects are an exciting component that makes excellent use of MATLAB skills to test full understanding and application of principles. They are a prime element of the SciTech Shareware Program.

## Shareware in Electromagnetics (SWEM) Program

As the name implies, Shareware in electromagnetics has been set up by SciTech Publishing as a means for instructors, teaching assistants, and even students to share their ideas and methods for learning, appreciating, and applying electromagnetic principles. Some items are accessible only by our textbook adopters and buyers (exam sets and their solutions, for example), while others are set up to be shared publicly, the only stipulation being registration and a contribution, however modest, to SWEM. Think of it as a neighborhood block party for the electromagnetics teaching and learning community. Bring your "covered dish, salad, or dessert" and share in the overall goodies, fun, and collegiality. For major SWEM items such as Projects, Applications, and Optional Topics, custom submission forms have been created on the website for ease of a contribution. For general ideas and items, such as a cool web link or figures, a generic form may be used. In any case, one can simply submit an email to SWEM@scitechpub.com. Acknowledgements will always be given for submissions unless anonymity is requested. See the available shareware and submission forms under "Instructors" at **www.scitechpub.com/lonngren2e.htm**.

We recognize that there are several different approaches to teaching electromagnetics in the usual engineering curriculum. We have tried to make our Second Edition adaptable to every approach. Every course will quickly review mathematical techniques and background material from previous math and physics courses. After that, curriculum sequences move down different paths.

*Historical Approach* – One traditional sequence is to follow a historical approach, where topics are covered in a sequence similar to the historical development of the subject matter, paralleling the experiments that revealed electromagnetic phenomena from ancient times. Thus, the usual course starts with electrostatics, then covers magnetostatics, introduces Maxwell's equations, wave phenomena, and follows up with applications such as transmission lines, waveguides, antennas, etc. This has probably been the most common curriculum sequence used until recently, and our book maintains this traditional approach.

*Maxwell's Equations Approach* – Another approach is to present Maxwell's equations early, develop wave phenomenon from them and then cover electrostatics, magnetostatics, and applications. A common variation on this with physics departments is to show how Maxwell's equations follow from relativity. This approach tends to require more mathematical sophistication from students, but it is popular with some instructors because of its independence from experimentally derived laws.

*Transmission Lines Approach* – Because of the perceived higher level of mathematics associated with electromagnetics, an increasing number of instructors today prefer to build upon the subject matter with which the student is already familiar. Thus, they prefer to introduce transmission lines early as a logical extension of the circuit theory that students have studied prior to arriving in their electromagnetics courses. This approach has several advantages, beyond the obvious one of using familiar circuit analogies to help the student develop their physical insight. Using transmission lines, students are exposed to applications of the electromagnetics theory early in the class, providing them with a rationale for the need of this subject matter. Secondly, transmission lines naturally incorporate many of the concepts that students sometimes find difficult to visualize when talking about fields and waves, such as time delay, dispersion, attenuation, etc.

Students should appreciate that there is no one best way to learn (or teach) this material. Each of these approaches is equally valid, and chances are that the student can learn this material, whatever the approach, if a sustained effort is made. To support both the student and the instructor in this educational effort, we have tried to make this text flexible enough to be used with a variety of curriculum sequences with varying degrees of emphasis. For instance, we made Chapter 7 on transmission lines, independent of other chapters, so those instructors wishing to cover this material first can do so, with a minimum of backtracking required. On the other hand, the instructor and student can start at the beginning of the text and work forward through the chapter material in a more conventional sequencing of chapters. Additionally, instructors can choose to skip more advanced sections of the chapters, so they can cover more topics at the expense of depth of topic coverage. Thus, this text can be used for a one or two quarter format, or in a one or two semester format, especially when supplemented by the topics on the accompanying CD. Following are some suggestions for course syllabi, depending on what the instructor wishes to emphasize and how much time he or she has available.

- A traditional one-quarter course primarily emphasizing static fields could be covered using the first five chapters.

- A traditional one-semester course with reduced emphasis on static fields, but including transmission line applications would include Chapters 1, 2, 3, and portions of Chapters 4 and 5 (Sections 4-1 through 4-6 and Sections 5-1 through 5-5), followed by Chapter 6 and portions of Chapter 7 (Sections 7-1 through 7-8).

- A one-semester "transmission lines first" approach, consisting of Chapter 1, followed by Chapter 7 (Sections 7-1 through 7-8), then Chapters 2 and 3, followed by portions of Chapter 4 (Sections 4-1 through 4-6), Chapter 5 (Sections 5-1 through 5-5) and Chapter 6.

A second semester course can be developed from the remainder of the text, as well as selected supplemental topics from the student CD. For instance, a second semester course emphasizing EM waves and their applications would consist of a review of the material previously covered in Chapters 4–7, and additional selections from Chapter 4 (Sections 4-6 through 4-8), as well as Chapter 8, and CD selections on transmission lines, waveguides and antennas. The Instructor's Resource CD offers additional suggestions, and students and instructors should check the appropriate sections of the website for updates.

Of course, these are just suggestions; the actual course content will reflect the interests of the instructors and the programs for which they are preparing their students . It is for this reason that we have tried to enhance the flexibility of this text with supplemental material covering a variety of electromagnetic topics. Your suggestions and contributions of additional topics are invited and welcomed. Let us know your thoughts.

## Acknowledgments

We are extremely proud of this new edition and the improvements made to virtually every aspect of the book, its supplements, and the web support. In very large part these upgrades are due to our adopting professors, their students, and the editorial advisory board members who volunteered to assist. All share our passion for electromagnetics. Adopters who taught from the first edition and passed along many helpful corrections and suggestions were:

Jonathon Wu – Windsor University
David Heckmann – University of North Dakota
Perry Wheless, Jr. – University of Alabama
Fran Harackiewicz – Southern Illinois University at Carbondale

Advisors who have given conceptual suggestions, read chapters closely, and critiqued the supplements in detail are:

Sven Bilén – Pennsylvania State University
Don Dudley – University of Arizona
Chuck Bunting – Oklahoma State University
Jim West – Oklahoma State University
Kent Chamberlin – University of New Hampshire
Christos Christodoulou – University of New Mexico
Larry Cohen – Naval Research Laboratory
Atef Elsherbeni – The University of Mississippi
Cindy Furse – University of Utah
Mike Havrilla – Air Force Institute of Technology
Anthony Martin – Clemson University
Wilson Pearson – Clemson University

The publication of an undergraduate textbook represents a close collaboration among authors, technical advisors, and the publisher's staff. We consider ourselves extremely fortunate to be working with a highly personal, committed, and passionate publisher like SciTech Publishing. President and editor Dudley Kay has believed in our approach and the book's potential for wide acceptance through two editions. Susan Manning is our production director, and we have learned to appreciate her with awe and astonishment for all the bits and pieces of a textbook that must be managed. Production Assistant Robert Lawless, an engineering graduate student at North Carolina State University, provided incredible support, understanding, and suggestions for reader clarity. Bob Doran's unfailing good humor and optimism is just what every author wants in a sales director.

Special thanks to Michael Georgiev, Sava Savov's graduate student, for his early work on the illustrations program, its technical accuracy, and checks on its conversion to two-color renderings. Graphic designer Kathy Gagne's cover art, chapter openings, web pages, and advertising pages are all a remarkably cohesive effort that helps support our vision brilliantly. Dr. Bob Roth and daughter Anne Roth have extended Kathy's design to beautifully matching student support and instructor feedback web pages.

The authors would like to acknowledge Professor Louis A. Frank, Dr. John B. Sigwarth, and NASA for permission to use the picture on the cover of this book, Professor Er-Wei Bai for discussions concerning MATLAB, and Professor Jon Kuhl for his support of this project.

## Errors and Suggestions

Our publisher assures us that the perfect book has yet to be written. We can reasonably expect some errors, but we need not tolerate them. We also strive for increased clarity on a challenging subject. We invite instructors, students, and individual readers to contribute to the errata list and recommendations via the feedback forms on the website. SciTech will maintain a continually updated errata sheet and corrected page PDFs on the website, and the corrections and approved suggestions will be incorporated into each new printing of the book. We vow to make this project "organic" in the sense that it will be continually growing, shedding dead and damaged leaves, and becoming continuously richer and more palatable. If we are successful and realize our goal, every reader and user of the book and its supplements will have a hand in its evolving refinements. We accept all responsibility for any errors and shortcomings and invite direct comments at any time.

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# Editorial Advisory Board in Electromagnetics

# Notation Table

| Coordinate System | Coordinates |
|---|---|
| Cartesian | $(x, y, z)$ |
| Cylindrical | $(\rho, \phi, z)$ |
| Spherical | $(r, \theta, \phi)$ |

| Quantity | Symbol | SI Unit | Abbreviations | Dimensions |
|---|---|---|---|---|
| Amount of substance | | mole | mol | |
| Angle | $\phi$, $\theta$ | radian | rad | |
| Electric current | $I$; $i$ | ampere | A | I |
| Length | $\mathscr{L}$, $\ell$, ... | meter | m | L |
| Luminous intensity | | candela | cd | |
| Mass | $M$; $m$ | kilogram | kg | M |
| Thermodynamic temperature | | Kelvin | K | K |
| Time | $t$, $T$ | second | s | T |
| Admittance | $Y$ | Siemens | S | $I^2T^3/M/L^2$ |
| Capacitance | $C$ | farad | F | $I^2T^4/M/L^2$ |
| Charge; point charge | $Q$; $q$ | Coulomb | C | IT |
| Charge density, line | $\rho_\ell$ | coulomb/meter | C/m | IT/L |
| Charge density, surface | $\rho_s$ | coulomb/meter$^2$ | C/m$^2$ | IT/L$^2$ |
| Charge density, volume | $\rho_v$ | coulomb/meter$^3$ | C/m$^3$ | IT/L$^3$ |
| Conductance | $G$ | Siemens | S | $I^2T^3/M/L^2$ |
| Conductivity | $\sigma$ | Siemens/meter | S/m | $I^2T^3/M/L^3$ |
| Current density | $\mathbf{J}$ | ampere/meter$^2$ | A/m$^2$ | I/L$^2$ |
| Current density, surface | $\mathbf{J}_\ell$ | ampere/meter | A/m | I/L |
| Electric flux density | $\mathbf{D}$ | coulomb/meter$^2$ | C/m$^2$ | IT/L$^2$ |
| Electric dipole moment | $\mathbf{p}$ | coulomb • meter | C • m | ITL |
| Electrical field intensity | $\mathbf{E}$ | Volt/meter | V/m | ML/I/T$^3$ |
| Electric potential; Voltage | $V$ | Volt | V | ML$^2$/I/T$^3$ |

| Quantity | Symbol | SI Unit | Abbreviations | Dimensions |
|---|---|---|---|---|
| Energy, Work | $W$ | joule | J | $ML^2/T^2$ |
| Energy density, volume | $w$ | joule/meter$^3$ | $J/m^3$ | $M/L/T^2$ |
| Force | $\mathbf{F}$ | Newton | N | $ML/T^2$ |
| Force density, volume | $\mathbf{f}$ | Newton/meter$^3$ | $N/m^3$ | $M/L^2/T^2$ |
| Frequency | $f$ | Hertz | Hz | 1/T |
| Impedance | $Z$ | Ohm | Ω | $ML^2/I^2/T^3$ |
| Inductance | $L$; $M$ | Henry | H | $ML^2/I^2/T^2$ |
| Magnetic dipole moment | $\mathbf{m}$ | ampere • meter$^2$ | A • $m^2$ | $IL^2$ |
| Magnetic field intensity | $\mathbf{H}$ | ampere/meter | A/m | I/L |
| Magnetic flux | $\Phi$ | Weber | Wb | $ML^2/I/T^2$ |
| Magnetic flux density | $\mathbf{B}$ | Tesla | T | $M/I/T^2$ |
| Magnetic vector potential | $\mathbf{A}$ | Tesla • meter | T • m | $ML/I/T^2$ |
| Magnetic voltage | $V_m$ | ampere | A | I |
| Magnetization | $\mathbf{M}$ | ampere/meter | A/m | I/L |
| Period | T | second | s | T |
| Permeability | $\mu$ | Henry/meter | H/m | $ML/I^2/T^2$ |
| Permittivity | $\varepsilon$ | farad/meter | F/m | $I^2T^4/M/L^3$ |
| Polarization | $\mathbf{P}$ | coulomb/meter$^2$ | $C/m^2$ | $IT/L^2$ |
| Power | $P$ | watt | W | $ML^2/T^3$ |
| Power density, volume | $p$ | watt/meter$^3$ | $W/m^3$ | $M/L/T^3$ |
| Power density, surface | S | watt/meter$^2$ | $W/m^2$ | $M/L/T^2$ |
| Propagation constant | $\gamma$ | 1/meter | 1/m | 1/L |
| Reluctance | $\Gamma_m$ | 1/Henry | 1/H | $I^2T^2/M/L^2$ |
| Resistance | $R$ | Ohm | Ω | $ML^2/I^2/T^3$ |
| Torque | $\mathbf{T}$ | Newton • meter | N • m | $ML^2/T^2$ |
| Velocity | $\mathbf{v}$ | meter/second | m/s | L/T |
| Wavelength | $\lambda$ | meter | m | L |

CHAPTER 5

# Time-Varying Electromagnetic Fields



The subject of time-varying electromagnetic fields will be the central theme throughout the remainder of this text. Here and in the following chapters we will generalize to the time-varying case the static electric and magnetic fields that were reviewed in Chapter 2 and Chapter 3. In doing this, we must first appreciate the insight of the great nineteenth century theoretical physicist James Clerk Maxwell who was able to write down a set of equations that described electromagnetic fields. These equations have survived unblemished for almost two centuries of experimental and theoretical questioning. The equations are now considered to be on an equal footing with the equations of Isaac Newton and many of Albert Einstein's thoughts on relativity.[1] We will concern ourselves here and now with what was uncovered and explained at the time of Maxwell's life.

## 5.1 Faraday's Law of Induction

The first time-varying electromagnetic phenomenon that usually is encountered in an introductory course dedicated to the study of time-varying electrical circuits is the determination of the electric potential across an inductor

[1] We can only speculate about what these three giants would say and write if they met today at a café and had only one "back of an envelope" between them.

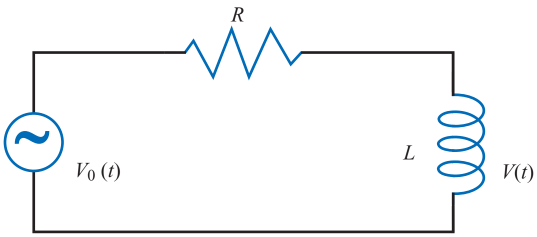



**FIGURE 5–1**

A simple electrical circuit consisting of an AC voltage source $V_0(t)$, an inductor, and a resistor. The voltage across the inductor is $V(t)$.

that is inserted in an electrical circuit. A simple circuit that exhibits this effect is shown in Figure 5–1.

The voltage across the inductor is expressed with the equation

$$V(t) = L\frac{dI(t)}{dt} \tag{5.1}$$

where $L$ is the inductance, the units of which are henries, $V(t)$ is the time-varying voltage across the inductor, and $I(t)$ is the time-varying current that passes through the inductor. The actual dependence that these quantities have on time will be determined by the voltage source. For example, a sinusoidal voltage source in a circuit will cause the current in that circuit to have a sinusoidal time variation or a temporal pulse of current will be excited by a pulsed voltage source. In Chapter 3, we recognized that a time-independent current could create a time-independent magnetic field and that a time-independent voltage was related to an electric field. These quantities are also related for time-varying cases as will be shown here.

The actual relation between the electric and the magnetic field components is computed from an experimentally verified effect that we now call ***Faraday's law***. This is written as

$$V(t) = -\frac{d\Psi_m(t)}{dt} \tag{5.2}$$

where $\Psi_m(t)$ is the total time-varying magnetic flux that passes through a surface. This law states that a voltage $V(t)$ will be induced in a closed loop that completely surrounds the surface through which the magnetic field passes. The voltage $V(t)$ that is induced in the loop is actually a voltage or potential difference $V(t)$ that exists between two points in the loop that are separated by an infinitesimal distance. The distance is so small that we can actually think of the loop as being closed. The polarity of the induced voltage will be such that it opposes the change of the magnetic flux, hence a minus sign appears in (5.2). The voltage can be computed from the line integral

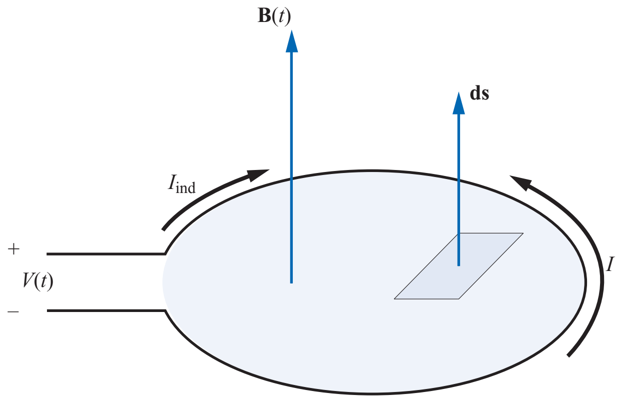



**FIGURE 5–2**

A loop through which a time-varying magnetic field passes.

of the electric field between the two points. This effect is also known as ***Lenz's law***.

A schematic representation of this effect is shown in Figure 5–2. Small loops as indicated in this figure, and which have a cross-sectional area $\Delta s$, are used to detect and plot the magnitude of time-varying magnetic fields in practice. We will assume that the loop is sufficiently small or that we can let it shrink in size so that it is possible to approximate $\Delta s$ with the differential surface area $|\mathbf{ds}|$. The vector direction associated with **ds** is normal to the plane containing the differential surface area. If the stationary orientation of the loop **ds** is perpendicular to the magnetic flux density $\mathbf{B}(t)$, zero magnetic flux will be captured by the loop and $V(t)$ will be zero. By rotating this loop about a known axis, it is also possible to ascertain the vector direction of $\mathbf{B}(t)$ by correlating the maximum detected voltage $V(t)$ with respect to the orientation of the loop. Recall from our discussion of magnetic circuits that we used the total magnetic flux $\Psi_m$. This is formally written in terms of an integral. In particular, the magnetic flux that passes through the loop is given by

$$\Psi_m(t) = \int_{\Delta s} \mathbf{B} \bullet \mathbf{ds} \tag{5.3}$$

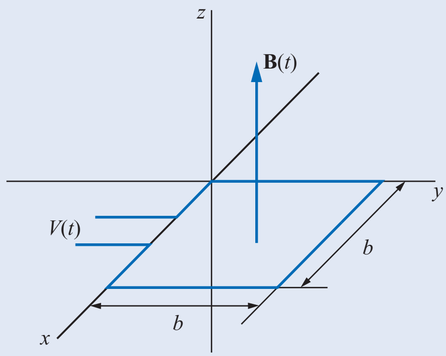
As we will see later, either the magnetic flux density or the surface area or both could be changing in time. The scalar product reflects the effects arising from an arbitrary orientation of the loop with respect to the orientation of the magnetic flux density. It is important to realize that the loops that we are considering may not be wire loops. The loops could just be closed paths.

## EXAMPLE 5.1

Let a stationary square loop of wire lie in the $x$–$y$ plane that contains a spatially homogeneous time-varying magnetic field.

$$\mathbf{B}(\mathbf{r}, t) = B_0 \sin \omega t \mathbf{u_z}$$

Find the voltage $V(t)$ that could be detected between the two terminals that are separated by an infinitesimal distance.




**Answer.** The magnetic flux that is enclosed within the loop is given by

$$\Psi_m = \int_{\Delta s} \mathbf{B} \bullet \mathbf{ds} = \int_{y=0}^{b} \int_{x=0}^{b} (B_0 \sin \omega t \mathbf{u_z}) \bullet (dxdy\mathbf{u_z}) = B_0 b^2 \sin \omega t$$

The induced voltage in the loop is found from Faraday's law (5.2)

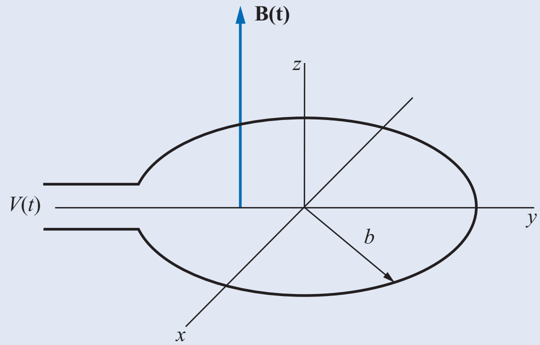
$$V(t) = -\frac{d\Psi_m(t)}{dt} = -\omega B_0 b^2 \cos \omega t$$

## EXAMPLE 5.2

Let a stationary loop of wire lie in the $x$–$y$ plane that contains a spatially inhomogeneous time-varying magnetic field given by

$$\mathbf{B}(r, t) = B_0 \cos\left(\frac{\pi\rho}{2b}\right) \cos \omega t \mathbf{u_z}$$

where the amplitude of the magnetic flux density is $\mathbf{B}_0 = 2$ T, the radius of the loop is $b = 0.05$ m, and the angular frequency of oscillation of the time-varying magnetic field is $\omega = 2\pi f = 314$ s$^{-1}$. The center of the loop is at the point $\rho = 0$. Find the voltage $V(t)$ that could be detected between the two terminals that are separated by an infinitesimal distance.




**Answer.** The magnetic flux $\Psi_m$ that is enclosed within the "closed" loop is given by

$$\Psi_m = \int_{\Delta s} \mathbf{B} \bullet \mathbf{ds} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{b} \left[B_0 \cos\left(\frac{\pi\rho}{2b}\right) \cos \omega t \mathbf{u_z}\right] \bullet [\rho d\rho d\phi \mathbf{u_z}]$$

The integral over $\phi$ yields a factor of $2\pi$ while the integral over $\rho$ is solved via integration by parts. The result is

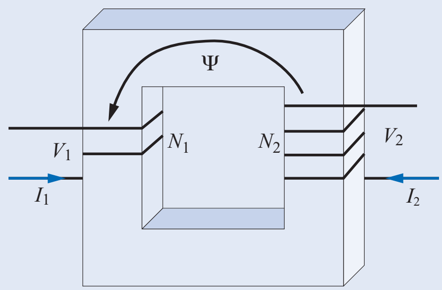
$$\Psi_m = (B_0 \cos \omega t)(2\pi)\left[\frac{4b^2}{\pi^2}\left(\frac{\pi}{2} - 1\right)\right] = \frac{8b^2}{\pi}\left(\frac{\pi}{2} - 1\right)B_0 \cos \omega t$$

The induced voltage in the loop is obtained from Faraday's law (5.2)

$$V(t) = -\frac{d\Psi_m(t)}{dt} = \frac{8b^2}{\pi}\left(\frac{\pi}{2} - 1\right)B_0 \omega \sin \omega t = 228.2 \sin(314t) \text{ V}$$

## EXAMPLE 5.3

Another application of Faraday's law is the explanation of how an ideal ***transformer*** operates. Find the voltage that is induced in loop two if a time-varying voltage $V_1$ is connected to loop one.



**Answer.** To solve this problem we will first assume that we are dealing with an ideal transformer. This implies that the core has infinite permeability, or $\mu_r = \infty$. This will cause all of the magnetic flux to be confined to the core.

Application of voltage $V_1$ to the primary (left hand) side of the transformer will cause a current $I_1$ to flow, also causing the flux to circulate within the core, according to the right hand rule. The voltage on the primary side and the resulting flux, $\Psi_m$, are related by the equation

$$V_1 = -N_1 \frac{d\Psi_m}{dt}$$

This flux induces a voltage in the windings on the secondary (right hand) side equal to

$$V_2 = -N_2 \frac{d\Psi_m}{dt}$$

Dividing the second equation by the first equation yields

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

Because this is an ideal lossless transformer, all of the instantaneous power delivered to the primary will be available at the secondary, or $P_1 = P_2$, which means $(I_1)(V_1) = (I_2)(V_2)$.

Given this relationship, as well as the relationship of the third equation, we can also show that

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

Finally, we can use the relationships for primary and secondary resistance, that is

$$R_{\text{pri}} = \frac{V_1}{I_1} \quad \text{and} \quad R_{\text{sec}} = \frac{V_2}{I_2}$$

to show that the ratio of the primary resistance to the secondary resistance equals the square of the turns ratio or

$$\frac{R_{\text{pri}}}{R_{\text{sec}}} = \left(\frac{N_1}{N_2}\right)^2$$

Note that even though a real transformer has some loss, it is a highly efficient device, typically having an efficiency of 95–98%, as a properly designed transformer has very low core losses. Also, the resistance of the primary winding is normally very small, as is the secondary winding, unless we are using the transformer to match impedances. Thus, compared to other electrical devices, transformers are some of the most efficient devices available.

Since the two terminals in Figure 5–2 are separated by a very small distance, we will be permitted to assume that they actually are touching, at least in a mathematical sense even though they must be separated physically. This will allow us to consider the loop to be a closed in various integrals that follow but still permit us to detect a potential difference between the two terminals. The magnetic flux $\Psi_m(t)$ can be written in terms of the magnetic flux density

$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}$ and the voltage $V(t)$ can be written in terms of the electric field $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}$. This yields the result

$$\oint \mathbf{E} \bullet \mathbf{dl} = -\frac{d}{dt} \int_{\Delta s} \mathbf{B} \bullet \mathbf{ds} \tag{5.4}$$

It is worth emphasizing the point that this electric field is the component of the electric field that is tangential to the loop since this is critical in our argument. In addition, (5.4) includes several possible mechanisms in which the magnetic flux could change in time. Either the magnetic flux density changes in time, the cross-sectional area changes in time, or there is a combination of the two mechanisms. These will be described below.

Although both the electric and magnetic fields depend on space and time, we will not explicitly state this fact in every equation that follows. This will conserve time, space, and energy if we now define and later understand that $\mathbf{E}(\mathbf{r}, t) \equiv \mathbf{E}$ and $\mathbf{B}(\mathbf{r}, t) \equiv \mathbf{B}$ in the equations. This short-hand notation will also allow us to remember more easily the important results in the following material. In this notation, the independent spatial variable $\mathbf{r}$ refers to a three-dimensional position vector where

$$\mathbf{r} = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z} \tag{5.5}$$

in Cartesian coordinates. The independent variable $t$ refers to time. We must keep this notation in our minds in the material that follows.

The closed-line integral appearing in (5.4) can be converted into a surface integral using Stokes's theorem. We obtain for the left side of (5.4)

$$\oint \mathbf{E} \bullet \mathbf{dl} = \int_{\Delta s} \nabla \times \mathbf{E} \bullet \mathbf{ds} \tag{5.6}$$

Let us assume initially that the surface area of the loop does not change in time. This implies that $\Delta s$ is a *constant*. In this case, the time derivative can then be brought inside the integral

$$\int_{\Delta s} (\nabla \times \mathbf{E}) \bullet \mathbf{ds} = -\int_{\Delta s} \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds} \tag{5.7}$$

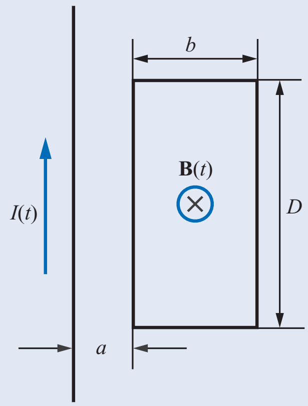
The two integrals will be equal over any arbitrary surface area if and only if the two integrands are equal. This means that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.8}$$

The integral representation (5.4) and the differential representation (5.8) are equally valid in describing the physical effects that are included in Faraday's law. These equations are also called ***Faraday's law of induction*** in honor of their discoverer. Faraday stated the induction's law in 1831 after making the assumption that a new phenomenon called an ***electromagnetic field*** would surround every electric charge.

## EXAMPLE 5.4

A small rectangular loop of wire is placed next to an infinitely long wire carrying a time-varying current. Calculate the current $i(t)$ that flows in the loop if the conductivity of the wire is $\sigma$. To simplify the calculation, we will neglect the magnetic field created by the current $i(t)$ that passes through the wire of the loop.




**Answer.** From the left-hand side of (5.4), we write the induced current $i(t)$ as

$$\oint \mathbf{E} \cdot \mathbf{dl} = \oint \frac{\mathbf{J} \cdot \mathbf{dl}}{\sigma} = \frac{i(2D + 2b)}{\sigma A} = i(t)R$$

where $A$ is the cross-sectional area of the wire and $R$ is the resistance of the wire. The current density is assumed to be constant over the cross-section of the wire. The magnetic flux density of the infinite wire is found from Ampere's law

$$B(t) = \frac{\mu_0 I(t)}{2\pi\rho}$$

The right-hand side of (5.4) can be written as

$$\begin{aligned}
-\frac{d}{dt}\int_{\Delta s} \mathbf{B} \cdot \mathbf{ds} &= -\frac{d}{dt}\left[\int_{z=0}^{D} \int_{\rho 5a}^{a+b} \frac{\mu_0 I(t)}{2\pi\rho} d\rho dz\right] \\
&= -\frac{d}{dt}\left[D\frac{\mu_0 I(t)}{2\pi}\ln\left(\frac{b+a}{a}\right)\right] \\
&= -\left[\frac{\mu_0 D}{2\pi}\ln\left(\frac{b+a}{a}\right)\right]\frac{dI(t)}{dt}
\end{aligned}$$

Hence, the current $i(t)$ that is induced in the loop is given by

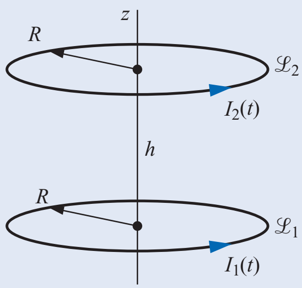
$$i(t) = -\frac{1}{R}\left[\frac{\mu_0 D}{2\pi}\ln\left(\frac{b+a}{a}\right)\right]\frac{dI(t)}{dt}$$

The term in the brackets corresponds to a term that is called the ***mutual inductance*** $M$ between the wire and the loop. If we had included the effects of the magnetic field created by the current $i(t)$ in the loop, an additional term proportional to $di(t)/dt$ would appear in the equation. In this case, a term called the ***self-inductance*** $L$ of the loop would be found. Hence a differential equation for $i(t)$ would have to be solved in order to incorporate the effects of the self-inductance of the loop.

It is also important to calculate the mutual inductance between two coils. This can be accomplished by assuming that a time-varying current in one of the coils would produce a time-varying magnetic field in the region of the second coil. This will be demonstrated in the following example.

## EXAMPLE 5.5

Find the normalized mutual inductance $M/\mu_0$ of the system consisting of two similar parallel wire loops with a radius $R = 50$ cm if their centers are separated by a distance $h = 2R$ (Helmholtz's coils).




**Answer.** The magnetic flux density can be replaced with the magnetic vector potential. Using $\mathbf{B} = \nabla \times \mathbf{A}$ and Stokes's theorem, we can reduce the surface integral for the magnetic flux to the following line integral:

$$\Psi_{m12} = \int_{\Delta s_2} (\nabla \times \mathbf{A}) \bullet \mathbf{ds}_2 = \oint_{\mathscr{L}_2} \mathbf{A} \bullet \mathbf{dl}_2$$

The magnetic vector potential is a solution of the vector Poisson's equation. With the assumption that the coil has a small cross-section, the volume integral can be converted to the following line integral:

$$\mathbf{A} = \frac{\mu_0 I_1}{4\pi} \oint_{\mathscr{L}_1} \frac{\mathbf{dl}_1}{R_{12}}$$

where $R_{12}$ is a distance between the source (1) and the observer (2). The definition of the mutual inductance

$$M = \frac{\Psi_{m12}}{I_1}$$

yields the following double integral for the normalized mutual inductance

$$\frac{M}{\mu_0} = \frac{1}{4\pi} \oint_{\mathscr{L}_2} \oint_{\mathscr{L}_1} \frac{\mathbf{dl}_1 \bullet \mathbf{dl}_2}{R_{12}}$$

For the case of two identical parallel loops, we obtain

$$\frac{M}{\mu_0} = \frac{R^2}{2} \int_0^{2\pi} \frac{\cos\phi d\phi}{\sqrt{h^2 + 2R^2 - 2R^2 \cos\phi}}$$

For the special case ($h = 2R$), this single integral can be further simplified.

$$\frac{M}{\mu_0} = R \times \left( \frac{1}{2\sqrt{2}} \int_0^{2\pi} \frac{\cos\phi d\phi}{\sqrt{3 - \cos\phi}} \right) = R \times \text{constant}$$

This shows that the mutual inductance increases linearly with the increasing of the wire radius $R$. The integral for the constant $C$ can be solved *numerically* by applying the MATLAB function ***quad***. Assuming a radius $R = 0.5$ m and a constant $C = 0.1129$, the normalized mutual inductance is computed to be $M/\mu_0 = 0.0564$.

In the derivation of (5.7), an assumption was made that the area $\Delta s$ of the loop did not change in time and only a time-varying magnetic field existed in space. This assumption need not always be made in order for electric fields to be generated by magnetic fields. We have to be thankful for the fact that the effect can be *generalized* since much of the conversion of electric energy to mechanical energy or mechanical energy to electrical energy is based on this phenomenon.

In particular, let us assume that a conducting bar moves with a velocity $\mathbf{v}$ through a uniform time-independent magnetic field $\mathbf{B}$ as shown in Figure 5–3. The wires that are connected to this bar are parallel to the magnetic field and are connected to a voltmeter that lies far beneath the plane of the moving bar. From the Lorentz force equation, we can calculate the force $\mathbf{F}$ on the freely mobile charged particles in the conductor. Hence, one end of the bar will become positively charged, and the other end will have an excess of negative charge.

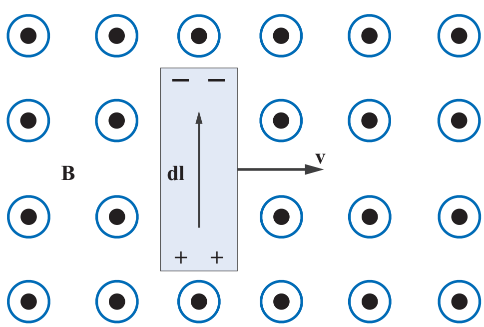



**FIGURE 5–3**

A conducting bar moving in a uniform time-independent magnetic field that is directed out of the paper. Charge distributions of the opposite sign appear at the two ends of the bar.

Since there is a charge separation in the bar, there will be an electric field that is created in the bar. Since the net force on the bar is equal to zero, the electric and magnetic contributions to the force cancel each other. This results in an electric field that is

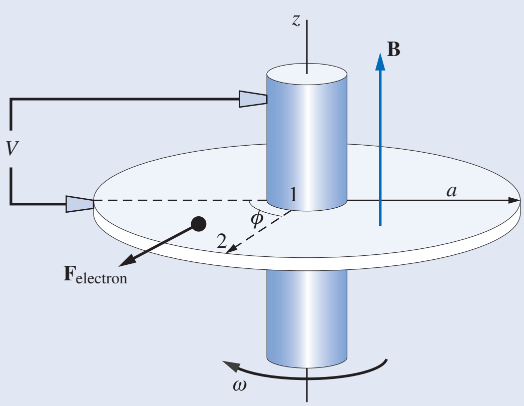
$$\mathbf{E} \equiv \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \tag{5.9}$$

This electric field can be interpreted to be an induced field acting in the direction along the conductor that produces a voltage $V$, and it is given by

$$V = \int_a^b (\mathbf{v} \times \mathbf{B}) \bullet \mathbf{dl} \tag{5.10}$$

## EXAMPLE 5.6

A ***Faraday disc generator*** consists of a circular metal disc rotating with a constant angular velocity $\omega = 600\ \text{s}^{-1}$ in a uniform time-independent magnetic field. A magnetic flux density $\mathbf{B} = B_0\mathbf{u_z}$ where $B_0 = 4$ T is parallel to the axis of rotation of the disc. Determine the induced open-circuit voltage that is generated between the brush contacts that are located at the axis and the edge of the disc whose radius is $a = 0.5$ m.




**Answer.** An electron at a radius $\rho$ from the center has a velocity $\omega\rho$ and therefore experiences an outward directed radial force $-q\omega\rho B_0$. The Lorentz force acting on the electron is

$$-q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] = 0$$

At ***equilibrium***, we find that the electric field can be determined from the Lorentz force equation to be directed radially inward and it has a magnitude $\omega\rho B_0$. Hence we write

$$V = -\int_1^2 (\mathbf{v} \times \mathbf{B}) \bullet \mathbf{dl} = -\int_0^a [(\omega\rho\mathbf{u}_\phi) \times B_0\mathbf{u_z}] \bullet d\rho\mathbf{u}_\rho$$

$$= \omega B_0 \int_a^0 \rho d\rho = -\frac{\omega B_0 a^2}{2} = -300 \text{ V}$$

which is the potential generated by the Faraday disc generator.

If the bar depicted in Figure 5–3 were moving through a time-dependent magnetic field instead of a constant magnetic field, then we would have to add together the potential caused by the motion of the bar and the potential caused by the time-varying magnetic field. This implies that the principle of superposition applies for this case. This is a good assumption in a vacuum or in any linear medium.

## EXAMPLE 5.7

A rectangular loop rotates through a time-varying magnetic flux density $\mathbf{B} = B_0 \cos(\omega t)\mathbf{u_y}$. The loop rotates with the same angular frequency $\omega$. Calculate the induced voltage at the terminals.

**Answer.** Due to the rotation of the loop, there will be two components to the induced voltage. The first is due to the motion of the loop, and the second is due to the time-varying magnetic field.

The voltage due to the rotation of the loop is calculated from

$$V_{\text{rotation}} = \int_{b/2}^{-b/2} \mathbf{v} \times \mathbf{B} \bullet \mathbf{dl}\Big|_{\text{bottom edge}} + \int_{-b/2}^{b/2} \mathbf{v} \times \mathbf{B} \bullet \mathbf{dl}\Big|_{\text{top edge}}$$

The contributions from either end will yield zero. We write

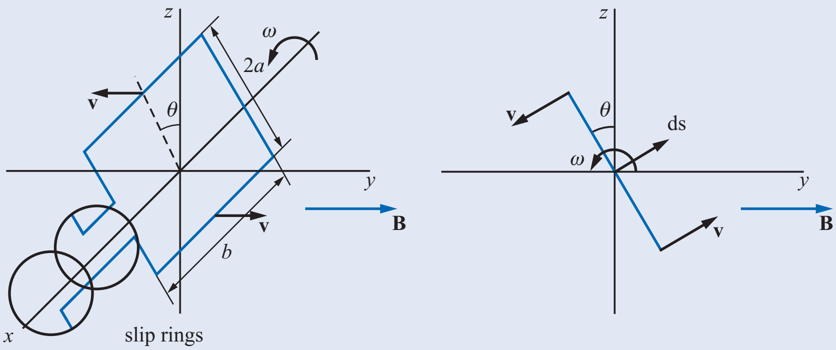
$$V_{\text{rotation}} = \int_{b/2}^{-b/2} vB_0 \cos(\omega t) \sin\theta(-\mathbf{u_x}) \bullet (dx\mathbf{u_x})$$

$$+ \int_{-b/2}^{b/2} vB_0 \cos(\omega t) \sin\theta(\mathbf{u_x}) \bullet (dx\mathbf{u_x})$$

The angle $\theta = \omega t$, and the velocity $v = \omega a$. Hence the term of the induced voltage due to the rotation of the loop yields

$$V_{\text{rotation}} = \omega B_0 ba(2\omega t)$$

We recognize that the area of the loop is equal to $\Delta s = 2ab$.




(a) (b)

From (5.4), we can compute the voltage due to the time-varying magnetic field. In this case, we note that

$$\mathbf{s} = [\cos(\omega t)\mathbf{u_y} + \sin(\omega t)\mathbf{u_z}]dxa$$

Therefore, the voltage due to time variation of the magnetic field is

$$V_{\text{time varying}} = -\int_{\Delta s} \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds}$$

$$= \omega B_0 \int_{x5-b/2}^{b/2} \int_{z5-a}^{a} \sin(\omega t)\mathbf{u_y} \bullet [\cos(\omega t)\mathbf{u_y} + \sin(\omega t)\mathbf{u_z}]\, dz\, dx$$

The integration leads to

$$V_{\text{time varying}} = \omega B_0 ba \sin(2\omega t)$$

The total voltage is given by the sum of the voltage due to rotation and the voltage due to the time variation of the magnetic field.

$$V = V_{\text{rotation}} + V_{\text{time varying}} = 2\omega B_0 ba \sin(2\omega t)$$

This results in the generation of the second harmonic.

We can apply a repeated vector operation to Faraday's law of induction that is given in (5.8) to obtain an equation that describes another feature of time-varying magnetic fields. We take the divergence of both sides of (5.8) and interchange the order of differentiation to obtain

$$\nabla \bullet \nabla \times \mathbf{E} = -\nabla \bullet \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \bullet \mathbf{B}) \tag{5.11}$$

The first term $\nabla \bullet \nabla \times \mathbf{E}$ is equal to zero since the divergence of the curl of a vector is equal to zero by definition. This follows also from Figure 5–2 where the electric field is constrained to follow the loop since that is the only component that survives the scalar product of $\mathbf{E} \bullet \mathbf{dl}$. The electric field can neither enter nor leave the loop, which would be indicative of a nonzero divergence. Nature is kind to us in that it frequently lets us interchange the orders of differentiation without inciting any mathematical complications. This is a case where it can be done. Hence for any arbitrary time dependence, we again find that

$$\boxed{\nabla \bullet \mathbf{B} = 0} \tag{5.12}$$

This statement that is valid for time-dependent cases is the same result that was given in Chapter 3 as a postulate for static fields. It also continues to reflect the fact that we have not found magnetic monopoles in nature and time varying magnetic field lines are continuous.

Let us integrate (5.12) over an arbitrary volume $\Delta v$. This volume integral can be converted to a closed-surface integral using the divergence theorem

$$\int_{\Delta v} \nabla \bullet \mathbf{B}\, dv = \oint \mathbf{B} \bullet \mathbf{ds} \tag{5.13}$$

from which we write

$$\boxed{\oint \mathbf{B} \bullet \mathbf{ds} = 0} \tag{5.14}$$

Equation (5.14) is also valid for time-independent electromagnetic fields.

## 5.2 Equation of Continuity

Before obtaining the next equation of electromagnetics, it is useful to step back and derive the equation of continuity. In addition, we must understand the ramifications of this equation. The equation of continuity is fundamental at this point in developing the basic ideas of electromagnetic theory. It can also be applied in several other areas of engineering and science, so the process of understanding this theory will be time well spent.

In order to derive the equation of continuity, let us consider a model that assumes a stationary number of positive charges are located initially at the center of a transparent box whose volume is $\Delta v = \Delta x \Delta y \Delta z$. These charges are at the prescribed positions within the box for times $t \leqslant 0$. A one-dimensional view of this box consists of two parallel planes, and it is shown in Figure 5–4$a$. We will neglect the Coulomb forces between the individual

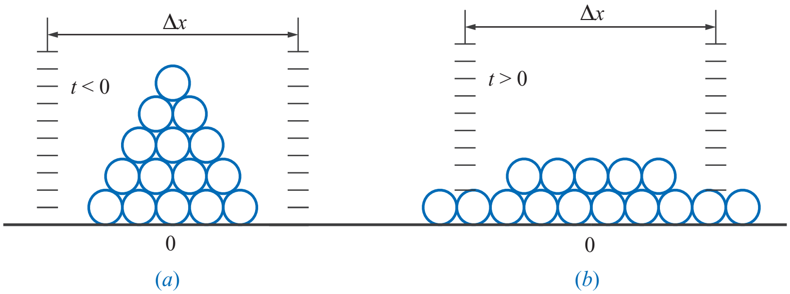



**FIGURE 5–4**

**(a)** Charges centered at $x = 0$ at time $t < 0$ are allowed to expand at $t = 0$.
**(b)** As time increases some of these charges may pass through the screens at $x = \pm\Delta x / 2$.

charges. If we are uncomfortable with this assumption of noninteracting charged particles, we could have assumed alternatively that the charges were just noninteracting gas molecules or billiard balls and derived a similar equation for these objects. The resulting equation could then be multiplied by a charge $q$ that would be impressed on an individual entity. For times $t \geqslant 0$, the particles or the charges can start to move and actually leave through the two screens as time increases. This is depicted in Figure 5–4*b*. The magnitude of the cross-sectional area of a screen is equal to $\Delta s = \Delta y \Delta z$. The charges that leave the "screened-in region" will be in motion. Hence, those charges that leave the box from either side will constitute a current that emanates from the box. Due to our choice of charges within the box having a positive charge, the direction of this current $I$ will be the same direction as the motion of the charge.

Rather than just examine the small number of charges depicted in Figure 5–4, let us assume that there is now a large number of them. We will still neglect the Coulomb force between charges. The number of charges will be large enough so it is prudent to describe the charge within the box with a charge density $\rho_v$ where $\rho_v = \Delta Q/\Delta v$ and $\Delta Q$ is the total charge within a volume $\Delta v$. Hence, the decrease of the charge density $\rho_v$ acts as a source for the total current $I$ that leaves the box as shown in Figure 5–5.

A temporal *decrease* of charge density within the box implies that charge leaves the box since the charge is neither destroyed nor does it recombine with charge of the opposite sign. The total current that leaves the box through any portion of the surface is due to a decrease of the charge within the box. The magnitude of this current is expressed by

$$I = -\frac{dQ}{dt} \tag{5.15}$$

These are real charges, and the current that we are describing is not the displacement current that we will encounter later.

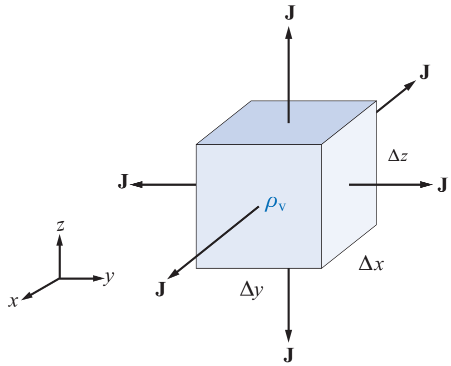



**FIGURE 5–5**

Charge within the box leaves through the walls.

Equation (5.15) can be rewritten as

$$\oint_{\Delta s} \mathbf{J} \bullet \mathbf{ds} = -\int_{\Delta v} \frac{\partial \rho_v}{\partial t} dv \tag{5.16}$$

where we have taken the liberty of summing up the six currents that leave the six sides of the box that surrounds the charges; this summation is expressed as a closed-surface integral. The closed-surface integral given in (5.16) can be converted to a volume integral using the divergence theorem. Hence (5.16) can be written as

$$\int_{\Delta v} \nabla \bullet \mathbf{J} dv = -\int_{\Delta v} \frac{\partial \rho_v}{\partial t} dv \tag{5.17}$$

Since this equation must be valid for any arbitrary volume, we are left with the conclusion that the two integrands must be equal, from which we write

$$\boxed{\nabla \bullet \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0} \tag{5.18}$$

Equation (5.18) is the ***equation of continuity*** that we are seeking. Note that this equation has been derived using very simple common sense arguments. However, we can show the same result by a more rigorous argument proving that this expression holds under all known circumstances. Although we have derived it using finite-sized volumes, the equation is valid at a point. Its importance will be noted in the next section where we will follow in the footsteps of James Clerk Maxwell.

We recall from our first course that dealt with circuits that the Kirchhoff's current law stated that the net current entering or leaving a node was equal to zero. Charge is neither created nor destroyed in this case. This is shown in Figure 5–6. The dashed lines represent a closed surface that surrounds the node. The picture shown in Figure 5–5 generalizes this node to three dimensions.

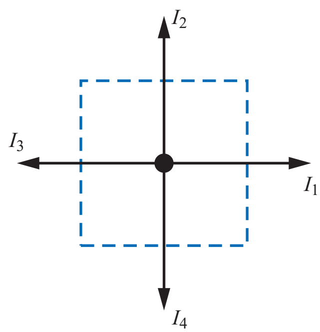



**FIGURE 5–6**

A closed surface (represented by dashed lines) surrounding a node.

## EXAMPLE 5.8

Charges are introduced into the interior of a conductor during the time $t < 0$. Calculate how long it will take for these charges to move to the surface of the conductor so the interior charge density $\rho_v = 0$ and interior electric field $\mathbf{E} = 0$.

**Answer.** Introduce Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ into the equation of continuity

$$\sigma(\nabla \cdot \mathbf{E}) = -\frac{\partial \rho_v}{\partial t}$$

The electric field is related to the charge density through Poisson's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}$$

Hence we obtain the differential equation

$$\frac{d\rho_v}{dt} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

whose solution is

$$\rho_v = \rho_{v0} e^{-\left(\frac{\sigma}{\varepsilon}\right)t}$$

The initial charge density $\rho_{v0}$ will decay to $[1/e \approx 37\%]$ of its initial value in a time $\tau = \varepsilon/\sigma$, which is called the ***relaxation time***. For copper, this time is

$$\tau = \frac{\varepsilon_0}{\sigma} = \frac{\frac{1}{36\pi} \times 10^{-9}}{5.8 \times 10^{7}} \approx 1.5 \times 10^{-19} \text{ s}$$

Other effects that are not described here may cause this time to be different. Relaxation times for insulators may be hours or days.

**EXAMPLE 5.9**

The current density is $\mathbf{J} = e^{-x^2}\mathbf{u_x}$. Find the time rate of increase of the charge density at $x = 1$.

**Answer.** From the equation of continuity (5.18), we write

$$\frac{\partial \rho_v}{\partial t} = -\nabla \bullet \mathbf{J} \Rightarrow \frac{\partial \rho_v}{\partial t} = -\frac{\partial J_x}{\partial x} = 2x\,e^{-x^2}\Big|_{x51} = 0.736\ \text{A/m}^3$$

**EXAMPLE 5.10**

The current density in a certain region may be approximated with the function

$$\mathbf{J} = J_0 \frac{e^{-t/\tau}}{r}\mathbf{u}_r$$

in spherical coordinates. Find the total current that leaves a spherical surface whose radius is $a$ at the time $t = \tau$. Using the equation of continuity, find an expression for the charge density $\rho_v(r, t)$.

**Answer.** The total current that leaves the spherical surface is given by

$$I = \oint \mathbf{J} \bullet d\mathbf{s}\Big|_{r5a,\ t5\tau} = 4\pi a^2 \left(\frac{J_0 e^{-t/\tau}}{a}\right)\Bigg|_{t5\tau} = 4\pi a J_0 e^{-1}\,(\text{A}).$$

In spherical coordinates, the equation of continuity that depends only upon the radius $r$ is written as

$$\frac{\partial \rho_v}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_0 \frac{e^{-t/\tau}}{r}\right) = -J_0 \frac{e^{-t/\tau}}{r^2}$$

Hence, after integration the charge density is given by

$$\rho_v = \int \frac{\partial \rho_v}{\partial t} dt \Rightarrow \rho_v = J_0 \frac{\tau e^{-t/\tau}}{r^2}\ \ (\text{C/m}^3)$$

where the arbitrary constant of integration is set equal to zero.

## 5.3 Displacement Current

Our first encounter with time-varying electromagnetic fields yielded Faraday's law of induction in equation (5.2). The next encounter will illustrate the genius of James Clerk Maxwell. Through his efforts in the nineteenth

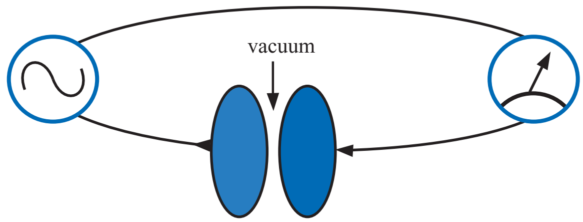



**FIGURE 5–7**

An elementary circuit consisting of an ideal parallel plate capacitor connected to an AC voltage source and an AC ammeter.

century, we are now able to answer a fundamental question that would arise when analyzing a circuit in the following *gedanken experiment*. Let us connect two wires to the two plates of an ideal capacitor consisting of two parallel plates separated by a vacuum and an AC voltage source as shown in Figure 5–7. An AC ammeter is also connected in series with the wires in this circuit, and it measures a constant value of AC current $I$. Two questions might enter our minds at this point.

1. How can the ammeter read any value of current since the capacitor is an open circuit and the current that passes through the wire would be impeded by the vacuum that exists between the plates?
2. What happens to the time-varying magnetic field that is created by the current and surrounds the wire as we pass through the region between the capacitor plates?

The answer to the first question will require that we first reexamine the equations that we have obtained up to this point and then interpret them, guided by the light that has been turned on by Maxwell. In particular, let us write the second postulate of steady magnetic fields—Ampere's law. This postulate states that a magnetic field **B** is created by a current **J**. It is written here

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.19}$$

Let us take the divergence of both sides of this equation. The term on the left-hand side

$$\nabla \bullet \nabla \times \mathbf{B} = 0 \tag{5.20}$$

by definition. Applying the divergence operation to the term on the right-hand of (5.19), we find that

$$\mu_0 \nabla \bullet \mathbf{J} = 0 \tag{5.21}$$

This, however, is not compatible with the equation of continuity (5.18) that we have just shown to be true under all circumstances.

To get out of this dilemma, Maxwell postulated the existence of another type of current in nature. This current would be in addition to the conduction current discussed in Chapter 3 and a convection current that would be created by charge passing through space with a constant drift velocity. The new current with a density $\mathbf{J_d}$ is called a ***displacement current***, and it is found by incorporating the equation for the displacement flux density $\mathbf{D}$ into the equation of continuity by use of Gauss's law. Hence

$$\nabla \bullet \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$$

$$\nabla \bullet \mathbf{J} + \frac{\partial(\nabla \bullet \mathbf{D})}{\partial t} = \nabla \bullet \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \tag{5.22}$$

where we have freely interchanged the order of differentiation. The displacement current density is identified as

$$\boxed{\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t}} \tag{5.23}$$

This is the current that passes between the two plates of the capacitor in our gedanken experiment that was performed at the beginning of this section.

The time-varying conduction current that passes through the wire causes a build-up of charges of opposite signs on the two plates of the capacitor. The time variation of these charges creates a time-varying electric field between the plates.[2] The time-varying displacement current will pass from one plate to the other, and an answer to the first question has been obtained. The conduction current in the wire becomes a displacement current between the plates. This displacement current does not exist in a time-independent system.

The postulate for magnetostatics will have to be modified to incorporate this new current and any possible time-varying magnetic fields. It becomes

$$\boxed{\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{5.24}$$

We can also answer the second question. With the inclusion of the displacement current that passes between the capacitor plates, we can assert that the time-varying magnetic field that surrounds the conduction current-carrying wire will be equal in magnitude and direction to the time-varying magnetic field that surrounds the capacitor.

Let us integrate both sides of (5.24) over the cross-sectional area specified by the radius $\rho$ at two locations in Figure 5–8. The first integral will be at a

[2] Recall that in a vacuum $\mathbf{D} = \varepsilon_0 \mathbf{E}$. If a dielectric is inserted between the plates, we must use $\mathbf{D} = \varepsilon \mathbf{E}$.

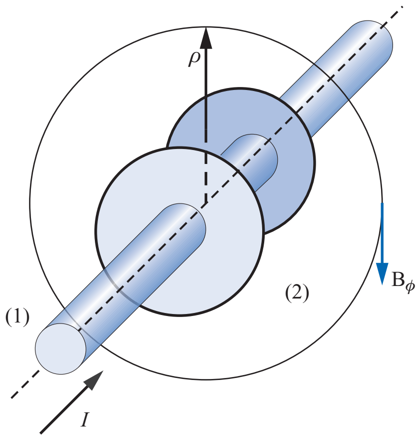



**FIGURE 5–8**

Two parallel plates in a capacitor separate two wires. The circle whose radius is $\rho$ could be surrounding the wire (1) at either edge or between the plates (2). The radius of the wire is $a$ and that of the plate is $b$.

location surrounding the wire and the second will be between the two capacitor plates

$$\int_{\Delta s}\left(\nabla \times \frac{\mathbf{B}}{\mu_0}\right) \bullet \mathbf{ds} = \int_{\Delta s}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \bullet \mathbf{ds} \tag{5.25}$$

or using Stokes's theorem, we write

$$\boxed{\oint \frac{\mathbf{B}}{\mu_0} \bullet \mathbf{dl} = \int_{\Delta s}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \bullet \mathbf{ds}} \tag{5.26}$$

The left-hand side of the integral in (5.26) yields

$$2\pi\rho\frac{B_\phi}{\mu_0}$$

At location (1) in Figure 5–8, the displacement current equals zero, and we are left with the integral

$$2\pi\rho\frac{B_\phi}{\mu_0} = \int_{\Delta s} \mathbf{J} \bullet \mathbf{ds} \tag{5.27}$$

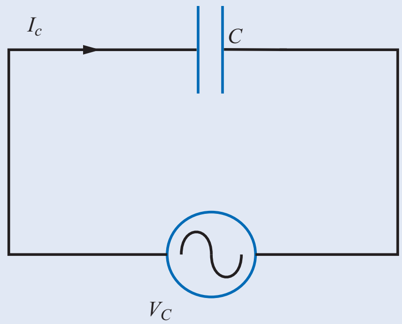
At location (2) in Figure 5–8, the conduction current equals zero, and we are left with the integral

$$2\pi\rho\frac{B_\phi}{\mu_0} = \int_{\Delta s} \frac{\partial \mathbf{D}}{\partial t} \bullet \mathbf{ds} \tag{5.28}$$

The next example will demonstrate that these currents are *identical,* hence the magnetic flux densities will be the same at the same radius $\rho$.

**EXAMPLE 5.11**

Verify that the conduction current in the wire equals the displacement current between the plates of the parallel plate capacitor in the circuit. The voltage source has $V_c = V_0 \sin \omega t$.




**Answer.** The conduction current in the wire is given by

$$I_c = C\frac{dV_c}{dt} = CV_0\omega\cos\omega t$$

The capacitance of the parallel plate capacitor is given by

$$C = \frac{\varepsilon A}{d}$$

where $A$ is the area of the plates that are separated by a distance $d$. The electric field between the plates is given by $E = V_c/d$. The displacement flux density equals

$$D = \varepsilon E = \varepsilon\frac{V_0}{d}\sin\omega t$$

The displacement current is computed from

$$I_d = \int_A \frac{\partial \mathbf{D}}{\partial t} \bullet \mathbf{ds} = \left(\frac{\varepsilon A}{d}\right)V_0\omega\cos\omega t = CV_0\omega\cos\omega t = I_c$$

**EXAMPLE 5.12**

The magnetic flux density in a vacuum is given by

$$\mathbf{B} = B_0\cos(2x)\cos(\omega t - \beta y)\mathbf{u_x} = B_x\mathbf{u_x}$$

Find the displacement current, the displacement flux density, and the volume charge density associated with this magnetic flux density.

**Answer.** We write

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$$

$$= \frac{1}{\mu_0}\begin{vmatrix} \mathbf{u_x} & \mathbf{u_y} & \mathbf{u_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & 0 & 0 \end{vmatrix} = -\frac{\beta B_0}{\mu_0}\cos(2x)\sin(\omega t - \beta y)\mathbf{u_z}$$

The displacement flux density **D** is found from the displacement current as

$$\mathbf{D} = \int \mathbf{J_d}\, dt = \int\left[-\frac{\beta B_0}{\mu_0}\cos(2x)\sin(\omega t - \beta y)\mathbf{u_z}\right]dt$$

$$= \frac{\beta B_0}{\omega\mu_0}\cos(2x)\cos(\omega t - \beta y)\mathbf{u_z}$$

Noting that the magnitude of the displacement flux density is not a function of $z$, we find the charge density given by

$$\rho_v = \nabla \bullet \mathbf{D} \Rightarrow \frac{\partial D_z}{\partial z} = 0$$

## EXAMPLE 5.13

In a lossy dielectric medium with a conductivity $\sigma$ and a relative permittivity $\varepsilon_r$, there is a time-harmonic electric field $E = E_0 \sin \omega t$. Compare the magnitudes of the following terms: (a) the conduction current density $J_c$ and (b) the displacement current density $J_d$.

**Answer.** The conduction current density can be found from Ohm's law $J_c = \sigma E = \sigma E_0 \sin \omega t$, while the displacement current density can be calculated from (5.23) $J_d = \partial D/\partial t = \varepsilon E_0 \omega \cos \omega t$. The ratio of their magnitudes is

$$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$

For materials that have a relative dielectric constant that is close to one, this fraction will depend mainly on the conductivity of the material and the frequency of the electromagnetic signal. The conduction current is dominant at low frequencies in a conductor, and the displacement current will be dominant in a dielectric at high frequencies. This latter effect will be discussed further in the next chapter.

## 5.4 Maxwell's Equations

Everything that we have learned up to this point can be summarized in Maxwell's[3] four differential equations, which are rewritten below as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.29}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{5.30}$$

$$\nabla \bullet \mathbf{D} = \rho_v \tag{5.31}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{5.32}$$

These four equations, along with a set of relations called the ***constitutive relations***

$$\left.\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{J} &= \sigma \mathbf{E} \end{aligned}\right\} \tag{5.33}$$

describe electromagnetic phenomena. The constitutive relations relate the electromagnetic fields to the material properties in which the fields exist. We will see that the propagation of electromagnetic waves such as light is described with Maxwell's equations.

Nonlinear phenomena can also be described with this set of equations through any nonlinearity that may exist in the constitutive relations. For example, certain optical fibers used in communication have a dielectric constant that depends nonlinearly on the amplitude of the wave that propagates in the fiber. It is possible to approximate the relative dielectric constant in the fiber with the expression $\varepsilon_r \approx [1 + \alpha|E|^2]$, where $\alpha$ is a constant that has the dimensions of $[\text{V/m}]^{-2}$. In writing (5.33), we have also assumed that the materials are isotropic and hysteresis can be neglected. The study of the myriad effects arising from these phenomena is of interest to a growing number of engineers and scientists throughout the world. However, we will not concern ourselves with these problems here other than to be aware of their existence.

[3] It is common in engineering talks to place a standard six-foot stick man next to a drawing of a machine in order to indicate its size. For example, a man standing adjacent to a truck in a modern coal mine would still be beneath the center of the axle, which would mean that this truck is huge. To emphasize the importance and size of these four equations, the reader could think of the symbol $\nabla$ as being an inverted pyramid arising out of the grains of sand of the remaining words and equations in this text and several others.

## EXAMPLE 5.14

Show that the two "divergence" equations are implied by the two "curl" equations and the equation of continuity.

**Answer.** To show this, we must remember the vector identity $\nabla \cdot \nabla \times \zeta \equiv 0$ where $\zeta$ is any vector. Hence

$$\nabla \cdot \nabla \times \mathbf{E} \equiv 0 = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B})$$

implies $\nabla \cdot \mathbf{B} =$ constant. This constant equals zero since there are no isolated sources nor sinks at which the magnetic flux density can originate nor terminate. This implies that magnetic monopoles do not exist in time varying systems either.

We write similarly

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = -\frac{\partial \rho_v}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D})$$

This implies that $\nabla \cdot \mathbf{D} = \rho_v$.

## EXAMPLE 5.15

In a conducting material, we may assume that the conduction current density is much larger than the displacement current density. Show that Maxwell's equations can be cast in the form of a ***diffusion equation*** in this material.

**Answer.** In this case, (5.29) and (5.30) are written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E}$$

where the displacement current has been neglected. Take the curl of the second equation

$$\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E}$$

Expand the left-hand side with a vector identity and substitute the first equation into the right-hand side.

$$\nabla\left(\nabla \cdot \frac{\mathbf{B}}{\mu_0}\right) - \nabla^2\left(\frac{\mathbf{B}}{\mu_0}\right) = -\sigma\frac{\partial \mathbf{B}}{\partial t}$$

From (5.32), the first term is zero, leaving

$$\nabla^2 \mathbf{B} = \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t}$$

This is a diffusion equation with a diffusion coefficient $D = (\mu_0 \sigma)^{-1}$. Since **B** is a vector, this corresponds to three scalar equations for the three components. The term $\nabla^2$ is the Laplacian operator described previously.

It may appear that all of this mathematical manipulation is to show off our math skills. However, showing that we can obtain the diffusion equation from Maxwell's equations is important, because the concept of diffusion provides important physical insight into what happens to free charges in various types of materials. During further study, we will see that the diffusion process is involved in many physical phenomena involving electric and magnetic fields.

Consider the case of heat flow. We know from experience that if there is a location with a higher concentration or quantity of heat, and a second location with a lower concentration or quantity of heat, then the heat will tend to flow toward the area of lower concentration. Furthermore, the greater the temperature difference between the two locations, the faster we observe an equalization of the two quantities taking place, at least at first. As the two temperatures equalize, the rate of heat flow slows, until the two locations are at equal temperatures, and heat flow stops, except for random thermal motion. Of course, we can also change the rate of flow by changing the conditions associated with the process. For instance, we could place an insulator between the two locations, which would keep heat from flowing as quickly, as we do to prevent heat from flowing out of our homes in the winter.

In a similar fashion, we can model the flow of free charges in materials, when there is a charge differential between two locations. Also, we will see the same type of behaviors, such as rate of charge flow, and the effects of electrical insulators, in the movement and control of charge in various situations. Thus, not only charge flow in conductors can be modeled, but also flow in such diverse items as components (resistors, capacitors, cables, etc.) and materials (conductors, insulators, and semiconductors). In fact, the entire semiconductor industry and its products are based on an ability to model and control the diffusion of charge carriers in semiconducting material. Thus, the ability to relate Maxwell's equations to diffusion equations is very important and useful indeed.

**EXAMPLE 5.16**

Solve the diffusion equation for the case of a magnetic flux density $B_x(z, t)$ near a planar vacuum–copper interface, assuming the following values for copper: $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m and $\sigma = 5.8 \times 10^7$ S/m. Plot the solution for the spatial profile of the magnetic field, assuming a 60-Hz time-harmonic electromagnetic signal is applied.

**Answer.** Assuming $e^{j\omega t}$ time-variation, the diffusion equation is transformed to the following ordinary differential equation for the spatial variation of the magnetic field.

$$\frac{d^2 B_x(z)}{dz^2} = j\mu_0 \sigma \omega B_x(z)$$

where $z$ is the coordinate normal to the vacuum–copper boundary. Assuming variation in the $z$ direction to be $B_x(z) = B_0 e^{(-\gamma z)}$, we write

$$\gamma^2 = j\omega\mu_0\sigma \Rightarrow \gamma = \alpha + j\beta = \sqrt{j\omega\mu_0\sigma}$$

We will encounter this expression for $\gamma^2$ again in the next chapter and study it in some detail, but for now, we will accept it as part of the solution for the preceding differential equation.

The magnitude of the magnetic flux density decays exponentially in the $z$ direction from the surface into the conductor.

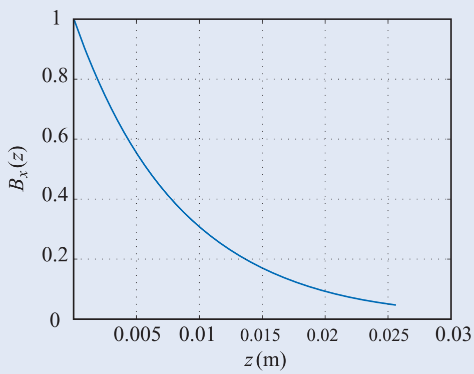
$$B_x(z) = B_0 e^{-\alpha z}$$

with

$$\alpha = \sqrt{\pi f \mu_0 \sigma} = \sqrt{\pi \times 60 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7} = 117.2 \text{ m}^{-1}$$

The quantity $\delta = 1/\alpha$ is called a "skin depth." Here it is $\delta = 8.5$ mm.

The plot of the spatial variation of the magnitude of the magnetic field inside the conductor is presented in the figure below.




The subject of the skin depth will be encountered again in the next chapter.

As written, Maxwell's equations in (5.29) to (5.32) are partial differential equations evaluated at a particular point in space and time. A *totally equivalent* way of writing them is to write these equations as integrals. This is accomplished by integrating both sides of the first two equations over the same cross-sectional area, applying Stokes's theorem to the terms involving the curl operations, integrating both sides of the second two equations over the same volume, and applying the divergence theorem. We summarize this as follows:

$$\oint_{\Delta L} \mathbf{E} \bullet \mathbf{dl} = -\int_{\Delta s} \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds} \tag{5.34}$$

$$\oint_{\Delta L} \mathbf{H} \bullet \mathbf{dl} = \int_{\Delta s} \left( \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \bullet \mathbf{ds} \tag{5.35}$$

$$\oint_{\Delta s} \mathbf{D} \bullet \mathbf{ds} = \int_{\Delta v} \rho_v dv \tag{5.36}$$

$$\oint_{\Delta s} \mathbf{B} \bullet \mathbf{ds} = 0 \tag{5.37}$$

Equation (5.34) states that the closed-line integral of the electric field around a closed loop is equal to the time rate of change of the magnetic flux that passes through the surface area defined by the closed loop. This is the meaning of Faraday's law. Equation (5.35) states that the closed-line integral of the magnetic field intensity is equal to the current that is enclosed within the loop. The current consists of the contribution due to the conduction current and the displacement current. This generalizes Ampere's circuital law, which we encountered earlier.

Equation (5.36) states that the total displacement flux $\Psi_e$ that leaves a closed surface is equal to the charge that is enclosed within the surface (Gauss's law). If the enclosed charge is negative, then the displacement flux $\Psi_e$ enters the closed surface and terminates on this negative charge. Equation (5.37) states that the magnetic flux density is continuous and cannot terminate nor originate from a magnetic charge, i.e., the nonexistence of magnetic monopoles or magnetic charges.

Equations (5.34) to (5.37) are the integral forms of Maxwell's equations, and they are of the same importance as the differential forms given in (5.29) to (5.32). Using the integral form of Maxwell's equations, we can derive easily the boundary conditions that relate the electromagnetic fields in one medium to those in another.

In examining either of the two forms of Maxwell's equations, we can make another useful observation. Looking at the first two equations of the differential form, (5.29) and (5.30), or the first two equations of the integral form, (5.34) and (5.35), we notice that all four of these equations have both electric field and magnetic field terms. Therefore, these are coupled equations. What this means is that when we perturb either the electric or magnetic field, we automatically affect the other field.

While this may seem like an obvious point, it has important implications both in understanding certain physical phenomena involving electromagnetic fields, as well as in the computational solutions for electromagnetic fields problems. For instance, in developing the three-dimensional version of the Finite Difference Time Domain method, we have to work with both the electric and magnetic fields, and maintain this coupled relationship. We will explore these issues in greater detail in other sections of this text.

Either of the two forms of Maxwell's equations can be used, although we will encounter the differential form more often in practice. An important derivation that describes the magnitude and direction of the flow of electromagnetic power will employ vector identities and the differential form of Maxwell's equations.

## 5.5 Poynting's Theorem

A frequently encountered problem in practice is to determine the direction that power is flowing if the electric and magnetic fields are measured independently in some experiment. This may not seem important in the laboratory where a signal generator can be separated from a resistive load impedance, and the direction of the flow of power can be clearly ascertained. This is not always so straight-forward in the case of electric and magnetic fields. For instance, it is clear that the sun radiates energy that is received by the Earth, and the amount of that energy can be measured. However, an investigator using a satellite floating in space may wish to determine the source of some anomalous extragalactic electromagnetic radiation in order to further map out the universe. Poynting's theorem will provide us with the method to accomplish this.

To obtain Poynting's theorem for an arbitrary volume depicted in Figure 5–9, we will require two of Maxwell's equations and a vector identity. The two equations that are required for this derivation are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.38}$$

and

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{5.39}$$

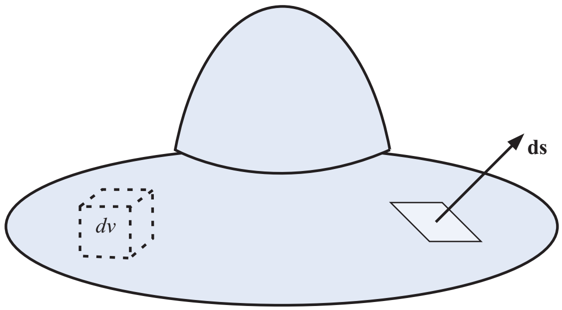



**FIGURE 5–9**

An arbitrarily shaped volume that contains a source of electromagnetic energy.

Let us take the scalar product of **E** with (5.39) and subtract it from the scalar product of **H** with (5.38). Performing this operation leads to

$$\mathbf{H} \bullet \nabla \times \mathbf{E} - \mathbf{E} \bullet \nabla \times \mathbf{H} = -\mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \bullet \left[\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right]$$

The left-hand side of this equation can be replaced using vector identity (A.9)

$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet \nabla \times \mathbf{A} - \mathbf{A} \bullet \nabla \times \mathbf{B}$$

Therefore, we obtain

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \bullet \mathbf{J} \tag{5.40}$$

After the introduction of the constitutive relations (5.33), the terms involving the time derivatives can be written as

$$-\mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}[\mu \mathbf{H} \bullet \mathbf{H} + \varepsilon \mathbf{E} \bullet \mathbf{E}] = -\frac{\partial}{\partial t}\frac{1}{2}[\mu H^2 + \varepsilon E^2] \tag{5.41}$$

Substitute (5.41) into (5.40) and integrate both sides of the resulting equation over the same volume $\Delta v$. This volume is enclosed completely by the surface $\Delta s$. Performing this integration leads to

$$\int_{\Delta v} \nabla \bullet (\mathbf{E} \times \mathbf{H}) dv = -\frac{\partial}{\partial t}\int_{\Delta v} \frac{1}{2}[\mu H^2 + \varepsilon E^2] dv - \int_{\Delta v} \mathbf{E} \bullet \mathbf{J} dv \tag{5.42}$$

The volume integral on the left-hand side of (5.42) can be converted to a closed-surface integral via the divergence theorem. With the substitution of Ohm's law $\mathbf{J} = \sigma \mathbf{E}$, we finally obtain

$$\oint_{\Delta s} (\mathbf{E} \times \mathbf{H}) \bullet \mathbf{ds} = -\frac{\partial}{\partial t}\int_{\Delta v} \frac{1}{2}[\mu H^2 + \varepsilon E^2] dv - \int_{\Delta v} \sigma E^2 dv \tag{5.43}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is called the ***Poynting vector***. It is the power density of the radiated electromagnetic fields in W/m$^2$. The direction of the radiated power is included in this vector.

Let us now give a physical interpretation to each of the three terms that appear in this equation. The units of the closed-surface integral are

$$\frac{\text{volts}}{\text{meter}} \times \frac{\text{amperes}}{\text{meter}} \times \text{meter}^2 = \text{watts}$$

or the closed-surface integral has the units of power. Using the definition of the scalar product and the fact that the notation **ds** refers to the outward normal of the surface that encloses the volume $\Delta v$, this term represents the *total power that leaves* or is radiated from the volume $\Delta v$.

The terms within the integrand of the first volume integral can be recognized as the stored magnetic energy density and the stored electric energy density that were previously described in static fields. The time derivative introduces a unit of s$^{-1}$. The units of this term are

$$\frac{1}{\text{second}} \times \frac{\text{joule}}{\text{meter}^3} \times \text{meter}^3 = \text{watts}$$

This term corresponds to the time derivative of the *stored electromagnetic energy* within the volume.

The units of the second volume integral correspond to Joule heating within the volume, and they are also in terms of watts.

$$\frac{1}{\text{ohms} \times \text{meter}} \times \frac{\text{volts}^2}{\text{meter}^2} \times \text{meter}^3 = \text{watts}$$

The reference to Joule heating indicates that electromagnetic power *is converted to heat* and this power cannot be recovered. A toaster uses Joule heating.

Hence Poynting's theorem states that the power that *leaves* a region is equal to the temporal decay in the energy that is stored within the volume minus the power that is dissipated as heat within it. A common-sense example will illustrate this theorem. Additional applications of this important theorem will be found in the chapter of this book that discusses radiation.

Equation (5.43), which can be considered to be a form of the conservation of energy equation, can also be written in differential form. Recalling that ***electromagnetic energy density*** is defined as

$$w = \frac{1}{2}[\mu H^2 + \varepsilon E^2] \tag{5.44}$$

and the ***power loss density*** is given by

$$p_L = \sigma E^2 \tag{5.45}$$

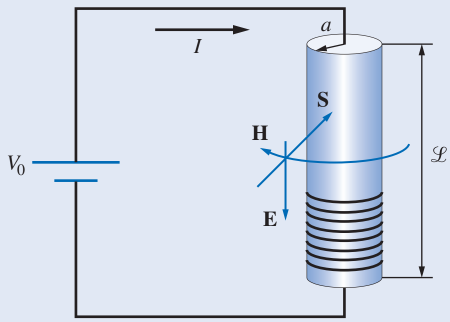
We can reinterpret (5.42) in the following differential form of the energy conservation of the system.

$$\nabla \cdot \mathbf{S} + \frac{\partial w}{\partial t} = -p_L \tag{5.46}$$

This equation is somewhat similar to the equation of continuity (5.18) with a "sink" term that corresponds to the Joule heating.

## EXAMPLE 5.17

Using Poynting's theorem, calculate the power that is dissipated in the resistor as heat. The electric energy is supplied by the battery. Neglect the magnetic field that is confined within the resistor and calculate its value only at the surface. In addition, assume that there are conducting surfaces at the top and bottom of the resistor so they are equipotential surfaces. Also, assume that the radius of the resistor is much less than its length.




**Answer.** The electric field has a magnitude of $E = V_0/\mathscr{L}$ and the magnitude of the magnetic field intensity at the outer surface of the resistor is $H = I/(2\pi a)$. The direction of the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is *into* the resistor. There is no energy stored in the resistor. The magnitude of the current density that is in the same direction as the electric field is $J = I/(\pi a^2)$. Therefore, the various terms in Poynting's theorem (5.43) are found to be

$$-\left(\frac{V_0}{\mathscr{L}}\right)\left(\frac{I}{2\pi a}\right)(2\pi a\mathscr{L}) = -\frac{d}{dt}\int_{\Delta v}[0+0]dv - \left(\frac{I}{\pi a^2}\right)\left(\frac{V_0}{\mathscr{L}}\right)(\pi a^2\mathscr{L})$$

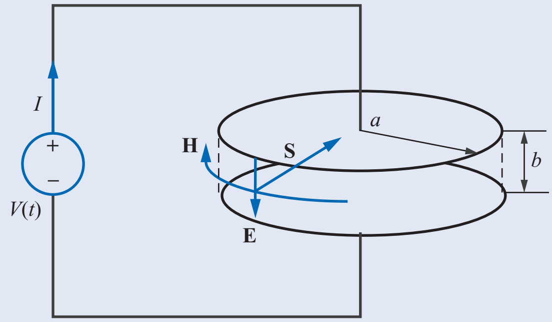
yielding

$$-V_0 I = -V_0 I$$

The electromagnetic energy of the battery is fully absorbed by the resistor.

## EXAMPLE 5.18

Using Poynting's theorem, calculate the power that is flowing through the surface area at the radial edge of a capacitor. Neglect the ohmic losses in the wires connecting the capacitor with the signal generator. Also assume that the radius of the capacitor is much greater than the separation distance between the plates, or $a >> b$.




**Answer.** Assuming the electric field **E** is confined between the plates and is uniform, we can find the total electric energy that is stored in the capacitor to be

$$W = \left(\frac{\varepsilon E^2}{2}\right)(\pi a^2 b)$$

The total magnetic energy that is stored in the capacitor is equal to zero.

The differentiation of this electric energy with respect to time yields

$$-\frac{dW}{dt} = -\varepsilon(\pi a^2 b)E\frac{dE}{dt}$$

This is the only term that survives on the right side of (5.43) since an ideal capacitor does not dissipate energy.

The left-hand side of (5.43) requires an expression for the time-varying magnetic field intensity in terms of the displacement current. Evaluating (5.26) at the radial edge of the capacitor, we write

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int_{\Delta s}\left(\varepsilon\frac{\partial \mathbf{E}}{\partial t}\right) \cdot \mathbf{ds}$$

There is no conduction current in this ideal capacitor, or $I = 0$. We obtain

$$H(2\pi a) = \varepsilon\frac{dE}{dt}(\pi a^2) \Rightarrow H = \frac{\varepsilon a}{2}\frac{dE}{dt}$$

Now we can write the Poynting vector power flow as

$$P_s = -(EH)(2\pi ab) = -\varepsilon(\pi a^2 b)E\frac{dE}{dt}$$

The minus sign arises since the direction of the Poynting vector is radially inward. Comparing both expressions, we find that they are equal, which implies that

$$P_s = -\frac{dW}{dt}$$

This states that *energy is conserved* in the circuit as should be expected.

In these two examples, we see that Poynting's theorem can be interpreted in terms of electrical circuit elements. In these examples, electromagnetic power was directed into the element. The radiation of electromagnetic power that is directed radially outward will be discussed later when antennas are described.

## 5.6 Time-Harmonic Electromagnetic Fields

In practice, we frequently will encounter electromagnetic fields whose temporal variation is harmonic. Maxwell's equations and the Poynting vector will assume a particular form since the fields can be represented as *phasors.* In particular, we write the fields as

$$\mathbf{E}(x, y, z, t) = \mathrm{Re}[\mathbf{E}(x, y, z)e^{j\omega t}] \tag{5.47}$$

and

$$\mathbf{H}(x, y, z, t) = \mathrm{Re}[\mathbf{H}(x, y, z)e^{j\omega t}] \tag{5.48}$$

where Re stands for the real part. There may be a phase angle $\phi$ between the electric and magnetic fields that will be absorbed into the terms $\mathbf{E}(x, y, z)$ and $\mathbf{H}(x, y, z)$.

In terms of the phasors $\mathbf{E}$ and $\mathbf{H}$, we write Maxwell's equations as

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{H}(\mathbf{r}) \tag{5.49}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \tag{5.50}$$

$$\nabla \bullet \mathbf{E}(\mathbf{r}) = \frac{\rho_v(\mathbf{r})}{\varepsilon} \tag{5.51}$$

$$\nabla \bullet \mathbf{B}(\mathbf{r}) = 0 \tag{5.52}$$

where the term representing the temporal variation $e^{j\omega t}$ that is common to both sides of these equations has been canceled. Hopefully, there will be little confusion in notation since we have not introduced any new symbols.

**EXAMPLE 5.19**

Compute the frequency at which the conduction current equals the displacement current.

**Answer.** Using (5.33) and (5.50), we write

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + j\omega\varepsilon\mathbf{E}(\mathbf{r}) = (\sigma + j\omega\varepsilon)\mathbf{E}(\mathbf{r})$$

The frequency is given by

$$\omega = \frac{\sigma}{\varepsilon}$$

For copper, the frequency $f = \omega/2\pi$ is

$$f = \frac{\sigma}{2\pi\varepsilon_0} = \frac{5.8 \times 10^7}{2\pi \times \frac{1}{36\pi} \times 10^{-9}} \approx 1.04 \times 10^{18}\ \text{Hz}$$

At frequencies much above this value, copper, which is thought to be a good conductor, acts like a dielectric.

The derivation of the Poynting vector requires some care when we are considering time-harmonic fields. This is because the Poynting vector involves the product $\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})$. Power is a real quantity, and we must be careful since

$$\text{Re}[\mathbf{E}(\mathbf{r})e^{j\omega t}] \times \text{Re}[\mathbf{H}(\mathbf{r})e^{j\omega t}] \neq \text{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})e^{j\omega t}] \tag{5.53}$$

To effect the derivation of the Poynting vector, we make use of the following relations:

$$\text{Re}[\mathbf{E}(\mathbf{r})] = \left(\frac{\mathbf{E}(\mathbf{r}) + \mathbf{E}^*(\mathbf{r})}{2}\right) \quad \text{and} \quad \text{Re}[\mathbf{H}(\mathbf{r})] = \left(\frac{\mathbf{H}(\mathbf{r}) + \mathbf{H}^*(\mathbf{r})}{2}\right) \tag{5.54}$$

where the star indicates the complex conjugate of the function. We write

$$\text{Re}[\mathbf{E}(\mathbf{r})] \times \text{Re}[\mathbf{H}(\mathbf{r})] = \left(\frac{\mathbf{E}(\mathbf{r}) + \mathbf{E}^*(\mathbf{r})}{2}\right) \times \left(\frac{\mathbf{H}(\mathbf{r}) + \mathbf{H}^*(\mathbf{r})}{2}\right)$$

$$= \frac{\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) + \mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) + \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) + \mathbf{E}^*(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})}{4}$$

The time variation $e^{j\omega t}$ cancels in two of the terms and it introduces a factor of $e^{\pm j2\omega t}$ in the remaining two terms. After taking a time average of this power, these latter terms will contribute nothing to the result. We finally obtain the time average power to be

$$\boxed{\mathbf{S}_{\mathbf{av}}(\mathbf{r}) = \frac{1}{2}\,\text{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})]\ \ (\text{W/m}^2)} \tag{5.55}$$

**EXAMPLE 5.20**

The field vectors in free space are given by

$$\mathbf{E} = 10\cos\left(\omega t + \frac{4\pi}{3}z\right)\mathbf{u_x}\,(\text{V/m}) \qquad \text{and} \qquad \mathbf{H} = \frac{\mathbf{u_z} \times \mathbf{E}}{120\pi}\,(\text{A/m})$$

The frequency $f = 500$ MHz. Determine the Poynting vector. The numerical value of $120\pi$ as a free-space impedance will become apparent in the next chapter.

**Answer.** In phasor notation, the fields are expressed as

$$\mathbf{E}(\mathbf{r}) = 10e^{j\left(\frac{4\pi}{3}\right)z}\mathbf{u_x} \qquad \text{and} \qquad \mathbf{H}(\mathbf{r}) = \frac{10}{120\pi}e^{j\left(\frac{4\pi}{3}\right)z}\mathbf{u_y}$$

and the Poynting vector is

$$\mathbf{S_{av}}(\mathbf{r}) = \frac{1}{2}\text{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})] = \frac{10^2}{2 \times 120\pi}\mathbf{u_z} = 0.133\mathbf{u_z}\,(\text{W/m}^2)$$

Having now manipulated the complex phasors to derive (5.55), let us apply this to the derivation of Poynting's theorem. In particular, we desire to explicitly obtain the terms $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}^*(\mathbf{r})$. The procedure that we will follow is to subtract the scalar product of $\mathbf{E}(\mathbf{r})$ with the complex conjugate of (5.50) from the scalar product of $\mathbf{H}^*(\mathbf{r})$ with (5.49), resulting in

$$\mathbf{H}^*(\mathbf{r}) \bullet \nabla \times \mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \bullet \nabla \times \mathbf{H}^*(\mathbf{r}) = -j\omega\mu\mathbf{H}(\mathbf{r}) \bullet \mathbf{H}^*(\mathbf{r})$$
$$+ j\omega\varepsilon\mathbf{E}(\mathbf{r}) \bullet \mathbf{E}^*(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \bullet \mathbf{J}(\mathbf{r})^* \qquad (5.56)$$

Employing the same vector identity that we used previously to derive (5.40), we recognize that (5.56) can be written as

$$\nabla \bullet (\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})) = -j\omega\mu H^2 + j\omega\varepsilon E^2 - \sigma E^2 \qquad (5.57)$$

where $\mathbf{H}(\mathbf{r}) \bullet \mathbf{H}^*(\mathbf{r}) = H^2$, $\mathbf{E}(\mathbf{r}) \bullet \mathbf{E}^*(\mathbf{r}) = E^2$, and $\mathbf{E}(\mathbf{r}) \bullet \mathbf{J}(\mathbf{r})^* = \sigma E^2$.

Following the procedure that has served us so well previously, we integrate the terms that appear in (5.57) over the volume of interest.

$$\int_{\Delta v} \nabla \bullet (\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}))dv = -j\omega\int_{\Delta v} [\mu H^2 - \varepsilon E^2]dv - \int_{\Delta v} \sigma E^2 dv \qquad (5.58)$$

The volume integral is converted to a closed-surface integral that encloses the volume $\Delta v$.

$$\oint_{\Delta s} (\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})) \bullet \mathbf{ds} = -j\omega\int_{\Delta v} [\mu H^2 - \varepsilon E^2]dv - \int_{\Delta v} \sigma E^2 dv \quad (5.59)$$

The closed-surface integral represents the total power that is radiated from within the volume enclosed by this surface. The last term represents the power that is dissipated within this volume. This power could have been turned into heat and would not be recovered. The remaining two terms are the time-average energy stored within the volume. The factor $j$ indicates that this is similar to the ***reactive energy*** stored in the capacitor or inductor in an *RLC* circuit.

## 5.7 Conclusion

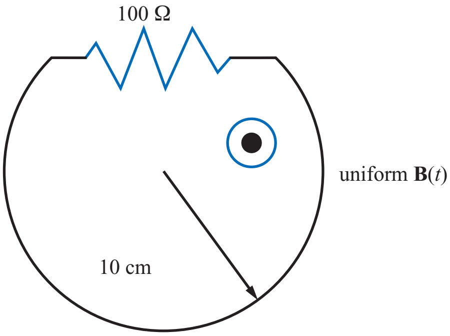
We have now come to the end of a long journey in order to obtain the set of four Maxwell's equations that describe electromagnetic phenomena. We have demonstrated that time-varying electric and magnetic fields can be determined from each other through these equations and that they are intimately intertwined. Faraday's law of induction and Ampere's circuital law with the introduction of a displacement current relate time-varying magnetic fields to time-varying electric fields. The term that represents the displacement current arises from the requirement that the equation of continuity must be satisfied. The boundary conditions that we encountered in static fields apply equally well in time-varying fields.

There is a T-shirt that paraphrases the book of Genesis by stating that "In the beginning, God said '. . .' and there was light" where these equations are included within the proclamation. The goal and accomplishment of thousands of graduate students since Maxwell first inscribed these equations on paper has been to pose a new electromagnetic problem, solve it starting from these equations, and write a thesis. Even after obtaining a graduate degree, this set of equations usually appears as "Equations 1 to 4" in many of their later scholarly articles that are then stored in dusty archives. You, as a student, are not expected to write these equations on a crib sheet and bring them to an examination or even memorize them for that inquiry. *You are expected to know them!* The intellectual and even the visceral understanding of these equations is what this course and much of electrical and computer engineering is about.

## 5.8 Problems

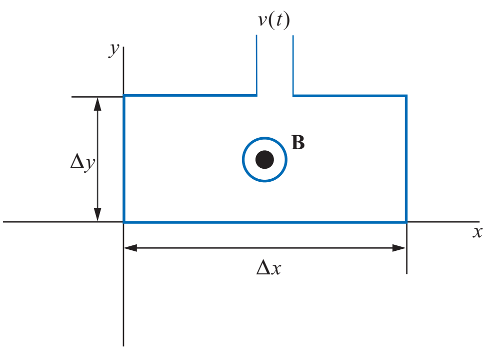
**5.1.1.** In a source-free region, we find that $\mathbf{B} = z\mathbf{u_y} + x\mathbf{u_z}$. Does **E** vary with time?

**5.1.2.** A perfect conductor joins two ends of a 100 Ω resistor, and the closed loop is in a region of uniform magnetic flux density $B = 10\,e^{(-t/10)}$ T.



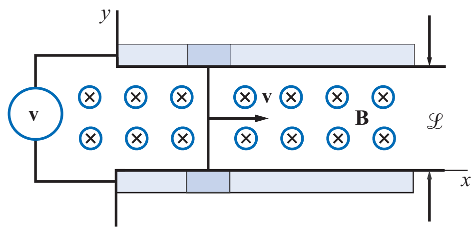

Neglecting the self-inductance of the loop, find and plot the voltage $V(t)$ that appears across the 100 Ω resistor. A device based on this principle is used to monitor time-varying magnetic fields in experiments and in biological studies.

**5.1.3.** A closed loop ($\Delta x = 30$ cm × $\Delta y = 20$ cm) of wire passes through a nonuniform time-independent magnetic field $\mathbf{B} = y\mathbf{u_z}$ T with a constant velocity $\mathbf{v}_0 = 5\mathbf{u_x}$ m/s. At $t = 0$, the loop's lower left corner is located at the origin. Find an expression for the voltage $V$, generated by the loop as a function of time. You may neglect the magnetic field created by the current in the loop.



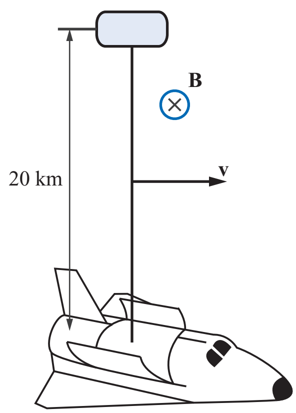

**5.1.4.** Repeat Problem 5.1.3 with the magnetic flux density being uniform in space $\mathbf{B} = 0.1\mathbf{u_z}$ T. Explain your result.

**5.1.5.** Find the generated voltage if the axle moves at a constant velocity $\mathbf{v} = v\mathbf{u_x} = 3\mathbf{u_x}$ m/s in a uniform magnetic field of $\mathbf{B} = B_0\mathbf{u_z} = 5\mathbf{u_z}$ T. At $t = 0$, the axle was at $x = 0$, $L = 40$ cm.

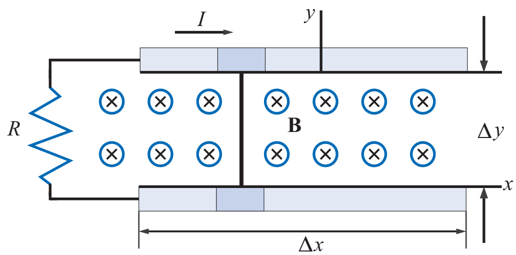



**5.1.6.** Repeat Problem 5.1.5 with the constraint that the rails separate with $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 x$. The wheels are free to slide on the "trombone-like" axle so they remain on the rails ($\mathscr{L}_0 = 0.4$ m, $\mathscr{L}_1 = 0.04$ m).

**5.1.7.** A tethered satellite is to be connected to the Shuttle to generate electricity as it passes through the ambient plasma. A plasma consists of a large number of positive charges and negative charges. Assuming that the Shuttle takes 1.5 hours to go around the Earth, find the expected voltage difference $\Delta V$ between the tether and the Shuttle. The Shuttle flies approximately 400 km above the earth where $\mathbf{B} \approx 10^{-5}$ T.

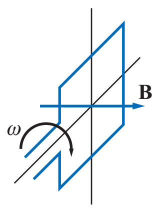


**5.1.8.** A conducting axle oscillates over two conducting parallel rails in a uniform magnetic field $\mathbf{B} = B_0\mathbf{u_z}$ ($B_0 = 4$ T). The position of the axle is given by $x = (\Delta x/2)\,[1 - \cos\omega t]$ ($\Delta x = 0.2$ m, $\omega = 500\ \text{s}^{-1}$). Find and plot the current $I(t)$ if the resistance is $R = 10\ \Omega$ and the distance between the rails is $\Delta y = 0.1$ m.

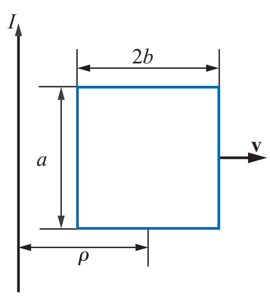



**5.1.9.** Repeat Problem 5.1.8 with the magnetic field also varying in time as $\mathbf{B} = B_0\cos\omega t\,\mathbf{u_z}$ with $B_0$ and $\omega$ having the same values.

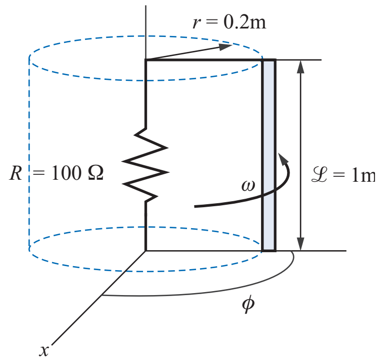
**5.1.10.** Calculate the voltage that is induced between the two nodes as the coil with dimensions 0.5 m × 0.5 m rotates in a uniform magnetic field with a flux density $B = 2$ T with a constant angular frequency $\omega = 1200\ \text{s}^{-1}$.




**5.1.11.** A square loop is adjacent to an infinite wire that carries a current $I$. The loop moves with a velocity $\mathbf{v} = v_0\mathbf{u}_\rho$. The center of the loop is at $\rho$, and the initial position is $\rho = b$. Determine the induced voltage $V(t)$ in the loop assuming dimensions $a \times 2b$.




**5.1.12.** The 1 m long wire shown in the figure rotates with an angular frequency $\omega = 40\pi\ \text{s}^{-1}$ in the magnetic field $\mathbf{B} = 0.5\cos\phi\,\mathbf{u}_\phi$ T. Find the current in the closed loop with a resistance 100 Ω.




**5.2.1.** The current density is $\mathbf{J} = \sin(\pi x)\mathbf{u_x}$. Find the time rate of increase of the charge density $\partial\rho_v/\partial t$ at $x = 1$.

**5.2.2.** The current density is $\mathbf{J} = e^{(-\rho^2)}\mathbf{u}_r$ in cylindrical coordinates. Find the time rate of increase of the charge density at $\rho = 1$.

**5.3.1.** Compare the magnitudes of the conduction and displacement current densities in copper ($\sigma = 5.8 \times 10^7$ S/m, $\varepsilon = \varepsilon_0$), sea water ($\sigma = 4$ S/m, $\varepsilon = 81\ \varepsilon_0$), and earth ($\sigma = 10^{-3}$ S/m, $\varepsilon = 10\ \varepsilon_0$) at 60 Hz, 1 MHz, and at 1 GHz.

**5.3.2.** Given the conduction current density in a lossy dielectric as $J_c = 0.2\sin(2\pi 10^9 t)\ \text{A/m}^2$, find the displacement current density if $\sigma = 10^{-3}$ S/m and $\varepsilon_r = 6.5$.

**5.4.1.** Show that the fields $\mathbf{B} = B_0\cos\omega t\,\mathbf{u_x}$ and $\mathbf{E} = E_0\cos\omega t\,\mathbf{u_z}$ do not satisfy Maxwell's equations in air $\varepsilon_r \approx 1$. Show that the fields $\mathbf{B} = B_0\cos(\omega t - ky)\mathbf{u_x}$ and $\mathbf{E} = E_0\cos(\omega t - ky)\mathbf{u_z}$ satisfy these equations. What is the value of $k$ in terms of the other stated parameters?

**5.4.2.** Given

$$\mathbf{E} = E_0\cos(\omega t - ky)\mathbf{u_z}$$

$$\text{and}\quad \mathbf{H} = \left(\frac{E_0}{Z_0}\right)\cos(\omega t - ky)\mathbf{u_x}$$

in a vacuum, find $Z_0$ in terms of $\varepsilon_0$ and $\mu_0$ so Maxwell's equations are satisfied.

**5.4.3.** Do the fields

$$\mathbf{E} = E_0 \cos x \cos(\omega t)\mathbf{u_y} \text{ and } \mathbf{H} = \left(\frac{E_0}{\mu_0}\right)\sin x \sin(\omega t)\mathbf{u_z}$$

satisfy Maxwell's equations?

**5.4.4.** Find a charge density $\rho_v$ that could produce an electric field in a vacuum $\mathbf{E} = E_0 \cos x \cos(\omega t)\mathbf{u_x}$.

**5.4.5.** Find the displacement current density flowing through the dielectric of a coaxial cable of radii $a$ and $b$ where $b > a$ if a voltage $V_0 \cos \omega t$ is connected between the two conducting cylinders.

**5.4.6.** Find the displacement current density flowing through the dielectric of two concentric spheres of radii $a$ and $b$ where $b > a$ if a voltage $V_0 \cos \omega t$ is connected between the two conducting spheres.

**5.4.7.** Starting from Maxwell's equations, derive the equation of continuity.

**5.4.8.** Write all of the terms that appear in Maxwell's equations in Cartesian coordinates.

**5.5.1.** If $\mathbf{E} = E_0 \cos(\omega t - \beta z)\mathbf{u_y}$ is a solution to Maxwell's equations, find **H**. Find $\mathbf{S_{av}}$.

**5.5.2.** If $\mathbf{H} = H_0 \cos(\omega t - \beta z)\mathbf{u_y}$ is a solution to Maxwell's equations, find **E**; find $\mathbf{S_{av}}$.

**5.5.3.** Compute the electric energy that is stored in a cube whose volume is 1 $m^3$ in which a uniform electric field of $10^4$ V/m exists. Compute the stored energies if the cube is empty and if it is filled with water that has $\varepsilon = 81\varepsilon_0$.

**5.6.1.** Write $\mathbf{E} = 120\pi \cos(3 \times 10^9 t - 10z)\mathbf{u_x}$ and $\mathbf{H} = 1 \cos(3 \times 10^9 t - 10z)\mathbf{u_y}$ in phasor notation.

**5.6.2.** Write the phasors $\mathbf{E} = 3e^{-j\beta z}\mathbf{u_x}$ and $\mathbf{H} = 0.4e^{-j45^\circ}e^{-j\beta z}\mathbf{u_y}$ in the time domain. The frequency of oscillation is $\omega$. Find the average Poynting vector $\mathbf{S_{av}}$.

**5.6.3.** At a frequency of $f = 1$ MHz, verify that copper ($\sigma = 5.8 \times 10^7$ S/m, $\varepsilon_r \approx 1$) is a good conductor, and quartz ($\sigma = 10^{-17}$ S/m, $\varepsilon_r = 4$) is a good insulator.

**5.6.4.** Find the frequency where quartz becomes a conductor.

**5.6.5.** Find the frequency where copper becomes an insulator.

# Index

## H



## I



## J



## K



## L



## M

## Vector Operations in the Three Coordinate Systems

*Cartesian*

$$\nabla a = \frac{\partial a}{\partial x}\mathbf{u_x} + \frac{\partial a}{\partial y}\mathbf{u_y} + \frac{\partial a}{\partial z}\mathbf{u_z}$$

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u_z}$$

$$\nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$$

*Cylindrical*

$$\nabla a = \frac{\partial a}{\partial \rho}\mathbf{u}_\rho + \frac{1}{\rho}\frac{\partial a}{\partial \phi}\mathbf{u}_\phi + \frac{\partial a}{\partial z}\mathbf{u_z}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\mathbf{u}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\mathbf{u_z}$$

$$\nabla^2 a = \frac{1}{\rho}\frac{\partial\left(\rho\frac{\partial a}{\partial \rho}\right)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 a}{\partial \phi^2} + \frac{\partial^2 a}{\partial z^2}$$

*Spherical*

$$\nabla a = \frac{\partial a}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial a}{\partial \theta}\mathbf{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial a}{\partial \phi}\mathbf{u}_\phi$$

$$\nabla \bullet \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{u}_r$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\mathbf{u}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\mathbf{u}_\phi$$

$$\nabla^2 a = \frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial a}{\partial r}\right)}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial a}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 a}{\partial \phi^2}$$

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## Fundamental Physical Constants

| Quantity | Symbol | Value | Unit |
|---|---|---|---|
| Avogadro constant | $N_A$ | $6.0221415 \times 10^{23}$ | $\text{mol}^{-1}$ |
| Boltzmann constant | $k$ | $1.3806505 \times 10^{-23}$ | J/K |
| Planck constant | $h$ | $6.6260693 \times 10^{-34}$ | J sec |
| electron charge | $e$ | $1.60217653 \times 10^{-19}$ | C |
| electron mass | $m_e$ | $9.1093826 \times 10^{-31}$ | kg |
| proton mass | $m_p$ | $1.67262171 \times 10^{-27}$ | kg |
| proton-electron mass ratio | $m_p/m_e$ | 1836.15267261 | |
| Gravitation constant | $G$ | $6.6742 \times 10^{-11}$ | $\text{m}^3/\text{kg sec}^2$ |
| standard acceleration of gravity | $g_n$ | 9.80665 | $\text{m/sec}^2$ |
| permittivity in vacuum | $\varepsilon_0$ | $8.854187817 \times 10^{-12}$ | F/m |
| Approx. value $\varepsilon_0$ | $\varepsilon_0$ | $(1/36\pi) \times 10^{-9}$ | F/m |
| permeability in vacuum | $\mu_0$ | $4\pi \times 10^{-7}$ | H/m |
| Approx. value $\mu_0$ | $\mu_0$ | $12.566370614 \times 10^{-7}$ | H/m |
| speed of light in vacuum | $c, c_0$ | 299, 792, 458 | m/sec |
| characteristic impedance in vacuum | $Z_0$ | 376.730313461 | $\Omega$ |

Additional physical constants can be found on the National Institute of Standards and Technology (NIST) website at: http://physics.nist.gov/cuu/Constants/

## SI Prefixes

| $10^n$ | Prefix | Symbol | $10^n$ | Prefix | Symbol |
|---|---|---|---|---|---|
| $10^{24}$ | yotta | Y | $10^{-1}$ | deci | d |
| $10^{21}$ | zetta | Z | $10^{-2}$ | centi | c |
| $10^{18}$ | exa | E | $10^{-3}$ | milli | m |
| $10^{15}$ | peta | P | $10^{-6}$ | micro | $\mu$ |
| $10^{12}$ | tera | T | $10^{-9}$ | nano | n |
| $10^{9}$ | giga | G | $10^{-12}$ | pico | p |
| $10^{6}$ | mega | M | $10^{-15}$ | femto | f |
| $10^{3}$ | kilo | k | $10^{-18}$ | atto | a |
| $10^{2}$ | hecto | h | $10^{-21}$ | zepto | z |
| $10^{1}$ | deca, deka | da | $10^{-24}$ | yocto | y |

## Useful Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{dx}{x} = \ln x$$

$$\int e^x dx = e^x$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2 + a^2}}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$$

$$\int e^{ax}\cos(c + bx)dx = \frac{e^{ax}(a\cos(bx + c) + b\sin(bx + c))}{a^2 + b^2}$$

## Trigonometric Relations

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x \pm 90^\circ) = \mp \sin x$$

$$\cos(-x) = \cos x$$

$$\sin(x \pm 90^\circ) = \pm \cos x$$

$$\sin(-x) = -\sin x$$

$$e^{jx} = \cos x + j\sin x$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

## Approximations for Small Quantities

For $|x| << 1$,

$$(1 \pm x)^n \simeq 1 \pm nx$$

$$(1 \pm x)^2 \simeq 1 \pm 2x$$

$$\sqrt{1 \pm x} \simeq 1 \pm \frac{x}{2}$$

$$\frac{1}{\sqrt{1 \pm x}} \simeq 1 \mp \frac{x}{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \simeq 1 + x$$

$$\ln(1 + x) \simeq x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \simeq x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \simeq 1 - \frac{x^2}{2}$$

$$\lim_{x \to 0}\frac{\sin x}{x} = 1$$

## Half-Angle Formulas

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

## Hyperbolic Functions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$