Fundamentals of Solid Mechanics

Course at the European School for Advanced Studies in Earthquake Risk Reduction

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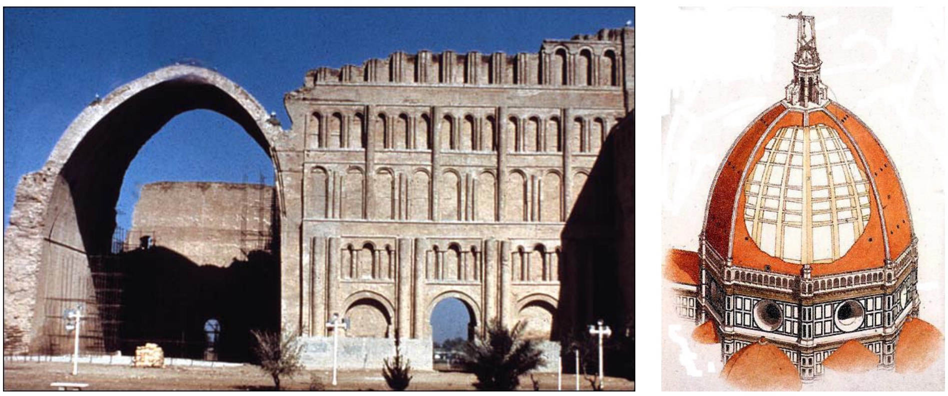
Contents

Introduction, historical sketch

Remnants of old civil engineering constructions prove that already in ancient times the human being was able to build very complex structures. The old masters had both experience and intuition with which they were able to create the most daring structures. However, they did not use in their work any theoretical models as we understand them today. Most likely, it was first in XVI century that notions necessary for such a modeling were invented.

Leonardo da Vinci (1452-1519) sketched in his notebooks a possible test of the tensile strength of a wire.



Arc of Ctesiphon (the winter palace of Sassanids, Iraq) constructed in 129 B.C. (left panel) and the dome of the Florence cathedral designed and built by Filippo Brunelleschi in 1425 (right panel) – two examples of the early ingenious constructions.

'In his book on mechanics[1] Galileo (Galileo Galilei, 1564-1642) also dealt with the strength of materials, founding that branch of science as well. He was the first to show

[1]Dialogo di Galileo Galilei Linceo Matematico Sopraordinario dello Studio di Pisa, in Florenza, Per Gio: Batista Landini MDCXXXII

that if a structure increased in all dimensions equally it would grow weaker – at least he was the first to explain the theoretical basis for this. This is what is known as the square-cube law. The volume increases as the cube of linear dimensions but the strength only as the square. For that reason larger animals require proportionally sturdier supports than small ones. A deer expanded to the size of an elephant and kept in exact proportion would collapse, its legs would have to be thickened out of proportion for proper support'[2].

Below we mention only a few scientists whose contributions are particularly important for the development of continuum mechanics at its early stage. Many further details can be found in the article of J. R. Rice[3].

Robert Hooke is considered to be the founder of linear elasticity. He discovered in 1660 (published in 1678), the observation that the displacement under a load was for many materials proportional to force. However, he was not aware yet of the necessity of terms of stress and strain. A similar discovery was made by E. Mariotte (France, 1680). He described as well the explanation for the resistance of beams to transverse loadings. He considered the existence of bending moments caused by transverse loadings and developing extensional and compressional deformations, respectively, in material fibers along upper and lower portions of beams. The first to introduce the relation between stresses and strains was the Swiss mathematician and mechanician Jacob Bernoulli (1654-1705). In his last paper of 1705 he indicated that the proper way of describing deformation was to give force per unit area, i.e. stress, as a function of the elongation per unit length, i.e. strain, of a material fiber under tension. Numerous most important contributions were made by the Swiss mathematician and mechanician Leonhard Euler (1707-1783), who was taught mathematics by Jacob' brother Johann Bernoulli (1667-1748). Among many other ideas he proposed a linear relation between stress σ and strain ε in the form $\sigma = E\varepsilon$ (1727). The coefficient E is now usually called Young's modulus after English naturalist Thomas Young who developed a similar idea in 1807.

Since the proposition of Jacob Bernoulli the notion that there is an internal tension acting across surfaces in a deformed solid has been commonly accepted. It was used, for example, by German mathematician and physicist Gottfried Wilhelm Leibniz in 1684. Euler introduced the idea that at a given section along the length of a beam there were internal tensions amounting to a net force and a net bending moment. Euler introduced the idea of compressive normal stress as the pressure in a fluid in 1752.

The French engineer and physicist Charles-Augustin Coulomb (1736-1806) was apparently the first to relate the theory of a beam as a bent elastic line to stress and strain in an actual beam. He discovered the famous expression $\sigma = My/I$ for the stress due to the pure bending of a homogeneous linear elastic beam; here M is the bending moment, y is the distance of a point from an axis that passes through the section centroid, parallel to the axis of bending, and I is the integral of y^2 over the section area. The notion of shear stress was introduced by French mathematician Parent in 1713. It was the work of Coulomb in 1773 to develop extensively this idea in connection with beams and with the stressing and failure of soil. He also studied frictional slip in 1779.

The most important and extensive contributions to the development of continuum

[2] quotation from I. Asimov's Biographical Encyclopedia of Science and Technology, Doubleday, 1964.

[3] J. R. Rice; *Mechanics of Solids*, published as a section of the article on Mechanics in the 1993 printing of the 15th edition of Encyclopaedia Britannica (volume 23, pages 734 - 747 and 773), 1993.

mechanics stem from the great French mathematician Augustin Louis Cauchy (1789-1857), originally educated as an engineer. 'In 1822 he formalized the stress concept in the context of a general three-dimensional theory, showed its properties as consisting of a 3 by 3 symmetric array of numbers that transform as a tensor, derived the equations of motion for a continuum in terms of the components of stress, and gave the specific development of the theory of linear elastic response for isotropic solids. As part of this work, Cauchy also introduced the equations which express the six components of strain, three extensional and three shear, in terms of derivatives of displacements for the case when all those derivatives are much smaller than unity; similar expressions had been given earlier by Euler in expressing rates of straining in terms of the derivatives of the velocity field in a fluid'[4].

In this book we present a modern version of those models in which not only linear elastic but also viscoelastic and plastic materials are included. The full nonlinear theory is not included but some of its basic notions such as Lagrangian and Eulerian descriptions are indicated.

In the presentation we avoid many mathematical details in order to be understandable for less mathematically skillful engineers and natural scientists. Those who would like to clean up some mathematical points we refer to the work of M. Gurtin [4]. Nonlinear problems are presented in many modern monographs. We quote here only four examples [2], [11], [21], [22].

Linear models of elasticity, viscoelasticity, plasticity, viscoplasticity and dislocations are presented. To keep the volume of the notes related to the extent of the one-semester course we have not included such important subjects as coupling to thermal effects, a theory of brittle materials (damage), some topics, such as critical states of soils or earthquake mechanics, are only indicated. The book almost does not contain exercises which we offer to students separately.

Some examples, auxiliary remarks and reminders which are not necessary for the systematic presentation of the material are confined by the signs ★...♣. References are made in two ways. I have selected a number of books and monographs – 24 to be exact – which were used extensively in preparing these notes and which may serve as a help in homework of students. Many references to particular issues and, especially, historical notes, are made in the form of footnotes. I would like to apology to these readers who do not speak Polish for some references in this language. I did it only occasionally when the English version is not available and when I wanted to pay a tribute to my collegues and masters for teaching me many years ago the subject of continuum mechanics.

[4] J. R. Rice; *Mechanics of Solids*, published as a section of the article on Mechanics in the 1993 printing of the 15th edition of Encyclopaedia Britannica (volume 23, pages 734 - 747 and 773), 1993.

Chapter 1

Modicum of vectors and tensors

1.1 Algebra

The most important notions of mechanics such as positions of points, velocities, accelerations, forces are vectors and many other important objects such as deformations, stresses, elasticity parameters form tensors. Therefore we begin our presentation with a brief overview of a vector calculus in Euclidean spaces. We limit the presentation to three-dimensional spaces as continuum mechanics does not require any more general approach.

Vectors are objects characterized by the length and the direction. A vector space $\mathcal{V}$ is defined by a set of axioms describing three basic operations on vectors belonging to this space: a multiplication by a real number, an addition, and a scalar product. The first operation

$$\forall_{\mathbf{a}\in\mathcal{V}}\forall_{\alpha\in\Re}\mathbf{b} = \alpha\mathbf{a} \in\mathcal{V} \tag{1.1}$$

defines, for any vector $\mathbf{a}$, a new vector $\mathbf{b} = \alpha\mathbf{a}$ whose direction is the same (it has the opposite orientation for $\alpha < 0$) as this of the vector $\mathbf{a}$ and the length is α times larger (or smaller for $|\alpha| < 1$) than this of $\mathbf{a}$. The second operation

$$\forall_{\mathbf{a},\mathbf{b}\in\mathcal{V}}\mathbf{a} + \mathbf{b} = \mathbf{c} \in\mathcal{V} \tag{1.2}$$

defines for two vectors $\mathbf{a}, \mathbf{b}$ a new vector $\mathbf{c}$ which is constructed according to the rule of triangle shown in Fig.1.1.

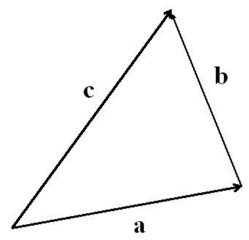


Fig. 1.1: Addition of vectors

The third operation

$$\forall_{\mathbf{a},\mathbf{b}} \alpha = \mathbf{a} \cdot \mathbf{b}, \quad \alpha \in \Re, \tag{1.3}$$

is a scalar product which for any two vectors $\mathbf{a}, \mathbf{b}$ defines a real number. For any vector $\mathbf{a}$

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2, \tag{1.4}$$

it defines its length $|\mathbf{a}|$, while for two vectors $\mathbf{a}, \mathbf{b}$ it specifies the angle $\varphi = (\mathbf{a}, \mathbf{b})$ between them

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \cos \varphi. \tag{1.5}$$

These three operations satisfy a set of axioms, such as associativity, commutativity, etc. which we do not specify here as in the analytical description these are replaced by similar axioms for operations on real numbers.

One should also mention that, by means of the above operations one can introduce the vector $\mathbf{0}$ whose length is 0 and direction is arbitrary as well as a number of linearly independent vectors which defines the dimension of the vector space $\mathcal{V}$. We say that the space $\mathcal{V}$ is three-dimensional if for any three different non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of different direction the relation

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 = \mathbf{0}, \tag{1.6}$$

is satisfied only if the numbers $\alpha_1, \alpha_2, \alpha_3$ are all equal to zero. This property allows to replace the above presented geometrical approach to vector calculus by an analytical approach. This was an ingenious idea of René Descartes (1596-1650). For Euclidean spaces which we use in these notes we select in the vector space $\mathcal{V}$ three linearly independent vectors $\mathbf{e}_i, i = 1, 2, 3$, which satisfy the following condition

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3, \tag{1.7}$$

where δ_{ij} is the so-called Kronecker delta. It is equal to one for $i = j$ and zero otherwise. For $i = j$ we have

$$\mathbf{e}_i \cdot \mathbf{e}_i = 1, \tag{1.8}$$

which means that each vector $\mathbf{e}_i$ has the unit length. Simultaneously, for $i \neq j$,

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad \Rightarrow \quad \varphi = \frac{\pi}{2}, \tag{1.9}$$

where φ is the angle between $\mathbf{e}_i$ and $\mathbf{e}_j$. It means that these vectors are perpendicular.

We call such a set of three vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the base (or basis) vectors of the space $\mathcal{V}$. Obviously, any linear combination

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathcal{V}, \tag{1.10}$$

is a non-zero vector for a_i not simultaneously equal to zero. It is easy to see that any vector from the space $\mathcal{V}$ can be written in this form. If we choose such a vector $\mathbf{a}$ then

$$\mathbf{a} \cdot \mathbf{e}_i = \sum_{j=1}^{3} (a_j \mathbf{e}_j) \cdot \mathbf{e}_i = \sum_{j=1}^{3} a_j \delta_{ij} = a_i. \tag{1.11}$$

The numbers a_i are called coordinates of the vector $\mathbf{a}$. Clearly, they may be written in the form

$$a_i = |\mathbf{a}| \, |\mathbf{e}_i| \cos\left(\left(\mathbf{a}, \mathbf{e}_i\right)\right) = |\mathbf{a}| \cos\left(\left(\mathbf{a}, \mathbf{e}_i\right)\right), \tag{1.12}$$

i.e. geometrically it is the length (with an appropriate sign depending on the choice of $\mathbf{e}_i$!) of projection of the vector $\mathbf{a}$ on the direction of the unit vector $\mathbf{e}_i$. Certainly, $(\mathbf{a}, \mathbf{e}_i)$ denotes the angle between the vectors $\mathbf{a}$ and $\mathbf{e}_i$.

The above rule of representation of an arbitrary vector allows to write the operations in the vector space in the following manner

$$\begin{aligned} \mathbf{b} &= \alpha\mathbf{a} = (\alpha a_i)\,\mathbf{e}_i, \quad \mathbf{b} = b_i \mathbf{e}_i, \quad b_i = \alpha a_i, \\ \mathbf{c} &= \mathbf{a} + \mathbf{b} = (a_i + b_i)\,\mathbf{e}_i, \quad \mathbf{c} = c_i \mathbf{e}_i, \quad c_i = a_i + b_i, \\ \alpha &= \mathbf{a} \cdot \mathbf{b} = a_i b_i, \quad \alpha = a_i b_i, \end{aligned} \tag{1.13}$$

where we have introduced the so-called Einstein convention that a repetition of an index in the product means the summation over all values of this index. For instance,

$$a_i b_i \equiv \sum_{i=1}^{3} a_i b_i. \tag{1.14}$$

Relations (1.13) allow for the replacement of geometrical rules of vector calculus by analytical rules for numbers denoting coordinates of the vector. A vector $\mathbf{a}$ in the three-dimensional space is in this sense equivalent to the matrix $(a_1, a_2, a_3)^T$ provided we choose a specific set of base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. However, in contrast to matrices which are collections of numbers, vectors satisfy certain rules of transformation between matrices specified by the rotations of base vectors. It means that two different matrices $(a_1, a_2, a_3)^T$ and $(a_{1'}, a_{2'}, a_{3'})^T$ may define the same vector if the coordinates a_i and $a_{i'}$ are connected by a certain rule of transformation. We can specify this rule immediately if we consider a rotation of the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the new base vectors $\{\mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{3'}\}$. We have

$$\begin{gathered} \mathbf{e}_{i'} = A_{i'i}\mathbf{e}_i, \quad A_{i'i} = \mathbf{e}_{i'} \cdot \mathbf{e}_i = \cos\left(\left(\mathbf{e}_{i'}, \mathbf{e}_i\right)\right), \\ \mathbf{e}_i = A_{ii'}\mathbf{e}_{i'}, \quad A_{ii'} = \mathbf{e}_i \cdot \mathbf{e}_{i'} = \cos\left(\left(\mathbf{e}_i, \mathbf{e}_{i'}\right)\right), \\ \mathbf{e}_{i'} = A_{i'i}A_{ij'}\mathbf{e}_{j'} \quad \Rightarrow \quad A_{i'i}A_{ij'} = \delta_{i'j'}. \end{gathered} \tag{1.15}$$

Hence the matrix $(A_{ii'})$ is inverse to the matrix $(A_{i'i})$ Obviously, these matrices of transformation $A_{i'i}$ and $A_{ii'}$ are square and formed of sine and cosine functions of angles between base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{3'}\}$.

★**Example:** Further we refer frequently to the rotation in the case of a two-dimensional space. In a three-dimensional space a rotation is defined by three angles (e.g. Euler angles of crystallography). In the two-dimensional case we need one angle, say ϕ, which we assume to be positive in the anticlockwise direction (see: Fig. 1.2).

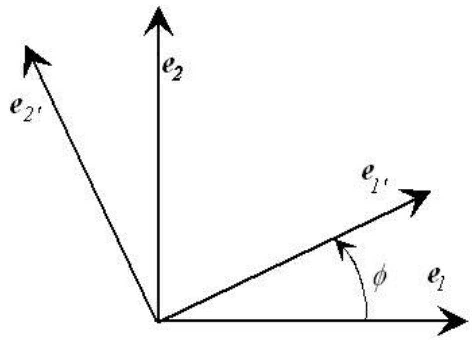


Fig. 1.2: Rotation of the basis on the plane perpendicular to $\mathbf{e}_3$

Then we have

$$\begin{aligned} A_{11'} &= \mathbf{e}_1 \cdot \mathbf{e}_{1'} = \cos\phi, \quad A_{12'} = e_1 \cdot \mathbf{e}_{2'} = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi, \\ A_{21'} &= \mathbf{e}_2 \cdot \mathbf{e}_{1'} = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi, \quad A_{22'} = \mathbf{e}_2 \cdot \mathbf{e}_{2'} = \cos\phi, \\ A_{33'} &= \mathbf{e}_3 \mathbf{e}_{3'} = 1, \end{aligned} \tag{1.16}$$

and the remaining components are zero. Similarly, we can find the components of the matrix $A_{i'i}$. Then we have

$$(A_{ii'}) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (A_{i'i}) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (A_{ii'})^T = (A_{i'i}). \tag{1.17}$$

The last property, which holds also in a three-dimensional case, together with (1.15) means that the matrix $(A_{i'i})$ is orthogonal. This is the general property of all matrices of rotation. We shall return to this property in the discussion of the deformation of continua.

As an example let us consider the two-dimensional transformation of the vector

$$\mathbf{a} = a_i \mathbf{e}_i, \quad (a_i) = (1, 2, 3), \tag{1.18}$$

with $\phi = \pi/6$. We have

$$\begin{aligned} \mathbf{a} &= a_{i'}\mathbf{e}_{i'}, \quad a_{i'} = A_{ii'}a_i \quad \Rightarrow \\ \Rightarrow & \quad (a_{i'}) = (\cos\phi + 2\sin\phi, -\sin\phi + 2\cos\phi, 3) = \\ &= \left(1+\sqrt{3}/2, \sqrt{3}-1/2, 3\right). \end{aligned} \tag{1.19}$$

♣

Bearing the above considerations in mind we can write for an arbitrary vector **a**

$$\mathbf{a} = a_i\mathbf{e}_i = a_{i'}\mathbf{e}_{i'} \quad \Rightarrow \quad a_{i'} = (a_i\mathbf{e}_i)\cdot\mathbf{e}_{i'} = A_{ii'}a_i, \tag{1.20}$$

In mathematics the rule of transformation (1.20) is considered to be the formal definition of the vector.

We return now to objects defined in a general case on three-dimensional vector spaces. One of the most important objects defined on these spaces is a tensor of the second rank which transforms an arbitrary vector into another vector. In addition this transformation should be linear and homogeneous. Formally, we can write

$$\mathbf{b} = \mathbf{t}(\mathbf{a}) = \mathbf{T}\mathbf{a}, \tag{1.21}$$

where **T** is independent of **a**. The first part of this relation means that the vector **b** is the value of the function **t** calculated for a chosen vector **a**. The second part means that the function **t** is linear. It should be invertible, i.e. the tensor $\mathbf{T}^{-1}$ should exist and be unique

$$\mathbf{a} = \mathbf{T}^{-1}\mathbf{b}, \quad \mathbf{T}\mathbf{T}^{-1} = \mathbf{1}, \tag{1.22}$$

where **1** is the unit tensor. These properties indicate that a representation of the tensor **T** in any set of base vectors is a square matrix.

For a chosen set of base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can introduce the operation of the tensor product $\otimes$ which defines the unit tensor **1** as the matrix (δ_{ij}) and we write

$$\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.23}$$

Clearly, in order to be a unit tensor it must possess the following property

$$\mathbf{a} = \mathbf{1a} \quad \Rightarrow \quad a_i\mathbf{e}_i = (\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(a_k\mathbf{e}_k), \tag{1.24}$$

which means that the tensor product operates in this way that we take the scalar product of the second unit vector in **1** (i.e. $\mathbf{e}_j$ in our case) with the vector **a** appearing after the unit tensor. Then (1.24) becomes

$$a_i\mathbf{e}_i = \delta_{ij}\mathbf{e}_i a_k \delta_{jk}, \tag{1.25}$$

which is, of course, an identity.

Making use of the tensor product we can write the following representation for an arbitrary tensor of the second rank

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.26}$$

Matrix (T_{ij}) is the representation of the tensor $\mathbf{T}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The numbers T_{ij} are called components of the tensor $\mathbf{T}$. Now the relation (1.21) can be written in the form

$$b_i \mathbf{e}_i = (T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j)(a_k \mathbf{e}_k) = T_{ij} a_k \mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_k = T_{ij} a_j \mathbf{e}_i \quad \Rightarrow \quad b_i = T_{ij} a_j. \tag{1.27}$$

The tensor $\mathbf{T}$ changes both the direction and the length of the vector $\mathbf{a}$. For the length we have the relation

$$\begin{aligned} |\mathbf{b}|^2 &= \mathbf{b} \cdot \mathbf{b} = (\mathbf{Ta}) \cdot (\mathbf{Ta}) = \\ &= T_{ij} a_j T_{il} a_l = a_j T_{ij} T_{il} a_l = \mathbf{a} \cdot \mathbf{T}^T \mathbf{Ta}. \end{aligned} \tag{1.28}$$

Usually $|\mathbf{b}| \neq |\mathbf{a}|$. However, if the tensor $\mathbf{T}$ possesses the property $\mathbf{T}^T \mathbf{T} = \mathbf{1}$ the length of the vector $\mathbf{b}$ remains the same as this of the vector $\mathbf{a}$. Such tensors are called orthogonal and we denote them usually by $\mathbf{O}$. Obviously, they have the property

$$\mathbf{O}^T = \mathbf{O}^{-1}, \tag{1.29}$$

which may also be used as the definition of orthogonality of the tensor. Orthogonal tensors yield only rotations of vectors.

Once we have the length of the vector $\mathbf{b}$ we can easily find the angle of rotation caused by the tensor $\mathbf{T}$. We have for the angle $\varphi = (\mathbf{a}, \mathbf{b})$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \cos\varphi \quad \Rightarrow \quad \cos\varphi = \frac{T_{ij} a_i a_j}{\sqrt{a_i a_i} \sqrt{(T_{ij} a_j T_{ik} a_k)}}. \tag{1.30}$$

The representation (1.26) allows to specify rules of transformation of components of an arbitrary tensor of the second rank $\mathbf{T}$. Performing the transformation of base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \rightarrow \{\mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{3'}\}$ we obtain

$$\begin{aligned} \mathbf{T} &= T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = T_{ij} (A_{ii'} \mathbf{e}_{i'}) \otimes (A_{jj'} \mathbf{e}_{j'}) = \\ &= A_{ii'} A_{jj'} T_{ij} \mathbf{e}_{i'} \otimes \mathbf{e}_{j'} = T_{i'j'} \mathbf{e}_{i'} \otimes \mathbf{e}_{j'}. \end{aligned}$$

Hence

$$T_{i'j'} = A_{ii'} A_{jj'} T_{ij}. \tag{1.31}$$

Similarly to vectors, this relation is used in tensor calculus as a definition of the tensor of the second rank.

Majority of second rank tensors in mechanics are symmetric. They possess six independent components instead of nine components of a general case. However, some particular problems, such as Cosserat media and couple stresses, interactions with electromagnetic fields require full nonsymmetric tensors. An arbitrary tensor can be always split into a symmetric and antisymmetric parts

$$\mathbf{T} = \mathbf{T}^a + \mathbf{T}^s, \quad \mathbf{T}^a = \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right), \quad \mathbf{T}^s = \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right), \tag{1.32}$$

and, obviously, $\mathbf{T}^a$ possesses only three off-diagonal non-zero components while $\mathbf{T}^s$ possesses six components. It is often convenient to replace the antisymmetric part by a vector. Usually, one uses the following definition

$$\mathbf{V} = V_k \mathbf{e}_k, \quad V_k = -\frac{1}{2}\epsilon_{ijk} T^a_{ij} \quad \Rightarrow \quad T^a_{ij} = -\epsilon_{ijk} V_k. \tag{1.33}$$

where $\mathbf{V}$ is called the axial vector and ϵ_{ijk} is the permutation symbol (Levi-Civita symbol). It is equal to one for the even permutation of indices $\{1,2,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, minus one for the odd permutation of indices $\{2,1,3\}$, $\{1,3,2\}$, $\{3,2,1\}$ and zero otherwise, i.e.

$$\epsilon_{ijk} = \frac{1}{2}(j-i)(k-i)(k-j). \tag{1.34}$$

This symbol appears in the definition of the so-called exterior or vector product of two vectors. The definition of this operation

$$\mathbf{b} = \mathbf{a}_1 \times \mathbf{a}_2, \tag{1.35}$$

is such that the vector $\mathbf{b}$ is perpendicular to both $\mathbf{a}_1$ and $\mathbf{a}_2$, its length is given by the relation

$$|\mathbf{b}| = |\mathbf{a}_1| \, |\mathbf{a}_2| \sin\left((\mathbf{a}_1, \mathbf{a}_2)\right), \tag{1.36}$$

and the direction is determined by the anticlockwise screw rule. For instance

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \tag{1.37}$$

for the base vectors used in this work. In general, we have for these vectors

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k. \tag{1.38}$$

Consequently, for the vector product of two arbitrary vectors $\mathbf{a}_1 = a^1_i \mathbf{e}_i, \mathbf{a}_2 = a^2_i \mathbf{e}_i$, we obtain

$$\mathbf{b} = b_k \mathbf{e}_k = \mathbf{a}_1 \times \mathbf{a}_2 = \left(a^1_i \mathbf{e}_i\right) \times \left(a^2_j \mathbf{e}_j\right) = a^1_i a^2_j \epsilon_{ijk} \mathbf{e}_k \quad \Rightarrow \quad b_k = \epsilon_{ijk} a^1_i a^2_j. \tag{1.39}$$

The vector product is well defined within a theory of 3-dimensional vector spaces. However, for example for two-dimensional spaces of vectors tangent to a surface at a given point, the result of this operation is a vector which does not belong any more to the vector space. It is rather a vector locally perpendicular to the surface.

The permutation symbol can be also used in the evaluation of determinants. For a tensor $\mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, we can easily prove the relation

$$\det \mathbf{T} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k}. \tag{1.40}$$

In mechanics of isotropic materials we use also the following identity ("contracted epsilon identity")

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{1.41}$$

A certain choice of base vectors plays a very important role in the description of properties of tensors of the second rank. This is the contents of the so-called eigenvalue problem. We proceed to present its details.

First we shall find a direction $\mathbf{n}$ defined by unit vector, $|\mathbf{n}| = 1$, whose transformation by the tensor $\mathbf{T}$ into the vector $\mathbf{Tn}$ is extremal in the sense that the length of its projection on $\mathbf{n}$ given by $\mathbf{n} \cdot \mathbf{Tn}$ is largest or smallest with respect to all changes of $\mathbf{n}$. This is the variational problem

$$\delta \left(\mathbf{n} \cdot \mathbf{Tn} - \lambda \left(\mathbf{n} \cdot \mathbf{n} - \mathbf{1}\right)\right) = 0, \tag{1.42}$$

for an arbitrary small change of the direction, $\delta\mathbf{n}$. Here λ is the Lagrange multiplier which eliminates the constraint on the length of the vector $\mathbf{n} : \mathbf{n} \cdot \mathbf{n} = 1$. Obviously, the problem can be written in the form

$$\delta\mathbf{n} \cdot \left[\left(\mathbf{T}^s - \lambda \mathbf{1}\right) \mathbf{n}\right] = \mathbf{0}, \tag{1.43}$$

for arbitrary variations $\delta\mathbf{n}$.

Let us notice that

$$\mathbf{n} \cdot \mathbf{Tn} = \mathbf{n} \cdot \left(\mathbf{T}^a + \mathbf{T}^s\right) \mathbf{n} = \mathbf{n} \cdot \mathbf{T}^s \mathbf{n}. \tag{1.44}$$

Consequently, the antisymmetric part $\mathbf{T}^a$ has no influence on the solution of the problem.

Bearing the above remark in mind, we obtain from (1.43)

$$\left(\mathbf{T}^s - \lambda \mathbf{1}\right) \mathbf{n} = \mathbf{0}, \tag{1.45}$$

or, in components,

$$\left(T^s_{ij} - \lambda \delta_{ij}\right) n_j = 0. \tag{1.46}$$

It means that λ are the eigenvalues of the symmetric tensor $\mathbf{T}^s$ and $\mathbf{n}$ are the corresponding eigenvectors.

The existence of nontrivial solutions of the set of three equations (1.46) requires that its determinant is zero

$$\det \left(\mathbf{T}^s - \lambda \mathbf{1}\right) = 0, \tag{1.47}$$

i.e.

$$\begin{vmatrix} T^s_{11} - \lambda & T^s_{12} & T^s_{13} \\ T^s_{12} & T^s_{22} - \lambda & T^s_{23} \\ T^s_{13} & T^s_{23} & T^s_{33} - \lambda \end{vmatrix} = 0. \tag{1.48}$$

This can be written in the explicit form

$$\lambda^3 - I\lambda^2 + II\lambda - III = 0, \tag{1.49}$$

where

$$I = \operatorname{tr} \mathbf{T}^s = T^s_{ii}, \quad II = \frac{1}{2}\left(I^2 - \operatorname{tr} \mathbf{T}^{s2}\right) = \frac{1}{2}\left(\left(T^s_{ii}\right)^2 - T^s_{ij} T^s_{ij}\right), \quad III = \det \mathbf{T}^s, \tag{1.50}$$

are the so-called principal invariants of the tensor $\mathbf{T}^s$. Obviously, the cubic equation (1.49) possesses three roots. For symmetric real matrices they are all real. It is customary in mechanics to order them in the following manner

$$\lambda^{(1)} \geq \lambda^{(2)} \geq \lambda^{(3)}. \tag{1.51}$$

However, in more general mathematical problems, in particular when the spectrum of eigenvalues is infinite, the smallest eigenvalue is chosen as the first and then the eigenvalue sequence is growing.

Clearly

$$\begin{aligned} I &= \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}, \\ II &= \lambda^{(1)}\lambda^{(2)} + \lambda^{(1)}\lambda^{(3)} + \lambda^{(2)}\lambda^{(3)}, \\ III &= \lambda^{(1)}\lambda^{(2)}\lambda^{(3)}. \end{aligned} \tag{1.52}$$

Once we have these values we can find the corresponding eigenvectors $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$ from the set of equations (1.45). Only two of these equations are independent which means that we can normalize the eigenvectors by requiring $\left|\mathbf{n}^{(\alpha)}\right| = 1$, $\alpha = 1, 2, 3$. It is easy to show that these vectors are perpendicular to each other. Namely we have for $\alpha, \beta = 1, 2, 3$

$$\mathbf{n}^{(\beta)} \cdot \left[\left(\mathbf{T}^s - \lambda^{(\alpha)}\mathbf{1}\right)\mathbf{n}^{(\alpha)}\right] = 0 \quad \Rightarrow \quad \left(\lambda^{(\alpha)} - \lambda^{(\beta)}\right)\mathbf{n}^{(\alpha)} \cdot \mathbf{n}^{(\beta)} = 0. \tag{1.53}$$

If the eigenvalues are distinct, i.e. $\lambda^{(\alpha)} \neq \lambda^{(\beta)}$ for $\alpha \neq \beta$ we obtain

$$\mathbf{n}^{(\alpha)} \cdot \mathbf{n}^{(\beta)} = 0, \tag{1.54}$$

and, consequently, the vectors $\mathbf{n}^{(\alpha)}, \mathbf{n}^{(\beta)}$ are perpendicular (orthogonal). The proof can be easily extended on the case of twofold and threefold eigenvalues.

The above property of eigenvectors allows to use them as a special set of base vectors $\left\{\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right\}$. Then the tensor $\mathbf{T}^s$ can be written in the form

$$\mathbf{T}^s = \sum_{\alpha=1}^{3} \lambda^{(\alpha)}\mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}. \tag{1.55}$$

We call this form the spectral representation of the tensor $\mathbf{T}^s$. Obviously, the tensor in this representation has the form of diagonal matrix

$$\left(T^s_{\alpha\beta}\right) = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}. \tag{1.56}$$

1.2 Analysis

Apart from algebraic properties of vectors and tensors which we presented above, problems of continuum mechanics require differentiation of these objects with respect to spatial variables and with respect to time. Obviously, such problems appear when vectors and tensors are functions of coordinates in space and time. Then we speak about vector or tensor fields.

One of the most important properties of the vector fields defined on three-dimensional domains is the Helmholtz (1821-1894) decomposition[1]. For every square-integrable vector

[1] C. Amrouche, C. Bernardi, M. Dauge, V. Girault;. Vector potentials in three dimensional non-smooth domains, *Mathematical Methods in the Applied Sciences,* **21**, 823–864, 1998.

field $\mathbf{v}$ the following orthogonal decomposition holds

$$\mathbf{v}(\mathbf{x}) \equiv \mathbf{v}(x_1, x_2, x_3) = \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\psi}, \quad \mathbf{x} = x_i \mathbf{e}_i, \tag{1.57}$$

where x_i are Cartesian coordinates of the point $\mathbf{x}$ of the three-dimensional Euclidean space $\mathcal{E}^3$,

$$\operatorname{grad}\varphi = \frac{\partial \varphi}{\partial x_i}\mathbf{e}_i, \quad \operatorname{rot}\boldsymbol{\psi} = \epsilon_{ijk}\frac{\partial \psi_k}{\partial x_j}\mathbf{e}_i, \tag{1.58}$$

and $\varphi, \boldsymbol{\psi}$ are called scalar and vector potential, respectively.

The operator rot (it is identical with curl which is used in some texts to denote the same operation) can be easily related to the integration along a closed curve. Namely G. Stokes (1819-1903) proved the Theorem that for all fields $\mathbf{v}$ differentiable on an oriented surface S with the normal vector $\mathbf{n}$ the following relation holds

$$\int_S (\operatorname{rot}\mathbf{v}) \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{v} \cdot d\mathbf{x}, \tag{1.59}$$

$$\text{i.e.} \quad \int_S \epsilon_{ijk}\frac{\partial v_k}{\partial x_j} n_i dS = \oint_{\partial S} \mathbf{v}_i dx_i,$$

where ∂S is the boundary curve of the surface S.

On the other hand, integration over a closed surface is related to the volume integration. This is the subject of the Gauss (1777-1855) Theorem discovered by J. L. Lagrange in 1762.

For a compact domain $V \subset \mathcal{E}^3$ with a piecewise smooth boundary ∂V the following relation for a continuously differentiable vector field $\mathbf{v}$ holds

$$\int_V \operatorname{div}\mathbf{v} dV = \oint_{\partial V} \mathbf{v} \cdot \mathbf{n} dS, \tag{1.60}$$

where ∂V denotes the boundary of the domain of volume integration V and $\mathbf{n}$ is a unit vector orthogonal to the boundary and oriented outwards.

G. Leibniz (1646-1716) Theorem for volume integrals which we use in analysis of balance equations of mechanics describes the time differentiation of integrals whose domain is time-dependent. It has the following form

$$\frac{d}{dt}\int_{V(t)} f(t, \mathbf{x}) dV = \int_V \frac{\partial f}{\partial t}(t, \mathbf{x}) dV + \oint_{\partial V} f(t, \mathbf{x}) \mathbf{v} \cdot \mathbf{n} dS, \tag{1.61}$$

where t denotes time, $\mathbf{x}$ is the point within V, $\mathbf{v}$ is the velocity of boundary points of the domain V. Instead of a rigorous proof it is useful to observe the way in which the above structure of the time derivative arises. The right-hand side consists of two contributions: the first one arises in the case of time independent domain of integration, V, and due to time differentiation there is only a contribution of the integrand f while the second one arises when the function f is time independent (i.e. time t is kept constant in f) and the volume V changes. They add due to the linearity of the operator of differentiation as in the case of differentiation of the product $d(fg)/dt = gdf/dt + fdg/dt$. The structure of the second contribution is explained in Fig.1.3.

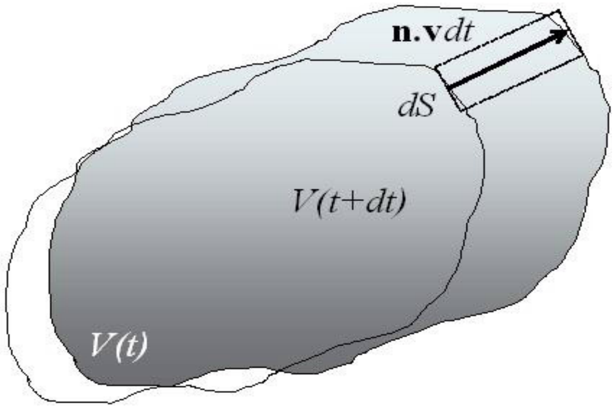


Fig. 1.3: Interpretation of Leibniz Theorem

Locally the change of the volume can be written in the form $dV = dS\mathbf{n} \cdot \mathbf{v}dt$, where $\mathbf{v}$ is the velocity of the point of the boundary. Consequently, the change of the domain of integration for the increment of time dt has the form

$$\int\limits_{V(t+dt)} f(t)\, dV - \int\limits_{V(t)} f(t)\, dV = \left(\oint\limits_{\partial V(t)} f(t)\, \mathbf{v} \cdot \mathbf{n} dS \right) dt. \tag{1.62}$$

Now the second contribution of (1.61) easily follows.

As a particular case of the last Theorem we have for $f = 1$

$$\frac{d}{dt} \int\limits_V dV = \oint\limits_{\partial V} \mathbf{v} \cdot \mathbf{n} dS = \int\limits_V \operatorname{div} \mathbf{v} dV \quad \Rightarrow \quad \frac{dV}{dt} = \int\limits_V \operatorname{div} \mathbf{v} dV. \tag{1.63}$$

This relation indicates that $\operatorname{div} \mathbf{v}$ describes the local time changes of the volume. This interpretation is useful in the analysis of the mass balance equation of continuum mechanics.

More details of the vector calculus and its applications in mechanics can be found in numerous books on continuum mechanics and thermodynamics (e.g. [22]). In some exercises and examples we use curvilinear coordinates rather than Cartesian coordinates applied in the above presentation. We demonstrate their properties and appropriate rules of transformations in these examples.

Chapter 2

Geometry and kinematics of continua

2.1 Preliminaries

As mentioned in Chapter 1 a theoretical description of mechanical behaviour of structures requires continuous models. It means that a collection of points of structures form a three-dimensional continuum of a certain mathematical construct (the so-called differentiable manifold). The points $\mathbf{X}$ of the set $\mathcal{B}_0$ of such a continuum move in a three-dimensional space of motion $\mathcal{E}^3$. The main purpose of continuum mechanics is to determine this motion for any given set of external agents (forces, or given displacements of boundaries of structure, or a mixture of both). It means that one has to solve the set of governing equations in order to find a current position of an arbitrary point $\mathbf{X} \in \mathcal{B}_0$ of the material body $\mathcal{B}_0$. A collection $\mathcal{B}_t$ of positions of all points from the material body $\mathcal{B}_0$ at a given instant of time t is called the current configuration of the body. Once a particular model is selected (elasticity, viscoelasticity, plasticity, viscoplasticity, etc.) the knowledge of these current configurations allows to calculate deformations, stresses, dissipation of energy, work done on the system or any other quantity which may be of practical interest. In some cases one is interested only, for instance, in the distribution of stresses in the system. As we know from the classical linear elasticity such problems may be solved by a transformation of the governing set of equations of motion into equations for stresses. In the linear elasticity we call them Beltrami-Michell equations. However, in general such a transformation is difficult if possible at all and, therefore we limit our attention in these notes to the formulation of governing equations of motion only marginally referring to other approaches.

Engineering problems arise usually in connection with a particular geometry of structures which may lead to considerable simplifications of mechanical models. This is related to the fact that one or two spacial dimensions of a structure are much smaller than the remaining dimension. This yields models of shells, plates, rods, bars and their combinations such as fibrous media, nets and so on. We refer to models of such structures in some examples but, due to a limited volume of these notes we shall not go into any details of their modeling.

A configuration $\mathcal{B}_t$ as a collection of points in $\mathcal{E}^3$ occupied by points $\mathbf{X} \in \mathcal{B}_0$ of the material body is in some cases of practical interest not sufficient to describe a geometry of the system. For instance, in the description of suspensions or liquid crystals one may need additional local degrees of freedom related to rotations of microparticles. In some other cases one may even need additional tensors as a natural space in which the motion appears is not Euclidean. This appears in the description of a continuous distribution of dislocations important in the theory of plasticity. Sometimes an appropriate extension of the classical continuum necessary to describe such systems can be done by the so-called microstructural variables. We mention some of them further in this book. However, it is not always the case (e.g. a nonlinear theory of dislocations, or diffusion processes of mixtures). Such problems shall not be considered in these notes.

2.2 Reference configuration and Lagrangian description

The choice of the reference configuration $\mathcal{B}_0$ with respect to which the motion of the body is measured is arbitrary. If possible we choose a natural stress-free configuration. This is not always convenient for solids (e.g. for prestressed structures) and it is never possible for fluids. These cases will be discussed separately but, in principle, the analysis presented below can be taken over also for such cases. In those many systems in which we can choose a natural configuration we assume its deformation to be zero. The motion of the body, i.e. the function which describes the geometry of all subsequent configurations is described by the function of two variables: time, t, and the point, $\mathbf{X}$, of the reference configuration $\mathcal{B}_0$. The latter specifies the particle which is described. In chosen Cartesian coordinate systems we have

$$\mathbf{x} = \mathbf{f}(\mathbf{X}, t), \quad \text{i.e.} \quad x_k = f_k(X_1, X_2, X_3, t), \quad \mathbf{X} \in \mathcal{B}_0, \quad k = 1, 2, 3, \tag{2.1}$$

where

$$\mathbf{x} = x_k \mathbf{e}_k, \quad \mathbf{X} = X_K \mathbf{e}_K, \tag{2.2}$$

are position vectors in the current configuration and reference configuration, respectively. As the reference configuration is one of the configurations appearing in the real motion of the body, for instance, the one for $t = 0$, we can, certainly, choose the same coordinate system for all configurations, i.e.

$$x_k = \delta_{kK} X_K, \quad \mathbf{e}_k = \delta_{kK} \mathbf{e}_K. \tag{2.3}$$

However, in the analysis of certain invariance properties (e.g. isotropy of the material, material objectivity, etc.), it is convenient to distinguish between these two systems as we did in the relation (2.1). The coordinate system with the base vectors $\mathbf{e}_K$ is called

Lagrangian and the coordinate system with the base vectors $\mathbf{e}_k$ is called Eulerian.

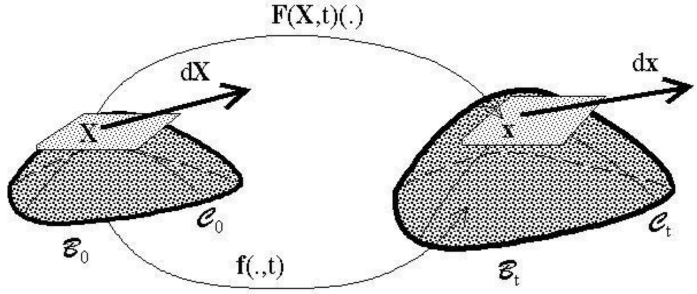


Fig. 2.1: Transformation from the reference configuration $\mathcal{B}_0$ to the current configuration $\mathcal{B}_t$. The function of motion f describes time changes of position of material points X $\in \mathcal{B}_0$ into x $\in \mathcal{B}_t$ while the deformation gradient F describes the transformation of material vectors (e.g. a tangent vector dX of the curve $\mathcal{C}_0$ into a tangent vector dx of the curve $\mathcal{C}_t$)

The vector function $\mathbf{f}(\mathbf{X}, t)$, the function of motion, is assumed to be twice continuously differentiable with respect to all variables. Consequently, it must be also continuous. This property of the theories of continuum eliminates many important processes from continuous models. Some examples are shown in Fig. 2.2. In practical applications, we overcome this difficulty by some additional sophisticated constructions (e.g. for cracks in solids or vorticities in fluids). We shall point out some of them further in these notes.

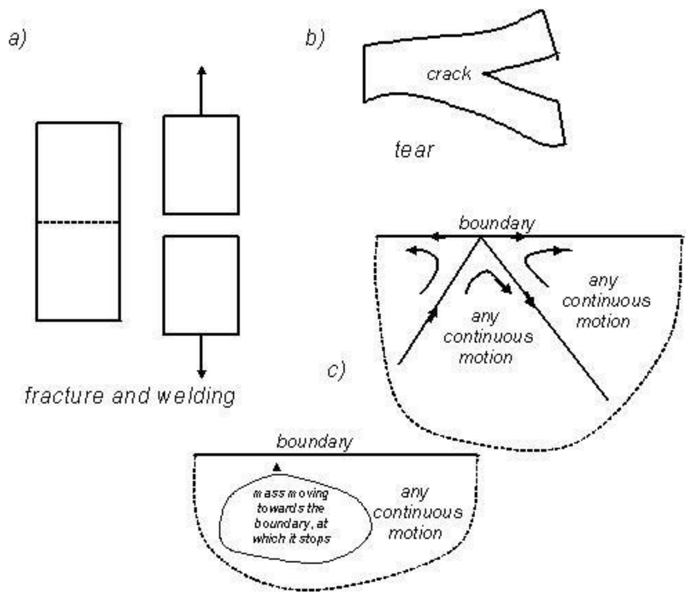


Fig. 2.2: Some motions (nontopological) which cannot be described by continuous functions of motion.

The function of motion $\mathbf{f}(\mathbf{X},t)$ specifies the local rule of transformation of the so-called material vectors. If we select a smooth curve $\mathcal{C}_0$ in the reference configuration $\mathcal{B}_0$, given by a parametric equation $\mathbf{X} = \mathbf{X}(S)$, where S is the parameter along this curve then in each point of this curve the tangent vector is given by the derivative $d\mathbf{X}/dS$. A corresponding infinitesimal vector $d\mathbf{X} = (d\mathbf{X}/dS)\,dS$ changes during the motion in the following way

$$d\mathbf{x} = (\mathrm{Grad}\,\mathbf{f})\,d\mathbf{X} \equiv (\mathrm{Grad}\,\mathbf{f})\,\frac{d\mathbf{X}}{dS}dS = \frac{d\mathbf{x}}{dS}dS, \tag{2.4}$$

or, in coordinates,

$$dx_k = \frac{\partial f_k}{\partial X_K}dX_K = \frac{\partial f_k}{\partial X_K}\frac{dX_K}{dS}dS = \frac{dx_k}{dS}dS. \tag{2.5}$$

Hence, in the current configuration, the tangent vector $d\mathbf{X}/dS$ of the curve $\mathcal{C}_0$ changes into the tangent vector $d\mathbf{x}/dS$ of the curve $\mathcal{C}_t$ given by the relation $\mathbf{x} = \mathbf{f}(\mathbf{X}(S),t)$ in the configuration $\mathcal{B}_t$ which is the image of the curve $\mathcal{C}_0$ of the initial configuration $\mathcal{B}_0$. This new vector has, in general, a different length and a different direction than the vector $d\mathbf{X}/dS$. All such vectors, $\mathbf{V}$, which fulfil the above indicated rule of transformation when changing the configuration $\mathcal{B}_0$ into $\mathcal{B}_t$

$$\begin{aligned} \mathbf{v} &= \mathbf{F}\mathbf{V}, \quad \mathbf{F} = \mathrm{Grad}\,\mathbf{f}, \\ \text{i.e.} \quad v_k &= F_{kK}V_K, \quad F_{kK} = \frac{\partial f_k}{\partial X_K}, \end{aligned} \tag{2.6}$$

are called material vectors. This transformation by means of the object $\mathbf{F}$ which is called the deformation gradient is the most important notion describing geometrical changes of a continuum during the deformation. Before we present the full analysis of the deformation gradient $\mathbf{F}$ let us consider two simple examples.

★We begin with the simplest case of a uniform extension of a cube in three perpendicular directions (see: Fig. 2.3.)

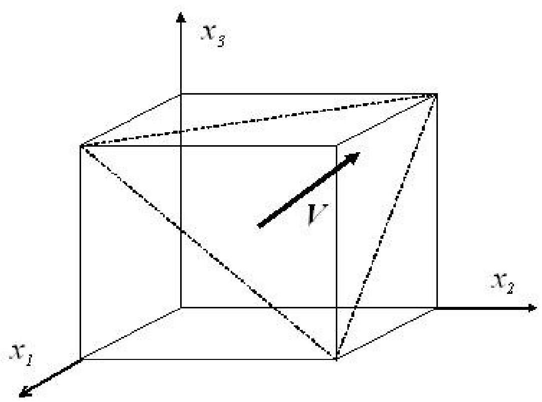


Fig. 2.3: Reference configuration of a cube with the vector V of the examples

described by the following function of motion in Cartesian coordinates

$$x_1 = X_1 (1 + \varepsilon_1), \quad x_2 = X_2 (1 + \varepsilon_2), \quad x_3 = X_3 (1 + \varepsilon_3), \tag{2.7}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are three constants. Then the deformation gradient is given by the relation

$$\mathbf{F} = F_{kK} \mathbf{e}_k \otimes \mathbf{e}_K, \quad (F_{kK}) = \begin{pmatrix} 1+\varepsilon_1 & 0 & 0 \\ 0 & 1+\varepsilon_2 & 0 \\ 0 & 0 & 1+\varepsilon_3 \end{pmatrix}. \tag{2.8}$$

Obviously, it is independent of coordinates, i.e. the deformation is homogeneous. We check the action of the deformation gradient on a chosen vector $\mathbf{V}$. An example of this vector is shown in Fig. 2.3. In coordinates indicated in this Figure the vector $\mathbf{V}$ has the following components

$$\mathbf{V} = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3.$$

Its image after the deformation is as follows

$$\begin{aligned} \mathbf{v} &= \mathbf{FV} = \\ &= ((1+\varepsilon_1)\, \mathbf{e}_1 \otimes \mathbf{e}_1 + (1+\varepsilon_2)\, \mathbf{e}_2 \otimes \mathbf{e}_2 + (1+\varepsilon_3)\, \mathbf{e}_3 \otimes \mathbf{e}_3)(-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \\ &= -(1+\varepsilon_1)\, \mathbf{e}_1 + (1+\varepsilon_2)\, \mathbf{e}_2 + (1+\varepsilon_3)\, \mathbf{e}_3. \end{aligned} \tag{2.9}$$

Hence the current image $\mathbf{v}$ of the vector $\mathbf{V}$ has a different length and a different direction

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = \sum_{i=1}^{3} (1+\varepsilon_i)^2, \quad |\mathbf{V}|^2 = \mathbf{V} \cdot \mathbf{V} = 3, \tag{2.10}$$

$$\cos((\mathbf{v}, \mathbf{V})) = \frac{\sum_{i=1}^{3} (1+\varepsilon_i)}{\sqrt{3}\sqrt{\sum_{i=1}^{3} (1+\varepsilon_i)^2}}.$$

Obviously, for $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ the angle between $\mathbf{v}$ and $\mathbf{V}$ is equal to zero.♣

★The second example describes the so-called simple shearing in the plane perpendicular to $\mathbf{e}_1$. Then

$$x_1 = X_1, \quad x_2 = X_2 + X_3 \tan\varphi, \quad x_3 = X_3, \tag{2.11}$$

and the corresponding deformation gradient is as follows (Lagrangian and Eulerian base vectors are identical $\mathbf{e}_k = \delta_{kK} \mathbf{e}_K$)

$$\mathbf{F} = F_{kK} \mathbf{e}_k \otimes \mathbf{e}_K, \quad (F_{kK}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan\varphi \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

The current image of the vector $\mathbf{V}$ is now given by the relation

$$\begin{aligned} \mathbf{v} &= \mathbf{FV} = \\ &= (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \tan\varphi \mathbf{e}_2 \otimes \mathbf{e}_3)(-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \\ &= -\mathbf{e}_1 + (1+\tan\varphi)\, \mathbf{e}_2 + \mathbf{e}_3. \end{aligned} \tag{2.13}$$

Also in this case the vector $\mathbf{v}$ has the different length and direction from the vector $\mathbf{V}$

$$|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = 2 + (1+\tan\varphi)^2, \quad |\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = 3, \tag{2.14}$$
$$\cos((\mathbf{v},\mathbf{V})) = \frac{3+\tan\varphi}{\sqrt{3}\sqrt{2+(1+\tan\varphi)^2}}.$$

♣

The deformation gradient $\mathbf{F}$ is, obviously, represented by a square matrix. However, it is not a tensor of the second rank. Clearly, if we change the Lagrangian basis $\mathbf{e}_K \to \mathbf{e}_{K'} = A_{K'K}\mathbf{e}_K$ (compare (1.15)) but keep unchanged the Eulerian basis $\mathbf{e}_k$ the deformation gradient transforms as follows

$$F_{kK'} = A_{KK'}F_{kK}, \tag{2.15}$$

i.e. it transforms as three vectors for $k = 1,2,3$ rather than a tensor. The same property possesses the first index under the transformation $\mathbf{e}_k \to \mathbf{e}_{k'} = A_{k'k}\mathbf{e}_k$, and $\mathbf{e}_K$ is kept unchanged

$$F_{k'K} = A_{kk'}F_{kK}. \tag{2.16}$$

This is one of the reasons why the notation for Lagrangian and Eulerian coordinates is different.

The deformation gradient can be written in a different form in which these transformation rules possess an obvious interpretation. This is the subject of the so-called polar decomposition Theorem. For a nonsingular deformation gradient $\mathbf{F}$, $\det\mathbf{F} > \mathbf{0}$ (we return to the justification of this property in the Chapter on the conservation of mass) there exist an orthogonal matrix $\mathbf{R}$ and a symmetric tensor $\mathbf{U}$ such that

$$\mathbf{F} = \mathbf{RU}, \quad \mathbf{R}^T = \mathbf{R}^{-1}, \quad \mathbf{U}^T = \mathbf{U}, \tag{2.17}$$
$$F_{kK} = R_{kL}U_{KL}, \quad (R_{kL})^T = (R_{kL})^{-1}, \quad U_{KL} = U_{LK},$$

and this decomposition is unique. $\mathbf{R}$ describes the local rotation and the right stretch tensor $\mathbf{U}$ the local deformation. We show the construction of these objects. Let us note that in the following product

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^2, \tag{2.18}$$
$$C_{KL} = F_{kK}F_{kL} = U_{KM}U_{ML},$$

the orthogonal part $\mathbf{R}$ does not appear. Hence, in order to find $\mathbf{U}$ we have to take "the square root" of $\mathbf{C}$. Obviously, it is not the same operation which we perform with numbers. We define it through the eigenvalues. Namely, we calculate first the eigenvalues and eigenvectors of the tensor $\mathbf{C}$

$$(\mathbf{C} - \lambda_C\mathbf{1})\,\mathbf{K}_C = \mathbf{0}. \tag{2.19}$$

Obviously

$$\det(C_{KL} - \lambda_C\delta_{KL}) = 0, \tag{2.20}$$

and the solution of this cubic equation gives three eigenvalues $\lambda_C^{(1)}, \lambda_C^{(2)}, \lambda_C^{(3)}$ of the tensor $\mathbf{C}$. They are all real and positive ($\det \mathbf{C} = (\det \mathbf{F})^2 > 0$!). From (2.19) one can find then unit eigenvectors $\mathbf{K}_C^{(1)}, \mathbf{K}_C^{(2)}, \mathbf{K}_C^{(3)}$. They are orthogonal (compare (1.54)). Hence, we can write the tensor $\mathbf{C}$ in the spectral representation

$$\mathbf{C} = \sum_{\alpha=1}^{3} \lambda_C^{(\alpha)} \mathbf{K}_C^{(\alpha)} \otimes \mathbf{K}_C^{(\alpha)}. \tag{2.21}$$

Simultaneously, the eigenvalue problem for $\mathbf{U}$ has the following form

$$(\mathbf{U} - \lambda_U \mathbf{1}) \, \mathbf{K}_U = 0. \tag{2.22}$$

If we multiply this equation by $\mathbf{U}$ from the left we have

$$(\mathbf{U}\mathbf{U} - \lambda_U \mathbf{U}) \, \mathbf{K}_U = \left(\mathbf{C} - \lambda_U^2 \mathbf{1}\right) \mathbf{K}_U = \mathbf{0}. \tag{2.23}$$

Consequently

$$\lambda_U = \sqrt{\lambda_C}, \quad \mathbf{K}_U = \mathbf{K}_C. \tag{2.24}$$

Hence, we obtain the following spectral representation for the right stretch $\mathbf{U}$

$$\mathbf{U} = \sum_{\alpha=1}^{3} \sqrt{\lambda_C^{(\alpha)}} \mathbf{K}_C^{(\alpha)} \otimes \mathbf{K}_C^{(\alpha)}. \tag{2.25}$$

This is what was meant by taking a square root of $\mathbf{C}$. Once $\mathbf{U}$ is given we can calculate $\mathbf{R}$ from (2.17)

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}. \tag{2.26}$$

The above presented procedure is simultaneously the proof of the polar decomposition Theorem.

★Before we proceed let us consider a simple example for the application of the above procedure. We find the polar decomposition of the deformation gradient in the simple shearing given by the relation (2.12). The Lagrangian and Eulerian base vectors are assumed to be identical $\mathbf{e}_k = \delta_{kK} \mathbf{e}_K$. We have

$$\mathbf{C} = C_{KL} \mathbf{e}_K \otimes \mathbf{e}_L, \quad (C_{KL}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan\varphi \\ 0 & \tan\varphi & 1 + \tan^2\varphi \end{pmatrix}. \tag{2.27}$$

The eigenvalue problem yields the following equation for eigenvalues λ_C

$$(1 - \lambda_C) \left[(1 - \lambda_C) \left((1 - \lambda_C) + \tan^2\varphi\right) - \tan^2\varphi\right] = 0. \tag{2.28}$$

The solution has the form

$$\lambda_C^{(1)} = 1, \quad \lambda_C^{(2)} = \left(\frac{1 - \sin\alpha}{\cos\alpha}\right)^2, \quad \lambda_C^{(3)} = \left(\frac{1 + \sin\alpha}{\cos\alpha}\right)^2, \tag{2.29}$$

where

$$\tan\alpha = \frac{1}{2}\tan\varphi. \tag{2.30}$$

The corresponding eigenvectors follow from (2.19). We obtain

$$\begin{aligned}
\mathbf{K}_C^{(1)} &= \mathbf{e}_1, \\
\mathbf{K}_C^{(2)} &= -\left(\frac{1}{\sqrt{2}}\frac{\cos\alpha}{\sqrt{1-\sin\alpha}}\right)\mathbf{e}_2 + \left(\frac{1}{\sqrt{2}}\sqrt{1-\sin\alpha}\right)\mathbf{e}_3, \\
\mathbf{K}_C^{(3)} &= \left(\frac{1}{\sqrt{2}}\frac{\cos\alpha}{\sqrt{1+\sin\alpha}}\right)\mathbf{e}_2 + \left(\frac{1}{\sqrt{2}}\sqrt{1+\sin\alpha}\right)\mathbf{e}_3.
\end{aligned} \tag{2.31}$$

According to the relation (2.25) we obtain for the tensor $\mathbf{U}$

$$\mathbf{U} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \cos\alpha \mathbf{e}_2 \otimes \mathbf{e}_2 + \sin\alpha\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \left(\frac{2}{\cos\alpha} - \cos\alpha\right)\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{2.32}$$

Its inverse has the form

$$\mathbf{U}^{-1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \left(\frac{2}{\cos\alpha} - \cos\alpha\right)\mathbf{e}_2 \otimes \mathbf{e}_2 - \sin\alpha\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \cos\alpha \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{2.33}$$

Consequently, according to the relation (2.26), we have for the orthogonal part

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \cos\alpha \mathbf{e}_2 \otimes \mathbf{e}_2 + \sin\alpha\left(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \cos\alpha \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{2.34}$$

For the linear theory which we consider further in this book it is useful to collect the above results in the case of the small angle φ. Then $\alpha \approx \varphi/2$ and we obtain

$$\begin{aligned}
\mathbf{U} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\varphi}{2}\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \mathbf{e}_3 \otimes \mathbf{e}_3, \\
\mathbf{U}^{-1} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{\varphi}{2}\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \mathbf{e}_3 \otimes \mathbf{e}_3, \\
\mathbf{R} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\varphi}{2}\left(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2\right) + \mathbf{e}_3 \otimes \mathbf{e}_3.
\end{aligned} \tag{2.35}$$

On the plane perpendicular to $\mathbf{e}_1$ these objects yield the following transformations of the edges of unit length of the rectangular prism

$$\begin{aligned}
\mathbf{U}\mathbf{e}_2 &= \mathbf{e}_2 + \frac{\varphi}{2}\mathbf{e}_3, \quad \mathbf{F}\mathbf{e}_2 = \mathbf{R}\mathbf{U}\mathbf{e}_2 = \mathbf{e}_2, \\
\mathbf{U}\mathbf{e}_3 &= \frac{\varphi}{2}\mathbf{e}_2 + \mathbf{e}_3, \quad \mathbf{F}\mathbf{e}_3 = \mathbf{R}\mathbf{U}\mathbf{e}_3 = \varphi\mathbf{e}_2 + \mathbf{e}_3.
\end{aligned} \tag{2.36}$$

Obviously, the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ along the edges are material.

We demonstrate these transformations in Fig. 2.4.

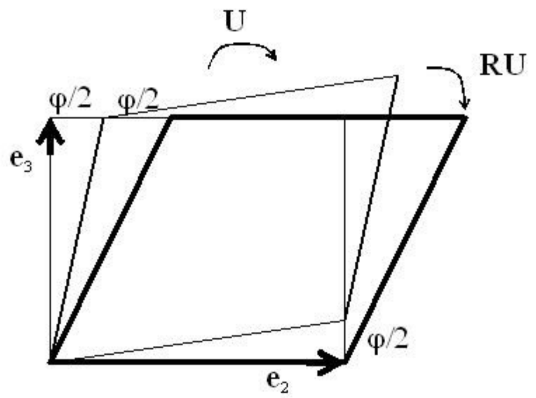


Fig. 2.4: Geometry of the linear simple shearing on the plane perpendicular to $\mathbf{e}_1$.

Obviously, the symmetric stretch tensor $\mathbf{U}$ yields the change of shape from the rectangle to the parallelogram with the symmetry axis of the declination $\pi/4$. The orthogonal tensor $\mathbf{R}$ rotates back the deformed parallelogram in such a way that the horizontal edge before the deformation becomes horizontal after the deformation as well.♣

The above presented example indicates that $\mathbf{U} = U_{KL}\mathbf{e}_K \otimes \mathbf{e}_L$ describes the true local deformation. This is the reason for calling it the right stretch tensor while its square $\mathbf{C} = \mathbf{U}^2 = C_{KL}\mathbf{e}_K \otimes \mathbf{e}_L$ is called the right Cauchy-Green deformation tensor.
As $\mathbf{R} = R_{kK}\mathbf{e}_k \otimes \mathbf{e}_K$ describes local rotations – due to orthogonality – it does not change the length of material vectors.

The polar decomposition Theorem can be also written in the dual form

$$\mathbf{F} = \mathbf{VR}, \quad \mathbf{V} = V_{kl}\mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{V}^T, \quad \mathbf{R} = R_{kK}\mathbf{e}_k \otimes \mathbf{e}_K, \quad \mathbf{R}^T = \mathbf{R}^{-1}, \qquad (2.37)$$
$$\mathbf{B} = \mathbf{FF}^T = \mathbf{V}^2 = \mathbf{B}^T.$$

Then $\mathbf{V}$ is called the left stretch tensor and $\mathbf{B}$ the left Cauchy-Green deformation tensor. For the latter we can easily prove

$$\lambda_B = \lambda_C, \quad \mathbf{k}_B = \frac{\mathbf{FK}_C}{|\mathbf{FK}_C|}, \quad \mathbf{B} = \frac{1}{|\mathbf{FK}_C|^2} \sum_{\alpha=1}^{3} \lambda_C^{(\alpha)} \left(\mathbf{FK}_C^{(\alpha)}\right) \otimes \left(\mathbf{FK}_C^{(\alpha)}\right), \qquad (2.38)$$

i.e. Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ have the same eigenvalues.

In Fig. 2.5 [5] we demonstrate schematically the interpretation of the polar decomposition.

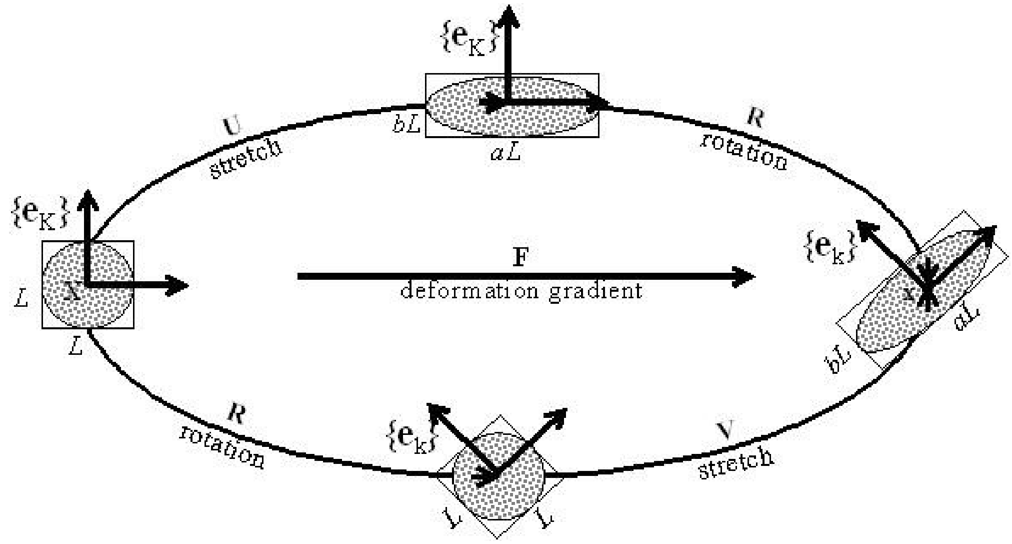


Fig. 2.5: Polar decomposition of the deformation gradient F as the composition of stretch V followed by rotation R (i.e. VR) or vice versa (RU)

2.3 Displacement, velocity, Eulerian description

The motion of the body can be described not only by the function of motion $\mathbf{f}$ but, as customary in the linear elasticity, by the displacement vector $\mathbf{u}$. Similarly to $\mathbf{f}$, it is defined with respect to a chosen reference configuration, say $\mathcal{B}_0$,

$$\mathbf{x} = \mathbf{f}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t). \tag{2.39}$$

Then the gradient of deformation has the form

$$\mathbf{F} = \mathbf{1} + \operatorname{Grad} \mathbf{u}. \tag{2.40}$$

Provided we identify the Lagrangian and Eulerian coordinate systems the Cauchy-Green deformation tensors have then the following form

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T\mathbf{F} = \mathbf{1} + \operatorname{Grad}\mathbf{u} + (\operatorname{Grad}\mathbf{u})^T + (\operatorname{Grad}\mathbf{u})^T \operatorname{Grad}\mathbf{u}, \qquad (2.41)\\ \text{i.e.} \quad C_{KL} &= \delta_{KL} + \delta_{Kk}\frac{\partial u_k}{\partial X_L} + \delta_{Lk}\frac{\partial u_k}{\partial X_K} + \frac{\partial u_k}{\partial X_K}\frac{\partial u_k}{\partial X_L}, \end{aligned}$$

$$\begin{aligned} \mathbf{B} &= \mathbf{F}\mathbf{F}^T = \mathbf{1} + \operatorname{Grad}\mathbf{u} + (\operatorname{Grad}\mathbf{u})^T + \operatorname{Grad}\mathbf{u}(\operatorname{Grad}\mathbf{u})^T, \qquad (2.42)\\ \text{i.e.} \quad B_{kl} &= \delta_{kl} + \delta_{kK}\frac{\partial u_l}{\partial X_K} + \delta_{lK}\frac{\partial u_k}{\partial X_K} + \frac{\partial u_k}{\partial X_K}\frac{\partial u_l}{\partial X_K}. \end{aligned}$$

These relations show that the displacement vector is not very convenient in nonlinear models. There are also other reasons for not using it in such theories (see: [22]). However, it is a very useful notion in linear models and it shall be extensively used in this book.

For our further considerations in these notes it is important to specify the above geometrical description under the assumption of small deformations. We proceed to do so.

We base our considerations on the analysis of the right Cauchy-Green deformation tensor $\mathbf{C}$. It is clear that rotations cannot be assumed to be small as even in the case of lack of deformation the system may rotate as a rigid body and this rotation, of course, cannot be small. For this reason, we cannot constrain the deformation gradient $\mathbf{F}$ and rather measures of deformations are appropriate tools. The undeformed configuration is characterized by the deformation gradient $\mathbf{F} = \mathbf{1}$. Consequently, this configuration is described by the deformation tensors $\mathbf{C} = \mathbf{1}$ and $\mathbf{B} = \mathbf{1}$. The spectral representation of the tensor $\mathbf{C}$ (2.21) indicates that this tensor differs a little from the unit tensor if its eigenvalues $\lambda_C^{(\alpha)}$ deviate a little from unity. These eigenvalues are called principal stretches. It is convenient to introduce a norm for the tensor $\mathbf{C} - \mathbf{1}$ rather than for $\mathbf{C}$. It is done by the following relation

$$\|\mathbf{C} - \mathbf{1}\| = \max_{\alpha=1,2,3} \left| \lambda_C^{(\alpha)} - 1 \right|. \tag{2.43}$$

Then, we say that the body undergoes small deformations if the norm of $\mathbf{C}$ satisfies the condition

$$\|\mathbf{C} - \mathbf{1}\| \ll 1. \tag{2.44}$$

Under this condition one does not have to distinguish between Lagrangian and Eulerian coordinates. For an arbitrary function $h(\mathbf{x}, t)$ we have

$$\frac{\partial h}{\partial X_K} \mathbf{e}_K = F_{kK} \frac{\partial h}{\partial x_k} \mathbf{e}_K \approx \frac{\partial h}{\partial x_k} \mathbf{e}_k. \tag{2.45}$$

It means that Eulerian coordinates can be treated for small deformations as Lagrangian coordinates.

For small deformations, it is convenient to introduced different measures of deformation. One defines for arbitrary deformations the following measures

– Green-St. Venant measure

$$\mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{1}), \quad \lambda_E = \frac{\lambda_C - 1}{2}, \tag{2.46}$$

– Almansi-Hamel measure

$$\mathbf{e} = \frac{1}{2} \left(\mathbf{1} - \mathbf{B}^{-1} \right), \quad \lambda_e = \frac{1 - 1/\lambda_C}{2}. \tag{2.47}$$

For small deformations

$$\lambda_e = \frac{\lambda_C - 1}{2\lambda_C} \approx \frac{\lambda_C - 1}{2} = \lambda_E. \tag{2.48}$$

Hence, both these measures, $\mathbf{E}$ and $\mathbf{e}$, are not distinguishable.

In terms of the displacement vector the Almansi-Hamel measure of small deformations can be written in the following form

$$\mathbf{e} \approx \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right) + O\left(\varepsilon^2\right), \quad \text{i.e.} \quad e_{kl} \approx \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + O\left(\varepsilon^2\right), \quad (2.49)$$

where

$$\varepsilon \equiv \|\operatorname{grad}\mathbf{u}\| \ll 1, \quad \|\operatorname{grad}\mathbf{u}\| = \sqrt{(\operatorname{grad}\mathbf{u}) \cdot (\operatorname{grad}\mathbf{u})} = \sqrt{\frac{\partial u_k}{\partial x_l}\frac{\partial u_k}{\partial x_l}}. \quad (2.50)$$

and $O\left(\varepsilon^2\right)$ are contributions of the order $\varepsilon^2 = (\operatorname{grad}\mathbf{u}) \cdot (\operatorname{grad}\mathbf{u})$ and higher.

This measure will be used in linear models discussed further.

In the description of fluids it is convenient to change the reference configuration. Obviously, in contrast to solids one cannot expect an existence of configurations which are stress-free. Consequently, any choice of the reference configuration will be not natural for fluids. The most suitable seems to be the current configuration and, indeed, it is chosen as the reference configuration in most works on fluid mechanics. Such a description does not identify particles. In any point $\mathbf{x}$ of the space of motion, occupied by the material one can specify the velocity $\mathbf{v}(\mathbf{x}, t)$ as a function of time. By means of this field one can find trajectories of particles and, consequently, identify them. In a particular case of the reference configuration $\mathcal{B}_0$ these trajectories are labeled by points $\mathbf{X} \in \mathcal{B}_0$. We proceed to describe the details of such a description.

The velocity of particles is defined in the Lagrangian description by the time derivative of the function of motion

$$\dot{\mathbf{x}} \equiv \mathbf{v} = \frac{\partial \mathbf{f}}{\partial t}(\mathbf{X}, t), \quad \text{i.e.} \quad v_k = \frac{\partial f_k}{\partial t}(X_1, X_2, X_3, t). \quad (2.51)$$

Consequently, if we define the trajectory of motion of the particle $\mathbf{X}$ as the curve $\mathbf{f}(\mathbf{X}, t)$ parametrized by the time t, the velocity is the vector tangent to the trajectory. In the Eulerian description we change the variables $\mathbf{X} \to \mathbf{x}$ by the function $\mathbf{X} = \mathbf{f}^{-1}(\mathbf{x}, t)$ inverse to $\mathbf{f}$ with respect to $\mathbf{X}$. Such a function exists because we assume the determinant $\det \mathbf{F} \neq 0$. Then the velocity in the Eulerian description is the following vector function

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{B}_t, \quad (2.52)$$

and it points in the direction tangent to the trajectory of the particle $\mathbf{X}$ which is instantaneously located in the point $\mathbf{x}$ of the configuration space. Consequently, the equation of this trajectory is given by the set of three ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}(t = 0) = \mathbf{X}. \quad (2.53)$$

For a given velocity field this is a highly nonlinear set which can be only seldom solved analytically.

In order to appreciate the details of the Eulerian description, we consider three configurations: $\mathcal{B}_0$ which has been discussed before and two current configurations $\mathcal{B}_t$, $\mathcal{B}_\tau$

for instances of time t and τ, respectively. Then the function of motion mapping these configurations on each other forms the diagram shown in Fig. 2.6.

In these mappings we have

$$\mathbf{x} = \mathbf{f}(\mathbf{X}, t), \quad \boldsymbol{\xi} = \mathbf{f}(\mathbf{X}, \tau) \quad \Rightarrow \quad \boldsymbol{\xi} = \mathbf{f}_t(\mathbf{x}, \tau) = \mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{x}, t), \tau\right). \tag{2.54}$$

Hence, the function of the relative motion $\mathbf{f}_t(., \tau)$ describes positions of points $\mathbf{x}$ of the configuration in the instant of time t at the new instant of time τ. Obviously, these functions specify the corresponding gradients of deformation

$$\begin{aligned} & d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}, \quad d\boldsymbol{\xi} = \mathbf{F}(\mathbf{X}, \tau)\, d\mathbf{X}, \quad \Rightarrow \\ & \Rightarrow \quad d\boldsymbol{\xi} = \mathbf{F}_t(\mathbf{x}, \tau)\, d\mathbf{x}, \quad \mathbf{F}_t(\mathbf{x}, \tau) = \mathbf{F}(\mathbf{X}, \tau)\, \mathbf{F}^{-1}(\mathbf{X}, t)\big|_{\mathbf{X}=\mathbf{f}^{-1}(\mathbf{x}, t)}. \end{aligned} \tag{2.55}$$

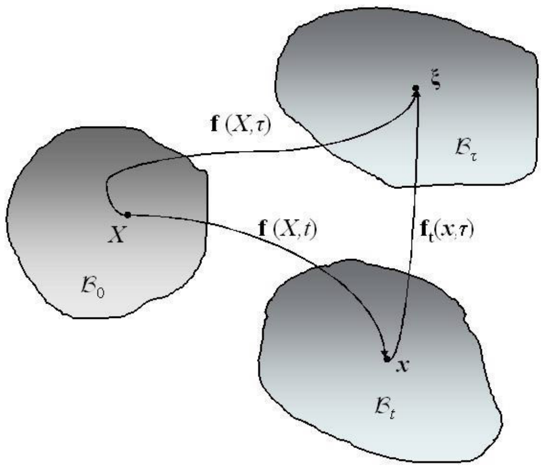


Fig. 2.6: Three configurations yielding Eulerian description

Consequently,

$$\mathbf{F}_t(\mathbf{x}, \tau = t) = \mathbf{1}, \tag{2.56}$$

and, for this reason we say that $\mathbf{F}_t(., \tau)$ is the relative deformation gradient with respect to the configuration at the instant of time t.

The above notion allows to introduce time derivatives of arbitrary quantities in the Eulerian description. As an example let us consider a material vector function $\mathbf{Q}(\mathbf{X}, t)$. Its current image is as follows

$$\mathbf{q}(\mathbf{X}, t) = \mathbf{F}(\mathbf{X}, t)\, \mathbf{Q}(\mathbf{X}, t). \tag{2.57}$$

Hence

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t}(\mathbf{X}, t) &= \mathbf{F}(\mathbf{X}, t) \frac{\partial \mathbf{Q}}{\partial t}(\mathbf{X}, t) + (\operatorname{Grad} \mathbf{v})\, \mathbf{Q}(\mathbf{X}, t), \quad \operatorname{Grad} \mathbf{v} \equiv \frac{\partial \mathbf{F}}{\partial t}(\mathbf{X}, t), \\ \text{i.e.} \quad \frac{\partial q_k}{\partial t} &= F_{kK} \frac{\partial Q_K}{\partial t} + \frac{\partial v_k}{\partial X_K} Q_K, \quad \frac{\partial v_k}{\partial X_K} = \frac{\partial F_{kK}}{\partial t} \equiv \frac{\partial^2 f_k}{\partial t \partial X_K}. \end{aligned} \tag{2.58}$$

These rules of differentiation are straightforward. It is not so in the Eulerian description. We have rather

$$\begin{aligned} \mathbf{q}(\mathbf{x},t) &= \mathbf{F}\left(\mathbf{f}^{-1}(\mathbf{x},t),t\right)\mathbf{Q}\left(\mathbf{f}^{-1}(\mathbf{x},t),t\right), \\ \mathbf{q}(\boldsymbol{\xi},\tau) &= \mathbf{F}\left(\mathbf{f}^{-1}(\boldsymbol{\xi},\tau),\tau\right)\mathbf{Q}\left(\mathbf{f}^{-1}(\boldsymbol{\xi},\tau),\tau\right) \quad \Rightarrow \\ &\Rightarrow \quad \mathbf{q}(\mathbf{x},t) = \mathbf{F}_t^{-1}(\boldsymbol{\xi},\tau)\,\mathbf{q}(\boldsymbol{\xi},\tau)\big|_{\boldsymbol{\xi}=\mathbf{f}_t^{-1}(\mathbf{x},\tau)}. \end{aligned} \tag{2.59}$$

The time derivative of $\mathbf{q}(\mathbf{x},t)$ is now defined in the same way as in the Lagrangian description in the limit $\tau \to t$. We obtain

$$\begin{aligned} \mathcal{L}_v\mathbf{q}(\mathbf{x},t) &= \left.\frac{d\left[\mathbf{F}_t^{-1}(\boldsymbol{\xi},\tau)\,\mathbf{q}(\boldsymbol{\xi},\tau)\right]}{d\tau}\right|_{\tau=t} = \\ &= \frac{\partial\mathbf{q}}{\partial t}(\mathbf{x},t) + (\mathbf{v}\cdot\operatorname{grad})\,\mathbf{q}(\mathbf{x},t) + \left.\frac{d\left[\mathbf{F}_t^{-1}(\boldsymbol{\xi},\tau)\right]}{d\tau}\right|_{\tau=t}\mathbf{q}(\mathbf{x},t). \end{aligned} \tag{2.60}$$

The last contribution can be transformed in the following way

$$\frac{d\left[\mathbf{F}_t^{-1}\mathbf{F}_t\right]}{d\tau} = \mathbf{0} = \frac{d\mathbf{F}_t^{-1}}{d\tau}\mathbf{F}_t + \mathbf{F}_t^{-1}\frac{d\mathbf{F}_t}{d\tau} \quad \Rightarrow \quad \frac{d\mathbf{F}_t^{-1}}{d\tau} = -\mathbf{F}_t^{-1}\frac{d\mathbf{F}_t}{d\tau}\mathbf{F}_t^{-1}. \tag{2.61}$$

Hence

$$\begin{aligned} \left.\frac{d\left[\mathbf{F}_t^{-1}(\boldsymbol{\xi},\tau)\right]}{d\tau}\right|_{\tau=t} &= -\frac{d\mathbf{F}\left(\mathbf{f}^{-1}(\mathbf{x},t),t\right)}{dt}\mathbf{F}^{-1}\left(\mathbf{f}^{-1}(\mathbf{x},t),t\right) = \\ &= -\operatorname{grad}\mathbf{v}(\mathbf{x},t), \end{aligned} \tag{2.62}$$

i.e.

$$\left.\frac{dF_{tkl}^{-1}}{d\tau}\right|_{\tau=t} = -\frac{dF_{kK}}{dt}F_{Kl}^{-1} = \frac{\partial v_k}{\partial X_K}\frac{\partial X_K}{\partial x_l} = \frac{\partial v_k}{\partial x_l}. \tag{2.63}$$

The quantity

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \tag{2.64}$$

is called the velocity gradient and it plays an important role in nonlinear fluid mechanics. Obviously it can be split into symmetric and antisymmetric parts

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right), \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right), \tag{2.65}$$

$$\text{i.e.} \quad \frac{\partial v_k}{\partial x_l} = D_{kl} + W_{kl}, \quad D_{kl} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right), \quad W_{kl} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right).$$

The tensor $\mathbf{D}$ is called the stretching and the tensor $\mathbf{W}$ is called the spin.

Substitution of (2.64) in (2.60) yields

$$\begin{aligned} \mathcal{L}_v\mathbf{q}(\mathbf{x},t) &= \frac{\partial\mathbf{q}}{\partial t}(\mathbf{x},t) + (\mathbf{v}\cdot\operatorname{grad})\,\mathbf{q}(\mathbf{x},t) - \mathbf{L}\mathbf{q}(\mathbf{x},t), \\ \text{i.e.} \quad \mathcal{L}_v q_k &= \frac{\partial q_k}{\partial t} + v_l\frac{\partial q_k}{\partial x_l} - \frac{\partial v_k}{\partial x_l}q_l. \end{aligned} \tag{2.66}$$

This is the so-called Lie derivative of $\mathbf{q}$ related to the velocity field $\mathbf{v}$. Obviously, additional contributions to the standard partial time derivative are nonlinear and they play an important role in nonlinear theories. This derivative of a vector field in Eulerian description as well as analogous derivatives of tensor fields have an important property that they are objective, i.e. invariant with respect to a change of observer. We shall not discuss this subject in these notes.

Let us complete this juxtaposition of Lagrangian and Eulerian description of geometry with the proof of the Euler-Piola-Jacobi identities which are frequently used by the transformation of balance equations. Namely

$$\operatorname{Div}\left(J\mathbf{F}^{-T}\right)(\mathbf{X},t)=0, \quad \operatorname{div}\left(J^{-1}\mathbf{F}^{T}\right)(\mathbf{x},t)=0. \tag{2.67}$$

We prove the first one. The dual identity follows in the similar manner. We write it in Cartesian coordinates

$$\begin{aligned}\frac{\partial\left(JF_{Kk}^{-1}\right)}{\partial X_K} &= \frac{\partial J}{\partial X_K}F_{Kk}^{-1}+J\frac{\partial F_{Kk}^{-1}}{\partial X_K}\\ &= JF_{Ll}^{-1}\frac{\partial F_{lL}}{\partial X_K}F_{Kk}^{-1}-JF_{Kl}^{-1}F_{Lk}^{-1}\frac{\partial F_{lL}}{\partial X_K},\end{aligned}$$

and (2.67) follows when we use the symmetry $\partial F_{lL}/\partial X_K=\partial F_{lK}/\partial X_L$. In the derivation we have used the identity

$$\begin{aligned}\frac{\partial\left(F_{Kk}^{-1}F_{kL}\right)}{\partial X_K} &= 0=\frac{\partial F_{Kk}^{-1}}{\partial X_K}F_{kL}+F_{Kk}^{-1}\frac{\partial F_{kL}}{\partial X_K}\\ \Longrightarrow \quad \frac{\partial F_{Kk}^{-1}}{\partial X_K} &= -F_{Kl}^{-1}F_{Lk}^{-1}\frac{\partial F_{lL}}{\partial X_K}.\end{aligned} \tag{2.68}$$

The Lagrangian description yields as well an identity which is very useful in wave analysis. Usually, it is proved as a part of the so-called Hadamard Theorem. We show this identity in the different way. For a given set of two fields: deformation gradient $\mathbf{F}(\mathbf{X},t)$ and the velocity field $\mathbf{v}=(\mathbf{X},t)$ one has to require the so-called integrability conditions

$$\frac{\partial F_{kK}}{\partial t}=\frac{\partial v_k}{\partial X_K}, \quad \frac{\partial F_{kK}}{\partial X_L}=\frac{\partial F_{kL}}{\partial X_K}, \tag{2.69}$$

for the function of motion $\mathbf{f}(\mathbf{X},t)$ to exist. If $\mathbf{F}$ and $\mathbf{v}$ are not given *a priori* but derived from $\mathbf{f}$ then the integrability conditions (2.69) are identically fulfilled. On the other hand, if $\mathbf{F}$ and $\mathbf{v}$ are not sufficiently smooth, for example they possess a singularity on a moving surface $\mathcal{S}$ then we can require the integrability conditions to be fulfilled only in a weaker form. We investigate the first one and write it as

$$\frac{d}{dt}\int_{\mathcal{P}}F_{kK}\,dV-\oint_{\partial\mathcal{P}}\mathbf{v}_kN_K\,dS=0, \tag{2.70}$$

for all subbodies $\mathcal{P}$. Obviously, if there are no singularities this condition is equivalent to (2.69). We present the local form of this relation in Chapter 4 on balance equations.

2.4 Infinitesimal strains

Now we return to the analysis of the deformation under the assumption (2.44), i.e. to the case of small deformations which is of the main concern in these notes.

The most important property of small deformation is the identity of the Lagrangian and Eulerian description, i.e. we can identify the systems of coordinates $x_k = \delta_{kK} X_K$, $\mathbf{e}_k = \delta_{kK}\mathbf{e}_K$ and the dependence on x_k and X_K is the same. As a consequence, we can write the right Cauchy-Green deformation tensor in the form

$$\mathbf{F} = \mathbf{1} + \operatorname{grad}\mathbf{u} \quad \Rightarrow \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \approx \operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T + \mathbf{1} \approx \mathbf{F}\mathbf{F}^T = \mathbf{B}, \qquad (2.71)$$
$$\text{i.e.} \quad C_{kl} = \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} + \delta_{kl},$$

as $\varepsilon = \|\operatorname{grad}\mathbf{u}\| \ll 1$.

Simultaneously, the Almansi-Hamel deformation tensor (2.48) has the following form

$$\begin{aligned}
\mathbf{e} &= \frac{1}{2}\left(\mathbf{1} - \mathbf{B}^{-1}\right) = \frac{1}{2}\left(\mathbf{1} - \sum_{\alpha=1}^{3} \frac{1}{\lambda_B^{(\alpha)}} \mathbf{k}_B^{(\alpha)} \otimes \mathbf{k}_B^{(\alpha)}\right) = \\
&= \frac{1}{2}\sum_{\alpha=1}^{3}\left(1 - \frac{1}{\lambda_B^{(\alpha)}}\right)\mathbf{k}_B^{(\alpha)} \otimes \mathbf{k}_B^{(\alpha)} = \frac{1}{2}\sum_{\alpha=1}^{3}\left(1 - \frac{1}{\lambda_C^{(\alpha)}}\right)\mathbf{k}_B^{(\alpha)} \otimes \mathbf{k}_B^{(\alpha)} \approx \qquad (2.72) \\
&\approx \sum_{\alpha=1}^{3} \frac{\lambda_C^{(\alpha)} - 1}{2} \mathbf{K}_C^{(\alpha)} \otimes \mathbf{K}_C^{(\alpha)} = \sum_{\alpha=1}^{3} \lambda_e^{(\alpha)} \mathbf{K}_C^{(\alpha)} \otimes \mathbf{K}_C^{(\alpha)}, \quad \left|\mathbf{F}\mathbf{K}_C^{(\alpha)}\right| \approx 1.
\end{aligned}$$

Hence, for small deformations, as already indicated (compare (2.49)),

$$\mathbf{e} \approx \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right), \qquad (2.73)$$
$$\text{i.e.} \quad e_{kl} \approx \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}.$$

This is the most commonly used measure of deformation in linear theories. If relation (2.73) is used as the definition of the deformation measure $\mathbf{e}$ and not as an approximation of Almansi-Hamel tensor, $\mathbf{e}$ is called the strain field. We proceed to investigate some of its properties.

We begin with a certain invariance problem which plays an important role in all branches of continuum mechanics. One should expect that the description of deformation should not change if we rotate very slowly the body as a whole. This rigid body rotation should be slow in this sense that we should not evoke inertial body forces such as centrifugal forces. These would, certainly, produce deformations. Therefore we investigate only a static problem. The rigid body rotation is then defined by the following relation in the Lagrangian description

$$\mathbf{x} = \mathbf{O}\mathbf{X}, \quad \mathbf{O}^{\mathbf{T}} = \mathbf{O}^{-\mathbf{1}}, \quad \mathbf{F}|_{\text{rigid}} = \mathbf{1} + \operatorname{Grad}\mathbf{u}|_{rigid} = \mathbf{O}, \qquad (2.74)$$

where $\mathbf{O}$ is the constant orthogonal matrix. Hence, in the case of the Almansi-Hamel deformation tensor defined by (2.73)

$$\mathbf{C}|_{\text{rigid}} = \mathbf{F}^T\mathbf{F} = \mathbf{O}^T\mathbf{O} = \mathbf{1}, \quad \mathbf{e}|_{\text{rigid}} = \frac{1}{2}\left(\mathbf{O}+\mathbf{O}^T\right) - \mathbf{1}, \tag{2.75}$$

which means that the rigid rotation yields undeformed body whose measure of deformation is the Cauchy-Green tensor $\mathbf{C}$ but it yields a deformation if it is defined by the strain field $\mathbf{e}$, i.e. by the simplified Almansi-Hamel tensor (2.73). Hence, it is not a proper measure for large deformations. The original definition (3.46) yields, of course, for the rigid rotation the vanishing deformation $\mathbf{e} = \frac{1}{2}\left(\mathbf{1}-\mathbf{B}^{-1}\right) = \frac{1}{2}\left(\mathbf{1}-\mathbf{O}\mathbf{O}^T\right) = \mathbf{0}$.

Displacements in the linear theory for the rigid rotation (rigid displacement) are given by the relation

$$\mathbf{w}(\mathbf{x}) = \mathbf{u}_0 + (\mathbf{O}_0 - \mathbf{1})(\mathbf{x}-\mathbf{x}_0), \tag{2.76}$$

where $\mathbf{u}_0$ is an arbitrary vector, $\mathbf{x}_0$ is the reference point and $\mathbf{x}$ is the position of the point of the body. $\mathbf{O}_0$ denotes a constant orthogonal matrix and, therefore, it describes an arbitrary time-independent rotation. As already mentioned, we do not introduce the time dependence in order to eliminate inertial effects. In the above relation, the difference $\mathbf{O}_0(\mathbf{x}-\mathbf{x_0}) - (\mathbf{x}-\mathbf{x}_0)$ is the displacement of point $\mathbf{x}$ due to the rotation around the point $\mathbf{x}_0$. On this displacement we superpose the displacement $\mathbf{u}_0 = \mathbf{u}(\mathbf{x}_0)$ of the point $\mathbf{x}_0$.

Consequently

$$\operatorname{grad}\mathbf{w} = \mathbf{O}_0 - \mathbf{1}. \tag{2.77}$$

In order to see the consequences of this relation in the case of the linear model, we consider the spectral representation of the matrix of rotation $\mathbf{O}_0$. We have

$$(\mathbf{O}_0 - \lambda_o\mathbf{1})\,\mathbf{k}_o = \mathbf{0} \quad \Rightarrow \quad \det(\mathbf{O}_0 - \lambda_o\mathbf{1}) = 0, \tag{2.78}$$

from which it follows

$$\left(\mathbf{1} - \lambda_o\mathbf{O}_0^T\right)\mathbf{k}_o = \mathbf{0} \quad \Rightarrow \quad \det\left(\mathbf{O}_0 - \frac{1}{\lambda_o}\mathbf{1}\right) = 0, \tag{2.79}$$

Hence

$$\lambda_o^2 = 1 \quad \Rightarrow \quad \lambda_o = \pm 1. \tag{2.80}$$

It means that the matrix $\mathbf{O}_0$ possesses two real eigenvalues. We skip the negative value which describes the mirror picture. Let us choose the reference system in such a way that one of the axes coincide with the eigenvector corresponding to $\lambda_o = 1$. Then the matrix $\mathbf{O}_0$ has the form (compare (1.17))

$$\left(O_{kl}^0\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{2.81}$$

Clearly, this matrix possesses, in addition to real eigenvalues, the complex eigenvalues as well

$$
\begin{aligned}
&(1-\lambda_o)\left[(\cos\varphi-\lambda_o)^2+(\sin\varphi)^2\right]=0 \quad \Rightarrow \\
&\Rightarrow \quad \lambda_o^{(1)}=1 \quad \text{or} \quad \lambda_o^2-2\cos\varphi\lambda_o+1=0, \\
&\text{i.e.} \quad \lambda_o^{(2,3)}=\cos\varphi\pm i\sin\varphi=e^{\pm i\varphi}.
\end{aligned} \tag{2.82}
$$

These eigenvalues determine the angle of rotation φ around the single real eigenvector corresponding to $\lambda_0 = 1$.

For small angles of rotation, the orthogonal matrix contains the nontrivial antisymmetric part

$$
\mathbf{O}_0=\mathbf{1}+\mathbf{\Omega}_0, \quad \left(\Omega_{kl}^0\right)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varphi \\ 0 & -\varphi & 0 \end{pmatrix}, \quad \mathbf{\Omega}_0^T=-\mathbf{\Omega}_0. \tag{2.83}
$$

The rigid displacement is characterized by the following Gurtin Theorem:

The following three statements are equivalent:

1. $\mathbf{w}$ is a rigid displacement field, i.e.

$$
\mathbf{w}=\mathbf{u}_0+\mathbf{\Omega}_0\left(\mathbf{x}-\mathbf{x}_0\right), \quad \mathbf{\Omega}_0^T=-\mathbf{\Omega}_0. \tag{2.84}
$$

2. The strain field $\mathbf{e}$ is vanishing on the domain $\mathcal{B}_t$.
3. $\mathbf{w}$ has the projection property on $\mathcal{B}_t$, i.e. for any pair of points $\mathbf{x},\mathbf{y}\in\mathcal{B}_t$

$$
\left(\mathbf{w}\left(\mathbf{x}\right)-\mathbf{w}\left(\mathbf{y}\right)\right)\cdot\left(\mathbf{x}-\mathbf{y}\right)=0. \tag{2.85}
$$

Namely, we have

$$
\left[\mathbf{u}_0+\mathbf{\Omega}_0\left(\mathbf{x}-\mathbf{x}_0\right)-\mathbf{u}_0-\mathbf{\Omega}_0\left(\mathbf{y}-\mathbf{x}_0\right)\right]\cdot\left(\mathbf{x}-\mathbf{y}\right)=0,
$$

due to the antisymmetry of $\mathbf{\Omega}_0$. Hence 1.⇒3. Now we take the derivative of (2.85) with respect to $\mathbf{x}$ and then with respect to $\mathbf{y}$. Evaluating it at $\mathbf{x}=\mathbf{y}$ we obtain

$$
\begin{aligned}
&\left[(\mathrm{grad})_x\,\mathbf{w}\left(\mathbf{x}\right)\right]^T\cdot\left(\mathbf{x}-\mathbf{y}\right)-\left(\mathbf{w}\left(\mathbf{x}\right)-\mathbf{w}\left(\mathbf{y}\right)\right)=\mathbf{0} \\
&\Rightarrow \quad \left[\mathrm{grad}\,\mathbf{w}\left(\mathbf{x}\right)\right]^T+\mathrm{grad}\,\mathbf{w}\left(\mathbf{x}\right)=2\mathbf{e}\left(\mathbf{x}\right)=\mathbf{0},
\end{aligned}
$$

i.e. 3.⇒2. Finally, for $\mathbf{e}=\mathbf{0}$ we have $\mathrm{grad}\left(\mathrm{grad}\,\mathbf{w}\right)=\mathbf{0}$. Hence $\mathbf{w}$ is a linear function of $\mathbf{x}$: $\mathbf{w}=\mathbf{u}_0+\mathbf{A}\left(\mathbf{x}-\mathbf{x}_0\right)$ and, as $\mathbf{e}=\mathbf{0}\Rightarrow\mathbf{A}+\mathbf{A}^T=0$, it must have the form of the rigid displacement which means 2.⇒1. This completes the proof.

The above Theorem implies immediately the Kirchhoff Theorem: if two displacement fields $\mathbf{u}$ and $\mathbf{u}'$ produce the same strain field $\mathbf{e}$ then

$$
\mathbf{u}=\mathbf{u}'+\mathbf{w}, \tag{2.86}
$$

where $\mathbf{w}$ is a rigid displacement. We have, obviously, $\mathrm{grad}\left(\mathbf{u}-\mathbf{u}'\right)+\left[\mathrm{grad}\left(\mathbf{u}-\mathbf{u}'\right)\right]^T=\mathbf{0}$ which yields, according to Gurtin's Theorem, that $\mathbf{w}$ is the rigid displacement.

An arbitrary displacement $\mathbf{u}(\mathbf{x})$ can be split into the displacement caused by the deformation and this caused by the rigid rotation. Then for small deformations

$$\operatorname{grad}\mathbf{u} = \mathbf{e} + \boldsymbol{\Omega}, \tag{2.87}$$

where

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^T\right), \tag{2.88}$$

and the vector

$$\boldsymbol{\omega} = \frac{1}{2}\operatorname{rot}\mathbf{u}, \quad \omega_k = -\frac{1}{2}\varepsilon_{kij}\Omega_{ij}, \tag{2.89}$$

is the rotation vector.

Bearing for subbodies $\mathcal{P}$ of small volumes V the relation (1.63) in mind as well as the relation (2.51) for the velocity under small deformations (i.e. identical Lagrange and Euler reference systems)

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) \tag{2.90}$$

we have

$$\frac{\delta V}{\delta t} = \int_{\mathcal{P}} \operatorname{div}\frac{\partial \mathbf{u}}{\partial t} dV = \frac{d}{dt}\int_{\mathcal{P}} \operatorname{div}\mathbf{u} dV \approx \frac{V \operatorname{tr}\mathbf{e}}{\delta t} \quad \Rightarrow \quad \frac{\delta V}{V} = \operatorname{tr}\mathbf{e}, \tag{2.91}$$

i.e. $\operatorname{tr}\mathbf{e}$ describes the dilatation.

A special important class of deformations is defined by the homogeneous displacement field

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{A}(\mathbf{x} - \mathbf{x}_0), \tag{2.92}$$

where the point $\mathbf{x}_0$, the vector $\mathbf{u}_0$ and the tensor $\mathbf{A}$ are independent of $\mathbf{x}$. If $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{A}$ is symmetric then $\mathbf{u}$ is called a pure strain from $\mathbf{x}_0$.

★The following deformations for homogeneous displacement fields should be mentioned:

1) simple extension of amount e in the direction $\mathbf{n}$, $|\mathbf{n}| = 1$,

$$\mathbf{u} = e\left((\mathbf{x} - \mathbf{x}_0)\cdot\mathbf{n}\right)\mathbf{n}, \quad \operatorname{grad}\mathbf{u} = e\mathbf{n}\otimes\mathbf{n}, \tag{2.93}$$

$$\mathbf{e} = e\mathbf{n}\otimes\mathbf{n} \quad \text{i.e.} \quad (e_{ij}) = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\Omega} = \mathbf{0},$$

for the base vectors $\{\mathbf{e}_1 = \mathbf{n}, \mathbf{e}_2, \mathbf{e}_3\}$;

2) uniform dilatation of amount e

$$\mathbf{u} = e(\mathbf{x} - \mathbf{x}_0), \quad \operatorname{grad}\mathbf{u} = e\mathbf{1}, \tag{2.94}$$

$$\mathbf{e} = e\mathbf{1} \quad \text{i.e.} \quad (e_{ij}) = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix},$$

$$\boldsymbol{\Omega} = \mathbf{0};$$

3) simple shear of amount κ with respect to the directions $\{\mathbf{m}, \mathbf{n}\}$, $\mathbf{m} \cdot \mathbf{n} = 0$ (compare (2.35)) and both these vectors are unit

$$\mathbf{u} = \kappa \mathbf{m} \cdot (\mathbf{x} - \mathbf{x}_0)\, \mathbf{n}, \quad \operatorname{grad} \mathbf{u} = \kappa \mathbf{m} \otimes \mathbf{n} \tag{2.95}$$

$$\mathbf{e} = \frac{1}{2} \kappa \left(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m} \right) \quad \text{i.e.} \quad (e_{ij}) = \begin{pmatrix} 0 & \kappa/2 & 0 \\ \kappa/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\Omega} = \frac{1}{2} \kappa \left(\mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m} \right),$$

for the base vectors $\{\mathbf{e}_1 = \mathbf{m}, \mathbf{e}_2 = \mathbf{n}, \mathbf{e}_3\}$.♣

2.5 Compatibility conditions

Provided $\mathbf{e}$ is given, the relation for the strain field

$$\mathbf{e} = \frac{1}{2} \left(\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^T \right), \tag{2.96}$$

can be considered to constitute the set of the first order partial differential equations for the displacement field $\mathbf{u}$. The uniqueness of solution is answered by the Kirchhoff Theorem – two solutions differ at most by a rigid displacement field $\mathbf{w}$. We answer now the question of existence of such solutions.

Let us define the rot operator for the tensor $\mathbf{e}$ in the following way

$$\forall_{\mathbf{a}=const} \left[\operatorname{rot} \mathbf{e} \right] \mathbf{a} = \operatorname{rot} \left[\mathbf{e}^T \mathbf{a} \right] \quad \text{i.e.} \quad \operatorname{rot} \mathbf{e} = \epsilon_{ijk} \frac{\partial e_{mk}}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_m. \tag{2.97}$$

Then, for the strain tensor

$$\operatorname{rot} \mathbf{e} = \operatorname{grad} \boldsymbol{\omega}, \tag{2.98}$$

where the rotation vector $\boldsymbol{\omega}$ is defined by the relation (2.89). It follows from the simple calculation

$$\operatorname{rot} \mathbf{e} = \frac{1}{2} \operatorname{rot} \left(\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^T \right) = \frac{1}{2} \operatorname{grad} \operatorname{rot} \mathbf{u} = \operatorname{grad} \boldsymbol{\omega}, \tag{2.99}$$

or, in coordinates

$$\epsilon_{ijk} \frac{\partial e_{mk}}{\partial x_j} = \frac{1}{2} \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right) = \frac{1}{2} \epsilon_{ijk} \frac{\partial^2 u_k}{\partial x_j \partial x_m} = \frac{\partial \omega_i}{\partial x_m}. \tag{2.100}$$

Hence

$$\operatorname{rot} \operatorname{rot} \mathbf{e} = \mathbf{0}, \tag{2.101}$$

where we have used the relation (2.99)

$$\operatorname{rot} \operatorname{rot} \mathbf{e} = \epsilon_{pnm} \frac{\partial^2 \omega_i}{\partial x_m \partial x_n} \mathbf{e}_p \otimes \mathbf{e}_i \equiv 0.$$

The relation (2.101) is the equation of compatibility.

It can be proved that the equation of compatibility is sufficient for the existence of solutions of the equation (2.96) provided the domain $\mathcal{B}_t$ is simply connected. We leave out a rather simple proof (e.g. [4]).

A similar but much more complicated relation can be shown for the nonlinear theory (see: Sect. 2.5 of [22]). It follows from the assumption that the space of motion is Euclidean which means that its tensor of curvature is zero. We shall not present this problem in these notes. However, we discuss further the case of the body with continuous distribution of dislocations for which this condition is violated.

The compatibility condition in the explicit form is as follows

$$\epsilon_{ijk}\epsilon_{lmn}\frac{\partial^2 e_{jm}}{\partial x_k \partial x_n} = 0. \tag{2.102}$$

It can be easily shown that for a particular case of plane displacements in which $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ the set of six independent equations (2.102) reduces to the single relation

$$2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 e_{11}}{\partial x_2 \partial x_2} + \frac{\partial^2 e_{22}}{\partial x_1 \partial x_1}. \tag{2.103}$$

This equation is used in the construction of the so-called Beltrami-Michell equations of linear elasticity which we have already mentioned at the beginning of this Chapter and which we present in Chapter 5.

Chapter 3

Balance of mass and momentum

Classical continuum mechanics is primarily concerned with the search for solutions for the function of motion. In the Lagrangian description current values of mass density are then given by a simple kinematical relation. Field equations for these fields follow from two fundamental equations of physics – conservation of mass and momentum. Some additional fields such as density of dislocations or plastic deformations require additional evolution equations which we discuss further in these notes. In this Chapter we present these two conservation laws of mechanics in both Lagrangian and Eulerian description. We mention as well the consequences of the third mechanical conservation law – moment of momentum conservation. It is a basis for the field equations in models of systems in which an additional local degree of freedom, the rotation (spin), plays a rule. This is, for instance, the case for liquid crystals. For classical continua considered in these notes it restricts the form of the stress tensor, an object appearing in the conservation law for momentum.

3.1 Conservation of mass

As in the mechanics of points the mass in continuum mechanics is a measure of inertia of subbodies of the body $\mathcal{B}_0$. Subbodies $\mathcal{P} \subset \mathcal{B}_0$ are subsets of a certain mathematical structure which we do not need to present in details. Each subbody has a prescribed mass $M(\mathcal{P}) > 0$ and we make the assumption (continuity) that this quantity can be described by the density ρ_0

$$M(\mathcal{P}) = \int_{\mathcal{P}} \rho_0(\mathbf{X})\, dV. \tag{3.1}$$

In contrast to material points of classical mechanics of points and rigid bodies the material point $\mathbf{X} \in \mathcal{B}_0$ possesses no mass, the mass density ρ_0 serves only the purpose of determining the mass of finite subbodies through the relation (3.1). In this sense, the mass density is not measurable. We measure in laboratories the mass of bodies and assuming the homogeneity we define the mass density as the fraction

$$\rho_0 \approx \frac{M(\mathcal{P})}{V(\mathcal{P})}. \tag{3.2}$$

According to the law of mass conservation the mass $M(\mathcal{P})$ does not change in any process

$$\frac{dM(\mathcal{P})}{dt} = 0. \tag{3.3}$$

We may have mass supplies from other components due to chemical reactions or phase changes but this requires the construction of continuum mechanics of mixtures of many components which we shall not present in these notes. This assumption on conservation does not mean that mass density remains constant in processes. Its changes are connected with changes of the volume occupied by the subbody $\mathcal{P}$ at any instant of time. On the other hand this volume is given by the function of motion. Namely, in a current configuration the points $\mathbf{X} \in \mathcal{P}$ occupy the following domain in the space of motion $\mathcal{E}^3$

$$\mathcal{P}_t = \{\mathbf{x} | \, \mathbf{x} = \mathbf{f}(\mathbf{X}, t), \mathbf{X} \in \mathcal{P}\}. \tag{3.4}$$

Such domains we call material. Hence the mass of $\mathcal{P}$ can be written in the form

$$M(\mathcal{P}) = \int_{\mathcal{P}_t} \rho dV_t, \tag{3.5}$$

where ρ is the current mass density and dV_t is the volume element in the current configuration. The element dV_t is the transformation of the element $dV = dX_1 dX_2 dX_3$ to the current configuration. In Cartesian coordinates we can write it in the following way

$$\begin{aligned} dV_t &= [(\mathbf{F}\mathbf{e}_1 dX_1) \times (\mathbf{F}\mathbf{e}_2 dX_2)] \cdot (\mathbf{F}\mathbf{e}_3 dX_3) = \\ &= [(F_{k1}\mathbf{e}_k) \times (F_{l2}\mathbf{e}_l)] \cdot (F_{m3}\mathbf{e}_m) \, dV = \\ &= \varepsilon_{klm} F_{k1} F_{l2} F_{m3} dV = J dV, \quad J = \det \mathbf{F}, \end{aligned} \tag{3.6}$$

where the relation (1.40) has been used and we have used the same base vectors in Lagrangian and Eulerian coordinates $\mathbf{e}_k = \delta_{kK}\mathbf{e}_K$. Consequently

$$\rho = \rho_0 J^{-1}. \tag{3.7}$$

Hence for a given deformation gradient $\mathbf{F}$ changes of the mass density are determined. For this reason, the field of mass density does not appear among unknown fields in the Lagrangian description. Obviously, in the reference configuration $J = 1$. Both the initial mass density ρ_0 and the current mass density must be positive. Consequently, we have the condition

$$J > 0. \tag{3.8}$$

This condition yields the local invertibility of the function of motion $\mathbf{x} = \mathbf{f}(\mathbf{X}, t)$: $\mathbf{X} = \mathbf{f}^{-1}(\mathbf{x}, t)$.

The relation (3.5) can be also written in the local form in time

$$\frac{d}{dt}\int_{\mathcal{P}_t}\rho dV_t = 0, \tag{3.9}$$

for all material domains $\mathcal{P}_t$. Making use of the Leibniz Theorem (1.56) we can write it in the following form

$$\int_{\mathcal{P}_t}\frac{\partial\rho}{\partial t}dV_t + \oint_{\partial\mathcal{P}_t}\rho\mathbf{v}\cdot\mathbf{n}dS_t = 0. \tag{3.10}$$

Bearing the Gauss Theorem (1.60) in mind, we obtain

$$\int_{\mathcal{P}_t}\left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right)dV_t = 0. \tag{3.11}$$

Hence, for almost all points of the body (except of points where ρ is discontinuous) the following local form of the mass conservation in Eulerian description follows

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{3.12}$$

$$\text{i.e.}\quad \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = 0.$$

Obviously, (3.7) fulfills this equation identically. In order to see it, we rewrite the above equation in the form

$$\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_k}v_k\right) + \rho\frac{\partial v_k}{\partial x_k} = 0. \tag{3.13}$$

The expression in the parenthesis is called the material time derivative. It is identical with the partial time derivative in the Lagrangian description. Namely we have

$$\frac{\partial\rho(f_k(\mathbf{X},t),t)}{\partial t} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_k}\frac{\partial f_k}{\partial t} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_k}v_k. \tag{3.14}$$

Hence, in Lagrangian coordinates, (3.13) yields

$$\frac{\partial J^{-1}}{\partial t} + J^{-1}\mathbf{F}^{-T}\cdot\frac{\partial\mathbf{F}}{\partial t} = 0, \quad \mathbf{F}^{-T}\cdot\frac{\partial\mathbf{F}}{\partial t} = \frac{\partial F_{kK}}{\partial t}F^{-1}_{Kk}. \tag{3.15}$$

On the other hand, the rule of differentiation of determinants yields

$$\frac{\partial J^{-1}}{\partial t} = -\frac{1}{J^2}\frac{\partial J}{\partial\mathbf{F}}\cdot\frac{\partial\mathbf{F}}{\partial t} = -\frac{1}{J^2}J\mathbf{F}^{-T}\cdot\frac{\partial\mathbf{F}}{\partial t}, \tag{3.16}$$

and (3.15) becomes the identity.

Apart from the continuity relation (3.12) the relation (3.10) yields also a very useful relation on moving surfaces on which the mass density possesses a finite discontinuity.

In order to find this relation, we consider first a descending family of subbodies $\{\mathcal{P}_i\}$ intersecting the surface $\mathcal{S}$ in the reference configuration. We assume that this surface moves in the direction of its unit normal $\mathbf{N}$ with the speed U, i.e. the velocity of points of the singular surface is $U\mathbf{N}$. The family of subbodies is descending in this sense that at each instant of time they have the same common part with the singular surface and their volume is diminishing to zero with the growing index i

$$\forall_{i\neq j}\mathcal{P}_i \cap \mathcal{S} = \mathcal{P}_j \cap \mathcal{S}, \quad \lim_{i\to\infty}(\mathrm{vol}\,\mathcal{P}_i) = 0, \quad \mathrm{vol}\,\mathcal{P}_i = \int\limits_{\mathcal{P}_i} dV. \tag{3.17}$$

This construction with an appropriate orientation of normal vectors is shown in Fig. 3.1.

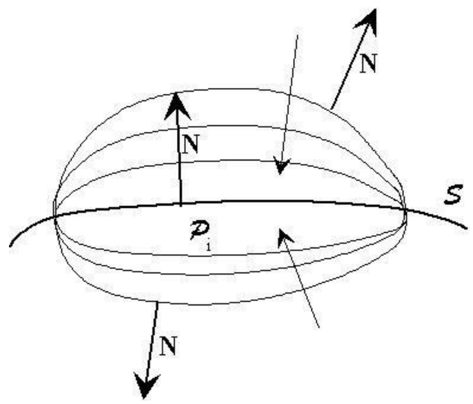


Fig. 3.1: A singular surface S intersecting the family of subbodies $\{\mathcal{P}_i\}$

Now the relation (3.3) can be written in the form

$$\frac{d}{dt}\int\limits_{\mathcal{P}_i}\rho_0 dV = \frac{d}{dt}\int\limits_{\mathcal{P}_i^+}\rho_0 dV + \frac{d}{dt}\int\limits_{\mathcal{P}_i^-}\rho_0 dV, \quad \mathcal{P}_i^+ \cup \mathcal{P}_i^- = \mathcal{P}_i, \tag{3.18}$$

where $\mathcal{P}_i^+, \mathcal{P}_i^-$ are the parts of $\mathcal{P}_i$ created by the intersection with $\mathcal{S}$. Obviously, ρ_0 is constant within each of these two subbodies but it does not have to possess the same value if the surface $\mathcal{S}$ is the surface of discontinuity for the mass density. Simultaneously, due to the motion of $\mathcal{S}$ the subbodies $\mathcal{P}_i^+, \mathcal{P}_i^-$ change in time. Hence, according to the Leibniz Theorem, we obtain

$$\frac{d}{dt}\int\limits_{\mathcal{P}_i}\rho_0 dV = -\int\limits_{\partial\mathcal{P}_i^+\cap\mathcal{S}}\rho_0^+ U dS + \int\limits_{\partial\mathcal{P}_i^-\cap\mathcal{S}}\rho_0^- U dS = 0, \tag{3.19}$$

where the opposite sign in the integrals follows from the opposite orientation of the outward oriented normal vector of $\mathcal{P}_i^+$ and the normal vector of the surface $\mathcal{S}$. The quantities ρ_0^+ and ρ_0^- are, of course, the values of the mass density on two sides of the surface $\mathcal{S}$. Hence, this surface is carrying the discontinuity of the mass density

$$[[\rho_0]] = \rho_0^+ - \rho_0^- . \tag{3.20}$$

Taking the limit $i \to \infty$ in (3.19) we obtain the following local form of this relation

$$U\,[[\rho_0]] = 0. \tag{3.21}$$

In a particular case of the surface $\mathcal{S}$ which is not moving – we say then that the surface is material – we have $U = 0$ and the jump of mass density $[[\rho_0]]$ may be arbitrary. If it is different from zero the surface $\mathcal{S}$ is the surface of contact of two different materials. Otherwise the initial mass density ρ_0 must be continuous $[[\rho_0]] = 0$.

The same considerations can be repeated for the current configuration. Instead of the surface $\mathcal{S}$ we have to consider its current image $\mathcal{S}_t$ moving with the speed c, i.e. its velocity is $c\mathbf{n}$. The subbodies in the reference configuration have to be transformed to the current configuration $\mathcal{P}_i(t) = \{\mathbf{x}|\,\mathbf{x} = \mathbf{f}(\mathbf{X}, t), \mathbf{X} \in \mathcal{P}_i\}$. Then (3.9) yields

$$\int\limits_{\mathcal{P}_i(t)} \frac{\partial \rho}{\partial t} dV + \oint\limits_{\partial \mathcal{P}_i^+(t)} \rho \mathbf{w} \cdot \mathbf{n} dS_t + \oint\limits_{\partial \mathcal{P}_i^-(t)} \rho \mathbf{w} \cdot \mathbf{n} dS_t = 0, \tag{3.22}$$

where $\mathbf{w} = \mathbf{v}$ for points of the surfaces beyond $\mathcal{S}$ and $\mathbf{w} = c\mathbf{n}$ for points on $\mathcal{S}$. Again taking the limit, we obtain

$$[[\rho(\mathbf{v}\cdot\mathbf{n} - c)]] \equiv \rho^+ \left(\mathbf{v}^+ \cdot \mathbf{n} - c\right) - \rho^- \left(\mathbf{v}^- \cdot \mathbf{n} - c\right) = 0 \tag{3.23}$$

where, obviously, $\mathbf{v}^\pm$ are the limits of the particle velocity on the surface $\mathcal{S}_t$. The relation (3.23) is Eulerian counterpart of Lagrangian relation (3.21). It is clear that for the material surface $\mathcal{S}_t$ we have $\mathbf{v}\cdot\mathbf{n} = c$ and the particle velocity is continuous.

The above presented analysis can be repeated for the equation (2.70) which has the formal structure of balance equation. We have

$$\begin{aligned} \frac{d}{dt}\int\limits_{\mathcal{P}_i} F_{kK} dV &= \frac{d}{dt}\int\limits_{\mathcal{P}_i^+} F_{kK} dV + \frac{d}{dt}\int\limits_{\mathcal{P}_i^-} F_{kK} dV = \\ = \int\limits_{\mathcal{P}_i} \frac{\partial F_{kK}}{\partial t} dV &- \int\limits_{\partial\mathcal{P}_i^+ \cap \mathcal{S}} F_{kK}^+ U dS + \int\limits_{\partial\mathcal{P}_i^- \cap \mathcal{S}} F_{kK}^- U dS = \\ &= \oint\limits_{\partial \mathcal{P}_i} v_k N_K dS, \end{aligned} \tag{3.24}$$

where $F_{kK}^\pm$ are the limits of the deformation gradient on both sides of the surface $\mathcal{S}$. Taking the limit described above we obtain

$$[[F_{kK}]]\,U + [[v_k]]\,N_K = 0, \quad [[F_{kK}]] = F_{kk}^+ - F_{kK}^-, \quad [[v_k]] = v_k^+ - v_k^- . \tag{3.25}$$

This is the kinematical compatibility condition which is the part of the Hadamard Theorem and which we have mentioned in Section 2.3. It says that the discontinuity of velocity on a singular surface – it may be, for instance, the front of the so-called shock wave – yields necessarily the discontinuity of deformation and *vice versa*.

3.2 Conservation of momentum

3.2.1 Lagrangian description

The momentum for continuum is again defined for subbodies rather than points as we know it from the classical mechanics of mass points. Namely, it refers to the motion of the center of gravity of small material portions and then for an arbitrary subbody $\mathcal{P}$ it has the form

$$\mathbf{M}(\mathcal{P}) = \int_{\mathcal{P}} \rho_0 \mathbf{v} dV. \tag{3.26}$$

Its time changes depend on interactions with the external world of the subbody. If these interactions are zero the momentum is conserved

$$\frac{d\mathbf{M}}{dt} = 0. \tag{3.27}$$

However, if we cut $\mathcal{P}$ from the body $\mathcal{B}_0$ which itself interacts with the external world it cannot be expected that the interactions of subbody vanish. They are transmitted through the surface $\partial\mathcal{P}$ as well as directly into the interior of $\mathcal{P}$. The latter may be, for instance, gravitational or they may be created by the rotational motion of the body in the form of centrifugal or Coriolis forces. Consequently, instead of (3.27) we have to write

$$\frac{d\mathbf{M}}{dt} = \oint_{\partial\mathcal{P}} \mathbf{t_N} dS + \int_{\mathcal{P}} \rho_0 \mathbf{b} dV, \tag{3.28}$$

where $\mathbf{t_N}$ are the so-called tractions, i.e. forces acting on a unit surface of $\partial\mathcal{P}$ in the reference configuration, and $\mathbf{b}$ are called body forces. They may result from the action of the gravity but they may be as well a consequence of the motion of the body as a whole which results, for instance, in centrifugal forces. A. L. Cauchy (1789-1857) has shown that the tractions are linear functions of the unit normal vector $\mathbf{N}$ of the surface $\partial\mathcal{P}$, i.e. [22]

$$\mathbf{t_N} = \mathbf{PN}, \quad \text{i.e.} \quad t_k^{(N)} = P_{kK} N_K, \quad \mathbf{t_N} = t_k^{(N)} \mathbf{e}_k, \quad \mathbf{P} = P_{kK} \mathbf{e}_k \otimes \mathbf{e}_K, \tag{3.29}$$

where $\mathbf{P}$ is independent of $\mathbf{N}$. This object is called Piola-Kirchhof stress tensor even though it is not a tensor. $t_k^{(N)}$ are components of the stress vector for a particular choice of the subbody $\mathcal{P}$, i.e. for a particular choice of the surface $\partial\mathcal{P}$.

Making use of the Gauss Theorem we can write (3.28) in the form

$$\int_{\mathcal{P}} \left(\rho_0 \frac{\partial \mathbf{v}}{\partial t} - \operatorname{Div} \mathbf{P} - \rho_0 \mathbf{b} \right) dV = 0, \tag{3.30}$$

which must hold for all subbodies of $\mathcal{B}_0$. Consequently, for almost all points of the body $\mathcal{B}_0$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{b}, \tag{3.31}$$

$$\text{i.e.} \quad \rho_0 \frac{\partial v_k}{\partial t} = \frac{\partial P_{kK}}{\partial X_K} + \rho_0 b_k.$$

This is the local form of the momentum conservation law in Lagrangian description. Obviously $\partial v_k/\partial t \equiv \partial^2 f_k/\partial t^2$ is the acceleration in this description.

The most important consequence of the above analysis is the existence of the stress tensor $\mathbf{P}$. However, the Lagrangian description has the disadvantage of referring to a nonphysical reference surface. Practical applications are mostly based on the Eulerian description. We return later to the comparison of both approaches.

3.2.2 Eulerian description

Let us make the transformation from the Lagrangian to Eulerian description in the momentum balance (3.30). We have

$$\int_{\mathcal{P}_t} \left(\rho_0 \frac{\partial \mathbf{v}}{\partial t} - \operatorname{Div} \mathbf{P} - \rho_0 \mathbf{b} \right) J^{-1} dV_t = 0, \tag{3.32}$$

where (3.6) has been used and the function in parenthesis still depends on $\mathbf{x}$ through the function of motion $\mathbf{f}^{-1}(\mathbf{x}, t)$. However

$$\operatorname{Div} \mathbf{P} = \frac{\partial P_{kK}}{\partial X_K} \mathbf{e}_k = \frac{\partial P_{kK}}{\partial x_l} \frac{\partial x_l}{\partial X_K} \mathbf{e}_k = \mathbf{F} \cdot (\operatorname{grad} \mathbf{P}), \tag{3.33}$$

Using the identity $(2.67)_2$, the relation (3.7) and making the transformation of variables $\mathbf{X} \to \mathbf{x}$ we have

$$\int_{\mathcal{P}_t} \left(\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \mathbf{v}) \right) - \operatorname{div} \mathbf{T} - \rho \mathbf{b} \right) dV_t = 0, \quad \mathbf{T} = J^{-1} \mathbf{P} \mathbf{F}^T, \tag{3.34}$$

$$\text{i.e.} \quad \sigma_{kl} = J^{-1} P_{kK} F_{lK}, \quad \mathbf{T} = \sigma_{kl} \mathbf{e}_k \otimes \mathbf{e}_l.$$

The new stress tensor $\mathbf{T}$ is called the Cauchy stress tensor. Components of the Cauchy stress tensor are often defined in Cartesian coordinates $x_1 = x,\ x_2 = y,\ x_3 = z$ and base vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. Then the components are denoted in the following way

$$(\sigma_{kl}) = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}. \tag{3.35}$$

The components $\sigma_x, \sigma_y, \sigma_z$ are called normal stresses and the components $\tau_{xy}, \tau_{xz}, \tau_{yz}$ are called shear stresses. Normal components are also described by a double index $\sigma_x = \sigma_{xx}, \sigma_y = \sigma_{yy}, \sigma_z = \sigma_{zz}$.

Now transforming the stress contribution from the volume to surface integral we obtain

$$\int_{\mathcal{P}_t} \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \mathbf{v}) \right) dV_t - \oint_{\partial \mathcal{P}_t} \mathbf{T} \mathbf{n} dS_t = \int_{\mathcal{P}_t} \rho \mathbf{b} dV_t. \tag{3.36}$$

Consequently, the stress vector (traction) in the current configuration is given by the relation

$$\mathbf{t}_{\mathbf{n}} = \mathbf{T} \mathbf{n} \quad \text{i.e.} \quad t_k^{(n)} = \sigma_{kl} n_l. \tag{3.37}$$

In contrast to the stress vector (traction) $\mathbf{t_N}$ appearing in the relation (3.28) the above stress vector describes the force of surface interaction per unit area of the current surface $\partial\mathcal{P}_t$.

The relation (3.37) is called the Cauchy relation. The local form of the momentum balance (3.34)

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\operatorname{grad}\mathbf{v})\right)=\operatorname{div}\mathbf{T}+\rho\mathbf{b}, \tag{3.38}$$

$$\text{i.e.}\quad \rho\left(\frac{\partial v_k}{\partial t}+v_l\frac{\partial v_k}{\partial x_l}\right)=\frac{\partial\sigma_{kl}}{\partial x_l}+\rho b_k.$$

is called the Cauchy equation.

In fluid mechanics the local momentum equation is also used in a different form

$$\rho\left(\frac{\partial v_k}{\partial t}+2v_lW_{kl}\right)=\left(\frac{\partial\sigma_{kl}}{\partial x_l}-\frac{1}{2}\rho\frac{\partial v^2}{\partial x_k}\right)+\rho b_k,\quad v^2=\mathbf{v}\cdot\mathbf{v}, \tag{3.39}$$

where W_{kl} is the spin (compare (2.65)) and which follows from the identity

$$v_l\frac{\partial v_k}{\partial x_l}=2v_lW_{kl}+\frac{1}{2}\frac{\partial v^2}{\partial x_k}. \tag{3.40}$$

Then for potential (irrotational) flows

$$\mathbf{v}=\operatorname{grad}\varphi,\quad\text{i.e.}\quad v_k=\frac{\partial\varphi}{\partial x_k}\quad\Rightarrow\quad\mathbf{W}=\mathbf{0}, \tag{3.41}$$

and the momentum balance becomes

$$\frac{\partial v_k}{\partial t}=\frac{1}{\rho}\frac{\partial\sigma_{kl}}{\partial x_l}-\frac{\partial}{\partial x_k}\left(\frac{1}{2}v^2+\Phi\right),\quad\mathbf{b}=-\operatorname{grad}\Phi, \tag{3.42}$$

which immediately yields the Bernoulli Theorems for ideal fluids (e.g. [21]). Namely, for ideal fluids which do not carry the shear stresses the stress tensor is reduced to the diagonal form

$$\mathbf{T}=-p\mathbf{1}\quad\text{i.e.}\quad\sigma_{kl}=-p\delta_{kl}, \tag{3.43}$$

where $p=-\frac{1}{3}\sigma_{kk}$ is the pressure. Then an arbitrary direction is principal for this stress tensor. Equation (3.42) has now the following form (it is assumed that $\rho=\rho(p)$!)

$$\frac{\partial v_k}{\partial t}=-\frac{\partial}{\partial x_k}\left(P+\frac{1}{2}v^2+\Phi\right),\quad P=\int\frac{dp}{\rho}+const., \tag{3.44}$$

where P is the so-called pressure function. Obviously, for slow motions in which one can neglect the acceleration this relation indicates that $P+\frac{1}{2}v^2+\Phi$ is only a function of time, and this, in turn, yields the Bernoulli Theorem. This form of momentum balance yields the theory of water waves.

As in the case of mass balance equation, the conservation of momentum yields conditions on a singular surface which may carry the discontinuities of mass and momentum.

In the Lagrangian description, we have to introduce again a descending family of subbodies $\{\mathcal{P}_i\}$ defined by (3.18) and then (3.28) can be written in the form

$$\int\limits_{\mathcal{P}_i} \rho_0 \frac{\partial \mathbf{v}}{\partial t} dV - \int\limits_{\partial\mathcal{P}_i^+ \cap \mathcal{S}} \rho_0^+ \mathbf{v}^+ U dS + \int\limits_{\partial\mathcal{P}_i^- \cap \mathcal{S}} \rho_0^- \mathbf{v}^- U dS = \oint\limits_{\partial\mathcal{P}_i} \mathbf{PN} dS + \int\limits_{\mathcal{P}_i} \rho_0 \mathbf{b} dV, \quad (3.45)$$

for each member $\mathcal{P}_i$ of this family. Again the minus sign follows from the opposite orientation of normal vectors. In the limit $i \to \infty$ we obtain

$$\rho_0 U [[\mathbf{v}]] + [[\mathbf{PN}]] = 0, \quad (3.46)$$

where $\mathbf{N}$ is the unit normal of the surface $\mathcal{S}$. Consequently, a discontinuity of the stress vector (traction) on a singular surface yields the discontinuity of the velocity and this, in turn, according to Hadamard Theorem (compare (3.25)), yields a discontinuity of the deformation.

For the Eulerian description we obtain the following dual relation

$$\int\limits_{\mathcal{P}_i(t)} \rho \frac{\partial \mathbf{v}}{\partial t} dV + \int\limits_{\partial\mathcal{P}_i^+(t)} \rho \mathbf{v} \mathbf{w} \cdot \mathbf{n} dS + \int\limits_{\partial\mathcal{P}_i^-(t)} \rho \mathbf{v} \mathbf{w} \cdot \mathbf{n} dS = \oint\limits_{\partial\mathcal{P}_i(t)} \mathbf{Tn} dS_t + \int\limits_{\mathcal{P}_i(t)} \rho \mathbf{b} dV, \quad (3.47)$$

Again, in the limit we obtain (compare (3.22), (3.23))

$$[[\rho (\mathbf{v} \cdot \mathbf{n} - c) \mathbf{v}]] - [[\mathbf{Tn}]] = \mathbf{0}. \quad (3.48)$$

Obviously, the expression $(\mathbf{v} \cdot \mathbf{n} - c)$ describes the motion of the particles with respect to the moving singular surface in the direction normal to this surface. If we account for the mass conservation (3.23) then

$$\rho (\mathbf{v} \cdot \mathbf{n} - c) [[\mathbf{v}]] - [[\mathbf{Tn}]] = \mathbf{0}. \quad (3.49)$$

where the value $\rho (\mathbf{v} \cdot \mathbf{n} - c)$ is taken on any side of the surface as they are equal. For material surfaces we obtain, of course,

$$[[\mathbf{Tn}]] = \mathbf{0}, \quad (3.50)$$

i.e. the continuity of tractions. This relation is fundamental for the formulation of boundary conditions in terms of stresses. Otherwise, if the traction is discontinuous on the nonmaterial surface it yields necessarily the discontinuity of velocity.

The relations for mass and momentum conservations on singular surfaces (3.23), (3.49) are called the dynamic compatibility conditions.

3.2.3 Moment of momentum

It can be easily shown that the moment of momentum conservation yields the symmetry of the Cauchy stress tensor. The classical definition of the moment of momentum for the subbody $\mathcal{P} \subset \mathcal{B}_0$ has the form

$$\mathbf{K}(t) = \int\limits_{\mathcal{P}} \rho_0 \mathbf{x} \times \mathbf{v} dV, \quad (3.51)$$

where $\mathbf{x} = \mathbf{f}(\mathbf{X}, t)$ is the position of the point $\mathbf{X} \in \mathcal{P}$ at the instant of time t. Its time changes are described by the relation

$$\frac{d}{dt} \int_{\mathcal{P}} \rho_0 \mathbf{x} \times \mathbf{v} dV = \oint_{\partial \mathcal{P}} \mathbf{x} \times (\mathbf{PN}) \, dS + \int_{\mathcal{P}} \rho_0 \mathbf{x} \times \mathbf{b} dV. \tag{3.52}$$

In Cartesian coordinates this relation can be written in the form

$$\int_{\mathcal{P}} \rho_0 \varepsilon_{ijk} x_j \frac{\partial v_k}{\partial t} dV = \oint_{\partial \mathcal{P}} \varepsilon_{ijk} x_j P_{kK} N_K dS + \int_{\mathcal{P}} \rho_0 \varepsilon_{ijk} x_j b_k dV. \tag{3.53}$$

In the first integral, the observation that the contribution with the derivative $\partial x_j / \partial t = v_j$ is identically zero has been used. The surface integral can be transformed in the following way

$$\oint_{\partial \mathcal{P}} \varepsilon_{ijk} x_j P_{kK} N_K dS = \int_{\mathcal{P}} \varepsilon_{ijk} \left(F_{jK} P_{kK} + x_j \frac{\partial P_{kK}}{\partial X_K} \right) dS. \tag{3.54}$$

Bearing (3.31) in mind, we obtain

$$\varepsilon_{ijk} F_{jK} P_{kK} = 0, \tag{3.55}$$

in almost all points of $\mathcal{B}_0$. Consequently, the definition of the Cauchy stress tensor (3.34) yields

$$\varepsilon_{ijk} \sigma_{jk} = 0, \quad \text{i.e.} \quad \mathbf{T} = \mathbf{T}^T. \tag{3.56}$$

Hence, the conservation of moment of momentum yields the symmetry of the Cauchy stress tensor.

3.2.4 Stress analysis

Due to the symmetry of the Cauchy stress tensor we can easily solve the problem of the biggest and smallest local stresses. We begin this analysis with the maximum and minimum of normal stresses, i.e. these stress components which are perpendicular to the surface of the cross-section of the system. If $\mathbf{n}$ denotes the unit vector perpendicular to the surface at the point $\mathbf{x}$ in the current configuration then we have to find this vector for which the stress vector projected on $\mathbf{n}$, i.e. $\mathbf{n} \cdot \mathbf{t_n} = \mathbf{n} \cdot (\mathbf{Tn})$ is maximum or minimum. We have solved this problem already for tensors of deformation (compare (2.19)). The direction $\mathbf{n}$ is given by the solution of the eigenvalue problem

$$(\mathbf{T} - \sigma \mathbf{1}) \, \mathbf{n} = \mathbf{0}, \tag{3.57}$$

where σ are eigenvalues of $\mathbf{T}$. These are given by the equation

$$\det (\mathbf{T} - \sigma \mathbf{1}) = 0. \tag{3.58}$$

In the explicit form it reads

$$\sigma^3 - I_\sigma \sigma^2 + II_\sigma \sigma - III_\sigma = 0, \tag{3.59}$$

where

$$
\begin{aligned}
I_\sigma &= \operatorname{tr}\mathbf{T} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}, \\
II_\sigma &= \frac{1}{2}\left(I_\sigma^2 - \operatorname{tr}\mathbf{T}^2\right) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{22} \end{vmatrix} = \\
&= \sigma^{(1)}\sigma^{(2)} + \sigma^{(1)}\sigma^{(3)} + \sigma^{(2)}\sigma^{(3)}, \\
III_\sigma &= \det\mathbf{T} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix} = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)},
\end{aligned}
\tag{3.60}
$$

are the principal invariants of the stress tensor $\mathbf{T}$. The name 'invariant' refers to the fact that these quantities remain the same in all frames of reference, i.e. they are independent of the choice of the base vectors. Once we have the three solutions $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$ of the equation (3.59), which are called principal values of the stress tensor $\mathbf{T}$, we can find from the set (3.57) three corresponding normalized eigenvectors $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$. They are called principal directions of the stress tensor $\mathbf{T}$. Obviously, they are orthogonal, i.e. $\mathbf{n}^{(\alpha)} \cdot \mathbf{n}^{(\beta)} = \delta_{\alpha\beta}$. According to the relation (3.57) we have

$$
\mathbf{n}^{(\alpha)} \cdot \mathbf{T}\mathbf{n}^{(\beta)} = 0 \quad \text{for} \quad \alpha \neq \beta. \tag{3.61}
$$

Hence, if we choose $\left\{\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right\}$ as the basis vectors we obtain the following spectral representation of stress tensor

$$
\mathbf{T} = \sum_{\alpha=1}^{3} \sigma^{(\alpha)} \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}. \tag{3.62}
$$

The graphical representations of the stress tensor $\mathbf{T}$ in an arbitrary Cartesian basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and in the principal directions $\left\{\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\right\}$ are shown in Fig. 3.2.

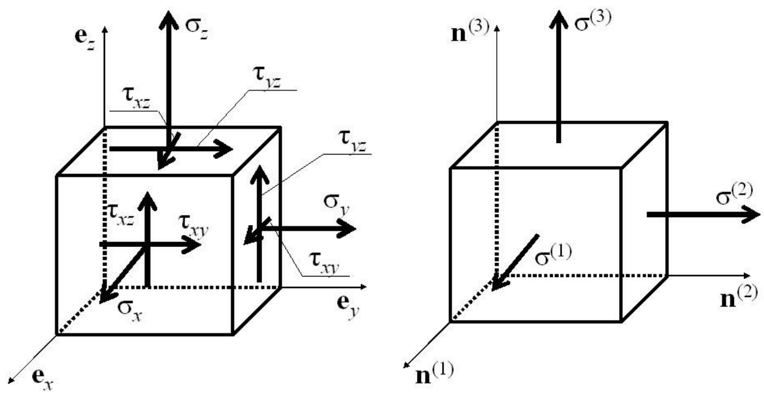


Fig. 3.2: Stress components in Cartesian coordinates – an arbitrary coordinate system (left panel) and the system of principal directions (right panel)

In order to appreciate the notions of stress tensors we present a few simple examples.

★Let us begin with an example of the Cauchy stress tensor for which we want to find principal stresses and the principal directions. In a chosen frame of Cartesian coordinates it has the form of the following matrix

$$(\sigma_{ij}) = \begin{pmatrix} 5 & 2 & -3 \\ 2 & 4 & 1 \\ -3 & 1 & -3 \end{pmatrix}, \quad \mathbf{T} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{3.63}$$

where the inputs are in MPa. The corresponding eigenvalue problem has the form

$$\begin{pmatrix} 5-\sigma & 2 & -3 \\ 2 & 4-\sigma & 1 \\ -3 & 1 & -3-\sigma \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.64}$$

and the eigenvector $\mathbf{n}$ should be of the unit length

$$\mathbf{n} = n_i\mathbf{e}_i, \quad \mathbf{n}\cdot\mathbf{n} = (n_1)^2 + (n_2)^2 + (n_3)^2 = 1. \tag{3.65}$$

Obviously, the determinant of the set of equations (3.64) must be zero. Hence

$$-\det(\mathbf{T} - \sigma\mathbf{1}) = \sigma^3 - I_\sigma\sigma^2 + II_\sigma\sigma - III_\sigma = 0. \tag{3.66}$$

Coefficients of this equation, the principal invariants of the stress tensor $\mathbf{T}$, are given by the relations

$$\begin{aligned} I_\sigma &= \operatorname{tr}\mathbf{T} = 6 \text{ MPa}, \quad II_\sigma = \frac{1}{2}\left(I_\sigma - \operatorname{tr}\mathbf{T}^2\right) = -21 \text{ (MPa)}^2, \\ III_\sigma &= \det\mathbf{T} = -101 \text{ (MPa)}^3. \end{aligned} \tag{3.67}$$

The solution of (3.66) (it has ben obtained by *Maple7*)has the form

$$\sigma^{(3)} = -4.3099 \text{ MPa}, \quad \sigma^{(2)} = 3.3832 \text{ MPa}, \quad \sigma^{(1)} = 6.9267\text{MPa}. \tag{3.68}$$

It is easy to check that the following identities are satisfied

$$\begin{aligned} I_\sigma &= \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}, \quad II_\sigma = \sigma^{(1)}\sigma^{(2)} + \sigma^{(1)}\sigma^{(3)} + \sigma^{(2)}\sigma^{(3)}, \\ III_\sigma &= \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}. \end{aligned} \tag{3.69}$$

Now, by means of the equations (3.64) and the normalization condition (3.65) we find the corresponding eigenvectors. They are as follows

$$\begin{aligned} \mathbf{n}^{(1)} &= 0.8394\mathbf{e}_1 + 0.5043\mathbf{e}_2 - 0.2029\mathbf{e}_3, \\ \mathbf{n}^{(2)} &= 0.4256\mathbf{e}_1 - 0.8419\mathbf{e}_2 - 0.3319\mathbf{e}_3, \\ \mathbf{n}^{(3)} &= 0.3382\mathbf{e}_1 - 0.1922\mathbf{e}_2 + 0.9212\mathbf{e}_3. \end{aligned} \tag{3.70}$$

Obviously, due to the relation $\mathbf{n}^{(\alpha)} \cdot \mathbf{e}_i = \cos\left(\left(\mathbf{n}^{(\alpha)}, \mathbf{e}_i\right)\right)$, the components of eigenvectors are cosines of the angles between eigenvectors and corresponding base vectors. We have

$$\begin{aligned} \left(\mathbf{n}^{(1)}, \mathbf{e}_1\right) &= 70.23^0, \quad \left(\mathbf{n}^{(1)}, \mathbf{e}_2\right) = 101.08^0, \quad \left(\mathbf{n}^{(1)}, \mathbf{e}_3\right) = 22.90^0, \\ \left(\mathbf{n}^{(2)}, \mathbf{e}_1\right) &= 64.81^0, \quad \left(\mathbf{n}^{(2)}, \mathbf{e}_2\right) = 147.34^0, \quad \left(\mathbf{n}^{(2)}, \mathbf{e}_3\right) = 109.38^0, \\ \left(\mathbf{n}^{(3)}, \mathbf{e}_1\right) &= 32.92^0, \quad \left(\mathbf{n}^{(3)}, \mathbf{e}_2\right) = 59.72^0, \quad \left(\mathbf{n}^{(3)}, \mathbf{e}_3\right) = 101.71^0. \end{aligned} \tag{3.71}$$

The spectral representation of the stress tensor has the form

$$\mathbf{T} = 6.9267\mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + 3.3832\mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} - 4.3099\mathbf{n}^{(3)} \otimes \mathbf{n}^{(3)}. \tag{3.72}$$

Hence, the biggest value of the stress is $\mathbf{n}^{(1)} \cdot \mathbf{T}\mathbf{n}^{(1)} = 6.9267$ MPa, it is tension (positive value!) in the direction $\mathbf{n}^{(1)}$. The smallest value is $\mathbf{n}^{(3)} \cdot \mathbf{T}\mathbf{n}^{(3)} = -4.3099$ MPa, it is compression (negative value!). On surfaces perpendicular to the principal directions $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$ shear stresses are equal to zero: $\mathbf{n}^{(\alpha)} \cdot \mathbf{T}\mathbf{n}^{(\beta)} = 0$ for $\alpha \neq \beta$.♣

★In the second example we calculate the stress vector $\mathbf{t_n}$ on the plane intersecting the cube which is loaded on the faces by the stress $\mathbf{T}$ given by the formula (3.63). The plane crosses the points $(1, 0, 0.5), (0, 1, 1), (1, 1, 0)$ as shown in Fig. 3.3.

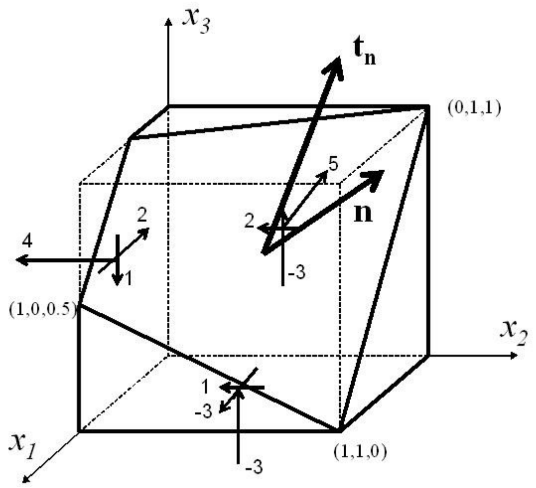


Fig. 3.3: Intersection of the cube discussed in the example

On the same Figure we indicate the components of stresses acting on the back faces of the cube. The plane intersecting the cube is, obviously, given by the equation

$$p(x_1, x_2, x_3) = x_1 + 0.5x_2 + x_3 - 1.5 = 0. \tag{3.73}$$

The vector perpendicular to this plane is parallel to $\operatorname{grad} p = \mathbf{e}_1 + 0.5\mathbf{e}_2 + \mathbf{e}_3$ and, after the normalization, it yields the following normal vector $\mathbf{n}$

$$\mathbf{n} = \frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 + \frac{2}{3}\mathbf{e}_3. \tag{3.74}$$

Now the stress vector on the plane is specified by the relation

$$\mathbf{t_n} = \mathbf{Tn} = \sigma_{ij} n_j \mathbf{e}_i = 2\mathbf{e}_1 + \frac{10}{3}\mathbf{e}_2 - 3\mathbf{e}_3. \tag{3.75}$$

The normal component of the stress acting in the intersection by the plane follows as

$$\sigma_{\mathbf{n}} = \mathbf{n} \cdot \mathbf{t_n} = \frac{4}{9} = 0.4444 \text{ MPa}. \tag{3.76}$$

Consequently, the shear stress on this plane is given by the vector

$$\boldsymbol{\tau}_{\mathbf{n}} = \mathbf{t_n} - \sigma_{\mathbf{n}}\mathbf{n} = \frac{1}{27}\left(46\mathbf{e}_1 + 86\mathbf{e}_2 - 89\mathbf{e}_3\right), \quad |\boldsymbol{\tau}_{\mathbf{n}}| = 4.8902 \text{ MPa}. \tag{3.77}$$

♣

★In this example we show the difference between the stress tensors of Piola-Kirchhoff and Cauchy. Let us consider again the deformation of the prism described in the example (2.7). We make the simplifying assumption that the material is incompressible, i.e. its volume remains constant during the deformation. This is practically fulfilled for many materials, e.g. for rubber-like materials. Then

$$J = 1 \quad \Rightarrow \quad (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) = 1. \tag{3.78}$$

Assuming that the prism is loaded by the force P in the direction $\mathbf{e}_3$ we conclude from symmetry $\varepsilon_1 = \varepsilon_2$. Hence

$$1+\varepsilon_1 = 1+\varepsilon_2 = \frac{1}{\sqrt{1+\varepsilon_3}}. \tag{3.79}$$

The reference area on which the external force P is distributed, say $A_0 = (a\mathbf{e}_1)\cdot(a\mathbf{e}_2) = a^2$, changes due to the deformation to $A = (a\mathbf{F}\mathbf{e}_1)\cdot(a\mathbf{F}\mathbf{e}_2) = a^2(1+\varepsilon_1)^2 = a^2/(1+\varepsilon_3)$. Consequently, the normal components of the Piola-Kirchhoff and Cauchy stress in the $\mathbf{e}_3$-direction, respectively, are as follows

$$P_{33} = \frac{P}{a^2}, \quad \sigma_{33} = \frac{P}{a^2}(1+\varepsilon_3). \tag{3.80}$$

Hence, even a moderate elongation of, say, $\varepsilon_3 = 0.1$ yields the 10% difference in these components. The component σ_{33} which is indeed measured in laboratory is bigger than the component of the Piola-Kirchhoff stress. This difference may even yield a change in the behaviour of stresses in function of deformation. There are cases where the growing Cauchy stress corresponds to a decaying Piola-Kirchhoff stress and *vice versa*. This may lead to erroneous conclusions concerning the so-called material stability.♣

There are two two-dimensional cases in which the above stress analysis possesses a simple geometrical interpretation. The first case – plane stresses – appears when the stress tensor consists of four non-zero components

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.81}$$

where $\sigma_x \equiv \sigma_{xx}, \sigma_y = \sigma_{yy}$.

The second case – plane strains – appears when the stress tensor consists of five non-zero components

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \tag{3.82}$$

with $\sigma_x \equiv \sigma_{xx}, \sigma_y = \sigma_{yy}, \sigma_z \equiv \sigma_{zz}$.

Then the eigenvalue problem

$$(\mathbf{T} - \sigma \mathbf{1})\,\mathbf{n} = \mathbf{0}, \tag{3.83}$$

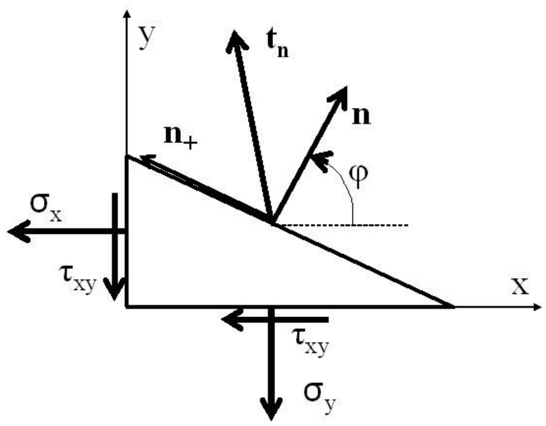


Fig. 3.4: Transformation of the stress tensor for 2D-case

yields $\sigma^{(3)} = 0$ and $\mathbf{n}^{(3)} = \mathbf{e}_z$ in the first case and $\sigma^{(3)} = \sigma_z$ and $\mathbf{n}^{(3)} = \mathbf{e}_z$ in the second case. For this reason it is sufficient to consider the problem on the plane perpendicular to $\mathbf{e}_z$. We consider the components of the stress on the plane perpendicular to the unit vector $\mathbf{n} = \cos\varphi \mathbf{e}_x + \sin\varphi \mathbf{e}_y$ (see Fig. 3.4.). It is convenient to introduce the vector perpendicular to $\mathbf{n}$: $\mathbf{n}_+ = -\sin\varphi \mathbf{e}_x + \cos\varphi \mathbf{e}_y$.

The stress vector $\mathbf{t_n}$ is obviously given by the relation

$$\mathbf{t_n} = \mathbf{Tn} = (\sigma_x \cos\varphi + \tau_{xy} \sin\varphi)\,\mathbf{e}_x + (\tau_{xy} \cos\varphi + \sigma_y \sin\varphi)\,\mathbf{e}_y. \tag{3.84}$$

Then the normal component $\sigma_{\mathbf{n}}$ and the tangential component $\tau_{\mathbf{n}}$ are as follows

$$\begin{aligned}\sigma_{\mathbf{n}} &= \mathbf{t_n}\cdot\mathbf{n} = \frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\varphi+\tau_{xy}\sin 2\varphi, \\ \tau_{\mathbf{n}} &= \mathbf{t_n}\cdot\mathbf{n}_+ = -\frac{\sigma_x-\sigma_y}{2}\sin 2\varphi+\tau_{xy}\cos 2\varphi.\end{aligned} \tag{3.85}$$

If we eliminate the angle φ from these relations we obtain the relation which should hold for an arbitrary intersection of the square on the plane perpendicular to $\mathbf{e}_z$. This equation has the form

$$\tau_{\mathbf{n}}^2+\left(\sigma_{\mathbf{n}}-\frac{\sigma_x+\sigma_y}{2}\right)^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2. \tag{3.86}$$

This is the equation of the circle on the plane of $(\sigma_{\mathbf{n}},\tau_{\mathbf{n}})$-variables whose center lies on the axis $\tau_{\mathbf{n}}=0$ at the point $\frac{1}{2}(\sigma_x+\sigma_y)$. It is called Mohr's circle. The radius of this circle is equal to $\sqrt{\frac{1}{4}(\sigma_x-\sigma_y)^2+\tau_{xy}^2}$. For a given tensor (3.81) or (3.82) the angle φ yields the values of the components of the stress vector on the plane of intersection as indicated in Fig. 3.5.

As a particular case of relations (3.85) we can locate the position of the principal direction and the principal values of stresses for this two-dimensional case. We use the property of the principal direction that the shear stress on the plane perpendicular to this direction is vanishing. Then $(3.85)_2$ yields

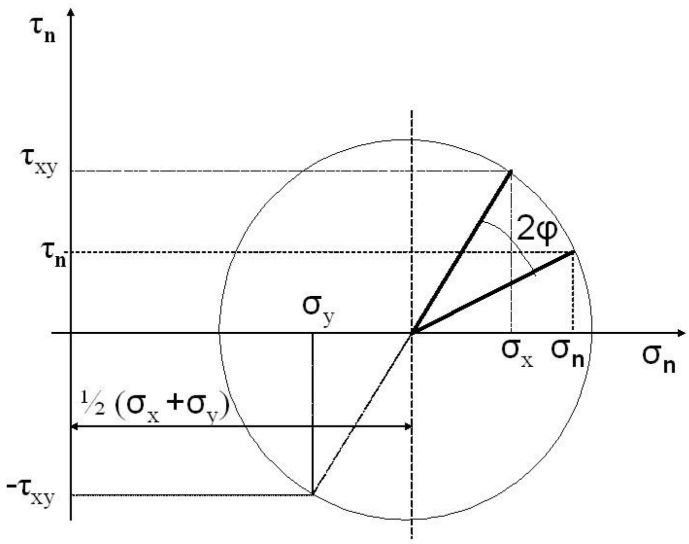


Fig. 3.5: Mohr's circle

$$\tan 2\varphi_0=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}. \tag{3.87}$$

Certainly, this corresponds to the point of intersection of the circle with the line $\tau_{\mathbf{n}} = 0$. The values of normal stresses for this angle are

$$\begin{aligned} \sigma^{(1),(2)} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \\ \sigma^{(1)} &= \max \sigma_{\mathbf{n}}, \quad \sigma^{(2)} = \min \sigma_{\mathbf{n}}, \end{aligned} \tag{3.88}$$

It follows as well from the Mohr circle that the maximum shear stresses are equal to the radius of the circle

$$\max \tau_{\mathbf{n}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \equiv \frac{\sigma^{(1)} - \sigma^{(2)}}{2}, \tag{3.89}$$

and they appear on the plane which forms angles $\pi/4$ and $3\pi/4$ with principal directions.

Let us mention that a similar construction can be also made for the general three-dimensional case. Below we present an example of such a construction. It is performed for three principal directions separately. Then for each direction one can perform a two-dimensional construction described above. This follows from the fact that in the principal direction shear stress is equal to zero which means that the stress distribution in the remaining directions is two-dimensional. This construction is useful, for instance, in motivation of various yield criteria. Applications in rock mechanics are discussed by Jaeger, Cook and Zimerman [6].

★We construct Mohr's circles for the stress tensor

$$(\sigma_{ij}) = \begin{pmatrix} 1 & -9 & 2 \\ -9 & 1 & 2 \\ 2 & 2 & 16 \end{pmatrix}. \tag{3.90}$$

Principal stresses and corresponding principal directions are as follows

$$\begin{aligned} \sigma^{(1)} &= 16.3288, \quad \mathbf{n}^{(1)} = -0.7024\mathbf{e}_1 - 0.7024\mathbf{e}_2 + 0.1155\mathbf{e}_3. \\ \sigma^{(2)} &= 10.00 \qquad \mathbf{n}^{(2)} = 0.7071\mathbf{e}_1 - 0.7071\mathbf{e}_2, \\ \sigma^{(3)} &= -8.3288, \quad \mathbf{n}^{(3)} = 0.08166\mathbf{e}_1 + 0.08166\mathbf{e}_2 + 0.9933\mathbf{e}_3, \end{aligned} \tag{3.91}$$

The construction of Mohr's circles is shown in Fig. 3.6. Each circle is constructed for two-dimensional coordinates on planes perpendicular to the corresponding principal direction. For instance, the left circle corresponds to stress distribution on the plane perpendicular to $\mathbf{n}^{(3)}$. Any state of stresses in an arbitrary cross-section is a point of the dashed area.

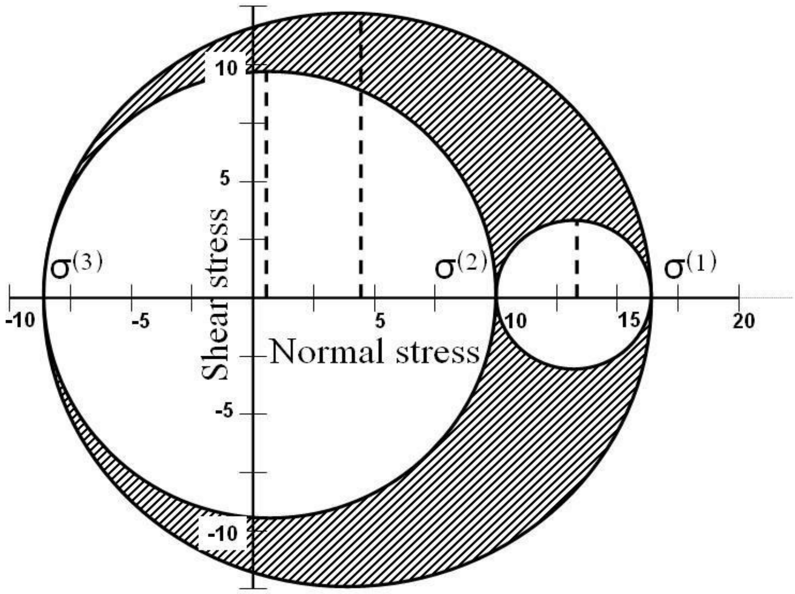


Fig. 3.6: An example of Mohr's circles for three-dimensional stress distribution.♣

However, in contrast to the two-dimensional case presented above, this three-dimensional construction does not seem to have any practical bearing. It shows only that the most important principal stresses $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$ determine the maximum shear stresses as $\left(\sigma^{(1)} - \sigma^{(3)}\right)/2$, $\left(\sigma^{(2)} - \sigma^{(3)}\right)/2$, $\left(\sigma^{(1)} - \sigma^{(2)}\right)/2$ and the orientations of planes on which they act are determined by the angle $\pi/4$ between the normals to those planes and the corresponding principal directions.

Chapter 4

Thermodynamics of solids

Continuum mechanics describes not only processes of deformations caused by the mechanical forces but it must account for various nonmechanical effects which are necessarily coupled to mechanical processes. For instance, any process in which a dissipation of energy appears, and such are processes in viscoelastic or plastic materials, must account for the nonmechanical transport of energy, for example by conduction. It is also natural to include stresses which appear due to changes of temperature (the so-called thermal stresses) or stresses caused by chemical reactions in structural elements. These phenomena require some thermodynamical considerations. For this reason, we present in this Chapter fundamental elements of continuum thermodynamics, in particular the first law, called the principle of conservation of energy and the second law, called the entropy inequality.

Even in the case of purely mechanical processes in which the dissipation does not appear the energy conservation law may be quite useful as an example of some variational principles of classical elasticity clearly shows. We return to these problems later.

4.1 Energy conservation law

We begin from the formulation of the law of the energy conservation in the Lagrangian description. Even though the origin of the subject goes back to the works of Fourier at the beginning of XIX century the formulation of this law can be found first in the work of J. R. Mayer in 1842. However the real modern development in this field began at the end of XIX century with works of Maxwell, Boltzmann and many others.

The classical systems without the so-called internal degrees of freedom are characterized by the total energy consisting of two contributions: the internal energy and the kinetic energy. The first one describes the accumulation of energy in the system due to interactions of particles which change their microstates as a consequence of deformation and temperature. If a subsystem is completely isolated from the external world its total energy remains constant. This is, of course, not the case if forces acting on the system perform working or the contact with other subbodies yields the exchange of energy caused by the difference of temperature. Bearing all these agents in mind, we can write the energy conservation law for any subbody $\mathcal{P} \subset \mathcal{B}_0$ in the form

$$\frac{d}{dt} \int_{\mathcal{P}} \rho_0 \left(\varepsilon + \frac{1}{2} v^2 \right) dV = \oint_{\partial \mathcal{P}} \left(-\mathbf{Q} \cdot \mathbf{N} + \mathbf{t_N} \cdot \mathbf{v} \right) dS + \int_{\mathcal{P}} \rho_0 \left(\mathbf{b} \cdot \mathbf{v} + r \right) dV, \tag{4.1}$$

where ε denotes the internal energy density per unit mass of the reference configuration and $\frac{1}{2}\rho_0 v^2 \equiv \frac{1}{2}\rho_0 \mathbf{v} \cdot \mathbf{v}$ is the density of the kinetic energy per unit volume of the initial configuration. The surface terms consist of the working of stresses $\mathbf{t_N} \cdot \mathbf{v}$ and of the nonmechanical transfer of energy per unit reference surface and unit time, $\mathbf{Q} \cdot \mathbf{N}$, in which $\mathbf{Q}$ is called the heat flux vector. The vectorial form of the heat flux (i.e. the linearity with respect to the normal vector $\mathbf{N}$) is the consequence of the Cauchy Theorem analogous to this which we have used in the construction of the stress tensors. The volume integral on the right-hand side describes the supply of energy by the working of body forces $\mathbf{b} \cdot \mathbf{v}$ and the radiation r. The last contribution plays no role for solids as the supply of energy in the form of radiation is essential only on boundaries of solids. Therefore we shall neglect it in further considerations.

The transformation to the local form requires the same steps as in the case of momentum conservation law. Bering the relation (3.29) in mind, we obtain for almost all points of the body $\mathcal{B}_0$

$$\rho_0 \frac{\partial}{\partial t} \left(\varepsilon + \frac{1}{2} v^2 \right) + \operatorname{Div} \left(\mathbf{Q} - \mathbf{P}^T \mathbf{v} \right) = \rho_0 \mathbf{b} \cdot \mathbf{v}. \tag{4.2}$$

Differentiation of the kinetic energy and power of stresses yields

$$\rho_0 \frac{\partial \varepsilon}{\partial t} + \operatorname{Div} \mathbf{Q} + \mathbf{v} \cdot \left(\rho_0 \frac{\partial \mathbf{v}}{\partial t} - \operatorname{Div} \mathbf{P} \right) - \mathbf{P} \cdot \operatorname{Grad} \mathbf{v} = \mathbf{v} \cdot \left(\rho_0 \mathbf{b} \right).$$

Bearing the momentum conservation (3.31) in mind we obtain

$$\rho_0 \frac{\partial \varepsilon}{\partial t} + \operatorname{Div} \mathbf{Q} = \mathbf{P} \cdot \operatorname{Grad} \mathbf{v}, \tag{4.3}$$

$$\text{i.e.} \quad \rho_0 \frac{\partial \varepsilon}{\partial t} + \frac{\partial Q_K}{\partial X_K} = P_{kK} \frac{\partial v_k}{\partial X_K} \equiv P_{kK} \frac{\partial F_{kK}}{\partial t}.$$

This is the so-called equation of balance of internal energy. Due to the working of stresses on the right-hand side it is not a conservation law.

On a singular surface the conservation of energy yields a condition important for contact problems – boundary conditions, phase transformations, etc. The same procedure which we have used for the conservation of mass and momentum leads to the following local relation

$$\rho_0 U \left[\left[\varepsilon + \frac{1}{2} \mathbf{v}^2 \right]\right] - \left[\left[\mathbf{Q} - \mathbf{P}^T \mathbf{v} \right]\right] \cdot \mathbf{N} = 0, \tag{4.4}$$

at each point of the singular surface. In a particular case of the continuous velocity, the relation (3.46) yields immediately

$$\rho_0 U \left[\left[\varepsilon \right]\right] - \left[\left[\mathbf{Q} \right]\right] \cdot \mathbf{N} = 0, \tag{4.5}$$

i.e. on material surfaces on which $U = 0$ the heat flux $\mathbf{Q} \cdot \mathbf{N}$ is continuous and this yields an important boundary condition commonly used, for instance, in physics of structures. Otherwise the relation (4.5) plays an important role in the theory of phase transformations.

Performing straightforward transformation we can write the above equations in the Eulerian form. The equation (4.1) becomes

$$\frac{d}{dt} \int\limits_{\mathcal{P}_t} \rho \left(\varepsilon + \frac{1}{2} v^2 \right) dV_t = \int\limits_{\mathcal{P}_t} \operatorname{div} \left(-J^{-1} \mathbf{F} \mathbf{Q} + \left(J^{-1} \mathbf{F} \mathbf{P}^T \right) \mathbf{v} \right) dV_t + \int\limits_{\mathcal{P}_t} \rho \left(\mathbf{b} \cdot \mathbf{v} + r \right) dV_t, \tag{4.6}$$

where we have used the Gauss Theorem and the Euler-Piola-Jacobi identities (2.67). Obviously

$$\mathbf{q} = J^{-1} \mathbf{F} \mathbf{Q}, \quad \text{i.e.} \quad q_k = J^{-1} F_{kK} Q_K, \tag{4.7}$$

is the current value of the heat flux vector which describes the nonmechanical transfer of energy per unit current surface and unit time.

Bearing the relation (3.34) for Cauchy stresses in mind, we obtain

$$\frac{d}{dt} \int\limits_{\mathcal{P}_t} \rho \left(\varepsilon + \frac{1}{2} v^2 \right) dV_t = \int\limits_{\mathcal{P}_t} \operatorname{div} \left(-\mathbf{q} + \mathbf{T} \cdot \mathbf{v} \right) dV_t + \int\limits_{\mathcal{P}_t} \rho \left(\mathbf{b} \cdot \mathbf{v} + r \right) dV_t. \tag{4.8}$$

Now, as in the previous cases we perform the differentiation with respect to time

$$\frac{d}{dt} \int\limits_{\mathcal{P}_t} \rho \left(\varepsilon + \frac{1}{2} v^2 \right) dV_t = \int\limits_{\mathcal{P}_t} \frac{\partial}{\partial t} \left[\rho \left(\varepsilon + \frac{1}{2} v^2 \right) \right] dV_t + \oint\limits_{\partial \mathcal{P}_t} \rho \left(\varepsilon + \frac{1}{2} v^2 \right) \mathbf{v} \cdot \mathbf{n} dS_t. \tag{4.9}$$

Accounting for the conservation of mass (3.12) we can transform this relation to the following local form

$$\begin{aligned} &\rho \frac{\partial}{\partial t} \left(\varepsilon + \frac{1}{2} v^2 \right) + \rho \left(\mathbf{v} \cdot \operatorname{grad} \right) \left(\varepsilon + \frac{1}{2} v^2 \right) = \\ &\quad = \operatorname{div} \left(-\mathbf{q} + \mathbf{T} \cdot \mathbf{v} \right) + \rho \left(\mathbf{b} \cdot \mathbf{v} + r \right), \\ \text{i.e.} \quad &\rho \frac{\partial}{\partial t} \left(\varepsilon + \frac{1}{2} v^2 \right) + \rho v_k \frac{\partial}{\partial x_k} \left(\varepsilon + \frac{1}{2} v^2 \right) = \\ &\quad = \frac{\partial}{\partial x_k} \left(-q_k + \sigma_{kl} v_l \right) + \rho \left(b_k v_k + r \right). \end{aligned} \tag{4.10}$$

This is the local form of the first law of thermodynamics, i.e. the law of conservation of energy.

As in the case of the Lagrangian description we can apply the law of conservation of momentum (3.38). Then it follows

$$\rho\left(\frac{\partial\varepsilon}{\partial t}+\mathbf{v}\cdot\operatorname{grad}\varepsilon\right)+\operatorname{div}\mathbf{q}=\mathbf{T}\cdot(\operatorname{grad}\mathbf{v})+\rho r, \tag{4.11}$$

$$\text{i.e.}\quad \rho\left(\frac{\partial\varepsilon}{\partial t}+v_k\frac{\partial\varepsilon}{\partial x_k}\right)+\frac{\partial q_k}{\partial x_k}=\sigma_{kl}\frac{\partial v_k}{\partial x_l}+\rho r.$$

This is the Eulerian form of the balance equation of internal energy ε. As before, it is not the conservation law due to the presence of the stress power $\partial\sigma_{kl}\left(\partial v_k/\partial x_l\right)$ which, as it is said in physics, does not have a divergence form.

Similarly to the previous considerations we can construct the energy conservation law on singular surfaces in the Eulerian description. We obtain the following counterpart of the Lagrangian relation (4.4)

$$\rho\left(\mathbf{v}\cdot\mathbf{n}-c\right)\left[\left[\varepsilon-\frac{1}{2}v^2\right]\right]+[[\mathbf{q}-\mathbf{T}\mathbf{v}]]\cdot\mathbf{n}=0, \tag{4.12}$$

where the mass balance (3.23) has been used. In a particular case of ideal fluids one can transform the last contribution in the following way

$$[[\mathbf{T}\mathbf{v}]]\cdot\mathbf{n}=-\left[[p\mathbf{v}\cdot\mathbf{n}]\right]=\rho\left(\mathbf{v}\cdot\mathbf{n}-c\right)\left[\left[\frac{p}{\rho}\right]\right],$$

where we have assumed $[[\mathbf{v}]]=0$ i.e. $[[p]]=0$. Consequently

$$r=[[h]]=-\frac{[[\mathbf{q}]]\cdot\mathbf{n}}{\rho\left(\mathbf{v}\cdot\mathbf{n}-c\right)},\quad h=\varepsilon+\frac{p}{\rho}, \tag{4.13}$$

where h is called the specific enthalpy and r denotes the so-called latent heat. This notion plays an important role in the theory of phase transformations (e.g. evaporation, condensation, melting, etc.) and chemical reactions (e.g. combustion).

4.2 Second law of thermodynamics

The first traces of the second law of thermodynamics can be found even in the works of Fourier whose relation for the heat flux was constructed at the end of XVIII century (published in 1808 and republished in part in his book: *Théorie analytique de la chaleur* in 1822) in such a way that the heat transfer was possible only from hotter to colder areas of the body otherwise not loaded. S. Carnot in 1824 constructed a procedure for the determination of the efficiency of heat engines which was directly related to the second law of thermodynamics. The most essential step was done by L. Boltzmann whose works in years 1868-1872 yielded the famous H-Theorem. This theorem which corresponds to the modern entropy inequality shows that the macroscopic irreversibility can be reflected by a single scalar function. The proof of Boltzmann was done for ideal gases and it was based on the hypothesis that macroscopic modelling always enhances an element of probability. For the details of the motivation of the modern form of the second law of thermodynamics we refer to the book of K. Wilmanski [22].

In these notes we do not need to go into any details of thermodynamics. The reference to the second law will be made occasionally but we shall not present any details of the strategy of constructing thermodynamic models. In this Chapter we demonstrate this strategy on the example of an ideal fluid in order to explain some basic notions.

The construction of nonequilibrium thermodynamics is based on the assumption of the existence of an entropy function which is a constitutive scalar satisfying in the Lagrangian description the balance law of the following form

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_0 \eta dV + \oint_{\partial\mathcal{P}} \mathbf{H}\cdot\mathbf{N} dS = \int_{\mathcal{P}} \hat{\eta} dV, \tag{4.14}$$

for any subbody $\mathcal{P} \subset \mathcal{B}_0$ where η is the specific entropy, $\mathbf{H}$ is the entropy flux and $\hat{\eta}$ is the entropy production density. It is assumed that for every process in the system the entropy production is nonnegative for any subbody $\mathcal{P} \subset \mathcal{B}_0$

$$\int_{\mathcal{P}} \hat{\eta} dV \geq 0. \tag{4.15}$$

This formulation of the second law of thermodynamics has two features. First of all the effect of radiation is neglected. We have already mentioned that the energy radiation has no practical bearing and the same assumption is made for the entropy. Secondly, the structure of balance law (4.14) is such that surfaces have no contributions to the entropy production. This may not be the case and it is assumed here only for simplicity.

For many models of classical continuum thermodynamics one can prove that the entropy flux $\mathbf{H}$ and the heat flux $\mathbf{Q}$ are proportional

$$\mathbf{H} = \frac{\mathbf{Q}}{T}, \tag{4.16}$$

where T is the absolute temperature. In many systems of practical importance, such as mixtures, this relation does not hold (e.g. [22]). However, it is sufficient for our purposes. Then the second law can be written in the form

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_0 \eta dV + \oint_{\partial\mathcal{P}} \frac{\mathbf{Q}\cdot\mathbf{N}}{T} dS \geq 0, \tag{4.17}$$

which is called Clausius-Duhem inequality.

We can easily construct its local counterparts

$$\rho_0 \frac{\partial \eta}{\partial t} + \mathrm{Div}\left(\frac{\mathbf{Q}}{T}\right) \geq 0 \quad \text{in regular points of } \mathcal{B}_0, \tag{4.18}$$

and this is called the entropy inequality as well as

$$\rho_0 U [[\eta]] - \left[\left[\frac{\mathbf{Q}\cdot\mathbf{N}}{T}\right]\right] = 0 \quad \text{in points of singular surfaces.} \tag{4.19}$$

As before, we can transform these laws to the Eulerian description. Then they have the form

$$\rho\left(\frac{\partial\eta}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\eta\right)+\operatorname{div}\left(\frac{\mathbf{q}}{T}\right)\geq 0\quad\text{in regular points of }\mathcal{B}_t,\tag{4.20}$$

and

$$\rho\,(\mathbf{v}\cdot\mathbf{n}-\mathbf{c})\,[[\eta]]+\left[\left[\frac{\mathbf{q}\cdot\mathbf{n}}{T}\right]\right]=0\quad\text{in points of singular surfaces.}\tag{4.21}$$

The last relation shows that indeed singular surfaces described by the second law following from the balance law (4.14) do not produce entropy. An extension of thermodynamics on processes in which it is not the case is complicated and not much has been done in this direction. However, a particular case of (4.21) plays an important role in the theory of phase transformations in spite of their irreversibility. If we assume that the velocity is continuous $[[\mathbf{v}]]=0$, and, additionally, that the temperature does not suffer a jump either, $[[T]]=0$, then we have

$$\rho\,(\mathbf{v}\cdot\mathbf{n}-\mathbf{c})\left[\left[\psi+\frac{p}{\rho}\right]\right]=0,\quad \psi=\varepsilon-T\eta,\tag{4.22}$$

where ψ is the so-called Helmholtz free energy function. This relation for surfaces separating two phases yields the equation for phase equilibrium line. In a particular case of evaporation and condensation it leads to the so-called Maxwell construction.

As we indicated above the entropy inequality must hold only for real processes in systems. We explain this limitation on a simple example of an ideal fluid.

★In order to see the consequences of the second law of thermodynamics, we have to construct field equations for a chosen set of fields. In the case of ideal fluids these fields are the mass density $\rho\,(\mathbf{x},t)$, velocity $\mathbf{v}\,(\mathbf{x},t)$ and temperature $T\,(\mathbf{x},t)$. They are described by the mass, momentum and energy conservation equations

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})&=0,\\ \rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\mathbf{v}\right)&=-\operatorname{grad}p,\\ \rho\left(\frac{\partial\varepsilon}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\varepsilon\right)+\operatorname{div}\mathbf{q}&=-p\operatorname{div}\mathbf{v},\end{aligned}\tag{4.23}$$

where the Cauchy stress tensor is reduced to the pressure $p=-\frac{1}{3}\operatorname{tr}\mathbf{T}$ as $\mathbf{T}=-p\mathbf{1}$. The above set becomes the set of field equations if we define additionally $p,\varepsilon,\mathbf{q}$ in terms of $\rho,\mathbf{v},T$ and their derivatives. For ideal fluids these constitutive relations have the form

$$p=p\,(\rho,T)\,,\quad \varepsilon=\varepsilon\,(\rho,T)\,,\quad \mathbf{q}=-K_T\operatorname{grad}T,\tag{4.24}$$

where K_T is the so-called heat conductivity. The relation for the heat flux is called the Fourier relation of heat conduction.

The entropy inequality (4.20), i.e.

$$\rho\left(\frac{\partial\eta}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\eta\right)+\operatorname{div}\left(\frac{\mathbf{q}}{T}\right)\geq 0,\quad \eta=\eta\,(\rho,T)\,,\tag{4.25}$$

must hold for all solutions of the set (4.23) with constitutive relations (4.24). It means that on the class of solutions of the entropy inequality we impose constraints in the form of field equations. Such constraints can be eliminated in the same way as we do for problems of mechanics with constraints, i.e. by means of Lagrange multipliers. This technique is now commonly used in continuum thermodynamics (e.g. [11], [21], [22]). In our simple example we can eliminate constraints directly. As the derivatives of the velocity $\partial \mathbf{v}/\partial t$ and $\operatorname{grad}\mathbf{v}$ do not enter the entropy inequality, the laws of mass and momentum conservation do not restrict the class of solution of this inequality. Hence, we have to account only for the energy conservation. We do so by eliminating the heat flux. It follows

$$\begin{aligned}\rho\left(\frac{\partial\psi}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\psi\right)+\eta\left(\frac{\partial T}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,T\right)-\\ -\frac{p}{\rho}\left(\frac{\partial\rho}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\,\rho\right)+\frac{1}{T}\mathbf{q}\cdot\operatorname{grad}T\leq 0,\end{aligned}\tag{4.26}$$

where, as already indicated in (4.22),

$$\psi=\varepsilon-T\eta=\psi\left(\rho,T\right),\tag{4.27}$$

denotes the Helmholtz free energy function and we have used the mass balance equation to eliminate $\operatorname{div}\mathbf{v}=\left(\partial\rho/\partial t+\mathbf{v}\cdot\operatorname{grad}\rho\right)/\rho$. The chain rule of differentiation yields

$$\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial t}+\frac{\partial\psi}{\partial T}\frac{\partial T}{\partial t},\tag{4.28}$$

and similarly for $\operatorname{grad}\psi$. The contributions containing the derivatives $\partial\rho/\partial t$, $\partial T/\partial t$ are linear with respect to these derivatives. As they are not constraint anymore and, consequently, can be chosen arbitrarily, their coefficients must be zero in order to fulfil the inequality. This yields the following results

$$\begin{aligned}p&=\rho^2\frac{\partial\psi}{\partial\rho},\quad \eta=-\frac{\partial\psi}{\partial T}\quad\Rightarrow\quad\varepsilon=\psi-T\frac{\partial\psi}{\partial T},\\ \mathcal{D}&=-\mathbf{q}\cdot\operatorname{grad}T\geq 0\quad\Rightarrow\quad K_T\geq 0.\end{aligned}\tag{4.29}$$

The identities for p,η,ε show that we need only a constitutive relation for the Helmholtz free energy ψ in order to reproduce the remaining relations. For this reason, such functions are called thermodynamical potentials. Simultaneously, the entropy inequality is reduced to the so-called residual inequality which restricts the function $\mathcal{D}$ called the dissipation function. In our simple case it yields the condition for the heat conduction which is equivalent to Fourier's assumption on the flow of energy in the heat conductor from hotter to colder regions. In irreversible mechanical processes the dissipation is related to the viscosity (viscoelastic materials) and to the plastic working (viscoplastic materials).

Let us complete this example with a relation which follows from the thermodynamic identities. We have

$$\begin{aligned}d\psi=d\varepsilon-\eta dT-Td\eta=\frac{\partial\psi}{\partial T}dT+\frac{\partial\psi}{\partial\rho}d\rho=-\eta dT+\frac{p}{\rho^2}d\rho\quad\Rightarrow\\ \Rightarrow\quad d\eta=\frac{1}{T}\left(d\varepsilon-\frac{p}{\rho^2}d\rho\right).\end{aligned}\tag{4.30}$$

This is the so-called Gibbs equation for ideal fluids.♣

Chapter 5

Elastic materials

Relations which we presented in previous Chapters describe properties of any continuous system. In this sense they are universal. However, in order to find the behaviour of a particular material system they are not sufficient. We have to perform the so-called closure which means that we have to add certain relations which yield field equations for a chosen set of fields. Together with initial and boundary conditions these field equations form a mathematical problem of partial differential equations. Solutions of the latter may be analytical and we show some of them in these notes or they may be approximations following, for example, from numerical procedures. This problem shall not be considered in this booklet.

We begin the demonstration of the closure procedure with a class of materials called elastic. In the next Subsection we present briefly equations of nonlinear elasticity and then we discuss extensively linear problems for isotropic materials.

5.1 Non-linear elasticity

The fundamental field which we want to find in continuum mechanics is the function of motion $\mathbf{f}$. For the so-called elastic materials it is the only unknown field as thermal problems are ignored. This means that the temperature is assumed to be constant, processes are isothermal.

The governing equation for the function of motion follows from the momentum conservation (3.31), i.e.

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{b}, \quad \mathbf{v} = \frac{\partial \mathbf{f}}{\partial t}, \quad \mathbf{x} = \mathbf{f}(\mathbf{X}, t), \quad \mathbf{P} = \mathbf{P}(\mathbf{X}, t), \quad \mathbf{X} \in \mathcal{B}_0, \tag{5.1}$$

where the body force $\mathbf{b}(\mathbf{X}, t)$ is assumed to be given.

In order to transform (5.1) into the field equation for $\mathbf{f}$ we have to specify the Piola-Kirchhoff stress tensor $\mathbf{P}$. We do so with the help of the second law of thermodynamics. Elimination of the heat flux contribution $\operatorname{Div} \mathbf{Q}$ by means of the energy balance (4.3) and the assumption of the constant temperature yields the condition that for all isothermal processes the following inequality must be satisfied

$$\rho_0 \frac{\partial \psi}{\partial t} \leq \mathbf{P} \cdot \operatorname{Grad} \mathbf{v}, \quad \psi = \varepsilon - T\eta. \tag{5.2}$$

We now make the basic constitutive assumption which defines the class of nonlinear elastic materials. Namely, we assume that the Helmholtz free energy function ψ depends on the motion of continuum solely through the deformation gradient

$$\psi = \psi(\mathbf{F}). \tag{5.3}$$

It means that, for instance, rates of deformation do not have any influence on the free energy. Substitution of this assumption in the inequality (5.2) yields

$$\left(\rho_0 \frac{\partial \psi}{\partial \mathbf{F}} - \mathbf{P}\right) \cdot \frac{\partial \mathbf{F}}{\partial t} \leq 0, \tag{5.4}$$

and this condition should hold for all derivatives $\partial \mathbf{F} / \partial t$. Consequently, we have to require

$$\mathbf{P} = \rho_0 \frac{\partial \psi}{\partial \mathbf{F}}, \tag{5.5}$$

and the inequality (5.4) is identically satisfied as equality. This means that the dissipation in such processes is zero, i.e. all processes in elastic materials are reversible.

Many applications of such a model are based on the isotropy assumption. It means that reactions of the material on a given external loading are independent of the orientation of a sample. Simultaneously, we require that local rotations do not influence the reaction of the material. It means that the Helmholtz free energy depends on the deformation gradient $\mathbf{F} = \mathbf{RU}$ in such a way that $\mathbf{R}$ does not appear in the constitutive relation. Hence, we can replace the dependence on $\mathbf{F}$ by the dependence on the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2$. Simultaneously, the isotropy of the material means that the constitutive relation should not change by an arbitrary change of the reference coordinates. This is possible if the constitutive dependence of a scalar function on $\mathbf{C}$ is reduced to the dependence on the invariants of $\mathbf{C}$. Finally, we have

$$\psi = \psi(I, II, III), \quad I = \operatorname{tr} \mathbf{C}, \quad II = \frac{1}{2}\left(I^2 - \operatorname{tr} \mathbf{C}^2\right), \quad III = \det \mathbf{C}. \tag{5.6}$$

Substitution in (5.5) and the chain rule of differentiation yield

$$\begin{aligned} \mathbf{P} &= 2\rho_0 \mathbf{F}\left(\frac{\partial \psi}{\partial I}\frac{\partial I}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial II}\frac{\partial II}{\partial \mathbf{C}} + \frac{\partial \psi}{\partial III}\frac{\partial III}{\partial \mathbf{C}}\right) = \\ &= 2\rho_0 \mathbf{F}\left(\frac{\partial \psi}{\partial I}\mathbf{1} + \frac{\partial \psi}{\partial II}(I\mathbf{1} - \mathbf{C}) + \frac{\partial \psi}{\partial III} III \mathbf{C}^{-1}\right) = \\ &= 2\rho_0 \left(\frac{\partial \psi}{\partial I}\mathbf{B} + \frac{\partial \psi}{\partial II}\left(I\mathbf{B} - \mathbf{B}^2\right) + \frac{\partial \psi}{\partial III} III \mathbf{1}\right)\mathbf{F}^{-T}, \end{aligned} \tag{5.7}$$

where, for convenience, we have used the left Cauchy-Green tensor, defined by the relation (2.37): $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. This is equivalent to $\mathbf{C}$ because both deformation measures $\mathbf{C}$ and $\mathbf{B}$

have the same invariants (compare (2.38)). Now, using the definition (3.34) of the Cauchy stress $\mathbf{T} = J^{-1}\mathbf{PF}^T$ we obtain

$$\mathbf{T} = \daleth_0 \mathbf{1} + \daleth_1 \mathbf{B} + \daleth_{-1} \mathbf{B}^{-1}, \tag{5.8}$$

where

$$\begin{aligned} \daleth_0 &= 2\rho \left(II \frac{\partial \psi}{\partial II} + III \frac{\partial \psi}{\partial III} \right), \\ \daleth_1 &= 2\rho \frac{\partial \psi}{\partial I}, \quad \daleth_{-1} = -2\rho III \frac{\partial \psi}{\partial II}, \quad \rho = \rho_0 J^{-1} \equiv \frac{\rho_0}{\sqrt{III}}, \end{aligned} \tag{5.9}$$

(compare (3.7)). The coefficients $\daleth_0, \daleth_1, \daleth_{-1}$ are called elasticities or response coefficients and they are functions of invariants I, II, III. The relation (5.8) defines the so-called compressible Mooney-Rivlin material. Many other constitutive relations for non-linear elastic materials are presented, for instance, in [2], [14] or [22]. Their main applications serve the purpose of description of such materials as rubber, many polymeric materials, biological tissues, etc. We shall not discuss them any further in these notes.

5.2 Linear elasticity, isotropic and anisotropic materials

5.2.1 Governing equations

Let us summarize geometrical and dynamic relations for a linear model which we have presented in Chapters 2 and 3.

As indicated the convenient way to describe the motion of the linear material is through the displacement field $\mathbf{u}$ which is the function of Lagrangian coordinates and time. This is the so-called displacement approach. As in a linear model we do not distinguish between Lagrangian and Eulerian reference systems, the displacement function which is the main field of the linear elasticity is the following sufficiently smooth function

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \quad \text{i.e.} \quad u_k = u_k(x_1, x_2, x_3, t), \quad \mathbf{u} = u_k \mathbf{e}_k, \quad \mathbf{x} = x_k \mathbf{e}_k. \tag{5.10}$$

This function defines the velocity $\mathbf{v}$ and the strain field $\mathbf{e}$ of the linear model

$$\begin{aligned} \mathbf{v} &= \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{e} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right), \\ \text{i.e.} \quad v_k &= \frac{\partial u_k}{\partial t}, \quad e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right). \end{aligned} \tag{5.11}$$

In some applications (e.g. theory of acoustic waves) it is convenient to introduce the velocity $\mathbf{v}$ and the strain field $\mathbf{e}$ as unknown fields instead of the displacement $\mathbf{u}$. Then these fields must satisfy the following integrability condition

$$\frac{\partial \mathbf{e}}{\partial t} = \frac{1}{2}\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\right) \quad \text{i.e.} \quad \frac{\partial e_{kl}}{\partial t} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right), \tag{5.12}$$

which is, obviously, satisfied if the displacement field $\mathbf{u}$ is given.

Additionally, the strain field must fulfil the compatibility condition (2.102) which follows from the Euclidean character of the space of motion, i.e.

$$\epsilon_{ijk}\epsilon_{lmn}\frac{\partial^2 e_{jm}}{\partial x_k \partial x_n} = 0. \tag{5.13}$$

As already indicated there are six independent relations which follow from (5.13). Namely

$$\begin{aligned}
2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2}, \\
2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} &= \frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2}, \\
2\frac{\partial^2 e_{13}}{\partial x_1 \partial x_3} &= \frac{\partial^2 e_{11}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_1^2}, \\
\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} &= \frac{\partial^2 e_{12}}{\partial x_3 \partial x_1} - \frac{\partial^2 e_{23}}{\partial x_1^2} + \frac{\partial^2 e_{13}}{\partial x_1 \partial x_2}, \\
\frac{\partial^2 e_{22}}{\partial x_1 \partial x_3} &= \frac{\partial^2 e_{23}}{\partial x_1 \partial x_2} - \frac{\partial^2 e_{13}}{\partial x_2^2} + \frac{\partial^2 e_{12}}{\partial x_2 \partial x_3}, \\
\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} &= \frac{\partial^2 e_{13}}{\partial x_2 \partial x_3} - \frac{\partial^2 e_{12}}{\partial x_3^2} + \frac{\partial^2 e_{23}}{\partial x_1 \partial x_3}.
\end{aligned} \tag{5.14}$$

If we substitute the strain-stress relations in these compatibility conditions we obtain equations for stresses. If the boundary value problem is also formulated in stresses then this set can be solved. This approach is called the stress approach. We show further some examples of such a formulation.

In the displacement approach, purely mechanical problems (isothermal processes) require the field equation for the displacement $\mathbf{u}$. This follows from the linear form of the momentum conservation

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \tag{5.15}$$

where $\rho \approx const$, provided we specify the Cauchy stress tensor $\mathbf{T}$ in terms of the displacement and its derivatives. As in the nonlinear case, the constitutive dependence of $\mathbf{T}$ is given by a function of the strain field $\mathbf{e}$. Assuming that initial stresses (i.e. stresses in the configuration in which $\mathbf{e} = \mathbf{0}$) are zero the most general linear form of such a relation is as follows

$$\sigma_{kl} = c_{klij}e_{ij}, \quad \mathbf{T} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{e} = e_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{5.16}$$

where c_{ijkl} are 81 constants ($= 3^4$). However, the symmetry of the Cauchy stresses $\sigma_{ij} = \sigma_{ji}$ and of the strain field $e_{ij} = e_{ji}$ reduces the number of independent constants to 21. This can be easily seen in a Voigt notation which is commonly used in the crystallography. It replaces the components of the second rank tensors by six-dimensional vectors

$$\begin{aligned}
(e_{11}, e_{22}, e_{33}, e_{23}, e_{13}, e_{12}) &= (e_1, e_2, e_3, e_4, e_5, e_6), \\
(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) &= (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6).
\end{aligned} \tag{5.17}$$

Then the relation (5.16) can be replaced by the following matrix relation

$$\sigma_\alpha = C_{\alpha\beta} e_\beta, \quad \alpha, \beta = 1, ..., 6. \tag{5.18}$$

One can easily find the correspondence between c_{ijkl} and $C_{\alpha\beta}$ which we do not quote here. Obviously, the 6×6 matrix $(C_{\alpha\beta})$ is symmetric which means that it possesses $6 + 30/2 = 21$ independent components. This is the maximum number of independent material parameters which may appear in the relation (5.16). Materials which require this number of constants are called anisotropic. They do not possess any particular symmetry properties which means that samples cut from such a material in different directions yield different response to the same external loading.

Some particular cases of material symmetry are of practical importance. We call the material orthotropic if it possesses three orthogonal planes of symmetry defined by the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then in this reference system the stress-strain relation reduces to the following relation in Voigt's notation

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}, \tag{5.19}$$

i.e. it is described by 9 material parameters. The material is transversely isotropic if it is symmetric with respect to a rotation about an axis of symmetry. If $\mathbf{e}_3$ is such an axis then in Vogt's notation the stress-strain relation is as follows

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}, \tag{5.20}$$

i.e. it is described by 5 material parameters.

In spite of many anisotropic materials appearing in practical applications (e.g. wood, composites, many rock structures) in these notes we concentrate on isotropic materials which were already investigated in the nonlinear case. This limitation is connected with technical difficulties.

Linear isotropic elastic material is characterized by two material parameters. For instance, for such materials, we can write the relation (5.16) in the form

$$\begin{aligned} \mathbf{T} =& \lambda \left(\operatorname{tr} \mathbf{e}\right) \mathbf{1} + 2\mu \mathbf{e} \quad \text{i.e.} \quad \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}, \\ & \text{i.e.} \quad c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}\right), \end{aligned} \tag{5.21}$$

where λ, μ are the so-called Lamé constants and the relation (5.21) is called Hooke's law.

This law corresponds to the constitutive relation for the Helmholtz free energy

$$\rho\psi = \frac{1}{2}\mathbf{T}\cdot\mathbf{e} = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}\left(\lambda\left(\operatorname{tr}\mathbf{e}\right)^2 + 2\mu\mathbf{e}\cdot\mathbf{e}\right) = \frac{1}{2}\left(\lambda\left(e_{kk}\right)\left(e_{ll}\right) + 2\mu e_{kl}e_{kl}\right), \quad (5.22)$$

Namely, the nonlinear relation (5.5) can be easily transformed to the thermodynamic relation of the linear model

$$\mathbf{T} = \rho\frac{\partial\psi}{\partial\mathbf{e}}. \quad (5.23)$$

★We have

$$\begin{aligned}
\mathbf{T} &= J^{-1}\mathbf{P}\mathbf{F}^T = J^{-1}\rho_0\frac{\partial\psi}{\partial\mathbf{F}}\mathbf{F}^T = \\
&= \rho\left(\frac{\partial\psi}{\partial C_{KL}}\frac{\partial C_{KL}}{\partial F_{mM}}\right)F_{lM}\mathbf{e}_m\otimes\mathbf{e}_l = \\
&= 2\rho\frac{\partial\psi}{\partial C_{KL}}F_{kK}\delta_{mk}\delta_{LM}F_{lM}\mathbf{e}_m\otimes\mathbf{e}_l = \\
&= 2\rho F_{kK}\frac{\partial\psi}{\partial C_{KL}}F_{lL}\mathbf{e}_m\otimes\mathbf{e}_l = \\
&= 2\rho F_{kK}\frac{\partial\psi}{\partial E_{MN}}\frac{\partial E_{MN}}{\partial C_{KL}}F_{lL}\mathbf{e}_m\otimes\mathbf{e}_l = \\
&= \rho F_{kK}\frac{\partial\psi}{\partial E_{MN}}\delta_{MK}\delta_{NL}F_{lL}\mathbf{e}_m\otimes\mathbf{e}_l \approx \rho\frac{\partial\psi}{\partial e_{kl}}\mathbf{e}_k\otimes\mathbf{e}_l.
\end{aligned} \quad (5.24)$$

♣

The Hooke law yields in particular the following relation for the volume changes given in the linear theory by the trace of the strain field, $\operatorname{tr}\mathbf{e}$,

$$p = -\frac{1}{3}\operatorname{tr}\mathbf{T} = -K\operatorname{tr}\mathbf{e}, \quad K = \lambda + \frac{2}{3}\mu, \quad (5.25)$$

where K is the so-called compressibility (or bulk) modulus. It is often used in soil mechanics together with $G = \mu$ and the latter is called the shear (Kirchhoff) modulus.

Substitution of (5.25) in (5.21) yields immediately the following inverse relations

$$\mathbf{e} = \frac{1}{2\mu}\left(\mathbf{T} - \frac{\lambda}{3\lambda + 2\mu}\left(\operatorname{tr}\mathbf{T}\right)\mathbf{1}\right), \quad (5.26)$$

$$\text{i.e.} \quad e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right).$$

Coefficients $1/2\mu$ and $\lambda/\left[2\mu\left(3\lambda + 2\mu\right)\right]$ are called compliances. They form the following isotropic compliance matrix

$$c'_{ijkl} = \frac{1}{2\mu}\delta_{ik}\delta_{jl} - \frac{\lambda}{2\mu\left(3\lambda + 2\mu\right)}\delta_{ij}\delta_{kl} \quad \Rightarrow \quad e_{ij} = c'_{ijkl}\sigma_{kl}, \quad (5.27)$$

which is the inverse to the isotropic elasticity matrix c_{ijkl}.

The relation (5.26) is usually written in the following explicit form

$$\begin{aligned} e_{11} &= \frac{1}{E}\left(\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right), \\ e_{22} &= \frac{1}{E}\left(\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right), \\ e_{33} &= \frac{1}{E}\left(\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right), \\ e_{12} &= \frac{1}{2\mu}\sigma_{12}, \quad e_{13} = \frac{1}{2\mu}\sigma_{13}, \quad e_{23} = \frac{1}{2\mu}\sigma_{23}, \end{aligned} \tag{5.28}$$

and the comparison with (5.26) yields

$$\begin{array}{ccc} E = \dfrac{\mu\left(3\lambda + 2\mu\right)}{\lambda + \mu} & & \lambda = \dfrac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)} \\ & \Rightarrow & \\ \nu = \dfrac{\lambda}{2\left(\lambda + \mu\right)} & & \mu = \dfrac{E}{2\left(1+\nu\right)} \end{array}, \tag{5.29}$$

where E is called Young (elasticity) modulus and ν is the Poisson number. Then it follows for the compressibility modulus

$$K = \frac{E}{3(1-2\nu)}. \tag{5.30}$$

Values of material parameters are not quite arbitrary and we shall demonstrate various limits which they have to fulfil as we proceed with the presentation of particular problems. We shall see, for example that the compressibility modulus K and the shear modulus μ are nonnegative. It means that, due to $(5.29)_4$, Poisson's number is: $\nu > -1$ and, according to $(5.29)_3$, $\nu \leq 0.5$. As Poisson's number ν, by means of (5.28), assigns the expansion of the material under the loading in the perpendicular (lateral) direction (e.g. $\sigma_{22} = \sigma_{33} = 0$, $\sigma_{11} < 0 \quad \Rightarrow \quad e_{22} = e_{33} = -\nu\sigma_{11}/E$; e_{22}, e_{33} would be expected to be positive for positive E) it is often speculated that $\nu > 0$. It has been shown that it must not be the case[1].

[1] e.g. R. Lakes; Foam structures with a negative Poisson's ratio, *Science* 235:1038–1040, 1987,
G. W. Milton; Composite materials with Poisson's ratios close to -1. *Journal of the Mechanics and Physics of Solids,* 40(5):1105–1137, 1992.

In Fig. 5.1. we show a two-dimensional structure[2] which possesses the property $\nu < 0$.

Table: *Elastic constants for chosen materials at temperature* 20^0C.

	$\lambda\ [10^{10}\text{Pa}]$	$\mu\ [10^{10}\text{Pa}]$	$E\ [10^{10}\text{Pa}]$	ν	$K\ [10^{10}\text{Pa}]$
aluminium	5.63	2.60	6.98	0.34	7.36
brass	8.90	3.60	9.76	0.36	11.30
copper	10.63	4.55	12.29	0.35	13.66
duralumin	5.78	2.70	7.24	0.34	7.58
ice (-4^0C)	0.70	0.36	0.96	0.33	0.94
iron	10.49	8.20	21.00	0.28	15.96
lead	4.07	0.57	1.64	0.44	4.45
marble	4.15	2.70	7.04	0.30	5.95
plexiglass	0.28	0.12	0.32	0.35	0.36
polystyrene	0.28	0.12	0.32	0.35	0.36
steel	11.78	8.0	20.76	0.30	17.11

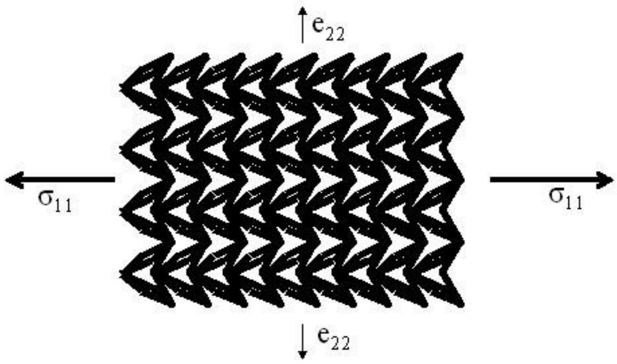


Fig. 5.1: A structure modelled by a linear elastic continuum with the negative Poisson number ν

In the other limit $\nu = 0.5$ there are no volume changes

$$\operatorname{tr}\mathbf{e} = \frac{1-2\nu}{E}\operatorname{tr}\mathbf{T} \quad \Rightarrow_{\nu=0.5} \quad \operatorname{tr}\mathbf{e} = 0 \quad \text{and} \quad K = \infty, \tag{5.31}$$

i.e. $J = 1$ and $\rho = \rho_0$. Such materials are called incompressible.

[2]U. D. Larsen, O. Sigmund, S. Bouwstra; Design and fabrication of compliant micro-mechanisms and structures with negative Poisson's ratio, *Journal of Microelectromechanical Systems,* 6:99–106, 1997.

5.2.2 Navier-Cauchy equations, Green functions, displacement potentials

Substitution of the Hooke law (5.21) in the momentum balance (5.15) yields immediately the governing equations for the displacement $\mathbf{u}$

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\mu\,\mathrm{div}\,\mathrm{grad}\,\mathbf{u}+\rho\mathbf{b}\equiv \equiv (\lambda+\mu)\,\mathrm{grad}\,\mathrm{tr}\,\mathbf{e}+\mu\nabla^2\mathbf{u}+\rho\mathbf{b}, \tag{5.32}$$

or in Cartesian coordinates

$$\rho\frac{\partial^2 u_k}{\partial t^2} = (\lambda+\mu)\frac{\partial^2 u_l}{\partial x_k\partial x_l}+\mu\frac{\partial^2 u_k}{\partial x_l\partial x_l}+\rho b_k. \tag{5.33}$$

These are Navier-Cauchy equations. They are also called Lamé equations.

★In many practical problems it is convenient to use curvilinear coordinates. For the purpose of these notes we need only cylindrical and spherical coordinates. We shall not go into any details and present below the basic relations for these two systems and for the so-called physical components of the quantities. Let us only briefly explain the latter notion.

We assume that curvilinear coordinates are introduced in the three-dimensional Euclidean space, i.e.

$$y^\alpha = y^\alpha(x_1,x_2,x_3),\quad \alpha=1,2,3,\quad \Rightarrow\quad x_k = x_k\left(y^1,y^2,y^3\right),\quad k=1,2,3, \tag{5.34}$$

are equations of parametric lines. Then the covariant and contravariant base vectors are given by the following relations

$$\mathbf{g}_\alpha = \frac{\partial\mathbf{r}}{\partial y^\alpha},\quad \mathbf{r}=x_k\mathbf{e}_k,\quad \mathbf{g}^\alpha\cdot\mathbf{g}_\beta=\delta^\alpha_\beta, \tag{5.35a}$$

i.e. $\mathbf{g}_\alpha$ is tangent to the y^α-parametric line, and $\mathbf{g}^\alpha$ is perpendicular to the parametric surface $y^\alpha = const.$

Both cylindrical and spherical coordinates are orthogonal, i.e. the metric tensor

$$g_{\alpha\beta} = \mathbf{g}_\alpha\cdot\mathbf{g}_\beta, \tag{5.36}$$

is diagonal. One can introduce unit base vectors

$$\mathbf{e}_{(\alpha)} = \mathbf{g}_\alpha\frac{1}{\sqrt{g_{\alpha\alpha}}}\quad \text{(do not add!)},\quad \mathbf{e}_{(\alpha)}\cdot\mathbf{e}_{(\beta)}=\delta_{\alpha\beta}. \tag{5.37}$$

Then the displacement vector $\mathbf{u}$ and the deformation tensor $\mathbf{e}$ have the following components

$$\mathbf{u} = u^{(\alpha)}\mathbf{e}_{(\alpha)},\quad u^\alpha=\mathbf{u}\cdot\mathbf{g}^\alpha,\quad u^{(\alpha)}=u^\alpha\sqrt{g_{\alpha\alpha}}\quad \text{(do not add!)}, \tag{5.38}$$

$$\mathbf{e} = e^{(\alpha)(\beta)}\mathbf{e}_{(\alpha)}\otimes\mathbf{e}_{(\beta)},\quad e^{\alpha\beta}=\mathbf{g}^\alpha\cdot\mathbf{e}\mathbf{g}^\beta,\quad e^{(\alpha)(\beta)}=e^{\alpha\beta}\sqrt{g_{\alpha\alpha}}\sqrt{g_{\beta\beta}}\quad \text{(do not add!)},$$

which are called physical components.

Differentiation of an arbitrary vector, say – the displacement vector $\mathbf{u}$, yields

$$\frac{\partial \mathbf{u}}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha}\left(u^\mu \mathbf{g}_\mu\right) = \frac{\partial u^\mu}{\partial y^\alpha}\mathbf{g}_\mu + u^\mu \frac{\partial \mathbf{g}_\mu}{\partial y^\alpha} = \left(\frac{\partial u^\mu}{\partial y^\alpha} + \Gamma^\mu_{\alpha\beta}u^\beta\right)\mathbf{g}_\mu, \tag{5.39}$$

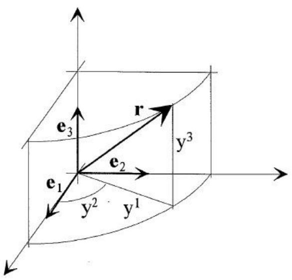
$$\text{where}\quad \Gamma^\mu_{\alpha\beta} = \mathbf{g}^\mu \cdot \frac{\partial \mathbf{g}_\beta}{\partial y^\alpha} \equiv \mathbf{g}^\mu \cdot \frac{\partial^2 \mathbf{r}}{\partial y^\alpha \partial y^\beta}, \quad \frac{\partial u^\mu}{\partial y^\alpha} + \Gamma^\mu_{\alpha\beta}u^\beta = \nabla_\alpha u^\mu,$$

where $\Gamma^\mu_{\alpha\beta}$ are called Christoffel symbols and ∇_α is the covariant derivative of the vector. Similar relations follow for tensors of the second order.

In the relations quoted below for cylindrical and spherical coordinates we have replaced partial derivatives of the Cartesian coordinates by covariant derivatives of the corresponding curvilinear coordinates and components of the vectors and tensors are physical components in the corresponding coordinate systems.♣

★1) Cylindrical coordinates.

These coordinates are defined by the following transformation of Cartesian coordinates



$$\begin{aligned} &x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \\ &x_3 = z, \qquad \mathbf{r} = \mathbf{e}_i x_i, \\ &r = y^1 = \sqrt{(x_1)^2 + (x_2)^2}, \quad \theta = y^2 = \arctan\left(\frac{x_2}{x_1}\right), \\ &z = y^3 = x_3. \end{aligned} \tag{5.40}$$

The corresponding unit base vectors are

$$\mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \quad \mathbf{e}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \quad \mathbf{e}_z = \mathbf{e}_3. \tag{5.41}$$

Then the strain function $\mathbf{e}$ is as follows

$$\begin{aligned} &e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \\ &e_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right), \quad e_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right), \\ &e_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \end{aligned} \tag{5.42}$$

The momentum balance equations become

$$\begin{aligned} \rho\frac{\partial^2 u_r}{\partial t^2} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r, \\ \rho\frac{\partial^2 u_\theta}{\partial t^2} &= \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \rho b_\theta, \\ \rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z. \end{aligned} \tag{5.43}$$

The displacement equations are in this case

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + \\
&+ (\lambda+\mu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right] + \rho b_r, \\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) + \\
&+ (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right] + \rho b_\theta, \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= (\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right] + \rho b_z.
\end{aligned} \tag{5.44}$$

In addition, the Laplace operator in cylindrical coordinates has the following form

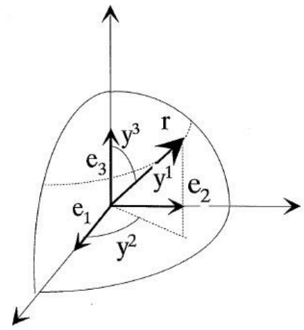
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{5.45}$$

while the volume changes are

$$\mathrm{tr}\,\mathbf{e} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}. \tag{5.46}$$

♣

★2) Spherical coordinates.
We have here



$$\begin{aligned}
&x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \\
&x_3 = r\cos\theta, \qquad \mathbf{r} = x_i\mathbf{e}_i, \\
&r = y^1 = \sqrt{x_k x_k}, \quad \varphi = y^2 = \arctan\left(\frac{x_2}{x_1}\right), \\
&\theta = y^3 = \arccos\frac{x_3}{\sqrt{x_k x_k}},
\end{aligned} \tag{5.47}$$

and the corresponding base vectors

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\varphi\mathbf{e}_1 + \sin\theta\sin\varphi\mathbf{e}_2 + \cos\theta\mathbf{e}_3, \\
\mathbf{e}_\varphi &= -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2, \\
\mathbf{e}_\theta &= \cos\theta\cos\varphi\mathbf{e}_1 + \cos\theta\sin\varphi\mathbf{e}_2 - \sin\theta\mathbf{e}_3.
\end{aligned} \tag{5.48}$$

The strain function $\mathbf{e}$ has the form

$$\begin{gathered} e_{rr}=\frac{\partial u_r}{\partial r},\quad e_{\varphi\varphi}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta,\\ e_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad e_{r\varphi}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}-\frac{u_\varphi}{r}+\frac{\partial u_\theta}{\partial r}\right),\\ e_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right),\quad e_{\varphi\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\theta}-\frac{u_\varphi}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right). \end{gathered} \tag{5.49}$$

The momentum balance equations are as follows

$$\begin{aligned} \rho\frac{\partial^2 u_r}{\partial t^2}&=\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{2\sigma_{rr}-\sigma_{\varphi\varphi}+\sigma_{r\theta}\cot\theta}{r}+\rho b_r,\\ \rho\frac{\partial^2 u_\varphi}{\partial t^2}&=\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}\frac{\partial\sigma_{\varphi\theta}}{\partial\theta}+\frac{3\sigma_{r\varphi}+2\sigma_{\varphi\theta}\cot\theta}{r}+\rho b_\varphi,\\ \rho\frac{\partial^2 u_\theta}{\partial t^2}&=\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\theta}}{\partial\varphi}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{3\sigma_{r\theta}+(\sigma_{\theta\theta}-\sigma_{\varphi\varphi})}{r}+\rho b_z. \end{aligned} \tag{5.50}$$

The displacement equations are in this case

$$\begin{aligned} \rho\frac{\partial^2 u_r}{\partial t^2}&=\mu\left\{\nabla^2 u_r-\frac{2}{r^2}\left[u_r+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\right\}+\\ &+(\lambda+\mu)\frac{\partial\operatorname{tr}\mathbf{e}}{\partial r}+\rho b_r,\\ \rho\frac{\partial^2 u_\varphi}{\partial t^2}&=\mu\left\{\nabla^2 u_\varphi+\frac{2}{r^2\sin\theta}\left[\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\theta}{\partial\theta}\cot\theta-\frac{u_\varphi}{2\sin\theta}\right]\right\}+\\ &+(\lambda+\mu)\frac{1}{r\sin\theta}\frac{\partial\operatorname{tr}\mathbf{e}}{\partial\varphi}+\rho b_\varphi,\\ \rho\frac{\partial^2 u_\theta}{\partial t^2}&=\mu\left\{\nabla^2 u_\theta-\frac{2}{r^2}\left[\frac{\partial u_\theta}{\partial\theta}-\frac{u_\theta}{2\sin^2\theta}-\frac{\partial u_\varphi}{\partial\varphi}\frac{\cos\theta}{\sin^2\theta}\right]\right\}+\\ &+(\lambda+\mu)\frac{1}{r}\frac{\partial\operatorname{tr}\mathbf{e}}{\partial\varphi}+\rho b_\theta, \end{aligned} \tag{5.51}$$

where the Laplace operator has the form

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{5.52}$$

and the volume changes are

$$\operatorname{tr}\mathbf{e}=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(u_r r^2\sin\theta\right)+\frac{\partial}{\partial\theta}\left(u_\theta r\sin\theta\right)+\frac{\partial}{\partial\varphi}\left(u_\varphi r\right)\right]. \tag{5.53}$$

♣

The set of displacement equations can be solved provided we specify initial and boundary conditions. The latter may be given in terms of displacements or their derivatives.

In Fig. 5.2. we show a domain $\mathcal{B}_t$ on which the equation (5.33) is defined and whose part of the boundary $\partial\mathcal{B}_t^\sigma$ is loaded by a given traction $\mathbf{t_n}$ and on the remaining part of the boundary $\partial\mathcal{B}_t^u$, $\partial\mathcal{B}_t^\sigma \cup \partial\mathcal{B}_t^u = \partial\mathcal{B}_t$, the displacement $\mathbf{u}_b$ is given.

Hence

$$\begin{gathered}\mathbf{u}(\mathbf{x},t) = \mathbf{u}_b(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_t^u, \\ \lambda(\operatorname{div}\mathbf{u})\,\mathbf{n} + \mu\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right)\mathbf{n}(\mathbf{x},t) = \mathbf{t_n}(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_t^\sigma,\end{gathered} \tag{5.54}$$

i.e.

$$\begin{gathered}u_k(\mathbf{x},t) = u_k^b(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_t^u, \\ \lambda\frac{\partial u_l}{\partial x_l}n_k + \mu\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)n_l(\mathbf{x},t) = t_k^n(\mathbf{x},t) \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_t^\sigma.\end{gathered} \tag{5.55}$$

In the sequel we show a few typical examples of this boundary value problem.

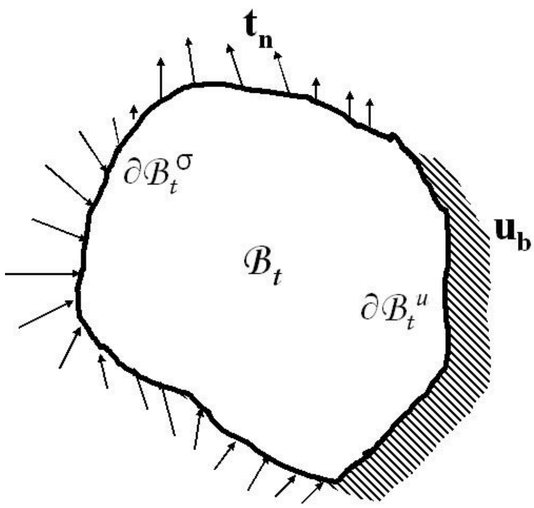


Fig. 5.2: Boundary conditions of linear elasticity

One of the important general features of the set (5.32) is its hyperbolicity. We shall not go into a detailed mathematical definition of this notion. We rather use its physical interpretation that hyperbolic systems describe the propagation of waves of weak discontinuity (acoustic waves). We shall show this property farther in a few different ways but it can also be immediately seen when we use Helmholtz decomposition (1.57)

$$\begin{gathered}\mathbf{u} = \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\psi}, \\ \text{i.e.} \quad u_k = \frac{\partial\varphi}{\partial x_k} + \epsilon_{klm}\frac{\partial\psi_m}{\partial x_l},\end{gathered} \tag{5.56}$$

where $\varphi, \boldsymbol{\psi}$ are scalar and vector displacement potentials. Substitution in (5.33) yields

$$\frac{\partial}{\partial x_k}\left[\rho\frac{\partial^2\varphi}{\partial t^2} - (\lambda + 2\mu)\nabla^2\varphi\right] + \epsilon_{kpq}\frac{\partial}{\partial x_p}\left[\rho\frac{\partial^2\psi_q}{\partial t^2} - \mu\nabla^2\psi_q\right] = 0, \qquad (5.57)$$
$$\nabla^2 = \frac{\partial^2}{\partial x_m \partial x_m},$$

if we neglect for simplicity the body forces. The contributions in square brackets should be zero independently as a differentiation of (5.57) shows. Consequently

$$\begin{aligned} \frac{\partial^2\varphi}{\partial t^2} &= c_L^2\nabla^2\varphi, \quad c_L^2 = \frac{\lambda + 2\mu}{\rho} = \frac{K + \frac{4}{3}\mu}{\rho}, \\ \frac{\partial^2\boldsymbol{\psi}}{\partial t^2} &= c_T^2\nabla^2\boldsymbol{\psi}, \quad c_T^2 = \frac{\mu}{\rho} < c_L^2. \end{aligned} \qquad (5.58)$$

The inequality follows for $K > 0, \mu > 0$ which we justify further. These are two linear wave equations of the second order. In the one-dimensional special case they have the following form

$$\begin{aligned} \left(\frac{\partial}{\partial t} - c_L\frac{\partial}{\partial x_1}\right)\left(\frac{\partial\varphi}{\partial t} + c_L\frac{\partial\varphi}{\partial x_1}\right) &= 0, \\ \left(\frac{\partial}{\partial t} - c_T\frac{\partial}{\partial x_1}\right)\left(\frac{\partial\boldsymbol{\psi}}{\partial t} + c_T\frac{\partial\boldsymbol{\psi}}{\partial x_1}\right) &= 0, \end{aligned} \qquad (5.59)$$

with the following d'Alambert solutions (e.g. [21])

$$\begin{aligned} \varphi &= \varphi_-(x_1 + c_L t) + \varphi_+(x_1 - c_L t), \\ \boldsymbol{\psi} &= \boldsymbol{\psi}_-(x_1 + c_T t) + \boldsymbol{\psi}_+(x_1 - c_T t), \end{aligned} \qquad (5.60)$$

where $\varphi_-, \varphi_+, \boldsymbol{\psi}_-, \boldsymbol{\psi}_+$ are arbitrary twice differentiable functions, i.e. they indeed describe waves with speeds of propagation c_L and c_T. As we show further the scalar potential φ describes the so-called longitudinal wave, i.e. a wave in which the motion of particles is in the same direction as the propagation of the wave. Its speed of propagation is $c_L = \sqrt{(\lambda + 2\mu)/\rho}$. The vector potential $\boldsymbol{\psi}$ describes the so-called transversal wave with the motion of particles perpendicular to the direction of propagation and its speed of propagation is $c_T = \sqrt{\mu/\rho}$. We present properties of these dynamic solutions later in many details.

Some solutions of the set of displacement equations for the infinite medium can be constructed by means of Green functions. These are functions specifying the displacement in an arbitrary point in the case of a single force $\mathbf{f} = f_k\mathbf{e}_k$ acting at the point $\mathbf{x} = \mathbf{0}$. This is the method well-known in many branches of physics as well as structural mechanics where it is called the influence function method [12]. In the static case (i.e. for the acceleration $\partial^2\mathbf{u}/\partial t^2$ identically zero) the displacement $\mathbf{u}(\mathbf{x})$ can be written in the following form

$$u_i = G_{ik}f_k, \qquad (5.61)$$

where

$$G_{ik}(\mathbf{x})=\frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\left(\frac{\delta_{ik}}{r}+\frac{\lambda+\mu}{\lambda+3\mu}\frac{x_i x_k}{r^3}\right),\quad r=\sqrt{x_p x_p}. \tag{5.62}$$

This solution has been found by W. Thomson (lord Kelvin) in 1848. Once we have this solution we can find the displacement in an infinite medium for an arbitrary set of loading forces using the superposition. For instance, for the force $P\delta(r)\delta(z)$ acting in the direction of the z-axis this function yields the following form of the displacement in cylindrical coordinates $\{r,\theta,z\}$

$$\mathbf{u}=\frac{P}{4\pi\mu\sqrt{r^2+z^2}}\left[\frac{1}{4(1-\nu)}\frac{rz}{(r^2+z^2)}\mathbf{e}_r+\left(1-\frac{1}{4(1-\nu)}\frac{r^2}{r^2+z^2}\right)\mathbf{e}_z\right]. \tag{5.63}$$

Hence, the component of the displacement u_z in the direction of the force diminishes as $\sqrt{r^2+z^2}$. There is also an additional component u_r in the radial direction $\mathbf{e}_r$. Obviously, the solution is singular in the point $r=0, z=0$, i.e. in the point of action of the force.

The similar procedure can be applied in the full dynamic case. We obtain

$$\begin{aligned}G_{ij}(\mathbf{x},t)=\frac{1}{4\pi\rho}&\left\{\delta\left(t-\frac{r}{c_T}\right)\left(\frac{\delta_{ij}}{c_T^2 r}-\frac{x_i x_j}{c_T^2 r^3}\right)+\right.\\&+\delta\left(t-\frac{r}{c_L}\right)\frac{x_i x_j}{c_L^2 r^3}+\\+&\left.\left[H\left(t-\frac{r}{c_L}\right)-H\left(t-\frac{r}{c_T}\right)\right]\frac{t}{r^3}\left(3\frac{x_i x_j}{r^2}-\delta_{ij}\right)\right\},\end{aligned} \tag{5.64}$$

where the Dirac-δ of the argument $t-r/c_T$ activates the first contribution when the shear wave of the speed c_T arrives to a chosen point, the Dirac-δ of the argument $t-r/c_L$ activates the second contribution, when the longitudinal wave of the speed c_L arrives to a chosen point, and the Heaviside function contributions $H\left(t-\frac{r}{c_L}\right)=\begin{cases}1 & \text{for}\quad t>\frac{r}{c_L}\\ 0 & \text{for}\quad t<\frac{r}{c_L}\end{cases}$ and $H\left(t-\frac{r}{c_T}\right)=\begin{cases}1 & \text{for}\quad t>\frac{r}{c_T}\\ 0 & \text{for}\quad t<\frac{r}{c_T}\end{cases}$ of the third contribution activate this term between both waves.

Derivation of the above static Green function as well as the dynamic Green function can be found in the Appendix. In their derivation we use integral Fourier transform method.

Inspection of the relation (5.64) shows that it consists of contributions which possess different properties when $r\to 0$ and when $r\to\infty$. In the first case we can neglect the first two contributions because they are proportional to $1/r$ in contrast to the last term which is proportional to $1/r^3$. Consequently, this term dominates for small r. We call it the near field approximation. On the other hand, these are the first two terms which dominate for large r and we can neglect the last contribution. We call this case the far field approximation. We shall use these approximations in estimates of the action of dislocations in the last Chapter of this book.

The method of Green function can be also extended on finite domains and this method is also indicated in the Appendix. However, construction of solutions of boundary value problems for finite domains by means of Green functions requires a rather involved integrations. These can be performed numerically but analytical solutions can be found easier using other methods.

An extensive class of methods has been proposed for static problems (compare the presentation of many examples in the classical books of Timoshenko and Goodier [18] and Landau and Lifshitz [8]). They lead to solutions of simpler Laplace or Poisson equations for functions called potentials. We present some of them.

Some elements of the solution by means of one scalar and one vector function can be found already in the early paper of J. Boussinesq (1878)[3],[4]. It has been fully developed by P. F. Papkowich (1932) and H. Neuber (1934). Namely, they have shown that the following relation

$$\mathbf{u} = \mathbf{\Phi} - \frac{1}{4(1-\nu)} \operatorname{grad}\left[\Phi_0 + \mathbf{x}\cdot\mathbf{\Phi}\right], \tag{5.65}$$

satisfies identically the displacement equations provided the scalar and vector Papkovich-Neuber potentials, $\Phi_0, \mathbf{\Phi}$ fulfil Poisson's equations

$$\nabla^2\Phi_0 - \frac{\rho\mathbf{x}\cdot\mathbf{b}}{\mu} = 0, \quad \nabla^2\mathbf{\Phi} + \frac{\rho\mathbf{b}}{\mu} = 0, \tag{5.66}$$

i.e. for $\mathbf{b} = \mathbf{0}$ they are harmonic functions. The proof is by substitution.

The procedure of solution is as follows. We solve the equations (5.66) with boundary conditions formulated in terms of functions $\Phi_0, \mathbf{\Phi}$. These follow either from relations (5.65) if they are given for displacements on the boundary or from the relations for the stress vector on the boundary. Then we have to use the following relation for stresses

$$\sigma_{ij} = 2\mu\left[\frac{\partial^2\Psi}{\partial x_i\partial x_j} - 2(1-\nu)\left(\frac{\partial\Phi_i}{\partial x_j} + \frac{\partial\Phi_j}{\partial x_i}\right) - \delta_{ij}\nu\nabla^2\Psi\right], \tag{5.67}$$

where

$$\Psi = \Phi_0 + x_k\Phi_k, \quad \nabla^2\Psi = 2\frac{\partial\Phi_k}{\partial x_k}, \tag{5.68}$$

which is the consequence of the Hooke law (5.21).

In many problems it is sufficient to introduce three functions. For instance, for the half-space we choose Φ_0 and $\mathbf{\Phi} = (\Phi_1, \Phi_2, 0)$. Displacements in the space caused by the body forces are determined by means of $\Phi_0 = 0, \mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$.

For axial symmetric cases one applies Boussinesq representation. For $\mathbf{b} = \mathbf{0}$, we have in cylindrical coordinates

$$u_r = \frac{\partial\Psi}{\partial r}, \quad u_z = \frac{\partial\Psi}{\partial z} - 4(1-\nu)\Phi_r, \quad \Psi = \Phi_0 + z\Phi_z. \tag{5.69}$$

[3] Valentin Joseph Boussinesq, 1842-1929.

[4] J. Boussinesq; Équilibre d'élasticité d'un solide isotrope sans pesanteur, supportant différents poids, C. R. Acad. Sci., Paris, **86**, 1260-1263, 1878.

P. F. Papkovich; Solution Générale des équations differentielles fondamentales d'élasticité exprimée par trois fonctions harmoniques, *Compt. Rend. Acad. Sci. Paris* 195: 513–515 , 1932.

H. Neuber; Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie, *Z. Angew. Math. Mech.* 14: 203–212, 1934.

Another class of potentials was introduced also by J. Boussinesq and later rediscovered by Somigliana and Galerkin[5]. This Boussinesq-Somigliana-Galerkin solution has the form

$$\mathbf{u} = \nabla^2 \mathbf{g} - \frac{1}{2(1-\nu)} \operatorname{grad} \operatorname{div} \mathbf{g}, \tag{5.70}$$

where

$$\nabla^2 \nabla^2 \mathbf{g} = -\frac{1}{\mu} \rho \mathbf{b}, \tag{5.71}$$

i.e. the vector potential $\mathbf{g}$ is biharmonic for $\mathbf{b} = \mathbf{0}$.

All these classes give rise to complete solutions of the displacement equations.

A different harmonic potential, particularly useful for half-space problems, was introduced by E. Trefftz[6]. Namely

$$\begin{aligned} u_1 &= \frac{\partial \Phi}{\partial x_1} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^2 \Phi}{\partial x_1 \partial x_3}, \\ u_2 &= \frac{\partial \Phi}{\partial x_2} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^2 \Phi}{\partial x_2 \partial x_3}, \quad \nabla^2 \Phi = 0. \\ u_3 &= -\frac{\lambda+2\mu}{\mu} \frac{\partial \Phi}{\partial x_3} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^2 \Phi}{\partial x_3^2}. \end{aligned} \tag{5.72}$$

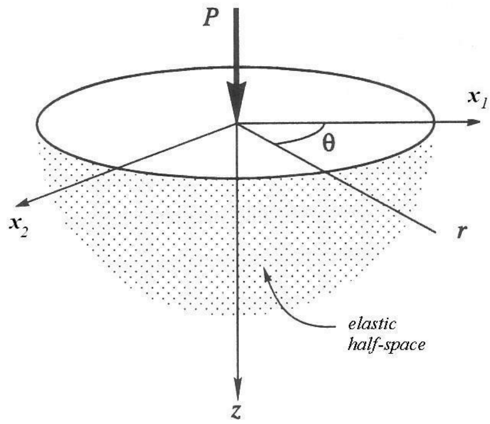


Fig. 5.3: Boussinesq problem

[5] J. Boussinesq; Application des potentiels à l'étude de l'équilibre et des mouvements des solides élasiques, Paris: Gauthier-Villars, 1885.

C. Somigliana; Sulle equazioni della elasticità, *Ann. Math.*, (2)**17**, 37.64, 1889.

B. Galerkin; On an investigation of stresses and deformations in elastic isotropic solids (in Russian), *Dokl. Akad. Nauk SSSR*, 353-358, 1930.

[6] E. Trefftz; *Mathematische Elastizitätstheorie*, in: *Handbuch der Physik,* Bd. VI, Berlin, Springer, 1928.

Substitution of these relations in (5.32) for the static case and without body forces yields the identity. By means of this potential we can construct the solution of the Boussinesq problem shown in Fig. 5.3.

The stress tensor in terms of potential Φ is given in the form

$$\begin{aligned}
\frac{\sigma_{11}}{2\mu} &= \frac{\partial^2\Phi}{\partial x_1^2} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_1^2 \partial x_3} - \frac{\lambda}{\mu}\frac{\partial^2\Phi}{\partial x_3^2}, \\
\frac{\sigma_{22}}{2\mu} &= \frac{\partial^2\Phi}{\partial x_2^2} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_2^2 \partial x_3} - \frac{\lambda}{\mu}\frac{\partial^2\Phi}{\partial x_3^2}, \\
\frac{\sigma_{33}}{2\mu} &= -\frac{\lambda+\mu}{\mu}\frac{\partial^2\Phi}{\partial x_3^2} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_3^3}, \\
\frac{\sigma_{12}}{2\mu} &= \frac{\partial^2\Phi}{\partial x_1 \partial x_2} + \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_1 \partial x_2 \partial x_3}, \\
\frac{\sigma_{23}}{2\mu} &= \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_2 \partial x_3^2}, \\
\frac{\sigma_{13}}{2\mu} &= \frac{\lambda+\mu}{\mu} x_3 \frac{\partial^3\Phi}{\partial x_1 \partial x_3^2}.
\end{aligned} \tag{5.73}$$

Consequently, the shear stresses σ_{23} and σ_{13} are zero on the plane $x_3 = 0$. Hence, the potential defined by relations (5.72) can be used only for half-spaces loaded in the direction perpendicular to this surface.

★Now we specify the potential Φ describing the load in the form of the force $P\mathbf{e}_3$ as indicated in Fig. 5.3. This solution is known as the Boussinesq problem. It is easy to check that the following form of the potential

$$\Phi = -\frac{P}{4\pi(\lambda+\mu)} \ln(x_3 + r), \quad r = \sqrt{x_k x_k}, \tag{5.74}$$

is a harmonic function, i.e. it satisfies the Laplace equation (5.72). Differentiation with respect to x_3 yields

$$\begin{aligned}
\frac{\partial\Phi}{\partial x_3} &= -\frac{P}{4\pi(\lambda+\mu)}\frac{1}{r}, \quad \frac{\partial^2\Phi}{\partial x_3^2} = \frac{P}{4\pi(\lambda+\mu)}\frac{x_3}{r^3}, \\
\frac{\partial^3\Phi}{\partial x_3^3} &= \frac{P}{4\pi(\lambda+\mu)}\left(\frac{1}{r^3} - 3\frac{x_3^2}{r^5}\right).
\end{aligned} \tag{5.75}$$

Substitution in $(5.73)_3$ yields the following relation for the normal stress in x_3-direction

$$\sigma_{33} = -\frac{3P}{2\pi}\frac{x_3^3}{r^5}, \quad r = \sqrt{x_k x_k}. \tag{5.76}$$

This component of stresses is zero on the plane $x_3 = 0$ except of the point $r = 0$ where it is singular. However, if we transform stresses to spherical coordinates and integrate over an arbitrary half-sphere of the radius r it becomes equal to P. In this sense, we satisfy the boundary conditions.

It is a straightforward calculation to find the displacement. For the vertical component u_3 we obtain

$$u_3 = -\frac{P}{4\pi(\lambda+\mu)}\frac{1}{r}\left(1-\frac{\lambda+\mu}{\mu}\left(1+\frac{x_3^2}{r^2}\right)\right). \tag{5.77}$$

This function of x_3 and $R = \sqrt{x_1^2 + x_2^2}$ (i.e. $r^2 = R^2 + x_3^2$) is shown in Fig. 5.4. Units are arbitrary and we have chosen $\lambda = 4.15 \times 10^{10}$ Pa and $\mu = 2.7 \times 10^{10}$ Pa. Obviously, there is a singularity at the point of action of the force $r = \sqrt{R^2 + x_3^2} = 0$.

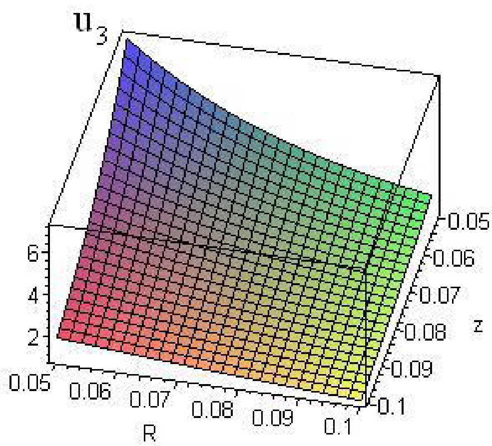


Fig. 5.4: Vertical displacement in the Boussinesq problem as a function of vertical distance $z = x_3$ from the surface and horizontal distance R from the axis x_3♣

The Boussinesq problem can be also solved by means of another Trefftz potentials which are defined in the following way

$$u_i = \varphi_i + x_3\chi_i. \tag{5.78}$$

It is a convenient method for problems of the half-space because the function φ_i satisfies for $x_3 = 0$ the same boundary condition as the displacement u_i. Both functions, φ_i and χ_i, are harmonic

$$\nabla^2\varphi_i = 0, \quad \nabla^2\chi_i = 0. \tag{5.79}$$

The function χ_i is connected to φ_i through the compatibility condition with the static displacement equations. Substitution in (5.33) with $\partial^2 u_i/\partial t^2 = 0, b_i = 0$ yields

$$2\frac{\partial\chi_3}{\partial x_i} + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial x_i}\left(\chi_3 + \frac{\partial\varphi_k}{\partial x_k} + x_3\frac{\partial\chi_k}{\partial x_k}\right) = 0. \tag{5.80}$$

In many cases it is sufficient to assume that χ_i possesses a potential

$$\chi_i = \frac{\partial\psi}{\partial x_i}, \quad \nabla^2\psi = 0. \tag{5.81}$$

Then (5.80) becomes

$$2\frac{\partial^2\psi}{\partial x_i \partial x_3} + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial x_i}\left(\frac{\partial\psi}{\partial x_3} + \frac{\partial\varphi_k}{\partial x_k}\right) = 0. \tag{5.82}$$

We can integrate once this equation with respect to x_i. Assuming the constant to be zero we obtain

$$\frac{\partial\psi}{\partial x_3} = -\frac{1}{3-4\nu}\frac{\partial\varphi_k}{\partial x_k}, \quad \frac{1}{3-4\nu} \equiv \frac{\lambda+\mu}{\lambda+3\mu}. \tag{5.83}$$

Now, solving the boundary value problem for φ_i we can find from the above relation the function ψ and then the displacement from the definition of the Trefftz potential which has now the form

$$u_i = \varphi_i + x_3\frac{\partial\psi}{\partial x_i}. \tag{5.84}$$

This representation yields not only the solution of the Boussinesq problem but also the solution of the so-called Cerrutti problem[7] in which the boundary x_3 is loaded by the tangential force P in the x_1-direction. We present the displacement for both problems in juxtaposition in the Table below. Both solutions were combined by R. D. Mindlin[8]. Numerous solutions of similar problems can be found in the book of K. L. Johnson[9].

Table: *Solutions of Boussinesq and Cerrutti problems for half-space*

	J. Boussinesq	V. Cerrutti
$u_1 =$	$\frac{Px_1}{4\pi\mu}\left(\frac{x_3}{r^3} - \frac{1-2\nu}{r(r+x_3)}\right)$	$\frac{Px_1}{4\pi\mu}\left(\frac{1}{r} + \frac{x_1^2}{r^3} + (1-2\nu)\left(\frac{1}{r+x_3} - \frac{x_1^2}{r(r+x_3)^2}\right)\right)$
$u_2 =$	$\frac{Px_2}{4\pi\mu}\left(\frac{x_3}{r^3} - \frac{1-2\nu}{r(r+x_3)}\right)$	$\frac{Px_1x_2}{4\pi\mu}\left(\frac{1}{r^3} - (1-2\nu)\frac{1}{r(r+x_3)^2}\right)$
$u_3 =$	$\frac{P}{4\pi\mu}\left(\frac{x_3^2}{r^3} + \frac{1-2\nu}{r}\right)$	$\frac{P}{4\pi\mu}\left(\frac{x_1x_3}{r^3} + (1-2\nu)\frac{x_1}{r(r+x_3)}\right)$

The above presented displacement formulation shall be also used further in the wave analysis.

[7] Valentino Cerrutti, 1850-1909

[8] R. D. Mindlin; Force at a point in the interior of a semi-infinite solid, Office of Naval Research Project NR-064-388 Contract Nonr-266(09), Technical Report No. HCU-9-s:K,>NR-266(09)-CE, May 1953.

see also: I. A. Okumura; On the generalization of Cerrutti's problem in an elastic half-space, *Structural Engn./Earthquake Eng.,* **12**, 2, 17-26, 1995,

D. A. Pozharskii; Generalization of the Cerruti Problem, *Doklady Physics*, **53**, No. 5, pp. 283–286, Pleiades Publishing, Ltd, 2008.

[9] K. L. Johnson; *Contact Mechanics,* (ninth printing) *Cambridge University Press*, 2003.

5.2.3 Beltrami-Michell equations

Static problems of linear elasticity can be also solved in a different way. Namely, if the boundary conditions prescribe tractions then we can directly find distributions of stresses. This can be done by use of the compatibility conditions (5.13) or rather (5.14) as there are only six independent compatibility conditions. These independent conditions may be also obtained by the contraction in (5.13) and then written in the form

$$\frac{\partial^2 e_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 e_{kk}}{\partial x_i \partial x_j} - \frac{\partial^2 e_{ik}}{\partial x_j \partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_j \partial x_k} = 0. \tag{5.85}$$

Substitution of the inverted Hooke law (5.26) yields

$$\begin{aligned} \nabla^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} - \frac{\lambda}{3\lambda+2\mu} \delta_{ij} \nabla^2 \sigma_{kk} - \\ - \left(\frac{\partial^2 \sigma_{kj}}{\partial x_i \partial x_k} + \frac{\partial^2 \sigma_{ki}}{\partial x_j \partial x_k} \right) = 0. \end{aligned} \tag{5.86}$$

Bearing the momentum balance equation (i.e. the equilibrium condition in the static case!) in mind

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0, \tag{5.87}$$

we reduce the system of equations (5.86) to the following form

$$\begin{aligned} \nabla^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} - \frac{\lambda}{3\lambda+2\mu} \delta_{ij} \nabla^2 \sigma_{kk} + \\ + \rho \frac{\partial b_i}{\partial x_j} + \rho \frac{\partial b_j}{\partial x_i} = 0. \end{aligned} \tag{5.88}$$

The trace of this relation yields

$$\nabla^2 \sigma_{kk} = -\frac{3\lambda+2\mu}{\lambda+2\mu} \rho \frac{\partial b_k}{\partial x_k}. \tag{5.89}$$

Hence, we can eliminate this Laplace operator contribution in (5.88). We obtain

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = -\rho \left(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i} \right) - \frac{\nu}{1-\nu} \delta_{ij} \rho \frac{\partial b_k}{\partial x_k}. \tag{5.90}$$

These are Beltrami-Michell stress equations. Together with boundary conditions for tractions they form the well-posed problem for the determination of stresses. Many examples of applications of these equations extended by the contribution of pore pressure can be found in geomechanics (e.g. [20]).

In a particular case of potential body forces

$$\rho b_i = -\frac{\partial \Gamma}{\partial x_i}, \quad \nabla^2 \Gamma = 0, \tag{5.91}$$

the set of equations (5.90) has the form

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial x_i\partial x_j} = -2\frac{\partial^2\Gamma}{\partial x_i\partial x_j}. \tag{5.92}$$

Therefore for such external forces the pressure is a harmonic function

$$\nabla^2 p = 0, \quad p = -\frac{1}{3}\sigma_{kk}. \tag{5.93}$$

Simultaneously, the application of Laplace operator to (5.92) yields

$$\nabla^2\nabla^2\sigma_{ij} = 0, \tag{5.94}$$

i.e. all components of stresses are biharmonic functions.

5.2.4 Plane strain and plane stress

We complete these considerations with a brief presentation of two special cases of statics: plane strain and plane stress systems (compare (3.81), (3.82). Then all functions depend only on two variables, say, $x_\alpha, \alpha = 1, 2$.

For a system extended to infinity in the x_3-direction we have the plane strains – u_3-component of displacement is identically zero and, consequently

$$e_{3k} = 0. \tag{5.95}$$

Constitutive relations reduce to the form

$$\begin{aligned}
\sigma_{\alpha\beta} &= \lambda\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta} + \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right), \\
\sigma_{33} &= \lambda\frac{\partial u_\gamma}{\partial x_\gamma} \quad \alpha, \beta, \gamma = 1, 2.
\end{aligned} \tag{5.96}$$

The displacement equations have then the following form

$$(\lambda+\mu)\frac{\partial^2 u_\beta}{\partial x_\alpha\partial x_\beta} + \mu\frac{\partial^2 u_\alpha}{\partial x_\beta\partial x_\beta} + \rho b_\alpha = 0. \tag{5.97}$$

They can be solved, as in a general case, by means of various potentials which satisfy either Laplace or Poisson equation. The most important of them are:

1) Galerkin function $\mathbf{F} = (F_1, F_2, 0)$

$$u_\alpha = \frac{\lambda+2\mu}{\mu}\frac{\partial^2 F_\alpha}{\partial x_\beta\partial x_\beta} - \frac{\lambda+\mu}{\mu}\frac{\partial^2 F_\beta}{\partial x_\alpha\partial x_\beta} \quad \Rightarrow \tag{5.98}$$

$$\Rightarrow \quad (\lambda+2\mu)\frac{\partial^4 F_\alpha}{\partial x_\beta\partial x_\beta\partial x_\gamma\partial x_\gamma} + \rho b_\alpha = 0 \quad \text{i.e.} \quad \nabla^2\nabla^2 F_\alpha + \frac{\rho b_\alpha}{\lambda+2\mu} = 0.$$

In many cases it is sufficient to introduce only one component of this function.

The stress in the direction of x_3-axis is as follows

$$\sigma_{33} = \frac{2\mu\nu}{1-2\nu}\frac{\partial^2}{\partial x_\alpha \partial x_\alpha}\frac{\partial F_\beta}{\partial x_\beta} = \frac{2\mu\nu}{1-2\nu}\frac{\partial}{\partial x_\beta}\nabla^2 F_\beta. \tag{5.99}$$

2) Papkovich-Neuber potentials

$$u_\alpha = \frac{\partial\left(\Phi_0 + x_\beta \Phi_\beta\right)}{\partial x_\alpha} - 4\left(1-\nu\right)\Phi_\alpha, \tag{5.100}$$

with the following equations for potentials

$$4\mu\left(1-\nu\right)\frac{\partial^2 \Phi_0}{\partial x_\beta \partial x_\beta} + \rho x_\beta b_\beta = 0,$$
$$4\mu\left(1-\nu\right)\frac{\partial^2 \Phi_\alpha}{\partial x_\beta \partial x_\beta} - \rho b_\alpha = 0, \quad \alpha = 1, 2. \tag{5.101}$$

3) Airy function. Compatibility conditions (5.85) reduce in the plane case to the single equation

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_2} + \frac{\partial^2 e_{22}}{\partial x_1 \partial x_1} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}. \tag{5.102}$$

In terms of stresses this condition and the equilibrium conditions without body forces have the form

$$\frac{\partial^2 \sigma_{11}}{\partial x_2 \partial x_2} + \frac{\partial^2 \sigma_{22}}{\partial x_1 \partial x_1} - \frac{\lambda}{2\left(\lambda+\mu\right)}\frac{\partial^2}{\partial x_\beta \partial x_\beta}\left(\sigma_{11}+\sigma_{22}\right) = 2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2},$$
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0. \tag{5.103}$$

The following Airy function F

$$\sigma_{\alpha\beta} = -\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} + \delta_{\alpha\beta}\frac{\partial^2 F}{\partial x_\gamma \partial x_\gamma}, \tag{5.104}$$

satisfies identically the equilibrium conditions and yields the following form of the compatibility condition

$$\nabla^2\nabla^2 F = \frac{\partial^4 F}{\partial x_\alpha \partial x_\alpha \partial x_\beta \partial x_\beta} \equiv \frac{\partial^4 F}{\partial x_1^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 F}{\partial x_2^4} = 0, \tag{5.105}$$

i.e. it is a biharmonic function.

Now we consider the case of plane stresses. It appears in membranes, i.e. systems whose one dimension, say in the x_3-direction, is much smaller than in the remaining two directions. Then we can use an approximation

$$\sigma_{3k} \approx 0. \tag{5.106}$$

Simultaneously

$$\sigma_{\alpha\beta} = 2\mu e_{\alpha\beta} + \frac{2\mu\lambda}{\lambda + 2\mu}\delta_{\alpha\beta} e_{\gamma\gamma}, \quad \alpha, \beta, \gamma = 1, 2, \tag{5.107}$$

which implies the following set of displacement equations

$$\mu\left(\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta} + \frac{3\lambda + 2\mu}{\lambda + 2\mu}\frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\beta}\right) + \rho b_\alpha = 0. \tag{5.108}$$

If we invert the constitutive relations (5.107)

$$e_{\alpha\beta} = \frac{1}{2\mu}\left(\sigma_{\alpha\beta} - \frac{\lambda}{3\lambda + 2\mu}\delta_{\alpha\beta}\sigma_{\gamma\gamma}\right), \tag{5.109}$$

then the compatibility relation (5.102) yields the following equation

$$\frac{\partial^2 \sigma_{11}}{\partial x_2 \partial x_2} + \frac{\partial^2 \sigma_{22}}{\partial x_1 \partial x_1} - \frac{\lambda}{3\lambda + 2\mu}\frac{\partial^2}{\partial x_\beta \partial x_\beta}(\sigma_{11} + \sigma_{22}) = 2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2}. \tag{5.110}$$

Hence, we can again introduce the Airy function F by the relation (5.104) which satisfies identically equilibrium conditions and it is again a biharmonic function satisfying the equation (5.105).

By means of this Airy function one solves in the linear elasticity the Flamant (1892) problem (stresses and displacements in a linear elastic wedge loaded by point forces at its sharp end; in particular the solution for the half-plane), punch problems and many others. We present here a simple example of a solution for a membrane with a hole. This solution indicates an important property of mechanical systems with imperfections that they yield stress concentration.

★In order to appreciate an influence of structure discontinuities on the distribution of stresses we consider a simple example of an infinite membrane with hole (cavity) of radius **a** (see: Fig. 5.5.)[10].

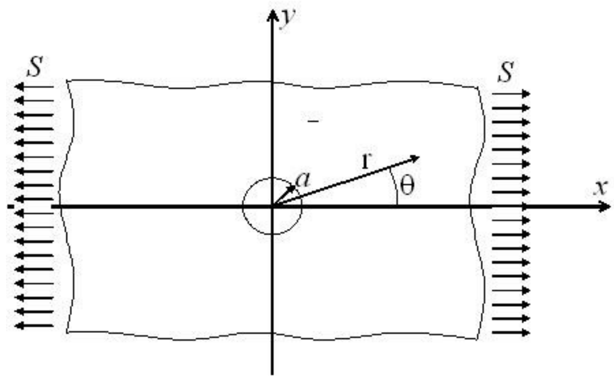


Fig. 5.5: Extension of a membrane with the circular cavity

[10] Chi-Teh Wang; *Applied Elasticity*, McGraw-Hill, N. Y., 1953

The membrane is loaded uniformly in the x-direction by the load of intensity S. Clearly, if the hole is not there the stress in the membrane has the following components

$$\sigma_x = S, \quad \sigma_y = \tau_{xy} = 0. \tag{5.111}$$

This solution corresponds to the Airy function

$$F_0 = \frac{1}{2} S y^2 = \frac{1}{4} S r^2 \left(1 - \cos 2\theta\right). \tag{5.112}$$

This relation implies the following components of stresses in the polar coordinates

$$\begin{aligned}
\sigma^0_{rr} &= \frac{1}{r}\frac{\partial F_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_0}{\partial \theta^2} = \frac{1}{2} S \left(1 + \cos 2\theta\right), \\
\sigma^0_{\theta\theta} &= \frac{\partial^2 F_0}{\partial r^2} = \frac{1}{2} S \left(1 - \cos 2\theta\right), \\
\tau^0_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_0}{\partial \theta}\right) = -\frac{1}{2} S \sin 2\theta.
\end{aligned} \tag{5.113}$$

We have made this transformation of coordinates as the problem with the cavity is easier in polar coordinates. For the problem with the cavity the following boundary conditions must be fulfilled

$$\begin{aligned}
\sigma_{rr} &= \tau_{r\theta} = 0 \quad \text{for} \quad r = a, \\
\sigma_{rr} &= \sigma^0_{rr}, \quad \tau_{r\theta} = \tau^0_{r\theta}, \quad \sigma_{\theta\theta} = \sigma^0_{\theta\theta} \quad \text{for} \quad r \to \infty.
\end{aligned} \tag{5.114}$$

The structure of the function F_0 suggests that we can try to find the Airy function for the more general case in the following form

$$F = f_1(r) + f_2(r) \cos 2\theta. \tag{5.115}$$

This function must be biharmonic. Substitution of (5.115) in the equation for Airy function in polar coordinates (compare (5.45))

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0, \tag{5.116}$$

yields two equations as the general equation should hold for arbitrary angles θ. They have the following form

$$\begin{aligned}
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) &= 0, \\
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2} + \frac{1}{r}\frac{df_2}{dr} - \frac{4 f_2}{r^2}\right) &= 0.
\end{aligned} \tag{5.117}$$

The solutions of these simple ordinary differential equations have the form

$$\begin{aligned}
f_1(r) &= c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4, \\
f_2(r) &= c_5 r^2 + c_6 r^4 + \frac{c_7}{r^2} + c_8.
\end{aligned} \tag{5.118}$$

This yields the Airy function in the form

$$F = \left(c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4\right) + \left(c_5 r^2 + c_6 r^4 + \frac{c_7}{r^2} + c_8\right) \cos 2\theta. \tag{5.119}$$

Hence relations for stress components are as follows

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = c_1\left(1 + 2\ln r\right) + 2c_2 + \frac{c_3}{r^2} - \left(2c_5 + \frac{6c_7}{r^4} + \frac{4c_8}{r^2}\right)\cos 2\theta,\\
\sigma_{\theta\theta} &= \frac{\partial^2 F}{\partial r^2} = c_1\left(3 + 2\ln r\right) + 2c_2 - \frac{c_3}{r^2} + \left(2c_5 + 12c_6 r^2 + \frac{6c_7}{r^4}\right)\cos 2\theta,\\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) = \left(2c_5 + 6c_6 r^2 - \frac{5c_7}{r^4} - \frac{2c_8}{r^2}\right)\sin 2\theta.
\end{aligned} \tag{5.120}$$

Stresses should be finite in infinity which means that constants c_1 and c_6 must be identically zero. The remaining boundary conditions lead to the following solution

$$\begin{aligned}
\sigma_r &= \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta,\\
\sigma_\theta &= \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta,\\
\tau_{r\theta} &= -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta.
\end{aligned} \tag{5.121}$$

It is seen that for $r = a$ and $\theta = \pi/2$ and $\theta = 3\pi/2$ the circumferential stress $\sigma_{\theta\theta}$ has the maximum value equal to $3S$. This is three times more than in the case without the hole. In Fig. 5.6. we show the behaviour of $\sigma_{\theta\theta}$ for these two values of the angle.

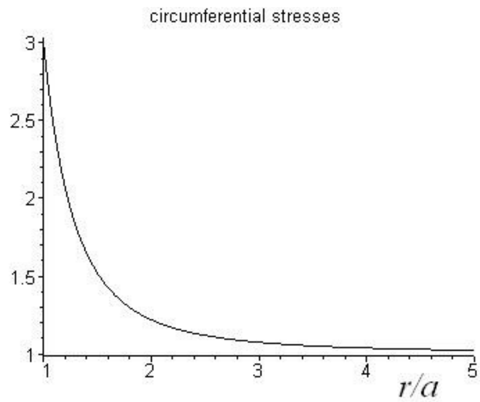


Fig. 5.6: Circumferential stresses $\sigma_{\theta\theta}/S$ in function of the distance from the hole r/a.♣

The problem of the concentration of stresses in the vicinity of various holes has a very extensive literature due to its practical bearing.

5.2.5 Waves in linear elastic materials

We return to the analysis of the displacement equations (5.32). We begin with a proof of existence of two waves described by these equations. This is based on Hadamard Theorem which we sketch for the purpose of the linear theory. We consider a point $\mathbf{x}_A$ on a singular surface on which both the strain $\mathbf{e}$ and the velocity $\mathbf{v}$ are continuous but their higher derivatives such as the acceleration $\partial\mathbf{v}/\partial t$ or the gradient of strain grad $\mathbf{e}$ may suffer a finite discontinuity. The point $\mathbf{x}_A$ changes its position with the moving singular surface and, say, after a small time increment δt is located in a point $\mathbf{x}_B = \mathbf{x}_A + \delta\mathbf{x}$. We assume that this change happens in the direction $\mathbf{n}$ perpendicular to the surface and with the speed c, i.e. $\delta\mathbf{x} = c\mathbf{n}\delta t$. Then the change of the value of the gradient of displacement between these two points can be calculated on the path ahead of the singular surface and behind this surface and, consequently, along the paths on which the gradient of displacement possesses continuous derivatives. We have

$$\begin{aligned} \left.\frac{\partial u_i}{\partial x_j}\right|_B &= \left.\frac{\partial u_i}{\partial x_j}\right|_A + \left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_A^+ \delta x_k + \left.\frac{\partial^2 u_i}{\partial x_j \partial t}\right|_A^+ \delta t = \\ &= \left.\frac{\partial u_i}{\partial x_j}\right|_A + \left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_A^- \delta x_k + \left.\frac{\partial^2 u_i}{\partial x_j \partial t}\right|_A^- \delta t, \end{aligned} \tag{5.122}$$

where the signature "+" and "−" indicates the limits on both sides of the surface. Subtracting these relations, we easily arrive at the following compatibility condition

$$\left[\left[\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right]\right] = -\frac{1}{c}\left[\left[\frac{\partial^2 u_i}{\partial x_j \partial t}\right]\right] n_k, \tag{5.123}$$

where $[[...]] = (...)^+ - (...)^-$ is the difference of limits on both sides of the surface. In the same way we prove the identity for the acceleration

$$\left[\left[\frac{\partial^2 u_i}{\partial x_j \partial t}\right]\right] = -\frac{1}{c}\left[\left[\frac{\partial^2 u_i}{\partial t^2}\right]\right] n_j. \tag{5.124}$$

These identities form the contents of Hadamard Theorem. They can be combined to give the following relation

$$\left[\left[\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right]\right] = \frac{1}{c^2}\left[\left[\frac{\partial^2 u_i}{\partial t^2}\right]\right] n_j n_k. \tag{5.125}$$

Now we form the jump of the displacement equations on the singular surface described above. We obtain

$$\rho\left[\left[\frac{\partial^2 u_i}{\partial t^2}\right]\right] = (\lambda+\mu)\left[\left[\frac{\partial^2 u_k}{\partial x_i \partial x_k}\right]\right] + \mu\left[\left[\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right]\right]. \tag{5.126}$$

Substitution of (5.125) yields

$$\left(c^2\delta_{ik} - \frac{\lambda+\mu}{\rho} n_i n_k - \frac{\mu}{\rho}\delta_{ik}\right)\left[\left[\frac{\partial^2 u_k}{\partial t^2}\right]\right] = 0. \tag{5.127}$$

Consequently, we obtain the eigenvalue problem in which the discontinuity of acceleration is the eigenvector. The eigenvalues can be found by the separation of longitudinal and transversal contributions. If we multiply the relation (5.127) by a unit vector $\mathbf{t}$ perpendicular to $\mathbf{n}$ (i.e. $t_i n_i = 0$) then we obtain

$$\left(c^2 - \frac{\mu}{\rho}\right)\left(\left[\left[\frac{\partial^2 u_k}{\partial t^2}\right]\right] t_k\right) = 0. \qquad (5.128)$$

Hence, either the projection of discontinuity on the direction perpendicular to $\mathbf{n}$ is zero and then we obtain the identity, or it is different from zero and then

$$c^2 = c_T^2 = \frac{\mu}{\rho}. \qquad (5.129)$$

This is the square of the speed of propagation of the front of the wave on which the acceleration suffers the transversal discontinuity. We call such waves transversal (shear waves). Obviously, in order to be real the speed of propagation yields the condition

$$\mu > 0, \qquad (5.130)$$

which is one of the limitations of material parameters mentioned at the beginning of this Chapter.

Now we multiply the equation (5.127) by the vector $\mathbf{n}$. It follows

$$\left(c^2 \delta_{ik} - \frac{\lambda + 2\mu}{\rho}\right)\left(\left[\left[\frac{\partial^2 u_k}{\partial t^2}\right]\right] n_k\right) = 0. \qquad (5.131)$$

Hence, either the projection of discontinuity on the direction $\mathbf{n}$ is zero and then we obtain the identity, or it is different from zero and then

$$c^2 = c_L^2 = \frac{\lambda + 2\mu}{\rho}. \qquad (5.132)$$

This is the square of the speed of propagation of the front of the wave on which the acceleration suffers the longitudinal discontinuity. We call such waves longitudinal. As before, for existence of these waves we have to require

$$\lambda + 2\mu > 0, \qquad (5.133)$$

which is the second limitation of material parameters. When both conditions (5.130) and

(5.133) are satisfied we say that the set of displacement equations (5.32) is hyperbolic.

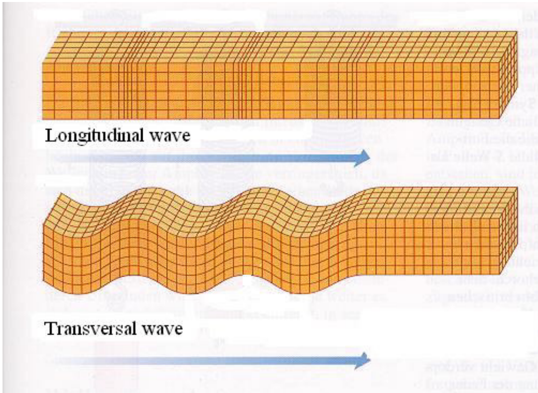


Fig. 5.7: Schematic picture of longitudinal and transversal waves

In Fig. 5.7. we demonstrate schematically the motion of particles by the transition of longitudinal and transversal waves.

Any dynamic solution of the displacement equations (5.32) describes the propagation of the wave front which divides the domain $\mathcal{B}_t$ at the instant of time t into a part which is not yet disturbed by the loading and the part behind the front where the dynamic displacement evolves. Exact solutions of this art can be constructed by the use of dynamic Green function for finite domains. However, technical difficulties in construction of such solutions are so extensive that it pays off to consider a local structure of dynamic disturbances. This is usually done by means of the Fourier analysis of plane waves. The latter assumption means that we replace the three-dimensional propagation by a one-dimensional local approximation. The solution is assumed to have the form

$$\mathbf{u} = \operatorname{Re}\left\{\mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\omega}\mathbf{t})}\right\}, \tag{5.134}$$

where $\mathbf{A}$ is a complex constant amplitude, ω is the so-called frequency of the wave. We assume that it is given. Such waves are called monochromatic. The vector $\mathbf{k}$ has the structure $\mathbf{k} = k\mathbf{n}$, $\mathbf{n}\cdot\mathbf{n} = 1$ and the unit vector $\mathbf{n}$ is the direction of propagation of the wave. It is assumed to be constant which means that the wave is plane. k is the so-called wave number and it may be complex. Consequently the function (5.134) can be written in the form

$$\mathbf{u} = e^{-\operatorname{Im}k(\mathbf{n}\cdot\mathbf{x})}\operatorname{Re}\left(\mathbf{A}e^{i\operatorname{Re}k(\mathbf{n}\cdot\mathbf{x}-c_{ph}t)}\right), \quad c_{ph} = \frac{\omega}{\operatorname{Re}k}, \tag{5.135}$$

and c_{ph} is called the phase speed. $\operatorname{Im}k$ describes the damping of the wave. The above relation can be also written in the form of a real function

$$\mathbf{u} = e^{-\operatorname{Im}k(\mathbf{n}\cdot\mathbf{x})}\mathbf{A}_0\cos\left(\operatorname{Re}k\left(\mathbf{n}\cdot\mathbf{x}-c_{ph}t\right)+\phi\right), \quad \mathbf{A} = \mathbf{A}_0e^{i\phi}, \tag{5.136}$$

and $\mathbf{A}_0$ is the real amplitude. $[\operatorname{Re}k\left(\mathbf{n}\cdot\mathbf{x}-c_{ph}t\right)+\phi]$ is called the phase and ϕ is called the phase shift.

It is convenient to change phase in the following manner

$$\operatorname{Re} k\left(\mathbf{n}\cdot\mathbf{x}-c_{ph}t\right)+\phi=2\pi\left(\frac{\operatorname{Re} k}{2\pi}\mathbf{n}\cdot\mathbf{x}-\frac{\omega}{2\pi}t\right)+\phi=$$
$$=2\pi\left(\frac{\mathbf{n}\cdot\mathbf{x}}{l}-ft\right)+\phi,\quad l=\frac{2\pi}{\operatorname{Re} k},\quad f=\frac{\omega}{2\pi}. \tag{5.137}$$

The quantity f is called the technical frequency and l is the wave length of the monochromatic wave of frequency ω.

The full solution of the displacement equations (5.32) can be constructed by means of the combination of monochromatic waves which form then contributions to a Fourier series. This representation of waves is called spectral.

We do not need to go into all details of the spectral analysis and present only solutions of the form (5.134). Substitution of this relation in the displacement equations yields

$$\rho\left(-i\omega\right)^2 A_i=\left(\lambda+\mu\right)A_k\left(ik_k\right)\left(ik_i\right)+\mu A_i\left(ik_k\right)\left(ik_k\right),$$

i.e.

$$\left[\rho\omega^2\delta_{ik}-\left(\lambda+\mu\right)k_ik_k-\mu k^2\delta_{ik}\right]A_k=0,\quad k=\sqrt{k_kk_k}. \tag{5.138}$$

This is again the eigenvalue problem. As in the case of the wave front which we have discussed above we separate the tangential and longitudinal components. Scalar multiplication by the unit vector $\mathbf{t}$ perpendicular to $\mathbf{k}$ yields

$$\left[\rho\omega^2-\mu k^2\right]A_kt_k=0. \tag{5.139}$$

Hence, the projection of the amplitude on the direction perpendicular to the direction of propagation $\mathbf{n}=\mathbf{k}/k$ different from zero yields

$$k^2=\frac{\omega^2}{c_T^2}. \tag{5.140}$$

This is the so-called dispersion relation for transversal monochromatic waves. It shows that the wave number is real in this case and that the phase speed is equal to the speed of propagation of transversal waves

$$c_{ph}=c_T, \tag{5.141}$$

i.e. the phase speeds are independent of the frequency of the monochromatic wave. Such waves are called non-dispersive. As $\operatorname{Im} k=0$ they are not attenuated (damping is zero).

Now scalar multiplication of the equation (5.138) by $\mathbf{k}$ yields

$$\left[\rho\omega^2-\left(\lambda+2\mu\right)k^2\right]A_kn_k=0. \tag{5.142}$$

Again for the projection of the amplitude on the direction of propagation different from zero we obtain

$$k^2=\frac{\omega^2}{c_L^2}. \tag{5.143}$$

This dispersion relation for longitudinal monochromatic waves yields the phase speed equal to the speed c_L

$$c_{ph} = c_L, \tag{5.144}$$

and this is again independent of the frequency. As transversal waves, also longitudinal monochromatic waves are non-dispersive and not attenuated.

The two sorts of waves which we presented above are called bulk waves because they propagate inside of the body. In the case of a boundary the situation changes. We consider here a simple case of a half-space with the plane boundary which yields the so-called surface waves. In the case under consideration they were discovered by J. W. Rayleigh (1887)[11]. On the boundary perpendicular to the x_3-axis we assume the boundary conditions

$$\mathbf{Tn}|_{x_3} = 0 \quad \text{i.e.} \quad \sigma_{k3}|_{x_3=0} = 0, \quad \mathbf{n} = -\mathbf{e}_3, \tag{5.145}$$

i.e. the boundary is stress-free. It means that the wave has been created far away from the origin of coordinates and its source will be ignored in the analysis. Simultaneously, we have the following Sommerfeld condition

$$\mathbf{u}|_{x_3 \to \infty} = \mathbf{0}. \tag{5.146}$$

It is easier to seek the solution of the problem when we make the following decomposition of the displacement vector

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T, \quad \operatorname{rot} \mathbf{u}_L = 0, \quad \operatorname{div} \mathbf{u}_T = 0, \tag{5.147}$$

where $\mathbf{u}_L$ is called the potential part and $\mathbf{u}_T$ is the solenoidal part. Obviously, it is directly connected with the Helmholtz decomposition (1.57): $\mathbf{u}_L = \operatorname{grad} \varphi$, and $\mathbf{u}_T = \operatorname{rot} \boldsymbol{\psi}$. These two parts must satisfy equations

$$\frac{\partial^2 \mathbf{u}_L}{\partial t^2} = c_L^2 \nabla^2 \mathbf{u}_L, \quad \frac{\partial^2 \mathbf{u}_T}{\partial t^2} = c_T^2 \nabla^2 \mathbf{u}_T, \tag{5.148}$$

following directly from the displacement equations (5.32) (compare (5.58)).

We seek the solution in the form of the following ansatz

$$\begin{aligned} \mathbf{u}_L &= A_L e^{-\gamma x_3} e^{i(kx_1 - \omega t)} \mathbf{e}_1 + B_L e^{-\gamma x_3} e^{i(kx_1 - \omega t)} \mathbf{e}_3, \\ \mathbf{u}_T &= A_T e^{-\beta x_3} e^{i(kx_1 - \omega t)} \mathbf{e}_1 + B_T e^{-\beta x_3} e^{i(kx_1 - \omega t)} \mathbf{e}_3. \end{aligned} \tag{5.149}$$

Hence, we consider the plane problem. We anticipate a progressive wave solution in the x_1-direction and the decay of the solution in the x_3-direction provided the coefficients γ, β are positive. If such a solution does not exist it means that the surface wave does not appear.

Substitution of the ansatz (5.149) in wave equations (5.148) leads to the compatibility conditions

$$\frac{\gamma^2}{k^2} = 1 - \frac{c_R^2}{c_L^2}, \quad \frac{\beta^2}{k^2} = 1 - \frac{c_R^2}{c_T^2}, \quad c_R = \frac{\omega}{k}, \tag{5.150}$$

[11] J. W. (Strutt) Rayleigh; On waves propagated along the plane surface of an elastic solid, *Proc. London Math. Soc.*, 17:4-11, 1887.

where c_R is the phase speed of the wave. Now we use the properties of the potential and solenoidal parts. We have

$$\begin{aligned}
\epsilon_{231}\frac{\partial u_1^L}{\partial x_3}+\epsilon_{213}\frac{\partial u_3^L}{\partial x_1} &= 0 \quad\Rightarrow\quad B_L=i\frac{\gamma}{k}A_L, \\
\frac{\partial u_1^T}{\partial x_1}+\frac{\partial u_3^T}{\partial x_3} &= 0 \quad\Rightarrow\quad B_T=i\frac{k}{\beta}A_T.
\end{aligned} \tag{5.151}$$

Consequently

$$\begin{aligned}
\mathbf{u}_L &= \left(\mathbf{e}_1+i\frac{\gamma}{k}\mathbf{e}_3\right)A_L e^{-\gamma x_3}e^{i(kx_1-\omega t)}, \\
\mathbf{u}_T &= \left(\mathbf{e}_1+i\frac{k}{\beta}\mathbf{e}_3\right)A_T e^{-\beta x_3}e^{i(kx_1-\omega t)}.
\end{aligned} \tag{5.152}$$

Obviously, the Sommerfeld condition is satisfied by these functions if $\gamma,\beta>0$. The stress components which we need in boundary conditions (5.145) can be written in the form

$$\begin{aligned}
\frac{1}{\rho}\sigma_{33} &= \left(c_L^2-2c_T^2\right)\frac{\partial u_1}{\partial x_1}+c_L^2\frac{\partial u_3}{\partial x_3}, \\
\frac{1}{\rho}\sigma_{13} &= c_T^2\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right).
\end{aligned} \tag{5.153}$$

Hence, for $x_3=0$ the substitution of (5.152) leads to the set of two equations

$$\begin{aligned}
\left(2-\frac{c_R^2}{c_T^2}\right)A_L+2A_T &= 0, \\
2\frac{\gamma\beta}{k^2}A_L+\left(2-\frac{c_R^2}{c_T^2}\right)A_T &= 0.
\end{aligned} \tag{5.154}$$

This is the homogeneous set of equations for amplitudes A_L, A_T. It possesses nontrivial solutions if the determinant is equal to zero

$$\left(2-\frac{c_R^2}{c_T^2}\right)^2-4\sqrt{1-\frac{c_R^2}{c_T^2}}\sqrt{1-\frac{c_R^2}{c_L^2}}=0, \tag{5.155}$$

where relations (5.150) have been used. This equation for c_R is called Rayleigh dispersion relation. It is clear that the speed of Rayleigh waves c_R is independent of the frequency ω. Consequently, Rayleigh waves are non-dispersive.

The solution of the equation (5.155) is shown in Fig. 5.8. As

$$\frac{c_T}{c_L}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1}{2}\frac{1-2\nu}{1-\nu}}<1, \tag{5.156}$$

the speed of Rayleigh waves is smaller than the speed of transversal waves. It has a physical interpretation in terms of the so-called constructive interference of longitudinal and transversal waves which happens after these waves are reflected from the boundary and so create the surface wave.

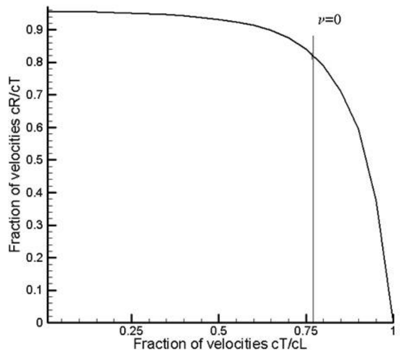


Fig. 5.8: Dimensionless speed of Rayleigh waves c_R/c_T as a function of the fraction of transversal and longitudinal speeds c_T/c_L.

Let us inspect the relations for components of displacements. According to (5.147) they have the form

$$\begin{aligned} u_1 &= \left(A_L e^{-\gamma x_3} + A_T e^{-\beta x_3}\right) e^{i(kx_1 - \omega t)}, \\ u_3 &= i\left(\frac{\gamma}{k} A_L e^{-\gamma x_3} + \frac{k}{\beta} A_T e^{-\beta x_3}\right) e^{i(kx_1 - \omega t)}. \end{aligned} \tag{5.157}$$

We choose the real part of these relations with $\cos(kx_1 - \omega t)$. Then by eliminating the time from these relations, we obtain

$$\frac{(\operatorname{Re} u_1)^2}{\alpha_1^2} + \frac{(\operatorname{Re} u_3)^2}{\alpha_3^2} = 1, \tag{5.158}$$

where

$$\begin{aligned} \alpha_1 &= A_L e^{-\gamma x_3} + A_T e^{-\beta x_3}, \\ \alpha_3 &= \frac{\gamma}{k} A_L e^{-\gamma x_3} + \frac{k}{\beta} A_T e^{-\beta x_3}. \end{aligned} \tag{5.159}$$

Hence the orbits of particles are ellipses with semiaxes $|\alpha_1|, |\alpha_3|$. One can easily show that the motion is anticlockwise. This is opposite to the direction of motion of water particles in shallow water waves. The size of these ellipses diminishes exponentially with the depth x_3. This is the reason for calling such a wave the surface wave. It can be shown that the elliptic motion of particles in planes $x_2 = const$ is the only motion possible for Rayleigh waves. Transversal Rayleigh waves do not exist.

The lack of dispersion in the plane case presented above explains the disastrous action of surface waves in earthquakes. In contrast to bulk waves which move from the point source (hypocenter) in approximately spherical form and are the first two arrivals, the surface wave moves from epicenter in approximately cylindrical form and arrives as the last one ($c_R < c_T < c_L$). Hence the energy which carries the wave is distributed on a much larger surfaces for bulk waves than for surface waves.

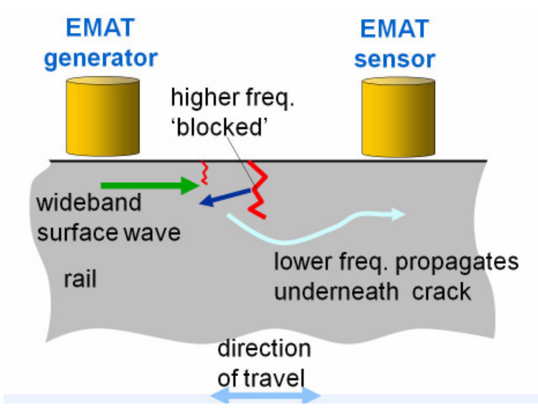


Fig. 5.9: An application of surface waves: testing of rails

The nondispersive character of three waves which we have considered above is rather exceptional. Whenever the problem comprises a characteristic length waves become dispersive. This is the case for surface waves in boreholes, tunnels, layers, or media with microstructure such as porous materials (characteristic length – dimensions of channels). We consider here the simplest example of Love waves[12].

A. E. H. Love (1911) has solved the problem of propagation of waves in a layer of thickness H on an elastic half-space. The plane of contact of these two media (interface) is perpendicular to the upward oriented x_3-axis and the origin of the coordinates lies on this plane. We distinguish the material properties by a 'prime', i.e. ρ', c'_T are the mass density and the speed of transversal waves in the layer while ρ, c_T are the mass density and the speed of transversal waves in the half-space. We assume that the motion of particles is perpendicular to the (x_1, x_3)-plane of propagation of waves. Then the problem is described by two wave equations

$$\begin{aligned} \frac{\partial^2 u'_2}{\partial t^2} &= c'^2_T \nabla^2 u'_2 \quad \text{for} \quad 0 < x_3 < H, \\ \frac{\partial^2 u_2}{\partial t^2} &= c^2_T \nabla^2 u_2 \quad \text{for} \quad x_3 < 0. \end{aligned} \tag{5.160}$$

One can show that longitudinal surface waves (i.e. waves with a displacement $(u'_1, 0, u'_3)$ in the plane $x_2 = const$) for this configuration do not exist.

[12] C. G. Lai, K. Wilmanski; (eds), *Surface Waves in Geomechanics: Direct and Inverse Modelling for Soils and Rocks*, Springer, 2005.

We seek the solution of the system (5.160) in the form of a monochromatic wave of the frequency ω

$$\begin{aligned} u_2' &= \left(A'e^{iks'x_3} + B'e^{-iks'x_3}\right)e^{i(kx_1-\omega t)} \equiv \\ &\equiv 2\left(\operatorname{Re}A'\cos ks'x_3 - \operatorname{Im}A'\sin ks'x_3\right)e^{i(kx_1-\omega t)}, \\ u_2 &= Be^{ksx_3}e^{i(kx_1-\omega t)}. \end{aligned} \tag{5.161}$$

This solution should satisfy the boundary conditions

1. Shear stress on the plane $x_3 = H$ is equal to zero, i.e.

$$\left.\frac{\partial u_2'}{\partial y}\right|_{x_3=H} = 0, \tag{5.162}$$

2. shear stress and the displacement must be continuous on the interface $x_3 = 0$

$$\begin{aligned} \rho' c_T'^2 \left.\frac{\partial u_2'}{\partial x_3}\right|_{x_3=0} &= \rho c_T^2 \left.\frac{\partial u_2}{\partial x_3}\right|_{x_3=0}, \quad \rho' c_T'^2 = \mu', \quad \rho c_T^2 = \mu, \\ \left.u_2'\right|_{x_3=0} &= \left.u_2\right|_{x_3=0}. \end{aligned} \tag{5.163}$$

Substitution of (5.161) in (5.160) yields the compatibility conditions

$$s'^2 = \frac{c^2}{c_T'^2} - 1, \quad s^2 = 1 - \frac{c^2}{c_T^2}, \quad c = \frac{\omega}{k}. \tag{5.164}$$

The boundary condition (5.162) leads to the following displacement in the layer

$$u_2' = 2\operatorname{Re}A' \frac{\cos\left(ks'\left(H - x_3\right)\right)}{\cos\left(ks'H\right)} e^{i(kx_1-\omega t)}. \tag{5.165}$$

Then the remaining two conditions (5.163) give rise to two equations for the constants $\operatorname{Re}A', B$. As this system of equations is homogeneous the determinant must be zero and we obtain the following Love dispersion relation

$$\omega = \frac{c}{Hs'}\left[\arctan\left(\frac{\rho c_T^2 s}{\rho' c_T'^2 s'}\right) + n\pi\right], \quad n = 1, 2, 3, ..., \tag{5.166}$$

where s, s' must be real and s must be positive for the amplitude of the wave to decay in the half-space. Consequently, according to (5.164),

$$c_T' \leq c \leq c_T. \tag{5.167}$$

Hence, the Love waves exist only in layers which are softer than the foundation. Simultaneously, the dispersion relation has infinitely many solutions, the so-called modes, and the corresponding speeds of propagation depend on the frequency ω. Love waves are dispersive. This means that packages of waves of different frequency become broader during the propagation – some of their monochromatic contributions are slower than the others. For this reason, one introduces also an "average" speed of propagation which is called the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{\frac{d\omega}{dc_{ph}} c_{ph}^2}{\frac{d\omega}{dc_{ph}} c_{ph} - \omega}, \tag{5.168}$$

where the second part of the relation follows immediately from the definitions. Consequently, we can find the group velocity immediately substituting the dispersion relation (5.166). We show here only a simple numerical example for the following data (compare examples in [1])

$$c_T = 5\ \frac{\text{km}}{\text{s}}, \quad c'_T = 3\ \frac{\text{km}}{\text{s}}, \quad \frac{\rho'}{\rho} = 0.875, \quad H = 10\ \text{km}. \tag{5.169}$$

Results are plotted in Fig. 5.10.

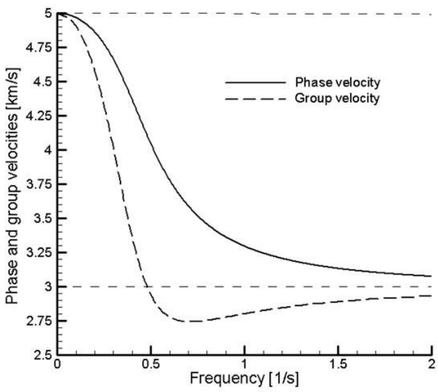


Fig. 5.10: Phase and group velocity of the first mode of Love wave for the data (5.169)

In contrast to the phase velocity the group velocity is not a monotonous function of the frequency. This property yields the existence of the so-called Airy phase which has an important bearing in description of seismic waves[13].

5.2.6 Principle of virtual work

Motivated by the principle of conservation of energy we present now one of the most important procedures in linear elasticity which forms both a method for mathematical proofs of existence and uniqueness of solutions as well as a foundation for numerous approximate methods of solutions.

[13] for some details see: C. G. Lai, K. Wilmanski; (eds), *Surface Waves in Geomechanics: Direct and Inverse Modelling for Soils and Rocks*, Springer, 2005.

Let us consider first the static problem. The body $\mathcal{B}_t$ is loaded by external forces $\rho\mathbf{b}$ and tractions $\mathbf{t_n}$ given on the surface $\partial\mathcal{B}_t^{\sigma}$ with the prescribed displacement $\mathbf{u}_b$ on the remaining part of the boundary $\partial\mathcal{B}_t^{u}$. This yields displacements $\mathbf{u}(\mathbf{x})$, stresses $\mathbf{T}(\mathbf{x})$ and strain $\mathbf{e}(\mathbf{x})$ in the body $\mathcal{B}_t$.

We impose on these displacements a virtual displacement $\delta\mathbf{u}$ which is small, continuously differentiable and admissible. It means that it has to comply to conditions limiting the motion of the body. For instance, it must be zero on the boundary $\partial\mathcal{B}_t^{u}$.

The principle of virtual work says that the work done by virtual displacements of external loadings is equal to the work of internal forces, i.e.

$$\int_{\mathcal{B}_t} \rho\mathbf{b}\cdot\delta\mathbf{u}dV + \int_{\partial\mathcal{B}_t} \mathbf{t_n}\cdot\delta\mathbf{u}dS = \int_{\mathcal{B}_t} \mathbf{T}\cdot\delta\mathbf{e}dV, \tag{5.170}$$

where

$$\delta\mathbf{e} = \frac{1}{2}\left((\operatorname{grad}\delta\mathbf{u}) + (\operatorname{grad}\delta\mathbf{u})^T\right) = \frac{1}{2}\delta\left((\operatorname{grad}\mathbf{u}) + (\operatorname{grad}\mathbf{u})^T\right). \tag{5.171}$$

We shall see that this statement is related to the energy conservation (4.1).

In order to derive (5.170) we multiply the equilibrium conditions by $\delta\mathbf{u}$ and integrate over the body $\mathcal{B}_t$. We have

$$\int_{\mathcal{B}_t} (\operatorname{div}\mathbf{T} + \rho\mathbf{b})\cdot\delta\mathbf{u}dV = 0. \tag{5.172}$$

This yields

$$\begin{aligned}\int_{\mathcal{B}_t} \operatorname{div}\mathbf{T}\cdot\delta\mathbf{u}dV &= \int_{\mathcal{B}_t} \left[\operatorname{div}(\mathbf{T}\delta\mathbf{u}) - \mathbf{T}\cdot\operatorname{grad}\delta\mathbf{u}\right]dV = \\ &= \int_{\partial\mathcal{B}_t} \mathbf{t_n}\cdot\delta\mathbf{u}dS - \int_{\mathcal{B}_t} \mathbf{T}\cdot\delta\mathbf{e}dV,\end{aligned} \tag{5.173}$$

where we have used $\delta\mathbf{u} = \mathbf{0}$ on $\partial\mathcal{B}_t^{u}$. This relation indicates (5.170).

Conversely the principle (5.170) yields local equilibrium conditions.

For isotropic elastic solids we can also write

$$\mathbf{T}\cdot\delta\mathbf{e} = (\lambda e_{kk}\delta_{ij} + 2\mu e_{ij})\,\delta e_{ij} = \frac{1}{2}\lambda\delta(e_{ii}e_{jj}) + \mu\delta(e_{ij}e_{ij}) = \frac{1}{2}\delta(\sigma_{ij}e_{ij}). \tag{5.174}$$

Consequently

$$\int_{\mathcal{B}_t} \rho\mathbf{b}\cdot\delta\mathbf{u}dV + \int_{\partial\mathcal{B}_t} \mathbf{t_n}\cdot\delta\mathbf{u}dS = \delta E, \quad E = \int_{\mathcal{B}_t} \rho\varepsilon dV, \quad \rho\varepsilon = \frac{1}{2}\sigma_{ij}e_{ij}. \tag{5.175}$$

Obviously, the quantity $\rho\varepsilon$ is the potential energy of the linear elastic material. The quantity E is called the work of deformations.

As the body forces $\mathbf{b}$ and tractions $\mathbf{t_n}$ are given their variations are zero. Hence, we can write

$$\delta\Pi_e = 0, \quad \Pi_e = E - \int_{\mathcal{B}_t} \rho\mathbf{b}\cdot\mathbf{u}dV - \int_{\partial\mathcal{B}_t} \mathbf{t_n}\cdot\mathbf{u}dS, \tag{5.176}$$

where Π_e is the potential energy of the displacement field. The above condition shows that this potential possesses an extremum in real motion. It is easy to see that it is minimum. We have to compare the potentials Π_e' and Π_e for displacements $\mathbf{u}+\delta\mathbf{u}$ and $\mathbf{u}$. We have

$$\begin{aligned} \rho\varepsilon\left(e_{ij}+\delta e_{ij}\right) &= \rho\varepsilon\left(e_{ij}\right) + \rho\frac{\partial\varepsilon}{\partial e_{ij}}\delta e_{ij} + \frac{1}{2!}\frac{\partial^2\varepsilon}{\partial e_{ij}\partial e_{kl}}\delta e_{ij}\delta e_{kl} + ..., \\ \Rightarrow \quad \rho\varepsilon\left(e_{ij}+\delta e_{ij}\right) - \rho\varepsilon\left(e_{ij}\right) &= \sigma_{ij}\delta e_{ij} + \frac{1}{2}\frac{\partial\sigma_{ij}}{\partial e_{kl}}\delta e_{ij}\delta e_{kl} + ... \end{aligned} \tag{5.177}$$

Therefore

$$\begin{aligned} \Pi_e' - \Pi_e &= \int_{\mathcal{B}_t} \sigma_{ij}\delta e_{ij}dV - \int_{\mathcal{B}_t} \rho\mathbf{b}\cdot\delta\mathbf{u}dV - \int_{\partial\mathcal{B}_t} \mathbf{t_n}\cdot\delta\mathbf{u}dS + \int_{\mathcal{B}_t} \frac{1}{2}\frac{\partial\sigma_{ij}}{\partial e_{kl}}\delta e_{ij}\delta e_{kl}dV = \\ &= \int_{\mathcal{B}_t} \frac{1}{2}\frac{\partial\sigma_{ij}}{\partial e_{kl}}\delta e_{ij}\delta e_{kl}dV = \int_{\mathcal{B}_t} \frac{1}{2}\left(\lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}\right)\delta e_{ij}\delta e_{kl}dV = \\ &= \int_{\mathcal{B}_t} \frac{1}{2}\left(\lambda\delta e_{ii}\delta e_{jj} + 2\mu\delta e_{ij}\delta e_{ij}\right)dV = \\ &= \int_{\mathcal{B}_t} \frac{1}{2}\left(\left(\lambda + \frac{2}{3}\mu\right)\delta e_{ii}\delta e_{jj} + 2\mu\delta e_{ij}^D\delta e_{ij}^D\right)dV, \end{aligned} \tag{5.178}$$

where

$$\delta e_{ij}^D = \delta e_{ij} - \frac{1}{3}\delta e_{kk}\delta_{ij}, \tag{5.179}$$

is the deviatoric part of δe_{ij}. The quantity (5.178) is positive if each contribution to the sum is positive. This follows from the fact that δe_{ii} and δe_{ij}^D are independent and arbitrary. Consequently, the potential Π_e possesses the minimum in equilibrium if and only if the material parameters satisfy the conditions

$$\lambda + \frac{2}{3}\mu = K > 0, \quad \mu > 0. \tag{5.180}$$

We have mentioned these limitations before. They indicate as well $\lambda+2\mu > 0$, i.e. (5.130) and (5.133) which means that the minimum condition implies the hyperbolicity of the displacement equations.

We can easily extend the above principle on the full dynamic case. Then the principle of virtual work (5.176) must be modified in the following way. We consider the motion of the body between two instances of time, t_1 and t_2. We consider the variation of real displacement $\delta\mathbf{u}$ which satisfies the conditions

$$\delta\mathbf{u}\left(\mathbf{x},t_1\right) = \delta\mathbf{u}\left(\mathbf{x},t_2\right) = 0. \tag{5.181}$$

Instead of (5.172) we have

$$\int_{t_1}^{t_2} dt \int_{\mathcal{B}_t} \left(\operatorname{div} \mathbf{T} + \rho \mathbf{b} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) \cdot \delta \mathbf{u} dV = 0. \tag{5.182}$$

We have to transform the contribution of acceleration. If we introduce the kinetic energy

$$\mathcal{K} = \int_{\mathcal{B}_t} \frac{1}{2} \rho \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} dV, \tag{5.183}$$

then

$$\int_{t_1}^{t_2} \delta \mathcal{K} dt = \int_{t_1}^{t_2} dt \int_{\mathcal{B}_t} \rho \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \delta \mathbf{u} \right) dV - \int_{t_1}^{t_2} dt \int_{\mathcal{B}_t} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \delta \mathbf{u} dV dt. \tag{5.184}$$

Due to conditions (5.181) the first integral on the right-hand side vanishes. Consequently,

$$\delta \int_{t_1}^{t_2} (E - \mathcal{K}) \, dt = \int_{t_1}^{t_2} dt \int_{\mathcal{B}_t} \rho \mathbf{b} \cdot \mathbf{u} dV + \int_{\partial \mathcal{B}_t} \mathbf{t_n} \cdot \delta \mathbf{u} dS, \tag{5.185}$$

or

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0, \quad \mathcal{L} = \mathcal{K} - \Pi_e. \tag{5.186}$$

This is the Hamilton principle. The functional $\mathcal{L}$ is called Lagrangian of the system. The Hamilton principle says that Lagrangian has an extremum in the interval of time $t_1 < t < t_2$, where in the endpoints of this interval the state of the body is known.

Hamilton's principle has a very extensive physical literature as it is the main tool in the derivation of model equations for numerous reversible processes of classical, relativistic and quantum mechanics. The method based on the construction of Lagrangian and various invariance principles yields equations of motion and conservation laws (see the classical reference of Landau and Lifschitz [7]). Attempts to extend the method on irreversible processes such as heat conduction or theory of dislocations in applications to plasticity were not successful because the so-called Lagrange-Euler equations which follow as the equations of motion in this approach are invariant with respect to time reversal, i.e. they must be reversible.

5.3 Thermoelasticity

All processes in linear elastic materials which we have been discussing were assumed to be isothermal. This is almost never the case and an influence of temperature difference may have a very substantial influence on the distribution of stresses. We show the simplest possible extension of the linear elasticity on processes in which the temperature is variable as well.

In the thermodynamic construction of a model we have to deal with at least two fields in nonisothermal processes: displacement $\mathbf{u}$ and temperature T. For the displacement we expect as before that it follows from the field equations constructed on the basis of the momentum conservation law. On the other hand, the temperature T should satisfy the field equation which we assume to follow from the energy conservation law. Consequently, we choose the equations (3.38) and (4.11) from which we construct the field equations. In the linear problems they are as follows

$$\rho \frac{\partial v_k}{\partial t} = \frac{\partial \sigma_{kl}}{\partial x_l} + \rho b_k, \qquad (5.187)$$
$$\rho \frac{\partial \varepsilon}{\partial t} + \frac{\partial q_k}{\partial x_k} = \sigma_{kl} \frac{\partial v_k}{\partial x_l},$$

where we have neglected the radiation and the mass density ρ is constant.

For thermoelastic materials we assume that the stresses σ_{kl}, the internal energy ε and the heat flux q_k are functions of the strain e_{kl}, temperature T and temperature gradient $\partial T / \partial x_k$. These constitutive functions should satisfy the second law of thermodynamics (4.20) which in the linear form is as follows

$$\rho \frac{\partial \eta}{\partial t} + \operatorname{div} \left(\frac{\mathbf{q}}{T} \right) \geq 0, \qquad (5.188)$$

for all processes. The entropy density η is also a function of the above listed constitutive variables. We simplify the considerations by assuming these constitutive laws in the form

$$\sigma_{kl} = \sigma_{kl} \left(e_{ij}, T \right), \quad \varepsilon = \varepsilon \left(e_{ij}, T \right), \quad \eta = \eta \left(e_{ij}, T \right),$$
$$q_k = -K_T \frac{\partial T}{\partial x_k}, \qquad (5.189)$$

where K_T is the thermal conductivity coefficient.The last relation is called Fourier's law. The detailed justification of these assumptions can be found in books on continuum thermodynamics (e.g. [22]).

Now we exploit the inequality (5.188). The procedure is the same as for ideal gases in Sec. 5.2. The momentum balance (5.187) does not impose any restrictions on the inequality (5.188) due to the presence of acceleration. The elimination of the heat flux from the inequality eliminates as well the constraint imposed by the energy conservation. We obtain

$$\rho \frac{\partial \psi}{\partial t} + \rho \eta \frac{\partial T}{\partial t} + \frac{1}{T} q_k \frac{\partial T}{\partial x_k} - \sigma_{kl} \frac{\partial e_{kl}}{\partial t} \leq 0, \qquad (5.190)$$
$$\psi = \varepsilon - T\eta = \psi \left(e_{ij}, T \right),$$

where ψ is the Helmholtz free energy function. This inequality should hold for all fields of displacement and temperature. Due to the symmetry of the stress tensor, we have made the following replacement

$$\sigma_{kl} \frac{\partial v_k}{\partial x_l} = \frac{1}{2} \sigma_{kl} \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) = \sigma_{kl} \frac{\partial e_{kl}}{\partial t}. \qquad (5.191)$$

Now, the chain rule of differentiation and the linearity of the inequality with respect to the derivatives $\partial T/\partial t, \partial e_{kl}/\partial t$ yield the following identities

$$\sigma_{kl} = \rho \frac{\partial \psi}{\partial e_{kl}}, \quad \eta = -\frac{\partial \psi}{\partial T}, \quad \varepsilon = \psi - T\frac{\partial \psi}{\partial T}, \tag{5.192}$$

and the residual inequality defining the dissipation

$$\mathcal{D} = -q_k \frac{\partial T}{\partial x_k} = K_T \left(\frac{\partial T}{\partial x_k} \frac{\partial T}{\partial x_k} \right) \geq 0 \quad \Rightarrow \quad K \geq 0. \tag{5.193}$$

These relations immediately imply the following Gibbs equation of linear thermoelasticity

$$d\eta = \frac{1}{T} \left(d\varepsilon - \frac{1}{\rho} \sigma_{kl} de_{kl} \right). \tag{5.194}$$

Now, we are in the position to formulate the linear isotropic model. We assume that the current temperature deviates only a little from the homogeneous initial temperature T_0, i.e.

$$\left| \frac{T - T_0}{T_0} \right| \ll 1. \tag{5.195}$$

Assuming in addition that the undeformed state ($e_{kl} = 0$) is stress-free we write the Helmholtz free energy in the form of quadratic function with respect to the deviation from the initial natural configuration

$$\rho\psi = -\frac{1}{2}\rho\frac{c_v}{T_0}(T - T_0)^2 - \gamma(T - T_0)e_{kk} + \frac{1}{2}c_{ijkl}e_{ij}e_{kl}, \tag{5.196}$$
$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The function must be quadratic due to relations (5.192) which imply that derivatives of ψ must be linear. The coefficients in the above relation possess the following interpretation. The internal energy follows in the form

$$\varepsilon = \frac{c_v}{2T_0}\left(T^2 - T_0^2\right) + \frac{\gamma}{\rho}T_0 e_{kk} + \frac{1}{2\rho}c_{ijkl}e_{ij}e_{kl}. \tag{5.197}$$

Consequently

$$\frac{\partial \varepsilon}{\partial T} = c_v \frac{T}{T_0} \approx c_v. \tag{5.198}$$

Hence the coefficient c_v is the specific heat by constant volume. Incidentally, the specific heats by constant volume and constant pressure, respectively, are practically identical for solids, in contrast to gases.

We proceed to the coefficient γ. For the stress tensor we obtain

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma(T - T_0)\delta_{ij}. \tag{5.199}$$

The first part is, obviously, identical with Hooke's law for isothermal processes. The trace of this equation leads to

$$e_{kk} = \frac{\sigma_{kk}}{3K} - \frac{\gamma}{K}(T - T_0), \quad K = \lambda + \frac{2}{3}\mu. \tag{5.200}$$

Hence the coefficient γ describes volume changes caused by the temperature difference. It is called volumetric thermal expansion coefficient for solids. It is sometimes denoted by α_V. In experiments usually the linear thermal expansion coefficient for solids $\alpha = \gamma/3K$ is measured. Some values of this coefficient are shown in the Table below.

Table: *Linear thermal expansion coefficient* $\alpha = \gamma/3K$ $\left[10^{-6}/°\mathrm{K}\right]$

aluminium	23.8	cast iron	11.8
asphalt	200	limescale	20
ice (0°C)	0.502	marble	11
iron	12.1	polystyrene	60÷80
pyrex glass	3.2	porcelain	3÷4
crystal glass	0.45	sandstone	5
granite	3÷8	firebrick	5

★In order to appreciate the order of magnitude of stresses created by the temperature difference we calculate the stress in a thin bar along x_1-axis fixed on both ends and heated uniformly from the temperature T_0 to T, $T - T_0$ =100°. We choose steel as the material for which

$$\begin{aligned} \alpha &= 11.8 * 10^{-6} 1/°\mathrm{K}, \\ \lambda &= 11.78 * 10^{10}\mathrm{Pa}, \\ \mu &= 8.00 * 10^{10}\mathrm{Pa}. \end{aligned} \tag{5.201}$$

Then we have

$$\begin{aligned} e_{11} &= 0 \quad \Rightarrow \quad \sigma_{22} + \sigma_{33} = 2(\lambda + \mu) e_{kk} - 2\gamma (T - T_0) \approx 0, \\ \text{i.e.} \quad e_{kk} &= \frac{\gamma}{\lambda + \mu} (T - T_0) = \alpha \frac{3\lambda + 2\mu}{\lambda + \mu} (T - T_0). \end{aligned} \tag{5.202}$$

Hence, for our data,

$$\sigma_{11} = \lambda e_{kk} - \gamma (T - T_0) = -\alpha\mu \frac{3\lambda + 2\mu}{\lambda + \mu} (T - T_0) = 245 \text{ MPa}. \tag{5.203}$$

As the yield limit (the limit of elastic behavior) for construction steel is app. 250 MPa we see from the above example that relatively small temperature difference may create already an irreparable damage in the material. It may, of course, yield the buckling as well. One should keep in mind that during a construction fire the temperature difference is app. 900° and the temperature of magma is from 700°C to 1300°C, i.e. the rocks shortly before melting reach the temperature app. 600°C and this temperature difference yields enormous thermal stresses.★

By means of the above constitutive relations we can write the field equations. They have the form

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} &= (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \nabla^2 \mathbf{u} - \gamma \operatorname{grad} T + \rho \mathbf{b}, \\ \frac{\partial T}{\partial t} &= \frac{K_T}{\rho c_v} \nabla^2 T - \frac{\gamma T_0}{\rho c_v} \frac{\partial}{\partial t} \operatorname{div} \mathbf{u}. \end{aligned} \tag{5.204}$$

Clearly, due to the thermal expansion the equations are coupled. In many cases, one can neglect the coupling in the equation for the temperature. Then the distribution of temperature is determined as in the so-called rigid heat conductors, i.e. in an undeformed body and then this temperature field can be introduced to the displacement equations as an external force.

5.4 Poroelasticity

Appearance of porous materials in nature is so common that we will not list even examples. Three of them are shown in Fig. 5.11. From the point of view of mechanics the main issue in description of such materials is the coupling between the motion of the solid skeleton and fluids in pores and channels. A proper continuum thermodynamics of such systems requires a multicomponent modeling which is called the theory of immiscible mixtures. We shall not enter this field of research in these notes and refer to many monographs on the subject[14].

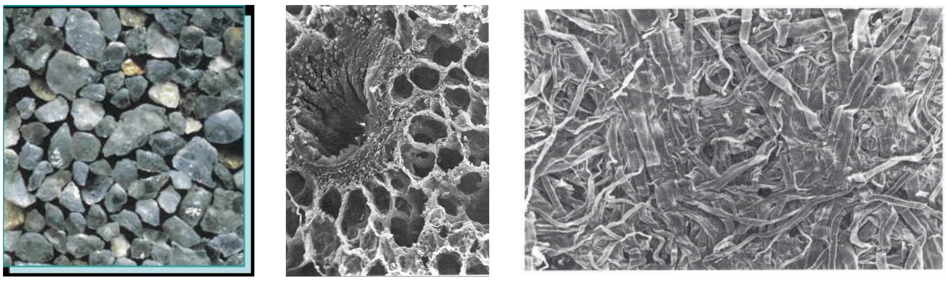


Fig. 5.11: Examples of porous materials: sand, bronchus, toilet paper

In this Subsection we show the modeling initiated by K. von Terzaghi (1883-1963) who proposed an extension of the classical linear elasticity by the diffusion equation for pore pressure[15]. The couplings between the pore pressure and stresses in the solid skeleton are similar to these in thermoelasticity. Consequently, the model belongs to the class of one-component models. Its motivation and modern developments within geomechanics can be found in the book of H. F. Wang [20].

The main field is again the displacement $\mathbf{u}(\mathbf{x},t)$ but, in addition, we have to consider volume changes of pore spaces. Macroscopically these changes are described by a quantity ε which is analogous to the volume changes of the skeleton given by $e = \operatorname{tr}\mathbf{e}$, $\mathbf{e} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right)$. M. Biot has proposed in 1941[16] a concept of the variation

[14] e.g. R. M. Bowen; Diffusion models implied by the theory of mixtures, in: C. Truesdell, *Rational Thermodynamics, Second Edition*, Springer, 237-263, N. Y., 1984,
J. Bear; *Dynamics of Fluids in Porous Media*, Dover, N.Y., 1972.,
as well as the monographs of Wilmanski [21], [22].

[15] K. von Terzaghi; *Erdbaumechanik auf bodenphysikalischer Grundlage*, Deuticke, Wien, 1925,
K. von Terzaghi; *Theoretical Soil Mechanics*, J. Wiley and Sons, New York, 1943.

[16] M. A. Biot; General theory of three-dimensional consolidation, *J. Appl. Physic*, **12**, 155-164, 1941.

in water content, ζ, which is defined in terms of the volume changes by the relation

$$\zeta = n_0 (e - \varepsilon), \tag{5.205}$$

where n_0 denotes the initial value of porosity. This is the fraction of voids to the total volume of the porous material, provided we choose a small domain for this definition. These small domains are called Representative Elementary Volumes (REV). We shall not discuss them in these notes. It can be shown that in a thermodynamical equilibrium $\zeta = 0$ which means that contributions of this variable to the model are irreversible. They are related to the relative motion of solid and fluid components, i.e. to the diffusion. We can write

$$\frac{\partial \zeta}{\partial t} = n_0 \frac{\partial}{\partial t} (\operatorname{div} \mathbf{u} - \varepsilon) = -n_0 \operatorname{div} \mathbf{v}_{seep}, \quad \mathbf{v}_{seep} = \mathbf{v}^F - \frac{\partial \mathbf{u}}{\partial t}, \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \tag{5.206}$$

where $\mathbf{v}^F$ is the velocity of the fluid and $\mathbf{v}_{seep}$ is the so-called seepage velocity.

For this reason, one cannot principally construct a variational formalism for such models in spite of many publications in which it is though attempted.

Once we have the additional field ζ we can introduce a conjugate dynamic variable, p, which is called pore pressure. The fundamental relations defining the couplings between the skeleton and the fluid can be written in principal coordinates in the following form

$$\begin{aligned}
e^{(1)} &= \frac{1}{E}\left(\sigma^{(1)} - \nu\left(\sigma^{(2)} + \sigma^{(3)}\right)\right) + \frac{p}{3H}, \\
e^{(2)} &= \frac{1}{E}\left(\sigma^{(2)} - \nu\left(\sigma^{(1)} + \sigma^{(3)}\right)\right) + \frac{p}{3H}, \\
e^{(3)} &= \frac{1}{E}\left(\sigma^{(3)} - \nu\left(\sigma^{(1)} + \sigma^{(2)}\right)\right) + \frac{p}{3H}, \\
\zeta &= \frac{1}{3H}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right) + \frac{p}{R},
\end{aligned} \tag{5.207}$$

where $1/H, 1/R$ are the so-called Biot moduli: $1/H$ is the so-called poroelastic expansion coefficient, while $1/R$ is the unconstrained specific storage coefficient. Their detailed presentation can be found in the book of H. Wang [20]. It should be mentioned that M. Biot, the founder of a systematic approach to the subject of poromechanics of saturated materials, was changing his notation many times and, for this reason, some care is required in reading his papers.

Certainly, the transformation of (5.207) to arbitrary coordinates yields the extended stress-strain relations. They have the form

$$\begin{aligned}
\sigma_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \alpha p \delta_{ij}, \quad \alpha = \frac{K}{H}, \\
\zeta &= \alpha e_{kk} + \frac{\alpha}{K_u B} p, \quad B = \frac{\alpha R}{K},
\end{aligned} \tag{5.208}$$

where α is called the Biot-Willis coefficient, B is the Skempton coefficient,

$$K_u = \frac{K}{1 - \alpha B}, \quad K = \lambda + \frac{2}{3}\mu, \tag{5.209}$$

and K_u is called the undrained bulk modulus, i.e. the bulk modulus which corresponds to $\zeta = 0$. Clearly, in addition to the Lamé constants λ, μ, the model contains two additional material parameters, for instance, α and B, or α and K_u.

In addition to the above described coupling properties between stresses, σ_{ij} and p, and strains e_{ij} and ζ, we have to describe the flow of the fluid through the porous materials. This is the subject of Darcy's law. It relates the gradient of the pore pressure to the relative velocity of the fluid and the skeleton, i.e. the seepage velocity, $\mathbf{v}_{seep} = \mathbf{q}/n_0$. It can be written in the following form

$$\mathbf{q} = -\frac{k}{\eta}\,\mathrm{grad}\,p, \tag{5.210}$$

where k is the so-called intrinsic permeability and η denotes the fluid viscosity. The ratio k/η is called the mobility. Some typical values of these parameters for water in pores (i.e. for $\eta = 10^{-3}$ Pa·s, $\rho = 1000$ kg/m^3, $g = 10$ m/s^2) are shown in the Table.

Table: *Permeability for a few rock types* [20]

Rock type	permeability k [m^2]	permeability [Darcy]	hydraulic conductivity [m/s]
sand or sandstone	10^{-12}	1	10^{-5}
sandstone or limestone	10^{-15}	10^{-3}	10^{-8}
granite or shale	10^{-18}	10^{-6}	10^{-11}

Once we have the constitutive relations (5.207) we can construct field equations using the classical momentum conservation law and an additional equation for the pore pressure which follows from the definition of the variation in water content (5.205) and the kinematic relation (5.206) combined with Darcy's law (5.210)

$$\begin{aligned} \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} &= (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mu\nabla^2\mathbf{u} - \alpha\,\mathrm{grad}\,p + \rho\mathbf{b}, \\ \frac{\partial p}{\partial t} &= c\nabla^2 p - K_u B\frac{\partial}{\partial t}\,\mathrm{div}\,\mathbf{u}, \end{aligned} \tag{5.211}$$

where

$$c = \frac{K_u B}{\alpha}\frac{k}{\eta}, \tag{5.212}$$

is called the diffusivity.

Comparison of equations of linear thermoelasticity (5.204) and linear poroelasticity (5.211) shows a full mathematical analogy of these models. Therefore many solution available in thermoelasticity such as Green's functions can be taken over to poroelasticity.

Let us mention that mechanics of poroelastic materials based on the theory of mixtures reveals many additional features related to porous structures which cannot be described by the very simplistic model presented above. One of those features is the existence of an additional bulk wave, the so-called P2-wave, discovered theoretically already by J. Frenkel (1944) and confirmed experimentally during the last two decades. This mode of propagation cannot be described by equations (5.211) because, like thermoelasticity equations (5.204), the contribution of the pore pressure is described by the parabolic equation $(5.211)_2$, corresponding to the heat conduction equation $(5.204)_2$ which is parabolic as well.

Chapter 6

Viscoelastic materials

6.1 Viscoelastic fluids and solids

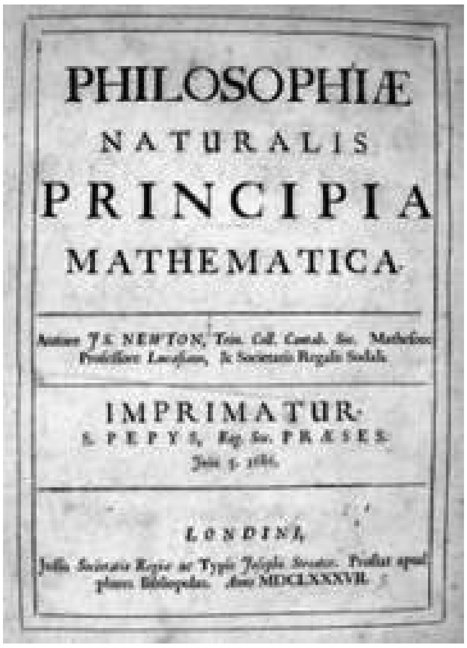
To the class of viscoelastic materials belong practically all elastic materials if they are observed in sufficiently long times. The description of such materials contains a characteristic relaxation time (or many, maybe even infinitely many of them) and this may have values from milliseconds to millions of years. In the latter case structural materials are obviously considered to be elastic. Over the lengths of time required to build mountain ranges, however, rocks appear to deform as very viscous fluids via a process known as slow creeping flow. Many materials such as polymers possess relaxation times of some hours to some years and then, of course, their viscous properties must be incorporated in modelling as well. Models of viscoelastic materials developed from the combination of elastic materials and viscous fluids.

Development of models of viscous fluids was initiated in times of Isaac Newton. In Book II of his Principia (1687) Newton formulated laws of resistance to the motion of fluids.

The next essential step, definitions of various rheological materials including viscoelastic materials was made first at the end of XIXth century. Maxwell, Boltzmann, Kelvin, Voigt and many others introduced first simple and then more sophisticated models of materials which were combinations of springs and dash-pots. We show some of them in the next Subsection. At the end of 50th of XXth century the development of continuous models began. This was initiated by works of A. E. Green, J. Ericksen and R. S. Rivlin[1] and continued by B. D. Coleman, N. Gurtin, W. Noll, C. Truesdell. Viscoelastic

[1]R. S. Rivlin, J. L. Ericksen; Stress-deformation relations for isotropic materials, *J. Rat. Mech. Anal.*, **4**, 323-425, 1955.
A. E. Green, R. S. Rivlin; The mechanics of non-linear materials with memory, I, *Arch. Rat. Mech. Anal.*, **1**, 1-21, 1957.
A. E. Green, R. S. Rivlin; The mechanics of non-linear materials with memory, III, *Arch. Rat. Mech. Anal.*, **1**, 387-404, 1960.
A. E. Green, R. S. Rivlin, A. J. M. Spencer; The mechanics of non-linear materials with memory, II, *Arch. Rat. Mech. Anal.*, **1**, 82-90, 1959.
B. D. Coleman, W. Noll; Foundations of linear viscoelasticity, *Rev. Mod. Phys.*, **33**, 239-249, 1961.

materials belong to the broader class of materials with memory developed in these works.



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Julii 5. 1686.

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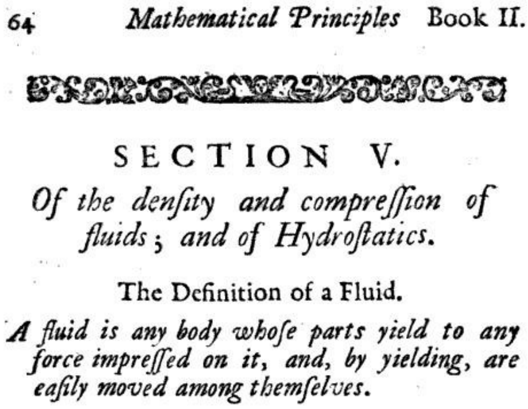
Mathematical Principles of Natural Philosophy, London, 1687



64 *Mathematical Principles* Book II.

SECTION V.

Of the denſity and compreſſion of fluids; and of Hydroſtatics.

The Definition of a Fluid.

A fluid is any body whoſe parts yield to any force impreſſed on it, and, by yielding, are eaſily moved among themſelves.

Fig. 6.1: The beginning of Section V, Book II of Isaac Newtons's Mathematical Principles of Natural Philosophy (London, 1687), containing the model of viscous fluids

In this Chapter, we present the main features of the viscoelasticity of solids. We follow two classical books on the subject R. M. Christensen [3] and A. C. Pipkin [15].

In Fig. 6.2. we show schematically a distinction between viscoelastic solids and viscoelastic (non-Newtonian) fluids. In the simple shear experiment we apply to a rectangle the shear deformation $\kappa(t) = \kappa_0 H(t)$, where $H(t)$ is the Heaviside function. The nonzero component of stretching (compare (2.65)) is then $\partial\kappa/\partial t = \kappa_0 \delta(t)$. This means, as shown in Part B) of the Fig. 6.2., that the shear stress in the linear elastic material remains constant and in the viscous fluid becomes infinite at the initial instant of time and then it is zero.

In the case of viscoelastic solid (see Part C) of Fig. 6.2.) the stress would relax after a long time to a finite value smaller than the initial value but different from zero (stress relaxation). For a viscoelastic fluid the stress would begin with the same value as in the case of viscoelastic solid but it would relax to zero as in the case of viscous fluid.

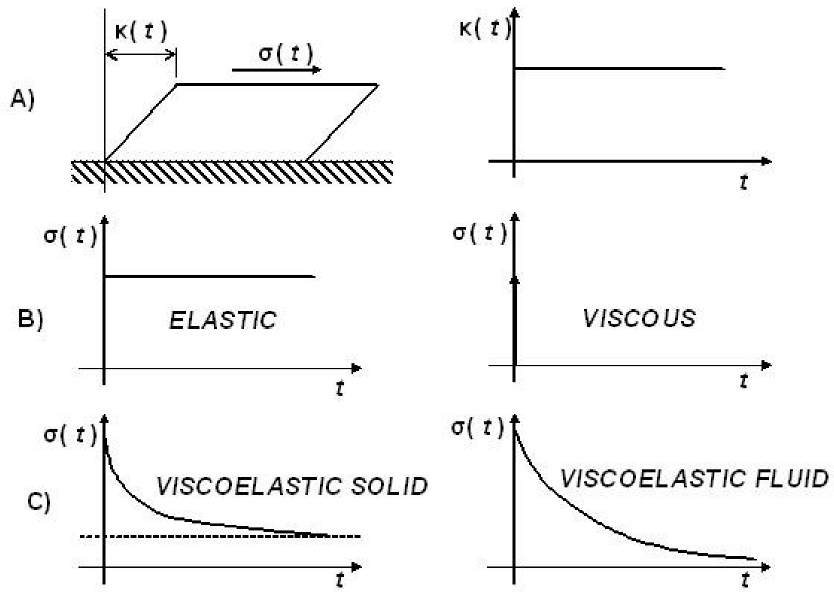


Fig. 6.2: Viscoelastic solids vs. fluids – stress relaxation in viscoelastic solids and fluids

These observations will be justified on simple rheological models in the next Subsection and they will be incorporated in the construction of a standard model of viscoelastic solids.

The most important effect appearing in viscoelastic materials is creep. We explain this notion on a simple example of a slab subjected to a one-step stress history $\sigma(t) = \sigma_0 H(t)$. We show the behaviour schematically in Fig. 6.3. The response of an elastic solid would be $\kappa(t) = \kappa_0 H(t)$, i.e. constant shear for positive time t. In a viscous fluid, the shear would increase at a constant rate, $\kappa(t) = \sigma t/\eta$, where η denotes the viscosity. In a viscoelastic material, the shear at first jumps so that the instantaneous response is elastic. The shear then continues to increase with a decreasing rate and it approaches a finite limit $\kappa(\infty)$. This is the phenomenon of creep. Otherwise, if the shear increases linearly in long times it is characteristic for a viscoelastic fluid.

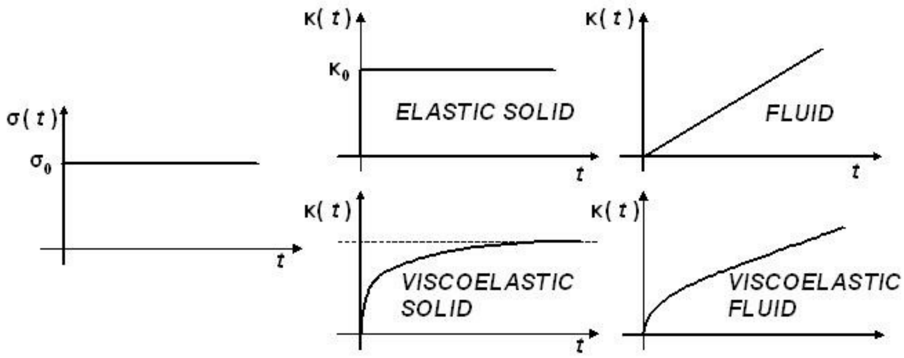


Fig. 6.3: Behaviour of various materials under a one-step stress history (schematic) – creep in viscoelastic solids and fluids

The above mentioned creep effects yield the notion of memory in the material behaviour. We quote here R. M. Christensen [3]: 'It is instructive to consider a situation which represents a generalization of the response to a single suddenly applied change of surface traction. Suppose a material having instantaneous elasticity and creep characteristics described above is subjected to two nonsimultaneously applied sudden changes in uniform stress, superimposed upon each other. After the first application of stress, but before the second, the material responds in some time dependent manner which depends upon the magnitude of the first stress state. But now consider the situation that exists at an arbitrary small interval of time after the sudden application of the second stress state. The material not only experiences the instantaneous response to the second change in surface traction but also it experiences a continuing time dependent response to the first applied level of stress. An elastic material would respond only to the total stress level at every instant of time. Thus, this more general type of material possesses a characteristic which can be descriptively referred to as a memory effect. That is, the material response is not only determined by the current state of stress, but is also determined by all past states of stress'.

Let us mention in passing that the theory of non-newtonian (viscoelastic) fluids has a very extensive literature which we do not quote in these notes.

6.2 Rheological models

We present here a few very simple models reflecting memory effects arising in viscoelastic solids. Equations describing their behaviour follow as special cases of constitutive relations for viscoelastic materials. R. M. Christensen in his book [3] presents first these general constitutive relations and then simple rheological elements appear as illustration. It seems to be more appealing to proceed the other way around.

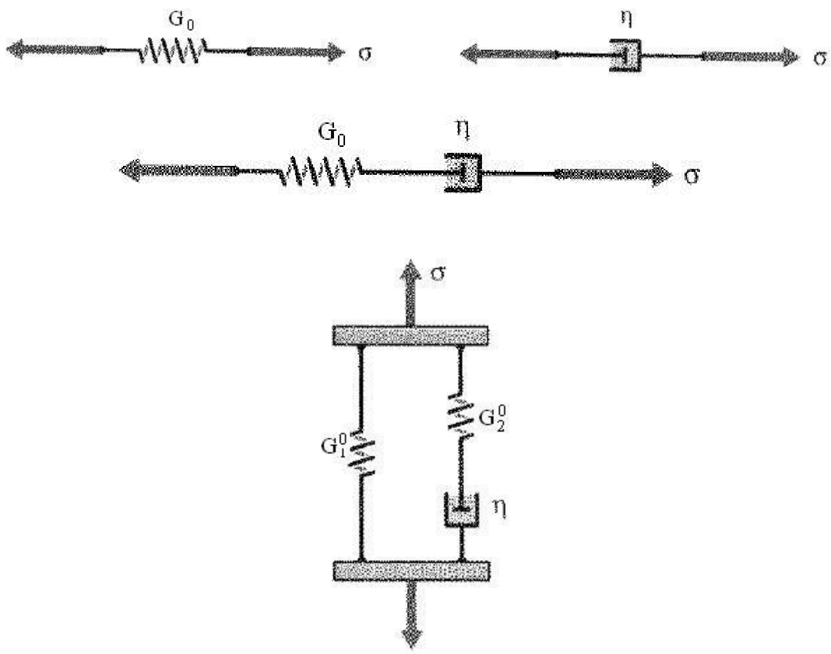


Fig. 6.4.: Some rheological elements: spring and dashpot (upper panel, on top), Maxwell element (upper panel, on bottom), standard element (lower panel).

The one-dimensional models are constructed by the combination of springs and dashpot elements. The spring is an ideal elastic element obeying the linear force-extension relation. The force and extension are analogs for stress and strain. This element fulfils the relation

$$\sigma = G_0 \kappa. \tag{6.1}$$

The dashpot is an ideal viscous element that extends at the rate proportional to the applied force

$$\dot{\kappa} = \frac{d\kappa}{dt} = \frac{\sigma}{\eta}, \tag{6.2}$$

where η is the viscosity.

Various combinations of these two elements yield simple rheological models of viscoelastic materials. The simplest one is Maxwell's model presented in Fig. 6.4. (upper panel, on bottom). It is described by the conditions

$$\begin{aligned} \sigma &= \sigma_{el} = \sigma_{vis}, \quad \kappa = \kappa_{el} + \kappa_{vis}, \\ \sigma_{el} &= G_0 \kappa_{el}, \quad \sigma_{vis} = \eta \dot{\kappa}_{vis}, \end{aligned} \tag{6.3}$$

with an obvious meaning of the notation. The elasticity constant G_0, [Pa], is called in the theory of viscoelasticity the relaxation modulus, and η, $[\mathrm{Pa}\cdot\mathrm{s}]$ is the viscosity. Relations (6.3) yield the following constitutive equation for Maxwell's model

$$\dot{\sigma}+\frac{1}{\tau}\sigma=G_0\dot{\kappa}, \quad \tau=\frac{\eta}{G_0}. \tag{6.4}$$

Obviously, it is an evolution equation for stresses σ and τ is the relaxation time. The formal solution of this equation has the following form

$$\sigma=\sigma_0 e^{-t/\tau}-G_0\int\limits_0^t \frac{d\kappa}{ds}\left(t-s\right)e^{-s/\tau}ds, \quad \sigma_0=\sigma\left(t=0\right). \tag{6.5}$$

For the constant stress $\sigma = const$ the equation (6.4) yields the constant rate of strain This corresponds to the fluid-like behaviour in Fig. 6.3. The same conclusion follows in the case of a constant strain $\dot{\kappa}=0$ which yields the decay of stresses in time to zero. In this case the solution of the equation (6.4) has the form $\sigma=\sigma_0 e^{-t/\tau}$.

Another model which can be simply constructed from the spring and dashpot is the parallel connection. This is called the Kelvin model and it is described by the conditions

$$\begin{aligned} \sigma &= \sigma_{el}+\sigma_{vis}, \quad \kappa=\kappa_{el}=\kappa_{vis}, \\ \sigma_{el} &= G_0\kappa_{el}, \quad \sigma_{vis}=\eta\dot{\kappa}_{vis}. \end{aligned} \tag{6.6}$$

This corresponds to the evolution equation for the strain κ

$$\dot{\kappa}+\frac{1}{\tau}\kappa=\frac{\sigma}{\eta}. \tag{6.7}$$

Hence, for the constant strain we obtain the elastic behaviour. Otherwise, the formal solution of the equation (6.7) has the form

$$\kappa=\kappa_0 e^{-t/\tau}+\frac{1}{\eta}\int\limits_0^t \sigma\left(t-s\right)e^{-s/\tau}ds. \tag{6.8}$$

For the constant stresses σ_0, this yields $\kappa\left(t=0\right)=\kappa_0$ and $\kappa\left(t\to\infty\right)=\sigma_0\tau/\eta=\sigma_0/G_0$. This is the behaviour of the simplest viscoelastic solid (compare Fig. 6.3).

Another combination of springs and dashpots is the parallel combination of a spring of elasticity G_1 and a Maxwell model with parameters G_2, η. This is the so-called standard rheological model. It is described by the relations

$$\begin{aligned} \sigma &= \sigma_1+\sigma_2, \quad \kappa=\kappa_1=\kappa_M, \\ \sigma_1 &= G_1\kappa_1, \quad \dot{\sigma}_2+\frac{\eta}{G_2}\sigma_2=G_2\dot{\kappa}_M. \end{aligned} \tag{6.9}$$

These relations lead to the rate equation

$$\dot{\sigma}+\frac{1}{\tau_2}\sigma=\eta\frac{\tau_1+\tau_2}{\tau_1\tau_2}\left(\dot{\kappa}+\frac{1}{\tau_1+\tau_2}\kappa\right), \quad \tau_1=\frac{\eta}{G_1}, \quad \tau_2=\frac{\eta}{G_2}. \tag{6.10}$$

This equation contains two relaxation times τ_1, τ_2.

In practical applications the above presented models are much too simple. In order to incorporate more relaxation times parallel arrangements of N Maxwell models are constructed. This is called a generalized Maxwell model. Similarly, by a series order of N Kelvin models one can construct a generalized Kelvin model. The analysis of properties of such models can be performed by the Laplace transform. We shall not present any further details concerning this subject and refer rather to the book of D. R. Bland[2].

★It is instructive to investigate the form of the second law of thermodynamics for rheological materials. In contrast to the full viscoelastic models consequences of the entropy inequality (4.20) can be easily found for those rheological cases. We consider Maxwell's model as an example. In the one dimensional linear case under considerations the balance of internal energy (4.11) and the entropy inequality reduce to the following form

$$\rho\frac{\partial\varepsilon}{\partial t}+\frac{\partial q}{\partial x}-\sigma\frac{\partial\kappa}{\partial t}=0,\quad \rho\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q}{T}\right)\geq 0, \tag{6.11}$$

where q is the x-component of the heat flux and η is the entropy density (not viscosity in this exercise!). Combination of these two relations yields

$$\rho\frac{\partial\psi}{\partial t}+\rho\eta\frac{\partial T}{\partial t}+\frac{1}{T}q\frac{\partial T}{\partial x}-\sigma\frac{\partial\kappa}{\partial t}\leq 0,\quad \psi=\varepsilon-T\eta. \tag{6.12}$$

We have to write additional constitutive relations for the Helmholtz free energy ψ and the heat flux q. As we are interested in isothermal processes, the contributions $\partial T/\partial t$ and $\partial T/\partial x$ vanish from the problem and we assume only that ψ is of the following form

$$\psi=\psi\left(\kappa,\sigma\right). \tag{6.13}$$

It is important to notice that both κ and σ are governed by differential equations and, for this reason, should be treated as two independent fields. Now the inequality (6.12) can be written in the form

$$\rho\left(\frac{\partial\psi}{\partial\kappa}-\sigma\right)\frac{\partial\kappa}{\partial t}+\rho\frac{\partial\psi}{\partial\sigma}\left(G_0\frac{\partial\kappa}{\partial t}-\frac{\sigma}{\tau}\right)\leq 0. \tag{6.14}$$

where we have made use of (6.4). This inequality must hold for arbitrary derivatives $\partial\kappa/\partial t$. Hence

$$\frac{\partial\psi}{\partial\kappa}+G_0\frac{\partial\psi}{\partial\sigma}=\frac{\sigma}{\rho},\quad \mathcal{D}=\frac{\rho}{\tau}\frac{\partial\psi}{\partial\sigma}\sigma\geq 0. \tag{6.15}$$

The first relation can be considered to be the differential equation for ψ. It is easy to find its solution using the method of characteristics. It has the form

$$\psi=\psi_{el}\left(\sigma-G_0\kappa\right)+\frac{\sigma^2}{2\rho G_0}, \tag{6.16}$$

[2] D. F. Bland; *The Theory of Linear Viscoelasticity*, Pergamon Press, Oxford, 1960.

where independent variables are now $\xi = \sigma - G_0\kappa$ and σ. The first part remains constant along the straight lines on (σ, κ)-planes: $\xi = \sigma - G_0\kappa = const$. This is an elastic part of the solution. The second part of (6.15), the dissipation inequality, yields $\tau G_0 \geq 0$, i.e. the viscosity must be positive.

The above calculations are typical for systems with an evolution equation describing additional fields in the model. In Maxwell's model the stress σ is such a field.

Let us mention in passing that the exploitation of the entropy inequality $(6.11)_2$ was so easy because the constitutive relation for stresses has in the Maxwell model the form of differential equation. As we see further, general three-dimensional models may not have this form. Usually they appear as functional dependencies. Then the evaluation of thermodynamical restrictions requires special techniques, in particular the so-called linear extensions of functionals (Fréchet derivatives). This problem has been extensively investigated in 60ties and it is sometimes called Coleman's method from the name of its founder. We shall not enter this subject in these notes.♣

★Laplace transforms

We recall here the basic definitions of the Laplace transform needed in the theory of viscoelasticity. It is well-known that many simple functions do not possess a Fourier transform as the defining integrals fail to converge for infinite limits. Therefore it is convenient to consider not the transform of the function $f(t)$ but rather of the function $f(t)\exp(-rt)$. Then the transform

$$\bar{f}(z) = \int_{-\infty}^{\infty} f(t)\exp(-zt)\,dt, \quad z = r + i\omega, \tag{6.17}$$

is called two-sided Laplace transform of $f(t)$. Obviously the inverse follows by multiplication of (6.17) by $\exp(rt)$ and integration

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(r + i\omega)\exp[(r + i\omega)t]\,d\omega. \tag{6.18}$$

If we are only interested in values of $f(t)$ for t positive, it is sufficient to transform the function $f(t)H(t)$ rather then $f(t)$. In this manner we avoid problems with divergence of the above integral for the lower limit of $\exp(-rt)$. Then the Laplace transform is the cut off of the two-sided Laplace transform

$$\bar{f}(z) = \int_{0}^{\infty} f(t)\exp(-zt)\,dt. \tag{6.19}$$

Its inverse has the form

$$f(t)H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(r + i\omega)\exp[(r + i\omega)t]\,d\omega = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \bar{f}(z)\exp(zt)\,dz. \tag{6.20}$$

The first integral clears the meaning of the second integral. If the integral (6.19) converges for some damping factor $r = \operatorname{Re} z$, it also converges for all larger damping factors. Consequently, the integral converges in some half-plane $\operatorname{Re} z > r_0$ in the complex $z-$plane. The line $r = r_0$ is called the abscissa of convergence. All values of r which lie to the right of r_0 are big enough for convergence. This is the very important property because the function $\bar{f}(z)$ has no singularity in the half-plane of convergence. However, it may have it on the abscissa of convergence. As a consequence the integrating $\bar{f}(z)$ around any closed contour yields zero. It means that $\bar{f}(z)$ is an analytic function of z in the half-plane of convergence. This property is the basis for the calculation of inverse integrals (6.20).

Frequently used in application to viscoelasticity is the Laplace transform of the convolution integral. It is as follows[3]

$$\int_0^\infty \left[\underbrace{\int_0^t f(s)\, g(t-s)\, ds}_{\text{convolution integral}} \right] \exp(-zt)\, dt = \bar{f}(z)\, \bar{g}(z)\,. \tag{6.21}$$

Example: we consider the Maxwell model described by the evolution equation (6.4). The Laplace transform of this equation yields

$$-\sigma_0 + z\bar{\sigma} + \frac{1}{\tau}\bar{\sigma} = G_0\left(-\kappa_0 + z\bar{\kappa}\right). \tag{6.22}$$

At the instant of the time $t = 0$ we have the elastic reaction $\sigma_0 = G_0\kappa_0$. This is the deformation of the spring in the Maxwell model before the dashpot had time to start moving. This yields the solution of the problem in the transformation $z-$plane

$$\bar{\sigma} = G_0 \frac{z}{z + 1/\tau}\bar{\kappa}. \tag{6.23}$$

Hence, the abscissa is crossing the point $r_0 = -1/\tau, \omega = 0$. The half-plane to the right of the vertical line $r = r_0$ contains no singular points of $\bar{\sigma}$. The inverse yields the solution (6.5)♣

6.3 Three-dimensional viscoelastic model

Motivated by the above rheological considerations we construct now a constitutive model for the three-dimensional viscoelastic continuum. We expect the stress tensor to depend on the history of strain. We can formally postulate the following relation

$$\sigma_{ij}(t) = \underset{s=0}{\overset{\infty}{\Psi_{ij}}} \left(e_{kl}(t-s), e_{kl}(t)\right), \tag{6.24}$$

[3] R. Bracewell; *The Fourier Transform and Its Applications,* 2nd Edition, McGraw–Hill, 1986 .

where $\overset{\infty}{\underset{s=0}{\Psi_{ij}}}$ denotes a linear tensor valued functional mapping the strain history $e_{ij}(t)$, $-\infty \leq t \leq \infty$, into the stress history $\sigma_{ij}(t)$. In addition, the functional possesses a parametric dependence upon the current value of strain $e_{ij}(t)$ which describes the instantaneous elastic response mentioned in the above presented properties of viscoelastic materials. We do not include a dependence on the spatial variable $\mathbf{x}$ as the material is assumed to be homogeneous. The above functional has an integral representation for continuous histories of strain. It follows from the Riesz representation Theorem[4]. Namely, it has the form of the Stieltjes integral

$$\sigma_{ij} = \int_0^\infty e_{kl}(t-s)\, dG_{ijkl}(s), \tag{6.25}$$

where each component of the fourth order tensor G_{ijkl} is of bounded variation. The components of this tensor are called relaxation functions. The above convolution of the constitutive law implies that it is invariant with respect to arbitrary shifts in the time scale. This invariance is related to the energy conservation but we shall not discuss it any further in these notes.

Integral constitutive relations of this type are called the Boltzmann integrals (the Boltzmann superposition principle).

The tensor G_{ijkl} possesses obvious symmetries following from the symmetry of the stress and strain tensors

$$G_{ijkl}(t) = G_{jikl}(t) = G_{ijlk}(t). \tag{6.26}$$

Assuming additionally the continuity of the first derivative of the tensor G_{ijkl} and $e_{ij}(t) = 0$ for $t < 0$ we can write the relation (6.25) in the form

$$\sigma_{ij} = G_{ijkl}(0)\, e_{kl}(t) + \int_0^t e_{kl}(t-s)\, \frac{dG_{ijkl}(s)}{ds} ds. \tag{6.27}$$

This form exposes the instantaneous elastic reaction of the material. Bearing the continuity of $e_{ij}(t)$ in mind, we can integrate (6.27) by parts. It follows

$$\sigma_{ij} = \int_0^t G_{ijkl}(t-s)\, \frac{de_{kl}(s)}{ds} ds. \tag{6.28}$$

Under the weaker assumption of a step discontinuity at $t = 0$ one can generalize the above relation[5] and obtain the following relation

$$\sigma_{ij}(t) = G_{ijkl}(t)\, e_{kl}(0) + \int_0^t G_{ijkl}(t-s)\, \frac{de_{kl}(s)}{ds} ds. \tag{6.29}$$

[4]e.g. N. Dunford, J. T. Schwartz; *Linear Operators. Part I: General Theory*, Interscience Publ., New York, 1958

[5]M. E. Gurtin, E. Sternberg; On the linear theory of viscoelasticity, *Arch. Rat. Mech. Anal.*, **11**, 291-356, 1962.

In spite of the discontinuity at $t = 0$ the lower limit can be shifted from 0 to $-\infty$ due to the above mentioned shift invariance provided $e_{ij}(t \to -\infty) \to 0$. Integration by parts yields then

$$\sigma_{ij}(t) = \int_{-\infty}^{t} G_{ijkl}(t-s) \frac{de_{kl}(s)}{ds} ds. \tag{6.30}$$

An alternative to the above constitutive relation is the inverse

$$e_{ij}(t) = \int_{-\infty}^{t} J_{ijkl}(t-s) \frac{d\sigma_{kl}(s)}{ds} ds, \tag{6.31}$$

where

$$J_{ijkl}(t) = J_{jikl}(t) = J_{ijlk}(t), \quad J_{ijkl}(t) = 0 \quad \text{for} \quad -\infty < t < 0. \tag{6.32}$$

These functions are assumed to possess continuous first derivatives and they are called creep functions.

We limit all further considerations to isotropic materials. The most general isotropic representation of the fourth order tensor contains two independent parameters (compare (5.21)). It is convenient to write it in the form

$$G_{ijkl}(t) = \frac{1}{3}\left[G_2(t) - G_1(t)\right]\delta_{ij}\delta_{kl} + \frac{1}{2}\left[G_1(t)\right]\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{6.33}$$

where $G_1(t)$ and $G_2(t)$ are independent relaxation functions. If we separate spherical and deviatoric parts of stress and strain tensors

$$\begin{aligned} \sigma_{ij} &= \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma_{ij}^D, \quad \sigma_{kk}^D = 0, \\ e_{ij} &= \frac{1}{3}e_{kk}\delta_{ij} + e_{ij}^D, \quad e_{kk}^D = 0, \end{aligned} \tag{6.34}$$

then the relation (6.30) splits in the following way

$$\begin{aligned} \sigma_{ij}^D(t) &= \int_{-\infty}^{t} G_1(t-s) \frac{de_{ij}^D(s)}{ds} ds, \\ \sigma_{kk} &= \int_{-\infty}^{t} G_2(t-s) \frac{de_{kk}(s)}{ds} ds. \end{aligned} \tag{6.35}$$

Similarly, the inverse relations obtain the form

$$\begin{aligned} e_{ij}^D(t) &= \int_{-\infty}^{t} J_1(t-s) \frac{d\sigma_{ij}^D(s)}{ds} ds, \\ e_{kk} &= \int_{-\infty}^{t} J_2(t-s) \frac{d\sigma_{kk}(s)}{ds} ds. \end{aligned} \tag{6.36}$$

where $J_1(t), J_2(t)$ are two independent isotropic creep functions. Obviously, the functions G_1, J_1 are appropriate for shear processes and G_2, J_2 for dilatation processes.

The relaxation and creep functions are, of course, related to each other. The easiest way to find this relation is to perform the Laplace transformation on relations (6.35) and (6.36)[6]. We obtain

$$\begin{aligned} \bar{\sigma}_{ij}^D &= z\bar{G}_1\bar{e}_{ij}^D, \quad \bar{\sigma}_{kk} = z\bar{G}_2\bar{e}_{kk}, \\ \bar{e}_{ij}^D &= z\bar{J}_1\bar{\sigma}_{ij}^D, \quad \bar{e}_{kk} = z\bar{J}_2\bar{\sigma}_{kk}. \end{aligned} \tag{6.37}$$

These relations imply

$$J_\alpha = \left(z^2 G_\alpha\right)^{-1}, \quad \alpha = 1, 2. \tag{6.38}$$

The isotropic relations for elastic materials (5.21), (5.26) can be written in the form

$$\begin{aligned} \sigma_{kk} &= 3Ke_{kk}, \quad \sigma_{ij}^D = 2\mu e_{ij}^D, \\ e_{kk} &= \frac{1}{3K}\sigma_{kk}, \quad e_{ij}^D = \frac{1}{2\mu}\sigma_{ij}^D. \end{aligned} \tag{6.39}$$

They would suggest that $J_\alpha(t) = [G_\alpha(t)]^{-1}$. Relations (6.38) show that this is not correct. However, it can be shown using properties of the Laplace transform that limit values indeed satisfy such relations

$$\lim_{t\to 0} J_\alpha(t) = \lim_{t\to 0}[G_\alpha(t)]^{-1}, \quad \lim_{t\to\infty} J_\alpha(t) = \lim_{t\to\infty}[G_\alpha(t)]^{-1}. \tag{6.40}$$

Incidentally, to be consistent with the linear elasticity of isotropic materials relations (6.39) suggest the following notation for relaxation and creep functions

$$\mu(t) = G_1(t)/2, \quad K(t) = G_2(t)/3. \tag{6.41}$$

The above presented results suggest a useful short-hand notation for Stieltjes convolution integrals which has been introduced by Gurtin and Sternberg in the earlier quoted paper. Namely, we write instead of (6.25) the following relation

$$\sigma_{ij} = e_{kl} * dG_{ijkl}, \tag{6.42}$$

i.e. we write for two arbitrary functions f, g

$$f * dg = \int_{-\infty}^{t} f(t-s)\, dg(s), \quad g(t\to -\infty) = 0, \tag{6.43}$$

[6] It should be mentioned that, instead of the classical Laplace transform (6.19) it may be more convenient to use a modification which is called Laplace-Carson transform. It is defined by the relation

$$\bar{f}(z) = z\int_0^\infty f(t)\exp(-zt)\, dt.$$

It is applied in the presentation of viscoelasticity by Lemaitre and Chaboche [9]. We follow here rather the older approach of Christensen [3] and Pipkin [15].

where $f(t)$ is continuous for $0 \leq t \leq \infty$. If $f(t) = 0$ for $t < 0$ then one can show the commutativity relation

$$f * dg = g * df. \tag{6.44}$$

Consequently,

$$\sigma_{ij} = G_{ijkl} * de_{kl},$$

which is the counterpart of (6.30).

The above notation leads to the following useful identities

$$\begin{aligned} f * d(g * dh) &= (f * dg) * dh = f * dg * dh, \\ f * d(g + h) &= f * dg + f * dh. \end{aligned} \tag{6.45}$$

We shall not enhance the subject of a general theory of materials with memory. This can be found in classical monographs on the subject[7].

We proceed to investigate an example which helps to clear the distinction between the viscoelastic fluid and the viscoelastic solid. We consider the case of the simple shear defined by the relation (2.95). The amount of shear $\kappa = \kappa_0 H(t)$ is assumed to be the step function in time and it yields the stretching (2.65) in the linear theory to be identical with the time derivative of strain which is given by the rate of shearing $\dot{\kappa} = \kappa_0 \delta(t)$

$$D_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{de_{ij}}{dt}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{\dot{\kappa}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{6.46}$$

Hence,

$$\sigma_{12}^D = \frac{\kappa_0}{2}\int_0^t G_1(t-s)\,\delta(s)\,ds = \frac{\kappa_0}{2}G_1(t), \quad G_1(t) = 0 \text{ for } t < 0. \tag{6.47}$$

It follows from the definition of a linear isotropic viscoelastic solid that the following condition must hold

$$\lim_{t\to\infty} G_1(t) \to \text{nonzero constant} \quad \Rightarrow \quad \text{solids}. \tag{6.48}$$

On the other hand, for viscoelastic (non-Newtonian) fluid

$$\lim_{t\to\infty} G_1(t) = 0 \quad \Rightarrow \quad \text{fluids}. \tag{6.49}$$

The latter condition is necessary but not sufficient. In addition, the relaxation function G_1 must fulfil a condition for the steady state flow. Then the viscoelastic fluid at large values of time, where the steady state will be achieved, must behave like a viscous fluid of the viscosity η. Hence, the relaxation time must have the property

$$\eta = \frac{1}{2}\int_0^\infty G_1(s)\,ds \quad \text{(fluids)}. \tag{6.50}$$

[7] C. Truesdell, W. Noll; *The Non-Linear Field Theories of Mechanics,* Encyclopedia of Physics, vol. III/3, S. Flügge (ed.), Springer, Berlin, 1965,
C. Truesdell; *A First Course in Rational Continuum Mechanics,* The Johns Hopkins University Press, Baltimore, 1972 (Chapter XIII).

6.4 Differential constitutive relations

We have seen on examples the simple rheological models that constitutive relations of these models may have the form of evolution equations (rate-type constitutive relations). This can be taken over to the description of a three-dimensional continuum. Let us consider the following differential operator

$$p_0\sigma_{ij}^D + p_1\frac{d\sigma_{ij}^D}{dt} + p_2\frac{d^2\sigma_{ij}^D}{dt^2} + ... = q_0 e_{ij}^D + q_1\frac{de_{ij}^D}{dt} + q_2\frac{d^2 e_{ij}^D}{dt^2} + ..., \tag{6.51}$$

or, in the compact form,

$$P(D)\,\sigma_{ij}^D = Q(D)\,e_{ij}^D, \quad P(D) = \sum_{k=0}^{N} p_k D^k, \quad Q(D) = \sum_{k=0}^{N} q_k D^k, \quad D^k = \frac{d^k}{dt^k}. \tag{6.52}$$

This type of operators appear, for instance, for generalized Maxwell and Kelvin models mentioned before. In order to see the significance of such models for viscoelasticity we take the Laplace transform of (6.52)

$$\bar{P}(z)\,\bar{\sigma}_{ij}^D - \frac{1}{z}\sum_{k=1}^{N} p_k \sum_{r=1}^{N} z^r \left[\frac{d^{k-r}\sigma_{ij}^D}{dt^{k-r}}(t=0)\right] = \tag{6.53}$$

$$= \bar{Q}(z)\,\bar{e}_{ij}^D - \frac{1}{z}\sum_{k=1}^{N} q_k \sum_{r=1}^{N} z^r \left[\frac{d^{k-r}e_{ij}^D}{dt^{k-r}}(t=0)\right],$$

where

$$\bar{P}(z) = \sum_{k=0}^{N} p_k z^k, \quad \bar{Q}(z) = \sum_{k=0}^{N} q_k z^k. \tag{6.54}$$

These relations follow easily by integration by parts. If we compare these relations with (6.37) then they specify the relaxation function by the formula

$$z\bar{G}_1 = \bar{Q}(z)/\bar{P}(z), \tag{6.55}$$

provided the initial conditions are constraint by the relations

$$\sum_{r=k}^{N} p_r \left[\frac{d^{k-r}\sigma_{ij}^D}{dt^{k-r}}(t=0)\right] = \sum_{r=k}^{N} q_r \left[\frac{d^{k-r}e_{ij}^D}{dt^{k-r}}(t=0)\right], \quad k = 1, 2, ..., N. \tag{6.56}$$

Consequently the relaxation function G_1 is specified in terms of $2(N+1)$ parameters $p_0, ..., p_N$, $q_0, ..., q_N$ which are related to a sequence of relaxation times. Similar relations can be introduced for the dilatational part of stress and strain

$$L(D)\,\sigma_{kk}(t) = M(D)\,e_{kk}(t), \tag{6.57}$$

and these are again specified by a finite sequence of parameters.

The question if such models can be indeed physically plausible is not simple. Some models of this art, introduced for non-Newtonian fluids (the so-called Rivlin-Ericksen fluids) show that there appear problems of stability[8] and convergence[9] for dynamic processes and these models seem to work well for steady state flows. There are claims that non-Newtonian fluids require always the spectrum of infinitely many relaxation times and, consequently, such polynomial models as (6.52) are physically useless.

For viscoelastic solids a simple differential model is often based on the simplest differential equation describing the evolution of stresses. This is analogous to the Maxwell model (6.4) constructed within the classical rheology. It is based on the equation for stresses

$$\tau \frac{d\sigma_{ij}^{D}}{dt} + \sigma_{ij}^{D} = 2\eta \frac{de_{ij}^{D}}{dt}, \tag{6.58}$$

where τ is the relaxation time and η is the viscosity. It is the so-called standard linear viscoelastic solid [21]

6.5 Steady state processes and elastic-viscoelastic correspondence principle

Now we investigate a class of problems which appear in spectral analysis of waves in which we seek solutions in the form of monochromatic waves. It is then important to know the behaviour of the constitutive relation (6.35) if the time dependence of the strain is harmonic. We denote representatives of deviatoric and spherical strains and stresses by $\tilde{e}$ and $\tilde{\sigma}$, respectively, and assume

$$\tilde{e} = \tilde{e}_0 e^{i\omega t}, \tag{6.59}$$

where ω is the frequency and $\tilde{e}_0$ an amplitude. We write the typical contribution to (6.35) in the following form

$$\tilde{\sigma} = \int_{-\infty}^{t} G_\alpha (t-s) \frac{d\tilde{e}}{ds} ds, \tag{6.60}$$

where $\alpha = 1$ or 2 in dependence of the choice of $\tilde{\sigma}$. It is convenient to split the relaxation modulus into two parts: G_α^0 and G_α^1, where the first part is equal to the limit of the relaxation modulus for $t \to \infty$ and, consequently, $G_\alpha^1 (t \to \infty) = 0$. Both parts are zero for the time smaller than 0. Hence

$$\tilde{\sigma} = G_\alpha^0 \tilde{e}_0 e^{i\omega t} + i\omega \tilde{e}_0 \int_{-\infty}^{t} G_\alpha^1 (t-s) e^{i\omega s} ds. \tag{6.61}$$

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[8] D. Joseph; Instability of the rest state of fluids of arbitrary grade greater than one, *Arch. Rat. Mech. Anal.,* **75,** 251-256, 1981.

[9] R. S. Rivlin, K. Wilmanski; The passage from memory functionals to Rivlin-Ericksen constitutive equations, ZAMP, **38**, 624-629, 1987
as well as K. Wilmanski [21]

Changing the variables $\eta = t - s$ we obtain

$$\tilde{\sigma} = \left[G_\alpha^0 + \omega \int\limits_0^\infty G_\alpha^1(\eta) \sin \omega\eta d\eta + i\omega \int\limits_0^\infty G_\alpha^1(\eta) \cos \omega\eta d\eta \right] \tilde{e}_0 e^{i\omega t}. \tag{6.62}$$

Consequently, the stress $\tilde{\sigma}$ is given by the complex modulus G_α^*

$$\begin{aligned} \tilde{\sigma} &= G_\alpha^* \tilde{e}_0 e^{i\omega t}, \\ \operatorname{Re} G_\alpha^* &= G_\alpha^0 + \omega \int\limits_0^\infty G_\alpha^1(\eta) \sin \omega\eta d\eta, \\ \operatorname{Im} G_\alpha^* &= \omega \int\limits_0^\infty G_\alpha^1(\eta) \cos \omega\eta d\eta. \end{aligned} \tag{6.63}$$

The real part is called the storage modulus and the imaginary part the loss modulus.

The integration by parts yields the frequency limit behaviour of the above moduli. We have

$$\begin{aligned} \operatorname{Re} G_\alpha^*(\omega) &= G_\alpha^0 + G_\alpha^1(0) + \int\limits_0^\infty \frac{dG_\alpha^1(\eta)}{d\eta} \cos \omega\eta d\eta, \\ \operatorname{Im} G_\alpha^*(\omega) &= -\int\limits_0^\infty \frac{dG_\alpha^1(\eta)}{d\eta} \sin \omega\eta d\eta. \end{aligned} \tag{6.64}$$

Hence

$$\operatorname{Re} G_\alpha^*(\omega = 0) = G_\alpha^0 = G_\alpha(t)|_{t\to\infty}, \quad \operatorname{Im} G_\alpha^*(\omega = 0) = 0. \tag{6.65}$$

For the other limit, changing the variables $\omega\eta = \tau$ we easily obtain

$$\operatorname{Re} G_\alpha^*(\omega \to \infty) = G_\alpha^0 + G_\alpha^1(t = 0) = G_\alpha(t)|_{t\to 0}, \quad \operatorname{Im} G_\alpha^*(\omega \to \infty) = 0. \tag{6.66}$$

These relations show that for very high frequency the viscoelastic solid behaves as an elastic solid. The same concerns very low frequencies and this differs solids from viscous fluids.

The above considerations determine, obviously, the Fourier transforms of the constitutive relations. With the definitions for an arbitrary function f

$$\bar{f}(\omega) = \int\limits_{-\infty}^\infty f(t) e^{-i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int\limits_{-\infty}^\infty \bar{f}(\omega) e^{i\omega t} d\omega, \tag{6.67}$$

we have

$$\begin{aligned} \bar{\sigma}_{ij}^D(\omega) &= G_1^*(i\omega) \bar{e}_{ij}^D(\omega), \\ \bar{\sigma}_{kk} &= G_2^*(i\omega) \bar{e}_{kk}(\omega). \end{aligned} \tag{6.68}$$

The similarity of these relations to the elastic relations (6.39) is called the elastic-viscoelastic correspondence principle . It has been first observed by W. T. Read in 1950[10]. With respect to the important practical aspects of this principle we present it as well for the Laplace transform.

Let us consider the full set of governing equations describing the boundary value problem of a linear isotropic viscoelastic material. We have

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{6.69}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0, \quad \text{for } \mathbf{x} \in \mathcal{B}_t \tag{6.70}$$

$$\sigma_{ij}^D(t) = 2\int_{-\infty}^{t} \mu(t-s)\frac{\partial e_{ij}^D}{\partial s} ds, \quad \sigma_{kk} = 3\int_{-\infty}^{t} K(t-s)\frac{\partial e_{kk}}{\partial s} ds, \tag{6.71}$$

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma_{ij}^D, \quad e_{ij} = \frac{1}{3}e_{kk}\delta_{ij} + e_{ij}^D, \tag{6.72}$$

$$\sigma_{ij}(t) n_j = t_i^n \quad \text{for } \mathbf{x} \in \partial\mathcal{B}_t^\sigma, \quad u_i(t) = u_i^n \quad \text{for } \mathbf{x} \in \partial\mathcal{B}_t^u, \tag{6.73}$$

$$u_i(t) = e_{ij}(t) = \sigma_{ij}(t) = 0 \quad \text{for } -\infty < t < 0, \tag{6.74}$$

where we have used the notation (6.41) for the relaxation moduli.

Laplace transform of these equations has the form

$$\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{6.75}$$

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \rho \bar{b}_i = 0, \quad \text{for } \mathbf{x} \in \mathcal{B}_t \tag{6.76}$$

$$\bar{\sigma}_{ij}^D(t) = 2z\bar{\mu}(z)\bar{e}_{ij}^D \quad \bar{\sigma}_{kk} = 3z\bar{K}(z)\bar{e}_{kk}, \tag{6.77}$$

$$\bar{\sigma}_{ij} = \frac{1}{3}\bar{\sigma}_{kk}\delta_{ij} + \bar{\sigma}_{ij}^D, \quad \bar{e}_{ij} = \frac{1}{3}\bar{e}_{kk}\delta_{ij} + \bar{e}_{ij}^D, \tag{6.78}$$

$$\bar{\sigma}_{ij}(z) n_j = \bar{t}_i^n \quad \text{for } \mathbf{x} \in \partial\mathcal{B}_t^\sigma, \quad \bar{u}_i(z) = \bar{u}_i^n \quad \text{for } \mathbf{x} \in \partial\mathcal{B}_t^u, \tag{6.79}$$

where z is the transformation variable and bars denote Laplace transforms.

Obviously the set (6.75)-(6.79) has a form identical with equations of linear elasticity except of complex moduli $z\bar{K}(z), z\bar{\mu}(z)$ which replace real moduli K, μ of the elasticity theory. This correspondence reveals the possibility of converting numerous static solutions of elasticity into quasi-static solutions of viscoelasticity. The main problem is now the inversion of the Laplace transform.

One more general remark is appropriate for quasi-static problems of viscoelastic materials. Before we formulate it, let us collect the material functions corresponding to the material constants of elasticity. We have already the relations (6.41), i.e. $\mu(t) = G_1(t)/2$,

[10] W. T. Read; Stress analysis for compressible viscoelastic materials, *J, Appl. Phys.*, 21, 671-674, 1950.

$K(t) = G_2(t)/3$ and they yield in the transformed form (compare (5.29))

$$\begin{aligned} \bar{\lambda}(z) &= \bar{K}(z) - \frac{2}{3}\bar{\mu}(z) = \frac{1}{3}\left(\bar{G}_2(z) - \bar{G}_1(z)\right), \\ \bar{E}(z) &= = \frac{3\bar{\mu}(z)\bar{K}(z)}{\bar{K}(z) + \frac{1}{3}\bar{\mu}(z)} = \frac{3\bar{G}_1(z)\bar{G}_2(z)}{2\bar{G}_2(z) + \bar{G}_1(z)}, \\ \bar{\nu}(z) &= \frac{\bar{\lambda}(z)}{2\left(\bar{\lambda}(z) + \bar{\mu}(z)\right)} = \frac{\bar{G}_2(z) - \bar{G}_1(z)}{2\bar{G}_2(z) + \bar{G}_1(z)}. \end{aligned} \tag{6.80}$$

The question arises if we can apply the method of separation of variables in a quasi-static problems of linear viscoelasticity. It means that, for instance, the displacement should have the form

$$u_i(\mathbf{x}, t) = \check{u}_i(\mathbf{x})\, u(t), \tag{6.81}$$

where $u(t)$ is a common function for all components of the displacement. Hence, if we neglect the acceleration (quasi-static problem!) and body forces, the field equations for displacements have the form

$$\frac{\partial^2 \check{u}_i}{\partial x_k \partial x_k} \int\limits_{-\infty}^{t} \mu(t-s)\frac{du(s)}{ds}ds + \frac{\partial^2 \check{u}_k}{\partial x_i \partial x_k} \int\limits_{-\infty}^{t} \left[\lambda(t-s) + \mu(t-s)\right]\frac{du(s)}{ds}ds = 0. \tag{6.82}$$

Obviously, the time contributions must be eliminated from this equation and this yields

$$\lambda(t) + \mu(t) = \beta\mu(t), \quad \beta = const. \tag{6.83}$$

This yields immediately that Poisson's ratio ν must be independent of time. Bearing the last relation (6.80) in mind, we obtain the restriction

$$\frac{G_2(t)}{G_1(t)} = \frac{1+\nu}{1-2\nu} = const. \tag{6.84}$$

In the similar manner we can prove that the ratio of creep functions is a constant

$$\frac{J_2(t)}{J_1(t)} = \frac{1-2\nu}{1+\nu} = const. \tag{6.85}$$

These two conditions are necessary for the applicability of the method of separation of variables.

★We demonstrate the application of the correspondence principle on a simple example. We consider the axial symmetric problem of the cylinder under the given radial loading on both lateral surfaces. The outer surface is pressurized by an elastic case[11]. The Laplace transform of the radial displacement $\bar{u}_r(r, z)$ must fulfil the equation (see: (5.44))

$$\frac{\partial^2 \bar{u}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_r}{\partial r} - \frac{\bar{u}}{r^2} = 0, \tag{6.86}$$

[11] R. M. Christensen, R. N. Schreiner; Response to pressurization of a viscoelastic cylinder with an eroding internal boundary, *AIAA J.*, **3**, 1451, 1965.

with the solution

$$\bar{u}_r = \bar{C}(z)\, r + \frac{\bar{D}(z)}{r}. \tag{6.87}$$

In order to apply the boundary conditions, we have to write stress-strain relations in the transformed form. We have by the correspondence principle

$$\begin{aligned}
\bar{\sigma}_{rr} &= 2z\bar{\mu}\left(\frac{\partial \bar{u}_r}{\partial r} + \frac{z\bar{\nu}}{1-2z\bar{\nu}}\bar{e}\right), \quad \bar{e} = \frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r}, \\
\bar{\sigma}_{\theta\theta} &= 2z\bar{\mu}\left(\frac{\bar{u}_r}{r} + \frac{z\bar{\nu}}{1-2z\bar{\nu}}\bar{e}\right), \\
\bar{\sigma}_{zz} &= \frac{2z^2\bar{\nu}\bar{\mu}}{1-2z\bar{\nu}}.
\end{aligned} \tag{6.88}$$

The boundary conditions for the pressurized cylinder have the form

$$\begin{aligned}
\sigma_{rr}(r=a,t) &= -p(t), \quad \sigma_{rr}(r=b,t) = -q(t), \\
u(r=b,t) &= q(t)\left[\frac{b^2\left(1-\nu_c^2\right)}{E_c h}\right],
\end{aligned} \tag{6.89}$$

where E_c, ν_c are elastic properties of the case, h is its thickness, b the outer radius and a the inner radius of the viscoelastic cylinder.

Easy calculations yield the following form of the transformed stresses

$$\bar{\sigma}_{rr} = C_1 - \frac{b^2}{r^2}C_2, \quad \bar{\sigma}_{\theta\theta} = C_1 + \frac{b^2}{r^2}C_2, \quad \bar{\sigma}_{zz} = 2z\bar{\nu}C_1, \tag{6.90}$$

where

$$\begin{aligned}
C_1 &= \frac{-\bar{p}(S - z\bar{\mu})}{(S - z\bar{\mu}) + (b^2/a^2)\left[S(1-2z\bar{\nu}) + z\bar{\mu}\right]}, \\
C_2 &= \frac{\bar{p}\left[S(1-2z\bar{\nu}) + z\bar{\mu}\right]}{(S - z\bar{\mu}) + (b^2/a^2)\left[S(1-2z\bar{\nu}) + z\bar{\mu}\right]}, \\
S &= \frac{E_c h}{2b\left(1-\nu_c^2\right)}.
\end{aligned} \tag{6.91}$$

Inverse transformations performed in the quoted work of Christensen and Schreiner were made under the assumption that the modulus $\bar{\mu}$ is a polynomial in z and the Poisson number ν is constant (see above). The polynomial form of $\bar{\mu}$ follows from the following considerations suggested by rheological models. It is assumed that the modulus $\mu(t)$ has the form

$$\mu(t) = G_0 + \sum_{n=1}^{N} G_n e^{-t/\tau_n}, \tag{6.92}$$

where $G_0, G_n, \tau_n, n = 1, ..., N$ are constants. Obviously, τ_n have the interpretation of relaxation times. The values of these parameters are obtained by fitting to experimental data (e.g. for harmonic torsional loading experiments of cylindrical samples). Laplace transform of (6.92) yields

$$z\bar{\mu}(z) = \frac{A(z)}{\prod_{n=1}^{N}(z + 1/\tau_n)}, \tag{6.93}$$

where $A(z)$ is an Nth grade polynomial in z determined by coefficients G_n.

Now the inverse transformation can be made by the technique of integration of the function $\bar{f}(z) = P(z)/Q(z)$ whose denominator yields simple pole singularities (i.e. zeros of $Q(z)$) in the complex domain. We shall not quote rather complex final results.♣

Chapter 7

Plasticity

7.1 Introduction

Various plasticity models of mechanics are developed to describe a class of permanent deformations. These deformations are generated during loading processes and remain after the removal of the load. In this Chapter we present a few aspects of the classical linear plasticity. This model is based on the assumption on the additive separation of elastic and plastic deformation increments. In nonlinear models it is the deformation gradient $\mathbf{F}$ in which these permanent deformations are separated

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{7.1}$$

where the plastic deformation is described by $\mathbf{F}^p$ and the elastic part is $\mathbf{F}^e$. Only the product of these two objects is indeed the gradient of the function of motion $\mathbf{f}$. Neither $\mathbf{F}^e$ nor $\mathbf{F}^p$ can be written in such a form – they are not integrable. In spite of this problem, material vectors transformed by $\mathbf{F}^p$ form a vector space for each material point $\mathbf{X} \in \mathcal{B}_0$ and these spaces are sometimes called intermediate configurations. We shall not elaborate these issues of nonlinear models[1]. However, it should be mentioned that the assumption (7.1) indicates the additive separation of increments of deformation in the linear model. Namely, the time derivative of the deformation gradient has, obviously, the form

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p, \tag{7.2}$$

which yields for small strains

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}^e + \dot{\mathbf{e}}^p, \tag{7.3}$$

where $\dot{\mathbf{e}}^e$ is the elastic strain rate and $\dot{\mathbf{e}}^p$ is the plastic strain rate. In some older models it is even assumed that this additive decomposition concerns strains themselves: $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$ which is obviously much stronger than (7.3) and yields certain general doubts.

[1] e.g. see: Albrecht Bertram; *Elasticity and Plasticity of Large Deformations*, Springer Berlin, 2008.

The aim of the elastoplastic models in the displacement formulation is to find the displacement vector **u** whose gradient defines the strain **e** – as in the case of linear elasticity, and the plastic strain $\mathbf{e}^p$ which becomes an additional field.

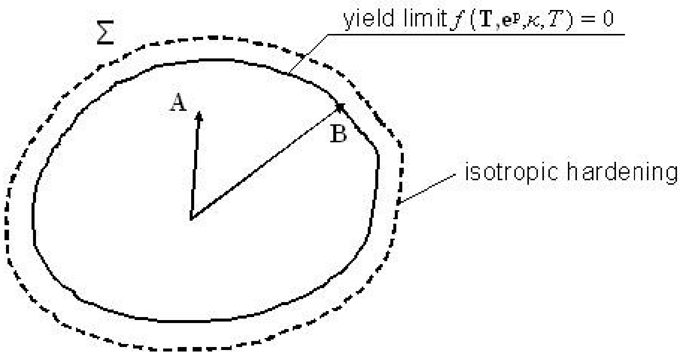

Fig. 7.1: States of material in the stress space Σ. A **– the elastic state,** B **– the plastic state.**

The most fundamental characteristic feature of classical plasticity is the distinction of an elastic domain in the space of stresses $\Sigma = \{\mathbf{T}\}, \mathbf{T} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. All paths of stresses which lie in the elastic domain produce solely elastic deformations, i.e. after inverting the process of loading the material returns to its original state. This is schematically shown in Fig. 7.1.

The elastic domain lies within the bounding yield surface also called the yield limit or the yield locus. Stress states which lie beyond this limit are attainable only by moving the whole yield surface. Such processes are called hardening. In Fig. 7.1. we demonstrate the so-called isotropic hardening. We return to this notion in the sequel. Increments of plastic strains are described by stresses whose direction points in the outward direction of the yield surface. This is related to the so-called Drucker[2] stability postulate which we present further.

The above described way of construction of plasticity is sometimes called stress space formulation and it was motivated by properties of metals. There is an alternative which has grown up from soil mechanics[3]. Such materials as rocks, soils and concrete reveal softening behaviour which violates Drucker's postulate. In order to avoid this problem, the so-called strain space formulation[4] was developed in which, instead of Drucker's postulate one applies the Ilyushyn[5] postulate. The detailed discussion of these stability problems can be found, for instance, in the book of Wu [24].

[2] D. C. Drucker; A more fundamental approach to plastic stress-strain relations, in: *Proc. 1st Nat. Congress Appl. Mech., ASME,* 487, 1951.

[3] This formulation has been initiated by the work: Z. Mróz; Non-associated flow laws in plasticity, *Journ. de Mecanique*, **2**, 21-42, 1963.

[4] J. Casey, P. M. Naghdi; On the nonequivalence of the stress space and strain space formulations of plasticity theory, *J. Appl. Mech.*, **50**, 350, 1983.

[5] A. A. Ilyushin; On the postulate of plasticity, PMM, **25**, 503, 1961.

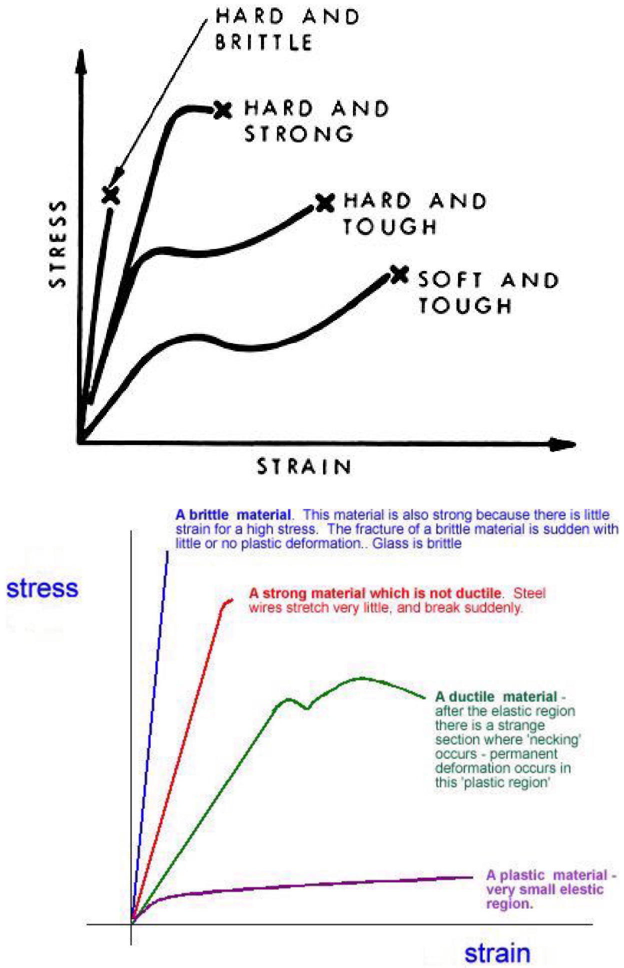

Fig. 7.2: Schematic plastic behaviour of various materials

In Fig. 7.2. we show schematically strain-stress curves for different types of materials. The steepest curve in both pictures correspond to the so-called brittle materials which practically do not reveal any plastic deformations prior to failure. Their deformations under high loading are small and they absorb only a little energy before breaking. It should be underlined that many materials may behave this way in low temperatures whereas their properties are very different in high temperatures. This transition explains mysterious catastrophes of Liberty ships in 40th of the XXth century.

The curves for ductile materials in Fig. 7.2. correspond to materials for which the classical plasticity was developed. They possess relative large irreversible deformations and by failure absorb a large amount of energy. Therefore they are called tough.

Many damages and accidents of cargo vessels were occurred, and especially for Liberty Ships. The vast majority of the sea accidents were related to brittle fracture. By 1st of April 1946, 1441 cases of damage had been reported for 970 cargo vessels, 1031 of which were to Liberty Ships. Total numbers of 4720 damages were reported. Seven ships were broken in two, e.g. "Schenectady".

7.2 Plasticity of ductile materials

We proceed to specify the yield surface in the stress space. As mentioned above this stress formulation was primarily motivated by plastic deformations of metals. In such materials the pressure p has practically no influence on plastic strains which means that the yield surface should be described only by the stress deviator: $\sigma_{ij}^D = \sigma_{ij} + p\delta_{ij},\ p = -\frac{1}{3}\sigma_{kk}$. The eigenvalues of the stress deviator follow from the eigenvalue problem

$$\left(\sigma_{ij}^D - s\delta_{ij}\right) n_j = 0, \tag{7.4}$$

and the solutions must satisfy the condition

$$I_s = s^{(1)} + s^{(2)} + s^{(3)} = 0. \tag{7.5}$$

The eigenvalues $s^{(\alpha)}$ and the eigenvalues $\sigma^{(\alpha)}$ of the full stress tensor σ_{ij} are, of course, connected by the relation

$$\sigma^{(\alpha)} = s^{(\alpha)} - p, \quad \alpha = 1, 2, 3. \tag{7.6}$$

For the purpose of formulation of various hypotheses for the yield surface, it is convenient to calculate invariants of the stress deviator and the maximum shear stresses. As presented in Subsection 3.2.4 (compare the three-dimensional Mohr circles), the extremum values of shear stress are given by the differences of three principal values of the stress tensor (radii of Mohr's circles)

$$\tau^{(1)} = \frac{\sigma^{(2)} - \sigma^{(3)}}{2}, \quad \tau^{(2)} = \frac{\sigma^{(1)} - \sigma^{(3)}}{2}, \quad \tau^{(3)} = \frac{\sigma^{(1)} - \sigma^{(2)}}{2}. \tag{7.7}$$

Hence, we have as well

$$\tau^{(1)} = \frac{s^{(2)} - s^{(3)}}{2}, \quad \tau^{(2)} = \frac{s^{(1)} - s^{(3)}}{2}, \quad \tau^{(3)} = \frac{s^{(1)} - s^{(2)}}{2}. \tag{7.8}$$

Further we use the sum of squares of these quantities. In terms of invariants of the stress tensor and of the stress deviator it has the form

$$\sum_{\alpha=1}^{3}\left(\tau^{(\alpha)}\right)^2=\frac{3}{2}\left[\frac{1}{3}I_\sigma^2-II_\sigma\right]=-\frac{3}{2}II_s=\frac{1}{2}\sigma_{eq}^2,\quad \sigma_{eq}=\sqrt{\frac{3}{2}\sigma_{ij}^D\sigma_{ij}^D},\tag{7.9}$$

where $\sigma_{eq}=\sigma^{(1)}$ in the uniaxial tension/compression for which $\sigma^{(2)}=\sigma^{(3)}=0$. It is clear that the second invariant of the deviatoric stresses must be negative. These relations follow from the definitions of the invariants

$$\begin{aligned} I_\sigma &= \sigma_{kk}=-3p,\quad II_\sigma=\frac{1}{2}\left(I_\sigma^2-\sigma_{ij}\sigma_{ij}\right),\quad III_\sigma=\det\left(\sigma_{ij}\right), \\ I_s &= 0,\quad II_s=-\frac{1}{2}\sigma_{ij}^D\sigma_{ij}^D=-J_2=-\frac{\sigma_{eq}^2}{3},\quad III_s=J_3=\det\left(\sigma_{ij}^D\right). \end{aligned}\tag{7.10}$$

The quantity $\sigma_{eq}=\sqrt{3J_2}$ is called the equivalent (effective) stress.

Now, we are in the position to define the elastic domain in the space of stresses. It is convenient to represent it by a domain in the three-dimensional space of principal stresses. In this space we choose the principal stresses $\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}$ as coordinates. The assumption that the pressure does not influence plastic strains means that yield surfaces in this space must be cylindrical surfaces with generatrix perpendicular to surfaces $s^{(1)}+s^{(2)}+s^{(3)}=0$, i.e. $\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}+3p=0$. The axis of those cylinders is, certainly, the straight line $\sigma^{(1)}=\sigma^{(2)}=\sigma^{(3)}$. This line is called the hydrostatic axis. In general, we can write the equation of the yield surface in the form

$$f\left(J_2,J_3,e_{ij}^p,\kappa,T\right)=0,\tag{7.11}$$

with the parametric dependence on the plastic strain e_{ij}^p, temperature T and the hardening parameter κ. We return to these parameters later. Two examples of yield surfaces, discussed further in some details, are shown in Fig. 7.3.

It is also convenient to introduce a normal (perpendicular) vector to the yield surface in the stress space given by its gradient in this space, i.e.

$$\mathbf{N}=\frac{\frac{\partial f}{\partial \mathbf{T}}}{\left|\frac{\partial f}{\partial \mathbf{T}}\right|},\quad \text{i.e.}\quad N_{ij}=\frac{\frac{\partial f}{\partial \sigma_{ij}}}{\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}}.\tag{7.12}$$

Then we can introduce local coordinates in which the yield function f identifies the elastic domain of the Σ-space assuming there negative values, i.e. for all elastic processes $f<0$.

We skip here the presentation of the history of the definition of yield surfaces which goes back to Galileo Galilei. There are two fundamental forms of this surface which are still commonly used in the linear plasticity of solids. The older one was proposed by H. Tresca in 1864 and it is called Tresca-Guest surface. Its equation has the form

$$\max_\alpha\left|\tau^{(\alpha)}\right|=\sigma_0\quad\Rightarrow\quad\sigma^{(1)}-\sigma^{(3)}=2\sigma_0>0,\tag{7.13}$$

where σ_0 is the material parameter and we have ordered the principal stresses $\sigma^{(1)}\geq\sigma^{(2)}\geq\sigma^{(3)}$. It means that the beginning of the plastic deformation appears in the point of the maximum shear stress. In the space of principal stresses it is a prism of six sides and infinite length (see: Fig. 7.3.). The parameter σ_0 may be dependent on all parameters listed in the general relation (7.11).

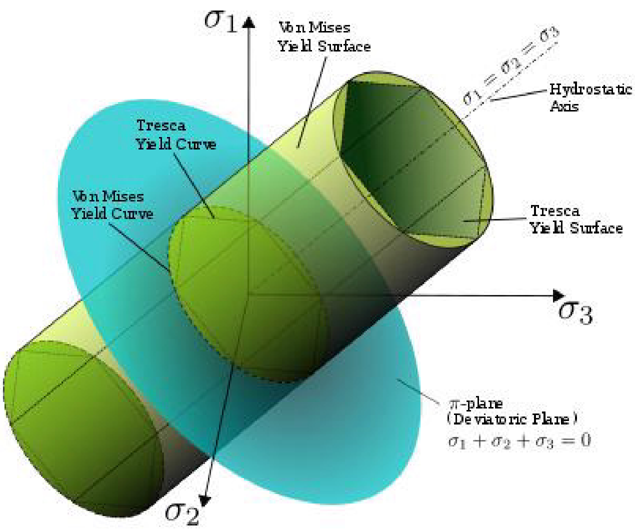

Fig. 7.3: Yield surfaces in the space of principal stresses

The second yield surface was proposed in 1904 by M. T. Huber[6] and then rediscovered in 1913 by R. von Mises and H. von Hencky. It says that the limit of elastic deformation is reached when the energy of shape changes (distortion energy) reaches the limit value $\rho\varepsilon_Y$. The distortion energy $\rho\varepsilon_D$ is defined as a part of the full energy of deformation $\rho\varepsilon$ reduced by the energy of volume changes $\rho\varepsilon_V$ (e.g. compare (5.175)). We have

$$\rho\varepsilon = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}\left(\sigma_{ij}\frac{\sigma_{kk}}{9K}\delta_{ij} + \sigma_{ij}\frac{\sigma_{ij}^D}{2\mu}\right) =$$

$$= \frac{1}{2K}\left(\frac{\sigma_{kk}}{3}\right)^2 + \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right) \quad \Rightarrow \quad \rho\varepsilon_V = \frac{p^2}{2K}, \quad \rho\varepsilon_D = \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right), \tag{7.14}$$

$$\text{i.e.} \quad \rho\varepsilon_D = \rho\varepsilon_Y \quad \Rightarrow \quad \rho\varepsilon_Y = \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right) = \frac{\sigma_{eq}^2}{6\mu}.$$

Making use of the identity, following from (7.5),

$$3\left(s^{(1)}s^{(2)} + s^{(1)}s^{(3)} + s^{(2)}s^{(3)}\right) = -\frac{1}{2}\left[\left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2\right], \tag{7.15}$$

[6] M. T. Huber; Przyczynek do podstaw wytrzymałości, *Czasop. Techn.*, Lwów, **22**, 1904. Due to the publication of this work in Polish it remained unknown until the hypothesis was rediscovered by von Mises and von Hencky.

we obtain

$$\sigma_{ij}^D \sigma_{ij}^D = \frac{1}{3}\left[\left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2\right]. \tag{7.16}$$

Consequently, bearing (7.6) in mind, the yield limit is reached when the principal stresses fulfil the condition

$$3\sigma_{ij}^D \sigma_{ij}^D = \left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2 = 2\sigma_Y^2, \quad \text{i.e.} \quad \sigma_{eq} = \sigma_Y, \tag{7.17}$$

where

$$\sigma_Y = \sqrt{6\mu\rho\varepsilon_Y}, \tag{7.18}$$

is the yield limit ($\sigma^{(1)} = \sigma_Y$ in uniaxial tension, i.e. for $\sigma^{(2)} = \sigma^{(2)} = 0$). It means that for processes in which $\sigma_{eq} < \sigma_Y$ all states are elastic ($f < 0$) and otherwise the system develops plastic deformations. Clearly, the relation (7.17) defines a circular cylinder in the space of principal stresses. Its axis is again identical with the line $\sigma^{(1)} = \sigma^{(2)} = \sigma^{(3)}$, it is extended to infinity and it has common generatrix with the prism of Tresca as shown in Fig. 7.3.

★In order to compare analytically both definitions of the yield surface we show that the yield stress σ_Y calculated by means of the distortion energy of the Huber-Mises-Hencky hypothesis (7.17) is not bigger than the material parameter $2\sigma_0$ of the Tresca hypothesis (7.13). Let us write (7.17) in the following form

$$\begin{aligned}
\sigma_Y &= \frac{1}{\sqrt{2}}\sqrt{\left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2} = \\
&= \frac{\left|\sigma^{(1)} - \sigma^{(3)}\right|}{\sqrt{2}}\sqrt{\left(\frac{\sigma^{(1)} - \sigma^{(2)}}{\sigma^{(1)} - \sigma^{(3)}}\right)^2 + \left(\frac{\sigma^{(2)} - \sigma^{(3)}}{\sigma^{(1)} - \sigma^{(3)}}\right)^2 + 1} = \\
&= \frac{\left|\sigma^{(1)} - \sigma^{(3)}\right|}{\sqrt{2}}\sqrt{\left(\frac{1-\mu_\sigma}{2}\right)^2 + \left(\frac{1+\mu_\sigma}{2}\right)^2 + 1} = \\
&= \left|\sigma^{(1)} - \sigma^{(3)}\right|\sqrt{\frac{3+\mu_\sigma^2}{4}},
\end{aligned} \tag{7.19}$$

where

$$\mu_\sigma = \frac{2\sigma^{(2)} - \left(\sigma^{(1)} + \sigma^{(3)}\right)}{\sigma^{(1)} - \sigma^{(3)}}, \tag{7.20}$$

is the so-called Lode parameter which describes an influence of the middle principal stress $\sigma^{(2)}$. Obviously $-1 \leq \mu_\sigma \leq 1$ which corresponds to $\sigma^{(2)} = \sigma^{(3)}$ for the lower bound, and $\sigma^{(2)} = \sigma^{(1)}$ for the upper bound. It plays an important role in the theory of civil engineering structures. Hence

$$\sigma_Y \leq \left|\sigma^{(1)} - \sigma^{(3)}\right| = 2\sigma_0. \clubsuit \tag{7.21}$$

★ We demonstrate on a simple example an application of the notion of the yield stress. We consider a circular ring of a constant thickness with external and internal radii a and b, respectively, and an external loading by the pressures p_a and p_b on these circumferences. We check when the material of the ring reaches in all points the yield stress according to the Huber-Mises-Hencky hypothesis. This is the so-called state of the load-carrying capacity of this structure.

This is the axial symmetric problem of plane stresses. Consequently, the principal stresses in cylindrical coordinates are given by $\sigma^{(1)} = \sigma_{rr}$, $\sigma^{(2)} = \sigma_{\theta\theta}$, $\sigma^{(3)} = 0$. These components of stresses must fulfil the equilibrium condition (momentum balance (5.43))

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{7.22}$$

and, according to (7.17), at each place of the ring

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr})^2 + (\sigma_{\theta\theta})^2 = 2\sigma_Y^2. \tag{7.23}$$

By means of (7.22) we eliminate the component $\sigma_{\theta\theta}$ of stresses and obtain the following equation

$$\left(r\frac{ds}{dr}\right)^2 + s\left(r\frac{ds}{dr}\right) + s^2 - 1 = 0, \quad s = \frac{\sigma_{rr}}{\sigma_Y\sqrt{2}}. \tag{7.24}$$

Solution of this quadratic equation with respect to the derivative ds/dr yields

$$\frac{ds}{dr} = -\frac{s}{2r} \pm \frac{1}{2r}\sqrt{4 - 3s^2}. \tag{7.25}$$

Consequently

$$\frac{ds}{-s \pm \sqrt{4 - 3s^2}} = \frac{dr}{2r}. \tag{7.26}$$

As we have to require $|s| < 2/\sqrt{3}$ we can change the variables

$$s = \frac{2}{\sqrt{3}}\sin\varphi, \tag{7.27}$$

and this yields

$$-\frac{d\varphi}{\tan\varphi \mp \sqrt{3}} = \frac{dr}{2r}. \tag{7.28}$$

Hence, we obtain two solutions but only one of them is real and it has the form

$$r = C\sqrt{\frac{1 + \dfrac{3s^2}{4 - 3s^2}}{\sqrt{\dfrac{s\sqrt{3}}{4 - 3s^2} + \sqrt{3}}}}\exp\left[-\frac{\sqrt{3}}{2}\arctan\frac{s\sqrt{3}}{\sqrt{4 - 3s^2}}\right], \tag{7.29}$$

where C is the constant of integration.

The construction of solution is shown in Fig. 7.4. in arbitrary units. For the radius b the value of the radial stress is given by $\sigma_r = -p_b$. We adjust the curve described by the relation (7.29) in such a way that it intersects the point $(-p_b, b)$ indicated by the circle in Fig. 7.4. This yields the value of the constant C in the solution. Then for the given value of the radius a we find the value of the pressure p_a which yields the limit value for the load of this structure, i.e. its load-carrying capacity.

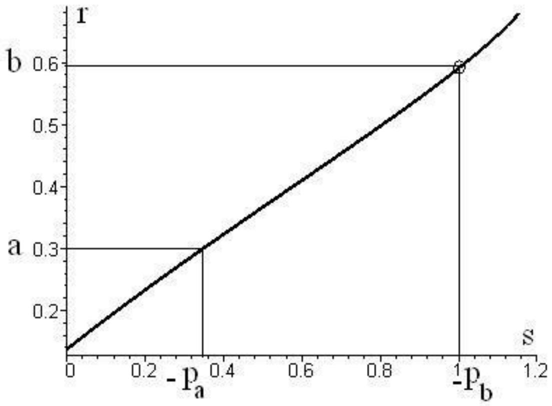

Fig. 7.4: Construction of solution for the load-carrying capacity of the ring♣

Yield surfaces impose conditions on elastic solutions under which the system possesses only elastic strains. In some design problems this is already sufficient. However, many engineering structures admit some plastic deformations – for example, in the case of concrete it is the rule – and then we have to find a way to describe the evolution of plastic strains $\mathbf{e}^p$. We proceed to develop such models.

First of all, we have to define not only the shape of the yield surface in the stress space, as we did above, but also its dependence on parameters listed in (7.11). We indicate here a few important examples. It is convenient to write the yield function in the form

$$f\left(J_2, J_3, e^p_{ij}, \kappa, T\right) = F\left(J_2, J_3\right) - \sigma_Y\left(e^p_{eq}, \kappa, T\right) = 0, \tag{7.30}$$

where σ_Y is the yield limit in the uniaxial tension/compression test for which

$$\begin{aligned} \sigma_{22} &= \sigma_{33} = \sigma^{(2)} = \sigma^{(3)} = 0 \quad \Rightarrow \quad \sigma_{eq} = \sigma^{(1)} = \sigma_{11}, \\ \dot{e}^p_{11} &= -2\dot{e}^p_{22} = -2\dot{e}^p_{33}, \quad \dot{e}^p_{ij} = 0 \text{ for } i \neq j \quad \Rightarrow \quad \dot{e}^p_{eq} = \dot{e}^p_{11}. \end{aligned} \tag{7.31}$$

where σ_{eq} is given by (7.9), and it is assumed to be given in terms of arguments listed in (7.30). e^p_{eq} is the equivalent plastic strain obtained from the integration in time of the effective rate of plastic deformation

$$\dot{e}^p_{eq} = \sqrt{\frac{2}{3}\dot{e}^p_{ij}\dot{e}^p_{ij}}. \tag{7.32}$$

Obviously, for isotropic materials we expect $\dot{e}_{ij}^{p}$ to be deviatoric. This yields relations (7.31).

Let us begin with the simplest case. In a particular case of ideal plasticity we consider materials without hardening. Then the function (7.30) has the form

$$f = \sqrt{\frac{3}{2}\sigma_{ij}^{D}\sigma_{ij}^{D}} - \sigma_{Y} = 0, \tag{7.33}$$

with the constant yield limit σ_Y. Of course, the stress tensor must be such that elastic processes remain within the elastic domain which is characterized by $f < 0$. Plastic deformations may develop for stresses which belong to the yield surface. Their changes yielding plastic deformation must remain on this surface which means that the increments of stresses described by $\dot{\sigma}_{ij}$ must be tangent to the yield surface. Hence, for such processes

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = 0, \tag{7.34}$$

as the gradient $\partial f/\partial \sigma_{ij}$ is perpendicular to the yield surface.

Now we make the fundamental constitutive assumption, specifying the rate of plastic strain $\dot{e}_{ij}^{p}$ and require that this quantity follows from a potential $G(\sigma_{ij})$ defined on the stress space

$$\dot{e}_{ij}^{p} = \dot{\lambda}\frac{\partial G}{\partial \sigma_{ij}}, \tag{7.35}$$

where $\dot{\lambda}$ is a scalar function following from the so-called Prager consistency condition. We demonstrate it further.

Let us introduce the notion of the outward normal vector to the surface $f = 0$ (compare (7.12)). Clearly

$$N_{ij} = \frac{\partial f}{\partial \sigma_{ij}}\left\|\frac{\partial f}{\partial \sigma_{kl}}\right\|^{-1}, \quad \left\|\frac{\partial f}{\partial \sigma_{kl}}\right\| = \sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}, \tag{7.36}$$

is such a vector. For the yield surface (7.33) it becomes

$$N_{ij} = \frac{\sigma_{ij}^{D}}{\left\|\sigma_{kl}^{D}\right\|}, \quad \left\|\sigma_{kl}^{D}\right\| = \sqrt{\sigma_{kl}^{D}\sigma_{kl}^{D}} = \sqrt{\frac{2}{3}}\sigma_{Y}. \tag{7.37}$$

Obviously, we have the following classification

$$\dot{e}_{ij}^{p} = \begin{cases} 0 \quad \text{for} \quad f < 0 \text{ or } f = 0 \text{ and } N_{ij}\dot{\sigma}_{ij} < 0. \\ \dot{\lambda}\frac{\partial G}{\partial \sigma_{ij}} \quad \text{for} \quad f = 0 \text{ and } N_{ij}\dot{\sigma}_{ij} = 0. \end{cases} \tag{7.38}$$

The condition $N_{ij}\dot{\sigma}_{ij} < 0$ means that the process yields the unloading – as N_{ij} is orthogonal to the yield surface, $\dot{\sigma}_{ij}$ must point in the direction of the elastic domain and, consequently, the process must be elastic.

In a particular case when the potential G and the yield function are identical we obtain

$$\dot{e}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}. \tag{7.39}$$

This is the so-called associated flow rule. In this case, the rate of plastic deformation $\dot{e}_{ij}^p$ is perpendicular to the yield surface, i.e. it is parallel to the normal vector N_{ij}.

It is appropriate to make here the following remark concerning the geometry of the yield surface. If this surface were not convex then in points in which it is concave tangent changes of the stress tensor $\dot{\sigma}_{ij}$, i.e. $(\partial f / \partial \sigma_{ij})\, \dot{\sigma}_{ij} = 0$, would yield stresses in the interior of the yield surface, i.e. in the range $f < 0$. This would be related to the development of pure elastic deformations in contrast to the assumption that tangent changes of stress yield plastic deformations. Therefore, in the classical plasticity nonconvex yield surfaces are not admissible. This is the subject of the so-called Drucker stability postulate. In the local form for the associated flow rules (7.39) it can be written as

$$\dot{e}_{ij}^p \dot{\sigma}_{ij} > 0. \tag{7.40}$$

It is also sometimes postulated in the global form

$$\int \left(\sigma_{ij} - \sigma_{ij}^0\right) de_{ij}^p > 0, \tag{7.41}$$

which shows that the postulate imposes a restriction on the work of plastic deformations between an arbitrary initial state of stress σ_{ij}^0 and an arbitrary finite state of stress σ_{ij}.

For the Huber-Mises-Hencky yield function (7.33) we obtain the associated flow rule

$$\dot{e}_{ij}^p = \dot{\lambda} \sqrt{\frac{3}{2}} \frac{\sigma_{ij}^D}{\left\|\sigma_{kl}^D\right\|} = \dot{\lambda} \sqrt{\frac{3}{2}} N_{ij}. \tag{7.42}$$

In the simple uniaxial tension/compression test we have then

$$\dot{e}_{eq}^p = \dot{e}_{11}^p = \dot{\lambda} \quad \Rightarrow \quad \dot{e}_{ij}^p \sigma_{ij} = \dot{\lambda} \sqrt{\frac{3}{2}} \left\|\sigma_{kl}^D\right\| = \dot{\lambda} \sigma_{11} = \dot{e}_{eq}^p \sigma_Y. \tag{7.43}$$

Hence for the rate of work (working) we obtain

$$\begin{aligned} \dot{W} = \dot{e}_{ij} \sigma_{ij} = \left(\dot{e}_{ij}^e + \dot{e}_{ij}^p\right) \sigma_{ij} \quad \Rightarrow \\ \Rightarrow \quad \dot{W}_p = \dot{e}_{ij}^p \sigma_{ij} = \dot{\lambda} \sqrt{\frac{3}{2}} \left\|\sigma_{kl}^D\right\| = \dot{e}_{eq}^p \sigma_{eq} = \dot{e}_{eq}^p \sigma_Y. \end{aligned} \tag{7.44}$$

The last expression – $\dot{e}_{eq}^p \sigma_Y$ – describes the plastic working in the one-dimensional test which is an amount of energy dissipated by the system per unit time due to the plastic deformation. Hence

$$\dot{\lambda} \geq 0, \tag{7.45}$$

and the equality holds only for elastic deformations. This statement follows easily from the second law of thermodynamics.

In order to construct an equation for plastic strains we account for the additive decomposition (7.3). For $f < 0$ we have elastic processes and then this property indicates (compare (5.26)) the following Prandl-Reuss equation for the rate of deformation

$$\dot{e}_{ij} = \left(\frac{\dot{\sigma}_{ij}^D}{2\mu} + \frac{\dot{\sigma}_{kk}}{9K}\delta_{ij} \right) + \dot{\lambda}\sqrt{\frac{3}{2}}N_{ij} = \tag{7.46}$$

$$= \left(\frac{\dot{\sigma}_{ij}^D}{2\mu} + \frac{\dot{\sigma}_{kk}}{9K}\delta_{ij} \right) + \dot{W}_p \frac{\sigma_{ij}^D}{\left\| \sigma_{kl}^D \right\|^2}. \tag{7.47}$$

This follows from the property of isotropic elastic materials for which the eigenvectors for the stress and for the strain are identical[7]. The spherical part is, obviously, purely elastic while the deviatoric part can be written in the form

$$\frac{d\sigma_{ij}^D}{dt} + \dot{W}_p \frac{4\mu}{3\sigma_Y^2}\sigma_{ij}^D = 2\mu \frac{de_{ij}^D}{dt}. \tag{7.48}$$

This equation is very similar to the evolution equation for stresses within the standard linear model of viscoelasticity (6.58) divided by the relaxation time τ. However, there is a very essential difference between these two models. It is easy to check that the equation of viscoelasticity (6.58) is not invariant with respect to a change of time scale $t \to \alpha t$, where α is an arbitrary constant. This indicates the rate dependence in the reaction of the material. It is not the case for the equation of plasticity (7.48). Differentiation with respect to time appears in all terms of this equation and for this reason the transformation parameter α cancels out. This is the reason for denoting the consistency parameter by $\dot{\lambda} = de_{eq}^p/dt$. It transforms: $t \to \alpha t \Rightarrow \dot{\lambda} \to \dot{\lambda}/\alpha$. Therefore, the classical plasticity is rate independent. The response of the system is the same for very fast and very slow time changes of the loading. In reality, metals do possess this property when the rate of deformation $\dot{e}_{eq}^p$ is approximately $10^{-6} - 10^{-4}$ 1/s. For higher rates one has to incorporate the rate dependence (compare the book of Lemaitre, Chaboche [9] for further details). This yields viscoplastic models presented further in these notes.

In the more general case of isotropic hardening and for isothermal processes σ_Y becomes a function of e_{eq}^p alone. For many materials it is also important to include the temperature dependence. Then σ_Y becomes the function of these two quantities. The model is similar to this which we have considered above but one has to correct the definition of the consistency parameter $\dot{\lambda}$. Finally, a dependence on the hardening parameter κ means that we account for the accumulation of plastic deformations in the material. The most common definitions of this parameter are as follows

a) the parameter accounting for the accumulation of the plastic energy

$$\kappa = \int_0^t \sigma_{ij}(\xi)\, \dot{e}_{ij}^p(\xi)\, d\xi, \tag{7.49}$$

[7] In order to prove it, it is sufficient to compare the eigenvalue problems for deviatoric stress and strain tensors.

b) Odqvist parameter accounting for the accumulation of the plastic deformation (compare (7.30) and (7.32))

$$\kappa = \int_0^t \sqrt{\frac{2}{3}\dot{e}^p_{ij}\dot{e}^p_{ij}}d\xi = \int_0^t \frac{de^p_{eq}}{d\xi}d\xi. \tag{7.50}$$

Then the relation (7.11) yields the consistency condition

$$\dot{f} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial e^p_{ij}}\dot{e}^p_{ij} + \frac{\partial f}{\partial T}\dot{T} + \frac{\partial f}{\partial \kappa}\dot{\kappa} = 0. \tag{7.51}$$

Simultaneously, $\dot{e}^p_{ij} \neq 0$ only in processes of loading which are defined by the relation

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} > 0. \tag{7.52}$$

In the limit case

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} = 0, \tag{7.53}$$

we say that the process is neutral. Finally, for the process of unloading,

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} < 0. \tag{7.54}$$

Consequently, for the evolution of plastic deformation we have the following relations

$$\dot{e}^p_{ij} = \begin{cases} 0 \quad \text{for either } f < 0 \text{ or } f = 0 \text{ and } \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} \leq 0 \\ \dot{\lambda}\dfrac{\partial f}{\partial \sigma_{ij}} \quad \text{for } f = 0 \text{ and } \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} > 0. \end{cases} \tag{7.55}$$

In the case of the hardening parameter (7.49) the consistency condition (7.51) can be written in the form

$$\dot{f} = \frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T} + \left[\frac{\partial f}{\partial e^p_{ij}} + + \frac{\partial f}{\partial \kappa}\sigma_{ij}\right]\dot{e}^p_{ij} = 0.$$

Hence, we obtain the following relation for the consistency parameter

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T}}{D}, \quad D = -\frac{\partial f}{\partial e^p_{ij}}\frac{\partial f}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \kappa}\frac{\partial f}{\partial \sigma_{ij}}\sigma_{ij}. \tag{7.56}$$

The quantity D is called the hardening function. We have

$$\dot{\lambda} > 0 \quad \Rightarrow \quad D > 0. \tag{7.57}$$

The flow rule can be now written in the form

$$\dot{e}^p_{ij} = \frac{1}{D}\frac{\partial f}{\partial \sigma_{ij}}\left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T}\right). \tag{7.58}$$

The right hand side is the homogeneous function of $\dot{\sigma}_{ij}$ and $\dot{T}$. Consequently, this flow rule possesses the same time invariance as the rule (7.39) for the model without hardening, i.e. the model is rate independent.

Apart from the above presented isotropic hardening materials reveal changes of the yield limit which can be attributed to the shift of the origin in the space of stresses. A typical example is the growth of the yield stress in tensile loading with the simultaneous decay of the yield stress for compression. In the uniaxial case it means that the whole stress-strain diagram will be shifted on a certain value of stresses. This is the Bauschinger effect.

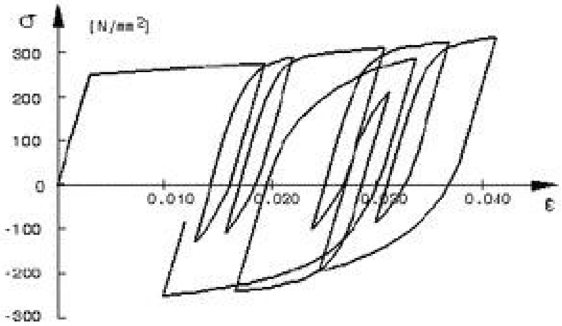

Fig. 7.5: An example of Bauschinger effect in cyclic loading

The corresponding hardening is called kinematical or anisotropic. It is described by the so-called back-stress $\mathbf{Z} = Z_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ which specifies the shift of the origin in the stress space. The yield function can be then written in the form

$$f(\mathbf{T}, \mathbf{Z}, \kappa) = F(\mathbf{T}, \mathbf{Z}) - \sigma_Y\left(e^p_{eq}, T\right) = \sqrt{\frac{3}{2}\bar{\sigma}^D_{ij}\bar{\sigma}^D_{ij}} - \sigma_Y = 0, \tag{7.59}$$

where

$$\bar{\sigma}^D_{ij} = \sigma^D_{ij} - Z_{ij}. \tag{7.60}$$

One has to specify an equation for the back-stress. It is usually assumed to have the form of the evolution equation, e.g.

$$\dot{Z}_{ij} = \dot{\beta}\left(\sigma_{ij} - Z_{ij}\right), \tag{7.61}$$

where $\dot{\beta}$ is a material parameter. We skip here the further details referring to numerous monographs on the subject[8].

[8] e.g. see the book [9] or
Albrecht Bertram; *Elasticity and Plasticity of Large Deformations*, Springer Berlin, 2008.
Michal Kleiber; *Handbook of Computational Solid Mechanics*, Springer, Heidelberg, 1998.
Gerard A. Maugin; *The Thermomechanics of Plasticity and Fracture,* Cambridge University Press, 1992.

7.3 Plasticity of soils

Theories of irrecoverable, permanent deformations of soils is very different from the plasticity of metals presented above. Metals produce plastic deformations primarily due to the redistribution and production of crystallographic defects called dislocations. Plastic behaviour of soils is mainly connected with the redistribution of grains and it is strongly influenced by fluids filling the voids (pores) of such a granular material. Strain due to the deformation of grains is often negligible in comparison to the amount of shear and dilatation caused by relative motions of grains. The behaviour is entirely different in the case of dry granular materials (frictional materials) than a material saturated by, for instance, water or oil where the cohesive forces play an important role. A detailed modern presentation of the problem of permanent deformations of soils can be found in the book of D. Muir Wood [23] (compare also a set of lectures von Verruijt [19]). Similar issues for rocks are presented in the classical book of Jaeger, Cook and Zimmerman[6]. We limit the attention only to few issues of this subject.

Attempts to describe the plasticity of granular materials stem from Coulomb, who formulated a simple relation between the normal stress σ_n on the surface with a normal vector $\mathbf{n}$ and the shear stress τ_n on this surface. It is a generalization of the law of friction between two bodies and has the form

$$|\tau_n| = c - \sigma_n \tan\varphi, \tag{7.62}$$

where φ is the so-called friction angle (angle of repose) and c denotes the cohesion intercept. This relation is called Mohr-Coulomb law. For dry granular materials the cohesion does not appear, $c = 0$, and then the angle of repose φ is the only material parameter. It is, for instance, the slope of natural sand hills and pits (Fig. 7.6).

Fig. 7.6: Sand pit trap of antlion in dry sand. Slope almost equal to φ

The above relation leads immediately to the yield function in terms of principal stresses $\sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)}$.. Namely

$$f\left(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\right) = \left(\sigma^{(1)} - \sigma^{(3)}\right) + \left(\sigma^{(1)} + \sigma^{(3)}\right)\sin\varphi - 2c\cos\varphi = 0. \tag{7.63}$$

The derivation from properties of Mohr's circle is shown in Fig. 7.7. Namely

$$|\tau_n| = \frac{\sigma^{(1)} - \sigma^{(3)}}{2} \cos\varphi, \quad \sigma_n = \frac{\sigma^{(1)} + \sigma^{(3)}}{2} - \frac{\sigma^{(1)} - \sigma^{(3)}}{2} \sin\varphi. \tag{7.64}$$

Substitution in (7.62) yields (7.63).

For $\varphi = 0$ and $c = \sigma_0$ the yield function (7.63) becomes the Tresca-Guest yield condition (7.13).

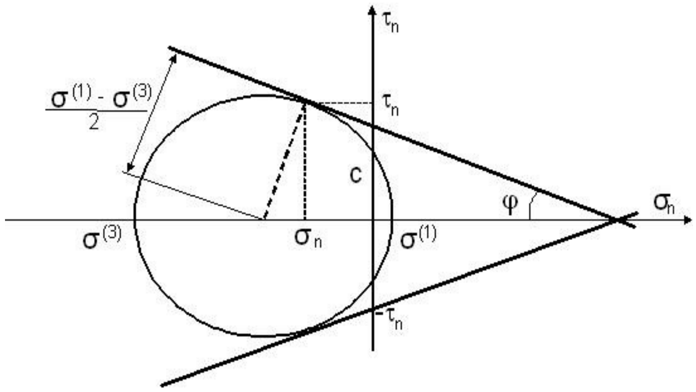

Fig. 7.7: Construction of Mohr-Coulomb yield function

Cohesive forces are influencing not only the relation between normal and shear stresses. Due to the porosity of granular materials a fluid in pores yields cohesive interactions as well as it carries a part of external loading. This observation was a main contribution of von Terzaghi to the theory of consolidation of soils[9]. He has made an assumption that the pore pressure p does not have an influence on the plastic deformation of soils. The meaning of p is here the same as in Subsection 6.4. and it should not be confused with the trace of the bulk stress σ_{ij}, i.e. $p \neq -\frac{1}{3}\sigma_{kk}$. It means that the stress appearing in yield functions must be reduced by subtracting the contribution of this pressure. If we define the effective stress

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}, \tag{7.65}$$

then the Mohr-Coulomb yield function becomes

$$\left(\sigma'^{(1)} - \sigma'^{(3)}\right) + \left(\sigma'^{(1)} + \sigma'^{(3)}\right) \sin\varphi - 2c\cos\varphi = 0. \tag{7.66}$$
$$\sigma'^{(\alpha)} = \sigma^{(\alpha)} + p, \quad \alpha = 1, 2, 3.$$

This function is shown in the upper panel of Fig. 7.8.

Further we distinguish by primes all quantities based on the effective stress.

[9] K. von Terzaghi; *Erdbaumechanik auf bodenphysikalischer Grundlage,* Franz Deuticke, Wien, 1925.

Incidentally, a similar notion of effective stresses appears in the theory of damage – it is related to changes of reference surface due to the appearance of cracks. Such models shall be not presented in these notes.

★**Remark.** There exists some confusion within the soil mechanics concerning the definition of positive stresses. Soils carry almost without exception only compressive loads (compare Fig. 7.7. and 7.8.) and, for this reason, in contrast to the classical continuum mechanics, a compressive stress is assumed to be positive. This is convenient in a fixed system of coordinates related to experimental setups such as triaxial apparatus. Then pressure p in the definition of effective stresses (7.65) would appear with the minus sign. Usually it is denoted in soil mechanics by u. In some textbooks[10] both conventions concerning the sign of stresses appear simultaneously. However, such a change of sign in a general stress tensor yields the lack of proper invariance with respect to changes of reference systems. It is also contradictory with the choice of the positive direction of vectors normal to material surfaces on which many mathematical problems of balance laws and the Cauchy Theorem rely. For these reasons, we work here with the same convention as in the rest of this book – tensile stress is positive.

In addition, one should be careful in the case of relation of such one-component models to models following from the theory of immiscible mixtures, for instance to Biot's model. Such models are based on partial quantities and then the pore pressure p is not the partial pressure p^F of a multicomponent model but rather $p = p^F/n$, where n is the porosity.♣

In soil mechanics, where the definitions of elastic domains described in the previous Subsection are not appropriate, it is convenient to introduce special systems of reference in the space of effective principal stresses. One of them is directly related to the set of invariants (7.10)

$$p' = \frac{1}{3}I_1' = -\frac{1}{3}I_\sigma', \quad q = \sqrt{3J_2'} = \sigma_{eq}, \quad r = 3\sqrt[3]{\frac{J_3'}{2}}. \tag{7.67}$$

Another one is a cylindrical system. One of the axes is the pressure $\left(-\frac{1}{3}\sigma'_{kk}\right)$, i.e. it is oriented along the line $\sigma'^{(1)} = \sigma'^{(2)} = \sigma'^{(3)}$. It is denoted by ξ and scaled $\xi = I_\sigma'/\sqrt{3}$. The other two coordinates are defined by the relations

$$\rho = \sqrt{2J_2'} \equiv \sqrt{\frac{2}{3}}\sigma'_{eq} \equiv \sqrt{\sigma'^D_{ij}\sigma'^D_{ij}}, \quad \cos(3\theta) = \left(\frac{r}{\sigma'_{eq}}\right)^3. \tag{7.68}$$

These are the so-called Haigh–Westergaard coordinates. The (ξ, ρ)- planes are called Rendulič planes and the angle θ is called the Lode angle. The transformation from these coordinates back to principal stresses is given by the relation

$$\begin{pmatrix} \sigma'^{(1)} \\ \sigma'^{(2)} \\ \sigma'^{(3)} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} \xi \\ \xi \\ \xi \end{pmatrix} + \sqrt{\frac{2}{3}}\rho \begin{pmatrix} \cos\theta \\ \cos\left(\theta - \frac{2}{3}\pi\right) \\ \cos\left(\theta + \frac{2}{3}\pi\right) \end{pmatrix}. \tag{7.69}$$

Mohr-Coulomb yield function in the Haigh–Westergaard coordinates has the following form

$$\left[\sqrt{3}\sin\left(\theta + \frac{\pi}{3}\right) - \sin\varphi\cos\left(\theta + \frac{\pi}{3}\right)\right]\rho - \sqrt{2}\xi\sin\varphi = \sqrt{6}c\cos\varphi. \tag{7.70}$$

[10] e.g. [23] or R. Lancellotta; *Geotechnical Engineering,* Balkema, Rotterdam, 1995.

Alternatively, in terms of the invariants (p', q, r) we can write

$$\left[\frac{1}{\sqrt{3}\cos\varphi}\sin\left(\theta+\frac{\pi}{3}\right)-\frac{1}{3}\tan\varphi\cos\left(\theta+\frac{\pi}{3}\right)\right]q-p'\tan\varphi=c, \tag{7.71}$$
$$\theta=\frac{1}{3}\arccos\left(\frac{r}{q}\right)^3.$$

As in the classical theory of plasticity, modifications of Mohr-Coulomb condition were introduced in order to eliminate corners in the yield surface. One of such modifications was introduced by D. C. Drucker and W. Prager. This condition for the limit state of soils has the following form

$$\sqrt{J_2'}-\frac{\sqrt{3}\cos\varphi}{\sqrt{3+\sin^2\varphi}}c-\frac{\sin\varphi}{\sqrt{3\left(3+\sin^2\varphi\right)}}I_\sigma'=0, \tag{7.72}$$

where the invariants J_2' and I_σ' are defined by relations for effective stress analogous to (7.10). This function is shown in the lower panel of Fig. 7.8. The dependence on the invariant I_σ' follows from the dependence of the yield in soils on volume changes, i.e. it describes an influence of dilatancy on the appearance of the critical limit state. For $\varphi=0$ and $c=\sigma_Y/\sqrt{3}$ this condition becomes identical with Huber-Mises-Hencky condition (7.17).

Due to its simplicity the Mohr-Coulomb yield surface is often used to model the plastic flow of geomaterials (and other cohesive-frictional materials). However, many such materials show dilatational behavior under triaxial states of stress which the Mohr-Coulomb model does not include. Also, since the yield surface has corners, it may be inconvenient to use the original Mohr-Coulomb model to determine the direction of plastic flow. Therefore it is common to use a non-associated plastic flow potential that is smooth. For example, one is using the function

$$g=\sqrt{\left(\alpha c_Y\tan\psi\right)^2+G^2\left(\varphi,\theta\right)q^2}-p'\tan\varphi, \tag{7.73}$$

where α is a parameter, c_Y is the value of c when the plastic strain is zero (also called the initial cohesion yield stress), ψ is the angle made by the yield surface in the Rendulič plane at high values of p' (this angle is also called the dilation angle), and $G(\varphi,\theta)$ is an appropriate function that is also smooth in the deviatoric stress plane.

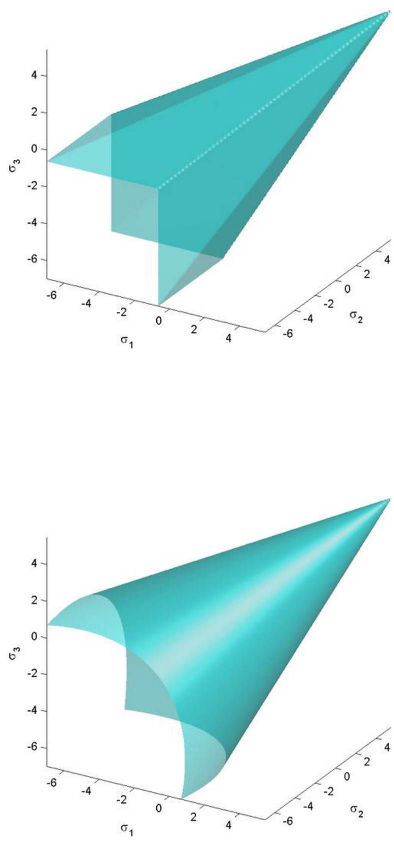

Fig. 7.8: Yield surfaces (7.66) **and** (7.72) **in the space of principal effective stresses** $\sigma_1 = \sigma'^{(1)}$, $\sigma_2 = \sigma'^{(2)}$, $\sigma_3 = \sigma'^{(3)}$.

We shall not expand this subject anymore. Due to the vast field of applications: soils, powders, avalanches, debris flows and many others, the number of models describing the critical behaviour of such materials is also very large. Cap plasticity models, Cam-Clay (CC) models, Modified-Cam-Clay (MCC) models, Mroz models, etc. are based on similar ideas as the models presented above. There exists also a class of hypoplasticity models in which the notion of the yield surface does not appear at all and which seem to fit well phenomena appearing in sands[11].

[11] compare articles of E. Bauer: Analysis of Shear Banding with a Hypoplastic Constitutive Model for a Dry and Cohesionless Granular Material, 335-350, and D. Kolymbas: The Importance of Sand in Earth Sciences, both in: B. Albers (ed.); *Continuous Media with Microstructure*, Springer, Berlin, 2010.

7.4 Viscoplasticity

There are many ways of extension of the classical plasticity to include rate dependence. Obviously, one of them would be to incorporate additionally some viscous properties as we did in Chapter 7. This kind of the model is developed since early works of P. Perzyna[12]. The other way, less ambitous, is to incorporate a rate dependence in the definition of the yield function. In principle, the classical yield function cannot exist in such models but one gets results by direct extension of plasticity models presented in this Chapter. For such models it is advocated in the books of Lemaitre, Chaboche [9] and Lemaitre, Desmorat [10].

We present here only a few hints to the model of the second kind. Namely, it is assumed that the yield criterion satisfies the relation

$$\begin{aligned} f &= 0, \quad \dot{f}=0 \quad \text{– plasticity}, \\ f &= \sigma_V > 0 \quad \text{– viscoplasticity}, \end{aligned} \tag{7.74}$$

with $f<0$ satisfied in the elastic domain. σ_V is a viscous stress given by a viscosity law. In both cases f can be chosen according to the rules discussed in previous Subsections. For instance, in the case of Huber-Mises-Hencky model with isotropic and kinematic hardening we have

$$\begin{aligned} f &= (\mathbf{T}-\mathbf{Z})_{eq} - \kappa - \sigma_Y, \qquad (7.75) \\ (\mathbf{T}-\mathbf{Z})_{eq} &= \sqrt{\frac{3}{2}\left(\sigma_{ij}^D - Z_{ij}^D\right)\left(\sigma_{ij}^D - Z_{ij}^D\right)}, \end{aligned}$$

where κ describes the isotropic hardening related to the size growth of the yield surface. It may be, for instance, assumed to have the exponential form

$$\kappa = \kappa_\infty \left[1-\exp\left(-b e_{eq}^p\right)\right], \tag{7.76}$$

where κ_∞, b are material parameters depending on temperature. Sometimes a power law $\kappa = K_p\left(e_{eq}^p\right)^{1/M}$ is sufficient.

Kinematic hardening described by the back-stresses Z_{ij} requires an evolution equation. It may have the form (7.61) or it may be the so-called Armstrong-Frederick law[13]

$$\frac{d}{dt}\left(\frac{Z_{ij}}{C}\right) = \frac{2}{3}\dot{e}_{ij}^p - \frac{\gamma}{C} Z_{ij}\dot{e}_{eq}^p, \tag{7.77}$$

for which the identification of parameters is easier [10].

The viscous stress σ_V is also given by various empirical relations. Two of them have the form

1) Norton power law

$$\sigma_V = K_N\left(\dot{e}_{eq}^p\right)^{1/N}, \tag{7.78}$$

[12] e.g. P. Perzyna; The constitutive equations for the rate sensitive plastic materials, *Quart. Appl. Math.*, **20**, 321-332, 1963.

[13] P. J. Armstrong, C. O. Frederick; A mathematical representation of the multiaxial Bauschinger effect, CEGB Report, RD/B/N731, Berkeley Nuclear Laboratories, 1966.

2) exponential law leading to the saturation at large plastic rates

$$\sigma_V = K_\infty \left[1 - \exp\left(-\frac{\dot{e}^p_{eq}}{n}\right)\right], \tag{7.79}$$

where K_N, K_∞, N and n are material parameters.

In Fig. 7.9 we show a comparison of results for various viscous models[14].

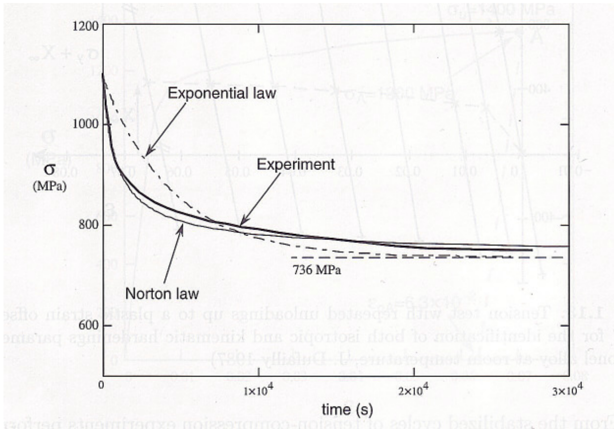

Fig. 7.9: Relaxation test for the identification of viscosity parameters – Inconel alloy at $\theta = 627^0$ C.

For the alloy investigated by Lemaitre and Dufailly the following parameters are appropriate

$$E = 160 \text{ GPa}, \quad K_N = 75 \text{ GPa/s}^{1/N}, \quad N = 2.4, \quad K_\infty = 10^4 \text{ GPa}, \quad n = 1.4 \times 10^{-2}\text{s}^{-1}.$$

Rate-dependent viscoplastic models must be used in cases of high deformation rates. For metals, the rates up to app. 10^{-3} 1/s do not influence substantially results in the plastic range of deformations. For higher rates the yield limit may grow even three times by the rate 100 1/s[15].

[14] J. Lemaitre, J. Dufailly; Damage measurements, *Engn. Fracture Mech.,* **28**, 1987

[15] P. Perzyna; *Thermodynamics of Inelastic Materials* (in Polish), PWN, Warsaw, 1978.

Chapter 8

Dislocations

8.1 Introduction

One of the difficult questions of materials science some 100 years ago was the elucidation of the mechanism of plastic deformation of crystalline bodies. Plasticity of ductile materials described in Subsection 8.2 is purely macroscopic and the range of its applicability can be explained only by means of microscopic properties of materials. In 30th of the XXth century it was shown by A. H. Cottrell, E. Orowan, M. Polanyi, J. W. Taylor, that dislocations, line defects in crystalline bodies, are the source of plastic deformation. Since this discovery a new branch of plasticity has been developed – crystal plasticity. It began with works of Schmid, Boas, Taylor and yielded important results in the field of evolution of plastic anisotropy, textures, cold rolling and forge techniques of metals.

In this Chapter, we present some properties of discrete dislocations as well as a continuum model of these defects. The theory of discrete dislocations found an application in modeling of rupture appearing by earthquakes. This application shall be briefly presented at the end of this Chapter.

We begin with the formal definition of the dislocation. There are two possibilities. One of them was proposed by C. Somigliana (1914) and it is based on the notion of the dislocation line. Another one was introduced by V. Volterra (1907) and it is using a notion of a singular surface on which the displacement vector $\mathbf{u}$ is discontinuous. Both definitions yield similar models but they are not equivalent[1]. In Fig. 8.1. and 8.2. we show a few schematic pictures of the dislocation in a crystal. In the left panel of Fig. 8.1. we show the result of a removal of half-planes of atoms from an infinite ideal cubic crystal. In result the upper half-space and the lower half-space possess a misfit. In order to correct it, in the vicinity of the horizontal cut the lattice constants (distance between lines indicated in the Figure) must be different. This yields the existence of infinite straight lines perpendicular to the page in which one of the atomic half-planes terminates and in their neighborhood the lattice is distorted. This line defect is called the

[1] see: introduction to the subject by T. Mura [13]. A detailed discussion of the problem can be found in Z. Mossakowska; Self-equilibrated stresses and dislocations (in Polish), in: *Technical Mechanics. Vol. IV: Elasticity*, M. Sokolowski (ed.), PWN, Warsaw, 1978,

edge dislocation. Obviously, if we try to complete a closed curve around such a line we have to make different number of lattice steps in one direction than we make backwards. This is seen even better in the cartoon of the screw dislocation in the right panel of this Figure.

The above described construction is demonstrated again in the Föll cartoons of Fig. 8.2. The right panel shows the combination of the edge and screw dislocations in which the vector **b** describing the misfit is neither perpendicular to the dislocation line (edge dislocation) nor parallel to this line (screw dislocation). It forms rather an angle of sixty degrees with this line. Obviously, depending on the combination, this angle may be arbitrary.

Dislocation lines carry both an accumulated energy and self-equilibrated stresses in the reference state of the body. They try to minimize this energy by minimizing the length. This means that in the infinite ideal crystal they form straight lines. However, in real finite crystals this is not possible. Consequently, they must either terminate on the boundaries or, which is mostly the case in reality, they must form closed circuits. This is also the reason for their motion. Under loading – shearing in the plane of misfit, this lines change the curvature and, in attempt to minimize the length, they are shifted along this plane.

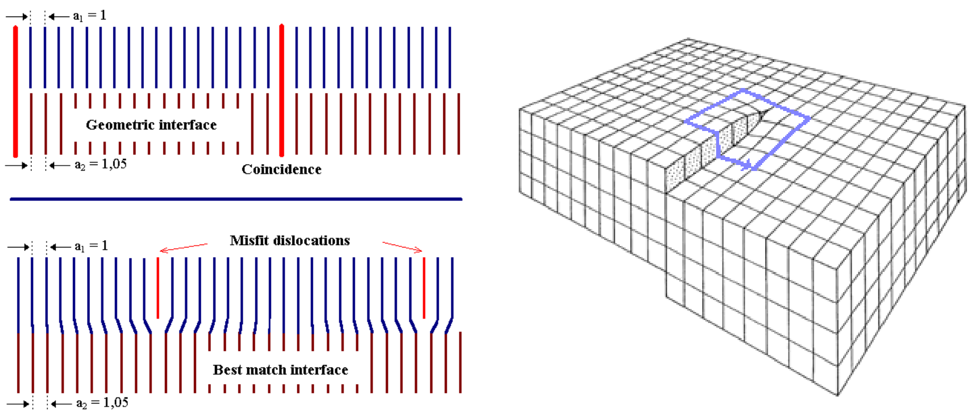

Fig. 8.1: Schematic picture of a two-dimensional misfit along the horizontal line yielding edge dislocation line every 20 steps (left panel) and a schematic picture of the screw dislocation (right panel)

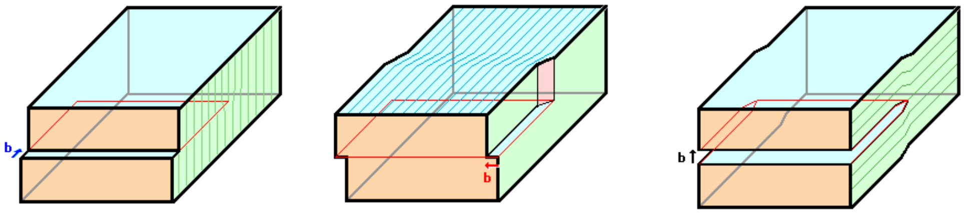

Fig. 8.2: Three characteristic types of dislocations: edge (left panel), screw (middle panel) and "sixty degree" (right panel)[2]

We proceed to the mathematical description of the dislocation.

8.2 Continuum with dislocations

The closed curve D is said to be the dislocation line (dislocation loop) in a continuum if a line integral along any sufficiently small closed circuit B circumventing once the curve D possesses the property

$$\oint_B d\mathbf{u} = \mathbf{b}, \tag{8.1}$$

for an arbitrary loading of the continuum. Obviously, $\mathbf{u}$ is the displacement vector and $\mathbf{b} \neq \mathbf{0}$ is called the Burgers vector of the dislocation line D. According to Stokes Theorem (1.59), for differentiable displacement field $\mathbf{u}(\mathbf{x}, t)$ we can write the integral in the form

$$\oint_B \frac{\partial u_i}{\partial x_j} dx_j = \int_{S_B} \epsilon_{kij} \frac{\partial^2 u_i}{\partial x_i \partial x_j} n_k dS = 0, \tag{8.2}$$

where S_B is a surface spanned on the curve B. Hence, the definition (8.1) is nontrivial only for displacement fields which are discontinuous on the surface S_B. This relates the Somigliana definition of the dislocation to the surface definition introduced by Volterra.

For energetic reasons $\mathbf{b}$ is usually the shortest translation vector of the lattice; e.g. $|\mathbf{b}| = a/2 < 110 >$ for the fcc lattice. The assumption that the Burgers contour B is sufficiently small means that it encloses only one dislocation line D and that it is intersecting only once a surface span by the line D. This is schematically shown in Fig. 8.3.. The sign of the Burgers vector $\mathbf{b}$ is defined by the right screw rule shown also in the Fig. 8.3..

[2] after HELMUT FÖLL; *Defects in crystals*, http://www.tf.uni-kiel.de/matwis/amat/def_en/index.html

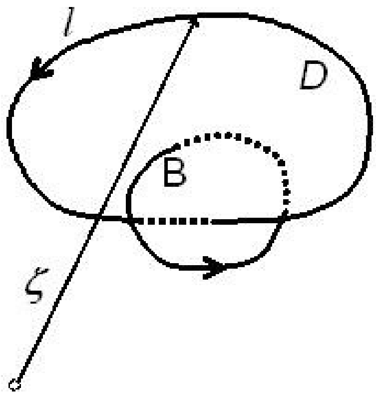

Fig. 8.3: Dislocation loop D**,**
Burgers contour B
and the sign convention

One of the surfaces related to the dislocation line is the cylinder for which the curve D is the directrix and whose generatrix are straight lines in the direction of the Burgers vector $\mathbf{b}$. This is the so-called gliding surface along which dislocations move most frequently because the resistance of the crystal to such a motion is in this surface the smallest (the conservative motion). Another possibility which requires much more energy is the so-called climbing of dislocations (the nonconservative motion). The latter requires some atomic diffusion processes. For details we refer to numerous books on the subject[3].

We proceed to describe the geometry of the dislocation line. It is described by the position vector

$$\mathbf{x} = \boldsymbol{\zeta}(l, t), \tag{8.3}$$

where l is a parameter along the line. Then the tangent vector and the velocity of dislocation are defined by the relations

$$\begin{aligned} d\zeta_k &= \frac{\partial \zeta_k}{\partial l} dl \quad \Rightarrow \quad t_k(\mathbf{x}, t) = \oint_D \delta(\mathbf{x} - \boldsymbol{\zeta}(l, t))\, d\zeta_k, \\ \dot{\zeta}_k &= \frac{\partial \zeta_k}{\partial t}, \end{aligned} \tag{8.4}$$

where the vector field $\mathbf{t}(\mathbf{x}, t)$ is given in the whole continuum but it is different from zero only on dislocation lines.

In order to describe the continuous fields in the presence of dislocation we introduce the tensor of distortion $\boldsymbol{\beta}$ which smears out the Burgers condition (8.1). This tensor

[3] e.g.:
J. Weertman, J. R. Weertman; *Elementary Dislocation Theory,* Macmillan, New York, 1967,
D. Hull; *Introduction to Dislocations,* Pergamon, 1975,
J. D. Eshelby; Continuum Theory of Lattice Defects, *Solid State Physics*, vol.3, 79, Academic Press, N.Y., 1956,
A. M. Kosevich; *Crystal Lattice: Photons, Solitons, Dislocations*, John Wiley, 1999.

coincides with the gradient of displacement in simply connected domains $\mathcal{P}$ which do not contain dislocation loops: $\mathcal{P} \cap D = \varnothing$. Hence we define

$$du_j = \beta_{ij} dx_i, \quad \text{with} \quad \beta_{ij}(\mathbf{x}, t) = \frac{\partial u_j}{\partial x_i} \quad \text{for} \quad \mathbf{x} \in \mathcal{P}. \tag{8.5}$$

Then the condition (8.1) has the form

$$b_j = \oint_B \beta_{ij} dx_i = \int_{S_B} \epsilon_{kli} \frac{\partial \beta_{ij}}{\partial x_l} n_k dS, \tag{8.6}$$

where the second relation follows from the Stokes Theorem for an arbitrary surface S_B spanned on the curve B.

This relation allows to smear out the field of distortion. Namely, for the dislocation loop D intersecting a surface S_B in the point $\mathbf{x}$, as shown in Fig. 8.4:

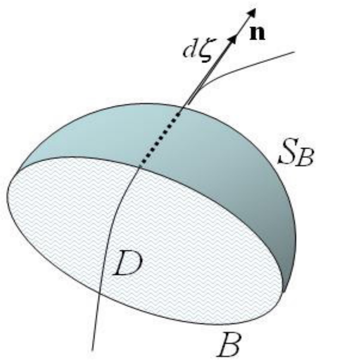

Fig. 8.4: Orientations of Burgers surface S_B and dislocation loop D

we use the following identity

$$\int_{S_B} \left(\oint_D \delta(\mathbf{x} - \boldsymbol{\zeta}) \, d\zeta_k \right) n_k dS = \begin{cases} 1 & \text{for the same orientation of } \mathbf{n} \text{ and } d\boldsymbol{\zeta}, \\ -1 & \text{for the opposite orientation of } \mathbf{n} \text{ and } d\boldsymbol{\zeta}, \\ 0 & \text{for } D \text{ not intersecting } S_B. \end{cases} \tag{8.7}$$

Then (8.6) can be written in the form

$$b_j \int_{S_B} \left(\oint_D \delta(\mathbf{x} - \boldsymbol{\zeta}) \, d\zeta_k \right) n_k dS = \int_{S_B} \epsilon_{kli} \frac{\partial \beta_{ij}}{\partial x_l} n_k dS. \tag{8.8}$$

This relation must hold for an arbitrary surface S_B spanned on the Burgers contour B. Consequently,

$$\epsilon_{kli} \frac{\partial \beta_{ij}}{\partial x_l} = b_j t_k, \tag{8.9}$$

where we have used the definition (8.4) of the vector $\mathbf{t}$ tangent to the dislocation loop D. This is the differential form of the Burgers relation (8.1). As the vector $\mathbf{t}$ is different from zero only on dislocation lines the above relation states that the distortion $\boldsymbol{\beta}$ has a vector potential beyond the dislocation line ($\operatorname{rot}\boldsymbol{\beta} = \mathbf{0}$), i.e. $\boldsymbol{\beta} = \operatorname{grad}\mathbf{u}$ as we have already mentioned before.

The tensor

$$\boldsymbol{\alpha} = \mathbf{b} \otimes \mathbf{t} \quad \text{i.e.} \quad \alpha_{ij} = b_i t_j, \tag{8.10}$$

is called the tensor of dislocation density. Obviously, it satisfies the relation

$$\epsilon_{ilk} \frac{\partial \beta_{kj}}{\partial x_l} - \alpha_{ij} = 0. \tag{8.11}$$

The above considerations allow to write the full set of equations which determine the distortion $\boldsymbol{\beta}$ and other fields of a linear elastic continuum caused by a given dislocation line. In the static case the problem is quite simple. We use the equilibrium condition

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{8.12}$$

which follows from (3.38) and the Hooke law (5.21)

$$\sigma_{ij} = c_{ijkl}\beta_{kl}, \quad c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{8.13}$$

We have to incorporate the relation (8.11) which describes the loading by the dislocation. To do so we differentiate the equilibrium condition and subsequently we substitute (8.13)

$$c_{ijkl} \frac{\partial^2 \beta_{kl}}{\partial x_j \partial x_p} = 0.$$

Now we multiply (8.11) by ϵ_{iql}, and use the identity

$$\epsilon_{kpl}\epsilon_{kqi} \frac{\partial \beta_{ij}}{\partial x_q} = \left(\delta_{pq}\delta_{il} - \delta_{qi}\delta_{qj}\right) \frac{\partial \beta_{ij}}{\partial x_q} = \frac{\partial \beta_{lj}}{\partial x_p} - \frac{\partial \beta_{pj}}{\partial x_l} = \epsilon_{kpl}\alpha_{kj}, \tag{8.14}$$

which follows from the contracted epsilon identity (1.41). Finally, we have

$$c_{ijkl} \frac{\partial^2 \beta_{pl}}{\partial x_i \partial x_k} = c_{ijkl}\epsilon_{qpk} \frac{\partial \alpha_{ql}}{\partial x_i}. \tag{8.15}$$

Together with the condition

$$e_{kl} = \frac{1}{2}\left(\beta_{kl} + \beta_{lk}\right), \tag{8.16}$$

the equation (8.15) fully describes the problem. Once we find β_{ij} we can find stresses from Hooke's law (8.13). As this field of stresses follows only from the presence of the dislocation without any external load we say that it is self-equilibrated.

Solutions of this equilibrium problem are very important because they determine the stress concentration in the vicinity of the line defect. They can be found by means of the

Green function of the linear elasticity. We quote here only the result of W. G. Burgers for the displacement **u** in the case of a dislocation loop D with the constant Burgers vector **b**. It has the form

$$\begin{aligned} u_k &= -\frac{1}{4\pi} b_k \oint\limits_D \frac{\epsilon_{ijl} r_j k_l}{r\,(r - r_i k_i)} d\zeta_i + \frac{1}{4\pi} \epsilon_{kij} b_i \oint\limits_D \frac{d\zeta_j}{r} + \frac{1}{4\pi} \frac{\lambda+\mu}{\lambda+2\mu} \epsilon_{ijl} b_j \frac{\partial}{\partial x_k} \oint\limits_D \frac{r_l}{r} d\zeta_i, \\ r_k &= x_k - \zeta_k\,(l)\,, \quad r = \sqrt{r_k r_k}, \end{aligned} \tag{8.17}$$

where **k** is the unit vector perpendicular to the plane of the loop D.

Only in exceptional cases one can perform analytically the integration in the above relation. It can be done for the straight line dislocations. In such a case, one obtains, for instance, the following components of stresses

1) screw dislocation given by $\mathbf{l} = (0, 0, 1)$ where **l** points in the direction of the dislocation line D, and $\mathbf{b} = (0, 0, b)$

$$\begin{gathered} \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \\ \tau_{xz} = \frac{\mu b}{2\pi} \frac{y}{x^2 + y^2}, \quad \tau_{yz} = -\frac{\mu b}{2\pi} \frac{x}{x^2 + y^2}, \end{gathered} \tag{8.18}$$

2) edge dislocation given by $\mathbf{l} = (0, 0, 1)\,,\ \mathbf{b} = (b, 0, 0)$

$$\begin{aligned} \tau_{xz} &= \tau_{yz} = 0, \\ \sigma_x &= \frac{b}{2\pi} \frac{2\mu\,(\lambda+\mu)}{\lambda+2\mu} \frac{y\left(3x^2 + y^2\right)}{\left(x^2+y^2\right)^2}, \\ \sigma_y &= -\frac{b}{2\pi} \frac{2\mu\,(\lambda+\mu)}{\lambda+2\mu} \frac{y\left(x^2 - y^2\right)}{\left(x^2+y^2\right)^2}, \\ \sigma_{xz} &= \frac{b}{2\pi} \frac{2\mu\lambda}{\lambda+2\mu} \frac{y}{x^2+y^2}, \\ \sigma_{xy} &= -\frac{b}{2\pi} \frac{2\mu\,(\lambda+\mu)}{\lambda+2\mu} \frac{x\left(x^2 - y^2\right)}{\left(x^2+y^2\right)^2}. \end{aligned} \tag{8.19}$$

Solutions for dislocations in some anisotropic media can be found in the explicit form as well.

We do not present details of the dynamic theory of dislocations[4]. The Burgers condition holds true also in this general case but one has to cope with the problem of elimination of the acceleration term in the equation of motion. One can derive an additional equation for the evolution of the distortion $\boldsymbol{\beta}$

$$\frac{\partial \beta_{kl}}{\partial t} - \frac{\partial v_l}{\partial x_k} = J_{kl}, \tag{8.20}$$

[4] some aspects of this theory can be found in the earlier quoted work of Z. Mossakowska as well as: H. Zorski; Theory of discrete defects, *Arch. Mech. Stos.,* **18**, 3, 1966.

where J_{kl} is the dislocation flux given by the relation

$$J_{kl} = b_l \epsilon_{kij} \oint_{D(t)} \dot{\zeta}_i \delta \left(\mathbf{x} - \boldsymbol{\zeta} \left(l, t \right) \right) d\zeta_j . \tag{8.21}$$

Some universal solutions are known also in this case but we shall not quote them in these notes.

8.3 On plasticity of metals

As we have mentioned at the beginning of this Chapter, the vehement research of the continua with dislocations was connected with the discovery that plastic deformations of metals are related to the redistribution and production of dislocations. In low temperatures, i.e. temperatures below app. 70% of the temperature of melting point, dislocations are moving on characteristic crystallographic planes on which they require the least energy for the motion. During this motion they get stacked on boundaries of grains of polycrystals and on other obstacles. One of them may be a point in which more than one dislocation appear simultaneously and their Burgers vectors annihilate each other, i.e. $\sum_\alpha \mathbf{b}^{(\alpha)} = 0$, where α numbers the dislocations in this point. Such knots play an important role in the production of dislocations. Namely, the shear stress which acts in the slip plane of motion of a dislocation is bending a line of dislocation pinned to two such obstacles. The loop is trying to minimize the energy which leads to overhanging shown in Fig. 8.5. When the two sides (green in Figure) meet they annihilate because their Burgers vectors are identical but of the opposite sign. The loop becomes free to move and the rest of the virginal line of dislocation begins the process anew. This is the so-called Frank-Read source of dislocations.

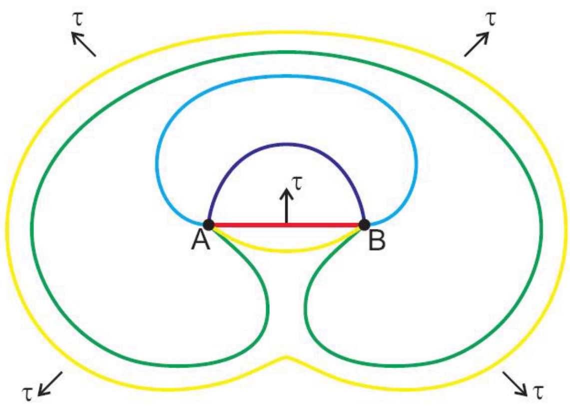

Fig. 8.5: Frank-Read source of dislocation

This and similar mechanisms yield the production of dislocations which is an irreversible process related to the increment of plastic strains. During plastic deformations the number density of dislocations may grow from some 10^{10} to 10^{20} $1/\mathrm{cm}^2$.

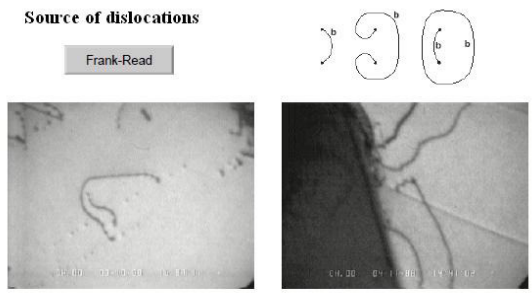

Fig. 8.6: Electron microscope picture of Frank-Read source. Black traces are dislocations on the surface of the sample

Modeling of such processes is based on certain microscopic observations transferred to the level of continuum. The fundamental role plays here the so-called Orovan equation which relates the rate of shearing to the Burgers vector, speed of dislocations and dislocation density. Together with the evolution equation for this density in which intensities of sources are incorporated one obtains a semistructural plasticity model, the so-called crystal plasticity which successfully solved many problems of mechanics of metals[5].

In high temperatures the process becomes more complicated because the defects pinned to the grain boundaries begin to move as well. In this range the theory of dislocations as presented above cannot be applied anymore.

8.4 Dislocations in geophysics

The origin of various models of dislocations goes back to the defective structure of crystalline bodies such as metals. However, we have seen that these models describe line defects in continua independently of a particular crystalline lattice. Only some indications concerning the Burgers vector bear on crystallography. Therefore one can apply such models in all cases in which a description of a discontinuous displacement field is needed. This is indeed the case in modeling of earthquakes. Most likely it was A. E. H. Love in 1945 who proposed to apply the Volterra dislocation model[6] in description of earthquakes. The problem of seismic sources was discussed by Vvedenskaya (1956), Steketee (1958) and others[7]. The modern presentation of the subject can be found in the book of Aki and Richards [1].

[5] for the introduction to crystal plasticity see the book of K. Wilmanski [22]. Many details can be found in the monograph R. W. K. Honeycombe; *The Plastic Deformation of Metals*, E. Arnold, 1968, and U. F. Kocks, A. Argon, M. Ashby; *Thermodynamics and Kinetics of Slip,* vol.19, Chalmers, B., Christian, J. W. and Massalski, T. W. (eds.), Pergamon Press, Oxford, 1975.

[6] V. Volterra, Sur l'équilibre des corps élastiques multiplement connexes, Ann. Sci. l'École Normale Supérieure, Paris, **24**, 401-517, 1907.

[7] A. V. Vvedenskaya; Determination of displacement fields for earthquakes by means of the dislocation theory (in Russian), *Izv. Akad. Nauk SSSR, Geofiz.,* **3**, 277-284, 1956,

The mechanism of earthquake rupture may be more complicated than this which can be described by the Volterra dislocation. It is related to crack formation and it is coupled to complicated tectonic processes which we do not discuss in this book. Reader interested in these problems is referred to an article of J. Rice [16]. We leave out the discussion of the structure of forces acting in the fault – according to Aki, Richard [1] the so-called double couple theory seems to be prevailing, and limit the attention to modeling a slip in the fault and its action on the vicinity.

A fault surface S_D with the boundary $\partial S_B = D$ lies in a linear isotropic medium and it is assumed to be perpendicular to the x_3-axis. A slip is presumed to take place in the direction of a unit vector $\mathbf{l} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2$. Then the displacement vector possesses a discontinuity

$$\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = \mathbf{b} = |\Delta\mathbf{u}|\,\mathbf{l}, \tag{8.22}$$

where $\mathbf{b}$ is the Burgers vector for the dislocation line D. Hence, Volterra dislocation allows to take over all results the theory of dislocations in the description of such a fault defect. Strictly speaking, the definition of Volterra dislocation requires as well that derivatives of $\mathbf{u}$ on S_D are continuous which means, of course, also the continuity of the stress. The field of displacement created by the dislocation yields a system of self-equilibrated stresses and, for this reason, an accompanying distortion may be considered as a field of initial deformations e^0_{ij} given by the relation

$$e^0_{ij} = -b_j \int\limits_{S_D} \delta(\mathbf{x} - \boldsymbol{\xi})\, n_i dS. \tag{8.23}$$

This relation yields immediately the notion of moment tensor density m_{ij} given by the Hooke's law for the initial deformation

$$m_{ij} = c_{ijkl} e^0_{kl}, \tag{8.24}$$

and, consequently, a definition of forces appearing in the equation for the real displacement $\mathbf{u}$

$$X^0_k = -\frac{\partial m_{kl}}{\partial x_l}, \quad \rho\frac{\partial^2 u_i}{\partial t^2} = c_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} + X^0_i. \tag{8.25}$$

Obviously, the tensor of material parameters for isotropic materials c_{ijkl} has the form (5.21). Now, the dynamic Green function (5.64) yields solutions for the displacement. For instance, for the source which is the Heaviside function $\mathbf{b}H(t)$ we obtain the displacement in the far field approximation in the following form

$$u^L_i = \frac{M_0}{4\pi\rho c^3_L r}(n_k l_l + n_l l_k)\frac{x_i x_k x_l}{r^3}\delta\left(t - \frac{r}{c_L}\right), \quad n_k\mathbf{e}_k = \mathbf{e}_3, \quad \mathbf{l} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2,$$
$$u^S_i = \frac{M_0}{4\pi\rho c^3_T r}(n_k l_l + n_l l_k)\left(\delta_{ik} - \frac{x_i x_k}{r^2}\right)\frac{x_l}{r}\delta\left(t - \frac{r}{c_T}\right), \tag{8.26}$$

J .A. Steketee; Some geophysical applications of the theory of dislocations, *Can. J. Phys.*, **36**, 1168-1198, 1958.

Some details can be found in the book of A. Udias: *Principles of Seismology*, Cambridge Univ. Press., 1999.

where $M_0 = \eta b\,(\text{area}\, S_D)$ is the seismic moment, η denotes the rigidity modulus. These are two arrivals, longitudinal and transversal, in a point with the distance r from the source.

The model indicated above specifies various notions of the earthquake such as seismic moment, its decomposition into various forces acting on the plane of the defect including the mentioned above double couples. However, in many respects it seems to be too simplified. For instance, it does not contain any criteria for the rupture. We shall not elaborate this subject any further.

Chapter 9

Appendix: Green functions for isotropic elastic materials

9.1 Purpose

Green's functions known also as fundamental solutions serve the purpose of construction of analytical solutions of linear differential equations. They also form the basis for at least two important procedures of approximation. The first one yields the so-called boundary element methods. The second one evaluates average macroscopic properties of materials with microstructure.

The Green function allows to solve the linear equation

$$\mathbf{L}\mathbf{u} + \mathbf{f} = 0, \tag{9.1}$$

where $\mathbf{L}$ is the linear operator, $\mathbf{u}$ an unknown vector function and $\mathbf{f}$ a given function. For the purpose of this Appendix we assume the operator $\mathbf{L}$ to be of the second order and the domain to be infinite. For finite domains one can obtain solutions by a simple transformation which we present further.

The formal solution of the equation (9.1) in the static case can be, for instance, written in the form of the following convolution integral[1]

$$\mathbf{u} = \mathbf{G} * \mathbf{f} = \int \mathbf{G}\left(\mathbf{x} - \mathbf{x}_1\right) \mathbf{f}\left(\mathbf{x}_1\right) dV_1, \tag{9.2}$$

where $\mathbf{G}$ is the Green tensor for the static problem.

In the following two Sections we present the construction of Green's functions for static and dynamic problems of linear elasticity. We follow here the presentation of T. D. Shermergor [17].

[1] e.g. R. de Wit; Continuum theory of stationary dislocations, *Solid State Physics,* **10,** 249, 1960.

9.2 Statics of isotropic elastic materials

In this Subsection we present the construction of Green's function for the linear elasticity in the static case. The operator $\mathbf{L}$ has in this case the following form

$$\mathbf{L} = L_{ik}\mathbf{e}_i \otimes \mathbf{e}_k, \quad L_{ik} = \frac{\partial}{\partial x_j} c_{ijkl} \frac{\partial}{\partial x_l}, \tag{9.3}$$

where c_{ijkl} is the tensor of elasticity. For heterogeneous materials it may be dependent on the point $\mathbf{x}$. We consider only homogeneous materials for which it consists of material constants. For isotropic materials it has the following explicit form

$$c_{ijkl} = \lambda\delta_{ik}\delta_{jl} + 2\mu\delta_{ij}\delta_{kl}, \tag{9.4}$$

where λ, μ are Lamé constants.

Substitution of (9.2) in (9.1) yields

$$\mathbf{L}(\mathbf{x}) \int \mathbf{G}(\mathbf{x} - \mathbf{x}_1) \mathbf{f}(\mathbf{x}_1) \, dV_1 = -\mathbf{f}(\mathbf{x}). \tag{9.5}$$

Consequently, the Green function must satisfy the equation

$$\mathbf{L}(\mathbf{x}) \mathbf{G}(\mathbf{x} - \mathbf{x}_1) = -\mathbf{1}\delta(\mathbf{x} - \mathbf{x}_1), \quad \mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{9.6}$$

and $\delta(\mathbf{x} - \mathbf{x}_1)$ is the Dirac function.

Obviously, in the linear elasticity the vector $\mathbf{u}$ denotes the displacement. Then the components of the Green function $G_{ij}(\mathbf{x} - \mathbf{x}_1)$ define the components of displacement $u_i(\mathbf{x})$ at the point $\mathbf{x}$ of the infinite medium caused by the unit force acting at the point $\mathbf{x}_1$ in the direction $\mathbf{e}_j$. If the medium is finite the Green function specifies the displacement $\mathbf{u}$ by the relation[2]

$$u_m(\mathbf{x}) = \int_V G_{im}(\mathbf{x} - \mathbf{x}_1) f_i(\mathbf{x}_1) \, dV_1 + \tag{9.7}$$

$$+ \int_{\partial V} \left[u_i(\mathbf{x}_1) c_{ijkl} \frac{\partial G_{km}}{\partial x_l}(\mathbf{x} - \mathbf{x}_1) + G_{im}(\mathbf{x} - \mathbf{x}_1) \sigma_{ij}(\mathbf{x}_1) \right] n_j dA_1,$$

where ∂V is the surface of the domain V and n_i are components of the normal vector of this surface.

The form of the Green tensor G_{ij} follows from the equation (9.6). It is usually found by means of the Fourier integral transform. We have

$$\bar{\mathbf{G}}(\mathbf{k}) = \int \mathbf{G}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{x}}, \quad \mathbf{G}(\mathbf{x}) = \frac{1}{8\pi^3} \int \bar{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{k}}. \tag{9.8}$$

Application of the Fourier transform to the equation (9.6) for $\mathbf{x}_1 = 0$ (this assumption is immaterial as for the infinite domain we can always shift the origin of the coordinates to the point $\mathbf{x}_1$), i.e. to the equation

$$c_{ijkl} \frac{\partial^2 G_{kn}}{\partial x_j \partial x_l}(\mathbf{x}) = -\delta_{in}\delta(\mathbf{x})$$

[2] R. de Wit; Continuum theory of stationary dislocations, *Solid State Physics,* **10,** 249, 1960.

yields

$$\bar{\zeta}_{ik}\bar{G}_{kn}(\mathbf{k}) = \delta_{in}, \quad \bar{\zeta}_{ik} = c_{ijkl}k_j k_l. \tag{9.9}$$

This is the set of algebraic relations which can be solved by inverting the matrix $\bar{\zeta}_{ik}$. In the case of isotropic materials given by (9.4) it is immediate. It can be done as well for materials with the hexagonal symmetry[3]. In general some approximate methods such as the method of perturbation must be applied[4], [5].

For isotropic materials we have

$$\bar{\zeta}_{ik} = \mu k^2 \delta_{ik} + (\lambda + \mu) k_i k_k, \quad k^2 = k_i k_i. \tag{9.10}$$

Consequently

$$\bar{G}_{ik} = \bar{\zeta}_{ik}^{-1} = \frac{1}{\mu}\left(\frac{1}{k^2}\delta_{ik} - \kappa \frac{k_i k_k}{k^2}\right), \quad \kappa = \frac{\lambda + \mu}{\lambda + 2\mu}. \tag{9.11}$$

It remains to invert the Fourier transform. Before we do so let us quote an identity which follows from considerations of electrostatics. In such a case the Maxwell equations describing the electromagnetic field reduce to the following two equations

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \quad \operatorname{rot}\mathbf{E} = 0, \tag{9.12}$$

where $\mathbf{E}$ the electric field and ρ is the electric charge density. The second relation implies the existence of the potential φ such that

$$\mathbf{E} = -\operatorname{grad}\varphi. \tag{9.13}$$

The choice of signs is the matter of tradition. Consequently, the equation determining the potential φ has the form of the Poisson equation

$$\nabla^2\varphi = -4\pi\rho, \quad \nabla^2 = \frac{\partial^2}{\partial x_k \partial x_k}, \tag{9.14}$$

[3]see: E. M. Lifshitz, L. N. Rosenzweig; O postrojenii tensora Grina dla osnownogo urawnienia teorii uprugosti w sluczaje nieograniczenoj uprugo-anisotropnoj sredy (in Russian), JETF, **17**, 9, 783, 1947,

E. Kröner;, Das Fundamentalintegral der anisotropen elastischen Differentialgleichungen, *Z. Phys.*, **151**, 4, 504, 1958,

L. Leicek; The Green function of the theory of elasticity in an anisotropic hexagonal medium, *Czechosl. J. Phys.*, **B19**, 6, 799, 1969.

[4]e.g. T. Mura; *Micromechanics of Defects in Solids*, 2nd ed., Martinus-Nijhoff, Dordrecht, 1987.

[5]e.g. the quotation of the Abstract of the paper of L. J. Gray, D. Ghosh, T. Kaplan; Evaluation of the anisotropic Green's function in three dimensional elasticity, *Computational Mechanics*, **17**, 4, 1996:

A perturbation expansion technique for approximating the three dimensional anisotropic elastic Green's function is presented. The method employs the usual series for the matrix $(\text{I–A})^{-1}$ to obtain an expansion in which the zeroth order term is an isotropic fundamental solution. The higher order contributions are expressed as contour integrals of matrix products, and can be directly evaluated with a symbolic manipulation program. A convergence condition is established for cubic crystals, and it is shown that convergence is enhanced by employing Voigt averaged isotropic constants to define the expansion point. Example calculations demonstrate that, for moderately anisotropic materials, employing the first few terms in the series provides an accurate solution and a fast computational algorithm. However, for strongly anisotropic solids, this approach will most likely not be competitive with the Wilson-Cruse interpolation algorithm.

where the last relation for the Laplace operator holds for Cartesian coordinates. An easy argument based on the balance equation of charge[6] yields for the charge density $\rho = e\delta(\mathbf{x})$ the following solution of (9.14)

$$\mathbf{E} = \frac{e\mathbf{x}}{r^3} \quad \Rightarrow \quad \varphi = \frac{e}{r}, \quad r^2 = \mathbf{x}\cdot\mathbf{x}, \tag{9.15}$$

where e is the electric charge. Obviously, it is the Coulomb law. Now the substitution of φ in (9.12) yields a representation of the Dirac δ-function by a function regular beyond the point $\mathbf{x} = \mathbf{0}$. We have

$$\delta(\mathbf{x}) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right). \tag{9.16}$$

We use this identity in the derivation of the Green function.

In addition to the above identity we have

$$\nabla^2 r = \frac{\partial^2 r}{\partial x_k \partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{x_k}{r}\right) = \frac{2}{r}. \tag{9.17}$$

Hence, bearing (9.16) in mind,

$$-8\pi\delta(\mathbf{x}) = \frac{\partial^4 r}{\partial x_k \partial x_k \partial x_l \partial x_l}. \tag{9.18}$$

We apply to this relation the Fourier transform. It follows

$$\int \frac{\partial^4 r}{\partial x_k \partial x_k \partial x_l \partial x_l} e^{-i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{x}} = \tag{9.19}$$

$$= \int \frac{\partial}{\partial x_k}\left(\frac{\partial^3 r}{\partial x_k \partial x_l \partial x_l} e^{-i\mathbf{k}\cdot\mathbf{x}}\right) dV_{\mathbf{x}} + ik_k \int \frac{\partial^3 r}{\partial x_k \partial x_l \partial x_l} e^{-i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{x}} = \dots$$

$$\dots = k^4 \int r e^{-i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{x}},$$

where we have used the Gauss divergence theorem and accounted for the fact that surface integrals must vanish for the infinite domain. Application of the inverse transform yields

$$r = -\frac{1}{\pi^2}\int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^4} dV_{\mathbf{k}}. \tag{9.20}$$

Differentiation of this relation leads to the following identities

$$\frac{\partial^2 r}{\partial x_k \partial x_l} = \frac{1}{\pi^2}\int \frac{k_k k_l}{k^4} e^{i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{k}}, \quad \nabla^2 r = \frac{1}{\pi^2}\int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2} dV_{\mathbf{k}}. \tag{9.21}$$

Now we are in the position to find the Fourier inverse of the relation (9.11). Bearing (9.8) in mind we obtain immediately

$$G_{ik}(\mathbf{x}) = \frac{1}{8\pi\mu}\left(\delta_{ik}\nabla^2 r - \kappa\frac{\partial^2 r}{\partial x_i \partial x_k}\right) = \tag{9.22}$$

$$= \frac{2-\kappa}{8\pi\mu}\left(\frac{\delta_{ik}}{r} + \frac{\kappa}{2-\kappa}\frac{x_i x_k}{r}\right), \quad \kappa = \frac{\lambda+\mu}{\lambda+2\mu} \equiv \frac{1}{2(1-\nu)}.$$

[6] e.g. L. D. Landau, E. M. Lifshitz; *Course of Theoretical Physic*, vol. 2: *The Classical Theory of Fields*, 4th ed., Oxford, Butterworth-Heinemann, 1980.

This is the Green function for static equations of the linear isotropic elasticity.

9.3 Dynamic Green function for isotropic elastic materials

In the dynamic case we have to include the inertial force in the momentum balance equation. Hence for the linear elasticity the relation (9.3) for the operator $\mathbf{L}$ must be replaced by the following one

$$\mathbf{L} = L_{ik}\mathbf{e}_i \otimes \mathbf{e}_k, \quad L_{ik} = -\delta_{ik}\rho\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial x_j}c_{ijkl}\frac{\partial}{\partial x_l}, \tag{9.23}$$

where ρ is the constant mass density.

For isotropic materials defined by (9.4), we have

$$L_{ik} = -- \delta_{ik}\rho\frac{\partial^2}{\partial t^2} + (\lambda + \mu)\frac{\partial^2}{\partial x_i \partial x_k} + \mu\frac{\partial^2}{\partial x_l \partial x_l}\delta_{ik}. \tag{9.24}$$

Green's tensor for the above operator will be sought again by the Fourier transformation. In the dynamic case we have to perform also the transformation with respect to time. To this aim we could use Laplace transform or, as we do below, we have to cut the Fourier transform to the range of nonnegative time. This reflects the principle of causality (determinism) of classical mechanics. Consequently, the Green function has to satisfy the following equations

$$\begin{aligned} L_{il}G_{lj}(\mathbf{x},t) &= \delta_{ij}\delta(\mathbf{x})\,\delta(t) \quad \text{for} \quad t \geq 0, \\ G_{ij} &= 0 \qquad\qquad\quad \text{for} \quad t < 0. \end{aligned} \tag{9.25}$$

Obviously, the Green function G_{ij} determines the displacement $u_i(\mathbf{x},t)$ in the instant of time t and at the point $\mathbf{x}$ caused by the unit force acting in the $\mathbf{e}_j$-direction in the instant of time $t = 0$ at the point $\mathbf{x} = \mathbf{0}$. Causality physically means that the displacement cannot be caused by incoming waves which should not exist yet before the force was applied at the point $\mathbf{x} = \mathbf{0}$.

As before, the displacement $u_i(\mathbf{x},t)$ for an arbitrary given external force $f_j(\mathbf{x},t)$ is then specified by the convolution integral

$$u_i = G_{il} * f_j. \tag{9.26}$$

Obviously the time integration in (9.25) yields immediately that the static Green function $G_{ij}(\mathbf{x})$, calculated in the previous Section should satisfy the relation

$$G_{ij}(\mathbf{x}) = \int_{-\infty}^{\infty} G_{ij}(\mathbf{x},t)\,dt. \tag{9.27}$$

As already mentioned we find the Green function for the infinite medium by the double Fourier transform

$$\begin{aligned}\bar{\mathbf{G}}(\mathbf{k},\omega) &= \int\int \mathbf{G}(\mathbf{x},t)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\omega}\mathbf{t})} dV_{\mathbf{x}} dt, \\ \mathbf{G}(\mathbf{x},t) &= \frac{1}{16\pi^4}\int\int \bar{\mathbf{G}}(\mathbf{k},\omega)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\omega}\mathbf{t})} dV_{\mathbf{k}} d\omega.\end{aligned} \tag{9.28}$$

We call $\mathbf{k}$ the wave vector and the scalar ω the frequency. Then after the Fourier transform the operator $L_{ij}(\mathbf{x},t)$ for isotropic materials has the following form

$$\bar{L}_{ij}(\mathbf{k},\omega) = \rho\omega^2\delta_{ij} - (\lambda+\mu)\, k_i k_j - \mu k^2 \delta_{ij}, \tag{9.29}$$

and the equation (9.25) becomes purely algebraic

$$\bar{L}_{ik}\bar{G}_{kj} = -\delta_{ij}. \tag{9.30}$$

We have to invert the matrix (9.29). Hence, after easy calculations the Fourier transform of the dynamic Green function is as follows

$$\bar{G}_{ij} = \frac{1}{\mu k^2 - \rho\omega^2}\left[\delta_{ij} - \frac{(\lambda+\mu)\, k_i k_j}{(\lambda+2\mu)\, k^2 - \rho\omega^2}\right]. \tag{9.31}$$

Obviously, the static transform of the Green function (9.11) follows from (9.31) by the substitution $\omega = 0$.

It is convenient to write the above relation by means of the speeds of propagation of longitudinal and transversal waves in a linear elastic material

$$c_L^2 = \frac{\lambda+2\mu}{\rho}, \quad c_T^2 = \frac{\mu}{\rho}. \tag{9.32}$$

We obtain

$$\rho\bar{G}_{ij} = \frac{1}{c_L^2 k^2 - \omega^2}\left[\delta_{ij} - \frac{\left(c_L^2 - c_T^2\right) k_i k_j}{c_L^2 k^2 - \omega^2}\right]. \tag{9.33}$$

The inverse of the above relation is given by the double integration prescribed by $(9.28)_2$. We perform first the integration with respect to the wave vector $\mathbf{k}$. We have

$$G_{ij}(\mathbf{x},\omega) = \frac{1}{8\pi^3}\int \bar{G}_{ij}(\mathbf{k},\omega)\, e^{-i\mathbf{k}\cdot\mathbf{x}} dV_{\mathbf{k}}. \tag{9.34}$$

Hence

$$G_{ij}(\mathbf{x},\omega) = I^0\delta_{ij} - \frac{\partial I}{\partial x_i \partial x_j}, \tag{9.35}$$

where

$$\begin{aligned}I^0 &= \frac{1}{8\pi^3}\int \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{c_T^2 k^2 - \omega^2} dV_{\mathbf{k}} = \frac{1}{4\pi r c_T^2}\exp\left(-\frac{i\omega r}{c_T}\right), \\ I &= -\frac{c_L^2 - c_T^2}{8\pi^3}\int \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{(c_L^2 k^2 - \omega^2)(c_T^2 k^2 - \omega^2)} dV_{\mathbf{k}} \\ &= \frac{1}{4\pi r\omega^2}\left[\exp\left(-\frac{i\omega r}{c_L}\right) - \exp\left(-\frac{i\omega r}{c_T}\right)\right].\end{aligned} \tag{9.36}$$

Calculations of integrals I^0 and I are made using the method of residua for complex functions. In order to use this method we assume formally that both speeds of propagation are complex. This would indeed be the case for viscoelastic materials. We demonstrate the calculations on the example of the integral I^0. Then

$$c_T^2(\omega) = \operatorname{Re} c_T^2(\omega) + i \operatorname{Im} c_T^2(\omega) = [1 + i\beta(\omega)] \operatorname{Re} c_T^2(\omega). \tag{9.37}$$

This extension yields the following form of the integral I^0

$$\begin{aligned} I^0 &= \frac{1}{8\pi^3} \int \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{c_T^2(\omega) k^2 - \omega^2} dV_{\mathbf{k}} = \\ &= \frac{1}{8\pi^3} \int_0^\infty \int_0^\pi \frac{e^{ikr\cos\theta} 2\pi \sin\theta k^2}{c_T^2(\omega) k^2 - \omega^2} dk d\theta = \\ &= \frac{1}{4\pi^2 ir} \int_0^\infty \frac{k\left(e^{ikr} - e^{-ikr}\right)}{c_T^2(\omega) k^2 - \omega^2} dk = \frac{1}{4\pi^2 ir} \int_{-\infty}^\infty \frac{k e^{ikr}}{c_T^2(\omega) k^2 - \omega^2} dk. \end{aligned} \tag{9.38}$$

This integral can be evaluated by the method of residua. Obviously, it possesses two poles

$$k = \pm \frac{\omega}{\{[1 + i\beta(\omega)] \operatorname{Re} c_T^2(\omega)\}^{0.5}} = \pm \frac{\omega \exp\left[-\frac{i}{2} \arctan\beta(\omega)\right]}{\left\{\sqrt{1 + \beta^2(\omega)} \operatorname{Re} c_T^2(\omega)\right\}^{0.5}}. \tag{9.39}$$

The pole with the minus sign lies in the second quadrant of the complex plane while the other pole lies in the fourth quadrant. Hence, we choose as the path of integration the real axis and the semicircle of the infinite radius in the upper part of the complex plane. We obtain

$$I^0 = \frac{1}{4\pi r c_T^2} \exp\left[-\frac{ir\omega}{c_T(\omega)}\right]. \tag{9.40}$$

The transition $c_T(\omega) \to c_T$ gives the desired result. In a similar manner one can calculate the integral I. We have to use obvious identities when differentiating in (9.35)

$$\frac{\partial r}{\partial x_i} = n_i, \quad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{1}{r}\left(\delta_{ij} - n_i n_j\right), \quad n_i n_i = 1. \tag{9.41}$$

It follows

$$G_{ij}(\mathbf{x}, \omega) = \frac{1}{r}\left[h(\omega r)\delta_{ij} + g(\omega r) n_i n_j\right], \tag{9.42}$$

where

$$\begin{aligned} h(\omega r) &= \frac{1}{4\pi r^2 \rho\omega^2}\left\{\left[\left(1 + \frac{ir\omega}{c_L}\right) e^{-i\omega r/c_L} - \left(1 + \frac{ir\omega}{c_T}\right) e^{-i\omega r/c_T}\right] + \frac{r^2\omega^2}{c_T^2} e^{-i\omega r/c_T}\right\}, \\ g(\omega r) &= -\frac{1}{4\pi r^2 \rho\omega^2}\left\{\left[3\left(1 + \frac{ir\omega}{c_L}\right) - -\frac{r^2\omega^2}{c_L^2}\right] e^{-i\omega r/c_L} - \right. \\ &\quad \left. - \left[3\left(1 + \frac{ir\omega}{c_T}\right) - \frac{r^2\omega^2}{c_T^2}\right] e^{-i\omega r/c_T}\right\} \end{aligned} \tag{9.43}$$

This is, obviously, the Green function for monochromatic waves of the given frequency ω. The time dependence of solution is then given by the factor $\exp(i\omega t)$.

It remains to perform the second inverse transformation. We use here the following relations

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega &= \delta(t), \\
\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\omega} e^{i\omega t} d\omega &= H(t) = \begin{cases} 1 \text{ for } t>0, \\ 0 \text{ for } t<0, \end{cases} \\
-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2} e^{i\omega t} d\omega &= \Psi(t) = \begin{cases} t \text{ for } t>0, \\ 0 \text{ for } t<0, \end{cases} \\
\frac{\partial \Psi}{\partial t} &= H(t), \quad \frac{\partial H}{\partial t} = \delta(t).
\end{aligned} \tag{9.44}
$$

Bearing these relation in mind as well as

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) &= \frac{3n_i n_j - \delta_{ij}}{r^3}, \qquad (9.45)\\
-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2}\left(1 + \frac{ir\omega}{c_L}\right) e^{i\omega(t-r/c_L)} d\omega &= \Psi\left(t - \frac{r}{c_L}\right) + \frac{r}{c_L} H\left(t - \frac{r}{c_L}\right),
\end{aligned}
$$

and similarly for c_T, we finally obtain the dynamic Green function

$$
\begin{aligned}
4\pi\rho G_{ij}(\mathbf{x}, t) = \delta\left(t - \frac{r}{c_T}\right)\left(\frac{\delta_{ij}}{c_T^2 r} - \frac{x_i x_j}{c_T^2 r^3}\right) + \\
+\delta\left(t - \frac{r}{c_L}\right)\frac{x_i x_j}{c_L^2 r^3} + \\
+\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) t \left(H\left(t - \frac{r}{c_L}\right) - H\left(t - \frac{r}{c_T}\right)\right).
\end{aligned} \tag{9.46}
$$

The first contribution describes the transversal part of the impulse which arrives with the speed c_T, the second contribution is the longitudinal part of the impulse which arrives with the speed c_L and the third contribution is the evolution of the impulse between the arrival of the longitudinal and transversal parts.

Easy integration in the relation (9.27) yields the static Green function given by the relation (9.22).

Bibliography

[1] K. Aki, P. G. Richards; *Quantitative Seismology*, University Science Books, Sausalito, 2002.

[2] Romesh C. Batra; *Elements of Continuum Mechanics*, AIAA Education Series, Reston, 2006.

[3] R. M. Christensen; *Theory of Viscoelasticity. An Introduction*, Academic Press, N. Y., 1971.

[4] M. E. Gurtin; *The Linear Theory of Elasticity*, in: C. Truesdell (ed.), Encyclopedia of Physics, vol. VIa/2 (Mechanics of Solids), Springer, Berlin, 1972.

[5] K. Hutter, K. Jöhnk; *Continuum Methods of Physical Modeling. Continuum Mechanics, Dimensional Analysis, Turbulence*, Springer, Berlin, 2004.

[6] J. C. Jaeger, N. G. W. Cook, R. W. Zimmerman; *Fundamentals of Rock Mechanics,* Blackwell, 2007.

[7] L. D. Landau, E. M. Lifshitz; *Mechanics,* Third Edition, Butterworth-Heinenann, Oxford, 1976.

[8] L. D. Landau, E. M. Lifshitz; *Theory of Elasticity*, 3rd ed., Oxford, Butterworth-Heinemann, 1986.

[9] J. Lemaitre, J.-L. Chaboche; *Mechanics of Solid Materials*, Cambridge University Press, 1990.

[10] J. Lemaitre, R. Desmorat; *Engineering Damage Mechanics,* Springer, Berlin, 2005.

[11] I-Shih Liu; *Continuum Mechanics*, Springer, Berlin, 2002.

[12] Y. A. Melnikov; *Influence Function Approach: Selected Topics of Structural Mechanics,* WIT Press, 2008.

[13] T. Mura; The continuum theory of dislocations, *Adv. Mater. Res.,* **3**, 1, 1968.

[14] R. W. Ogden; *Non-Linear Elastic Deformations*, Dover, Mineola, N. Y., 1984.

[15] A. C. Pipkin; *Lectures on Viscoelastic Theory*, Springer, New York, 1972.

[16] J. R. Rice; The Mechanics of Earthquake Rupture, in: *Physics of the Earth's Interior*, A. M. Dziewonski, E. Boschi (eds.), North Holland, 1980.

[17] T. D. Shermergor; *Teoria uprugosti mikroneodnorodnych sred,* (in Russian), "Nauka", Moscow, 1977.

[18] S. Timoshenko, J. N. Goodier; *Theory of Elasticity*, McGraw-Hill, N. Y., 1951.

[19] A. Verruijt; *Soil Mechanics,* Delft University of Technology, 2006.

[20] H. F. Wang; *Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology*, Princeton University Press, 2000.

[21] K. Wilmanski; *Thermomechanics of Continua*, Springer, Berlin, 1998.

[22] K. Wilmanski; *Continuum Thermodynamics. Part I: Foundations,* World Scientific, 2008.

[23] D. Muir Wood; *Soil Behaviour and Critical State Soil Mechanics,* Cambridge University Press, 1990.

[24] Han-Chin Wu; *Continuum Mechanics and Plasticity*, Chapman&Hall/CRC, 2005.

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