CHAPTER 6
Truss Analysis using Stiffness Method

Objectives

- เข้าใจวิธีของ **stiffness method**
- ใช้วิธี **stiffness method** กับ **Truss, BM & Frame** จะพูดถึงในบทหน้า

Fundamentals of the stiffness method

- The stiffness method:
- Is a disp method of analysis
- Can be used to analyse both statically determinate and indeterminate structures
- Yields the disp & forces directly
- It is generally much easier to formulate the necessary matrices for the computer using the stiffness method

Fundamentals of the stiffness method

- Application of the stiffness method requires subdividing the structure into a series of discrete finite elements & identifying their end points as nodes
- For truss analysis, the finite elements are represented by each of the members that compose the truss & the nodes represent the joints
- The force-disp properties of each element are determined & then related to one another using the force eqm eqn written at the nodes

Fundamentals of the stiffness method

- These relationships for the entire structure are then grouped together into the structure stiffness matrix, **K**
- The unknown disp of the nodes can then be determined for any given loading on the structure
- When these disp are known, the external & internal forces in the structure can be calculated using the force-disp relations for each member

Member stiffness matrix

- To establish the stiffness matrix for a single truss member using local x' and y' coordinates as shown When a +ve disp d_N is imposed on the near end of the member while the far end is held pinned
- The forces developed at the ends of the members are:

$$q'_N = \frac{AE}{L} d_N ; \quad q'_F = -\frac{AE}{L} d_N$$

Member stiffness matrix

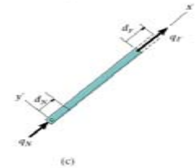
- Likewise, a +ve disp dF at the far end, keeping the near end pinned and results in member forces

$$q''_N = -\frac{AE}{L} d_F ; \quad q''_F = \frac{AE}{L} d_F$$

- By superposition, the resultant forces caused by both disp are

$$q_N = \frac{AE}{L} d_N - \frac{AE}{L} d_F$$

$$q_F = \frac{AE}{L} d_F - \frac{AE}{L} d_N$$



Member stiffness matrix

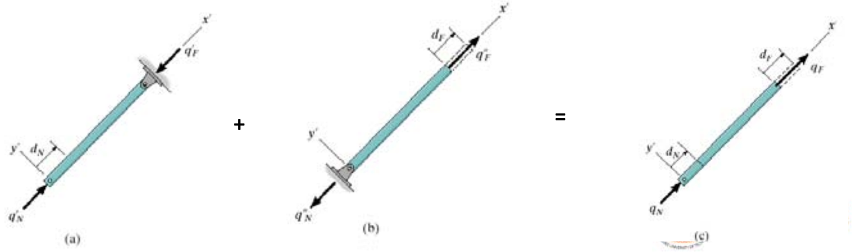
- These load-disp eqn may be written in matrix form as:

$$\begin{bmatrix} q_N \\ q_F \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_N \\ d_F \end{bmatrix}$$

$$q = k' d$$

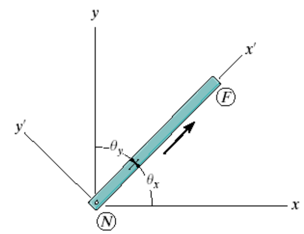
$$k' = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- This matrix, k' is called the member stiffness matrix



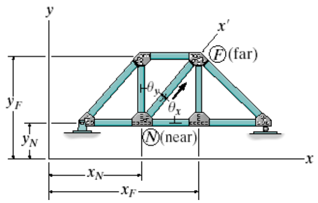
Displacement & Force Transformation matrices

- Since a truss is composed of many members, we will develop a method for transforming the member forces q and disp d defined in local coordinates to global coordinates
- Global coordinates convention: +ve x to the right and +ve y upward
- θ_x and θ_y as shown



Displacement & Force Transformation matrices

- matrix analysis as follows
- These will be identified as $\lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$
- For e.g. consider member NF of the truss as shown
- The coordinates of N & F are (x_N, y_N) and (x_F, y_F)



Displacement & Force Transformation matrices

$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L}$$

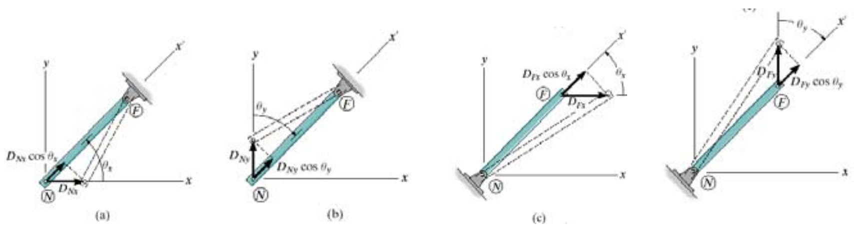
$$= \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L}$$

$$= \frac{y_F - y_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

Displacement & Force Transformation matrices

- Disp Transformation matrix
  - In global coordinates each end of the member can have 2 degrees of freedom or independent disp; namely joint N has D_{Nx} and D_{Ny}
  - Joint F has D_{Fx} and D_{Fy}



Displacement & Force Transformation matrices

- Disp Transformation matrix
  - When the far end is held pinned & the near end is given a global disp, the corresponding disp along member is $D_{Nx}\cos\theta_x$
  - A disp D_{ny} will cause the member to be displaced $D_{Ny}\cos\theta_y$ along the x' axis

$$d_N = D_{N_x}\cos\theta_x + D_{N_y}\cos\theta_y$$

$$d_F = D_{F_x}\cos\theta_x + D_{F_y}\cos\theta_y$$

Displacement & Force Transformation matrices

- **Disp Transformation matrix**

Let $\lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$

$d_N = D_{N_x}\lambda_x + D_{N_y}\lambda_y; \quad d_F = D_{F_x}\lambda_x + D_{F_y}\lambda_y$

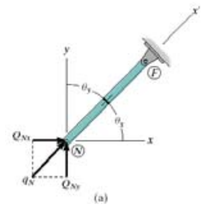
In matrix form,

$$\begin{bmatrix} d_N \\ d_F \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix} \begin{bmatrix} D_{N_x} \\ D_{N_y} \\ D_{F_x} \\ D_{F_y} \end{bmatrix}$$

$$d = TD$$

Displacement & Force Transformation matrices

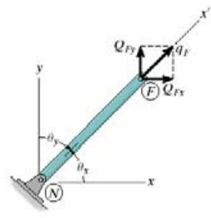
- **Force Transformation matrix**



$$Q_{N_x} = q_N \cos\theta_x; \; Q_{N_y} = q_N \cos\theta_y$$

  - **If q_F is applied to the bar, the global force components at F are:**

$$Q_{F_x} = q_F \cos\theta_x; \; Q_{F_y} = q_F \cos\theta_y$$



  - **Using** $\lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$

$$Q_{N_x} = q_N\lambda_x; \; Q_{N_y} = q_N\lambda_y$$
$$Q_{F_x} = q_F\lambda_x; \; Q_{F_y} = q_F\lambda_y$$

Displacement & Force Transformation matrices

- **Force Transformation matrix**
  - **In matrix form**

$$\begin{bmatrix} Q_{N_x} \\ Q_{N_y} \\ Q_{F_x} \\ Q_{F_y} \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \begin{bmatrix} q_N \\ q_F \end{bmatrix}$$

$$Q = T^T q$$

Displacement & Force Transformation matrices

- **Force Transformation matrix**
  - **In this case, TT transforms the 2 local forces q acting at the ends of the member into 4 global force components Q**
  - **This force transformation matrix is the transpose of the disp transformation matrix**

Member global stiffness matrix

- We can determine the member's forces q in terms of the global disp D at its end points

$$q = k'TD$$

- Substitution yields the final result:

$$Q = T^T k'TD$$

$$\text{or } \mathrm{Q} = \mathrm{kD}$$

$$k = T^T k'T$$

Member global stiffness matrix

- Performing the matrix operation yields:

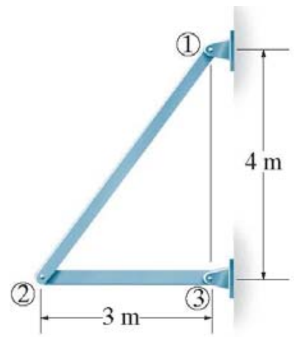
$$k = \frac{AE}{L}\begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} N_x & N_y & F_x & F_y \end{array} & \\ \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix} & \begin{array}{c} N_x \\ N_y \\ F_x \\ F_y \end{array} \end{array}$$

Truss stiffness matrix

- Once all the member stiffness matrices are formed in the global coordinates, it becomes necessary to assemble them in the proper order so that the stiffness matrix K for the entire truss can be found
- This is done by designating the rows & columns of the matrix by the 4 code numbers used to identify the 2 global degrees of freedom that can occur at each end of the member
- The structure stiffness matrix will then have an order that will be equal to the highest code number assigned to the truss since this rep the total no. of degree of freedom for the structure
- This method of assembling the member matrices to form the structure stiffness matrix will now be demonstrated by numerical e.g.
- This process is somewhat tedious when performed by hand but is rather easy to program on computer

Example 1

Determine the structure stiffness matrix for the 2 member truss as shown. AE is constant.

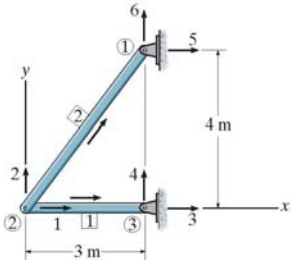


Example 1 cont'd

Member 1

$$\lambda_x = \frac{3-0}{3} = 1; \quad \lambda_y = \frac{0-0}{3} = 0$$

Dividing each element by L = 3m, we have:



Example 1 cont'd

Member 2

$$\lambda_x = \frac{3-0}{5} = 0.6; \quad \lambda_y = \frac{4-0}{5} = 0.8$$

Dividing each element by L = 5 m, we have:

$$k_2 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Example 1 cont'd

- This matrix has an order of 6x6 since there are 6 designated degrees of freedom for the truss.

Application of the stiffness method for truss analysis

- The global force components Q acting on the truss can then be related to its global displacements D using

$$Q = KD$$

- This eqn is referred to as the structure stiffness eqn

$$\boxed{\begin{bmatrix} \mathbf{Q}_k \\ \mathbf{Q}_u \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{D}_u \\ \mathbf{D}_k \end{bmatrix}}$$

Application of the stiffness method for truss analysis

- Expanding yields

$$Q_k = K_{11}D_u + K_{12}D_k$$
$$Q_u = K_{21}D_u + K_{22}D_k$$

- Often D_k = 0 since the supports are not displaced
- Thus becomes

$$Q_k = K_{11}D_u$$

Application of the stiffness method for truss analysis

- Since the elements in the partitioned matrix K_{11} represent the total resistance at a truss joint to a unit disp in either the x or y direction, then the above eqn symbolizes the collection of all the force eqm eqn applied to the joints where the external loads are zero or have a known value Q_k
- Solving for D_u, we have:

$$D_u = [K_{11}]^{-1} Q_k$$

Application of the stiffness method for truss analysis

- With D_k =0 yields $\quad Q_u = K_{21}D_u$

- The member forces can be determined

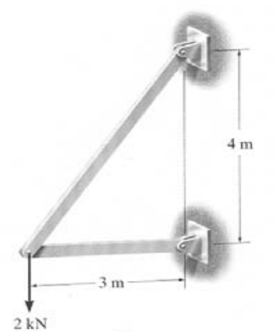
$$\begin{bmatrix} q_N \\ q_F \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}\begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{bmatrix}$$

- Since with $q_N = -q_F$ for eqm,

$$[q_F] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\lambda_x & -\lambda_y & \lambda_x & \lambda_y \end{bmatrix}\begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{bmatrix}$$

Example 2

Determine the force in each member of the 2-member truss as shown. AE is constant.



(a)

Example 2 cont'd

The origin of x,y and the numbering of the joints & members are shown.

By inspection, it is seen that the known external disp are $D_3=D_4=D_5=D_6=0$

Also, the known external loads are $Q_1=0$, $Q_2=-2kN$.

Hence,

$$D_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \quad Q_k = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Example 2 cont'd

Q = KD for the truss we have

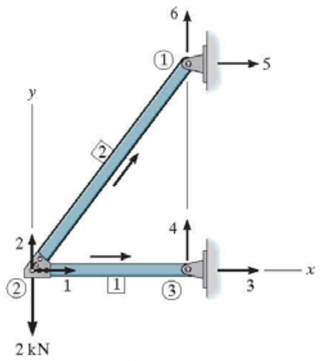
We can now identify K_{11} and thereby determine D_u

By matrix multiplication,

Example 2 cont'd

By inspection one would expect a rightward and downward disp to occur at joint 2 as indicated by the +ve & -ve signs of the answers.

Using these results,



Example 2 cont'd

Expanding & solving for the reactions

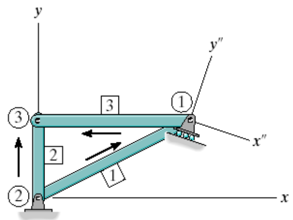
$$Q_3 = -1.5\ kN$$
$$Q_4 = 0\ kN$$
$$Q_5 = 1.5\ kN$$
$$Q_6 = 2.0\ kN$$

The force in each member can be found.

Using the data for λ_x and λ_y in example 14.1, we have:

Nodal Coordinates

- A truss can be supported by a roller placed on a incline
- When this occurs, the constraint of zero deflection at the support (node) cannot be directly defined using a single horizontal & vertical global coordinate system
- Consider the truss
- The condition of zero disp at node 1 is defined only along the y” axis

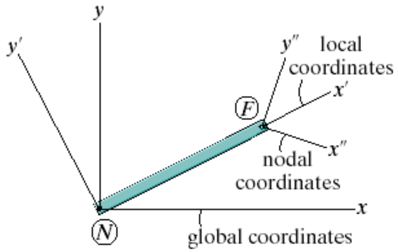


Nodal Coordinates

- Because the roller can displace along the x” axis this node will have disp components along both global coordinates axes x & y
- To solve this problem, so that it can easily be incorporated into a computer analysis, we will use a set of nodal coordinates x”, y” located at the inclined support
- These axes are oriented such that the reactions & support disp are along each of the coordinate axes
- To determine the global stiffness eqn for the truss, it becomes necessary to develop force & disp transformation matrices for each of the connecting members at this support so that the results can be summed within the same global x, y coordinate system

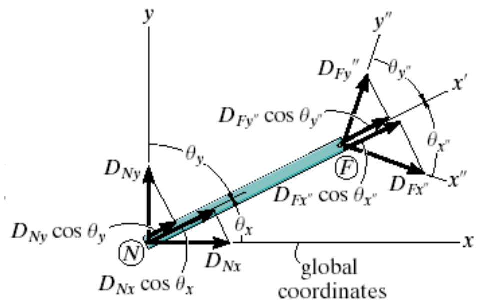
Nodal Coordinates

- Consider truss member 1 having a global coordinate system x, y at the near node and a nodal coordinate system x”, y” at the far node



Nodal Coordinates

- When disp D occur so that they have components along each of these axes as shown



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Nodal Coordinates

- This eqn can be written in matrix form as

$$\begin{bmatrix} d_N \\ d_F \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix} \begin{bmatrix} D_{N_x} \\ D_{N_y} \\ D_{F_{x''}} \\ D_{F_{y''}} \end{bmatrix}$$

$$Q_{N_x} = q_N \cos\theta_x \qquad Q_{N_y} = q_N \cos\theta_y$$

$$Q_{F_{x''}} = q_F \cos\theta_{x''} \qquad Q_{F_{y''}} = q_F \cos\theta_{y''}$$

Nodal Coordinates

- This can be expressed as:

$$\begin{bmatrix} Q_{N_x} \\ Q_{N_y} \\ Q_{F_{x''}} \\ Q_{F_{y''}} \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \begin{bmatrix} q_N \\ q_F \end{bmatrix}$$

- The disp & force transformation matrices in the above eqn are used to develop the member stiffness matrix for this situation
- We have $k = T^T k' T$

$$k = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_{x''} & \lambda_{y''} \end{bmatrix}$$

Nodal Coordinates

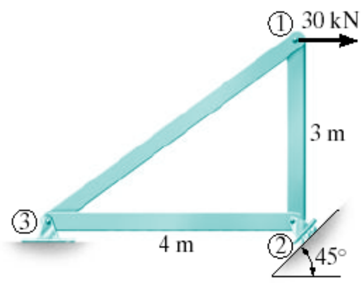
- Performing the matrix operation yields:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$

- This stiffness matrix is used for each member that is connected to an inclined roller support
- The process of assembling the matrices to form the structure stiffness matrix follows the standard procedure

Example 3

Determine the support reactions for the truss as shown.



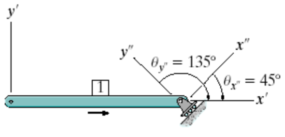
Example 3 cont'd

Determine the support reactions for the truss as shown.

The stiffness matrices for members developed.

Member 1,

$$\lambda_x = 1,\ \lambda_y = 0,\ \lambda_{x''} = 0.707,\ \lambda_{y''} = -0.707$$



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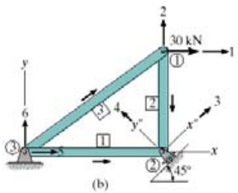

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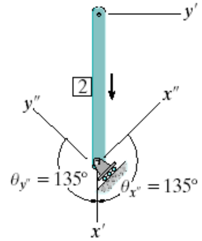
Example 3 cont'd

Member 2,

$$\lambda_x = 0,\ \lambda_y = -1,\ \lambda_{x''} = -0.707,\ \lambda_{y''} = -0.707$$



Example 3 cont'd

Member 3,

$$\lambda_x = 0.8,\ \lambda_y = 0.6$$

Example 3 cont'd

To determine the structure stiffness matrix, we have:

$$\begin{bmatrix} 30 \\ 0 \\ 0 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = AE \begin{bmatrix} 0.128 & 0.096 & 0 & 0 & -0.128 & -0.096 \\ 0.096 & 0.4053 & -0.2357 & -0.2357 & -0.096 & -0.072 \\ 0 & -0.2357 & 0.2917 & 0.0417 & -0.17675 & 0 \\ 0 & -0.2357 & 0.0417 & 0.2917 & 0.17675 & 0 \\ -0.128 & -0.096 & -0.17675 & 0.17675 & 0.378 & 0.096 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 3 cont'd

Carrying out the matrix multiplication of the upper partitioned matrices, the three unknown disp D are determined from solving the resulting simultaneous eqn.

The unknown reactions Q are obtained from the multiplication of the lower partitioned matrices.

SURANAREE
UNIVERSITY OF TECHNOLOGY