STUDIES IN *FUZZINESS* AND *SOFT COMPUTING*

John N. Mordeson
Premchand S. Nair

# Fuzzy Mathematics

## An Introduction for Engineers and Scientists

Second Edition

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Fuzzy Mathematics

# Studies in Fuzziness and Soft Computing

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With 30 Figures
and 9 Tables

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# FOREWORD

In the mid-1960's I had the pleasure of attending a talk by Lotfi Zadeh at which he presented some of his basic (and at the time, recent) work on fuzzy sets. Lotfi's algebra of fuzzy subsets of a set struck me as very nice; in fact, as a graduate student in the mid-1950's, I had suggested similar ideas about continuous-truth-valued propositional calculus (inf for "and", sup for "or") to my advisor, but he didn't go for it (and in fact, confused it with the foundations of probability theory), so I ended up writing a thesis in a more conventional area of mathematics (differential algebra). I especially enjoyed Lotfi's discussion of fuzzy convexity; I remember talking to him about possible ways of extending this work, but I didn't pursue this at the time.

I have elsewhere told the story of how, when I saw C.L. Chang's 1968 paper on fuzzy topological spaces, I was impelled to try my hand at fuzzifying algebra. This led to my 1971 paper "Fuzzy groups", which became the starting point of an entire literature on fuzzy algebraic structures.

In 1974 King-Sun Fu invited me to speak at a U.S.-Japan seminar on Fuzzy Sets and their Applications, which was to be held that summer in Berkeley. I wasn't doing any work in fuzzy mathematics at that time, but I had a long-standing interest in fuzzifying some of the basic ideas of pattern recognition, so I put together a paper dealing with fuzzy relations on fuzzy sets, treating them from the viewpoint of fuzzy graphs. This doesn't seem to have led to a flood of papers on fuzzy graph theory, but the topic does have important applications, and perhaps the prominence given to it in this book will lead to renewed activity in this area. [My 1976 paper "Scene labeling by relaxation operations" (with R.A. Hummel and S.W. Zucker; *IEEE Trans.* SMC-6, 420–433) included a treatment of the fuzzy version of the constraint satisfaction problem, but this too hasn't been widely followed up.]

Over the past 20 years I have made many excursions into various areas of fuzzy geometry, digital and otherwise. My 1978 paper "A note on the use of local min and max operators in digital picture processing" (with Y. Nakagawa; *IEEE Trans.* **SMC-8**, 632–635) was the first published treatment of fuzzy mathematical morphology (local max=dilation, local min=erosion). My 1979 paper "Thinning algorithms for gray-scale pictures" (with C.R. Dyer; *IEEE Trans.* **PAMI-1**, 88–89) presented a fuzzy version of the thinning process in which local min is applied only where it does not weaken fuzzy connectedness, as defined in my paper "Fuzzy digital topology" (published the same year).

The number of papers on fuzzy (digital) geometry grew steadily during the 1980's. As early as 1984 I was able to survey the subject in my paper "The fuzzy geometry of image subsets" (*Pattern Recognition Letters* **2**, 311–317). A subsequent survey appeared in the Proceedings of the First IEEE International Conference on Fuzzy Systems (San Diego, March 8–12, 1992), pp. 113–117. An updated version of this survey was presented at the opening session of the Joint Conference on Information Sciences in Wrightsville Beach, NC on September 28, 1995 and is to appear in Information Sciences. [An early reference on fuzzy geometry, inadvertently omitted from these surveys is J.G. Brown, "A note on fuzzy sets," *Info. Control* **18**, 1971, 32–39.]

At the end of 1980 I attempted to publish a tutorial paper on fuzzy mathematics in the American Mathematical Monthly, but the editors apparently didn't like the topic. I then tried the Mathematical Intelligencer; they eventually published a heavily revised version of that paper ("How many are few? Fuzzy sets, fuzzy numbers, and fuzzy mathematics", **2**, 1982, 139–143), but most of my discussion of fuzzy mathematical structures was dropped. The original version was University of Maryland Computer Science Technical Report 991, December 1980; it finally saw print in Paul Wang's book *Advances in Fuzzy Theory and Technology I,* Bookwrights Press, Durham, NC, 1993, 1–8.

I was delighted to hear that Profs. Mordeson and Nair were publishing this book-length introduction to fuzzy mathematics. I hope the book is successful and stimulates increased interest in the subject.

*Azriel Rosenfeld*
*University of Maryland*

# PREFACE

We eagerly accepted the invitation of Physica-Verlag to prepare a second edition of our book. The second edition contains an expanded version of the first. The first four chapters remain essentially the same. Chapter 5 is expanded to contain the work of Rosenfeld and Klette dealing with the degree of adjacency and the degree of surroundness. The work of Pal and Rosenfeld on image enhancement and thresholding by optimization of fuzzy compactness is also included. Rosenfeld's results on Hausdorff distance between fuzzy subsets is also included. We expand the geometry of Buckley and Eslami concerning points and lines in fuzzy plane geometry and include their new work on circles and polygons in fuzzy plane geometry. In Chapter 6, we add the latest results on the solution of nonlinear systems of fuzzy intersection equations of fuzzy singletons.

The book deals with fuzzy graph theory, fuzzy topology, fuzzy geometry, and fuzzy abstract algebra. The book is based on papers that have appeared in journals and conference proceedings. Many of the results that appear in the book are based on the work of Azriel Rosenfeld. The purpose of the book is to present the concepts of fuzzy mathematics from these areas which have applications to engineering, science, and mathematics. Some specific application areas are cluster analysis, digital image processing, fractal compression, chaotic mappings, coding theory, automata theory, and nonlinear systems of fuzzy equations. The style is geared to an audience more general than the research mathematician. In particular, the book is written with engineers and scientists in mind. Consequently, many theorems are stated without proof and many examples are given. Crisp results of the more abstract areas of mathematics are reviewed as needed, e. g., topology and abstract algebra. However some mathematical sophistication is required of the reader. Even though the book is not directed solely to mathematicians, it involves current mathematical results and so serves as a research book to those wishing to do research in fuzzy mathematics.

In Chapter 1, basic concepts of fuzzy subset theory are given. The notion of a fuzzy relation and its basic properties are presented. The concept of a fuzzy relation is fundamental to many of the applications given, e. g., cluster analysis and pattern classification. Chapter 1 is based primarily on the work of Rosenfeld and Yeh and Bang.

Chapter 2 deals with fuzzy graphs. Here again most of the results of this chapter are based on the work of Rosenfeld and Yeh and Bang. Applications of fuzzy graphs to cluster analysis and database theory are presented.

Chapter 3 concerns fuzzy topology. We do not attempt to give anywhere close to a complete treatment of fuzzy topology. There are two books devoted entirely to fuzzy topology and as a combination give an extensive study of fuzzy topology. These two books are by Diamond and Kloeden and by Liu and Luo. Their exact references can be found at the end of Chapter 3. In Chapter 3, we review some basic results of topology. We then feature the original paper on fuzzy topology by C. L. Chang. For the remainder of the chapter, we concentrate on results from fuzzy topology which have applications. These results deal with metric spaces of fuzzy subsets.

In Chapter 4, we present the work of Rosenfeld on fuzzy digital topology. An application to digital image processing is given. The chapter also treats nontopological concepts such as (digital) convexity.

Chapter 5 is on fuzzy geometry. Once again the work of Rosenfeld is featured. The fuzzy theory developed in this chapter is applicable to pattern recognition, computer graphics, and imaging processing. We also present the geometry currently under development by Buckley and Eslami. The presentation of their geometry is not complete since the book goes to press before their geometry is completed. We have expanded this and the next chapter as described above.

Chapter 6 deals with those results from fuzzy abstract algebra which have known applications. Rosenfeld is the father of fuzzy abstract algebra. He published only one paper on the subject. However this paper led to hundreds of research papers on fuzzy algebraic substructures of various algebraic structures. Here, as in the chapter on fuzzy topology, we review crisp concepts which are needed for the understanding of the chapter.

Of course the whole notion of fuzzy set theory is due to Lotfi Zadeh. His classic paper in 1965 has opened up new insights and applications in a wide range of scientific areas. A large part of Zadeh's orginal paper on fuzzy sets deals with fuzzy convexity. This notion plays an important role in this text.

*John N. Mordeson*
*Premchand S. Nair*

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The authors are grateful to the editorial and production staffs of Physica-Verlag, especially Janusz Kacprzyk, Martina Bihn, and Gabriele Keidel. We are indebted to Paul Wang of Duke University for his constant support of fuzzy mathematics. We are also appreciative of the support of Fr. Michael Proterra, Dean, Creighton College of Arts and Sciences. We thank Dr. and Mrs. George Haddix for their support of the center. We also thank Dr. Mark Wierman for showing us many important features of LaTeX. The first author dedicates the book to his youngest grandchildren Jessica and David. The second author is very grateful to his wife, Suseela, and his parents, Mr. Sukumaran Nair and Ms. Sarada Devi, for supporting his dreams.

# CONTENTS

# 1

# FUZZY SUBSETS

In this chapter we explore fuzziness as tool to capture uncertainty. Let $S$ be a set and let $A$ and $B$ be subsets of $S$. We use the notation $A \cup B$, $A \cap B$ to denote the union of $A$ and $B$ and intersection of $A$ and $B$, respectively. Let $B \backslash A$ denote the relative complement of $A$ in $B$. The (relative) complement of $A$ in $S$, $S \backslash A$, is sometimes denoted by $A^c$ when $S$ is understood.

Let $x$ be an element of $S$. If $x$ is an element of $A$, we write $x \in A$; otherwise we write $x \notin A$. We use the notation $A \subseteq B$ or $B \supseteq A$ to denote that $A$ is a subset of $B$. If $A \subseteq B$, but there exists $x \in B$ such that $x \notin A$, then we write $A \subset B$ or $B \supset A$ and say that $A$ is a proper subset of $B$. The cardinality of $A$ is denoted by $|A|$ or card$(A)$. The power set of $A$, written $\wp(A)$, is defined to be the set of all subsets of $A$. i. e., $\wp(A) = \{U | U \subseteq A\}$. A *partition* of $S$ is a set $\mathcal{P}$ of nonempty subsets of $S$ such that $\forall U, V \in \mathcal{P}$, either (1) $U = V$ or $U \cap V = \emptyset$, the empty set, and (2) $S = \bigcup_{U \in \mathcal{P}} U$.

We let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers.

Let $X$ and $Y$ be sets. If $x \in X$ and $y \in Y$, then $(x, y)$ denotes the ordered pair of $x$ with $y$. The *Cartesian cross product* of $X$ with $Y$ is defined to be the set $\{(x, y) | x \in X, y \in Y\}$ and is denoted by $X \times Y$. At times we write $X^2$ for $X \times X$. In fact, for $n \in \mathbb{N}, n \geq 2$, we let $X^n$ denote the set of all ordered $n$-tuples of elements from $X$. A *relation* $R$ of $X$ into $Y$ is a subset of $X \times Y$. Let $R$ be such a relation. Then the *domain* of $R$, written Dom$(R)$, is $\{x \in X | \exists y \in Y$ such that $(x, y) \in R\}$ and the *image* of $R$, written Im$(R)$, is $\{y \in Y | \exists x \in X$ such that $(x, y) \in R\}$. If $(x, y) \in R$, we sometimes write $xRy$ or $R(x) = y$. If $R$ is a relation from $X$ into $X$, we say that $R$ is a relation on $X$. A relation $R$ on $X$ is called

(i) *reflexive* if $\forall x \in X$, $(x, x) \in R$;

(ii) *symmetric* if $\forall x, y \in X$, $(x, y) \in R$ implies $(y, x) \in R$;

(iii) *transitive* if $\forall x, y, z \in X$, $(x, y)$ and $(y, z) \in R$ implies $(x, z) \in R$.

If $R$ is a relation on $X$ which is reflexive, symmetric, and transitive, then $R$ is called an *equivalence relation.* If $R$ is an equivalence relation on $X$, we let $[x]$ denote the equivalence class of $x$ with respect to $R$ and so $[x] = \{a \in X|aRx\}$. If $R$ is an equivalence relation on $X$, then $\{[x]|x \in X\}$ is a partition of $X$. Also if $\mathcal{P}$ is a partition of $X$ and $R$ is the relation on $X$ defined by $\forall x, y \in X$, $(x, y) \in R$ if $\exists U \in \mathcal{P}$ such that $x, y \in U$, then $R$ is an equivalence relation on $X$ whose equivalence classes are exactly those members of $\mathcal{P}$.

A relation $R$ on $X$ is called *antisymmetric* if $\forall x, y \in X$, $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$. If $R$ is a reflexive, antisymmetric, and transitive relation on $X$, then $R$ is called a *partial order* on $X$ and $X$ is said to be *partially ordered* by $R$.

Let $R$ be a relation of $X$ into $Y$ and $T$ a relation of $Y$ into a set $Z$. Then the *composition* of $R$ with $T$, written $T \circ R$, is defined to be the relation $\{(x, z) \in X \times Z|\exists y \in Y$, such that $(x, y) \in R$ and $(y, z) \in T\}$.

If $f$ is a relation of $X$ into $Y$ such that $\mathrm{Dom}(f) = X$ and $\forall x, x' \in X$, $x = x'$ implies $f(x) = f(x')$, then $f$ is called a *function* of $X$ into $Y$ and we write $f : X \to Y$. Let $f$ be a function of $X$ into $Y$. Then $f$ is sometimes called a *mapping* and $f$ is said to *map* $X$ into $Y$. If $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$, then $f$ is said to be *onto* $Y$ or to map $X$ *onto* $Y$. If $\forall x, x' \in X$, $f(x) = f(x')$ implies that $x = x'$, then $f$ is said to be *one-to-one* and $f$ is called an *injection.* If $f$ is a one-to-one function of $X$ onto $Y$, then $f$ is called a *bijection.* If $g$ is a function of $Y$ into a set $Z$, then the composition of $f$ with $g$, $g \circ f$, is a function of $X$ into $Z$ which is one-to-one if $f$ and $g$ are and which is onto $Z$ if $f$ is onto $Y$ and $g$ is onto $Z$. If $\mathrm{Im}(f)$ is finite, the we say that $f$ is *finite-valued.* We say that an infinite set $X$ is *countable* if there exists a one-to-one function of $X$ onto $\mathbb{N}$; otherwise we call $X$ *uncountable.*

Fuzzy theory holds that many things in life are matters of degree. A black and white photo is not just black and white; there are many levels of gray shades which can be observed in a typical picture. Computer scientists and engineers have long accepted this fact. As an example, a pixel can have a brightness value between 0 and 255. The 0 value stands for black, 255 stands for white and every number between 0 and 255 stands for a certain gray level.

Let $S$ be a set. A *fuzzy subset* of $S$ is a mapping $\tilde{A} : S \to [0, 1]$. We think of $\tilde{A}$ as assigning to each element $x \in S$ a degree of membership,

$0 \leq \tilde{A}(x) \leq 1$. Let $\tilde{A}$ be a fuzzy subset of $S$. We let $\tilde{A}^t = \{x \in S | \tilde{A}(x) \geq t\}$ for all $t \in [0,1]$. The sets $\tilde{A}^t$ are called *level sets* or *t-cuts* of $\tilde{A}$. We let $\text{supp}(\tilde{A}) = \{x \in S | \tilde{A}(x) > 0\}$. We call $\text{supp}(\tilde{A})$ the *support* of $\tilde{A}$. The set of all fuzzy subsets of $S$ is denoted by $\mathfrak{F}\wp(S)$ and is called the *fuzzy power set* of $S$.

**Example 1.1** *Let $S = \{a, b, c, d\}$. Then $A = \{a, b\}$ is a subset of $S$. On the other hand the mapping $\tilde{A} : S \to [0, 1]$ such that $\tilde{A}(a) = 1, \tilde{A}(b) = 1, \tilde{A}(c) = 0, \tilde{A}(d) = 0$ is a fuzzy subset of $S$. Similarly, $B = \{a, c, d\}$ is a subset of $S$ and the mapping $\tilde{B} : S \to [0, 1]$ such that $\tilde{B}(a) = 1, \tilde{B}(b) = 0, \tilde{B}(c) = 1, \tilde{B}(d) = 1$ is a fuzzy subset of $S$. We see that corresponding to a subset $X$ of $S$, there is always a fuzzy subset $\tilde{X}$ of $S$ with the following property.*

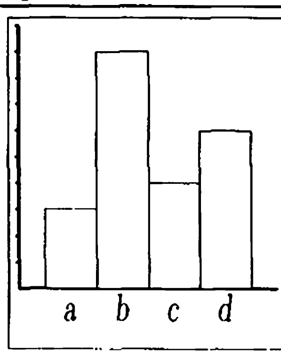
**(i)** *$x \in X$ if and only if $\tilde{X}(x) = 1$*

**(ii)** *$x \notin X$ if and only if $\tilde{X}(x) = 0$*

*On the other hand the mapping $\tilde{C} : S \to [0, 1]$ which assigns $\tilde{C}(a) = 0.3, \tilde{C}(b) = 0.9, \tilde{C}(c) = 0.4, \tilde{C}(d) = 0.625$ is a fuzzy subset of $S$. Corresponding to the fuzzy subset $\tilde{C}$ there are five level subsets of $S$, as shown below.*

$$\tilde{C}^t = \begin{cases} S & 0 \leq t \leq 0.3 \\ \{b, c, d\} & 0.3 < t \leq 0.4 \\ \{b, d\} & 0.4 < t \leq 0.625 \\ \{b\} & 0.625 < t \leq 0.9 \\ \emptyset & 0.9 < t \leq 1.0 \end{cases}$$

**FIGURE 1.1** Graphical representation of the fuzzy set $\tilde{C}$.




We see that a finite-valued fuzzy subset $\tilde{C}$ determines a "chain of subsets of $S$". Conversely, given a finite chain of subsets $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_k = S, k > 0$, there exists a fuzzy subset $\tilde{C}$ such that its level sets are

$C_1, C_2, \ldots, C_k = S$. The construction of $\tilde{C}$ can be outlined as follows. Choose $k$ numbers $0 \le t_1 < t_2 < \ldots < t_k \le 1$. For $s \in S$, define $\tilde{C}(s)$ as follows:

$$\tilde{C}(s) = \begin{cases} t_i & \text{if } s \in C_{k-i+1} \setminus C_{k-i}, i = 1, 2, \ldots, k-1, \\ t_k & \text{if } s \in C_1. \end{cases}$$

Let $A$ be a subset of a set $S$. Define $\chi_A : S \to [0, 1]$ by $\chi_A(a) = 1$ if $a \in A$ and $\chi_A(a) = 0$ if $a \in S \setminus A$. Then $\chi_A$ is called the *characteristic function* of $A$ in $S$.

**Definition 1.1** *Let $\tilde{A}, \tilde{B}$ be any two fuzzy subsets of $S$. Then*

**(i)** $\tilde{A} \subseteq \tilde{B}$ *if* $\tilde{A}(x) \le \tilde{B}(x)$ *for all* $x \in S$,

**(ii)** $\tilde{A} \subset \tilde{B}$ *if* $\tilde{A}(x) \le \tilde{B}(x)$ *for all* $x \in S$ *and there exists at least one* $x \in S$ *such that* $\tilde{A}(x) < \tilde{B}(x)$,

**(iii)** $\tilde{A} = \tilde{B}$ *if* $\tilde{A}(x) = \tilde{B}(x)$ *for all* $x \in S$.

We now proceed to define the union and intersection of fuzzy subsets as well as the complement of a fuzzy subset. We use the notation $\vee$ for supremum and $\wedge$ for infimum.

**Definition 1.2** *Let $\tilde{A}, \tilde{B}$ be any two fuzzy subsets of $S$. Then $\tilde{A} \cup \tilde{B}$ is the fuzzy subset of $S$ defined by*

$$(\tilde{A} \cup \tilde{B})(x) = \tilde{A}(x) \vee \tilde{B}(x) \text{ for all } x \in S$$

*and $\tilde{A} \cap \tilde{B}$ is the fuzzy subset of $S$ defined by*

$$(\tilde{A} \cap \tilde{B})(x) = \tilde{A}(x) \wedge \tilde{B}(x) \text{ for all } x \in S.$$

**Definition 1.3** *Let $\tilde{A}$ be any fuzzy subset of $S$. Then $\tilde{A}^c$ is the fuzzy subset of $S$ defined by*

$$\tilde{A}^c(x) = 1 - \tilde{A}(x) \text{ for all } x \in S.$$

**Example 1.2** *Let $S = \{a, b, c, d\}$. Let $\tilde{A} : S \to [0, 1]$ be such that $\tilde{A}(a) = 0.3, \tilde{A}(b) = 0.9, \tilde{A}(c) = 0.4, \tilde{A}(d) = 0.6$ and let $\tilde{B} : S \to [0, 1]$ be such that $\tilde{B}(a) = 0.3, \tilde{B}(b) = 0.5, \tilde{B}(c) = 0.7, \tilde{B}(d) = 0.2$. Then*

$$\tilde{A}^t = \begin{cases} S & 0 \le t \le 0.3 \\ \{b, c, d\} & 0.3 < t \le 0.4 \\ \{b, d\} & 0.4 < t \le 0.6 \\ \{b\} & 0.6 < t \le 0.9 \\ \emptyset & 0.9 < t \le 1.0 \end{cases} \quad \text{and } \tilde{B}^t = \begin{cases} S & 0 \le t \le 0.2 \\ \{a, b, c\} & 0.2 < t \le 0.3 \\ \{b, c\} & 0.3 < t \le 0.5 \\ \{c\} & 0.5 < t \le 0.7 \\ \emptyset & 0.7 < t \le 1.0 \end{cases}$$

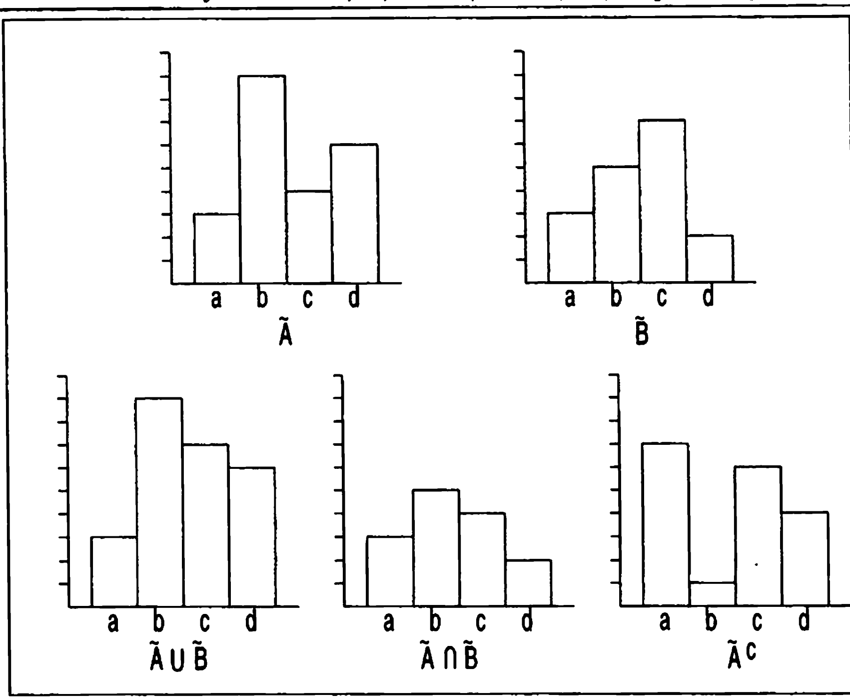
*Now* $(\tilde{A} \cup \tilde{B})(a) = 0.3, (\tilde{A} \cup \tilde{B})(b) = 0.9, (\tilde{A} \cup \tilde{B})(c) = 0.7$, *and* $(\tilde{A} \cup \tilde{B})(d) = 0.6$. *Also* $(\tilde{A} \cap \tilde{B})(a) = 0.3, (\tilde{A} \cap \tilde{B})(b) = 0.5, (\tilde{A} \cap \tilde{B})(c) = 0.4, (\tilde{A} \cap \tilde{B})(d) = 0.2$. *Thus*

$$(\tilde{A}\cup\tilde{B})^t = \begin{cases} S & 0 \le t \le 0.3 \\ \{b,c,d\} & 0.3 < t \le 0.6 \\ \{b,c\} & 0.6 < t \le 0.7 \\ \{b\} & 0.7 < t \le 0.9 \\ \emptyset & 0.9 < t \le 1.0 \end{cases} \quad (\tilde{A}\cap\tilde{B})^t = \begin{cases} S & 0 \le t \le 0.2 \\ \{a,b,c\} & 0.2 < t \le 0.3 \\ \{b,c\} & 0.3 < t \le 0.4 \\ \{b\} & 0.4 < t \le 0.5 \\ \emptyset & 0.5 < t \le 1.0 \end{cases}$$

*Further,*

$$\tilde{A}^c = \begin{cases} S & 0 \le t \le 0.1 \\ \{a,c,d\} & 0.1 < t \le 0.4 \\ \{a,c\} & 0.4 < t \le 0.6 \\ \{a\} & 0.6 < t \le 0.7 \\ \emptyset & 0.7 < t \le 1.0 \end{cases}$$

**FIGURE 1.2** Fuzzy subsets $\tilde{A}, \tilde{B}, \tilde{A}\cup\tilde{B}, \tilde{A}\cap\tilde{B}, \tilde{A}^c$, respectively.




If $\mathcal{S}$ is a collection of fuzzy subsets of $S$, we define the fuzzy subset $\tilde{I} = \bigcap_{\tilde{C}\in\mathcal{S}} \tilde{C}$ (intersection) of $\mathcal{S}$ by $\forall x \in S$, $\tilde{I}(x) = \wedge\{\tilde{C}(x) | \tilde{C} \in \mathcal{S}\}$ and the fuzzy subset $\tilde{U} = \bigcup_{\tilde{C}\in\mathcal{S}} \tilde{C}$ (union) of $\mathcal{S}$ by $\forall x \in S$, $\tilde{U}(x) = \vee\{\tilde{C}(x) | \tilde{C} \in \mathcal{S}\}$.

Algebraic properties of fuzzy subset operators can be summarized as follows. Now $\chi_S(x) = 1$ for all $x \in S$ and $\chi_\emptyset(x) = 0$ for all $x \in S$. Let $\tilde{A}, \tilde{B}$, and $\tilde{C}$ be fuzzy subsets of $S$. Then we have the following properties.

| | |
|---|---|
| (1) $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$ | (9) $\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$ |
| (2) $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$ | (10) $\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$ |
| (3) $\tilde{A} \cup \chi_\emptyset = \tilde{A}$ | (11) $\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$ |
| (4) $\tilde{A} \cap \chi_\emptyset = \chi_\emptyset$ | (12) $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$ |
| (5) $\tilde{A} \cup \chi_S = \chi_S$ | (13) $(\tilde{A} \cup \tilde{B})^c = \tilde{A}^c \cap \tilde{B}^c$ |
| (6) $\tilde{A} \cap \chi_S = \tilde{A}$ | (14) $(\tilde{A} \cap \tilde{B})^c = \tilde{A}^c \cup \tilde{B}^c$ |
| (7) $\tilde{A} \cup \tilde{A} = \tilde{A}$ | (15) $(\tilde{A}^c)^c = \tilde{A}$ |
| (8) $\tilde{A} \cap \tilde{A} = \tilde{A}$ | |

It is important to note that the properties $\tilde{A} \cap \tilde{A}^c = \chi_\emptyset$ and $\tilde{A} \cup \tilde{A}^c = \chi_S$ do not hold in general. In logic, the former property is known as the law of contradiction while the latter is known as the law of the excluded middle.

For the interested reader, the bibliography at the end of the chapter contains many excellent textbooks on fuzzy subsets.

## 1.1 Fuzzy Relations

Let $S$ and $T$ be two sets and let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of $S$ and $T$, respectively. Then a *fuzzy relation* $\tilde{R}$ from the fuzzy subset $\tilde{A}$ into the fuzzy subset $\tilde{B}$ is a fuzzy subset $\tilde{R}$ of $S \times T$ such that $\tilde{R}(x,y) \leq \tilde{A}(x) \wedge \tilde{B}(y), \forall x \in S$ and $y \in T$. In other words, for $\tilde{R}$ to be a fuzzy relation, we require that the degree of membership of a pair of elements never exceed the degree of membership of either of the elements themselves. If we think of the elements as computers and pairs as the communication links between the computers, this amounts to requiring that the strength of the communication link can never exceed the strengths of its connecting computers. More generally, the amount of flow of a quantity from a source to a sink is limited by the capability to transmit and receive. Also, the restriction $\tilde{R}(x,y) \leq \tilde{A}(x) \wedge \tilde{B}(y), \forall x \in S$ and $y \in T$ allows $\tilde{R}^t$ to be a relation from $\tilde{A}^t$ into $\tilde{B}^t$ for all $t \in [0,1]$ and for $\text{supp}(\tilde{R})$ to be a relation from $\text{supp}(\tilde{A})$ into $\text{supp}(\tilde{B})$.

There are three special cases of fuzzy relations which are extensively found in the literature.

**case 1:** $S = T$ and $\tilde{A} = \tilde{B}$. In this case, $\tilde{R}$ is said to be a fuzzy relation on $\tilde{A}$. Note that $\tilde{R}$ is a fuzzy subset of $S \times S$ such that $\tilde{R}(x,y) \leq \tilde{A}(x) \wedge \tilde{A}(y)$.

**case 2:** $\tilde{A}(x) = 1.0$ for all $x \in S$ and $\tilde{B}(y) = 1.0$ for all $y \in T$. In this case, $\tilde{R}$ is said to be a fuzzy relation from $S$ into $T$.

**case 3:** $S = T$, $\tilde{A}(x) = 1.0$ for all $x \in S$ and $\tilde{B}(y) = 1.0$ for all $y \in T$. In this case, $\tilde{R}$ is said to be a fuzzy relation on $S$.

There are many applications in which a fuzzy relation on a fuzzy subset is quite useful. Further, any result we obtain is clearly true for fuzzy relations on a set. Therefore, we devote the first three sections of this chapter to fuzzy relations on a fuzzy subset. The last section of this chapter is devoted to the study of case 2 and case 3.

**Example 1.3** *Let $S = \{a, b, c, d\}$. Define $\tilde{A}(a) = 0.3$, $\tilde{A}(b) = 0.7$, $\tilde{A}(c) = 0.4$ and $\tilde{A}(d) = 0.5$. Then $\tilde{A}$ is a fuzzy subset of $S$. Let $\tilde{R}$ be a fuzzy subset of $S \times S$ defined as follows:*

| | a | b | c | d |
|---|---|---|---|---|
| a | 0.1 | 0.2 | 0.0 | 0.3 |
| b | 0.2 | 0.7 | 0.4 | 0.3 |
| c | 0.3 | 0.6 | 0.2 | 0.4 |
| d | 0.2 | 0.5 | 0.3 | 0.5 |

*Since $\tilde{R}(c, b) \not\leq \tilde{A}(c) \wedge \tilde{A}(b)$, $\tilde{R}$ is not a fuzzy relation on $\tilde{A}$. However, if we redefine $\tilde{R}(c, b) = 0.4$ then $\tilde{R}$ is a fuzzy relation on $\tilde{A}$. Alternatively, we could redefine $\tilde{A}(c) = 0.6$ and meet the constraint.*

Let $\tilde{R}$ be a fuzzy relation on $\tilde{A}$. Then $\tilde{R}$ is called the *strongest fuzzy relation* on $\tilde{A}$ if and only if for all fuzzy relations $\tilde{Q}$ on $\tilde{A}$, $\forall x, y \in S$, $\tilde{Q}(x, y) \leq \tilde{R}(x, y)$.

**Example 1.4** *Let $S = \{a, b, c, d\}$. Define $\tilde{A}(a) = 0.3$, $\tilde{A}(b) = 0.5$, $\tilde{A}(c) = 0.7$ and $\tilde{A}(d) = 0.9$. Then $\tilde{A}$ is a fuzzy subset of $S$. The strongest fuzzy relation on $\tilde{A}$ is given below.*

| | a | b | c | d |
|---|---|---|---|---|
| a | 0.3 | 0.3 | 0.3 | 0.3 |
| b | 0.3 | 0.5 | 0.5 | 0.5 |
| c | 0.3 | 0.5 | 0.7 | 0.7 |
| d | 0.3 | 0.5 | 0.7 | 0.9 |

The converse problem may also arise in practice. That is, we know the strength of the pairs and we want to compute the minimum strength required for the elements themselves. For a given fuzzy subset $\tilde{R}$ of $S \times S$, the *weakest fuzzy subset* $\tilde{A}$ of $S$ on which $\tilde{R}$ is a fuzzy relation is defined by $\tilde{A}(x) = \vee\{\tilde{R}(x, y) \vee \tilde{R}(y, x) | y \in S\}$ for all $x \in S$. That is, if $\tilde{B}$ is a fuzzy subset of $S$ and $\tilde{R}$ is a fuzzy relation on $\tilde{B}$, then $\tilde{A} \subseteq \tilde{B}$.

**Example 1.5** *Let $S = \{a, b, c, d\}$. Let $\tilde{R}$ be a fuzzy subset of $S \times S$ defined as follows:*

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.1 | 0.2 | 0.0 | 0.3 |
| $b$ | 0.2 | 0.7 | 0.4 | 0.3 |
| $c$ | 0.3 | 0.6 | 0.2 | 0.4 |
| $d$ | 0.2 | 0.5 | 0.3 | 0.5 |

*The largest value on the first row and the first column is 0.3. Therefore we define* $\tilde{A}(a) = 0.3$. *Similarly, the largest value on the second row and the second column is 0.7. Therefore we define* $\tilde{A}(b) = 0.7$. *By the same reasoning,* $\tilde{A}(c) = 0.6$ *and* $\tilde{A}(d) = 0.5$. *Clearly,* $\tilde{R}$ *is a fuzzy relation on* $\tilde{A}$. *In fact,* $\tilde{A}$ *is the weakest fuzzy subset on which* $\tilde{R}$ *is a fuzzy relation.*

Given a fuzzy relation $\tilde{R}$ on a fuzzy set $\tilde{A}$, it is possible to apply a threshold value to both of them to obtain a relation $\tilde{R}^t$ on the subset $\tilde{A}^t$.

**Example 1.6** *Let* $S = \{a, b, c, d\}$. *Define* $\tilde{A}(a) = 0.4$, $\tilde{A}(b) = 0.7, \tilde{A}(c) = 0.6$ *and* $\tilde{A}(d) = 0.5$. *Then* $\tilde{A}$ *is a fuzzy subset of* $S$. *Let* $\tilde{R}$ *be a fuzzy subset of* $S \times S$ *defined as follows:*

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.1 | 0.2 | 0.0 | 0.3 |
| $b$ | 0.2 | 0.7 | 0.4 | 0.3 |
| $c$ | 0.3 | 0.6 | 0.2 | 0.4 |
| $d$ | 0.2 | 0.5 | 0.3 | 0.5 |

*It is clear that* $\tilde{R}$ *is a fuzzy relation on* $\tilde{A}$. *Let us choose* $t = 0.55$ *as our threshold value. Then* $\tilde{R}^t = \{(b, b), (c, b)\}$ *is a relation on* $\tilde{A}^t = \{b, c\}$.

## 1.2 Operations on Fuzzy Relations

Consider two computer networks, sharing the same computers, but different connecting lines. If we have the choice of selecting either of the connecting lines, the maximum strength possible as well as the minimum strength that can be guaranteed between any pair of computers is an interesting problem. Let $\tilde{R}$ and $\tilde{Q}$ be two fuzzy relations on a fuzzy subset $\tilde{A}$ of $S$. Then $\tilde{R} \cup \tilde{Q}$ is the relation on $\tilde{A}$ defined by $(\tilde{R} \cup \tilde{Q})(x, y) = \tilde{R}(x, y) \vee \tilde{Q}(x, y), \forall x, y \in S$. Clearly, $\tilde{R} \cup \tilde{Q}$ represents the maximum strength possible between any two computers. $\tilde{R} \cap \tilde{Q}$ is the relation on $\tilde{A}$ defined by $(\tilde{R} \cap \tilde{Q})(x, y) = \tilde{R}(x, y) \wedge \tilde{Q}(x, y), \forall x, y \in S$. The relation $\tilde{R} \cap \tilde{Q}$ represents the minimum strength that can be guaranteed between any two pairs in the event of at most one of the connecting lines are down at any time.

Suppose we want to send a message from computer $x$ to computer $z$. Let us say, we prefer to use the "$R$" network in the geographical area of the computer $x$ and the "$Q$" network in the geographical area of the computer $z$. Further we are willing to use exactly one computer, say $y$, to route our message. In such a situation, we want to find the maximum strength possible between $x$ and $z$. Fuzzy theory provides an operation called composition to provide the answer.

**Definition 1.4** *Let $\tilde{R} : S \times T \rightarrow [0,1]$ be a fuzzy relation from a fuzzy subset $\tilde{A}$ of $S$ into a fuzzy subset $\tilde{B}$ of $T$ and $\tilde{Q} : T \times U \rightarrow [0,1]$ be a fuzzy relation from a fuzzy subset $\tilde{B}$ of $T$ into a fuzzy subset $\tilde{C}$ of $U$. Define $\tilde{R} \circ \tilde{Q} : S \times U \rightarrow [0,1]$ by*

$$\tilde{R} \circ \tilde{Q}(x,z) = \vee\{\tilde{R}(x,y) \wedge \tilde{Q}(y,z) | y \in T\}$$

*for all $x \in S, z \in U$. The fuzzy relation $\tilde{R} \circ \tilde{Q}$ is called the* composition *of $\tilde{R}$ with $\tilde{Q}$.*

Note that composition of $\tilde{R}$ with $\tilde{Q}$ is a fuzzy relation from a fuzzy subset $\tilde{A}$ of $S$ into a fuzzy subset $\tilde{C}$ of $U$. A closer look at the definition of the composition operation reveals that $\tilde{R} \circ \tilde{Q}$ can be computed similar to matrix multiplication, where the addition is replaced by $\vee$ and the multiplication is replaced by $\wedge$. Since composition is associative,we use the notation $\tilde{R}^2$ to denote the composition $\tilde{R} \circ \tilde{R}, \tilde{R}^k$ to denote $\tilde{R}^{k-1} \circ \tilde{R}, k > 1$. Define $\tilde{R}^{\infty}(x,y) = \vee\{\tilde{R}^k(x,y) | k = 1, 2, \ldots\}$ for all $x, y \in S$. Finally, it is convenient to define $\tilde{R}^0(x,y) = 0$ if $x \neq y$ and $\tilde{R}^0(x,y) = \tilde{A}(x)$ otherwise, for all $x, y \in S$.We have introduced three binary operations. We now introduce a unary operation on a fuzzy relation. Given a fuzzy relation $\tilde{R}$ on a fuzzy subset $\tilde{A}$ of $S$, define the fuzzy relation $\tilde{R}^c$ on $\tilde{A}$ by $\tilde{R}^c(x,y) = 1 - \tilde{R}(x,y)$ for all $x, y \in S$.

**Definition 1.5** *Let $\tilde{R} : S \times T \rightarrow [0,1]$ be a fuzzy relation from a fuzzy subset $\tilde{A}$ of $S$ into a fuzzy subset $\tilde{B}$ of $T$. Define the fuzzy relation $\tilde{R}^{-1} : T \times S \rightarrow [0,1]$ of $\tilde{B}$ into $\tilde{A}$ by $\tilde{R}^{-1}(y,x) = \tilde{R}(x,y)$ for all $(y,x) \in T \times S$.*

**Example 1.7** *Let $S = \{a, b, c, d\}$. Define $\tilde{A}(a) = 0.4, \tilde{A}(b) = 0.7, \tilde{A}(c) = 0.6$ and $\tilde{A}(d) = 0.5$. Then $\tilde{A}$ is a fuzzy subset of $S$. Let $\tilde{R}$ and $\tilde{Q}$ be fuzzy relations defined on $\tilde{A}$ as follows:*

$\tilde{R}$:

| | a | b | c | d |
|---|---|---|---|---|
| a | 0.1 | 0.2 | 0.0 | 0.3 |
| b | 0.2 | 0.7 | 0.4 | 0.3 |
| c | 0.3 | 0.6 | 0.2 | 0.4 |
| d | 0.2 | 0.5 | 0.3 | 0.5 |

$\tilde{Q}$:

| | a | b | c | d |
|---|---|---|---|---|
| a | 0.3 | 0.1 | 0.3 | 0.4 |
| b | 0.4 | 0.5 | 0.3 | 0.5 |
| c | 0.2 | 0.4 | 0.6 | 0.3 |
| d | 0.3 | 0.3 | 0.4 | 0.2 |

*Then the relations $\tilde{R} \cup \tilde{Q}, \tilde{R} \cap \tilde{Q}$ and $\tilde{R} \circ \tilde{Q}$ are as below:*

$\tilde{R} \cup \tilde{Q}$:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.3 | 0.2 | 0.3 | 0.4 |
| $b$ | 0.4 | 0.7 | 0.4 | 0.5 |
| $c$ | 0.3 | 0.6 | 0.6 | 0.4 |
| $d$ | 0.3 | 0.5 | 0.4 | 0.5 |

$\tilde{R} \cap \tilde{Q}$:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.1 | 0.1 | 0.0 | 0.3 |
| $b$ | 0.2 | 0.5 | 0.3 | 0.3 |
| $c$ | 0.2 | 0.4 | 0.2 | 0.3 |
| $d$ | 0.2 | 0.3 | 0.3 | 0.2 |

$\tilde{R} \circ \tilde{Q}$:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.3 | 0.3 | 0.3 | 0.2 |
| $b$ | 0.4 | 0.5 | 0.4 | 0.5 |
| $c$ | 0.4 | 0.5 | 0.4 | 0.5 |
| $d$ | 0.4 | 0.5 | 0.4 | 0.5 |

$\tilde{R}^c$:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0.9 | 0.8 | 1.0 | 0.7 |
| $b$ | 0.8 | 0.3 | 0.6 | 0.7 |
| $c$ | 0.7 | 0.4 | 0.8 | 0.6 |
| $d$ | 0.8 | 0.5 | 0.7 | 0.5 |

### *Algebraic Properties of Fuzzy Relation Operators*

Let $\tilde{T}, \tilde{P}, \tilde{R}$ and $\tilde{Q}$ be any four fuzzy relations on a fuzzy subset $\tilde{A}$ of a set $S$. Then we have the following.

1. $\tilde{R} \cup \tilde{Q} = \tilde{Q} \cup \tilde{R}$
2. $\tilde{R} \cap \tilde{Q} = \tilde{Q} \cap \tilde{R}$
3. $\tilde{R} = (\tilde{R}^c)^c$
4. $\tilde{P} \cup (\tilde{R} \cup \tilde{Q}) = (\tilde{P} \cup \tilde{R}) \cup \tilde{Q}$
5. $\tilde{P} \cap (\tilde{R} \cap \tilde{Q}) = (\tilde{P} \cap \tilde{R}) \cap \tilde{Q}$
6. $\tilde{P} \circ (\tilde{R} \circ \tilde{Q}) = (\tilde{P} \circ \tilde{R}) \circ \tilde{Q}$
7. $\tilde{P} \cap (\tilde{R} \cup \tilde{Q}) = (\tilde{P} \cap \tilde{R}) \cup (\tilde{P} \cap \tilde{Q})$
8. $\tilde{P} \cup (\tilde{R} \cap \tilde{Q}) = (\tilde{P} \cup \tilde{R}) \cap (\tilde{P} \cup \tilde{Q})$
9. $(\tilde{R} \cup \tilde{Q})^c = \tilde{Q}^c \cap \tilde{R}^c$
10. $(\tilde{R} \cap \tilde{Q})^c = \tilde{Q}^c \cup \tilde{R}^c$
11. For all $t \in [0,1], (\tilde{R} \cup \tilde{Q})^t = \tilde{R}^t \cup \tilde{Q}^t$
12. For all $t \in [0,1], (\tilde{R} \cap \tilde{Q})^t = \tilde{R}^t \cap \tilde{Q}^t$
13. For all $t \in [0,1], (\tilde{R} \circ \tilde{Q})^t \supseteq \tilde{R}^t \circ \tilde{Q}^t$ and if $S$ is finite, $(\tilde{R} \circ \tilde{Q})^t = \tilde{R}^t \circ \tilde{Q}^t$
14. If $\tilde{T} \subseteq \tilde{R}$ and $\tilde{P} \subseteq \tilde{Q}$ then $\tilde{T} \cup \tilde{P} \subseteq \tilde{R} \cup \tilde{Q}$
15. If $\tilde{T} \subseteq \tilde{R}$ and $\tilde{P} \subseteq \tilde{Q}$ then $\tilde{T} \cap \tilde{P} \subseteq \tilde{R} \cap \tilde{Q}$
16. If $\tilde{T} \subseteq \tilde{R}$ and $\tilde{P} \subseteq \tilde{Q}$ then $\tilde{T} \circ \tilde{P} \subseteq \tilde{R} \circ \tilde{Q}$

## 1.3 Reflexivity, Symmetry and Transitivity

Throughout this section $\tilde{R}$ and $\tilde{Q}$ are fuzzy relations on a fuzzy subset $\tilde{A}$ of $S$. As we have seen, it is quite natural to represent a fuzzy relation in the form of a matrix. In this section, we shall use the matrix representation of a fuzzy relation to explain the properties of a fuzzy relation. In particular, we shall use the term "diagonal" to represent the principal diagonal of the matrix.

### *Reflexivity*

We call $\tilde{R}$ *reflexive* if $\tilde{R}(x,x) = \tilde{A}(x)$ for all $x \in S$. If $\tilde{R}$ is reflexive, then $\tilde{R}(x,y) \le \tilde{A}(x) \wedge \tilde{A}(y) \le \tilde{A}(x) = \tilde{R}(x,x)$ and it follows that "any diagonal element of $\tilde{R}$ is larger than or equal to any element in its row". Similarly, "any diagonal element is larger than or equal to any element in its column". Conversely, given a fuzzy relation $\tilde{R}$ on $\tilde{A}$ such that "any diagonal element is larger than or equal to any element in its row and column", define a fuzzy subset $\tilde{B}$ of $S$ as $\tilde{B}(x) = \tilde{R}(x,x), \forall x \in S$. Then $\tilde{B}$ is the weakest fuzzy subset of $S$ such that $\tilde{R}$ is a fuzzy relation on $\tilde{B}$. Further, $\tilde{R}$ is reflexive on $\tilde{B}$.

Fuzzy reflexive relations have some interesting algebraic properties. Let $\tilde{R}$ and $\tilde{Q}$ be any two fuzzy relations on a fuzzy subset $\tilde{A}$ of $S$. Then we have the following.

1. If $\tilde{R}$ is reflexive, $\tilde{Q} \subseteq \tilde{Q} \circ \tilde{R}$ and $\tilde{Q} \subseteq \tilde{R} \circ \tilde{Q}$.
2. If $\tilde{R}$ is reflexive, $\tilde{R} \subseteq \tilde{R}^2$.
3. If $\tilde{R}$ is reflexive, $\tilde{R}^0 \subseteq \tilde{R} \subseteq \tilde{R}^2 \subseteq \tilde{R}^3 \subseteq \ldots \subseteq \tilde{R}^\infty$.
4. If $\tilde{R}$ is reflexive, $\tilde{R}^0(x,x) = \tilde{R}(x,x) = \tilde{R}^2(x,x) = \tilde{R}^3(x.x) = \ldots = \tilde{R}^\infty(x,x) = \tilde{A}(x), \forall x \in S$.
5. If $\tilde{R}$ and $\tilde{Q}$ are reflexive, so is $\tilde{R} \circ \tilde{Q}$ and $\tilde{Q} \circ \tilde{R}$.
6. If $\tilde{R}$ is reflexive, for any $0 \le t \le 1, \tilde{R}^t$ is a reflexive relation on $\tilde{A}^t$.

### *Symmetry*

We call $\tilde{R}$ *symmetric* if $\tilde{R}(x.y) = \tilde{R}(y,x)$, for all $x, y \in S$. In other words, $\tilde{R}$ is symmetric if the matrix representation of $\tilde{R}$ is symmetric (with respect to the diagonal). Let $\tilde{R}$ and $\tilde{Q}$ be any two fuzzy relations on a fuzzy subset $\tilde{A}$ of $S$. Then we have the following.

1. If $\tilde{R}$ and $\tilde{Q}$ are symmetric, then so is $\tilde{R} \circ \tilde{Q}$ if and only if $\tilde{R} \circ \tilde{Q} = \tilde{Q} \circ \tilde{R}$.
2. If $\tilde{R}$ is symmetric, then so is every power of $\tilde{R}$.

3. If $\tilde{R}$ is symmetric, then for any $0 \leq t \leq 1$, $\tilde{R}^t$ is a symmetric relation on $\tilde{A}^t$.

*Transitivity*

We call $\tilde{R}$ *transitive* if $\tilde{R}^2 \subseteq \tilde{R}$. It may be noted that $\tilde{R}^\infty$ is transitive for any fuzzy relation $\tilde{R}$. Let $\tilde{P}, \tilde{R}$ and $\tilde{Q}$ be any three fuzzy relations on a fuzzy subset $\tilde{A}$ of $S$. Then we have the following.

1. If $\tilde{R}$ is transitive, then so is every power of $\tilde{R}$.
2. If $\tilde{R}$ is transitive and $\tilde{P} \subseteq \tilde{R}, \tilde{Q} \subseteq \tilde{R}$, then $\tilde{P} \circ \tilde{Q} \subseteq \tilde{R}$.
3. If $\tilde{R}$ is transitive, $\tilde{Q}$ is reflexive and $\tilde{Q} \subseteq \tilde{R}$, then $\tilde{R} \circ \tilde{Q} = \tilde{Q} \circ \tilde{R} = \tilde{R}$.
4. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R}^2 = \tilde{R}$.
5. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R}^0 \subseteq \tilde{R} = \tilde{R}^2 = \tilde{R}^3 = \ldots = \tilde{R}^\infty$.
6. If $\tilde{R}$ and $\tilde{Q}$ are transitive and $\tilde{R} \circ \tilde{Q} = \tilde{Q} \circ \tilde{R}$, then $\tilde{R} \circ \tilde{Q}$ is transitive.
7. If $\tilde{R}$ is symmetric and transitive, then $\tilde{R}(x, y) \leq \tilde{R}(x, x)$ and $\tilde{R}(y, x) \leq \tilde{R}(x, x)$, for all $x, y \in S$.
8. If $\tilde{R}$ is transitive, then for any $0 \leq t \leq 1$, $\tilde{R}^t$ is a transitive relation on $\tilde{A}^t$.

A fuzzy relation $\tilde{R}$ on $S$ which is reflexive, symmetric, and transitive is called a *fuzzy equivalence relation* on $S$.

## 1.4 Pattern Classification Based on Fuzzy Relations

Let $S$ be a set of patterns. A *classification fuzzy relation* $\tilde{R}$ on $S$ is a fuzzy relation satisfying the following two conditions.

**C1** $\tilde{R}(x, x) = 1$ for all $x \in S$.

**C2** $\tilde{R}(x, y) = \tilde{R}(y, x)$ for all $x, y \in S$.

Note that the condition C1 states that a pattern $x$ is identical with itself. Thus the relation is reflexive. Condition C2 means that any relation used to classify patterns has to be symmetric. Since $\tilde{R}$ is reflexive, $\tilde{R} \subseteq \tilde{R}^2 \subseteq \tilde{R}^3 \subseteq \ldots \subseteq \tilde{R}^\infty$. Note that $\tilde{R}^\infty$ is a fuzzy equivalence relation. So for any $0 \leq t \leq 1$, $(\tilde{R}^\infty)^t$ is an equivalence relation on $S$. Let $\mathcal{P}^t$ be the partition of $S$ induced by the equivalence relation $(\tilde{R}^\infty)^t$. The next result holds since $(\tilde{R}^\infty)^t$ is transitive.

**Lemma 1.1** *For all $x, y, z \in S$,*
$\tilde{R}^{\infty}(x, z) \geq \tilde{R}^{\infty}(x, y) \wedge \tilde{R}^{\infty}(y, z)$. ■

**Theorem 1.2** *If $\tilde{R}^{\infty}(x, y) \neq 1$, for all $x, y \in S$ such that $x \neq y$, then $\rho(x, y) = 1 - \tilde{R}^{\infty}(x, y)$ satisfies the axioms of distance. That is, $\forall x, y, z \in S$,*

**(i)** $\rho(x, y) = 0$ *if and only if* $x = y$,

**(ii)** $\rho(x, y) = \rho(y, x)$,

**(iii)** $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

**Proof.** We show only property (iii). By Lemma 1.1, $\tilde{R}^{\infty}(x, z) \geq \tilde{R}^{\infty}(x, y) \wedge \tilde{R}^{\infty}(y, z)$. Thus $1 + \tilde{R}^{\infty}(x, z) \geq \tilde{R}^{\infty}(x, y) + \tilde{R}^{\infty}(y, z)$. The desired result now follows easily. ■

**Example 1.8** *Let $S = \{x_1, x_2, x_3, x_4, x_5\}$ and $\tilde{R}(x_i, x_j)$ be as follows:*

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | *1.0* | *0.8* | *0.0* | *0.1* | *0.2* |
| $x_2$ | *0.8* | *1.0* | *0.4* | *0.0* | *0.9* |
| $x_3$ | *0.0* | *0.4* | *1.0* | *0.0* | *0.0* |
| $x_4$ | *0.1* | *0.0* | *0.0* | *1.0* | *0.5* |
| $x_5$ | *0.2* | *0.9* | *0.0* | *0.5* | *1.0* |

*Now $\tilde{R}^{\infty} = \tilde{R}^3$ is given by*

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | *1.0* | *0.8* | *0.4* | *0.5* | *0.8* |
| $x_2$ | *0.8* | *1.0* | *0.4* | *0.5* | *0.9* |
| $x_3$ | *0.4* | *0.4* | *1.0* | *0.4* | *0.4* |
| $x_4$ | *0.5* | *0.5* | *0.4* | *1.0* | *0.5* |
| $x_5$ | *0.8* | *0.9* | *0.4* | *0.5* | *1.0* |

*and we have the following partitions of $S$:*
$\mathcal{P}^0 = \mathcal{P}^{0.3} = \{\{x_1, x_2, x_3, x_4, x_5\}\}$
$\mathcal{P}^{0.45} = \{\{x_1, x_2, x_4, x_5\}, \{x_3\}\}$
$\mathcal{P}^{0.55} = \{\{x_1, x_2, x_5\}, \{x_4\}, \{x_3\}\}$
$\mathcal{P}^{0.85} = \{\{x_1\}, \{x_2, x_5\}, \{x_4\}, \{x_3\}\}$
$\mathcal{P}^{1.0} = \{\{x_1\}, \{x_2\}, \{x_5\}, \{x_4\}, \{x_3\}\}$
*Thus there are many partitions possible and depending upon the level of detail, one could classify the patterns based on equivalence relations. Note that if $s \geq t$, then $\mathcal{P}^s$ is a refinement of $\mathcal{P}^t$.*

### *Experimental Result*

We now present an experiment done by Tamura, Higuchi and Tanaka [8]. Portraits obtained from 60 families were used in their experiment, each of which were composed of between four and seven members. The reason why they chose the portraits is that even if parents do not possess a facial resemblance, they may be connected through their children, and consequently they could classify the portraits into families. First, they divided the 60 families into 20 groups, each of which was composed of 3 families. Each group was, on the average, composed of 15 members. The portraits of each group were presented to a different student to give the values of the subjective similarity $\tilde{R}(x, y)$ between all pairs on a scale of 1 to 5. The reason why they used the 5 rank representation instead of a continuous value representation is that it had been proved that a human being cannot distinguish into more than 5 ranks. Twenty students joined in this experiment. An example of the experiment for one group with 16 portraits is shown in Table 1.1 and Table 1.2. The first column of the tables gives the portrait number. In Table 1.1, the 5 rank representations are converted to values in [0, 1], namely 0.2, 0.4, 0.6, and 1.0. In this example, the number of patterns is sufficiently large that they can not be classified by inspection. The classification requires the calculation of $\tilde{R}^{\infty}$. Since the levels of the subjective values are different according to individuals, the threshold was determined in each group as follows. As they lowered the threshold, the number of classes decreased. Hence, under the assumption that the number of classes c to be classified was known to be 3, while lowering the threshold they stopped at the value which divided the patterns into 3 classes and some nonconnected patterns. However, as in the present case, when some $\tilde{R}(x, y)$ take the same value, sometimes there is no threshold by which the patterns are divided into just c given classes. In such a case, they made it possible to divide them into just c classes by stopping the threshold at the value where the patterns are divided into less than c classes and separating some connections randomly that have a minimum $\tilde{R}(x, y)$ of connections that have the stronger relation than the threshold. The correctly classified rates, the misclassified rates, and the rejected rates of 20 groups were within the range of 50-94 percent, 0-33 percent, and 0-33 percent, respectively, and they obtained the correctly classified rate 75 percent of the time, the misclassified rate 13 percent, and the rejected rate 12 percent as the averages of the 20 groups. Here, since the classes made in this experiment have no label, they calculated these rates by making a one-to-one correspondence between 3 families and 3 classes, so as to have the largest number of correctly classified patterns.

Thus Tamura et al.[8] have studied pattern classification using subjective information and performed experiments involving classification of portraits. The method of classification proposed here is based on the procedure of finding a path connecting 2 patterns. Therefore, this method may be combined with nonsupervised learning and may also be applicable to information retrieval and path detection [8].

**TABLE 1.1** Representation of subjective similarities[2] $\tilde{R}$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | |
| 2 | 0 | 1 | | | | | | | | | | | | | | |
| 3 | 0 | 0 | 1 | | | | | | | | | | | | | |
| 4 | 0 | 0 | 1 | 1 | | | | | | | | | | | | |
| 5 | 0 | 8 | 0 | 0 | 1 | | | | | | | | | | | |
| 6 | .5[1] | 0 | 2 | 2 | 0 | 1 | | | | | | | | | | |
| 7 | 0 | .8 | 0 | 0 | 1 | 0 | 1 | | | | | | | | | |
| 8 | 4 | 2 | 2 | 5[1] | 0 | 8 | 0 | 1 | | | | | | | | |
| 9 | 0 | 4 | 0 | .8 | 4 | 2 | .4 | 0 | 1 | | | | | | | |
| 10 | 0 | 0 | 2 | 2 | 0 | 0 | .2 | 0 | .2 | 1 | | | | | | |
| 11 | 0 | 5[1] | .2 | 2 | 0 | 0 | 8 | 0 | 4 | 2 | 1 | | | | | |
| 12 | 0 | 0 | 2 | 8 | 0 | 0 | 0 | 0 | 1 | 8 | 0 | 1 | | | | |
| 13 | 8 | 0 | 2 | 1 | 0 | .4 | 0 | 4 | 0 | 0 | 0 | 0 | 1 | | | |
| 14 | 0 | 8 | 0 | 2 | .1 | 0 | 8 | 0 | 2 | 2 | 6 | 0 | 0 | 1 | | |
| 15 | 0 | 0 | 4 | 8 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | .2 | .2 | 0 | 1 | |
| 16 | .6 | 0 | 0 | 2 | 2 | 8 | 0 | .4 | 0 | 0 | 0 | 0 | .1 | 2 | 0 | 1 |

**TABLE 1.2** The relation[2] $\tilde{R}^{\infty}$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | |
| 2 | 4 | 1 | | | | | | | | | | | | | | |
| 3 | 4 | 4 | 1 | | | | | | | | | | | | | |
| 4 | 5 | 4 | 4 | 1 | | | | | | | | | | | | |
| 5 | 1 | 8 | 4 | 4 | 1 | | | | | | | | | | | |
| 6 | 6 | 4 | 4 | 5 | 4 | 1 | | | | | | | | | | |
| 7 | 1 | 8 | 4 | .4 | .8 | 4 | 1 | | | | | | | | | |
| 8 | 6 | 4 | 4 | 5 | 4 | .8 | 1 | 1 | | | | | | | | |
| 9 | 5 | 4 | 4 | 8 | .1 | 5 | 4 | 5 | 1 | | | | | | | |
| 10 | 5 | 1 | .4 | 8 | 1 | 5 | .4 | 5 | 8 | 1 | | | | | | |
| 11 | .4 | 8 | 4 | 4 | 8 | 1 | 8 | 4 | .4 | 1 | 1 | | | | | |
| 12 | 5 | 1 | .1 | 8 | 1 | 5 | 4 | .5 | 8 | 8 | 4 | 1 | | | | |
| 13 | .8 | 1 | .4 | .5 | 4 | 6 | .4 | .6 | 5 | 5 | 4 | .5 | 1 | | | |
| 14 | 1 | 8 | 4 | 4 | .8 | 4 | 8 | .4 | 4 | .4 | 8 | .4 | 4 | 1 | | |
| 15 | 5 | 1 | 4 | 8 | 4 | .5 | 1 | .5 | 8 | .8 | 1 | .8 | 5 | .4 | 1 | |
| 16 | 6 | 4 | 4 | 5 | .4 | .8 | .1 | .8 | 5 | 5 | .1 | .5 | .6 | 4 | 5 | 1 |

Table 1.2 displays the subjective similarities among family members. The portraits of family members for family one were 1, 6, 8, 13, and 16 with portraits 6 and 13 being of parents and 1, 8, and 16 being of the children. The portraits of family members for family two were 2,5,7,11, and 14, with portraits 5 and 11 being of parents and 2, 7, and 14 being of the children. For family three portrait 4 was that of a grand mother, portraits 10 and 15 of parents and 3, 9, and 12 of the children. The rate of correct classification was 15/16, the rate of misclassification was 0/16, and the rate of rejection was 1/16. In this experiment the portrait 3 was rejected.

[1] This value was converted from 0.6 to 0.5 for division into just three classes.

[2] Reprinted from : Tamura, S., Higuchi, S., and Tanaka, K., Pattern Classification Based on Fuzzy Relations, IEEE Trans. SMC-1, 61 66. 1971

## 1.5 Advanced Topics on Fuzzy Relations

In this section we will show that the concept of a similarity relation introduced by Zadeh [11] is derivable in much the same way as an equivalence relation. Throughout this section we shall be dealing with a fuzzy relation on a set. The results in this section are from [9].

**Definition 1.6** *Let $\tilde{R}$ be a fuzzy relation on a set $S$. We define the following notions:*

**(i)** $\tilde{R}$ *is* $\epsilon$-reflexive *if* $\forall\ x \in S, \tilde{R}(x,x) \geq \epsilon$, *where* $\epsilon \in [0,1]$.

**(ii)** $\tilde{R}$ *is* irreflexive *if* $\forall x \in S, \tilde{R}(x,x) = 0$.

**(iii)** $\tilde{R}$ *is* weakly reflexive *if for all $x$, $y$ in $S$ and for all $\epsilon \in [0,1]$*, $\tilde{R}(x,y) = \epsilon \Rightarrow \tilde{R}(x,x) \geq \epsilon$.

**Remark 1** *Note that the definition of a reflexive relation as a 1-reflexive relation coincides with the definition of a reflexive relation in Section 1.3.*

**Lemma 1.3** *If $\tilde{R}$ is a fuzzy relation from $S$ into $T$, then the fuzzy relation $\tilde{R} \circ \tilde{R}^{-1}$ is weakly reflexive and symmetric.* ■

Let $\tilde{R}$ be a weakly reflexive and symmetric fuzzy relation on $S$. Define a family of non-fuzzy subsets $F^{\tilde{R}}$ as follows:

$F^{\tilde{R}} = \{K \subseteq S | (\exists 0 < \epsilon \leq 1)(\forall x \in S)[x \in K \Leftrightarrow (\forall x' \in K)[\tilde{R}(x,x') \geq \epsilon]]\}$.

We note that if we let

$F_\epsilon^{\tilde{R}} = \{K \subseteq S | (\forall x \in S)[x \in K \Leftrightarrow (\forall x' \in K)[\tilde{R}(x,x') \geq \epsilon]]\}$,

then we see that $\epsilon_1 \leq \epsilon_2 \Rightarrow F_{\epsilon_2}^{\tilde{R}} \preccurlyeq F_{\epsilon_1}^{\tilde{R}}$ where "$\preccurlyeq$" denotes a covering relation, i.e., every element in $F_{\epsilon_2}^{\tilde{R}}$ is a subset of an element in $F_{\epsilon_1}^{\tilde{R}}$.

A subset $J$ of $S$ is called *$\epsilon$-complete* with respect to $\tilde{R}$ if $\forall\ x, x' \in J, \tilde{R}(x,x') \geq \epsilon$. A *maximal $\epsilon$-complete* set is one which is not properly contained in any other $\epsilon$-complete set.

**Lemma 1.4** *$F^{\tilde{R}}$ is the family of all maximal $\epsilon$-complete sets with respect to $\tilde{R}$ for $0 \leq \epsilon \leq 1$.* ■

**Lemma 1.5** *Whenever $\tilde{R}(x,x') > 0$, there is some $\epsilon$-complete set $K \in F^{\tilde{R}}$ such that $\{x, x'\} \subseteq K$.* ■

We remark here that sometimes a subclass of $F^{\tilde{R}}$, satisfying the condition of Lemma 1.5, will cover the set $S$. For example, let $\tilde{R}$ be the fuzzy relation on $S = \{a, b, c, d, e, f\}$ given by the following matrix.

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 1.0 | 0.3 | 0.4 | 0.0 | 0.4 | 0.3 |
| $b$ | 0.3 | 1.0 | 0.2 | 0.3 | 0.0 | 0.4 |
| $c$ | 0.4 | 0.2 | 1.0 | 0.3 | 0.5 | 0.0 |
| $d$ | 0.0 | 0.3 | 0.3 | 1.0 | 0.0 | 0.0 |
| $e$ | 0.4 | 0.0 | 0.5 | 0.0 | 1.0 | 0.0 |
| $f$ | 0.3 | 0.4 | 0.0 | 0.0 | 0.0 | 1.0 |

We see that the family consisting of the three maximal complete sets $\{a,b,f\}$, $\{b,c,d\}$, and $\{a,c,e\}$ satisfy the condition of Lemma 1.5, but it does not contain the maximal complete set $\{a,b,c\}$. Note for example that $\{a,b,f\}$ is maximal $\epsilon'$-complete $\forall\ 0 < \epsilon' \leq \epsilon$ where $\epsilon = 0.3$ since $\tilde{R}(a,d) = \tilde{R}(b,e) = \tilde{R}(f,c) = 0$. We have

$F_1^{\tilde{R}} = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\}$
$F_{0.5}^{\tilde{R}} = \{\{c,e\},\{a\},\{b\},\{d\},\{f\}\}\}$
$F_{0.4}^{\tilde{R}} = \{\{a,c,e\},\{b,f\},\{d\}\}$
$F_{0.3}^{\tilde{R}} = \{\{a,b,f\},\{a,c,e\},\{b,d\},\{c,d\}\}$
$F_{0.2}^{\tilde{R}} = \{\{a,b,c\},\{b,c,d\},\{a,b,f\},\{a,c,e\}\}$.

We also note that $R$ is not transitive: $\tilde{R}(b,c) = .2 \not\geq .3 = (.3 \wedge .4) \vee (1 \wedge .2) \vee (.2 \wedge 1) \vee (.3 \wedge .3)\ (0 \wedge .5) \vee (.4 \wedge 0) = \vee\{\tilde{R}(b,y) \wedge \tilde{R}(y,c) \mid y \in S\} = \tilde{R} \circ \tilde{R}(b,c)$.

In the next two results $\chi_\emptyset$ is the characteristic function of $\emptyset$ in $S \times S$.

**Lemma 1.6** *If $\tilde{R} \neq \chi_\emptyset$ is a weakly reflexive and symmetric fuzzy relation on $S$, then there exists a set $T$ and a fuzzy relation $\tilde{P}$ from $S$ into $T$ such that $\tilde{R} = \tilde{P} \circ \tilde{P}^{-1}$.* ■

Combining Lemmas 1.3 and 1.6, we have the following result.

**Theorem 1.7** *A fuzzy relation $\tilde{R} \neq \chi_\emptyset$ on a set $S$ is weakly reflexive and symmetric if and only if there is a set $T$ and a fuzzy relation $\tilde{P}$ from $S$ into $T$ such that $\tilde{R} = \tilde{P} \circ \tilde{P}^{-1}$.* ■

It may be noted given a weakly reflexive and symmetric fuzzy relation $\tilde{R} \neq \chi_\emptyset$ on a set $S$, the set $T$ and the fuzzy relation $\tilde{P}$ from $S$ into $T$ can be constructed as follows. Define $T$ as the set $\{K^* | K \in F^{\tilde{R}}\}$. In other words, $T$ is a set having the same cardinality as $F^{\tilde{R}}$ and assume that there is a one-to-one mapping from $F^{\tilde{R}}$ onto $T$. Now $\tilde{P}$ from $S$ into $T$ is defined as follows:

$$\tilde{P}(x, K^*) = \begin{cases} \alpha & \text{if } x \in K, \alpha \text{ is the largest number such that } K \in F_\alpha^{\tilde{R}}. \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of this section, we shall use the notation $\phi_{\tilde{R}}$ to denote the fuzzy relation $\tilde{P}$ defined above.

**Definition 1.7** *A cover $C$ on a set $S$ is a family of subsets $S_i, i \in I$, of $S$ such that $\bigcup_{i\in I} S_i = S$, where $I$ is a nonempty index set.*

**Definition 1.8** *Let $\tilde{R}$ be a fuzzy relation from $S$ into $T$. We define the following, where $\epsilon \in [0, 1]$:*

**(i)** *$\tilde{R}$ is* $\epsilon$-determinate *if for each $x \in S$, there exists at most one $y \in T$ such that $\tilde{R}(x, y) \geq \epsilon$.*

**(ii)** *$\tilde{R}$ is* $\epsilon$-productive *if for each $x \in S$, there exists at least one $y \in T$ such that $\tilde{R}(x, y) \geq \epsilon$.*

**(iii)** *$\tilde{R}$ is an* $\epsilon$-function *if it is both $\epsilon$-determinate and $\epsilon$-productive.*

**Lemma 1.8** *If $\tilde{R}$ is an $\epsilon$-reflexive fuzzy relation on $S$, then $\phi_{\tilde{R}}$ is $\epsilon$-productive and for each $\epsilon' \leq \epsilon$, $F_{\epsilon'}^{\tilde{R}}$ is a cover of $S$.* ■

In the sequel, we use the term *productive (determinate, reflexive, function)* for 1-productive (1-determinate, 1-reflexive, 1-function).

**Corollary 1.9** *If $\tilde{R}$ is reflexive, then $\phi_{\tilde{R}}$ is productive and each $F_{\epsilon}^{\tilde{R}}$ $(0 < \epsilon \leq 1)$ is a cover of $S$.* ■

The following result is a consequence of Theorem 1.7 and Corollary 1.9.

**Corollary 1.10** *$\tilde{R}$ is reflexive and symmetric relation on $S$ if and only if there is a set $T$ and a productive fuzzy relation $\tilde{P}$ from $S$ into $T$ such that $\tilde{R} = \tilde{P} \circ \tilde{P}^{-1}$.* ■

**Lemma 1.11** *Let $\tilde{R}$ be a weakly reflexive, symmetric and transitive fuzzy relation on $S$, and let $\phi_{\tilde{R}}^{\epsilon}$ denote the relation $\phi_{\tilde{R}}$ whose range is restricted to $F_{\epsilon}^{\tilde{R}}$. That is, $\phi_{\tilde{R}}^{\epsilon}$ equals $\phi_{\tilde{R}}$ on $S \times \{K^* | K \in F_{\epsilon}^{\tilde{R}}\}$. Then for each $0 < \epsilon \leq 1$, $\phi_{\tilde{R}}^{\epsilon}$ is $\epsilon$-determinate and the elements of $F_{\epsilon}^{\tilde{R}}$ are pairwise disjoint.* ■

**Definition 1.9** *A similarity relation $\tilde{R}$ on $S$ is a fuzzy relation on $S$ which is reflexive, symmetric and transitive. $\tilde{R}$ is called an* $\epsilon$-similarity relation *if it is $\epsilon$-reflexive for some $0 < \epsilon \leq 1$, symmetric, and transitive.*

A similarity relation on $S$ is merely a fuzzy equivalence relation on $S$.
Since clearly reflexivity implies weak reflexivity, we have the following consequence of Lemmas 1.8 and 1.11.

**Corollary 1.12** *If $\tilde{R}$ is a similarity relation on $S$, then for each $0 < \epsilon \leq 1$, $F_{\epsilon}^{\tilde{R}}$ is a partition of $S$.* ■

Note that Corollary 1.12 says that every similarity relation $\tilde{R}$ can be represented as $\bigcup_{\alpha} \alpha\tilde{R}_\alpha$, where $\tilde{R}_\alpha$ is the equivalence relation induced by the partition $F_\alpha^{\tilde{R}}$. Indeed, it was pointed out by Zadeh [11] that if the $\tilde{R}_\alpha, 0 < \alpha \le 1$, are a nested sequence of distinct equivalence relations on $S$, with $\alpha_1 > \alpha_2$ if and only if $\tilde{R}_{\alpha_1} \subseteq \tilde{R}_{\alpha_2}$, $\tilde{R}_{\alpha_1}$ is nonempty and the domain of $\tilde{R}_{\alpha_1}$ is equal to the domain of $\tilde{R}_{\alpha_2}$, then $\tilde{R} = \bigcup_{\alpha} \alpha\tilde{R}_\alpha$ is a similarity relation on $S$, where

$$\alpha\tilde{R}_\alpha(x,y) = \begin{cases} \alpha & \text{if } (x,y) \in \tilde{R}_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

The following result, which is a straightforward consequence of Theorem 1.7 and Corollary 1.12, yields another characterization of a similarity relation.

**Theorem 1.13** *A relation $\tilde{R}$ is an $\epsilon$-similarity $(0 < \epsilon \le 1)$ relation on a set $S$ if and only if there is another set $T$ and an $\epsilon$-function $\tilde{P}$ from $S$ into $T$ such that $\tilde{R} = \tilde{P} \circ \tilde{P}^{-1}$.* ■

**Example 1.9** *Let $\tilde{R}$ be the fuzzy relation on $S = \{a,b,c,d,e,f\}$ given by the following matrix, $M_{\tilde{R}}$.*

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 1.0 | 0.5 | 0.5 | 0.2 | 0.2 | 0.2 |
| $b$ | 0.5 | 1.0 | 0.5 | 0.2 | 0.2 | 0.2 |
| $c$ | 0.5 | 0.5 | 1.0 | 0.2 | 0.2 | 0.2 |
| $d$ | 0.2 | 0.2 | 0.2 | 1.0 | 0.4 | 0.4 |
| $e$ | 0.2 | 0.2 | 0.2 | 0.4 | 1.0 | 0.4 |
| $f$ | 0.2 | 0.2 | 0.2 | 0.4 | 0.4 | 1.0 |

*Now $M_{\tilde{R}}{}^2 = M_{\tilde{R}}$. Thus $\tilde{R}$ is transitive. Clearly, $\tilde{R}$ is reflexive and symmetric. We have*

$F_1^{\tilde{R}} = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\}$
$F_{0.5}^{\tilde{R}} = \{\{a,b,c\},\{d\},\{e\},\{f\}\}$
$F_{0.4}^{\tilde{R}} = \{\{a,b,c\},\{d,e,f\}\}$
$F_{0.2}^{\tilde{R}} = \{X\}$.

*Let $\epsilon = 0.4$. Then the $\epsilon - function$ $\tilde{P} : X \times \{K^* | K \in F_{0.4}^{\tilde{R}}\} \to [0,1]$, is defined as follows: $\tilde{P}(a,\{a,b,c\}^*) = \tilde{P}(b,\{a,b,c\}^*) = \tilde{P}(c,\{a,b,c\}^*) = 0.5, \tilde{P}(d,\{d,e,f\}^*) = \tilde{P}(e,\{d,e,f\}^*) = \tilde{P}(f,\{d,e,f\}^*) = 0.4$, and $\tilde{P}(x,K^*) = 0$ otherwise.*

## 1.6 References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# 2 FUZZY GRAPHS

Any relation $R \subseteq S \times S$ on a set $S$ can be regarded as defining a graph with vertex set $S$ and edge set $R$. That is, a *graph* is a pair $(S, R)$, where $S$ is a set and $R$ is a relation on $S$. Similarly, any fuzzy relation $\tilde{R}$ on a fuzzy subset $\tilde{A}$ of a set $S$ can be regarded as defining a weighted graph, or fuzzy graph, where the edge $(x, y) \in S \times S$ has weight or strength $\tilde{R}(x, y) \in [0, 1]$. In this chapter, we shall use graph terminology and introduce fuzzy analogs of several basic graph-theoretical concepts. For simplicity, we will consider only undirected graphs through out this chapter, except in Section 2.3. Therefore, all fuzzy relations are symmetric and all edges are regarded as unordered pairs of vertices, except for Section 2.3. We abuse notation by writing $(x, y)$ for an edge in an undirected graph $(S, R)$, where $x, y \in S$. (We need not consider loops, that is, edges of the form $(x, x)$; we can assume, if we wish, that our fuzzy relation is reflexive.) Formally, a *fuzzy graph* $G = (S, \tilde{A}, \tilde{R})$ is a nonempty set $S$ together with a pair of functions $\tilde{A} : S \to [0, 1]$ and $\tilde{R} : S \times S \to [0, 1]$ such that for all $x, y$ in $S$, $\tilde{R}(x, y) \leq \tilde{A}(x) \wedge \tilde{A}(y)$. We call $\tilde{A}$ the *fuzzy vertex set* of $G$ and $\tilde{R}$ the *fuzzy edge set* of $G$, respectively. Note that $\tilde{R}$ is a fuzzy relation on $\tilde{A}$. We will assume from now on that the underlying set is $S$ and that it is finite. Therefore, for the sake of notational convenience, we may omit $S$ for the rest of our discussion and use the abbreviated notation $G = (\tilde{A}, \tilde{R})$. Thus in the most general case, both vertices and edges have membership value. However, in the special case where $\tilde{A}(x) = 1$, for all $x \in S$, edges alone have fuzzy membership. So, in this case, we use the abbreviated notation $G = (S, \tilde{R})$. The fuzzy graph $H = (\tilde{B}, \tilde{T})$ is called a *partial fuzzy subgraph* of $G = (\tilde{A}, \tilde{R})$ if $\tilde{B} \subseteq \tilde{A}$ and $\tilde{T} \subseteq \tilde{R}$. Similarly, the fuzzy graph $H = (P, \tilde{B}, \tilde{T})$ is called a *fuzzy subgraph* of $G = (S, \tilde{A}, \tilde{R})$ induced by $P$ if $P \subseteq S, \tilde{B}(x) = \tilde{A}(x)$ for all $x \in P$ and $\tilde{T}(x,y) = \tilde{R}(x,y)$ for all $x, y \in P$. Whenever there is no confusion, for the sake of simplicity, we call $H$ a fuzzy subgraph of $G$. It is worth noticing that a fuzzy subgraph $(P, \tilde{B}, \tilde{T})$ of a fuzzy graph $(S, \tilde{A}, \tilde{R})$ is in fact a special case of a partial fuzzy subgraph obtained as follows.

$$\tilde{B}(x) = \begin{cases} \tilde{A}(x) & \text{if } x \in P \\ 0 & \text{if } x \in S \setminus P \end{cases}$$

$$\tilde{T}(x,y) = \begin{cases} \tilde{R}(x,y) & \text{if } (x,y) \in P \times P \\ 0 & \text{if } (x,y) \in S \times S \setminus P \times P \end{cases}$$

Thus a fuzzy graph can have only one fuzzy subgraph corresponding to a given subset $P$ of $S$. Hence we shall use the notation $\langle P \rangle$ to denote the fuzzy subgraph of $G$ induced by $P$. For any threshold $t, 0 \leq t \leq 1$, $\tilde{A}^t = \{x \in S | \tilde{A}(x) \geq t\}$ and $\tilde{R}^t = \{(x,y) \in S \times S | \tilde{R}(x,y) \geq t\}$. Since $\tilde{R}(x,y) \leq \tilde{A}(x) \wedge \tilde{A}(y)$ for all $x, y \in S$, we have $\tilde{R}^t \subseteq \tilde{A}^t \times \tilde{A}^t$, so that $(\tilde{A}^t, \tilde{R}^t)$ is a graph with the vertex set $\tilde{A}^t$ and edge set $\tilde{R}^t$ for all $t \in [0,1]$.

**Proposition 2.1** *Let $G = (\tilde{A}, \tilde{R})$ be a fuzzy graph. If $0 \leq u \leq t \leq 1$, then $(\tilde{A}^t, \tilde{R}^t)$ is a subgraph of $(\tilde{A}^u, \tilde{R}^u)$.* ∎

**Proposition 2.2** *Let $H = (\tilde{B}, \tilde{T})$ be a partial fuzzy subgraph of $G = (\tilde{A}, \tilde{R})$. For any threshold $t, 0 \leq t \leq 1$, $(\tilde{B}^t, \tilde{T}^t)$ is a subgraph of $(\tilde{A}^t, \tilde{R}^t)$.* ∎

We say that the partial fuzzy subgraph $(\tilde{B}, \tilde{T})$ *spans* the fuzzy graph $(\tilde{A}, \tilde{R})$ if $\tilde{A} = \tilde{B}$. In this case, the two graphs have the same fuzzy vertex set; they differ only in the edge weights. For any fuzzy subset $\tilde{B}$ of $S$ such that $\tilde{B} \subseteq \tilde{A}$, the partial fuzzy subgraph of $(\tilde{A}, \tilde{R})$ *induced* by $\tilde{B}$ is the maximal partial fuzzy subgraph of $(\tilde{A}, \tilde{R})$ that has fuzzy vertex set $\tilde{B}$. Evidently, this is just the partial fuzzy graph $(\tilde{B}, \tilde{T})$, where $\tilde{T}(x,y) = \tilde{B}(x) \wedge \tilde{B}(y) \wedge \tilde{R}(x,y)$ for all $x, y \in S$.

## 2.1 Paths and Connectedness

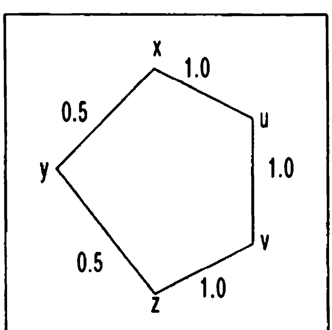
Let $G = (V, X)$ be a graph. A *walk* of $G$ is an alternating sequence of vertices and edges $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$, where $v_0, v_i \in V$, $x_i \in X$, $x_i = (v_{i-1}, v_i)$, $i = 1, \ldots, n$. A walk is sometimes denoted by $v_0 v_1 \ldots v_n$, where the edges are evident by context. A *path* is a walk with all vertices distinct. Let $v_0 v_1 \ldots v_n$ be a path. If $n \geq 2$ and $v_0 = v_n$, then the path is called a *cycle*. $G = (V, X)$ is said to be *complete* if $(u, v) \in X$ $\forall u, v \in V$, $u \neq v$. A *clique* of a graph is a maximal complete subgraph.

A *path* $\rho$ in a fuzzy graph $(\tilde{A}, \tilde{R})$ is a sequence of distinct vertices $x_0, x_1, \ldots, x_n$ such that $\tilde{R}(x_{i-1}, x_i) > 0$, $1 \leq i \leq n$. Here $n \geq 1$ is called the *length* of the path $\rho$. The consecutive pairs $(x_{i-1}, x_i)$ are called the *edges* of the path. The *strength* of $\rho$ is defined as $\wedge_{i=1}^{n} \tilde{R}(x_{i-1}, x_i)$. In other words, the strength of a path is defined to be the weight of the weakest edge of the path. A single vertex $x$ may also be considered as a path. In this case, the path is of length 0. If the path has length 0, it is convenient to define its strength to be $\tilde{A}(x_0)$. It may be noted that any path of length $n > 0$ can as well be defined as a sequence of edges $(x_{i-1}, x_i), 1 \leq i \leq n$, satisfying the condition $\tilde{R}(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$. A partial fuzzy subgraph $(\tilde{A}, \tilde{R})$ is said to be *connected* if $\forall x, y \in \text{supp}(\tilde{A}), \tilde{R}^{\infty}(x, y) > 0$.

We call $\rho$ a *cycle* if $x_0 = x_n$ and $n \geq 2$. Two vertices that are joined by a path are said to be *connected*. It is evident that "connected" is an equivalence relation. In fact, $x$ and $y$ are connected if and only if $\tilde{R}^{\infty}(x, y) > 0$. The equivalence classes of vertices under this relation are called *connected components* of the given fuzzy graph. They are just its maximal connected partial fuzzy subgraphs. A strongest path joining any two vertices $x, y$ has strength $\tilde{R}^{\infty}(x, y)$. We shall sometimes refer to this as the *strength of connectedness* between the vertices.

**Proposition 2.3** *If* $(\tilde{B}, \tilde{T})$ *is a partial fuzzy subgraph of* $(\tilde{A}, \tilde{R})$, *then* $\tilde{T}^{\infty} \subseteq \tilde{R}^{\infty}$. ■

Let $(\tilde{A}, \tilde{R})$ be a fuzzy graph. We now provide two popular ways of defining the distance between a pair of vertices. One way is to define the "distance" $dis(x, y)$ between $x$ and $y$ as the length of the shortest strongest path between them. This "distance" is symmetric and is such that $dis(x, x) = 0$ since by our definition of a fuzzy graph, no path from $x$ to $x$ can have strength greater than $\tilde{A}(x)$, which is the strength of the path of length 0. However, it does not satisfy the triangle property, as we see from the following example.




Here any path from $x$ to $y$ or from $y$ to $z$ has strength $\leq 1/2$ since it must involve either edge $(x, y)$ or edge $(y, z)$. Thus the shortest strongest paths between them have length 1. On the other hand, there is a path from $x$ to $z$, through $u$ and $v$, that has length 3 and strength 1. Thus $dis(x, z) = 3 > 1 + 1 = dis(x, y) + dis(y, z)$ in this case.

A better notion of distance in a fuzzy graph can be defined as follows: For any path $\rho = x_0, ..., x_n$, define the *$\tilde{R}$-length* of $\rho$ as the sum of the reciprocals of $\rho$'s edge weights, that is,

$$l(\rho) = \sum_{i=1}^{n} \frac{1}{\tilde{R}(x_{i-1}, x_i)}.$$

If $n = 0$, we define $l(\rho) = 0$. Clearly, for $n \geq 1$, we have $l(\rho) \geq 1$. For any two vertices $x, y$, we can now define their *$\tilde{R}$-distance* $\delta(x, y)$ as the smallest $\tilde{R}$-length of any path from $x$ to $y$. Thus $\delta(x,y) = \wedge\{l(\rho)|\rho$ is a path between $x$ and $y\}$ if $x$ and $y$ are connected. We define $\delta(x,y) = \infty$ if $x$ and $y$ are not connected.

**Proposition 2.4** *$\delta$ is a metric on $S$. That is, $\forall x, y, z \in S$,*

**(i)** $\delta(x,y) = 0$ *if and only if* $x = y$,

**(ii)** $\delta(x,y) = \delta(y,x)$,

**(iii)** $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$. ■

In the non-fuzzy case, $l(\rho)$ is just the length $n$ of $\rho$ since all the $\tilde{R}$'s are 1. Hence, $\delta(x, y)$ becomes the usual definition of distance, that is, the length of the shortest path between $x$ and $y$.

## 2.2 Clusters

In graph theory, there are several ways of defining "clusters" of vertices. One approach is to call a subset $C$ of $S$ a *cluster* of order $k$ if the following two conditions hold:

**(a)** for all vertices $x, y$ in $C$, $d(x,y) \leq k$;

**(b)** for all vertices $z \notin C$, $d(z,w) > k$ for some $w \in C$;

where $d(u, v)$ is the length of a shortest path between any two vertices $u$ and $v$.

Thus in a $k$-cluster $C$, every pair of vertices are within distance $k$ of each other, and $C$ is maximal with respect to this property. That is, no vertex outside $C$ is within distance $k$ of every vertex in $C$.

A 1-cluster is called a *clique*; it is a maximal complete subgraph. That is, a maximal subgraph in which each pair of vertices is joined by an edge. At the other extreme, if we let $k \to \infty$, a $k$-cluster becomes a connected component, that is, a maximal subgraph in which each pair of vertices is joined by a path (of any length).

These ideas can be generalized to fuzzy graphs as follows: In $G = (\tilde{A}, \tilde{R})$, we can call $C \subseteq S$ a *fuzzy cluster* of order $k$ if

$$\wedge\{\tilde{R}^k(x, y) \mid x, y \in C\} > \vee\{\wedge\{\tilde{R}^k(w, z) \mid w \in C\} \mid z \notin C\}.$$

Note that $C$ is an ordinary subset of $S$, not a fuzzy subset. If $G$ is an ordinary graph, we have $\tilde{R}^k(a, b) = 0$ or $1$ for all $a$ and $b$. Hence this definition reduces to

**(a)** $\tilde{R}^k(x, y) = 1$ for all $x, y$ in $C$,

**(b)** $\tilde{R}^k(w, z) = 0$ for all $z \notin C$ and some $w \in C$.

Property (a) implies that for all $x, y$ in $C$, there exists a path of length $\leq k$ between $x$ and $y$ and property (b) implies that for all $z \notin C$ and some $w \in C$, there does not exist a path of length $\leq k$. This is the same as the definition of a cluster of order $k$.

In fact, the $k$-clusters obtained using this definition are just ordinary cliques in graphs obtained by thresholding the $k$th power of the given fuzzy graph. Indeed, let $C$ be a fuzzy $k$-cluster, and let $\wedge\{\tilde{R}^k(x, y) \mid x, y \in C\} = t$. If we threshold $\tilde{R}^k$ (and $\tilde{A}$) at $t$, we obtain an ordinary graph in which $C$ is now an ordinary clique.

**Example 2.1** *Let*

$$V = \{x, y, z, u, v\}$$

*and*

$$X = \{(x, y), (x, z), (y, z), (z, u,), (u, v)\}.$$

*Let* $\tilde{A}(x) = \tilde{A}(y) = \tilde{A}(z) = \tilde{A}(u) = \tilde{A}(v) = 1$ *and* $\tilde{R}(x, y) = \tilde{R}(x, z) = \tilde{R}(y, z) = 1/2$, $\tilde{R}(z, u,) = \tilde{R}(u, v) = 1/4$. *Let* $C = \{x, y, z\}$. *Then* $\bigwedge_{c,d \in C} \tilde{R}^k(c, d) = 1/2$ *for* $k = 1, 2, \ldots$, $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^1(c, e)) = (1/4 \wedge 0 \wedge 0) \vee (0 \wedge 0 \wedge 0) = 0$, $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^2(c, e)) = (1/4 \wedge 1/4 \wedge 1/4) \vee (1/4 \wedge 0 \wedge 0) = 1/4$, *and* $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^k(c, e)) = (1/4 \wedge 1/4 \wedge 1/4) \vee (1/4 \wedge 1/4 \wedge 1/4) = 1/4$ *for* $k \geq 3$. *Hence* $C = \{x, y, z\}$ *is a fuzzy cluster of order* $k$ *for all* $k \geq 1$.

*Now let* $\tilde{R}(x, y) = \tilde{R}(x, z) = \tilde{R}(y, z) = 1/8$, $\tilde{R}(z, u) = \tilde{R}(u, v) = 1/4$. *Then* $\bigwedge_{c,d \in C} \tilde{R}^k(c, d) = 1/8$ *for* $k = 1, 2, \ldots$. $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^1(c, e)) = (1/4 \wedge 0 \wedge 0) \vee (0 \wedge 0 \wedge 0) = 0$, $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^2(c, e)) = (1/4 \wedge 1/8 \wedge 1/8) \vee (1/4 \wedge 0 \wedge 0) = 1/8$, *and* $\bigvee_{e \notin C} (\bigwedge_{c \in C} \tilde{R}^k(c, e)) = (1/4 \wedge 1/8 \wedge 1/8) \vee (1/4 \wedge 1/8 \wedge 1/8) = 1/8$ *for* $k \geq 3$. *Hence* $C$ *is a fuzzy cluster of order* 1, *but not of order* $k$ *for* $k \geq 2$.

## *Bridges and Cut Vertices*

Let $G = (\tilde{A}, \tilde{R})$ be a fuzzy graph, let $x, y$ be any two distinct vertices, and let $G'$ be the partial fuzzy subgraph of $G$ obtained by deleting the edge $(x,y)$. That is, $G' = (\tilde{A}, \tilde{R}')$, where $\tilde{R}'(x,y) = 0$ and $\tilde{R}' = \tilde{R}$ for all other pairs. We say that $(x,y)$ is a *bridge* in $G$ if $\tilde{R}'^{\infty}(u,v) < \tilde{R}^{\infty}(u,v)$ for some $u,v$. In other words, if deleting the edge $(x,y)$ reduces the strength of connectedness between some pair of vertices. Thus, $(x,y)$ is a bridge if and only if there exist vertices $u,v$ such that $(x,y)$ is an edge of every strongest path from $u$ to $v$.

**Theorem 2.5** *The following statements are equivalent:*

**(i)** $(x,y)$ *is a bridge;*

**(ii)** $\tilde{R}'^{\infty}(x,y) < \tilde{R}(x,y)$;

**(iii)** $(x,y)$ *is not the weakest edge of any cycle.*

**Proof.** (ii) $\Rightarrow$ (i): If $(x,y)$ is not a bridge, we must have $\tilde{R}'^{\infty}(x,y) = \tilde{R}^{\infty}(x,y) \geq \tilde{R}(x,y)$.

(i)$\Rightarrow$(iii): If $(x,y)$ is a weakest edge of a cycle, then any path involving edge $(x,y)$ can be converted into a path not involving $(x,y)$ but at least as strong, by using the rest of the cycle as a path from $x$ to $y$. Thus $(x,y)$ cannot be a bridge.

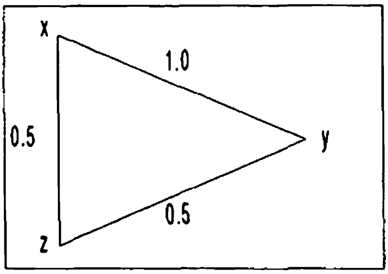
(iii)$\Rightarrow$(ii): If $\tilde{R}'^{\infty}(x,y) \geq \tilde{R}(x,y)$, there is a path from $x$ to $y$, not involving $(x,y)$, that has strength $\geq \tilde{R}(x,y)$, and this path together with $(x,y)$ forms a cycle of which $(x,y)$ is a weakest edge. ■

Let $w$ be any vertex and let $G^*$ be the partial fuzzy subgraph of $G$ obtained by deleting the vertex $w$. That is, $G^*$ is the partial fuzzy subgraph induced by $\tilde{A}^*$, where $\tilde{A}^*(w) = 0$; $\tilde{A}^* = \tilde{A}$ for all other vertices.

Note that in $G^* = (\tilde{A}^*, \tilde{R}^*)$, we must have $\tilde{R}^*(w,z) = 0$ for all $z$. We say that $w$ is a *cutvertex* in $G$ if $\tilde{R}^{*\infty}(u,v) < \tilde{R}^{\infty}(u,v)$ for some $u,v$ (other than $w$). In other words, if deleting the vertex $w$ reduces the strength of connectedness between some other pair of vertices. Evidently, $w$ is a cutvertex if and only if there exist $u,v$, distinct from $w$ such that $w$ is on every strongest path from $u$ to $v$. $G^*$ is called *nonseparable* (or sometimes: a *block* ) if it has no cut vertices. It should be pointed out that a block may have bridges (this cannot happen for non-fuzzy graphs). For example, in Figure 2.1 edge $(x,y)$ is a bridge since its deletion reduces the strength of connectedness between $x$ and $y$ from 1 to 0.5. However, it is easily verified that no vertex of this fuzzy graph is a cutvertex.

If between every two vertices $x,y$ of $G$ there exist two strongest paths that are disjoint (except for $x,y$ themselves), $G$ is evidently a block. This is analogous to the "if" of the non-fuzzy graph theorem that $G$ is a block (with at least three vertices) if and only if every two vertices of $G$ lie on a common cycle. The "only if", on the other hand, does not hold in the fuzzy case, as the example just given shows.

**FIGURE 2.1** Fuzzy graph with a bridge; but no cut vertices.

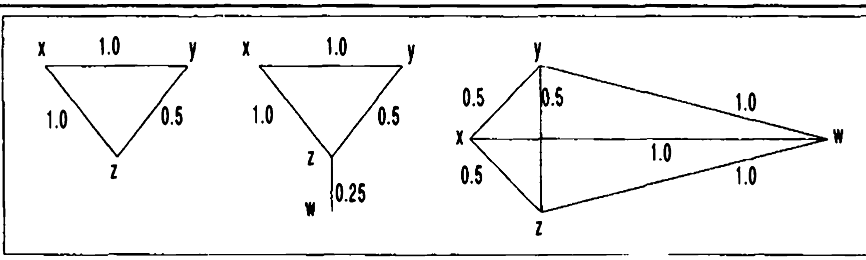



*Forests and Trees*

We recall that a graph that has no cycles is called *acyclic*, or a *forest;* and a connected forest is called a *tree*. We shall call a fuzzy graph a *forest* if the graph consisting of its nonzero edges is a forest, and a *tree* if this graph is also connected. More generally, we call the fuzzy graph $G = (\tilde{A}, \tilde{R})$ a *fuzzy forest* if it has a partial fuzzy spanning subgraph $F = (\tilde{A}, \tilde{T})$ which is a forest, where for all edges $(x, y)$ not in $F$ (i.e., such that $\tilde{T}(x, y) = 0$), we have $\tilde{R}(x, y) < \tilde{T}^{\infty}(x, y)$. In other words, if $(x, y)$ is in $G$ but $(x, y)$ is not in $F$, there is a path in $F$ between $x$ and $y$ whose strength is greater than $\tilde{R}(x, y)$. It is clear that a forest is a fuzzy forest.

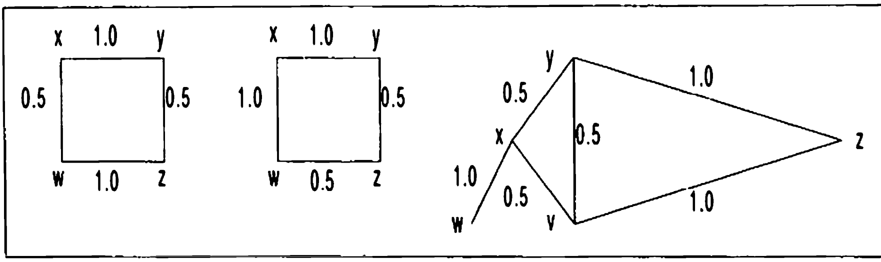
The fuzzy graphs in Figure 2.2 are fuzzy forests and the fuzzy graphs in Figure 2.3 are not fuzzy forests.

**FIGURE 2.2** Fuzzy forests.




If $G$ is connected, then so is $F$ since any edge of a path in $G$ is either in $F$, or can be diverted through $F$. In this case, we call $G$ a *fuzzy tree*. The examples of fuzzy forests given above are all fuzzy trees. Note that if we replaced $<$ by $\leq$ in the definition, then even the fuzzy graph $(S, \tilde{A}, \tilde{R})$, where $S = \{x, y, z\}$, $\tilde{A}(x) = \tilde{A}(y) = \tilde{A}(z) = 1, \tilde{R}(x, y) = \tilde{R}(x, z) = \tilde{R}(y, z) = 1$, would be a fuzzy forest since it has partial fuzzy spanning subgraphs such as $(S, \tilde{A}, \tilde{R}')$, where $\tilde{R}'(x, y) = \tilde{R}'(x, z) = 1$ and $\tilde{R}(y, z) = 0$.

**FIGURE 2.3** Fuzzy graphs; but not fuzzy forests.




**Theorem 2.6** *$G$ is a fuzzy forest if and only if in any cycle of $G$, there is an edge $(x,y)$ such that $\tilde{R}(x,y) < \tilde{R}'^{\infty}(x,y)$, where $G' = (\tilde{A}, \tilde{R}')$ is the partial fuzzy subgraph obtained by the deletion of the edge $(x,y)$ from $G$.*

**Proof.** Suppose $(x,y)$ is an edge, belonging to a cycle, which has the property of the theorem and for which $\tilde{R}(x,y)$ is smallest. (If there are no cycles, $G$ is a forest and we are done.) If we delete $(x,y)$, the resulting partial fuzzy subgraph satisfies the path property of a fuzzy forest. If there are still cycles in this graph, we can repeat the process. Note that at each stage, no previously deleted edge is stronger than the edge being currently deleted; hence the path guaranteed by the property of the theorem involves only edges that have not yet been deleted. When no cycles remain, the resulting partial fuzzy subgraph is a forest $F$. Let $(x,y)$ not be an edge of $F$; thus $(x,y)$ is one of the edges that we deleted in the process of constructing $F$, and there is a path from $x$ to $y$ that is stronger than $\tilde{R}(x,y)$ and that does not involve $(x,y)$ nor any of the edges deleted prior to it. If this path involves edges that were deleted later, it can be diverted around them using a path of still stronger edges; if any of these were deleted later, the path can be further diverted; and so on. This process eventually stabilizes with a path consisting entirely of edges of $F$. Thus $G$ is a fuzzy forest.

Conversely, if $G$ is a fuzzy forest and $\rho$ is any cycle, then some edge $(x,y)$ of $\rho$ is not in $F$. Thus by definition of a fuzzy forest we have $\tilde{R}(x,y) < \tilde{T}^{\infty}(x,y) \leq \tilde{R}'^{\infty}(x,y)$. ■

Note that if $G$ is connected, so is the constructed $F$ in the first part of the proof since no step of the construction disconnects.

**Proposition 2.7** *If there is at most one strongest path between any two vertices of $G$, then $G$ must be a fuzzy forest.*

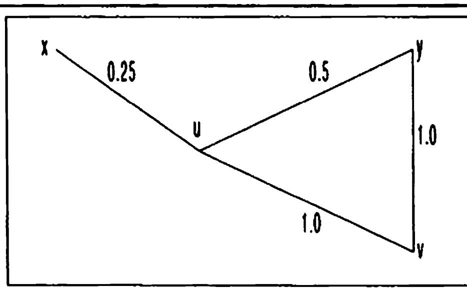
**Proof.** Suppose $G$ is not a fuzzy forest. Then by Theorem 2.6, there is a cycle $\rho$ in $G$ such that $\tilde{R}(x,y) \geq \tilde{R}'(x,y)$ for all edges $(x,y)$ of $\rho$. Thus $(x,y)$ is a strongest path from $x$ to $y$. If we choose $(x,y)$ to be a weakest edge of $\rho$, it follows that the rest of the $\rho$ is also a strongest path from $x$ to $y$, a contradiction. ■

The converse of Proposition 2.7 is false; $G$ can be a fuzzy forest and still have multiple strongest paths between vertices. This is because the strength of a path is that of its weakest edge, and as long as this edge lies in $F$, there is little constraint on the other edges. For example, the fuzzy graph in Figure 2.4 is a fuzzy forest. Here $F$ consists of all edges except $(u, y)$. The strongest paths between $x$ and $y$ have strength $1/4$, due to the edge $(x, u)$; both $x, u, v, y$ and $x, u, y$ are such paths, where the former lies in $F$ but the latter does not.

**Proposition 2.8** *If $G$ is a fuzzy forest, then the edges of $F$ are just the bridges of $G$.*

**Proof.** An edge $(x, y)$ not in $F$ cannot be a bridge since $\tilde{R}(x, y) < \tilde{T}^{\infty}(x, y) \leq \tilde{R}'^{\infty}(x, y)$. Suppose that $(x, y)$ is an edge in $F$. If it were not a bridge, we would have a path $\rho$ from $x$ to $y$, not involving $(x, y)$, of strength $\geq \tilde{R}(x, y)$. This path must involve edges not in $F$ since $F$ is a forest and has no cycles. However, by definition, any such edge $(u_i, v_i)$ can be replaced by a path $\rho_i$ in $F$ of strength $> \tilde{R}(u, v)$. Now $\rho_i$ cannot involve $(x, y)$ since all its edges are strictly stronger than $\tilde{R}(u, v) \geq \tilde{R}(x, y)$. Thus by replacing each $(u_i, v_i)$ by $\rho_i$, we can construct a path in $F$ from $x$ to $y$ that does not involve $(x, y)$, giving us a cycle in $F$, contradiction. ■

**FIGURE 2.4** A fuzzy forest with no multiple strongest paths between vertices.




## 2.3 Cluster Analysis and Modeling of Information Networks

In this section, fuzzy graphs will be analyzed from the connectedness viewpoint. The results will be applied to cluster analysis and modeling of information networks. We do not assume our (fuzzy) graphs are necessarily undirected in this section.

Let $G = (\tilde{A}, \tilde{R})$ be a fuzzy graph. We denote by $M_{\tilde{R}}$ the corresponding fuzzy matrix of a fuzzy graph $G$. In other words, $(M_{\tilde{R}})_{ij} = \tilde{R}(v_i, v_j)$.

**Theorem 2.9** *Let $G = (S, \tilde{A}, \tilde{R})$ be a fuzzy graph such that cardinality of $S$ is $n$. Then*

**(i)** *if $\tilde{R}$ is reflexive, there exists $k \leq n$ such that $M_{\tilde{R}} < M_{\tilde{R}}^2 < \ldots < M_{\tilde{R}^3}^k = M_{\tilde{R}^3}^{k+1}$;*

**(ii)** *if $\tilde{R}$ is irreflexive, the sequence $M_{\tilde{R}}, M_{\tilde{R}}^2, \ldots$ is eventually periodic.* ■

**Definition 2.1** *Let $G$ be a fuzzy graph. Let $0 \leq \epsilon \leq 1$. A vertex $v$ is said to be* $\epsilon$-reachable *from another vertex $u$ if there exists a positive integer $k$ such that $\tilde{R}^k(u,v) \geq \epsilon$. The reachability matrix of $G$, denoted by $M_{\tilde{R}^\infty}$, is the matrix of the fuzzy graph $(\tilde{A}, \tilde{R}^\infty)$. The $\epsilon$-reachability matrix of $G$, denoted by $M_{\tilde{R}^\infty}^\epsilon$, is defined as follows: $M_{\tilde{R}^\infty}^\epsilon(u,v) = 1$ if $\tilde{R}(u,v) \geq \epsilon$ and $M_{\tilde{R}^\infty}^\epsilon(u,v) = 0$, otherwise.*

The following algorithm can determine the reachability between any pair of vertices in a fuzzy graph $G$.

**Algorithm 2.1**. Determination of $M_{\tilde{R}^\infty}$

1. Let $R_i = (a_{i1}, \ldots, a_{in})$ denote the $i^{th}$ row.
2. Obtain the new $R_i$ by the following procedure:

$$a_{ij}(new) = \bigvee_j \{\bigvee_k \{a_{kj} \wedge a_{ik}(old)\}\}, a_{ij}(old)\}.$$

3. Repeat Step 2 until no further changes occur.
4. $M_{\tilde{R}^\infty}(i,j) = a_{ij}(new)$.

Note that a similar algorithm can be constructed for the determination of $M_{\tilde{R}^\infty}^\epsilon$, $0 \leq \epsilon \leq 1$.

**Definition 2.2** *Let $G$ be a fuzzy graph. The* connectivity *of a pair of vertices $u$ and $v$, denoted by $C(u,v)$ is defined to be $\tilde{R}^\infty(u,v) \wedge \tilde{R}^\infty(v,u)$. The* connectivity matrix *of $G$, denoted by $C_G$, is defined such that $C_G(u,v) = C(u,v)$. For $0 \leq \epsilon \leq 1$, the $\epsilon$-connectivity matrix of $G$, denoted by $C_G^\epsilon$, is defined as follows: $C_G^\epsilon(u,v) = 1$ if $C(u,v) \geq \epsilon$ and $C_G^\epsilon(u,v) = 0$ otherwise.*

**Algorithm 2.2**. Determination of $C_G$.

1. Construct $M_{\tilde{R}^\infty}$.
2. $C_G(i,j) = C_G(j,i) = M_{\tilde{R}^\infty}(i,j) \wedge M_{\tilde{R}^\infty}(j,i)$.

An algorithm for determining $C_G^\epsilon$ is similar to Algorithm 2.2.

**Definition 2.3** *Let $G$ be a fuzzy graph. $G$ is called* strongly $\epsilon$-connected *if every pair of vertices are mutually $\epsilon$-reachable. $G$ is said to be* initial $\epsilon$-connected *if there exists $v \in V$ such that every vertex $u$ in $G$ is $\epsilon$-reachable from $v$. A* maximal strongly $\epsilon$-connected fuzzy subgraph *($MS\epsilon CS$) of $G$ is a strongly $\epsilon$-connected fuzzy subgraph not properly contained in any other $MS\epsilon CS$.*

**TABLE 2.1** Fuzzy matrix and connectivity matrix of a fuzzy graph.

$$M_{\tilde{R}} = \begin{bmatrix} 1.0 & 0.6 & 0.4 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.2 & 0.6 & 0.3 \\ 0.0 & 0.8 & 1.0 & 0.0 & 0.9 \\ 0.2 & 0.7 & 0.3 & 1.0 & 0.2 \\ 0.4 & 0.0 & 0.5 & 0.3 & 1.0 \end{bmatrix} \qquad C_G^{0.5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Clearly strongly $\epsilon$-connectedness implies initial $\epsilon$-connectedness. Also, the following result is straightforward.

**Theorem 2.10** *A fuzzy graph $G$ is strongly $\epsilon$-connected if and only if there exists a vertex $u$ such that for any other vertex $v$ in $G, \tilde{R}^{\infty}(u,v) \geq \epsilon$ and $\tilde{R}^{\infty}(v,u) \geq \epsilon$.* ■

**Algorithm 2.3.** Determination of all $MS\epsilon CS$ in $G$.

1. Construct $C_G^{\epsilon}$.

2. The number of $MS\epsilon CS$ in $G$ is given by the number of distinct row vectors in $C_G^{\epsilon}$. For each row vector $\alpha$ in $C_G^{\epsilon}$, the vertices contained in the corresponding $MS\epsilon CS$ are the nonzero elements of the corresponding columns of $\alpha$.

**Example 2.2** *Let $G$ be a fuzzy graph whose corresponding fuzzy matrices $M_{\tilde{R}}$ and $C_G^{0.5}$ is given in Table 2.1. We see that the $MS0.5CS$'s of $G$ contain the following vertex sets, $\{1\}, \{2,4\}, \{3,5\}$, respectively.*

The previous result is now applied to clustering analysis. We assume that a data fuzzy graph $G = (V, \tilde{R})$ is given, where $V$ is a set of data and $\tilde{R}(u,v)$ is a quantitative measure of the similarity of the two data items $u$ and $v$. For $0 < \epsilon \leq 1$, an $\epsilon$-cluster in $V$ is a maximal subset $W$ of $V$ such that each pair of elements in $W$ is mutually $\epsilon$-reachable. Therefore, the construction of $\epsilon$-clusters of $V$ is equivalent to finding all maximal strongly $\epsilon$-connected fuzzy subgraphs of $G$.

**Algorithm 2.4.** Construction of $\epsilon$-clusters

1. Compute $\tilde{R}, \tilde{R}^2, \ldots, \tilde{R}^k$, where $k$ is the smallest integer such that $\tilde{R}^k = \tilde{R}^{k+1}$;

2. Let $\tilde{S} = \bigcup_{i=1}^{k} \tilde{R}^i$.

3. Construct $F_{\epsilon}^{\tilde{S}}$.

Then, each element in $F_{\epsilon}^{\tilde{S}}$ is an $\epsilon$-cluster.

We may also define an $\epsilon$-cluster in $V$ as a maximal subset $W$ of $V$ such that every element of $W$ is $\epsilon$-reachable from a special element $v$ in $W$. In this case, the construction of $\epsilon$-clusters is equivalent to finding all maximal initial $\epsilon$-connected fuzzy subgraphs of $G$. However, the relation induced by initial $\epsilon$-connected fuzzy subgraphs is not, in general, a similarity relation.

Another application is the use of fuzzy graphs to model information networks. Such a model was proposed in [34] utilizing the concepts of a directed graph. In [34] a measure of flexibility of a network was introduced. More specifically, let $N$ be a network with $m$ edges and $n$ vertices. Then the measure of flexibility on $N$, denoted by $Z(N)$, is defined as follows:

$$Z(N) = \frac{m-n}{n(n-2)}$$

The equation above is quite useful in classifying certain graph structures related to information networks. However, it is insensitive to certain classes of graphs. It seems that the use of fuzzy graphs is a more desirable model for information networks. The weights in each edge could be used as parameters such as number of channels between stations, costs for sending messages, etc. Thus, we propose here the use of a fuzzy graph to model an information network. Let $N$ have $n$ vertices. Define two measures of $N$ : flexibility and balancedness, denoted by $Z(N)$ and $B(N)$ respectively, as follows:

$$Z(N) = \frac{\sum_{i=1}^{n}\sum_{j\neq i}^{n} \tilde{R}(v_i, v_j)}{n(n-1)}, \quad B(N) = \frac{\sum_{i=1}^{n}\left|\sum_{j=1}^{n} \tilde{R}(v_i, v_j) - \sum_{k=1}^{n} \tilde{R}(v_k, v_i)\right|}{n(n-1)}$$

These two measures are much more sensitive to the structure of graphs as the one given in [34].

## 2.4 Connectivity in Fuzzy Graphs

In this section, connectivity of fuzzy graphs will be investigated. In this section and in fact for the remainder of the chapter, we assume all graphs are undirected. Let $G = (V, \tilde{R})$ be a fuzzy graph. Define the *degree* of a vertex $v$ to be $d(v) = \sum_{u\neq v} \tilde{R}(v, u)$. The *minimum degree* of $G$ is $\delta(G) = \wedge\{d(v) \mid v \in V\}$, and the *maximum degree* of $G$ is $\Delta(G) = \vee\{d(v) \mid v \in V\}$.

**Definition 2.4** *Let $G_i = (V_i, \tilde{R}_i), i = 1, 2$ be two fuzzy subgraphs of $G = (V, \tilde{R})$. The union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the fuzzy graph $(V', \tilde{R}')$, where $V' = V_1 \cup V_2$ and*

$$\tilde{R}'(u, v) = \begin{cases} \tilde{R}(u, v) & \text{if } \{u, v\} \subseteq V_1 \cup V_2 \\ 0 & \text{if } \{u, v\} \not\subseteq V_1 \cup V_2 \end{cases}$$

**Lemma 2.11** *Let $G = (V, \tilde{R})$ be a fuzzy graph and $G_i = (V_i, \tilde{R}_i), i = 1, \ldots, n$, be fuzzy subgraphs of $G$ such that $V_i \cap V_j = \emptyset$ for $i \neq j, 1 \leq i, j \leq n$, and $\bigcup_{i=1}^{n} G_i$ is connected. Then*

(i) $\delta(\bigcup_{i=1}^{n} G_i) \geq \wedge\{\delta(G_i) \mid i = 1, \ldots, n\}$,

(ii) $\Delta(\bigcup_{i=1}^{n} G_i) \geq \vee\{\delta(G_i) \mid i = 1, \ldots, n\}$. ■

Recall that $G$ is said to be connected if for each pair of vertices $u$ and $v$ in $V$, there exists a $k > 0$ such that $\tilde{R}^k(u, v) > 0$.

**Definition 2.5** *Let $G = (V, \tilde{R})$ be a fuzzy graph. $G$ is called* $\tau$-degree connected, *for some $\tau \geq 0$, if $\delta(G) \geq \tau$ and $G$ is connected. A* $\tau$-degree component *of $G$ is a maximal $\tau$-degree connected fuzzy subgraph of $G$.*

**Theorem 2.12** *For any $\tau > 0$, the $\tau$-degree components of a fuzzy graph are disjoint.*

**Proof.** Let $G_1$ and $G_2$ be two $\tau$-degree components of $G$ such that their vertex sets have at least one common element. Since $\delta(G_1 \cup G_2) \geq \delta(G_1) \wedge \delta(G_2)$ by Lemma 2.11, $G_1 \cup G_2$ is $\tau$-degree connected. Since $G_1$ and $G_2$ are maximal with respect to $\tau$-degree connectedness, we have that $G_1 = G_2$. ■

**Algorithm 2.5.** Determination of $\tau$-degree components of a finite fuzzy graph $G$.

1. Calculate the row sums of $M_{\tilde{R}}$.
2. If there are rows whose sums are less than $\tau$, then obtain a new reduced matrix by eliminating those vertices, and go to 1.
3. If there is no such row, then stop.
4. Each disjoint fuzzy subgraph of the graph induced by the vertices in the last matrix as well as each eliminated vertex is a maximal $\tau$-degree connected fuzzy subgraph.

**Definition 2.6** *Let $G$ be a fuzzy graph, and $\{V_1, V_2\}$ be a partition of its vertex set $V$. The set of edges joining vertices of $V_1$ and vertices of $V_2$ is called a* cut-set *of $G$, denoted by $(V_1, V_2)$, relative to the partition $\{V_1, V_2\}$. The* weight *of the cut-set $(V_1, V_2)$ is defined to be*

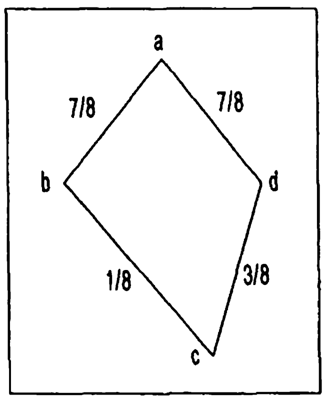
$$\sum_{u \in V_1, v \in V_2} \tilde{R}(u, v).$$

**Definition 2.7** *Let $G$ be a fuzzy graph. The* edge connectivity *of $G$, denoted by $\lambda(G)$, is defined to be the minimum weight of cut-sets of $G$. $G$ is called* $\tau$-edge connected *if $G$ is connected and $\lambda(G) \geq \tau$. A* $\tau$-edge component *of $G$ is a maximal $\tau$-edge connected subgraph of $G$.*

**Example 2.3** *Consider the fuzzy graph $G$ given below.*

**TABLE 2.2** Cut sets and their weights.

| $V_1$ | $V_2$ | *weight* |
|---|---|---|
| $\{a\}$ | $\{b,c,d\}$ | $\frac{7}{8}+\frac{7}{8}=\frac{7}{4}$ |
| $\{b\}$ | $\{a,c,d\}$ | $\frac{7}{8}+\frac{1}{8}=1$ |
| $\{c\}$ | $\{a,b,d\}$ | $\frac{1}{8}+\frac{3}{8}=\frac{1}{2}$ |
| $\{d\}$ | $\{a,b,c\}$ | $\frac{7}{8}+\frac{3}{8}=1\frac{1}{4}$ |
| $\{a,b\}$ | $\{c,d\}$ | $\frac{7}{8}+\frac{1}{8}=1$ |
| $\{a,c\}$ | $\{b,d\}$ | $\frac{7}{8}+\frac{7}{8}+\frac{1}{8}+\frac{3}{8}=2\frac{1}{4}$ |
| $\{a,d\}$ | $\{b,c\}$ | $\frac{7}{8}+\frac{3}{8}=1\frac{1}{4}$ |




*We summarize different cut-sets along with their weights in Table 2.2. We see that* $\lambda(G)=1/2$.

The following results can be proved similar to that of Lemma 2.11 and Theorem 2.12.

**Lemma 2.13** *Let* $G$ *be a fuzzy graph and* $G_i, i=1,\ldots,n$, *be fuzzy subgraphs of* $G$ *such that* $V_i \cap V_j = \emptyset$ *for all* $i,j, i\neq j, 1\leq i,j\leq n$ *and* $\bigcup_{i=1}^{n} G_i$ *is connected. Then* $\lambda(\bigcup_{i=1}^{n} G_i) \geq \bigwedge_{i=1}^{n} (\lambda(G_i))$. ■

**Theorem 2.14** *For* $\tau > 0$, *the* $\tau$*-edge components of a fuzzy graph are disjoint.*

The algorithm for determining $\tau$-edge components is based on a result of Matula [27]. In order to understand the algorithm we need to introduce the concept of a cohesive matrix and that of narrow slicing. ■

### *Cohesiveness*

Let $G = (V, \tilde{R})$ be a fuzzy graph. An element of $G$ is defined to be either a vertex or edge. That is, $e$ either a member of $V$ or $e$ is an unordered pair of members of $V$ such that $\tilde{R}(c) > 0$.

**Definition 2.8** *Let $e$ be an element of a fuzzy graph $G$. The* cohesiveness *of $e$, denoted by $h(e)$, is the maximum value of edge-connectivity of the subgraphs of $G$ containing $e$.*

**Lemma 2.15** *For any fuzzy graph $G$ and element $e$ and $0 < \tau \leq h(e)$, there exists a unique $\tau$-edge component of $G$ containing $e$.* ■

The unique $\tau$-edge component of $G$, for $\tau = h(e) > 0$, containing the element $e$ has the highest order of the maximum edge-connectivity subgraphs of $G$ containing $e$, and will be termed the *$h(e)$-edge component* of $e$, denoted by $H_e$.

**Example 2.4** *Consider the fuzzy graph $G$ given Example 2.3. We summarize the $\tau$-edge components of $G$ in the form a table. Recall that if $V_1$ is a subset of the set of vertices of $G$, $\langle V_1 \rangle$ denotes the fuzzy subgraph induced by $V_1$.*

| $\tau$ | $\tau -$ *edge components* |
|---|---|
| $(7/8, 1]$ | $\langle\{a\}\rangle, \langle\{b\}\rangle, \langle\{c\}\rangle, \langle\{d\}\rangle$ |
| $(1/2, 7/8]$ | $\langle\{c\}\rangle, \langle\{a, b, d\}\rangle$ |
| $[0, 1/2]$ | $\langle\{a, b, c, d\}\rangle$ |

The cohesiveness of an element may be determined from the knowledge of any subgraph of maximum edge-connectivity containing that given element, and clearly knowledge of the $\tau$-edge components of $G$ for all $\tau > 0$ is sufficient to determine $h(e)$ for all elements $e$ of $G$. The following theorem shows an important converse relation, that by utilizing the cohesiveness function it is possible to readily determine $H_e$ for any element $e$ with $h(e) > 0$.

**Theorem 2.16** *Let $e$ be an element of the fuzzy graph $G$ with $h(e) > 0$. Let $M_e$ be a maximal connected fuzzy subgraph of $G$ containing $e$ such that all elements of $M_e$ have cohesiveness at least $h(e)$. Then $M_e = H_e$.* ■

**Corollary 2.17** *For any fuzzy graph $G$ and any $\tau > 0$, the elements of $G$ of cohesiveness at least $\tau$ form a fuzzy graph whose components are $\tau-$edge components of $G$.* ■

**Corollary 2.18** *If $G'$ is an $\tau$-edge component of the fuzzy graph $G$ for some $\tau > 0$, then $G' = H_e$ for some element $e$ of $G$.* ■

### *Slicing in Fuzzy Graphs*

An ordered partition of the edges of the fuzzy graph $G, (C_1, C_2, \ldots, C_m)$, is a *slicing* of $G$ if each member

$$C_i \text{ is a cut-set } (A_i, \bar{A}_i) \text{ of } \begin{cases} G & \text{for } i = 1 \\ G \setminus \bigcup_{j=1}^{i-1} C_j & \text{for } 2 \le i \le m \end{cases}$$

A member of the slicing will also be termed a *cut of the slicing.* A slicing of $G$, which is minimal in the sense that there is no subpartition which is a slicing of $G$, is called a *minimal slicing* of $G$. Clearly each cut $C_i$ of a minimal slicing must be a minimal cut of some component of $G \setminus \bigcup_{j=1}^{i-1} C_j$. Further, a slicing of $G$ is a *narrow slicing* of $G$, if each cut $C_i$ is a minimum cut of some component of $G \setminus \bigcup_{j=1}^{i-1} C_j$. Note that the notion of slicing pertains only to graphs with at least one edge.

A slicing may be given a dynamic interpretation as a sequence of non-void cuts which separates $G$ into isolated vertices and a minimal (narrow) slicing effects this separation using only minimal (minimum) cuts at each step. This provides a simple way to compute the minimal (narrow) slicing. However, we want to make the observation that a narrow slicing is a minimal slicing but not vice versa.

**Algorithm 2.6.** Narrow slicing of connected fuzzy graph $G$.
1. $Z = \emptyset, G_1 = G, i = 1$.
2. While $G_i \neq \emptyset$ do
   $V =$ the vertex set of $G_i$.
   $v =$ a vertex in $G_i$ with minimum degree.
   $C_i = (\{v\}, V \setminus \{v\})$
   $Z = Z \cup \{C_i\}$
   $i = i + 1$
   $G_i =$ the fuzzy subgraph induced by $V \setminus \{v\}$.
3. $Z$ is a narrow slicing of $G$.

The following result is an important link between narrow slicing $Z$ and the cohesive function $h$ on a fuzzy graph.

**Theorem 2.19** *Let $Z = (C_1, C_2, \ldots, C_m)$ be a narrow slicing of $G$ obtained by successively removing one vertex at a time. Let $G_1 = G \supseteq G_2 \supseteq \ldots \supseteq G_m$ be the sequence of fuzzy subgraphs left after each slicing. Then $h(e) = \wedge\{\lambda(G_i) | e \in G_i, 1 \le i \le m\}$.* ■

**Example 2.5** *Let $G$ be a fuzzy graph such that*

$$M_{\tilde{R}} = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0.0 & 0.8 & 0.2 & 0.0 & 0.0 \\ b & 0.8 & 0.0 & 0.4 & 0.0 & 0.4 \\ c & 0.2 & 0.4 & 0.0 & 0.8 & 03 \\ d & 0.0 & 0.0 & 0.8 & 0.0 & 0.8 \\ e & 0.0 & 0.4 & 0.3 & 0.8 & 0.0 \end{array}$$

*As in the Algorithm 2.6, let $G_1$ denote the fuzzy graph $G$. Computing the sum along each row, we have*

| $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|
| 1.0 | 1.6 | 1.7 | 1.6 | 1.5 |

*The minimum value occurs at row $a$. So we set $C_1 = (\{a\}, \{b, c, d, e\})$ and let $G_2$ be the fuzzy subgraph induced by the vertex set $\{b, c, d, e\}$. Note that $\lambda(G_1) = 1.0$ and edges $(a, b)$ and $(a, c)$ appear only in $G_1$. It follows that $h(e) = 1.0$ for $e = (a, b), (a, c)$. Now the matrix associated with the fuzzy subgraph $G_2$ is given by*

$$\begin{array}{ccccc} & b & c & d & e \\ b & 0.0 & 0.4 & 0.0 & 0.4 \\ c & 0.4 & 0.0 & 0.8 & 03 \\ d & 0.0 & 0.8 & 0.0 & 0.8 \\ e & 0.4 & 0.3 & 0.8 & 0.0 \end{array}$$

*Computing the sum along each row, we have*

| $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|
| 0.8 | 1.5 | 1.6 | 1.5 |

*The minimum value occurs at row $b$. Hence $C_2 = (\{b\}, \{c, d, e\})$ and $G_3$ is the fuzzy subgraph generated by the vertex set $\{c, d, e\}$. Note that $\lambda(G_2) = 0.8$ and edges $(b, c)$ and $(b, e)$ appear in $G_1$ and $G_2$ and hence $h(e) = 1.0 \wedge 0.8 = 0.8$ for $e = (b, c), (b, e)$. Now the matrix associated with the fuzzy subgraph $G_3$ is given by*

$$\begin{array}{cccc} & c & d & e \\ c & 0.0 & 0.8 & 03 \\ d & 0.8 & 0.0 & 0.8 \\ e & 0.3 & 0.8 & 0.0 \end{array}$$

*Proceeding along these lines, we obtain the following cohesive matrix (where $i$th row $j$th column entry denote the cohesiveness of the edge $(i, j)$ if $i \neq j$ and the cohesiveness of the vertex $i$ if $i = j$, for $i, j \in \{a, b, c, d, e\}$)*

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $b$ | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 |
| $c$ | 1.0 | 1.0 | 0.0 | 1.1 | 1.1 |
| $d$ | 1.0 | 1.0 | 1.1 | 0.0 | 1.1 |
| $e$ | 1.0 | 1.0 | 1.1 | 1.1 | 0.0 |

*and the narrow slicing*
$((\{a\},\{b,c,d,e\}),\ (\{b\},\{c,d,e\}),\ (\{e\},\{c,d\}),\ (\{c\},\{d\}))$.

We are now ready to present an algorithm for the determination of $\tau$-edge components of a fuzzy graph $G$.

**Algorithm 2.7.** Determination of $\tau$-edge components of a fuzzy graph $G$.

1. Obtain the cohesive matrix $H$ of the $M_{\tilde{R}}$.
2. Obtain the $\tau$-threshold graph of $H$.
3. Each component of the graph is a maximal $\tau$-edge connected subgraph.

**Example 2.6** *Consider the fuzzy graph $G$ in the Example 2.5. The $\tau$-edge components of $G$ for various values $\tau$ can be summarized as follows.*

| $\tau$ | $\tau$-edge components |
|---|---|
| $(1.1,\infty)$ | $\langle\{a\}\rangle,\langle\{b\}\rangle,\langle\{c\}\rangle,\langle\{d\}\rangle,\langle\{e\}\rangle$ |
| $(1.0,1.1]$ | $\langle\{a\}\rangle,\langle\{b\}\rangle,\langle\{c,d,e\}\rangle$ |
| $[0,1.0]$ | $\langle\{a,b,c,d,e\}\rangle$ |

**Definition 2.9** *A* disconnection *of a fuzzy graph $G=(V,\tilde{A},\tilde{R})$ is a vertex set $D$ whose removal results in a disconnected or a single vertex graph. The* weight *of $D$ is defined to be* $\sum_{v\in D}\wedge\{\tilde{R}(v,u)|\tilde{R}(v,u)\neq 0, u\in V\}$.

**Definition 2.10** *The vertex connectivity of a fuzzy graph $G$, denoted by $\Omega(G)$, is defined to be the* minimum weight of disconnection *in $G$. $G$ is said to be $\tau$-vertex connected if $\Omega(G)\geq\tau$. A* $\tau$-vertex component *is a maximal $\tau$-vertex connected subgraph of $G$.*

Note that $\tau$-vertex components need not be disjoint as do $\tau$-degree and $\tau$-edge components. The following result is straightforward.

**Theorem 2.20** *Let $G$ be a fuzzy graph, then $\Omega(G)\leq\lambda(G)\leq\delta(G)$.* ∎

**Theorem 2.21** *For any three real numbers $a, b$, and $c$ such that $0<a\leq b\leq c$, there exists a fuzzy graph $G$ with $\Omega(G)=a,\lambda(G)=b$, and $\delta(G)=c$.*

**Proof.** Let $n$ be the smallest integer such that $c/n\leq 1$, and let $a'=a/n, b'=b/n$, and $c'=c/n$. Then $0<a'\leq b'\leq c'\leq 1$. Let $G$ be the fuzzy

graph constructed as follows. The vertex set is the union of three sets $A = \{u_0, u_1, u_2, \ldots, u_n\}, B = \{v_0, v_1, v_2, \ldots, v_n\}$, and $C = \{w_0, w_1, w_2, \ldots, w_n\}$ each containing $n+1$ vertices. Let $< X >$ denote the fuzzy subgraph induced by the set $X$, for $X = A, B, C$. In $C, d(w_0) = nc'$ and $d(w_i) = (n-1)+c'+b'$ for $1 \leq i \leq n$. In other words, $< C\backslash\{w_0\} >$ is 1.0-complete and $< C >$ is c′-complete. In $B, d(v_0) = n+1$ and $d(v_i) = n+(n-1)+a'+b'$ for $1 \leq i \leq n$. $< B >$ is 1.0-complete. In $A, d(u_0) = n+1$ and $d(u_i) = n+(n-1)+a'$ for $1 \leq i \leq n$. $< A >$ is 1.0-complete. Connections between subsets are as follows. Each $w_i$ is connected to $v_i$ with fuzzy value $b'$ for $1 \leq i \leq n$. And each $u_i (i \neq 0)$ is connected to $v_i$ with fuzzy value $a'$ and to $v_j$'s $(j \neq i, 0)$ with fuzzy value 1.0. Finally $u_0$ is connected to $v_0$ with fuzzy value 1. All other edges in the fuzzy graph have value 0. Now we will show that $G$ thus constructed satisfies the conditions imposed.

(1) From the process of the construction described above it is clear that $\delta(G) = d(w_0) = nc' = c$.

(2) The number of edges in any cut of the subgraphs $< A >, < B >$ or $< C >$ is greater than or equal to $n$ since $< A >, < B >$ and $< C >$ are $c'$-complete. Therefore the weight of a cut is greater than or equal to $nc'$, which means that the weight of any cut which contains a cut of $< A >, < B >$ or $< C >$ is greater than or equal to $nc'$. Only other cuts which do not contain a cut of $< A >, < B >$ or $< C >$ must contain the cut $(A, B \cup C)$ or $(A \cup B, C)$. The weight of the cut $(A, B \cup C)$ is $1+n(n-1)+na'$ and that of the cut $(A \cup B, C)$ is $nb'$. Now $nb' \leq nc'$ and $nb' \leq 1+n(n-1)+na'$. Hence $\lambda(G) = nb' = b$.

(3) Let us determine the minimum number of vertices in disconnection of $G$. Since $< A >, < B >$ and $< C >$ are at least c′-complete, they can be disconnected or become a single vertex by removing at least $n$ vertices. Only other possible ways to disconnect $G$ are disconnections between $A, B$, and $C$. Since $< (A\backslash\{u_0\}) \cup (B\backslash\{v_0\}) >$ *is* a $a'$-complete and $u_0$ and $v_0$ are connected to each other and to $< (A\backslash\{u_0\}) \cup (B\backslash\{v_0\}) >$, any disconnection must contain at least $n+1$ vertices. On the other hand, since $< B >$ and $< C >$ are connected by $n$ edges, at least $n$ vertices have to be removed to disconnect $< A \cup B >$ and $< C >$ . But since vertices on both sides of edges are all different, at least $n$ vertices have to be removed. Therefore, at least $n$ vertices have to be removed to disconnect the graph $G$. Then since $\wedge\{f(v) | v \in V\} = a'$ and actually $\{v_1, v_2, ..., v_n\}$ is a disconnection of $G$, the weight of the disconnection $\{v_1, v_2, ..., v_n\}$ specifies the vertex connectivity of the graph $G$, namely, $\Omega(G) = na' = a$. ■

## 2.5 Application to Cluster Analysis

The usual graph-theoretical approaches to cluster analysis involve first obtaining a threshold graph from a fuzzy graph and then applying various techniques to obtain clusters as maximal components under different connectivity considerations. These methods have a common weakness, namely, the weight of edges are not treated fairly in that any weight greater (less) than the threshold is treated as 1(0). In this section, we will extend these techniques to fuzzy graphs. It will be shown that the fuzzy graph approach is more powerful.

In Table 2.3, we provide a summary of various graph theoretical techniques for clustering analysis. This table is a modification of table II in Matula [26].

**TABLE 2.3** Cluster procedures.[1]

| *Cluster procedure* | *Graph theoretical interpretation of clusters* | *Cluster independence* | *Extent of chaining* |
|---|---|---|---|
| Single linkage | Maximal connected subgraphs | Disjoint | High |
| k-linkage | Maximal connected subgraphs of minimum degree | Disjoint | Moderate |
| k-edge connectivity | Maximal k-edge connected subgraph | Disjoint | Low |
| k-vertex connectivity | Maximal k-vertex connected subgraph and Cliques on k or less vertices | Limited overlap | Low |
| Complete linkage | Cliques | Considerable overlap | None |

In the following definition, clusters will be defined based on various connectivities of a fuzzy graph.

**Definition 2.11** *Let* $G = (V, \tilde{R})$ *be a fuzzy graph. A* cluster of type $k$ $(k = 1, 2, 3, 4)$ *is defined by the following conditions (i), (ii), (iii), and (iv) respectively.*

[1] Reprinted from : R.T. Yeh and S.Y. Bang, Fuzzy Sets and Their Applications, In: Zadeh, Fu, Tanaka, Shimura Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, 125 149, 1975.

(i) *maximal* $\epsilon$*-connected subgraphs, for some* $0 < \epsilon \leq 1$.

(ii) *maximal* $\tau$*-degree connected subgraphs.*

(iii) *maximal* $\tau$*-edge connected subgraphs.*

(iv) *maximal* $\tau$*-vertex connected subgraphs.*

Hierarchial cluster analysis is a method of generating a set of classifications of a finite set of objects based on some measure of similarity between a pair of objects. It follows from the previous definition that clusters of type (1), (2), and (3) are hierarchial with different $\epsilon$ and $\tau$, whereas clusters of type (4) are not due to the fact $\tau$-vertex components need not be disjoint.

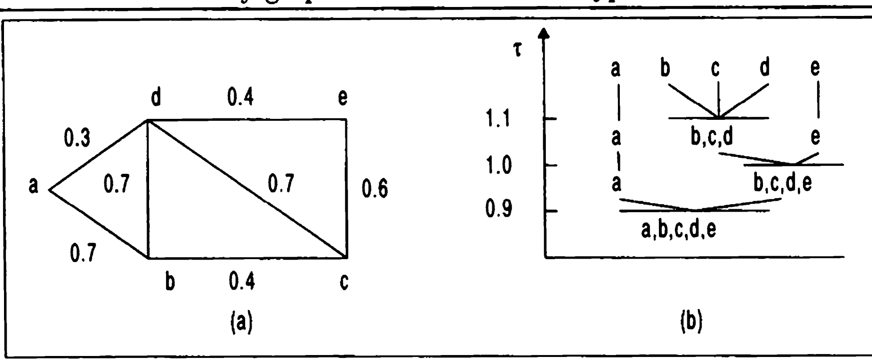
It is also easily seen that all clusters of type (1) can be obtained by the single-linkage procedure. The difference between the two procedures lies in the fact that $\epsilon$-connected subgraphs can be obtained directly from $M_{\tilde{R}^{\infty}}$ by at most $n-1$ matrix multiplications (where $n$ is the rank of $M_G$), whereas in the single-linkage procedure, it is necessary to obtain as many threshold graphs as the number of distinct fuzzy values in the graph.

Output of hierarchial clustering is called a *dendogram* which is a directed tree that describes the process of generating clusters.

In the following, we will show that not all clusters of types 2, 3 and 4 are obtainable by procedures of $k$-linkage, $k$-edge connectivity, and $k$-vertex connectivity, respectively.

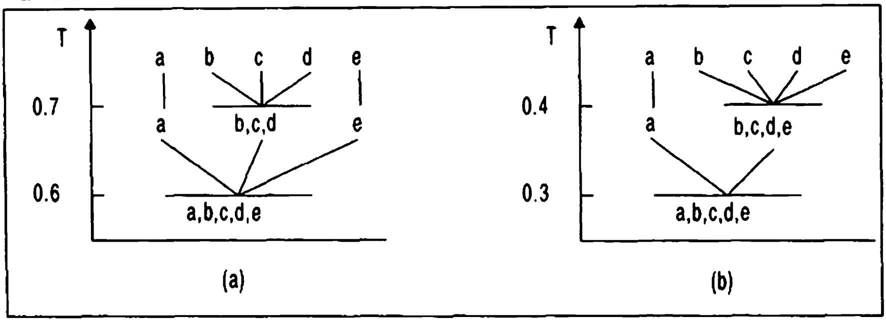
**Example 2.7** *Let $G$ be a fuzzy graph given in Figure 2.5(a). The dendrogram in Figure 2.5(b) indicates all the clusters of type 2.*

**FIGURE 2.5** A fuzzy graph and its clusters of Type 2[1] .




[1]Reprinted from R.T. Yeh and S.Y. Bang, Fuzzy Sets and Their Applications, In: Zadeh, Fu, Tanaka, Shimura Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, 125 149, 1975.

**FIGURE 2.6** Dendrograms for clusters obtained by $k$-linkage method for $k = 1$ and 2[1] .

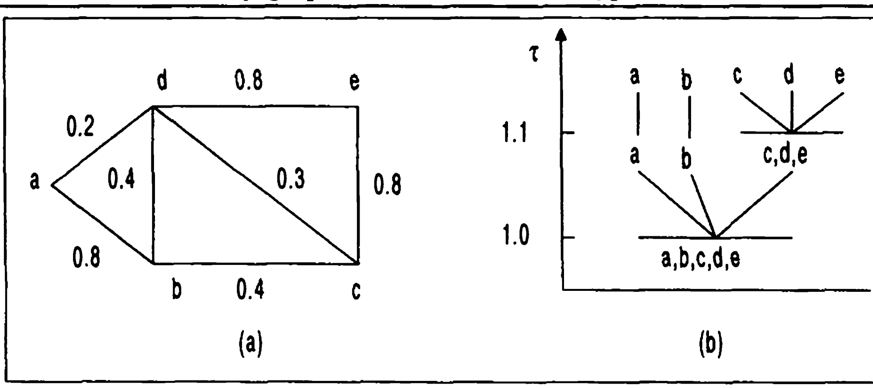



*It is easily seen from the threshold graphs of G that the same dendrogram cannot be obtained by the k-linkage procedure. Those for k = 1 and 2 are given in Figures 2.6(a) and 2.6(b), respectively.*

**Theorem 2.22** *The $\tau$-degree connectivity procedure for the construction of clusters is more powerful than the k-linkage procedure.*

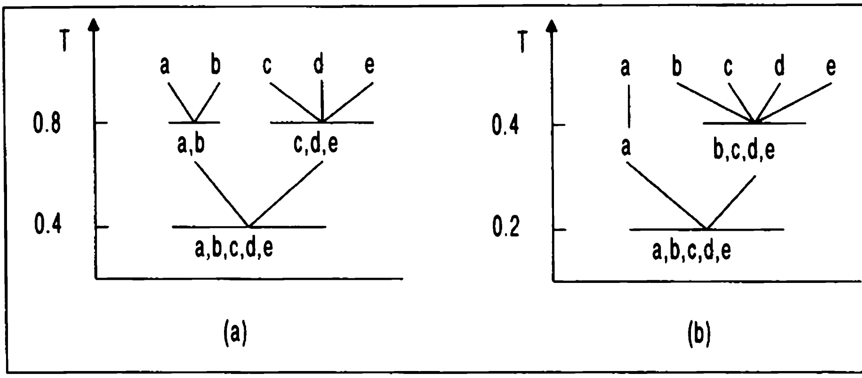
**Proof.** In light of Example 2.7, it is sufficient to show that all clusters obtainable by the $k$-linkage procedure are also obtainable by the $\tau$-degree connectivity procedure for some $\tau$. Let $G$ be a fuzzy graph. For $0 < \epsilon \leq 1$, let $G'$ be a graph obtained from $G$ by replacing those weights less than $\epsilon$ in $G$ by 0. For any $k$ used in the $k$-linkage procedure, set $\tau = k\epsilon$. It is easily seen that a set is a cluster obtained by applying the $k$-linkage procedure to $G$ if and only if it is a cluster obtained by applying the $\tau$-degree connectivity procedure to $G'$. ∎

**FIGURE 2.7** A fuzzy graph and its clusters of Type 3[1] .




**Example 2.8** *Let G be a fuzzy graph given in Figure 2.7(a). The dendrogram in Figure 2.7(b) gives all clusters of type 3.It is clear by examining*

**FIGURE 2.8** Dendrograms for clusters obtained from $k$-edge method for $k = 1$ and 2[1] .




*all the threshold graphs of G that the same dendrogram cannot be obtained by means of the k-edge connectivity technique for any k. Those for k = 1 and 2 are given in Figure 2.8.*

By Example 2.8 and following same proof procedure as in Theorem 2.22, we have the following result.

**Theorem 2.23** *The $\tau$-edge connectivity procedure for the construction of clusters is more powerful than the k-edge connectivity procedure.* ■

**Example 2.9** *Let G be a fuzzy graph given in Figure 2.9(a). The dendrogram in Figure 2.9(b) provides all clusters of type 4.*

**FIGURE 2.9** A symmetric graph and its clusters of Type 4[1] .

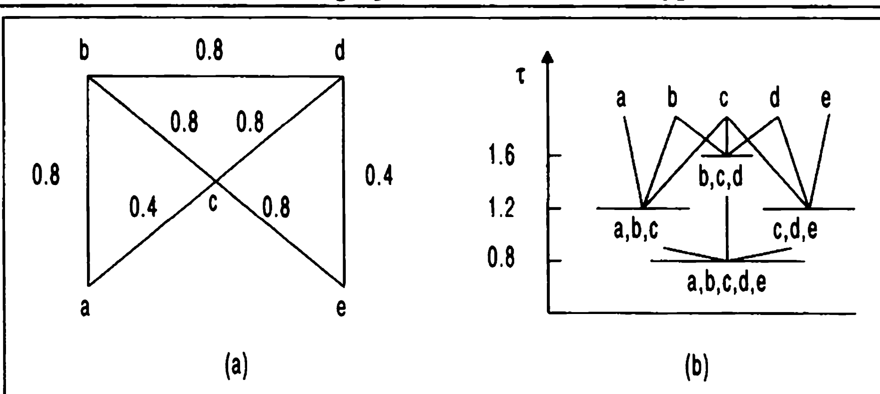
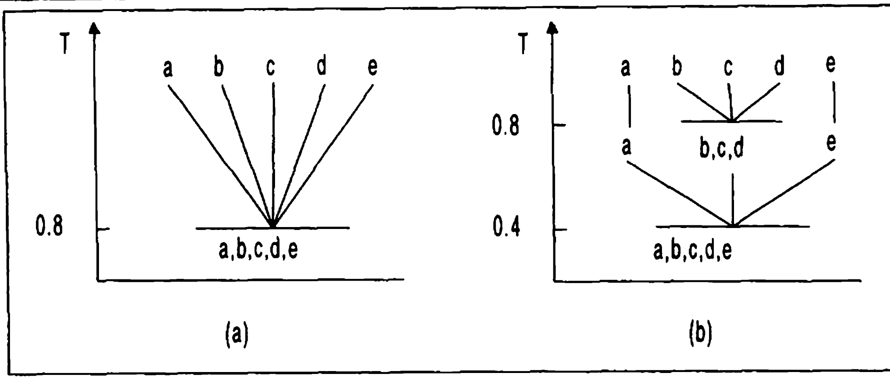


**FIGURE 2.10** Dendrograms for clusters obtained from $k$-vertex method for $k = 1$ and 2[1] .




*It is easily seen that the same dendrogram cannot be obtained by means of the $k$-vertex connectivity technique for any $k$. Those for $k = 1$ and 2 are given in Figure 2.10.*

Following the same proof procedure as in Theorems 2.22 and 2.23, we conclude with the result below.

**Theorem 2.24** *The $\tau -$vertex connectivity procedure for the construction of clusters is more powerful than the $k$-vertex connectivity procedure.*

## 2.6 Operations on Fuzzy Graphs

By a partial fuzzy subgraph of a graph $(V, X)$,we mean a partial fuzzy subgraph of the fuzzy graph $(\chi_V, \chi_X)$. If $G = (V, X)$ is a graph, a partial fuzzy subgraph of $G$ is an ordered pair $(\tilde{A}, \tilde{E})$ such that $\tilde{A}$ is a fuzzy subset of $V$ and $\tilde{E}$ is a fuzzy subset of $V \times V$. However, without any loss of generality, we could have defined $\tilde{E}$ as a fuzzy subset of $X$. Thus it is possible to interpret $(\tilde{A}, \tilde{E})$ as a partial fuzzy subgraph of $G$ and that is the interpretation we are going to follow for this section for the sake of clarity in presentation. Let $(\tilde{A}_i, \tilde{E}_i)$ be a partial fuzzy subgraph of the graph $G_i = (V_i, X_i)$ for $i = 1, 2$. We define the operations of Cartesian product, composition, union, and join on $(\tilde{A}_1, \tilde{E}_1)$ and $(\tilde{A}_2, \tilde{E}_2)$. Throughout this section we shall denote the edge between two vertices $u$ and $v$ by $uv$ rather than $(u, v)$. The motivation for this notational deviation is prompted by the fact that when we take the Cartesian product, the vertex of the graph

[1]Reprinted from : R.T. Yeh and S.Y. Bang, Fuzzy Sets and Their Applications, In: Zadeh, Fu, Tanaka, Shimura Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, 125 149, 1975

is, in fact, an ordered pair. If the graph $G$ is formed from $G_1$ and $G_2$ by one of these operations, we determine necessary and sufficient conditions for an arbitrary partial fuzzy subgraph of $G$ to also be formed by the same operation from partial fuzzy subgraphs of $G_1$ and $G_2$.

## *Cartesian Product and Composition*

Consider the *Cartesian product* $G = G_1 \times G_2 = (V, X)$ of graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$, [16]. Then

$V = V_1 \times V_2$

and

$X = \{(u, u_2)(u, v_2) | u \in V_1, u_2v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) | w \in V_2, u_1v_1 \in X_1\}$.

Let $\tilde{A}_i$ be a fuzzy subset of $V_i$ and $\tilde{E}_i$ a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\tilde{A}_1 \times \tilde{A}_2$ of $V$ and $\tilde{E}_1\tilde{E}_2$ of $X$ as follows:

$\forall (u_1, u_2) \in V, (\tilde{A}_1 \times \tilde{A}_2)(u_1, u_2) = \tilde{A}_1(u_1) \wedge \tilde{A}_2(u_2)$;

$\forall u \in V_1, \forall u_2v_2 \in X_2, \tilde{E}_1\tilde{E}_2((u, u_2)(u, v_2)) = \tilde{A}_1(u) \wedge \tilde{E}_2(u_2v_2)$,

$\forall w \in V_2, \forall u_1v_1 \in X_1, \tilde{E}_1\tilde{E}_2((u_1, w)(v_1, w)) = \tilde{A}_2(w) \wedge \tilde{E}_1(u_1v_1)$.

**Proposition 2.25** *Let $G$ be the Cartesian product of graphs $G_1$ and $G_2$. Let $(\tilde{A}_i, \tilde{E}_i)$ be a partial fuzzy subgraph of $G_i, i = 1, 2$. Then $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ is a partial fuzzy subgraph of $G$.*

**Proof.** $\tilde{E}_1\tilde{E}_2((u, u_2)(u, v_2)) = \tilde{A}_1(u) \wedge \tilde{E}_2(u_2v_2) \leq \tilde{A}_1(u) \wedge (\tilde{A}_2(u_2) \wedge \tilde{A}_2(v_2)) = (\tilde{A}_1(u) \wedge \tilde{A}_2(u_2)) \wedge (\tilde{A}_1(u) \wedge \tilde{A}_2(u_2)) = (\tilde{A}_1 \times \tilde{A}_2)(u, u_2) \wedge (\tilde{A}_1 \times \tilde{A}_2)(u, v_2)$. Similarly, $\tilde{E}_1\tilde{E}_2((u_1, w)(v_1, w)) \leq (\tilde{A}_1 \times \tilde{A}_2)(u_1, w) \wedge (\tilde{A}_1 \times \tilde{A}_2)(v_1, w)$. ■

The fuzzy graph $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ of Proposition 2.25 is called the *Cartesian product* of $(\tilde{A}_1, \tilde{E}_1)$ and $(\tilde{A}_2, \tilde{E}_2)$.

**Theorem 2.26** *Suppose that $G$ is the Cartesian product of two graphs $G_1$ and $G_2$. Let $(\tilde{A}, \tilde{E})$ be a partial fuzzy subgraph of $G$. Then $(\tilde{A}, \tilde{E})$ is a Cartesian product of a partial fuzzy subgraph of $G_1$ and a partial fuzzy subgraph of $G_2$ if and only if the following three equations have solutions for $x_i, y_j, z_{jk}$, and $w_{ih}$ where $V_1 = \{v_{11}, v_{12}, ..., v_{1n}\}$ and $V_2 = \{v_{21}, v_{22}, ..., v_{2m}\}$ :*

**(i)** $x_i \wedge y_j = \tilde{A}(v_{1i}, v_{2j}), i = 1, ..., n; j = 1, ..., m$;

**(ii)** $x_i \wedge z_{jk} = \tilde{E}((v_{1i}, v_{2j})(v_{1i}, v_{2k})), i = 1, ..., n; j, k$ *such that* $v_{2j}v_{2k} \in X_2$;

**(iii)** $y_j \wedge w_{ih} = \tilde{E}((v_{1i}, v_{2j})(v_{1h}, v_{2j})), j = 1, ..., m; i, h$ *such that* $v_{1i}v_{1h} \in X_1$.

**Proof.** Suppose that the solution exists. Consider an arbitrary, but fixed, $j, k$ in equations (ii) and $i, h$ in equations (iii). Let

$$\hat{z}_{jk} = \vee\{\tilde{E}((v_{1i}, v_{2j})(v_{1i}, v_{2k})) | i = 1, ..., n\}$$

and

$$\hat{w}_{ih} = \vee\{\tilde{E}((v_{1i}, v_{2j})(v_{1h}, v_{2j}))|j = 1, ..., m\}.$$

Set

$J = \{(j,k)|j,k$ are such that $v_{2j}v_{2k} \in X_2\}$

and

$I = \{(i,h)|i,h$ are such that $v_{1i}v_{1h} \in X_1\}$.

Now if $\{x_1, \ldots, x_n\} \cup \{z_{jk}|(j,k) \in J\} \cup \{y_1, \ldots, y_m\} \cup \{w_{ih}|(i,h) \in I\}$ is any solution to (i), (ii), and (iii), then $\{x_1, \ldots, x_n\} \cup \{\hat{z}_{jk}|(j,k) \in J\} \cup \{y_1, \ldots, y_m\} \cup \{\hat{w}_{ih}|(i,h) \in I\}$ is also a solution and, in fact, $\hat{z}_{jk}$ is the smallest possible $z_{jk}$ and $\hat{w}_{ih}$ is the smallest possible $w_{ih}$. Fix such a solution and define the fuzzy subsets $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ of $V_1, V_2, X_1$, and $X_2$, respectively, as follows:

$\tilde{A}_1(v_{1i}) = x_i$ for $i = 1, ..., n$;

$\tilde{A}_2(v_{2j}) = y_j$ for $j = 1, ..., m$;

$\tilde{E}_2(v_{2j}v_{2k}) = \hat{z}_{jk}$ for $j,k$ such that $v_{2j}v_{2k} \in X_2$;

$\tilde{E}_1(v_{1i}v_{1h}) = \hat{w}_{ih}$ for $i,h$ such that $v_{1i}v_{1h} \in X_1$.

For any fixed $j,k$, $\tilde{E}((v_{1i}, v_{2j})(v_{1i}, v_{2k})) \leq \tilde{A}(v_{1i}, v_{2j}) \wedge \tilde{A}(v_{1i}, v_{2k}) = (\tilde{A}_1(v_{1i}) \wedge \tilde{A}_2(v_{2j})) \wedge (\tilde{A}_1(v_{1i}) \wedge \tilde{A}_2(v_{2k})) \leq (\tilde{A}_2(v_{2j}) \wedge \tilde{A}_2(v_{2k})), i = 1, ..., n$.

Thus $\hat{z}_{jk} = \vee\{\tilde{E}((v_{1i}, v_{2j})(v_{1i}, v_{2k}))|i = 1, ..., n\} \leq \tilde{A}_2(v_{2j}) \wedge \tilde{A}_2(v_{2k})$. Hence $\tilde{E}_2(v_{2j}v_{2k}) \leq \tilde{A}_2(v_{2j}) \wedge \tilde{A}_2(v_{2k})$. Thus $(\tilde{A}_2, \tilde{E}_2)$ is a partial fuzzy subgraph of $G_2$. Similarly, $(\tilde{A}_1, \tilde{E}_1)$ is a partial fuzzy subgraph of $G_1$. Clearly, $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2$ and $\tilde{E} = \tilde{E}_1\tilde{E}_2$.

Conversely, suppose that $(\tilde{A}, \tilde{E})$ is the Cartesian product of partial fuzzy subgraphs of $G_1$ and $G_2$. Then solutions to equations (i),(ii) and (iii) exist by definition of Cartesian product. ■

**Remark 2** *Consider an arbitrary fixed solution to equations (i), (ii), and (iii) as determined in the proof of Theorem 2.26 (assuming one exists).*

**(i)** *Let $(j,k) \in J$ and let $I' = \{i_{jk} \in I|\hat{z}_{jk} = \tilde{E}((v_{1i_{jk}}, v_{2j})(v_{1i_{jk}}, v_{2k}))\}$ in Theorem 2.26. If $x_{i_{jk}} > \hat{z}_{jk}$ for some $i_{jk} \in I'$, then $z_{jk}$ is unique for these particular $x_1, \ldots, x_n$ and equals $\hat{z}_{jk}$; if $x_{i_{jk}} = \hat{z}_{jk} \forall i_{jk} \in I'$, then $\hat{z}_{jk} \leq z_{jk} \leq 1$ for these particular $x_1, \ldots, x_n$.*

**(ii)** *Let $(i,h) \in I$ and let $J' = \{j_{ih} \in J|\hat{w}_{ih} = \tilde{E}((v_{1i}, v_{2j_{ih}})(v_{1h}, v_{2j_{ih}}))\}$ in Theorem 2.26. If $y_{j_{ih}} > \hat{w}_{ih}$ for some $j_{ih} \in J'$, then $w_{ih}$ is unique for these particular $y_1, \ldots, y_m$ and equals $\hat{w}_{ih}$; if $y_{j_{ih}} = \hat{w}_{ih} \forall j_{ih} \in J'$, then $\hat{w}_{ih} \leq w_{ih} \leq 1$ for these particular $y_1, \ldots, y_m$.*

**Example 2.10** *Let $V_1 = \{v_{11}, v_{12}\}, V_2 = \{v_{21}, v_{22}\}, X_1 = \{v_{11}v_{12}\}$, and $X_2 = \{v_{21}v_{22}\}$. Let $\tilde{A}((v_{11}, v_{21})) = 1/4$, $\tilde{A}((v_{11}, v_{22})) = 1/2$, $\tilde{A}((v_{12}, v_{21})) = 1/8$, and $\tilde{A}((v_{12}, v_{22})) = 5/8$. Then $(\tilde{A}, \tilde{E})$ is not a Cartesian product of partial fuzzy subgraphs of $G_1$ and $G_2$ for any choice of $\tilde{E}$ since equations (i) of Theorem 2.26 are inconsistent:*

$$x_1 \wedge y_1 = 1/4, x_1 \wedge y_2 = 1/2, x_2 \wedge y_1 = 1/8, x_2 \wedge y_2 = 5/8$$

*is impossible.*

Examples are easily constructed where equations (i) have a solution, but either equations (ii) or (iii) are inconsistent.

We now consider the composition of two fuzzy graphs. Let $G_1[G_2]$ denote the composition of graph $G_1 = (V_1, X_1)$ with graph $G_2 = (V_2, X_2)$, [16]. Then $G_1[G_2] = (V_1 \times V_2, X^0)$ where $X^0 = \{(u, u_2)(u, v_2) \mid u \in V_1, u_2v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) \mid w \in V_2, u_1v_1 \in X_1\} \cup \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in X_1, u_2 \neq v_2\}$. Let $\tilde{A}_i$ be a fuzzy subset of $V_i$ and $\tilde{E}_i$ a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\tilde{A}_1 \circ \tilde{A}_2$ and $\tilde{E}_1 \circ \tilde{E}_2$ of $V_1 \times V_2$ and $X^0$, respectively, as follows:
$\forall (u_1, u_2) \in V_1 \times V_2$,

$$(\tilde{A}_1 \circ \tilde{A}_2)(u_1, u_2) = \tilde{A}_1(u_1) \wedge \tilde{A}_2(u_2);$$

$\forall u \in V_1, \forall u_2v_2 \in X_2$,

$$(\tilde{E}_1 \circ \tilde{E}_2)((u, u_2)(u, v_2)) = \tilde{A}_1(u) \wedge \tilde{E}_2(u_2v_2);$$

$\forall w \in V_2, \forall u_1v_1 \in X_1$,

$$(\tilde{E}_1 \circ \tilde{E}_2)((u_1, w)(v_1, w)) = \tilde{A}_2(w) \wedge \tilde{E}_1(u_1v_1);$$

$\forall (u_1, u_2)(v_1, v_2) \in X^0 \setminus X$,

$$(\tilde{E}_1 \circ \tilde{E}_2)((u_1, u_2)(v_1, v_2)) = \tilde{A}_2(u_2) \wedge \tilde{A}_2(v_2) \wedge \tilde{E}_1(u_1v_1).$$

where
$X = \{(u, u_2)(u, v_2) \mid u \in V_1, u_2v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) \mid w \in V_2, u_1v_1 \in X_1\}$.
We see that $\tilde{A}_1 \circ \tilde{A}_2 = \tilde{A}_1 \times \tilde{A}_2$ and that $\tilde{E}_1 \circ \tilde{E}_2 = \tilde{E}_1\tilde{E}_2$ on $X$.

**Proposition 2.27** *Let $G$ be the composition $G_1[G_2]$ of graph $G_1$ with graph $G_2$. Let $(\tilde{A}_i, \tilde{E}_i)$ be a partial fuzzy subgraph of $G_i, i = 1, 2$. Then $(\tilde{A}_1 \circ \tilde{A}_2, \tilde{E}_1 \circ \tilde{E}_2)$ is a partial fuzzy subgraph of $G_1[G_2]$.*

**Proof.** We have already seen in the proof of Proposition 2.25 that

$$(\tilde{E}_1 \circ \tilde{E}_2)((u_1, u_2)(v_1, v_2)) \leq (\tilde{A}_1 \circ \tilde{A}_2)((u_1, u_2)) \wedge (\tilde{A}_1 \circ \tilde{A}_2)((v_1, v_2))$$

for all $(u_1, u_2)(v_1, v_2) \in X$. Suppose that $(u_1, u_2)(v_1, v_2) \in X^0 \setminus X$ and so $u_1v_1 \in X_1, u_2 \neq v_2$. Then $(\tilde{E}_1 \circ \tilde{E}_2)((u_1, u_2)(v_1, v_2)) = \tilde{A}_2(u_2) \wedge \tilde{A}_2(v_2) \wedge \tilde{E}_1(u_1v_1) \leq \tilde{A}_2(u_2) \wedge \tilde{A}_2(v_2) \wedge \tilde{A}_1(u_1) \wedge \tilde{A}_1(v_1) = (\tilde{A}_1(u_1) \wedge \tilde{A}_2(u_2)) \wedge (\tilde{A}_1(v_1) \wedge \tilde{A}_2(v_2)) = (\tilde{A}_1 \circ \tilde{A}_2)((u_1, u_2)) \wedge (\tilde{A}_1 \circ \tilde{A}_2)((v_1, v_2))$. ■

The fuzzy graph $(\tilde{A}_1 \circ \tilde{A}_2, \tilde{E}_1 \circ \tilde{E}_2)$ of Proposition 2.27 is called the *composition* of $(\tilde{A}_1, \tilde{E}_1)$ with $(\tilde{A}_2, \tilde{E}_2)$.

**Theorem 2.28** *Suppose that $G$ is the composition $G_1[G_2]$ of two graphs $G_1$ and $G_2$. Let $(\tilde{A}, \tilde{E})$ be a partial fuzzy subgraph of $G$. Consider the following equations:*

**(i)** $x_i \wedge y_j = \tilde{A}(v_{1i}, v_{2j}), i = 1, ..., n; j = 1, ..., m;$

**(ii)** $x_i \wedge z_{jk} = \tilde{E}((v_{1i}, v_{2j})(v_{1i}, v_{2k})), i = 1, ..., n; j, k$ *such that* $v_{2j}v_{2k} \in X_2;$

**(iii)** $y_j \wedge w_{ih} = \tilde{E}((v_{1i}, v_{2j})(v_{1h}, v_{2j})), j = 1, ..., m; i, h$ *such that* $v_{1i}v_{1h} \in X_1;$

**(iv)** $y_j \wedge y_k \wedge w_{ih} = \tilde{E}((v_{1i}, v_{2j})(v_{1h}, v_{2k}))$, *where* $(v_{1i}, v_{2j})(v_{1h}, v_{2k}) \in X^0 \setminus X;$

*where $X$ is defined as above.*

*A necessary condition for $(\tilde{A}, \tilde{E})$ to be a composition of partial fuzzy subgraphs of $G_1$ and $G_2$ is that a solution to equations (i)–(iv) exists.*

*Suppose that a solution to equations (i)–(iv) exists. If*

$$\hat{w}_{ih} \geq \tilde{E}((v_{1i}, v_{2j})(v_{1h}, v_{2k})) \forall (i, h) \in I$$

*such that $(v_{1i}, v_{2j})(v_{1h}, v_{2j}) \in X^0 \setminus X$, then $(\tilde{A}, \tilde{E})$ is a composition of partial fuzzy subgraphs of $G_1$ and $G_2$.*

**Proof.** The necessary part of the theorem is clear. Suppose that a solution to equations (i)-(iv) exists. Then there exists a solution to equations (i)–(iv) as determined by in the proof of Theorem 2.26 for equations (i)–(iii) because every $w_{ih} \geq \hat{w}_{ih}$ and by the hypothesis concerning the $\hat{w}_{ih}$. Thus if $\tilde{A}_i, \tilde{E}_i, i = 1, 2,$are defined as in the proof of Theorem 2.26, we have that $(\tilde{A}_i, \tilde{E}_i)$ is a partial fuzzy subgraph of $G_i$, $i = 1, 2$, and $\tilde{A} = \tilde{A}_1 \circ \tilde{A}_2$ and $\tilde{E} = \tilde{E}_1 \circ \tilde{E}_2$. ∎

**Example 2.11** *Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be graphs and let $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ be fuzzy subsets of $V_1, V_2, X_1$, and, $X_2$, respectively. Then $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ is a partial fuzzy subgraph of $G_1 \times G_2$, but $(\tilde{A}_i, \tilde{E}_i)$ is not a partial fuzzy subgraph of $G_i, i = 1, 2$: Let $V_1 = \{u_1, v_1\}, V_2 = \{u_2, v_2\}, X_1 = \{u_1v_1\}$, and $V_2 = \{u_2v_2\}$. Define the fuzzy subsets $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ as follows: $\tilde{A}_1(u_1) = \tilde{A}_1(v_1) = \tilde{A}_2(u_2) = \tilde{A}_2(v_2) = 1/2$ and $\tilde{E}_1(u_1v_1) = \tilde{E}_2(u_2v_2) = 3/4$. Then $(\tilde{A}_i, \tilde{E}_i)$ is not a partial fuzzy subgraph of $G_i, i = 1, 2$. Now for $x \in V_1$ and $y \in V_2$, $\tilde{E}_1\tilde{E}_2((x, u_2)(x, v_2)) = \tilde{A}_1(x) \wedge \tilde{E}_2(u_2v_2) = 1/2 = \tilde{A}_1(x) \wedge \tilde{A}_2(u_2) \wedge \tilde{A}_2(v_2) = (\tilde{A}_1 \times \tilde{A}_2)((x, u_2)) \wedge (\tilde{A}_1 \times \tilde{A}_2)((x, v_2))$ and similarly, $\tilde{E}_1\tilde{E}_2((u_1, y)(v_1, y)) = (\tilde{A}_1 \times \tilde{A}_2)((u_1, y) \wedge (\tilde{A}_1 \times \tilde{A}_2)((v_1, y))$. Thus $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ is a partial fuzzy subgraph of $G_1 \times G_2$. Note that for $x_1y_1 \in X_1$ and $x_2, y_2 \in V_2$, $(\tilde{E}_1 \circ \tilde{E}_2)((x_1, x_2)(y_1, y_2)) = \tilde{A}_2(x_2) \wedge \tilde{A}_2(y_2) \wedge \tilde{E}_1(x_1y_1) = 1/2 = (\tilde{A}_1 \times \tilde{A}_2)((x_1, x_2)) \wedge (\tilde{A}_1 \times \tilde{A}_2)((y_1, y_2))$. Thus $(\tilde{A}_1 \circ \tilde{A}_2, \tilde{E}_1 \circ \tilde{E}_2)$ is a partial fuzzy subgraph of $G_1[G_2]$.*

We would also like to note that in Example 2.11, $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ satisfies the conditions in Theorem 2.26. Hence $(\tilde{A}_1 \times \tilde{A}_2, \tilde{E}_1\tilde{E}_2)$ is the Cartesian product of partial fuzzy subgraphs $(\tilde{B}_i, \tilde{F}_i)$ of $G_i, i = 1, 2$. In fact, these $\tilde{B}_i$ and $\tilde{F}_i$ $(i = 1, 2)$ are constant membership functions with membership value 1/2.

## *Union and Join*

Consider the *union* $G = G_1 \cup G_2$ of two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$, [16]. Then $G = (V_1 \cup V_2, X_1 \cup X_2)$. Let $\tilde{A}_i$ be a fuzzy subset of $V_i$ and $\tilde{E}_i$ a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\tilde{A}_1 \cup \tilde{A}_2$ of $V_1 \cup V_2$ and $\tilde{E}_1 \cup \tilde{E}_2$ of $X_1 \cup X_2$ as follows:

$(\tilde{A}_1 \cup \tilde{A}_2)(u) = \tilde{A}_1(u)$ if $u \in V_1 \setminus V_2$, $(\tilde{A}_1 \cup \tilde{A}_2)(u) = \tilde{A}_2(u)$ if $u \in V_2 \setminus V_1$, and $(\tilde{A}_1 \cup \tilde{A}_2)(u) = \tilde{A}_1(u) \vee \tilde{A}_2(u)$ if $u \in V_1 \cap V_2$;

$(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_1(uv)$ if $uv \in X_1 \setminus X_2$, $(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_2(uv)$ if $uv \in X_2 \setminus X_1$, and $(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_1(uv) \vee \tilde{E}_2(uv)$ if $uv \in X_1 \cap X_2$.

**Proposition 2.29** *Let $G$ be the union of the graphs $G_1$ and $G_2$. Let $(\tilde{A}_i, \tilde{E}_i)$ be a partial fuzzy subgraph of $G_i, i = 1, 2$. Then $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ is a partial fuzzy subgraph of $G$.*

**Proof.** Suppose that $uv \in X_1 - X_2$. We have three different cases to consider: (1) $u, v \in V_1 \setminus V_2$, (2) $u \in V_1 \setminus V_2, v \in V_1 \cap V_2$ and (3) $u, v \in V_1 \cap V_2$.

**(i)** Let $u, v \in V_1 \setminus V_2$. Then $(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_1(uv) \leq \tilde{A}_1(u) \wedge \tilde{A}_1(v) = (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)$.

**(ii)** Let $u \in V_1 \setminus V_2$ and $v \in V_1 \cap V_2$. Then $(\tilde{E}_1 \cup \tilde{E}_2)(uv) \leq (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1(v) \vee \tilde{A}_2(v)) = (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)$.

**(iii)** Let $u, v \in V_1 \cap V_2$. Then

$(\tilde{E}_1 \cup \tilde{E}_2)(uv) \leq (\tilde{A}_1(u) \vee \tilde{A}_2(u)) \wedge (\tilde{A}_1(v) \vee \tilde{A}_2(v)) = (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)$.

Similarly, if $uv \in X_2 \setminus X_1$, then $(\tilde{E}_1 \cup \tilde{E}_2)(uv) \leq (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)$. Suppose that $uv \in X_1 \cap X_2$. Then $(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_1(uv) \vee \tilde{E}_2(uv) \leq (\tilde{A}_1(u) \wedge \tilde{A}_1(v)) \vee (\tilde{A}_2(u) \wedge \tilde{A}_2(v)) \leq (\tilde{A}_1(u) \vee \tilde{A}_2(u)) \wedge (\tilde{A}_1(v) \vee \tilde{A}_2(v)) = (\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)$. ■

The fuzzy subgraph $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ of Proposition 2.29 is called the *union* of $(\tilde{A}_1, \tilde{E}_1)$ and $(\tilde{A}_2, \tilde{E}_2)$.

**Theorem 2.30** *If $G$ is a union of two fuzzy subgraphs $G_1$ and $G_2$, then every partial fuzzy subgraph $(\tilde{A}, \tilde{E})$ is a union of a partial fuzzy subgraph of $G_1$ and a partial fuzzy subgraph of $G_2$.*

**Proof.** Define the fuzzy subsets $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ of $V_1, V_2, X_1$, and $X_2$, respectively, as follows: $\tilde{A}_i(u) = \tilde{A}(u)$ if $u \in V_i$ and $\tilde{E}_i(uv) = \tilde{E}(uv)$ if $uv \in$

$X_i, i = 1, 2$. Then $\tilde{E}_i(u_iv_i) = \tilde{E}(u_iv_i) \leq \tilde{A}(u_i) \wedge \tilde{A}(v_i) = \tilde{A}_i(u_i) \wedge \tilde{A}_i(v_i)$ if $u_iv_i \in X_i, i = 1, 2$. Thus $(\tilde{A}_i, \tilde{E}_i)$ is a partial fuzzy subgraph of $G_i, i = 1, 2$. Clearly, $\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2$ and $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2$. ■

Consider the join $G = G_1 + G_2 = (V_1 \cup V_2, X_1 \cup X_2 \cup X')$ of graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ where $X'$ is the set of all edges joining the nodes of $V_1$ and $V_2$ and where we assume $V_1 \cap V_2 = \emptyset$, [16]. Let $\tilde{A}_i$ be a fuzzy subset of $V_i$ and $\tilde{E}_i$ a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\tilde{A}_1 + \tilde{A}_2$ of $V_1 \cup V_2$ and $\tilde{E}_1 + \tilde{E}_2$ of $X_1 \cup X_2 \cup X'$ as follows:

$(\tilde{A}_1 + \tilde{A}_2)(u) = (\tilde{A}_1 \cup \tilde{A}_2)(u) \forall u \in V_1 \cup V_2$;

$(\tilde{E}_1 + \tilde{E}_2)(uv) = (\tilde{E}_1 \cup \tilde{E}_2)(uv)$ if $uv \in X_1 \cup X_2$ and $(\tilde{E}_1 + \tilde{E}_2)(uv) = \tilde{A}_1(u) \wedge \tilde{A}_2(v)$ if $uv \in X'$.

**Proposition 2.31** *Let $G$ be the join of two graphs $G_1$ and $G_2$. Let $(\tilde{A}_i, \tilde{E}_i)$ be a partial fuzzy subgraph of $G_i, i = 1, 2$. Then $(\tilde{A}_1 + \tilde{A}_2, \tilde{E}_1 + \tilde{E}_2)$ is a partial fuzzy subgraph of $G$.*

**Proof.** Suppose that $uv \in X_1 \cup X_2$. Then the desired result follows from Proposition 2.29. Suppose that $uv \in X'$. Then $(\tilde{E}_1 + \tilde{E}_2)(uv) = \tilde{A}_1(u) \wedge \tilde{A}_2(v) = ((\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)) = ((\tilde{A}_1 + \tilde{A}_2)(u) \wedge (\tilde{A}_1 + \tilde{A}_2)(v))$. ■

The fuzzy subgraph $(\tilde{A}_1 + \tilde{A}_2, \tilde{E}_1 + \tilde{E}_2)$ of Proposition 2.31 is called the *join* of $(\tilde{A}_1, \tilde{E}_1)$ and $(\tilde{A}_2, \tilde{E}_2)$.

**Definition 2.12** *Let $(\tilde{A}, \tilde{E})$ be a partial fuzzy subgraph of a graph $G = (V, X)$. Then $(\tilde{A}, \tilde{E})$ is called a* strong partial fuzzy subgraph *of $G$ if $\tilde{E}(uv) = \tilde{A}(u) \wedge \tilde{A}(v)$ for all $uv \in X$.*

**Theorem 2.32** *If $G$ is the join of two subgraphs $G_1$ and $G_2$, then every strong partial fuzzy subgraph $(\tilde{A}, \tilde{E})$ of $G$ is a join of a strong partial fuzzy subgraph of $G_1$ and a strong partial fuzzy subgraph of $G_2$.*

**Proof.** Define the fuzzy subsets $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ of $V_1, V_2, X_1$, and $X_2$ as follows: $\tilde{A}_i(u) = \tilde{A}(u)$ if $u \in V_i$ and $\tilde{E}_i(uv) = \tilde{E}(uv)$ if $uv \in X_i, i = 1, 2$. Then $(\tilde{A}_i, \tilde{E}_i)$ is a fuzzy partial subgraph of $G_i, i = 1, 2$, and $\tilde{A} = \tilde{A}_1 + \tilde{A}_2$ as in the proof of Theorem 2.30. If $uv \in X_1 \cup X_2$, then $\tilde{E}(uv) = (\tilde{E}_1 + \tilde{E}_2)(uv)$ as in the proof of Theorem 2.30. Suppose that $uv \in X'$ where $u \in V_1$ and $v \in V_2$. Then $(\tilde{E}_1 + \tilde{E}_2)(uv) = \tilde{A}_1(u) \wedge \tilde{A}_2(v) = \tilde{A}(u) \wedge \tilde{A}(v) = \tilde{E}(uv)$ where the latter equality hold since $(\tilde{A}, \tilde{E})$ is strong. ■

**Example 2.12** *Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be graphs and let $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1$, and $\tilde{E}_2$ be fuzzy subsets of $V_1, V_2, X_1$, and $X_2$, respectively. Then $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$, but $(\tilde{A}_i, \tilde{E}_i)$ is not a partial fuzzy subgraph of $G_i, i = 1, 2$: Let $V_1 = V_2 = \{u, v\}$ and $X_1 = X_2 = \{uv\}$. Define the fuzzy subsets $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1, \tilde{E}_2$ of $V_1, V_2, X_1, X_2$, respectively, as follows: $\tilde{A}_1(u) = 1 = \tilde{A}_2(v), \tilde{A}_1(v) = 1/4 = \tilde{A}_2(u), \tilde{E}_1(uv) = 1/2 = \tilde{E}_2(uv)$. Then $(\tilde{A}_i, \tilde{E}_i)$ is not a partial fuzzy subgraph of $G_i, i = 1, 2$. Now*

$(\tilde{E}_1 \cup \tilde{E}_2)(uv) = \tilde{E}_1(uv) \vee \tilde{E}_2(uv) = 1/2 < 1 = (\tilde{A}_1(u) \vee \tilde{A}_2(u)) \wedge (\tilde{A}_1(v) \vee \tilde{A}_2(v)) = ((\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v))$. *Thus* $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ *is a partial fuzzy subgraph of* $G_1 \cup G_2$.

The above example can be extended to the case where $V_1 \not\subseteq V_2, V_2 \not\subseteq V_1$, $X_1 \not\subseteq X_2$, and $X_2 \not\subseteq X_1$ as follows: Let $V_1 = \{u, v, w\}, V_2 = \{u, v, z\}$, $X_1 = \{uv, uw\}$, $X_2 = \{uv, vz\}$, and $\tilde{A}_1(w) = \tilde{A}_2(z) = 1 = \tilde{E}_1(uw) = \tilde{E}_2(uz)$.

**Theorem 2.33** *Let* $G_1 = (V_1, X_1)$ *and* $G_2 = (V_2, X_2)$ *be graphs. Suppose that* $V_1 \cap V_2 = \emptyset$. *Let* $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1, \tilde{E}_2$ *be fuzzy subsets of* $V_1, V_2, X_1, X_2$, *respectively. Then* $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ *is a partial fuzzy subgraph of* $G_1 \cup G_2$ *if and only if* $(\tilde{A}_1, \tilde{E}_1)$ *and* $(\tilde{A}_2, \tilde{E}_2)$ *are partial fuzzy subgraphs of* $G_1$ *and* $G_2$, *respectively.*

**Proof.** Suppose that $(\tilde{A}_1 \cup \tilde{A}_2, \tilde{E}_1 \cup \tilde{E}_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$. Let $uv \in X_1$. Then $uv \notin X_2$ and $u, v \in V_1 - V_2$. Hence $\tilde{E}_1(uv) = (\tilde{E}_1 \cup \tilde{E}_2)(uv) \leq ((\tilde{A}_1 \cup \tilde{A}_2)(u) \wedge (\tilde{A}_1 \cup \tilde{A}_2)(v)) = (\tilde{A}_1(u) \wedge \tilde{A}_1(v))$. Thus $(\tilde{A}_1, \tilde{E}_1)$ is a partial fuzzy subgraph of $G_1$. Similarly, $(\tilde{A}_2, \tilde{E}_2)$ is a partial fuzzy subgraph of $G_2$. The converse is Proposition 2.29. ■

The following result follows from the proof of Theorem 2.33 and Proposition 2.31.

**Theorem 2.34** *Let* $G_1 = (V_1, X_1)$ *and* $G_2 = (V_2, X_2)$ *be graphs. Suppose that* $V_1 \cap V_2 = \emptyset$. *Let* $\tilde{A}_1, \tilde{A}_2, \tilde{E}_1, \tilde{E}_2$ *be fuzzy subsets of* $V_1, V_2, X_1, X_2$, *respectively. Then* $(\tilde{A}_1 + \tilde{A}_2, \tilde{E}_1 + \tilde{E}_2)$ *is a partial fuzzy subgraph of* $G_1 + G_2$ *if and only if* $(\tilde{A}_1, \tilde{E}_1)$ *and* $(\tilde{A}_2, \tilde{E}_2)$ *are partial fuzzy subgraphs of* $G_1$ *and* $G_2$, *respectively.* ■

**Definition 2.13** *Let* $(\tilde{A}, \tilde{E})$ *be a partial fuzzy subgraph of* $(V, T)$. *Define the fuzzy subsets* $\tilde{A}'$ *of* $V$ *and* $\tilde{E}'$ *of* $T$ *as follows:* $\tilde{A}' = \tilde{A}$ *and* $\forall uv \in T, \tilde{E}'(uv) = 0$ *if* $\tilde{E}(u, v) > 0$ *and* $\tilde{E}'(uv) = \tilde{A}(u) \wedge \tilde{A}(v)$ *if* $\tilde{E}(u, v) = 0$.

Clearly, $(\tilde{A}', \tilde{E}')$ is a fuzzy graph. $(\tilde{A}', \tilde{E}')$ is called the *complement* of $(\tilde{A}, \tilde{E})$. We also use the notation $G'$ for the complement of $G$.

**Definition 2.14** $(\tilde{A}, \tilde{E})$ *is said to be* complete *if* $X = T$ *and* $\forall uv \in X, \tilde{E}(uv) = \tilde{A}(u) \wedge \tilde{A}(v)$.

We use the notation $C_m(\tilde{A}, \tilde{E})$ for a complete fuzzy graph where $|V| = m$.

**Definition 2.15** $(\tilde{A}, \tilde{E})$ *is called a* fuzzy bigraph *if and only if there exists partial fuzzy subgraphs* $(\tilde{A}_i, \tilde{E}_i), i = 1, 2,$ *of* $(\tilde{A}, \tilde{E})$ *such that* $(\tilde{A}, \tilde{E})$ *is the join* $(\tilde{A}_1, \tilde{E}_1) + (\tilde{A}_2, \tilde{E}_2)$ *where* $V_1 \cap V_2 = \emptyset$ *and* $X_1 \cap X_2 = \emptyset$. *A fuzzy bigraph is said to be* complete *if* $\tilde{E}(uv) > 0$ *for all* $uv \in X'$.

We use the notation $C_{m,n}(\tilde{A}, \tilde{E})$ for a complete bigraph such that $|V_1| = m$ and $|V_2| = n$.

**Proposition 2.35** $C_{m,n}(\tilde{A}, \tilde{E}) = C_m(\tilde{A}_1, \tilde{E}_1)' + C_n(\tilde{A}_2, \tilde{E}_2)'$. ■

## 2.7 Fuzzy Intersection Equations

We give necessary and sufficient conditions for the solution of a system of fuzzy intersection equations. We also give an algorithm for the solution of such a system. We apply the results to fuzzy graph theory and to fuzzy commutative algebra.

In [22], Liu considered systems of intersection equations of the form

$$\begin{aligned} e_{11}x_1 \wedge \ldots \wedge e_{1n}x_n &= b_1 \\ &\vdots \\ e_{m1}x_1 \wedge \ldots \wedge e_{mn}x_n &= b_m \end{aligned} \tag{2.7.1}$$

where $e_{ij} \in \{0,1\}$ and $b_i, x_j \in L$ where $L$ is a complete distributive lattice, $i = 1, ..., m; j = 1, ..., n$. In this section, we consider systems of equations of the form (2.7.1), where $L$ is the closed interval $[0,1]$. Although this case is more restrictive, our approach is entirely different than that in [22]. The specificity of $[0,1]$ yields different types of results than those in [22]. We show that system (2.7.1) is equivalent to several independent systems of the type where $b_1 = \ldots = b_m$. Also our proofs concerning the existence of solutions are constructive in nature. In fact, we give an algorithm for the solution of a system of intersection equations. We also give two applications. One application is in the area of fuzzy graph theory. The other application is in the area of fuzzy commutative algebra. This latter application appears in Chapter 6.

### *Existence of Solutions*

We write the system (2.7.1) in the matrix form $E\bar{x} = \bar{b}$, where $E = [e_{ij}]$,

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \overline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We assume throughout that $\forall j = 1, ..., n, \exists i$ such that $e_{ij} = 1$. We also assume that the equations of (2.7.1) have been ordered so that $b_{q_1+1} = ... = b_{q_2} < b_{q_2+1} = ... = b_{q_3} < ... < b_{q_t+1} = ... = b_{q_{t+1}}$ where $0 = q_1 < q_2 < ... < q_{t+1} = m$. Let $I_r = \{q_r + 1, ..., q_{r+1}\}$ for $r = 1, ..., t$. For each $j = 1, ..., n$, let $i_j^*$ denote the maximum $i$ such that $e_{ij} = 1$. Let $e_{hj}^* = 0 \forall h \in \cup_{s=1}^{r-1} I_s$ and $e_{hj}^* = e_{hj} \forall h \in \cup_{s=r}^{t} I_s$, where $i_j^* \in I_r$. Let

$$E_j^* = \begin{bmatrix} e_{1j}^* \\ e_{2j}^* \\ \vdots \\ e_{mj}^* \end{bmatrix}.$$

Let $E^* = (E_1^*, \ldots, E_n^*)$.

**Theorem 2.36** *$E\bar{x} = \bar{b}$ and $E^*\bar{x} = \bar{b}$ are equivalent systems.*

**Proof.** Let $i$ be any row of $E$ and $j$ any column. Suppose that $e_{ij} = 1$. Let $i \in I_r$. Suppose $\exists h \in I_s, s < r$, such that $e_{hj} = 1$. Let $E'$ be the matrix $[e'_{ij}]$ where $e'_{uv} = e_{uv}$ if $(u, v) \neq (h, j)$ and $e'_{uv} = 0$ if $(u, v) = (h, j)$. That is, $E'$ is obtained from $E$ by replacing the $hj$-th component of $E$ with 0. It suffices to show that $E'\bar{x} = \bar{b}$ and $E\bar{x} = \bar{b}$ are equivalent. Now the $h$-th equations of $E'\bar{x} = \bar{b}$ and $E\bar{x} = \bar{b}$ are

$$e_{h1}x_1 \wedge \ldots \wedge 0x_j \wedge \ldots \wedge e_{hn}x_n = b_{q_s+1} \tag{2.7.2}$$

and

$$e_{h1}x_1 \wedge \ldots \wedge 1x_j \wedge \ldots \wedge e_{hn}x_n = b_{q_s+1}, \tag{2.7.3}$$

respectively. The other equations of $E'\bar{x} = \bar{b}$ and $E\bar{x} = \bar{b}$ are identical to each other. Since $e_{ij} = 1$, we have that $x_j \geq b_{q_r+1} > b_{q_s+1}$. Thus equation (2.7.2) and the $i$-th equation are equivalent to equation (2.7.3) and the $i$-th equation. Hence the desired result follows. ■

**Example 2.13** *Consider the following systems of intersection equations:*

$$x_1 \wedge x_3 \wedge x_4 = 1/2$$

$$x_2 \wedge x_3 = 3/4$$

*and*

$$x_1 \wedge x_4 = 1/2$$

$$x_2 \wedge x_3 = 3/4$$

*and*

$$x_1 \wedge x_3 = 1/2$$

$$x_1 \wedge x_2 \wedge x_3 = 3/4$$

*and*

$$x_1 = 1/2$$

$$x_1 \wedge x_2 \wedge x_3 = 3/4.$$

*The first two systems are equivalent while the last two systems are equivalent. The last two systems have no solution. In both pairs of systems, $j = 3$ and $i_j^* = 2$.*

**Theorem 2.37** *Consider the system $E\bar{x} = \bar{b}$.*

(i) *The system has a unique solution if and only if $\forall r = 1, ..., t$, the system*

$$E^*_{q_r+1}\bar{x} = b_{q_r+1}, ..., E^*_{q_{r+1}}\bar{x} = b_{q_{r+1}}$$

*has a unique solution.*

(ii) *The system is inconsistent if and only if $\exists i \in \{1, ..., m\}$ such that $b_i > 0$ and $e^*_{i1} = ... = e^*_{in} = 0$.*

**Proof.** (i) Suppose that $i \in I_r$ and $h \in I_s$ where $r \neq s$. Then $e^*_{ij} = 1$ implies $e^*_{hj} = 0$. That is, the $t$ systems

$$E^*_{q_r+1}\bar{x} = b_{q_r+1}, ..., E^*_{q_{r+1}}\bar{x} = b_{q_{r+1}}, r = 1, ..., t,$$

pairwise involve distinct unknowns.

(ii) Since the $t$ systems in (i) pairwise involve distinct unknowns,

$$E^*\bar{x} = \bar{b}$$

is inconsistent if and only if one of the $t$ systems is inconsistent. The desired result now follows by applying the condition in (ii) individually to the $t$ systems. ■

For the matrix $E$, let $E_i$ denote the $i$-th row of $E, i = 1, ..., m$. We write $E_g \leq E_h$ if and only if $\forall k = 1, ..., n, e_{gk} = 1$ implies $e_{hk} = 1$. We write $E_g < E_h$ if and only if $E_g \leq E_h$ and $E_g \neq E_h$. The addition of two rows of $E$ is componentwise with $0 + 0 = 0, 0 + 1 = 1 + 0 = 1 + 1 = 1$.

**Corollary 2.38** *Consider the system $E\bar{x} = \bar{b}$. Then $E\bar{x} = \bar{b}$ is inconsistent if and only if $\exists i, h_1, ..., h_k \in \{1, ..., m\}$ such that $b_i > 0, i \in I_r$ and $h_u \in I_{s_u}$ with $r < s_u$ for $u = 1, ..., k$ and $E_i \leq E_{h_1} + ... + E_{h_k}$.*

**Proof.** There exists $i \in \{1, ..., m\}$ such that $e^*_{i1} = ... = e^*_{in} = 0$ if and only if $\exists i, h_1, ..., h_k \in \{1, ..., m\}$ such that $i \in I_r$ and $h_u \in I_{s_u}$ with $r < s_u$ for $u = 1, ..., k$ and $E_i \leq E_{h_1} + ... + E_{h_k}$. ■

We now examine the case where $b_1 = ... = b_m$. Let $i \in \{1, ..., m\}$. Suppose that $\exists E_{i_1}, ..., E_{i_{k_i}} \leq E_i$. If $E_{i_1} + ... + E_{i_{k_i}} < E_i$, then let $c_{ij} = 0$ if $e_{i_r j} = 1$ for some $r = 1, ..., k_i$ and $c_{ij} = e_{ij}$ otherwise, $j = 1, ..., n$. If no such $E_{i_r}$ exist let $c_{ij} = e_{ij}, j = 1, ..., n$. Let $C_i = (c_{i1}, ..., c_{in})$ and $C = (C_1, ..., C_m)^T$, i.e., $C$ is the transpose of the matrix $(C_1, ..., C_m)$. (If $E_{i_1} + ... + E_{i_{k_i}} = E_i$, then the $i$-th equation may be deleted.)

**Theorem 2.39** *Suppose that $b_1 = ... = b_m = b$ in system (2.7.1). Let $C$ denote the matrix defined above. Then the systems $E\bar{x} = \bar{b}$ and $C\bar{x}R\bar{b}$ are equivalent where $R$ indicates that the relation in the $i$-th equation is either "$=$" or "$\geq$" depending upon whether $C_i = E_i$ or $C_i \neq E_i$, respectively.*

**Proof.** Let $S(0)$ denote the system $E\bar{x} = \bar{b}$ and let $S(i)$ denote the system obtained from $S(0)$ by replacing its $i$-th equation by $C_i\bar{x}R_ib$ where $R_i$ denotes "=" or "≥". Let $T(i)$ be the system $C_1\bar{x}R_1b, ..., C_i\bar{x}R_ib, E_{i+1}\bar{x} = b, ..., E_m\bar{x} = b$. It is easily seen that $E_{i_1}\bar{x} = b, ..., E_{i_{k_i}}\bar{x} = b, E_i\bar{x} = b$ and $E_{i_1}\bar{x} = b, ..., E_{i_{k_i}}\bar{x} = b, C_i\bar{x}R_ib$ are equivalent. Thus $S(0)$ and $S(i)$ are equivalent $\forall i = 1, ..., m$. Now $T(1) = S(1)$ and so $S(0)$ and $T(1)$ are equivalent. Assume that $S(0)$ and $T(i)$ are equivalent (the induction hypothesis). We now show that $S(0)$ and $T(i+1)$ are equivalent. Hence the result holds by induction. As noted above, $S(0)$ and $S(i+1)$ are equivalent. Let the $(i+1)$-st equation of $S(0)$ (and thus of $T(i)$) be denoted by $y_1 \wedge ... \wedge y_h \wedge z_1 \wedge ... \wedge z_k = b$ where $y_1, ..., y_h, z_1, ..., z_k \in \{x_1, ..., x_n\}$ and where the $(i+1)$-st inequality of $S(i+1)$ is $z_1 \wedge ... \wedge z_k \geq b$. Now $\{y_1, ..., y_h\} \cap \{z_1, ..., z_k\} = \emptyset$. Also, $y_1 \wedge ... \wedge y_h \wedge z_1 \wedge ... \wedge z_k = b$ is equivalent to ($y_1 \wedge ... \wedge y_h = b$ and $z_1 \wedge ... \wedge z_k \geq b$) or ($y_1 \wedge ... \wedge y_h \geq b$ and $z_1 \wedge ... \wedge z_k = b$). Since $T(i)$ and $S(i+1)$ are each equivalent to $S(0)$, $T(i)$ and $S(i+1)$ are equivalent. Hence the system $T(i)$ minus the $(i+1)$-st equation and the system $S(i+1)$ minus the $(i+1)$-st inequality individually imply $y_1 \wedge ... \wedge y_h = b$. Thus we have the equivalence of $T(i+1)$ and $S(i+1)$ and thus the equivalence of $T(i+1)$ and $S(0)$. ■

System (2.7.1) with $b_1 = ... = b_m$ is consistent if and only if $\forall i, \exists j$ such that $e_{ij} = 1$.

**Example 2.14** *Consider the following system* $S(0)$ :

$$x_1 \wedge x_2 \wedge x_3 = b$$

$$x_1 \wedge x_2 = b$$

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 = b.$$

*Let* $i = 1$. *Then* $E_2 < E_1$. *Applying Theorem 2.39, we obtain* $S(1)$ :

$$x_3 \geq b$$

$$x_1 \wedge x_2 = b$$

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 = b.$$

*Let* $i = 2$ *in* $S(0)$. *Then* $S(2) = S(0)$. *Let* $i = 3$ *in* $S(0)$. *Then* $E_1 + E_2 < E_3$. *Applying Theorem 2.39, we obtain* $S(3)$ :

$$x_1 \wedge x_2 \wedge x_3 = b$$

$$x_1 \wedge x_2 = b$$

$$x_4 \geq b.$$

*Thus $T(3)$ is the system*

$$x_3 \geq b$$

$$x_1 \wedge x_2 = b$$

$$x_4 \geq b.$$

**Theorem 2.40** *Suppose that $b_1 = ... = b_m = b$ in system (2.7.1). Let $C$ be the matrix as defined above. Suppose that $c_{hk} = c_{ik} = 1, h \neq i$, for some $i, k$ where $C_h\bar{x} \geq b$ and $C_i\bar{x} = b$. Suppose that $C_h \nleq C_i$. Let $d_{hk} = 0$ and $d_{uv} = c_{uv}$ if $(u, v) \neq (h, k)$. Let $D = [d_{ij}]$. Then $C_h\bar{x} \geq b, C_i\bar{x} = b$ are equivalent to $D_h\bar{x} \geq b, D_i\bar{x} = b$.*

**Proof.** Both systems force $x_k > b$.

If $C_h \leq C_i$, then drop the $h$-th equation. In fact, if $C_h\bar{x} \geq b, C_{h_1}\bar{x} = ... = C_{h_k}\bar{x} = b$ and $C_h \leq C_{h_1} + ... + C_{h_k}$, then drop the $h$-th equation.

We also note that $x_1 \wedge x_2 \geq b$ is equivalent to $x_1 \geq b$ and $x_2 \geq b$. ■

**Example 2.15** *The following systems are equivalent:*
$x_2 \wedge x_3 \geq b$
$x_1 \wedge x_2 = b$
*and*
$x_3 \geq b$
$x_1 \wedge x_2 = b.$
*Here $h = 1$ and $i = k = 2$.*

**Example 2.16** *The following systems are equivalent:*
$x_2 \wedge x_3 \geq b$
$x_1 \wedge x_2 = b$
$x_1 \wedge x_3 = b$
*and*
$x_1 \wedge x_2 = b$
$x_1 \wedge x_3 = b.$
*In the first system, $C_1 \leq C_2 + C_3$.*

To solve a general system of intersection equations, we may use the following algorithm. We use the notation $E_i'$ to denote the complement of $E_i$. We let $\theta$ denote the zero vector. We also assume that $b_1 > 0$.

**Algorithm 2.8.**

1. Sort the $E_i$ so that the $b_i$'s are in nondecreasing order.

2.1. Let Temp and Total each be a row of $n$ zeros

2.2. Let $c = b_m$

2.3. For $i = m$ down to 1 do

if $b_i = c$ then

$Temp = Temp + E_i$ and $E_i = E_i'$ NOR $Total$

if $c > b_i$ then

$c = b_i, Total = Total + Temp, Temp = E_i$

and $E_i = E_i'$ NOR $Total$

2.4. If $\exists i, 1 \leq i \leq m, E_i = \theta$, then

INCONSISTENT and STOP

3. For each distinct $b_k$

3.1. Let $E_j, i_1 \leq j \leq i_k$, be all rows such that $b_j = b_k$ where $1 \leq i_1, i_k \leq m$

3.2. Let $O_j$ be the number of 1's in $E_j$

3.3 Sort $E_j$'s such that $O_j$'s are in nondecreasing order

3.4. Let $T$ be a row of $n$ zeros

3.5. For $x = i_1$ to $i_k$ do

3.5.1. For $y = x - 1$ down to $i_1$ do

if $E_y'$ NOR $E_x = \theta$ then

$T = T + E_y$

3.5.2. If $T = E_x$ then

erase $E_x, R_x, b_x$

3.5.3 Else if $T \neq \theta$ then

$C_x = E_x$ XOR $T$ and $R_x =$ '$\geq$'

3.5.4 Else

$C_x = E_x$

4. For each distinct $b_k$

4.1. While $\exists C_i$ and $C_j$ such that

(1)$b_i = b_j = b_k$

(2)$R_i =$ '$=$'

(3)$R_j =$ '$\geq$' and

(4)$C_j'$ NOR $C_i \neq \theta$ do

$C_j = C_j'$ $NOR$ $C_i$

4.2. Let $T_k = \sum C_i \forall C_i$ such that $b_i = b_k$ and $R_i =$ '$=$'

4.3. If $\exists C_i$ such that (1)$R_i =$ '$\geq$' and (2)$C_i'$ $NOR$ $T_k = \theta$ then

erase $C_i$ and $b_i$ from matrices $C$ and $\bar{b}$, respectively.

The time complexity of the algorithm is easily seen to be $O(m^2 n)$. If each row in $E$ and $C$ is denoted as a binary number, then the time complexity becomes $O(m^2)$.

A unique minimal solution can be immediately determined.

The results of this section can be applied to those of Section 2.6 as we now describe. Suppose that $G$ is the Cartesian product of two graphs $G_1$ and $G_2$. Let $(\tilde{A}, \tilde{E})$ be a partial fuzzy subgraph of $G$. Then $(\tilde{A}, \tilde{E})$ is a Cartesian product of a partial fuzzy subgraph of $G_1$ and a partial fuzzy subgraph of $G_2$ if and only if the system of intersection equations as described in Theorem 2.26 has a solution.

The composition of fuzzy graphs is also defined in Section 2.6.1. If $(\tilde{A}, \tilde{E})$ is a partial fuzzy subgraph of the composition $G_1[G_2]$ of graphs $G_1$ and $G_2$, then necessary and sufficient conditions are given in Section 2.6.1 for $(\tilde{A}, \tilde{E})$ to be the composition of partial fuzzy subgraphs of $G_1$ and $G_2$ in terms of the existence of a solution to a system of fuzzy intersection equations.

In Chapter 6, we give an application to fuzzy commutative algebra.

## 2.8 Fuzzy Graphs in Database Theory

We now give an application of fuzzy graphs to database theory as developed in [18]. We examine fuzzy relations which store uncertain relationships between data. In classical relational database theory, design principles are based on functional dependencies. In this section, we generalize this notion for fuzzy relations and fuzzy functional dependencies. Results presented are useful for designing fuzzy relational databases.

**Definition 2.16** *Let $U = \{A_1, ..., A_n\}$ be the set of attributes and each $A_i$ is assigned to the set of possible values $DOM(A_i)$. A fuzzy subset $\tilde{R}$ of the Cartesian cross product $\times_{i=1}^{n} DOM(A_i)$ is called a* fuzzy relation *on $\times_{i=1}^{n} DOM(A_i)$*

In classical database theory, functional dependencies play important roles. A functional dependency '$X$ functionally determines $Y$ in $R$' means for any two tuples of the relation $R$, if the $X$ values are the same, then the $Y$ values are also same. In other words, $\chi_{X \to Y}$ is equivalent to

$$\forall t_1, t_2((R(t_1)\ .and.\ R(t_2)\ .and.\ t_1[X] = t_2[X]) \implies t_1[Y] = t_2[Y]).$$

For example, consider the relation $R$ given below:

| $A$ | $B$ | $C$ |
|---|---|---|
| $a$ | $b$ | $c$ |
| $a$ | $b$ | $d$ |
| $e$ | $f$ | $g$ |
| $e$ | $b$ | $d$ |

Note that $A$ functionally determines $B$ since for any two rows (known as tuples in database theory) $t_1$ and $t_2$ of $R$, if their values in column (known as attribute in database theory) $A$ are the same then those tuples have identical values in column $B$. However, $A$ does not functionally determine $C$ since considering the first two rows observe that while the column $A$ values are identical, the column $C$ values are not identical. It may be noted that $C$ functionally determines $B$ and $B$ does not functionally determine $A$.

We get a fuzzy version of the formula when we substitute the operators .and., $\forall$ with the operators min ($\wedge$), inf ($\wedge$) and .or., $\exists$ with max ($\vee$), sup ($\vee$), and $\Longrightarrow$ with $\rightarrow$, where the implication $\rightarrow$ is defined as follows:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ 1-(a-b), & \text{otherwise} \end{cases}$$

and finally .not. with $\neg$, where $\neg a = 1 - a$. In this way, we get that the truth value of the fuzzy relation $\tilde{R}$ satisfies a given functional dependency $X \rightarrow Y$ :

$$T_{\tilde{R}}(X,Y) = 1 - \vee\{\tilde{R}(t_1) \wedge \tilde{R}(t_2) \mid t_1[X] = t_2[X] \text{ but } t_1[Y] \neq t_2[Y]\},$$

where $t_1$ and $t_2$ are any two tuples of $\tilde{R}$. As in the classical database theory, we denote the union of attributes $X$ and $Y$ by $XY$.

**Example 2.17** *Consider the fuzzy relation $\tilde{R}$ on $DOM(A) \times DOM(B) \times DOM(C)$.*

| $A$ | $B$ | $C$ | $\tilde{R}(t)$ |
|---|---|---|---|
| $a$ | $b$ | $c$ | 1 |
| $a$ | $b$ | $f$ | 0.8 |
| $e$ | $d$ | $c$ | 0.7 |
| $e$ | $b$ | $f$ | 0.6 |

*The fuzzy relation $\tilde{R}$ generates the following truth values.*

$$\begin{array}{lll} T_{\tilde{R}}(A,B) = 0.4, & T_{\tilde{R}}(B,C) = 0.2, & T_{\tilde{R}}(C,A) = 0.3, \\ T_{\tilde{R}}(A,C) = 0.2, & T_{\tilde{R}}(B,A) = 0.4, & T_{\tilde{R}}(C,B) = 0.3, \\ T_{\tilde{R}}(AC,B) = 1, & T_{\tilde{R}}(BC,A) = 0.4, & T_{\tilde{R}}(AB,C) = 0.2, \\ T_{\tilde{R}}(AB,B) = 1, & T_{\tilde{R}}(AB,A) = 1. & \end{array}$$

Fuzzy functional dependency satisfies the following properties.

**A1** If $Y \subseteq X$, then $T_{\tilde{R}}(X,Y) = 1$,

**A2** $T_{\tilde{R}}(X,Y) \wedge T_{\tilde{R}}(Y,Z) \leq T_{\tilde{R}}(X,Z)$,

**A3** $T_{\tilde{R}}(X,Y) \leq T_{\tilde{R}}(XZ,YZ)$.

From these, other properties can be obtained:

**B1** $T_{\tilde{R}}(X,Y) \wedge T_{\tilde{R}}(Y,Z) \leq T_{\tilde{R}}(X,YZ)$,

**B2** $T_{\tilde{R}}(X,Y) \wedge T_{\tilde{R}}(WY,Z) \leq T_{\tilde{R}}(XW,Z)$,

**B3** if $Z \subseteq Y$, then $T_{\tilde{R}}(X,Y) \leq T_{\tilde{R}}(X,Z)$.

An important consequence is that $T_{\tilde{R}}(X,Y) = \wedge\{T_{\tilde{R}}(X,A) | A : A \in Y\}$.

Thus a fuzzy relation generates another a fuzzy relation $T_{\tilde{R}}(X,Y)$ on $U^2$ with the properties A1 - A3.

Moreover, if there is given an arbitrary fuzzy relation $\tilde{T}(X,Y)$ on $U^2$, then it defines the fuzzy relation $\tilde{T}^+(X,Y)$ which is the smallest fuzzy relation on $U^2$ that contains $\tilde{T}(X,Y)$ and has the properties A1 - A3. We call $\tilde{T}^+(X,Y)$ the closure of $\tilde{T}(X,Y)$. (Recall that $\tilde{T}_1(X,Y) \subseteq \tilde{T}_2(X,Y)$ if and only if $\tilde{T}_1(X,Y) \leq \tilde{T}_2(X,Y) \forall X,Y \subseteq U$.)

The closure is well defined because the fuzzy relation $\tilde{S}(X,Y) \equiv 1$ satisfies A1 - A3 and contains every fuzzy relation on $U^2$, and if $\tilde{T} \subseteq \tilde{S}_1, \tilde{T} \subseteq \tilde{S}_2$, where $\tilde{S}_1, \tilde{S}_2$ satisfy A1 - A3, then $\tilde{T} \subseteq \tilde{S}_1 \cap \tilde{S}_2$ and $\tilde{S}_1 \cap \tilde{S}_2$ also satisfies A1 - A3. ($\tilde{S}_1 \cap \tilde{S}_2(X,Y) := \tilde{S}_1(X,Y) \wedge \tilde{S}_2(X,Y)$ for all $X,Y \subseteq U$.)

Proof of the following result can be found in [18].

**Proposition 2.41** *$\tilde{T}^+(X,Y)$ is a closure, that is*

**(i)** $\tilde{T}(X,Y) \subseteq \tilde{T}^+(X,Y)$,

**(ii)** $\tilde{T}^{++}(X,Y) = \tilde{T}^+(X,Y)$,

**(iii)** *if* $\tilde{T}_1(X,Y) \subseteq \tilde{T}_2(X,Y)$, *then* $\tilde{T}_1^+(X,Y) \subseteq \tilde{T}_2^+(X,Y)$. ■

Now we extend $\tilde{T}^+(X,A)$ for fuzzy subsets $\tilde{X}$ as follows: Let $\tilde{X}$ be a fuzzy subset on $U$ and

$$T_f^+(\tilde{X},A) = \vee\{(T^+(Z,A) \wedge \lambda) \mid Z \subseteq U, \lambda \in [0,1], Z_\lambda \subseteq \tilde{X}\}$$

where for $\lambda \in [0,1]$ we define

$$Z_\lambda(A) = \begin{cases} \lambda, & \text{if } A \in Z \\ 0, & \text{otherwise.} \end{cases}$$

With the help of $\tilde{T}_f^+(\tilde{X},A)$, we define a closure set on $U$ as follows: Let $\tilde{X}$ be a fuzzy subset on $U$. Then $\tilde{X}^+$ is also a fuzzy set on $U$ and defined by $\tilde{X}^+(A) = \tilde{T}_f^+(\tilde{X},A)$ for all $A \in U$.

First note that $\tilde{T}_f^+(\tilde{X},A) = \tilde{T}^+(X,A)$ if $X$ is a crisp set, that is $X(A) = 1$ or 0 for all $A \in U$. This is true because $\tilde{T}^+(X,A)$ is an increasing function in the argument $X$.

Proof of the following result can be found in [18].

**Proposition 2.42** *$\tilde{X}^+$ is a closure on $U$, that is*

**(i)** $\tilde{X} \subseteq \tilde{X}^+$,

**(ii)** *if* $\tilde{X} \subseteq \tilde{Y}$, *then* $\tilde{X}^+ \subseteq \tilde{Y}^+$,

**(iii)** $\tilde{X}^{++} = \tilde{X}^+$. ■

*Representation of Dependency Structure $\tilde{T}(X,Y)$ by Fuzzy Graphs*

Let $\tilde{T}(X,Y)$ be a fuzzy relation on $U^2$. We correspond to $\tilde{T}(X,Y)$ a fuzzy graph $G_T = (\tilde{V},\tilde{E})$ as follows. The vertices are ordered pairs $(X,Y)$ such that $\tilde{V}(X,Y) = \tilde{T}(X,Y)$. Edges are ordered pairs of vertices such that $\tilde{E}((X,Y),(X,Z)) = \tilde{T}(Y,Z)$.

The following algorithm gives $\tilde{T}^+(X,Y)$ by modifying step by step the labels of the graph:

**Algorithm 2.9.**

1. For all $Y \subseteq X$ let $\tilde{V}((X,Y)) = 1$.
2. while (STAT1 is true or STAT2 is true) do
   (where STAT 1 is true means
   there exists an edge $e = (v_1, v_2)$ so that
   $\tilde{V}(v_2) < \tilde{V}(v_1) \wedge \tilde{V}(e)$,
   and STAT2 is true means
   there are vertices $v_1 = (X,Y)$ and $v_2 = (XZ,YZ)$ so that
   $\tilde{V}(v_2) < \tilde{V}(v_1)$)
   if (STAT1 is true) then
   $\tilde{V}(v_2) = \tilde{V}(v_1) \wedge \tilde{E}(e)$;
   for all edges $d = ((X,Y),(X,Z))$ where $v_2 = (Y,Z)$,
   $\tilde{E}(d) = \tilde{V}(v_2)$;
   if (STAT2 is true) then
   $\tilde{V}(v_2) = \tilde{V}(v_1)$;
   for all edges $d = ((W,XZ),(W,YZ))$ where $v_2 = (XZ,YZ)$,
   $\tilde{E}(d) = \tilde{V}(v_2)$;

3) $\tilde{T}^+(X,Y) = \tilde{V}(v)$, where $v = (X,Y)$.

**Proposition 2.43** *The algorithm is correct.* ■

Since $\tilde{X}^+(A)$ is defined by $\tilde{T}^+(X,A)$ when $X$ is a crisp set on $U$, it can be computed by this algorithm as well.

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# 3

# FUZZY TOPOLOGICAL SPACES

## 3.1 Topological Spaces

Topology has its roots in geometry and analysis. From a geometric point of view, topology was the study of properties preserved by a certain group of transformations, namely the homeomorphisms. Certain notions of topology are also abstractions of classical concepts in the study of real or complex functions. These concepts include open sets, continuity, connectedness, compactness, and metric spaces. They were a basic part of analysis before being generalized in topology.

In this section, we give a brief presentation of a few basic ideas concerning topological spaces.

**Definition 3.1** *Let $X$ be a nonempty set and let $\mathcal{T} \subseteq \wp(X)$. Then $\mathcal{T}$ is called a* topology *on $X$ if the following conditions hold:*

**(i)** $\emptyset, X \in \mathcal{T}$.

**(ii)** *The union of any collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$.*

**(iii)** *The intersection of any two members of $\mathcal{T}$ is a member of $\mathcal{T}$.*

*The members of $\mathcal{T}$ are called $\mathcal{T}$-*open sets *or simply* open sets. *The pair $(X, \mathcal{T})$ is called a* topological space.

**Example 3.1** *Let $\mathcal{U}$ denote those subsets of $\mathbb{R}$ which are arbitrary unions of open intervals of $\mathbb{R}$. (We recall that an open interval in $\mathbb{R}$ is the set*

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$, *where* $a, b \in \mathbb{R}$ *with* $a < b$.) *Then* $(\mathbb{R}, \mathcal{U})$ *is a topological space.* $\mathcal{U}$ *is called the* usual topology *on* $\mathbb{R}$.

**Example 3.2** *Let* $\mathcal{U}$ *denote those subsets of* $\mathbb{R}^2$ *which are arbitrary unions of open spheres of* $\mathbb{R}^2$. *(Here we recall that an open sphere in* $\mathbb{R}^2$ *is the set* $\{(x, y) \mid (x - h)^2 + (y - k)^2 < r^2\}$,*where* $h, k, r \in \mathbb{R}$ *with* $r > 0$.) *Then* $(\mathbb{R}^2, \mathcal{U})$ *is a topological space.* $\mathcal{U}$ *is called the* usual topology *on* $\mathbb{R}^2$.

**Example 3.3** *Let* $X = \{x, y, z, u, v\}$ *and* $\mathcal{T} = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u, v\}\}$. *Then* $(X, \mathcal{T})$ *is a topological space since* $\mathcal{T}$ *satisfies the conditions of Definition 3.1. However if we let*

$\mathcal{T}' = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u\}\}$
and
$\mathcal{T}'' = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{x, y, u, v\}\}$,
then neither $(X, \mathcal{T}')$ nor $(X, \mathcal{T}'')$ are topological spaces: $\{x, z, u\} \cup \{y, z, u\} = \{x, y, z, u\} \notin \mathcal{T}'$ and $\{x, z, u\} \cap \{x, y, u, v\} = \{x, u\} \notin \mathcal{T}''$.

**Example 3.4** *Let* $X$ *be any set and* $\mathcal{T} = \wp(X)$. *Then* $(X, \mathcal{T})$ *is a topological space. That is, every subset of* $X$ *is open.* $\mathcal{T}$ *is called the* discrete topology *on* $X$.

**Example 3.5** *Let* $X$ *be any set and* $\mathcal{T} = \{X, \emptyset\}$. *Then* $(X, \mathcal{T})$ *is a topological space. That is,* $X$ *and* $\emptyset$ *are the only open sets.* $\mathcal{T}$ *is called the* indiscrete topology *on* $X$.

**Example 3.6** *Let* $X$ *be any set and* $\mathcal{T} = \{A \mid A \subseteq X, |A^c| < \infty\} \cup \{\emptyset\}$, *where* $A^c$ *denotes the set complement of* $A$ *in* $X$. *If* $A, B \in \mathcal{T}$, *then* $(A \cap B)^c = A^c \cup B^c \in \mathcal{T}$. *Also if* $\mathcal{S} \subseteq \mathcal{T}$, *then* $(\bigcup_{A \in \mathcal{S}} A)^c = \bigcap_{A \in \mathcal{S}} A^c \in \mathcal{T}$. *Thus* $(X, \mathcal{T})$ *is a topological space.*

**Definition 3.2** *Let* $(X, \mathcal{T})$ *be a topological space. A subset* $\mathcal{B}$ *of* $\mathcal{T}$ *is called a* base *for* $\mathcal{T}$ *if every element of* $\mathcal{T}$ *is a union of members of* $\mathcal{B}$. *If* $\mathcal{B}$ *is a base for* $\mathcal{T}$, *then* $\mathcal{B}$ *is said to* generate $\mathcal{T}$.

**Example 3.7** *The open intervals form a base for the usual topology on* $\mathbb{R}$. *This follows since if* $U$ *is an open subset of* $\mathbb{R}$, *then* $\forall x \in U$, $\exists a_x, b_x \in \mathbb{R}$ *such that* $x \in (a_x, b_x) \subseteq U$ *and so* $U = \bigcup_{x \in U}(a_x, b_x)$.

**Theorem 3.1** *Let* $(X, \mathcal{T})$ *be a topological space and let* $A$ *be a subset of* $X$. *Let* $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. *Then* $(A, \mathcal{T}_A)$ *is a topological space.* $\mathcal{T}_A$ *is called the* relative topology *on* $A$. ■

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$ and $U \in \mathcal{T}$ be such that $x \in U$. Then $U$ is called an *open neighborhood* of $x$. A point $x \in X$ is called a *limit point* or *derived point* of a subset $A$ of $X$ if for all open neighborhoods $U$ of $x$, $(U \setminus \{x\}) \cap A \neq \emptyset$. The set of limit points of $A$, denoted by $A'$, is called the *derived set* of $A$.

**Example 3.8** *Let $X = \{x, y, z, u, v\}$ and $\mathcal{T} = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u, v\}\}$. Then $(X, \mathcal{T})$ is a topological space as noted in Example 3.3. Let $A = \{x, y, z\}$. Then $y$ is a limit point of $A$ since the open sets containing $y$ are $\{y, z, u, v\}$ and $X$, and each contains a point of $A$ different from $y$, namely $z$. The point $x$ is not a limit point of $A$ since the open set $\{x\}$ does not contain a point of $A$ different from $x$. Similarly, $u, v$ are limit points of $A$. But $z$ is not since the open set $\{z, u\}$ does not contain $x$ or $y$. Thus the derived set $A'$ of $A$ is $\{y, u, v\}$.*

**Definition 3.3** *Let $(X, \mathcal{T})$ be a topological space. A subset $A$ of $X$ is said to be* closed *if $A^c$ is open.*

**Example 3.9** *Consider again the topological space $(X, \mathcal{T})$, where $X = \{x, y, z, u, v\}$ and $\mathcal{T} = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u, v\}\}$. Then the closed subsets of $X$ are the complements of the members of $\mathcal{T}$, namely, $\emptyset, X, \{y, z, u, v\}, \{x, y, v\}, \{y, v\}, \{x\}$. We note that the sets $\{x\}, \{y, z, u, v\}$, $X$, and $\emptyset$ are each open and closed, while the set $\{x, y\}$ is neither open nor closed.*

Let $(X, \mathcal{T})$ be a topological space. Since $A^{cc} = A$ for a subset $A$ of $X$, $A$ is open if and only $A^c$ is closed.

**Theorem 3.2** *Let $(X, \mathcal{T})$ be a topological space. Then the following properties hold:*

**(i)** *$X$ and $\emptyset$ are closed sets.*

**(ii)** *The intersection of any collection of closed is a closed set.*

**(iii)** *The union of any two closed sets is closed.* ■

**Theorem 3.3** *Let $(X, \mathcal{T})$ be a topological space. Let $A$ be a subset of $X$. Then $A$ is closed if and only if $A' \subseteq A$, that is, $A$ contains all its limit points.* ■

**Definition 3.4** *Let $(X, \mathcal{T})$ be a topological space. Let $A$ be a subset of $X$. Then the* closure *of $A$, denoted by $\bar{A}$ (or $clA$), is defined to be the intersection of all closed subsets $X$ which contain $A$.*

Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. By Theorem 3.2, $\bar{A}$ is closed and in fact, $\bar{A}$ is the smallest closed subset of $X$ which contains $A$. It follows easily that $A$ is closed if and only if $A = \bar{A}$. We also have that $\bar{A} = A \cup A'$. A point $x \in X$ is called a *closure point* of $A$ if either $x \in A$ or $x \in A'$. Let $B$ be a subset of $X$.Then $A$ is said to be *dense* in $B$ if $B \subseteq \bar{A}$.

**Example 3.10** *Consider once again the topological space $(X, \mathcal{T})$, where $X = \{x, y, z, u, v\}$ and $\mathcal{T} = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u, v\}\}$. The closed subsets of $X$ are $\emptyset, X, \{y, z, u, v\}, \{x, y, v\}, \{y, v\}$, and $\{x\}$. It follows that $cl\{y\} = \{y, v\}$ since $\{y, v\}$ is the smallest closed set containing $\{y\}$. Similarly, $cl\{x, z\} = X$ and $cl\{y, u\} = \{y, z, u, v\}$.*

Let $A$ be a subset of a topological space $X$. A point $x \in A$ is called an interior point of $A$ if there exists an open set $U$ such that $x \in U \subseteq A$. Let $A^\circ$ denote the set of all interior points of $A$. Then $A^\circ$ is called the *interior* of $A$.

Example 3.1 provides us with a simple example of the interior of a set. Let $A$ denote the interval $(a, b]$. Then clearly $A^\circ = (a, b)$.

**Proposition 3.4** *Let $A$ be a subset of a topological space $X$. Then the following assertions hold.*

**(i)** *$A^\circ$ is open.*

**(ii)** *$A^\circ$ is the largest open subset of $A$.*

**(iii)** *$A$ is open if and only if $A = A^\circ$.* ■

Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. Let $\mathcal{C} \subseteq \wp(X)$. Then $\mathcal{C}$ is said to be a *cover* of $A$ if $A \subseteq \bigcup_{C \in \mathcal{C}} C$. If $\mathcal{C}$ is a cover of $A$ and every $C \in \mathcal{C}$ is open, then $\mathcal{C}$ is called an *open cover* of $A$. If $\mathcal{C}$ is a (open) cover of $A$ and $\mathcal{C}' \subseteq \mathcal{C}$ is also a cover of $A$, then $\mathcal{C}'$ is called a *(open) subcover* of $A$ contained in $\mathcal{C}$.

**Definition 3.5** *Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. Then $A$ is said to be* compact *if every open cover of $A$ contains a finite subcover. If $X$ is compact, then $(X, \mathcal{T})$ is said to be* compact .

The definition of compactness is motivated by the Heine-Borel Theorem of analysis. The following example is essentially this theorem.

**Example 3.11** *Consider the topological space $(\mathbb{R}, \mathcal{U})$ of Example 3.1. Then every closed and bounded interval $[a, b]$ of $\mathbb{R}$ is compact.*

**Example 3.12** *Consider again the topological space $(\mathbb{R}, \mathcal{U})$ of Example 3.1. Let $a, b \in \mathbb{R}$, $a < b$. Then $[a, b) \subseteq \bigcup_{n=1}^{\infty} (a - 1, b - 1/n)$. Thus $\mathcal{U} = \{(a - 1, b - 1/n) \mid n = 1, 2, \ldots\}$ is an open cover of $[a, b)$. However $\mathcal{U}$ contains no finite subcover of $[a, b)$. Hence $[a, b)$ is not compact. Similarly, $(a, b]$ and $(a, b)$ are not compact.*

**Example 3.13** *Let $(X, \mathcal{T})$ be a compact topological space and let $F$ be a finite subset of $X$. Then $F$ is compact.*

**Example 3.14** *Let $X$ be any set and $\mathcal{T} = \{A \mid A \subseteq X, |A^c| < \infty\} \cup \{\emptyset\}$, where $A^c$ denotes the set complement of $A$ in $X$. Then $(X, \mathcal{T})$ is a topological space as noted in Example 3.6. Let $\mathcal{U}$ be an open covering of $X$. Let $U$ be any member of $\mathcal{U}$ which is not empty. Set $F = U^c$. Then $F$ is finite. Now $\forall x \in X$, $\exists U_x \in \mathcal{U}$ such that $x \in U_x$. Since $F$ is finite, $\{U\} \cup \{U_x \mid x \in F\}$ is a finite subcovering of $X$.*

**Theorem 3.5** *Let $(X, \mathcal{T})$ be a compact topological space and let $F$ be a closed subset of $X$. Then $F$ is compact.* ■

**Theorem 3.6** *Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. Then $A$ is compact with respect to $\mathcal{T}$ if and only if $A$ is compact with respect to $\mathcal{T}_A$.* ■

**Definition 3.6** *Let $(X, \mathcal{T})$ be a topological space and let $A$ and $B$ be a subsets of $X$. Then $A$ and $B$ are said to be* separated *if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.*

**Example 3.15** *Consider the topological space $(\mathbb{R}, \mathcal{U})$ of Example 3.1. Let $A = (0,1)$, $B = (1,2)$, and $C = [2,3)$. Then $\bar{A} = [0,1]$ and $\bar{B} = [1,2]$. Hence, $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$. Thus $A$ and $B$ are separated. Now $B$ and $C$ are not separated even though $B \cap \bar{C} = \emptyset$ since $2 \in \bar{B} \cap C$.*

**Definition 3.7** *Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. Then $A$ is said to be* disconnected *if there exist open subsets $U$ and $V$ of $X$ such that $A = (A \cap U) \cup (A \cap V)$, $\emptyset = (A \cap U) \cap (A \cap V)$ and $(A \cap U) \neq \emptyset \neq (A \cap V)$. In this case, $U \cup V$ is called a* disconnection *of $A$. If $A$ is not disconnected, then $A$ is said to be* connected*. If $X$ is connected, then $(X, \mathcal{T})$ is said to be* connected.

If $(X, \mathcal{T})$ be a topological space, it follows immediately that $\emptyset$ and $\{x\}$ are connected subsets of $X$ for all $x \in X$.

**Example 3.16** *Consider the topological space $(\mathbb{R}, \mathcal{U})$ of Example 3.1. Let $A = (1,2] \cup [4,5)$. Let $U = (1,3)$ and $V = (3,5)$. Then $U \cup V$ is a disconnection of $A$.*

**Example 3.17** *Consider the topological space $(\mathbb{R}^2, \mathcal{U})$ of Example 3.2. Let $A = \{(x,y) \mid y^2 - x^2 \geq 4\}$. Let $U = \{(x,y) \mid y < -1\}$ and $V = \{(x,y) | y > 1\}$. Then $U \cup V$ is a disconnection of $A$.*

**Theorem 3.7** *Let $(X, \mathcal{T})$ be a topological space and let $A$ be a subset of $X$. Then $A$ is connected if and only if it is not the union of two nonempty separated subsets of $X$.* ■

**Theorem 3.8** *Let $(X, \mathcal{T})$ be a topological space. Then the following conditions are equivalent:*

**(i)** *$X$ is connected.*

**(ii)** *$X$ is not the union of two nonempty disjoint open sets.*

**(iii)** *$X$ and $\emptyset$ are the only subsets of $X$ which are both open and closed.* ■

**Theorem 3.9** *Let $(X, T)$ be a topological space and let $A$ be a subset of $X$. Then $A$ is connected with respect to $T$ if and only if $A$ is connected with respect to $T_A$.* ■

**Example 3.18** *Consider the topological space $(X, T)$, where $X = \{x, y, z, u, v\}$ and $T = \{X, \emptyset, \{x\}, \{z, u\}, \{x, z, u\}, \{y, z, u, v\}\}$. Since $\{x\}$ and $\{y, z, u, v\}$ are complements of each other and $X = \{x\} \cup \{y, z, u, v\}$, $X$ is disconnected. Let $A = \{y, u, v\}$. Then the relative topology on $A$ is $\{A, \emptyset, \{u\}\}$. Thus $A$ is connected by Theorem 3.8 since $A$ and $\emptyset$ are the only subsets of $A$ which are both open and closed in the relative topology.*

**Definition 3.8** *Let $(X, T)$ be a topological space and let $x \in X$. A subset $Y$ of $X$ is called a* neighborhood *of $x$ if the exists an open subset $U$ of $X$ such that $x \in U \subseteq Y$. The set $\mathcal{N}_x$ of all $Y$ such that $Y$ is a neighborhood of $x$ is called the* neighborhood system *of $x$.*

**Example 3.19** *Consider the topological space $(\mathbb{R}, \mathcal{U})$ of Example 3.1. Let $x \in \mathbb{R}$. Then $\forall \delta > 0$, the closed interval $[x - \delta, x + \delta]$ is a neighborhood of $x$ since $[x - \delta, x + \delta]$ contains the open interval $(x - \delta, x + \delta)$ and $x \in (x - \delta, x + \delta)$.*

**Example 3.20** *Consider the topological space $(\mathbb{R}^2, \mathcal{U})$ of Example 3.2. Let $(h, k) \in \mathbb{R}^2$ and let $r > 0$. Then $\{(x, y) \mid (x - h)^2 + (y - k)^2 \leq r\}$ is a neighborhood of $(h, k)$ since it contains the open sphere $\{(x, y) \mid (x - h)^2 + (y - k)^2 < r^2\}$.*

**Theorem 3.10** *Let $(X, T)$ be a topological space and let $x \in X$. Then the following properties hold $\forall x \in X$.*

**(i)** *$\mathcal{N}_x \neq \emptyset$ and $x$ belongs to every member of $\mathcal{N}_x$.*

**(ii)** *The intersection of any two members of $\mathcal{N}_x$ is a member of $\mathcal{N}_x$.*

**(iii)** *If $Z$ is a subset of $X$ and $\exists Y \in \mathcal{N}_x$ such $Z \supseteq Y$, then $Z \in \mathcal{N}_x$.*

**(iv)** *$\forall Z \in N_x$, $\exists Y \in \mathcal{N}_x$ such that $Z \supseteq Y$, where $Y \in \mathcal{N}_y$ $\forall y \in Y$.* ■

A topological space $(X, T)$ is called *locally compact* if every point of $X$ has a compact neighborhood. $\mathbb{R}$ with the usual topology is locally compact, but not compact.

**Definition 3.9** *Let $(X, \mathcal{T})$ be a topological space. A sequence of points $\{x_n \mid n = 1, 2, \ldots\}$ is said to* converge *to a point $x$ of $X$ if $\forall$ open subsets $U$ of $X$ such that $x \in U$, $\exists$ positive integer $n_0$ such that $\forall n \geq n_0$, $x_n \in U$. If the sequence $\{x_n \mid n = 1, 2, \ldots\}$ converges to $x$, then $x$ is said to be a* limit *of the sequence and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

**Example 3.21** *Let $\mathcal{T}$ be the discrete topology on the set $X$. That is, every subset of $X$ is open. Let $\{x_n \mid n = 1, 2, \ldots\}$ be a sequence of points in $X$. Suppose that $\{x_n \mid n = 1, 2, \ldots\}$ converges to a point $x \in X$. Then since $\{x\}$ is open, $\exists$ positive integer $n_0$ such that $\forall n \geq n_0$, $x_n \in \{x\}$. That is, $\exists$ positive integer $n_0$ such that $\forall n \geq n_0$, $x_n = x$.*

**Example 3.22** *Let $\mathcal{T}$ be the indiscrete topology on the set $X$. That is, $X$ and $\emptyset$ are the only open sets. Let $\{x_n \mid n = 1, 2, \ldots\}$ be a sequence of points in $X$. Since $X$ is the only open set which contains any point of $X$ and since $X$ contains all points of $X$, $\{x_n \mid n = 1, 2, \ldots\}$ converges to every point of $X$.*

Of course, many examples concerning sequences can be found from calculus.

**Definition 3.10** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Let $f$ be a function of $X$ into $Y$. Then $f$ is said to be* continuous relative *to $\mathcal{T}$ and $\mathcal{S}$ or simply* continuous *if $\forall V \in \mathcal{S}$, $f^{-1}(V) \in \mathcal{T}$.*

The definition of continuity of a function here is consistent with the one found in calculus.

**Example 3.23** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces defined as follows:*

$X = \{x, y, z, w\}$, $\mathcal{T} = \{X, \emptyset, \{x\}, \{x, y\}, \{x, y, z\}\}$

*and*

$Y = \{s, t, u, v\}$, $\mathcal{S} = \{Y, \emptyset, \{s\}, \{t\}, \{s, t\}, \{t, u, v\}\}$.

*Define the function $f$ of $X$ into $Y$ by $f(x) = t, f(y) = u, f(z) = v$, and $f(w) = u$. Then $f$ is continuous since the inverse image under $f$ of every member of $\mathcal{S}$ is in $\mathcal{T}$.*

**Example 3.24** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces defined in Example 3.23. Define the function $f$ of $X$ into $Y$ by $f(x) = s, f(y) = s, f(z) = u$, and $f(w) = v$. Then $f$ is not continuous since $\{t, u, v\} \in \mathcal{S}$, but $f^{-1}(\{t, u, v\}) = \{z, w\} \notin \mathcal{T}$.*

**Definition 3.11** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Let $f$ be a one-to-one function of $X$ onto $Y$. If $f$ and $f^{-1}$ are continuous, then $f$ is called a* homeomorphism *and $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are said to be* homeomorphic.

**Example 3.25** *Let $X = (-1, 1)$. The function $f : X \to \mathbb{R}$ defined by $f(X) = \tan(1/2)\pi x$ is a homeomorphism of $X$ onto $\mathbb{R}$, where $\mathbb{R}$ has the usual topology and $X$ has the corresponding relative topology.*

**Example 3.26** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces with the discrete topology. Then since every subset of $X$ is open and every subset of $Y$ is open, all functions from $X$ into $Y$ are continuous as are all functions from $Y$ into $X$. Hence $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are homeomorphic if and only if $X$ and $Y$ have the same number of elements.*

## 3.2 Metric Spaces and Normed Linear Spaces

In this section we give some basic ideas concerning the notion of a metric space. The notion of a metric space is simply an arbitrary set together with a distance function. The distance function is an abstraction of the notion of Euclidean distance. A distance function on a set which satisfies the properties of the following definition allows us to introduce spheres, neighborhoods, and the nearness relation.

**Definition 3.12** *Let $X$ be a nonempty set and $d$ a function from $X \times X$ into $\mathbb{R}$. Then $d$ is called a* metric *or* distance function *on $X$ if the following conditions hold: $\forall x, y, z \in X$,*

**(i)** $d(x, y) \geq 0$ *and* $d(x, x) = 0$;

**(ii)** $d(x, y) = d(y, x)$;

**(iii)** $d(x, z) \leq d(x, y) + d(y, z)$;

**(iv)** *if* $x \neq y$, *then* $d(x, y) > 0$.

The real number $d(x, y)$ is called the *distance* between $x$ and $y$. Condition (ii) in Definition 3.12 is called the *symmetric property.* Condition (iii) of Definition 3.12 is called the *triangle property.* It says that in $\mathbb{R}^2$, the sum of the lengths of two sides of a triangle is greater then or equal to the length of the remaining side.

If $X$ is a nonempty set and $d$ is a function from $X \times X$ into $\mathbb{R}$ which satisfies (i), (ii), and (iii) of Definition 3.12, then $d$ is called a *pseudometric* on $X$.

We now give some examples of metrics.

**Example 3.27** *Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\forall x, y \in \mathbb{R}, d(x, y) = |x - y|$. Then $d$ is a metric on $\mathbb{R}$.*

**Example 3.28** *Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\forall x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ . Then $d$ is a metric on $\mathbb{R}^2$.*

**Example 3.29** *Let $X$ be a nonempty set. Define $d : X \times X \to \mathbb{R}$ by $\forall x, y \in X$,*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{if } x \neq y. \end{cases}$$

*Then $d$ is a metric on $X$. The function $d$ is often called the* trivial *metric on $X$.*

**Example 3.30** *Let $C[a, b]$ denote the set of all continuous real-valued functions with domain the closed interval $[a, b]$. Define $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by $\forall f, g \in C[a, b]$*
$d(f, g) = \int_a^b |f(x) - g(x)| dx$.
*Then $d$ is a metric on $C[a, b]$.*

**Example 3.31** *Again let $C[a, b]$ denote the set of all continuous real-valued functions with domain the closed interval $[a, b]$. Define $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by $\forall f, g \in C[a, b]$*
$d(f, g) = \vee\{|f(x) - g(x)| x \in [a, b]\}$.
*Then $d$ is a metric on $C[a, b]$.*

**Example 3.32** *Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\forall x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,*
$d(x, y) = \vee\{|x_1 - y_1|, |x_2 - y_2|\}$.
*Then $d$ is a metric on $\mathbb{R}^2$.*

**Example 3.33** *Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\forall x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,*
$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$.
*Then $d$ is a metric on $\mathbb{R}^2$.*

**Definition 3.13** *Let $X$ be a set and $d$ a metric on $X$. $\forall x \in X$ and $\forall$ real numbers $r > 0$, let $S_d(x, r) = \{y \in X \mid d(x, y) < r\}$. We call $S_d(x, r)$ the* open sphere *or simply* sphere *with radius $r$ and center $x$.*

We sometimes write $S(x, r)$ for $S_d(x, r)$ when $d$ is understood.

**Theorem 3.11** *Let $X$ be a set and $d$ a metric on $X$. Let $\mathcal{B} = \{S_d(x, r) \mid x \in X, r \in \mathbb{R}, r > 0\}$. Then $\mathcal{B}$ is a base for a topology on $X$.* ■

**Definition 3.14** *Let $X$ be a set and $d$ a metric on $X$. The topology generated by $\mathcal{B}$ is called the* metric topology. *The pair $(X, d)$ is called a metric space .*

**Example 3.34** *Let $d$ be the metric on $\mathbb{R}$ defined by $d(x, y) = |x - y| \forall x, y \in \mathbb{R}$. Then the open spheres in $\mathbb{R}$ are exactly the finite open intervals. Thus $d$ induces the usual topology on $\mathbb{R}$. Similarly, the metric on $\mathbb{R}^2$ given by the distance formula induces the usual topology on $\mathbb{R}^2$.*

**Example 3.35** *Let $X$ be a set and let $d$ be the trivial metric on $X$ defined in Example 3.29. Then $\forall x \in X$ and $\forall r \in \mathbb{R}, 0 < r < 1$, $S(x, r) = \{x\}$. Thus $\forall x \in X$, $\{x\}$ is open. Hence every subset of $X$ is open. Thus the trivial metric on $X$ induces the discrete topology on $X$.*

**Definition 3.15** *Let $(X, d)$ and $(Y, e)$ be metric spaces. Let $f$ be a one-to-one function of $X$ onto $Y$. Then $f$ is said to preserve distances, $f$ is called an* isometry, *and $(X, d)$ and $(Y, e)$ are said to be an* isometric *if $\forall x, y \in X$, $d(x, y) = e(f(x), f(y))$.*

**Theorem 3.12** *If $(X, d)$ and $(Y, e)$ are isometric metric spaces, then they are homeomorphic.* ■

**Example 3.36** *Let $X$ be a nonempty set and let $d$ be the trivial metric on $X$. Let $Y$ be a nonempty set. Define $e : Y \times Y \to \mathbb{R}$ by $\forall x, y \in Y$,*

$$e(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y \end{cases}$$

*Then $e$ is a metric on $Y$ and $e$ induces the discrete topology on $Y$. Since $d$ also induces the discrete topology on $X$, $X$ and $Y$ are homeomorphic if and only if they have the same number of elements. However, even if $X$ and $Y$ have the same number of elements, they are not isometric since the distance between points is different.*

**Definition 3.16** *Let $(X, d)$ be a metric space. A sequence $\{a_n \mid n = 1, 2, ...\}$ in $X$ is said to be a* Cauchy sequence *if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n, m > n_0$, $d(a_n, a_m) < \epsilon$.*

**Definition 3.17** *Let $(X, d)$ be a metric space. A sequence $\{a_n \mid n = 1, 2, ...\}$ in $X$ is said to* converge *in $X$ if $\exists x \in X$ such that $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$, $d(x, a_n) < \epsilon$.*

**Definition 3.18** *Let $(X, d)$ be a metric space. Then $(X, d)$ is said to be* complete *if every Cauchy sequence in $X$ converges to a point of $X$.*

**Example 3.37** *$\mathbb{R}$ with the usual metric is complete. However $\mathbb{Q}$ with the same metric is not complete. The sequence $\{\sum_{i=1}^{n} 1/(i-1)! \mid n = 1, 2, ...\}$ converges to $e$ in $\mathbb{R}$ which is real, but not rational.*

**Example 3.38** *Let $X$ be a set and $d$ the trivial metric on $X$. Then a sequence $\{a_n \mid n = 1, 2, ...\}$ in $X$ is Cauchy if and only if $\exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$, $\exists x \in X$ such that $a_n = x$. Hence in this case, $\{a_n \mid n = 1, 2, ...\}$ converges to $x$. Thus $(X, d)$ is complete.*

**Example 3.39** *Let $\mathbb{R}$ have the usual metric. Let $X$ denote the open interval $(0, 1)$. If $X$ has the usual metric, then $X$ is not complete since the sequence $\{1/2, 1/3, ..., 1/n, ...\}$ does not converge in $X$. However it is interesting to note that $\mathbb{R}$ and $X$ are homeomorphic.*

**Definition 3.19** *Let $(X, d)$ be a metric space and let $f$ be a function of $X$ into itself. Then $f$ is called* contractive *or a* contraction map *if $\exists s \in [0,1)$ such that $\forall x, y \in X$, $d(f(x)), f(y)) \leq sd(x, y)$.*

**Example 3.40** *Consider the Euclidean space $\mathbb{R}^2$. Let $s \in (0,1)$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = s(x,y) = (sx, sy)\ \forall (x,y) \in \mathbb{R}^2$. Then* $d(f(x,y), f(u,v)) = d(s(x,y), s(u,v)) = \sqrt{(sx - su)^2 + (sy - sy)^2}$ $= sd((x,y),(u,v))$. *Hence $f$ is a contraction map.*

The following result is known as the "fixed point" theorem. We prove it in Section 3.7 of this chapter.

**Theorem 3.13** *Let $(X, d)$ be a complete metric space and let $f$ be a function of $X$ into itself. If $f$ is a contraction map, then $\exists$ unique fixed point for $f$, that is, $\exists$ unique $x \in X$ such that $f(x) = x$.* ■

**Definition 3.20** *A metric space $(X^*, d^*)$ is called a* completion *of a metric space $(X, d)$ if $X^*$ is complete and $X$ is isometric to a dense subset of $X^*$.*

**Example 3.41** *The set $\mathbb{R}$ with the usual metric is a completion of $\mathbb{Q}$ with the usual metric since $\mathbb{R}$ is complete and $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.*

Let $V$ be a vector space over $\mathbb{R}$.

**Definition 3.21** *Let $A$ be a subset of $V$. Then $A$ is said to be* convex *if $\forall v, u \in V$ and $\forall\ \lambda \in [0,1]$, $\lambda v + (1-\lambda)u \in A$.*

**Theorem 3.14** *The intersection of any collection of convex subsets of $V$ is convex.* ■

**Definition 3.22** *Let $A$ be a subset of $V$. Let $co(A)$ denote the intersection of all convex subsets of $V$ which contain $A$. Then $co(A)$ is called the* convex hull *of $A$.*

If $A$ a subset of $V$, then $co(A)$ is the smallest convex subset of $V$ which contains $A$.

**Theorem 3.15** *Let $A$ be a nonempty subset of $V$. Then $co(A) = \{\lambda v + (1-\lambda)u | v, u \in V, \lambda \in [0,1]\}$.* ■

**Proposition 3.16** *Let $A$ be a subset of $V$. Then $A \subseteq$ co$(A) = co(co(A))$. Moreover, if $A$ is closed (compact), then $co(A)$ is closed (compact).* ■

**Definition 3.23** *Let $|| \; ||$ be a function of $V$ into $\mathbb{R}$. Then $|| \; ||$ is called a* norm *on $V$ if the following conditions hold:*

(i) $\forall v \in V$. $||v|| \geq 0$ *and* $||v|| = 0 \Leftrightarrow v = 0$.

**(ii)** $\forall u, v \in V$, $||u + v|| \leq ||u|| + ||v||$.

**(iii)** $\forall a \in \mathbb{R}$, $\forall v \in V$, $||av|| = |a|||v||$.

*If* $||\ ||$ *is a norm on* $V$, *then* $V$ *is called a* normed linear space *and* $||v||$ *is called the norm of* $v$, *where* $v \in V$.

**Theorem 3.17** *Let* $V$ *be a normed linear space. Define* $d : V \times V \to \mathbb{R}$ *by* $\forall (u, v) \in V \times V$, $d(u, v) = ||u - v||$. *Then* $d$ *is a metric on* $V$ *and is called the* induced metric. ■

**Example 3.42** *Consider the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Define* $||\ || : \mathbb{R}^n \to \mathbb{R}$ *by* $\forall (a_1, ..., a_n) \in \mathbb{R}^n$,
$||(a_1, ..., a_n)|| = \sqrt{a_1{}^2 + ... + a_n{}^2}$.
*Then* $||\ ||$ *is a norm on* $\mathbb{R}^n$. $||\ ||$ *is called the Euclidean norm on* $\mathbb{R}^n$.
*Let* $p \geq 1$. *Define* $||\ ||$ *on* $\mathbb{R}^n$ *by* $\forall (a_1, ..., a_n) \in \mathbb{R}^n$,
$||(a_1, ..., a_n)|| = (a_1{}^p + ... + a_n{}^p)^{1/p}$.
*Then* $||\ ||$ *is a norm on* $\mathbb{R}^n$.

**Example 3.43** *Consider the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Define* $||\ || : \mathbb{R}^n \to \mathbb{R}$ *by* $\forall (a_1, ..., a_n) \in \mathbb{R}^n$,
$||(a_1, ..., a_n)|| = |a_1| \vee ... \vee |a_n|$.
*Then* $||\ ||$ *is a norm on* $\mathbb{R}^n$.

**Example 3.44** *Consider the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Define* $||\ || : \mathbb{R}^n \to \mathbb{R}$ *by* $\forall (a_1, ..., a_n) \in \mathbb{R}^n$,
$||(a_1, ..., a_n)|| = |a_1| + ... + |a_n|$.
*Then* $||\ ||$ *is a norm on* $\mathbb{R}^n$.

**Example 3.45** *Consider the vector space* $C[a, b]$ *of all real-valued continuous functions on the closed interval* $[a, b]$. *Define* $||\ || : C[a, b] \to \mathbb{R}$ *by* $\forall f \in C[a, b]$,
$||f|| = \int_a^b |f(x)| dx$.
*Then* $||\ ||$ *is a norm on* $C[a, b]$.
*Define* $||\ || : C[a, b] \to \mathbb{R}$ *by* $\forall f \in C[a, b]$,
$||f|| = (\int_a^b |f^2(x)| dx)^{1/2}$.
*Then* $||\ ||$ *is a norm on* $C[a, b]$. *This latter normed linear space is usually denoted by* $C^2[a, b]$.

**Example 3.46** *Consider again the vector space* $C[a, b]$ *of all real-valued continuous functions on the closed interval* $[a, b]$. *Define* $||\ || : C[a, b] \to \mathbb{R}$ *by* $\forall f \in C[a, b]$,
$||f|| = \vee \{|f(x)| | x \in [a, b]\}$.
*Then* $||\ ||$ *is a norm on* $C[a, b]$.

**Example 3.47** *Let $\mathbb{R}^\infty$ denote the set of all sequences $< x_n > = \{x_n \mid n = 1, 2, \ldots\}$ of real numbers such that $\sum\limits_{n=1}^{\infty} |x_n|^2 < \infty$. Then $\mathbb{R}^\infty$ is a vector space over $\mathbb{R}$. Define $\|\ \| : \mathbb{R}^\infty \to \mathbb{R}$ by $\forall\{x_n | n = 1, 2, \ldots\} \in \mathbb{R}^\infty$,*

$$\| < x_n > \| = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}.$$

*Then $\|\ \|$ is a norm on $\mathbb{R}^2$.*

**Definition 3.24** *Let $V$ be a normed linear space. Let $d$ be the metric induced by $\|\ \|$. Then $V$ is called a* Banach space *if the metric space $(V, d)$ is complete.*

The spaces given in Examples 3.42, 3.46, and 3.47 are Banach spaces.

Let $f$ be a function of $\mathbb{R}$ into itself. Then $f$ is said to be *lower semicontinuous* at $y \in \mathbb{R}$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $f(y) \leq f(x) + \epsilon\ \forall x \in \mathbb{R}$ such that $|x - y| < \delta$. *Upper semicontinuity* is defined in a similar manner.

Let $(X, d)$ be a metric space and let $\mathcal{F}$ be a set of functions of $X$ into $\mathbb{R}$. Then $\mathcal{F}$ is said to be *uniformly bounded* if $\exists M \in \mathbb{R}$ such that $\forall f \in \mathcal{F}, \forall x \in X, |f(x)| \leq M$. Also $\mathcal{F}$ is said to be *equicontinuous* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x, x') < \delta$ implies $|f(x) - f(x')| < \epsilon\ \forall f \in \mathcal{F}$. Here $\delta$ depends only on $\epsilon$ and not on any particular point or function. It is clear that if $\mathcal{F}$ is equicontinuous, then $\forall f \in \mathcal{F}$, $f$ is uniformly continuous. Ascoli's Theorem says that if $\mathcal{F}$ is a closed subset of the function space of Example 3.46, then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is uniformly bounded and equicontinuous.

## 3.3 Fuzzy Topological Spaces

We shall confine our attention in this section to the more basic concepts such as open set, closed set, neighborhood, interior set, continuity and compactness, following closely the definitions, theorems and proofs given in [17], the original paper on fuzzy topological spaces.

Let $X$ be a set. Recall that if $A \subseteq X$, then $\chi_A$ denotes the characteristic function of $A$ in $X$.

**Definition 3.25** *A* fuzzy topology *on $X$ is a family $\mathcal{FT}$ of fuzzy subsets of $X$ which satisfies the following conditions:*

**(i)** $\chi_\emptyset, \chi_X \in \mathcal{FT}$.

**(ii)** *If $\tilde{A}, \tilde{B} \in \mathcal{FT}$, then $\tilde{A} \cap \tilde{B} \in \mathcal{FT}$.*

**(iii)** *If $\tilde{A}_i \in \mathcal{FT}$ for each $i \in I$, then $\bigcup_{i \in I} \tilde{A}_i \in \mathcal{FT}$, where $I$ is an index set.*

*If $\mathcal{FT}$ is a fuzzy topology on $X$, then the pair $(X, \mathcal{FT})$ is called a* fuzzy topological space.

Let $(X, \mathcal{FT})$ be a fuzzy topological space. Then every member of $\mathcal{FT}$ is called a $\mathcal{FT}$*-open* fuzzy subset. A fuzzy subset is $\mathcal{FT}$*-closed* if and only if its complement is $\mathcal{FT}$-open. In the sequel, when no confusion is likely to arise, we shall call a $\mathcal{FT}$-open ($\mathcal{FT}$-closed) fuzzy subset simply an open (closed) fuzzy subset. As in (ordinary) topologies, the indiscrete fuzzy topology contains only $\chi_\emptyset$ and $\chi_X$, while the discrete fuzzy topology contains all fuzzy subsets of $X$. A fuzzy topology $\mathcal{FU}$ is said to be *coarser* than a fuzzy topology $\mathcal{FT}$ if $\mathcal{FU} \subseteq \mathcal{FT}$.

**Definition 3.26** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. A fuzzy subset $\tilde{Y}$ of $X$ is a* neighborhood, *or* nbhd *for short, of a fuzzy subset $\tilde{A}$ if there exists an open fuzzy subset $\tilde{U}$ of $X$ such that $\tilde{A} \subseteq \tilde{U} \subseteq \tilde{Y}$.*

The above definition differs somewhat from the ordinary one in that we consider here a nbhd of a fuzzy subset instead of a nbhd of a point.

**Theorem 3.18** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. A fuzzy subset $\tilde{A}$ of $X$ is open if and only if for each fuzzy subset $\tilde{B}$ of $X$ contained in $\tilde{A}$, $\tilde{A}$ is a neighborhood of $\tilde{B}$.*

**Proof.** It is immediate that if a fuzzy subset $\tilde{A}$ of $X$ is open, then for each fuzzy subset $\tilde{B}$ of $X$ contained in $\tilde{A}$, $\tilde{A}$ is a neighborhood of $\tilde{B}$. Conversely, suppose that for each fuzzy subset $\tilde{B}$ of $X$ contained in $\tilde{A}$, $\tilde{A}$ is a neighborhood of $\tilde{B}$. Then since $\tilde{A} \subseteq \tilde{A}$, there exists an open fuzzy subset $\tilde{U}$ such that $\tilde{A} \subseteq \tilde{U} \subseteq \tilde{A}$. Hence $\tilde{A} = \tilde{U}$ and $\tilde{A}$ is open. ■

**Definition 3.27** *Let $(X, \mathcal{FT})$ be a fuzzy topological space and let $\tilde{A}$ be a fuzzy subset of $X$. Then the* neighborhood system $\mathcal{N}$ *of $\tilde{A}$ is defined to be the set of all neighborhoods of $\tilde{A}$.*

**Theorem 3.19** *Let $(X, \mathcal{FT})$ be a fuzzy topological space and let $\tilde{A}$ be a fuzzy subset of $X$. Let $\mathcal{N}$ be the neighborhood system of $\tilde{A}$. If $\tilde{A}_1, ..., \tilde{A}_n \in \mathcal{N}$, then $\tilde{A}_1 \cap ... \cap \tilde{A}_n \in \mathcal{N}$. If $\tilde{B}$ is a fuzzy subset and $\exists\, \tilde{C} \in \mathcal{N}$ such that $\tilde{B} \supseteq \tilde{C}$, then $\tilde{B} \in \mathcal{N}$.*

**Proof.** If $\tilde{A}_1$ and $\tilde{A}_2$ are neighborhoods of a fuzzy subset $\tilde{A}$, there are open neighborhoods $\tilde{U}_1$ and $\tilde{U}_2$ contained in $\tilde{A}_1$ and $\tilde{A}_2$, respectively. Thus $\tilde{A}_1 \cap \tilde{A}_2$ contains the open neighborhood $\tilde{U}_1 \cap \tilde{U}_2$ and is hence a neighborhood of $\tilde{A}$. Thus the intersection of two (and hence of any finite number of) members of $\mathcal{N}$ is a member of $\mathcal{N}$. Hence, if a fuzzy subset $\tilde{B}$ contains a neighborhood $\tilde{C}$ of $\tilde{A}$, it contains an open neighborhood of $\tilde{A}$ since $\tilde{C}$ does and consequently is itself a neighborhood. ■

**Definition 3.28** *Let $(X, \mathcal{FT})$ be a fuzzy topological space and let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of $X$ such that $\tilde{A} \supseteq \tilde{B}$. Then $\tilde{B}$ is called an* interior fuzzy subset *of $\tilde{A}$ if $\tilde{A}$ is a neighborhood of $\tilde{B}$. The union of all interior fuzzy subsets of $\tilde{A}$ is called the interior of $\tilde{A}$ and is denoted by $\tilde{A}^\circ$.*

**Theorem 3.20** *Let $(X, \mathcal{FT})$ be a fuzzy topological space and let $\tilde{A}$ be a fuzzy subset of $X$. Then $\tilde{A}^\circ$ is open and is the largest open fuzzy subset contained in $\tilde{A}$. In fact, $\tilde{A}$ is open if and only if $\tilde{A} = \tilde{A}^\circ$.*

**Proof.** By Definition 3.28, $\tilde{A}^\circ$ is itself an interior fuzzy subset of $\tilde{A}$. Hence there exists an open fuzzy subset $\tilde{U}$ such that $\tilde{A}^\circ \subseteq \tilde{U} \subseteq \tilde{A}$. But $\tilde{U}$ is an interior fuzzy subset of $\tilde{A}$ and so $\tilde{U} \subseteq \tilde{A}^\circ$. Hence $\tilde{A}^\circ = \tilde{U}$. Thus $\tilde{A}^\circ$ is open and is the largest open fuzzy subset contained in $\tilde{A}$. If $\tilde{A}$ is open, then $\tilde{A} \subseteq \tilde{A}^\circ$ since $\tilde{A}$ is an interior fuzzy subset of $\tilde{A}$. Hence $\tilde{A} = \tilde{A}^\circ$. The converse is immediate. ■

## 3.4 Sequences of Fuzzy Subsets

**Definition 3.29** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. A sequence of fuzzy subsets, $\{\tilde{A}_n | n = 1, 2, \ldots\}$, is said to be* eventually contained *in a fuzzy subset $\tilde{A}$ if there is a positive integer $m$ such that if $n \geq m$, then $\tilde{A}_n \subseteq \tilde{A}$. The sequence is said to be* frequently contained *in $\tilde{A}$ if for each positive integer $m$ there is an integer $n$ such that $n \geq m$ and $\tilde{A}_n \subseteq \tilde{A}$. We say that the sequence converges to a fuzzy subset $\tilde{A}$ if it is eventually contained in each neighborhood of $\tilde{A}$.*

**Definition 3.30** *The sequence $\{\tilde{B}_i | i = 1, 2, \ldots\}$ is a* subsequence *of a sequence $\{\tilde{A}_n | n = 1, 2, \ldots\}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\tilde{B}_i = \tilde{A}_{f(i)}$ and for each integer $m$ there is an integer $n$ such that $f(i) \geq m$ whenever $i \geq n$.*

**Definition 3.31** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. A fuzzy subset $\tilde{A}$ of $X$ is called a* cluster fuzzy subset *of a sequence of fuzzy subsets if the sequence is frequently contained in every neighborhood of $\tilde{A}$.*

**Theorem 3.21** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. If the neighborhood system of each fuzzy subset of $X$ is countable, then the following assertions hold:*

**(i)** *A fuzzy subset $\tilde{A}$ is open if and only if each sequence of fuzzy subsets, $\{\tilde{A}_n \mid n = 1, 2, \ldots\}$, which converges to a fuzzy subset $\tilde{B}$ contained in $\tilde{A}$ is eventually contained in $\tilde{A}$.*

**(ii)** *If $\tilde{A}$ is a cluster fuzzy subset of a sequence $\{\tilde{A}_n \mid n = 1, 2, \ldots\}$ of fuzzy subsets, then there is a subsequence of the sequence converging to $\tilde{A}$.*

**Proof.** (i) Suppose that $\tilde{A}$ is open. Then $\tilde{A}$ is a neighborhood of $\tilde{B}$. Hence, $\{\tilde{A}_n \mid n = 1, ...\}$, is eventually contained in $\tilde{A}$. Conversely, let $\tilde{B} \subseteq \tilde{A}$ and let $\{\tilde{U}_1, ..., \tilde{U}_n, ...\}$ be the neighborhood system of $\tilde{B}$. Let $\tilde{V}_n = \bigcap_{i=1}^{n} \tilde{U}_i$. Then $\tilde{V}_1, ..., \tilde{V}_n, ...$ is a sequence which is eventually contained in each neighborhood of $\tilde{B}$, that is, $\tilde{V}_1, ..., \tilde{V}_n, ...$ converges to $\tilde{B}$. Hence, there is an $m$ such that for $n \geq m$, $\tilde{V}_n \subseteq \tilde{A}$ . The $\tilde{V}_n$ are neighborhoods of $\tilde{B}$. Therefore, by Theorem 3.18, $\tilde{A}$ is open.

(ii) Let $\{\tilde{R}_1, ..., \tilde{R}_n, ...\}$ be the neighborhood system of $\tilde{A}$ . Let $\tilde{S}_n = \bigcup_{i=1}^{n} \tilde{R}_i$. Then $\tilde{S}_1, ..., \tilde{S}_n, ...$ is a sequence such that $\tilde{S}_{n+1} \subseteq \tilde{S}_n$ for each $n$. For every nonnegative integer $i$, choose $f : \mathbb{N} \to \mathbb{N}$ such that $f(i) \geq i$ and $\tilde{A}_{f(i)} \subseteq \tilde{S}_i$. Then $\{\tilde{A}_{f(i)} \mid i = 1, 2, ...\}$ is a subsequence of the sequence $\{\tilde{A}_n \mid n = 1, 2, ...\}$. Clearly this subsequence converges to $\tilde{A}$. ■

## 3.5 $F$-Continuous Functions

In this section, we generalize the notion of continuity. We first establish several properties of fuzzy subsets induced by mappings.

**Definition 3.32** *Let $f$ be a function from a nonempty set $X$ into a nonempty set $Y$. Let $\tilde{B}$ be a fuzzy subset of $Y$. Then the* pre-image *of $\tilde{B}$ under $f$, written $f^{-1}(\tilde{B})$, is the fuzzy subset of $X$ defined by*

$$f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$$

*for all $x$ in $X$. Let $\tilde{A}$ be a fuzzy subset of $X$. The* image *of $\tilde{A}$ under $f$, written as $f(\tilde{A})$, is the fuzzy subset of $Y$ defined by*

$$f(\tilde{A})(y) = \begin{cases} \vee\{\tilde{A}(z) \mid z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \text{ is not empty,} \\ 0 & \text{otherwise,} \end{cases}$$

*for all $y$ in $Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.*

**Theorem 3.22** *Let $f$ be a function from $X$ into $Y$. Then the following assertions hold.*

**(i)** $f^{-1}(\tilde{B}^c) = (f^{-1}(\tilde{B}))^c$ *for any fuzzy subset $\tilde{B}$ of $Y$.*

**(ii)** $f(\tilde{A}^c) \supseteq (f(\tilde{A}))^c$ *for any fuzzy subset $\tilde{A}$ of $X$.*

**(iii)** $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow f^{-1}(\tilde{B}_1) \subseteq f^{-1}(\tilde{B}_2)$, *where $\tilde{B}_1, \tilde{B}_2$ are fuzzy subsets of $Y$.*

**(iv)** $\tilde{A}_1 \subseteq \tilde{A}_2 \Rightarrow f(\tilde{A}_1) \subseteq f(\tilde{A}_2)$, *where $\tilde{A}_1$ and $\tilde{A}_2$ are fuzzy subsets of $X$.*

**(v)** $\tilde{B} \supseteq f(f^{-1}(\tilde{B}))$ *for any fuzzy subset $\tilde{B}$ of $Y$.*

**(vi)** $\tilde{A} \subseteq f^{-1}(f(\tilde{A}))$ *for any fuzzy subset $\tilde{A}$ of $X$.*

**(vii)** *Let $f$ be a function from $X$ into $Y$ and $g$ be a function from $Y$ into $Z$. Then $(g \circ f)^{-1}(\tilde{C}) = f^{-1}(g^{-1}(\tilde{C}))$ for any fuzzy subset $\tilde{C}$ in $Z$, where $g \circ f$ is the composition of $g$ and $f$.*

**Proof.** (i) For all $x$ in $X$, $f^{-1}(\tilde{B}^c)(x) = \tilde{B}^c(f(X)) = 1 - \tilde{B}(f(x)) = 1 - f^{-1}(\tilde{B})(x) = (f^{-1}(\tilde{B}))^c(x)$.

(ii) For each $y \in Y$, if $f^{-1}(y)$ is not empty, then $f(\tilde{A}^c)(y) = \vee\{\tilde{A}^c(z) \mid z \in f^{-1}(y)\}\} = \vee\{1 - \tilde{A}(z) \mid z \in f^{-1}(y)\} = 1 - \wedge\{\tilde{A}(z) \mid z \in f^{-1}(y)\}$, and $(f^{-1}(\tilde{A}))^c(y) = 1 - f(\tilde{A})(y) = 1 - \vee\{\tilde{A}(z) \mid z \in f^{-1}(y)\}$. Thus $f(\tilde{A}^c)(y) \geq f(\tilde{A})^c(y)$.

(iii) Now $f^{-1}(\tilde{B}_1)(x) = \tilde{B}_1(f(x))$ and $f^{-1}(\tilde{B}_2)(x) = \tilde{B}_2(f(x))$ for any $x \in X$. Since $\tilde{B}_1 \subseteq \tilde{B}_2$, $f^{-1}(\tilde{B}_1)(x) \leq f^{-1}(\tilde{B}_2)(x)$ for all $x \in X$. Hence $f^{-1}(\tilde{B}_1) \subseteq f^{-1}(\tilde{B}_2)$.

(iv) $f(\tilde{A}_1)(y) = \vee\{\tilde{A}_1(z) \mid z \in f^{-1}(y)\}$ and $f(\tilde{A}_2)(y) = \vee\{\tilde{A}_2(z) \mid z \in f^{-1}(y)\}$. Since $\tilde{A}_1 \subseteq \tilde{A}_2$, $f(\tilde{A}_1)(y) \leq f(\tilde{A}_2)(y)$ for all $y \in Y$. Hence $f(\tilde{A}_1) \subseteq f(\tilde{A}_2)$.

(v) If $f^{-1}(y) \neq \emptyset$, then $f(f^{-1}(\tilde{B}))(y) = \vee\{f^{-1}(\tilde{B})(z) \mid z \in f^{-1}(y)\} = \vee\{\tilde{B}(f(z)) \mid z \in f^{-1}(y)\} = \tilde{B}(y)$. If $f^{-1}(y)$ is empty, $f(f^{-1}(\tilde{B}))(y) = 0$. Therefore, $f(f^{-1}(\tilde{B}))(y) \leq \tilde{B}(y)$ for all $y \in Y$.

(vi) $f(f^{-1}(\tilde{A}))(x) = f(\tilde{A})(f(x)) = \vee\{\tilde{A}(z) \mid z \in f^{-1}(f(x))\} \geq \tilde{A}(x)$ for all $x \in X$.

(vii) For all $x \in X$, $(g \circ f)^{-1}(\tilde{C})(x) = \tilde{C}(g \circ f(x)) = \tilde{C}(g(f(x))) = g^{-1}(\tilde{C})(f(x)) = f^{-1}(g^{-1}(\tilde{C}))(x)$. ■

**Definition 3.33** *Let $(X, \mathcal{FT})$ and $(Y, \mathcal{FU})$ be fuzzy topological spaces. A function $f$ from $X$ into $Y$ is said to be $F$-*continuous *if $f^{-1}(\tilde{U})$ is $\mathcal{FT}$-open for every $\mathcal{F}\tilde{U}$-open fuzzy subset $\tilde{U}$ of $Y$.*

Clearly, if $f$ is an $F$-continuous function from $X$ into $Y$ and $g$ is an $F$-continuous function from $Y$ into $Z$, then the composition $g \circ f$ is an $F$-continuous function of $X$ into $Z$, for $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ for each fuzzy subset $V$ of $Z$, and using the $F$-continuity of $g$ and $f$ it follows that if $V$ is open so is $(g \circ f)^{-1}(V)$.

**Theorem 3.23** *Let $(X, \mathcal{FT})$ and $(Y, \mathcal{FU})$ be fuzzy topological spaces. Let $f$ be a function of $X$ into $Y$. Then the conditions below are related as follows: (i) and (ii) are equivalent; (iii) and (iv) are equivalent; (i) implies (iii), and (iv) implies (v).*

**(i)** *The function $f$ is $F$-continuous.*

**(ii)** *The inverse under $f$ of every closed fuzzy subset of $Y$ is a closed fuzzy subset of $X$.*

**(iii)** *For each fuzzy subset $\tilde{A}$ of $X$, the inverse under $f$ of every neighborhood of $f(\tilde{A})$ is a neighborhood of $\tilde{A}$.*

**(iv)** *For each fuzzy subset $\tilde{A}$ of $X$ and each neighborhood $\tilde{V}$ of $f(\tilde{A})$, there is a neighborhood $\tilde{W}$ of $\tilde{A}$ such that $f(\tilde{W}) \subseteq \tilde{V}$.*

(v) *For each sequence of fuzzy subsets* $\{\tilde{A}_n \mid n = 1, 2, \dots\}$ *of* $X$ *which converges to a fuzzy subset* $\tilde{A}$ *of* $X$, *the sequence* $\{f(\tilde{A}_n) \mid n = 1, 2, \dots\}$ *converges to* $f(\tilde{A})$.

**Proof.** (i) $\Leftrightarrow$ (ii). Since $f^{-1}(\tilde{B}^c) = (f^{-1}(\tilde{B}))^c$ for every fuzzy subset $\tilde{B}$ of $Y$ by Theorem 3.22(i), the result is immediate.

(i) $\Rightarrow$ (iii). If $f$ is $F$-continuous, $\tilde{A}$ is a fuzzy subset of $X$, and $\tilde{V}$ is a neighborhood of $f(\tilde{A})$, then $\tilde{V}$ contains an open neighborhood $\tilde{W}$ of $f(\tilde{A})$. Since $f(\tilde{A}) \subseteq \tilde{W} \subseteq \tilde{V}$, $f^{-1}(f(\tilde{A})) \subseteq f^{-1}(\tilde{W}) \subseteq f^{-1}(\tilde{V})$. But $\tilde{A} \subseteq f^{-1}(f(\tilde{A}))$ by Theorem 3.22(vi) and $f^{-1}(\tilde{W})$ is open. Consequently, $f^{-1}(\tilde{V})$ is a neighborhood of $\tilde{A}$.

(iii) $\Rightarrow$ (iv). Since $f^{-1}(\tilde{V})$ is a neighborhood of $A$, we have $f(\tilde{W}) = f(f^{-1}(\tilde{V})) \subseteq \tilde{V}$, where $\tilde{W} = f^{-1}(\tilde{V})$.

(iv) $\Rightarrow$ (iii). Let $\tilde{V}$ be a neighborhood of $f(\tilde{A})$. Then there is a neighborhood $\tilde{W}$ of $\tilde{A}$ such that $f(\tilde{W}) \subseteq \tilde{V}$. Hence, $f^{-1}(f(\tilde{W})) \subseteq f^{-1}(\tilde{V})$. Furthermore, since $\tilde{W} \subseteq f^{-1}(f(\tilde{W}))$, $f^{-1}(\tilde{V})$ is a neighborhood of $\tilde{A}$.

(iv) $\Rightarrow$ (v). If $\tilde{V}$ is a neighborhood of $f(\tilde{A})$, there is a neighborhood $\tilde{W}$ of $\tilde{A}$ such that $f(\tilde{W}) \subseteq \tilde{V}$. Since $\{\tilde{A}_n \mid n = 1, 2, \dots\}$ is eventually contained in $\tilde{W}$, i.e., there is an $m$ such that for $n \geq m$, $\tilde{A}_n \subseteq \tilde{W}$, we have $f(\tilde{A}_n) \subseteq f(\tilde{W}) \subseteq \tilde{V}$ for $n \geq m$. Therefore $\{f(\tilde{A}_n) \mid n = 1, 2, \dots\}$ converges to $f(\tilde{A})$. ■

An $F$-continuous one-to-one function of a fuzzy topological space $(X, \mathcal{FT})$ onto a fuzzy topological space $(Y, \mathcal{FU})$ such that the inverse of the map is also $F$-continuous is called a *fuzzy homoemorphism*. If there exists a fuzzy homeomorphism of one fuzzy topological space onto another, the two fuzzy topological spaces are said to be *F-homeomorphic*. Two fuzzy topological spaces are called topologically *F-equivalent* if they are $F$-homeomorphic.

## 3.6 Compact Fuzzy Spaces

We now consider fuzzy compact topological spaces. If $\mathcal{A}$ is family of fuzzy subsets of a set $X$, we sometimes use the notation $\bigcup\{\tilde{A} \mid \tilde{A} \in \mathcal{A}\}$ for $\bigcup_{\tilde{A} \in \mathcal{A}} \tilde{A}$.

**Definition 3.34** *Let* $(X, \mathcal{FT})$ *be a fuzzy topological space. A family* $\mathcal{A}$ *of fuzzy subsets of* $X$ *is said to be a* cover *of a fuzzy subset* $\tilde{B}$ *of* $X$ *if* $\tilde{B} \subseteq \bigcup\{\tilde{A} | \tilde{A} \in \mathcal{A}\}$. *A cover* $\mathcal{A}$ *of* $\tilde{B}$ *is called an* open cover *of* $\tilde{B}$ *if each member of* $\mathcal{A}$ *is an open fuzzy subset of* $X$. *A subcover of* $\mathcal{A}$ *is a subfamily of* $\mathcal{A}$ *which is also a cover.*

**Definition 3.35** *A fuzzy topological space* $(X, \mathcal{FT})$ *is said to be* compact *if each open cover of* $\chi_X$ *has a finite subcover.*

**Definition 3.36** *A family $\mathcal{A}$ of fuzzy subsets of a set $X$ has the* finite intersection property *if the intersection of the members of each finite subfamily of $\mathcal{A}$ is nonempty.*

**Theorem 3.24** *Let $(X, \mathcal{FT})$ be a fuzzy topological space. Then $\chi_X$ is compact if and only if each family of closed fuzzy subsets of $X$ which has the finite intersection property has a nonempty intersection.*

**Proof.** If $\mathcal{A}$ is a family of fuzzy subsets of $X$, then $\mathcal{A}$ is a cover of $\chi_X$ if and only if $\bigcup\{\tilde{A} \mid \tilde{A} \in \mathcal{A}\} = \chi_X$, or if and only if $(\bigcup\{\tilde{A} \mid \tilde{A} \in \mathcal{A}\})^c = (\chi_X)^c = \chi_\emptyset$, or if and only if $\bigcap\{\tilde{A}^c \mid \tilde{A} \in \mathcal{A}\} = \chi_\emptyset$ by De Morgan's laws. Hence, $\chi_X$ is compact if and only if each family of open fuzzy subsets of $X$ such that no finite subfamily covers $\chi_X$, fails to be a cover, and this is true if and only if each family of closed fuzzy subsets which possesses the finite intersection property has a nonempty intersection. ■

**Theorem 3.25** *Let $(X, \mathcal{FT})$ and $(Y, \mathcal{FU})$ be fuzzy topological spaces. Let $f$ be an $F$-continuous function of $X$ onto $Y$. If $X$ is compact, then $Y$ is compact.*

**Proof.** Let $\mathcal{B}$ be an open cover of $\chi_Y$. Since $\bigcup_{\tilde{B} \in \mathcal{B}} f^{-1}(\tilde{B})(x) = \vee\{f^{-1}(\tilde{B})(x) \mid \tilde{B} \in \mathcal{B}\} = \vee\{\tilde{B}(f(x)) \mid \tilde{B} \in \mathcal{B}\} = 1$ for all $x \in X$, the family of all fuzzy subsets of the form $f^{-1}(\tilde{B})$, for $\tilde{B}$ in $\mathcal{B}$, is an open cover of $\chi_X$ which has a finite subcover. However, if $f$ is onto, then it is easily seen that $f(f^{-1}(\tilde{B})) = \tilde{B}$ for any fuzzy subset $\tilde{B}$ in $Y$. Thus, the family of images of members of the subcover is a finite subfamily of $\mathcal{B}$ which covers $\chi_Y$ and consequently $(Y, \mathcal{FU})$ is compact. ■

In [93] Lowen, finds the need to alter the definition of a fuzzy topological space in order to penetrate deeper into the structure of fuzzy topological spaces. Lowen replaces the condition that $\chi_\emptyset, \chi_X \in \mathcal{FT}$ in the definition of a fuzzy topological space to $\tilde{A} \in \mathcal{FT}$ for every fuzzy subset $\tilde{A}$ of $X$ such that $\exists t \in [0,1], \forall x \in X, \tilde{A}(x) = t$.

## 3.7 Iterated Fuzzy Subset Systems

In this section, we concentrate on the material from [14]. We first review some material from [9]. Let $(X, d)$ be a metric space and let $\mathcal{H}(X)$ denote the set whose points are nonempty compact subsets of $X$. Then the *Hausdorff distance* $h(d)$ (or simply $h$) between points $A$ and $B$ of $\mathcal{H}(X)$ is defined by $h(A, B) = d(A, B) \vee d(B, A)$. Then $h(d)$ is a metric on $\mathcal{H}(X)$ and $(\mathcal{H}(X), h(d))$ denotes the corresponding space of nonempty compact subsets of $X$ with the Hausdorff metric $h(d)$. $(\mathcal{H}(X), h(d))$ is sometimes referred to as the "space of fractals."

**Theorem 3.26** *(The Completeness of the Space of Fractals) Let $(X, d)$ be a complete metric space. Then $(\mathcal{H}(X), h)$ is a complete metric space. Moreover, if $\{A_n \in \mathcal{H}(X) | n = 1, 2, ...\}$ is a Cauchy sequence, then $A = \lim_{n\to\infty} A_n \in \mathcal{H}(X)$ can be characterized as follows:*
*$A = \{x \in X | \exists$ a Cauchy sequence $\{x_n \in A_n\}$ which converges to $x\}$.* ■

Let $(X, d)$ be a metric space. Recall from Definition 3.19 that a function $f : X \to X$ on $(X, d)$ is called contractive or a contraction map if there exists $s \in [0, 1)$ such that $d(f(x), f(y)) \leq sd(x, y)\ \forall x, y \in X$. Any such $s$ is called a *contractivity factor* for $f$.

**Example 3.48** *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (1/2)x + 1/2\ \forall x \in \mathbb{R}$. Then $f^n(x) = (1/2)^n x + (2^n - 1)/2^n$. We have that $|f(x) - f(y)| = (1/2)|x - y|$ and that $f(1) = 1$. Thus $f$ is a contraction map, $1/2$ is a contactivity factor for $f$, and $x_f = 1$ is the fixed point of $f$. Let $x = 0$. Then $\sum_{n=1}^{\infty} f^n(0) = \sum_{n=1}^{\infty}(2^n - 1)/2^n$ is a geometric series for $x_f = 1$. In fact, $\lim_{n\to\infty} f^n(x) = 1\ \forall\ x \in \mathbb{R}$.*

The properties in the next result help us think of fractals. That is, a fractal could be considered as a fixed point of a contractive mapping on $(\mathcal{H}(X), h(d))$, where the underlying metric space satisfies these properties.

**Proposition 3.27** *Let $(X, d)$ be a metric space and let $w : X \to X$.*

**(i)** *If $w$ is a contraction mapping, then $w$ is continuous.*

**(ii)** *If $w$ is continuous, then $w$ maps $\mathcal{H}(X)$ into itself.*

**(iii)** *Let $w$ be a contraction mapping with contractivity factor $s$. Then $w$ is a contraction mapping on $(\mathcal{H}(X), h(d))$ with contractivity factor $s$, where we consider $w : \mathcal{H}(X) \to \mathcal{H}(X)$ to be such that $w(B) = \{w(x) \mid x \in B\}$ for all $B \in \mathcal{H}(X)$.* ■

**Theorem 3.28** *(The Contraction Mapping Theorem) Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a contraction mapping on $(X, d)$. Then $f$ possesses exactly one fixed point $x_f \in X$. Moreover, $\forall x \in X$, the sequence $\{f^n(x) | n = 0, 1, 2...\}$ converges to $x_f$.*

**Proof.** Let $x \in X$. Let $s \in [0, 1)$ be a contractivity factor for $f$. Then

$$d(f^n(x), f^m(x)) \leq s^q d(x, f^{|n-m|}(x)), \tag{3.7.1}$$

where $q = n \wedge m$ and $n, m = 0, 1, 2, ....$ For $k = 0, 1, 2, ...,$we have that

$$\begin{aligned} d(x, f^k(x)) &\leq d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^{k-1}(x), f^k(x)) \\ &\leq (1 + s + s^2 + ... + s^{k-1})d(x, f(x)) \\ &\leq (1 + s)^{-1} d(x, f(x)). \end{aligned}$$

Substituting into equation (3.7.1), we obtain

$d(f^n(x), f^m(x)) \leq s^q(1-s)^{-1}d(x, f(x))$.
Hence it follows that $\{f^n(x) \mid n = 0, 1, 2, ...\}$ is a Cauchy sequence. Since $X$ is complete, there exists $x_f \in X$ such that $\lim_{n\to\infty} f^n(x) = x_f$. Since $f$ is contractive, it is continuous (Proposition 3.27) and hence
$f(x_f) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x_f$.
Thus $x_f$ is a fixed point of $f$. Let $y_f$ be any fixed point of $f$. Then
$d(x_f, y_f) = d(f(x_f), f(y_f)) \leq sd(x_f, y_f)$
and so $(1-s)d(x_f, y_f) \leq 0$. Thus $d(x_f, y_f) = 0$. Hence $x_f = y_f$. ■

The fixed points of a function are those which are not moved by a mapping. They restrict the motion of the space under nonviolent mappings of bounded deformation.

**Proposition 3.29** *Let $(X, d)$ be a metric space.*

**(i)** *$\forall B, C, D,$ and $E \in \mathcal{H}(X)$, $h(B \cup C, D \cup E) \leq h(B, C) \vee h(D, E)$, where $h$ is the usual Hausdorff metric.*

**(ii)** *Let $\{w_n \mid n = 1, 2, ..., N\}$ a set of contraction mappings on $(\mathcal{H}(X), h)$. Let $s_n$ denote the contractivity factor for $w_n$, $n = 1, 2, ..., N$. Define $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by $\forall B \in \mathcal{H}(X)$,*

$$W(B) = w_1(B) \cup ... \cup w_n(B).$$

*Then $W$ is a contraction mapping with contractivity factor $s = \vee\{s_n \mid n = 1, 2, ..., N\}$.* ■

**Definition 3.37** *A (hyperbolic)* iterated function system (IFS) *is a complete metric space $(X, d)$ together with a finite set of contraction mappings $w_n : X \to X$ with respective contractivity factors $s_n$ for $n = 1, 2, ..., N$. We use the notation $\{X : w_n, n = 1, ..., N\}$ and its contractivity factor is $s = \vee\{s_n \mid n = 1, 2, ..., N\}$.*

**Theorem 3.30** *Let $\{X : w_n, n = 1, ..., N\}$ be a hyperbolic iterated function system with contractivity factor $s$. Then the transformation $W : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $W(B) = \bigcup_{n=1}^{N} w_n(B)$ $\forall B \in \mathcal{H}(X)$, is a contraction map on the complete metric space $(\mathcal{H}(X), h(d))$ with contractivity factor $s$. That is,*

$$h(W(B), W(C)) \leq sh(B, C)$$

*$\forall B, C \in \mathcal{H}(X)$. Its unique fixed point, $A \in \mathcal{H}(X)$ satisfies $A = W(A) = \bigcup_{n=1}^{N} w_n(A)$, and is given by $A = lim_{n\to\infty} W^n(B)$ $\forall B \in \mathcal{H}(X)$.* ■

The fixed point $A \in \mathcal{H}(X)$ is called the *attractor* of the IFS.

**Example 3.49** *Let $X = \mathbb{R}$ and $d$ be the Euclidean metric on $X$. Consider the IFS $\{\mathbb{R}^2, w_1, w_2\}$, where $w_1, w_2 : X \to X$ are defined by $w_1(x) = (1/3)x$ and $w_2(x) = (1/3)x + 2/3$ $\forall x \in X$. It can be shown that the contractivity factor is $s = 1/3$. Let $B = [0,1]$ and $B_n = W^n(B)$ for $n = 1, 2, \ldots$. Then*

$B_1 = W(B) = w_1([0,1]) \cup w_2([0,1]) = \{(1/3)x | x \in [0,1]\} \cup \{(1/3)x + 2/3 | x \in [0,1]\} = [0,1/3] \cup [2/3,1]$

*and*

$B_2 = W(B_1) = w_1([0,1/3] \cup [2/3,1]) \cup w_2([0,1/3] \cup [2/3,1])$
$= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$.

*It follows that $A = \lim_{n\to\infty} B_n$ is the Cantor discontinuum. The Cantor discontinuum is usually described in the following manner. Take $[0,1]$ and remove the open middle third $(1/3, 2/3)$. From the remaining two closed intervals, remove their open middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$. By induction, we may continue this process and the remainder is the Cantor discontinuum.*

We now introduce a fuzzy subset approach for the construction, analysis, and approximation of sets and images which may exhibit fractal characteristics. We examine the inverse problem of encoding a target set or image with a relatively small number of parameters. Our method incorporates the technique of iterated function systems in its underlying structure. For each set $\{w_n \mid n = 1, \ldots, N\}$ of contraction maps, there a unique compact set $\mathcal{A} \subseteq X$, invariant under the "parallel action" $\bigcup_{n=1}^{N} w_n(\mathcal{A}) = \mathcal{A}$. For a given set of probabilities $\{p_n \mid n = 1, \ldots, N\}$ associated with $\{w_n \mid n = 1, \ldots, N\}, \sum_{i=1}^{N} p_n = 1$, there exists a unique invariant measure $\mu$ with support $\mathcal{A}$. We now summarize the approach used in [14].

(1) Let $\mathfrak{F}\wp(X)$ denote the set of all fuzzy subsets of $X$ and let $\mathcal{F}^*(X)$ be a particular subset of $\mathfrak{F}\wp(X)$. All images are considered as fuzzy subsets. This leads to two possible interpretations for $\tilde{A} \in \mathfrak{F}\wp(X)$ :

(a) in image representation, the value $\tilde{A}(x)$ of $\tilde{A}$ at a point $x \in X$ may be interpreted as the normalized grey level value associated with that point,

(b) in pattern recognition, the value $\tilde{A}(x)$ indicates the probability that the point $x$ is in the foreground of an image.

(2) Associated with each map $w_n$, $n = 1, \ldots, N$, is a grey level map $\phi_n : [0,1] \to [0,1]$, where $[0,1]$ is the grey level domain. The collection of maps $\{w_n \mid n = 1, \ldots, N\} \cup \{\phi_n \mid n = 1, \ldots, N\}$ is used to define an operator $T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ which is contractive with respect to the metric $d_\infty$ (defined below) on $\mathcal{F}^*(X)$. This metric is induced by the Hausdorff distance on the nonempty closed subsets of $X$. Starting with an arbitrary initial fuzzy subset $\tilde{A}_0 \in \mathcal{F}^*(X)$, the sequence $\tilde{A}_n \in \mathcal{F}^*(X)$ defined recursively by the iteration $\tilde{A}_{n+1} = \tilde{T}_s(\tilde{A}_n)$ converges in the $d_\infty$ metric to a unique and invariant fuzzy subset $\tilde{A}^* \in \mathcal{F}^*(X)$, that is, $T_s(\tilde{A}^*) = \tilde{A}^*$.

We let $(X, w, \Phi)$ denote the triple consisting of $X$, $w = \{w_n \mid n = 1, \ldots, N\}$, and $\Phi = \{\phi_n \mid n = 1, \ldots, N\}$. We call $(X, w, \Phi)$ an *iterated fuzzy subset system (IFZS)*. The compact space $X$ will be called the base set and the unique invariant fuzzy subset $\tilde{A}^*$ will be called an attractor for the IFZS. The support of $\tilde{A}^*$ is a subset of the attractor $\mathcal{A} \subseteq X$ of the underlying IFS defined by $\{w_n \mid n = 1, ..., N\}$. The approximation of the target image is accomplished in procedure (2) above. The fuzzy subsets $\tilde{A}_n$ represent grey level distributions on $X$ which converge to the grey level distribution $\tilde{A}^*$.

A black and white digitized image is a finite set $P$ of points or pixels $p_{ij}$, say in $\mathbb{R}^2$. Associated with each pixel $p_{ij}$ is a nonnegative grey level or brightness value, $t_{ij}$. We assume a normalized measure for grey levels, i. e., $0 \leq t_{ij} \leq 1$ ($0$ = black, the background; $1$ = white, the foreground). The function $h : P \to [0,1]$ defined by the grey level distribution of the image is called the *image function*. The digitized image is fully described by its image function $h$. This is also the situation in the more theoretical case where grey levels are distributed continuously on the base space $X$. At this point, one can see that an image as described by an image function is nothing but a fuzzy subset $\tilde{A} : X \to [0,1]$, even no probabilistic meaning is attached to the values $\tilde{A}(x)$ at each point $x \in X$.

It is usual to classify the pixels according to their grey levels by the $t$-cuts of $\tilde{A}$, where $t \in [0,1]$. The value $t$ is thought of as a threshold.

In the following, $(X, d)$ denotes a compact metric space and $\mathfrak{F}\wp(X)$ the set of all fuzzy subsets of $X$. We say that $\tilde{A} \in \mathfrak{F}\wp(X)$ is *normal* if $\tilde{A}(x_0) = 1$ for some $x_0 \in X$. We let $\mathcal{F}^*(X) = \{\tilde{A} \in \mathfrak{F}\wp(X) \mid \tilde{A}$ is normal and upper semicontinuous on $X\}$. If $\tilde{A} \in \mathfrak{F}\wp(X)$, we let $\tilde{A}^+$ denote the closure of $\{x \in X \mid \tilde{A}(x) > 0\}$.

**Proposition 3.31** *$\forall \tilde{A} \in \mathcal{F}^*(X)$ and $\forall t \in (0,1]$, $\tilde{A}^t$ is a nonempty compact subset of $X$ as is $\tilde{A}^+$.* ■

Let $\mathcal{H}(X)$ denote the set of all nonempty closed subsets of $X$ together with the Hausdorff distance function $h : \wp(X) \times \wp(X) \to \mathbb{R}$ defined by $\forall A, B \in \wp(X)$,

$h(A, B) = D(A, B) \vee D(B, A)$,

where $D : \wp(X) \times \wp(X) \to \mathbb{R}$ is such that

$D(A, B) = \vee\{\wedge\{d(x, y) \mid y \in B\} \mid x \in A\}$.

Then $(\mathcal{H}(X), h)$ is a compact metric space. In particular it contains the $t$-cuts $\tilde{A}^t$ $\forall \tilde{A} \in \mathcal{F}^*(X)$, $t \in [0,1]$. If we define $d_\infty : \mathcal{F}^*(X) \times \mathcal{F}^*(X) \to \mathbb{R}$ by $\forall \tilde{A}, \tilde{B} \in \mathcal{F}^*(X)$,

$d_\infty(u, v) = \vee\{h(\tilde{A}^t, \tilde{B}^t) \mid t \in [0,1]\}$,

then $d_\infty$ is a metric on $\mathcal{F}^*(X)$ and in fact $(\mathcal{F}^*(X), d_\infty)$ is a complete metric space.

We now introduce the IFS component of the IFZS. Then we are given $N$ contraction maps $w_i : X \to X$ such that for some $s \in [0,1)$,

$d(w_i(X), w_i(y)) \leq sd(x, y), \forall x, y \in X, i = 1, 2, ..., N$.

We call $s$ the contractivity factor. From [9, 10, 60], there exists a unique set $\mathcal{A} \in \mathcal{H}(X)$, the attractor of the IFS which satisfies:

$\mathcal{A} = \bigcup_{n=1}^{N} w_i(\mathcal{A})$, where $w_i(\mathcal{A}) = \{w_i(X) \mid x \in \mathcal{A}\}$. This represents the *self-tiling* property of IFS attractors. In other words, the map $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$\mathbf{w}(S) = \bigcup_{n=1}^{N} w_i(S), S \in \mathcal{H}(X)$

has an invariant set. This property is sometimes referred to as the "parallel action" of the $w_i$. We also have,

$h(\mathbf{w}^n(S), \mathcal{A}) \to 0$ as $n \to \infty, \forall S \in \mathcal{H}(X)$.

We now consider the selection of grey levels. For a general $N$-map IFS, $\mathbf{w} = \{w_i : X \to X \mid i = 1, ..., N\}$, it now remains to introduce and characterize the associated grey level maps $\Phi = \{\phi_i : [0,1] \to [0,1] \mid i = 1, ..., N\}$ to define the IFZS $\{X, \mathbf{w}, \Phi\}$. Since our objective is to construct an operator on the class of fuzzy subsets $\mathcal{F}^*(X)$, one condition to be satisfied by the functions $\phi_i$ is that they preserve upper semicontinuity when composed with functions of $\mathcal{F}^*(X)$, that is, $\phi_i \circ \tilde{A}$ is upper semicontinuous for $\tilde{A} \in \mathcal{F}^*(X)$. If the base space $X$ is finite, no conditions need to be imposed on the $\phi_i$. For the infinite case, however, the $\phi_i$ will have to be nondecreasing and right continuous.

**Lemma 3.32** *Let $\phi : [0,1] \to [0,1]$ and $X$ be an infinite and compact metric space. Then $\phi \circ \tilde{A}$ is upper semicontinuous $\forall \tilde{A} \in \mathcal{F}^*(X)$ if and only if $\phi$ is nondecreasing and right continuous.* ∎

We now summarize the conditions which should be satisfied by a set of grey level maps $\Phi = \{\phi_i : [0,1] \to [0,1] \mid i = 1, ..., N\}$ comprising an IFZS. For $i = 1, 2, ..., N$,

1. $\phi_i$ is nondecreasing,
2. $\phi_i$ is right continuous on $[0,1)$,
3. $\phi_i(0) = 0$,
4. $\exists\, j \in \{1, 2, ..., N\}$ such that $\phi_j(1) = 1$.

Properties 1 and 2, by Lemma 3.32 and Property 4, guarantee that the IFZS maps $\mathcal{F}^*(X)$ into itself. Property 3 is a natural assumption in the consideration of grey level functions: if the grey level of a point (pixel) $x \in X$ is zero, then it should remain zero after being acted upon by the $\phi_i$ maps.

We now introduce a general class of operators mapping $\mathfrak{F}_{\wp}(X)$ into itself, followed by a special class of operators which map $\mathcal{F}^*(X)$ into itself. The net result is the construction of an operator $T_s$, which is contractive on the compact metric space $(\mathcal{F}^*(X), d_\infty)$. The existence of a unique and attractive fixed point fuzzy subset/grey level distribution $\tilde{A} \in \mathcal{F}^*(X)$ will then be guaranteed.

Conforming to the extension principle for fuzzy subsets and by the same arguments that will justify our final choice for the operator $T : \mathcal{F}^*(X) \to$

$\mathcal{F}^*(X)$ (see Eq. (3.7.3) below), we define $\tilde{\tilde{A}}: \mathfrak{F}\wp(X) \to [0,1]$ as follows: $\forall$ $\tilde{A} \in \mathfrak{F}\wp(X)$ and $\forall B \subseteq X$,

$\tilde{\tilde{A}}\ (B) = \vee\{\tilde{A}(y) \mid y \in B\}$ if $B \neq \emptyset$

$\tilde{\tilde{A}}\ (\emptyset) = 0.$

Thus $\tilde{\tilde{A}}\ (\{x\}) = \tilde{A}(x)\ \forall x \in X$. Define $\tilde{\tilde{A}}_i: X \to [0,1]$ by

$\tilde{\tilde{A}}_i\ (x) = \tilde{\tilde{A}}\ (w_i^{-1}(x))$

$\forall w_i, i = 1, 2, ..., N$, and $\forall\ x \in X$,

where $w_i{}^{-1}(x) = \emptyset$ if $x \notin w(x)$. If $\tilde{A} \in \mathcal{F}^*(X)$, then each $\tilde{\tilde{A}}_i: X \to [0,1]$ is a fuzzy subset in $\mathcal{F}^*(X)$. For the upper semicontinuity of $\tilde{\tilde{A}}_i$ see Lemma 3.32; the normality is straightforward.

For a general IFZS $\{X, \mathbf{w}, \Phi\}$ consisting of $N$ IFS maps and $N$ grey level maps, consider the class of mappings $U_N : [0,1]^N \to [0,1]$ and the operator $T : \mathfrak{F}\wp(X) \to \mathfrak{F}\wp(X)$ that associates to each fuzzy subset $\tilde{A}$ the fuzzy subset $\tilde{B} = T\tilde{A}$ whose value at each $x \in X$ is given by

$$\tilde{B}(x) = (T\tilde{A})(x) = U_N(\tilde{Q}_1(x), \tilde{Q}_2(x), ..., \tilde{Q}_N(x)), \tag{3.7.2}$$

where $\tilde{Q}_i : X \to [0,1]$ is defined by

$$\tilde{Q}_i(x) = \phi_i(\tilde{A}(w_i{}^{-1}(x)))$$

for $i = 1, 2, ...., N$. In other words, the function $U_N$ operates on the modified grey levels of all possible pre-images of $x$ under the IFS maps $w_i$, the grey levels having been transformed by the appropriate $\phi_i$ maps.

It appears totally natural to assume $U_N$ symmetric in its arguments, i. e.,

$U_N(v_{i_1}, v_{i_2}, ..., v_{i_N}) = U_N(v_1, v_2, ..., v_N)$

for every permutation $(i_1, i_2, ..., i_N)$ of $\{1, 2, ..., N\}$. However it is convenient for computational purposes to assume the $U_N$ are defined as

$U_N = U_2(v_1, U_{N-1}(v_2, v_3, ..., v_N))$.

In particular,

$U_3(v_1, v_2, v_3) = U_2(v_1, U_2(v_2, v_3)) = U_3(v_3, v_1 v_2) = U_2(v_3, U_2(v_1, v_2)) = U_2(U_2(v_1, v_2), v_3)$. Hence we see that the function $U_2 : [0,1]^2 \to [0,1]$ is an associative binary operation on $[0,1]$. We shall let $S$ denote such a binary operation. We shall assume the following set of additional properties to be satisfied by $S$ :

1. $S : [0,1]^2 \to [0,1]$ is continuous.

2. For each $y \in [0,1]$, the mapping $x \to S(x,y)$ is nondecreasing; the brighter the pixel, the brighter its combination with another pixel.

3. 0 is an identity, that is, $S(0,y) = y\ \forall y \in [0,1]$; the combination of a pixel of brightness $y > 0$ with one of 0 brightness yield a pixel with brightness $y$.

4. For all $x \in [0,1]$, $S(x,x) \geq x$; the combination of two pixels of equal brightness should not result in a darker pixel.

**Theorem 3.33** *If $S : [0,1]^2 \to [0,1]$ satisfies properties (1) - (4) above, then there exists a sequence of disjoint open intervals $\{(a_r, b_r) | r = 1, 2, \ldots\}$ with $a_1 = 0 < b_1 \leq a_2 < b_2 \leq \ldots \leq 1$, and a sequence of increasing continuous functions $f_r : [a_r, b_r] \to [0, \infty]$ with $f_r(a_r) = 0$, such that*

$S(x,y) = g_r(f_r(x) + f_r(y)) \; \forall (x,y) \in [a_r, b_r]^2$,

*where $g_r$ (pseudoinverse of $f_r$) is defined as*

$$g_r(t) = \begin{cases} f_r^{-1}(t) & \text{if } t \in [0, f_r(b_r)] \\ b_r & \text{if } t \in [f_r(b_r), \infty] \end{cases}$$

*and finally*

$S(x,y) = x \vee y$ *if* $(x,y) \in [0,1]^2 \setminus \bigcup_{r=1}^{\infty} [a_r, b_r]^2$. ∎

Clearly, $S(a_r, a_r) = a_r$ and $S(b_r, b_r) = b_r$ for $r = 1, 2, \ldots$, that is, the $a_r$ and $b_r$ are idempotent for the operation $S$. Moreover, no element in the open intervals $(a_r, b_r)$ is an idempotent for $S$. It is possible that the sequence $\{(a_r, b_r) \mid r = 1, 2, \ldots\}$ may reduce to the single interval $(0,1)$: indeed this is the case when $S$ has 0 and 1 as its only idempotents, 0 being the identity, and 1 the annihilator. An example is given by the following operation, the $p$-norm, with $p$ a positive integer,

$$S(x,y) = \begin{cases} [x^p + y^p]^{1/2} & \text{if } x^p + y^p \leq 1 \\ 1 & \text{if } x^p + y^p \geq 1 \end{cases}$$

The functions $f$ and $g$ in Theorem 3.33 are given by

$f(s) = s^p$

$$g(t) = \begin{cases} t^{1/p} & \text{if } t \in [0,1], \\ 1 & \text{if } t \in [1, \infty]. \end{cases}$$

The other extreme case is when $S(x,x) = x$ for all $x \in [0,1]$. In this case,

$S(x,y) = x \vee y \; \forall (x,y) \in [0,1]^2$.

In fact, from properties (2) and (4),

$S(x,y) \geq S(x,0) = x$ and $S(x,y) \geq S(0,y) = y$

and so

$S(x,y) \geq x \vee y$.

On the other hand, if $x < y$,

$S(x,y) \leq S(y,y) = y = x \vee y$,

so that $S(x,y) = x \vee y$. Even though this operation represents an extreme case, it appears to be the most natural one for our particular applications: the combination of two pixels with equal brightness $t$ should result in a pixel with brightness $t$. As such, it will now be employed as the binary associative operation $U_2$ introduced at the beginning of this section.

We now investigate the properties of the resulting operator $T : \mathfrak{F}\wp(X) \to \mathfrak{F}\wp(X)$ in Eq. (3.7.3), when $U_2(v_1, v_2) = v_1 \vee v_2$, that is, when

$$(T\tilde{A})(x) = \vee\{\phi_1(\tilde{A}(w_1^{-1}(x))), \ldots, \phi_N(\tilde{A}(w_N^{-1}(x)))\} \equiv (T_s\tilde{A})(x). \quad (3.7.3)$$

It will then be shown that $T_s$ maps the class of fuzzy subsets $\mathcal{F}^*(X)$ into itself.

**Lemma 3.34** *For all $\tilde{A} \in \mathcal{F}^*(X)$ and $t \in [0,1]$ with $\tilde{Q}_i : X \to [0,1]$ $(1 \le i \le N)$ defined as in Eq. (3.7.2), we have*

(i) *$\tilde{Q}_i$ is upper semicontinous,*

(ii) $\tilde{Q}_i^t = w_i((\phi_i \circ \tilde{A})^t)$,

(iii) $(T_s\tilde{A})^t = \cup_{i=1}^N w_i((\phi_i \circ \tilde{A})^t)$. ■

We note that the $\tilde{Q}_i$ in Lemma 3.34 may not be normal and so some of their level sets may be empty.

We now state the main result.

**Theorem 3.35** *The operator $T_s$ is a contraction mapping on $(\mathcal{F}^*(X), d_\infty)$, i. e., $T_s$ maps $\mathcal{F}^*(X)$ into itself and for $0 \le s < 1$,*
$d_\infty(T_s\tilde{A}, T_s\tilde{B}) \le s d_\infty(\tilde{A}, \tilde{B})\ \forall \tilde{A}, \tilde{B} \in \mathcal{F}^*(X)$. ■

By virtue of the Contraction Mapping Principle over the complete metric space $(\mathcal{F}^*(X), d_\infty)$, we have the following important result.

**Corollary 3.36** *For each fixed IFZS $\{X, \mathbf{w}, \Phi\}$ there exists a unique fuzzy subset $\tilde{A}^* \in \mathcal{F}^*(X)$ such that $T_s\tilde{A}^* = \tilde{A}^*$.* ■

This gives a unique solution to the functional equation in the unknown $\tilde{A} \in \mathcal{F}^*(X)$,
$\tilde{A}(X) = \vee\{\phi_1(\tilde{A}(w_1{}^{-1}(X))), \ldots, \phi_N(\tilde{A}(w_N{}^{-1}(X)))\}$
for all $x \in X$. The solution fuzzy subset $\tilde{A}^*$ will be called the attractor of the IFZS since it follows from the Contraction Mapping Principle that
$d_\infty((T_s)^n\tilde{B}, \tilde{A}^*) \to 0$ as $n \to \infty$, $\forall \tilde{B} \in \mathcal{F}^*(X)$.

Another important consequence is the property

$$(\tilde{A}^*)^t = \bigcup_{i=1}^{N} w_i((\phi_i \circ \tilde{A}^*)^t), 0 \le t \le 1, \tag{3.7.4}$$

(cf. Lemma 3.34), which can be considered as a generalized self-tiling property of $t$-cuts of the fuzzy subset attractor $\tilde{A}^*$. Let us now show some properties of $\tilde{A}^*$.

It is easy to see that the operator $T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ is monotone, namely, $\tilde{A}, \tilde{B} \in \mathcal{F}^*(X)$, $\tilde{A} \subseteq \tilde{B}$ implies $T_s\tilde{A} \subseteq T_s\tilde{B}$.

**Proposition 3.37** *Let $A \in \mathcal{H}(X)$ be the attractor of the base space IFS $\{X, \mathbf{w}\}$ and let $\tilde{A}^* \in \mathcal{F}^*(X)$ denote the fuzzy subset attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$ with corresponding operator $T_s$. Then for $\tilde{B} \in \mathcal{F}^*(X)$ and $B \in \mathcal{H}(X)$.*

(i) $T_s\tilde{B} \subseteq \tilde{B} \Rightarrow \tilde{A}^* \subseteq \tilde{B}$.

(ii) $\mathbf{w}(B) \subseteq B \Rightarrow \mathcal{A} \subseteq B$.

(iii) $\tilde{B} \subseteq T_s\tilde{B} \Rightarrow \tilde{B} \subseteq \tilde{A}^*$.

(iv) $B \subseteq \mathbf{w}(B) \Rightarrow B \subseteq \mathcal{A}$.

The following theorem demonstrates the connection between the fuzzy subset attractor of an IFZS and the corresponding attractor of the base space IFS.

**Theorem 3.38** *Let $\mathcal{A} \in \mathcal{H}(X)$ be the attractor of the IFS $\{X.\mathbf{w}\}$ and let $\tilde{A}^* \in \mathcal{F}^*(X)$ denote the fuzzy subset attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$. Then supp$(\tilde{A}^*) \subseteq \mathcal{A}$, that is,*

$$\tilde{A}^{*+} \subseteq \mathcal{A}. \tag{3.7.5}$$

■

Note that equality holds in (3.7.5) for the following two cases:
$\forall i \in \{1, 2, ..., N\}$, $\phi_i(1) = 1$, then $\tilde{A}^* = \chi_A$
$\forall i \in \{1, 2, ..., N\}$, the $\phi_i$ are increasing at 0, i. e., ${\phi_i}^{-1}(0) = \{0\}$. Indeed, in this case $\tilde{A}^{*+} = \bigcup_{i=1}^{N} w_i((\phi_i \circ \tilde{A}^*)^+) = \bigcup_{i=1}^{N} w_i(\mathcal{A}) = w(\mathcal{A}) = \mathcal{A}$.

We also point out that in the case $\phi_j(0) > 0$ for one $j \in \{1, 2, ..., N\}$, the inclusion (3.7.5) is not true.

Another noteworthy consequence of the contractivity of the $T_s$ operator is given in the next theorem.

**Theorem 3.39** *(IFSZ Collage Theorem) Let $\tilde{B} \in \mathcal{F}^*(X)$ and suppose that there exists an IFZS $\{X, \mathbf{w}, \phi\}$ with contractivity factor $s$ so that*
$d_\infty(\tilde{B}, T_s\tilde{B}) < \epsilon$,
*where the operator $T_s$ is defined by Eq. (3.7.3). Then*
$d_\infty(\tilde{B}, \tilde{A}^*) < \epsilon/(1-s)$,
*where $\tilde{A}^* = T_s\tilde{A}^*$ is the invariant fuzzy subset of the IFZS.* ■

We now present some examples which illustrate the main features of the IFZS. In particular, the generality afforded by the grey level maps is shown.

**Example 3.50** *Let $X = [0, 1]$, $N = 4$, and $w_i(x) = 0.25x + 0.25(i-1)$, $i = 1, 2, 3, 4$. Here $\mathcal{A} = [0, 1]$. The following grey level maps were selected*

$$\phi_1(t) = \begin{cases} 0.25t & \text{if } 0 \le t < 0.25 \\ t - 0.18 & \text{if } 0.25 \le t \le 1 \end{cases}$$

$\phi_2(t) = t$, $t \in [0, 1]$,
$\phi_3(t) = 0.33t$,
$\phi_4(t) = \sin t$.

*The picture shown in [14, Figure 1, p. 93] is a representation of the graph of the IFZS attractor $\tilde{A}^*$ on $[0, 1]$.*

In the remaining examples, $X = [0, 1]^2$ is the base space. Photographs of the computer approximation to the IFZS invariant sets are shown in [14] as normalized grey level distributions: the brightness value $t_{ij}$ of a pixel $p_{ij}$ representing a point $x \in X$ obeys $0 \le t_{ij} \le 1$, with $t_{ij} = \tilde{A}^*(x)$; $t_{ij} = 0$ if $x$ is in the background.

**Example 3.51** *Let* $N = 4$, $w_1(x, y) = (0.5x, 0.5y)$, $w_2(x, y) = (0.5x + 0.5, 0.5y)$, $w_3(x, y) = (0.5x, 0.5y + 0.5)$, *and* $w_4(x, y) = (0.5x + 0.5, 0.5y + 0.5)$. *Here* $\mathcal{A} = X$. *The following grey level maps are given below*

$$\phi_1(t) = \begin{cases} 0 & \text{if } 0 \le t < 0.2505 \\ 0.25 & \text{if } 0.2505 \le t < 0.505 \\ 0.5 & \text{if } 0.505 \le t < 0.7505 \\ 0.75 & \text{if } 0.7505 \le t \le 1 \end{cases}$$

$\phi_2(t) = \phi_3(t) = \phi_4(t) = t$, $t \in [0, 1]$.

Note that $\tilde{A}^* \subset \mathcal{A}$ (strict inclusion). This is due to the fact that $\phi_1(0)$ is not strictly increasing (cf. Theorem 3.38).

Some comments on the properties of $\tilde{A}^*$, as evident from [14, Figure 2, p. 94], would be instructive here. The grey level distribution exhibits the generalized self-tiling property of level sets, as given by Eq. (3.7.4). Let $\mathcal{A}_i = w_i(\mathcal{A})$, $i = 1, 2, 3, 4$. One can see the effect of the transformation $\phi_1$ which is different from the other $\phi_i$. Given that $\phi_2, \phi_3, \phi_4$ are identity maps, the values of $\tilde{A}^*$ on $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ are the same. The presence of exactly four grey levels is due to the step function nature of $\phi_1$.

The flexibility afforded by the grey level maps $\phi_i$ should now be apparent. The dynamics of maps on the unit interval which are nondecreasing and right continuous may be exploited to affect the pointwise shading of the image/fuzzy subset in a rather controlled manner.

**Example 3.52** *Let* $N = 4$ *and the transformations* $w_i$ *taken from [10], define a base space IFS whose attractor* $\mathcal{A}$ *is a "leaf." We first consider identity grey level maps, that is,* $\phi_i(t) = t, i = 1, 2, 3, 4$. *Since* $\phi_i(1) = 1, i = 1, 2, 3, 4$, $\tilde{A}^* = \chi_{\mathcal{A}}$. *The attractor* $\tilde{A}^*$ *is shown in [14, Figure 3, p. 94].*

**Example 3.53** *The base space is as in Example 3.52, but with the grey level maps:*

$\phi_1(t) = 0.85t$, $\phi_2(t) = t$, $\phi_3(t) = 0.8\sqrt{t}$, *and* $\phi_4(t) = 0.4(t^2 + t)$, $t \in [0, 1]$. *The fuzzy subset attractor is shown in [14, Figure 4, p. 94].*

## 3.8 Chaotic Iterations of Fuzzy Subsets

For this section, most of the material is from [81]. We are interested in successive iterates $f^{n+1} = f^n \circ f$ of a function $f$ from a topological space $X$ into itself and a sequence of points

$$x_{n+1} = f(x_n), n = 0, 1, 2, \ldots \tag{3.8.1}$$

in $X$. In particular, we examine the chaotic behavior of such iterated sequences. A point $x \in X$ is called a *cycle point* of $f$ if $\exists n \in \mathbb{N}$ such that $x = f^n(x)$.

**Definition 3.38** *A function $f : [0,1] \to [0,1]$ is called* chaotic *if the following conditions hold:*

(i) *There exists a positive integer $K$ such that the iterative scheme (3.8.1) has a cycle of period $k$ for each $k \geq K$;*

(ii) *The iterative scheme (3.8.1) has a* scrambled set, *that is, an uncountable set $S \subset [0,1]$ containing no cycle points of $f$ such that*

  (a) $f(S) \subset S$,

  (b) *for every $x_0, y_0 \in S$ with $x_0 \neq y_0$,* $\limsup_{n\to\infty} |f^n(x_0) - f^n(y_0)| > 0$;

  (c) *for every $x_0 \in S$ and cyclic point $y_0$ of $f$,* $\limsup_{n\to\infty} |f^n(x_0) - f^n(y_0)| > 0$;

(iii) *There exists an uncountable subset $S_0 \subseteq S$ such that for all $x_0, y_0 \in S$,* $\liminf_{n\to\infty} |f^n(x_0) - f^n(y_0)| = 0$.

The following iterative system involves a function which displays chaotic behavior in the sense of Definition 3.38. The simple logistics equation
$x_{n+1} = 4x_n(1 - x_n)$, $0 \leq x_n \leq 1$,
describes the dynamics of a population with non-overlapping generations.

Another example is the baker's equation

$$x_{n+1} = \begin{cases} 2x_n & \text{if } 0 \leq x_n \leq 1/2, \\ 2(1 - x_n) & \text{if } 1/2 \leq x_n \leq 1, \end{cases} \tag{3.8.2}$$

modeling the mixing of a dye spot on a strip of dough which is repeatedly stretched and folded over on itself.

For further reading material, the reader may wish to consult [79, 80, 82, 88, 102].

**Theorem 3.40** *(Kloeden) Let $f$ be a continuous function of a Banach space $X$ into itself and suppose that there exist nonempty compact subsets $A$ and $B$ of $X$ and integers $n_1, n_2 \geq 1$ such that*

(i) *$A$ is homeomorphic to a convex subset of $X$,*

(ii) $A \subseteq f(A)$,

(iii) *$f$ is expanding on $A$, that is, there exists a constant $\lambda > 1$ such that* $\lambda\|x - y\| \leq \|f(X) - f(y)\|$ $\forall x, y \in A$,

**(iv)** $B \subset A$.

**(v)** $f^{n_1}(B) \cap A = \emptyset$,

**(vi)** $A \subseteq f^{n_1+n_2}(B)$,

**(vii)** $f^{n_1+n_2}$ *is one-to-one on* $B$.

*Then the mapping is chaotic in the sense of Definition 3.38.* ■

Consider the function of (3.8.2), $f : [0,1] \to [0,1]$ defined as follows:
$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ 2(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$
Now Theorem 3.40 applies to $f$ with $A = [9/16, 7/8]$, $B = [3/4, 7/8]$, $n_1 = 1$, and $n_2 = 1$.

Let $\mathcal{E}^n = \{\tilde{A} : \mathbb{R}^n \to [0,1] \mid \tilde{A}$ is normal, $\tilde{A}$ is fuzzy convex, $\tilde{A}$ is upper semicontinuous, and $\tilde{A}^+$ is compact$\}$ . In Theorem 3.41 below, we give an analogue of Theorem 3.40. Theorem 3.41 provides sufficient conditions for a mapping on fuzzy subsets to be chaotic, where the definition of chaotic here is entirely similar to that of Definition 3.38. We consider an iterative scheme of fuzzy subsets

$$\tilde{A}_{n+1} = f(\tilde{A}_n),\ n = 0, 1, 2, \ldots,$$
where $f$ is a continuous function of $\mathcal{E}^n$ into itself. The proof uses the following result by Kaleva [68]: Let $f : \mathcal{E}^n \to \mathcal{E}^n$ be a continuous function and let $C$ be compact convex subset of $\mathcal{E}^n$ such that $f(C) \subseteq C$. Then $f$ has a fixed point $\tilde{A} = f(\tilde{A}) \in C$.

**Theorem 3.41** *Let* $f : \mathcal{E}^n \to \mathcal{E}^n$ *be a continuous function. Suppose that there exist nonempty compact subsets* $\mathcal{A}$ *and* $\mathcal{B}$ *of* $\mathcal{E}^n$ *and integers* $n_1, n_2 \ge 1$ *such that*

**(i)** $\mathcal{A}$ *is homeomorphic to a convex subset of* $\mathcal{E}^n$,

**(ii)** $\mathcal{A} \subseteq f(\mathcal{A})$,

**(iii)** $f$ *is expanding on* $\mathcal{A}$, *that is, there exists a constant* $\lambda > 1$ *such that* $\lambda d_\infty(\tilde{A}, \tilde{B}) \le d_\infty(f(\tilde{A}), f(\tilde{B}))\ \forall \tilde{A}, \tilde{B} \in \mathcal{A}$,

**(iv)** $\mathcal{B} \subset \mathcal{A}$,

**(v)** $f^{n_1}(\mathcal{B}) \cap \mathcal{A} = \emptyset$,

**(vi)** $\mathcal{A} \subseteq f^{n_1+n_2}(\mathcal{B})$,

**(vii)** $f^{n_1+n_2}$ *is one-to-one on* $\mathcal{B}$.

*Then* $f$ *is chaotic.* ■

We now illustrate Theorem 3.41 with an example. First note that for each $\tilde{A} \in \mathcal{E}^l$, there exists $a, b : [0,1] \to \mathbb{R}$ such that the $t$-cuts of $\tilde{A}$ are the intervals $[a(t), b(t)]$. Moreover, $a$ is nondecreasing, $b$ is nonincreasing, and $a(1) \leq b(1)$.

**Example 3.54** *Consider the following subsets of* $\mathcal{E}^l$ :

**(i)** $\mathcal{E}_0{}^l = \{\tilde{A} \in \mathcal{E}^l \mid a(0) = 0\}$,

**(ii)** $\mathcal{T}_0{}^l = \{\tilde{A} \in \mathcal{E}_0{}^l \mid a(t) = (1/2)t(b(0) - L)$ *and* $b(t) = b(0) - (1/2)t(b(0) - L)$ *for some* $L$, $0 \leq L \leq b(0)\}$,

**(iii)** $\Delta_0{}^l = \{\tilde{A} \in \mathcal{T}_0{}^l \mid L = 0\}$.

*For any* $\tilde{A} \in \mathcal{E}_0{}^l$, *the support* $\tilde{A}^+$ *is a nonnegative interval anchored on* $x = 0$. *The endograph of any* $\tilde{A} \in \mathcal{T}_0{}^l$ *is a symmetric trapezium centered on* $x = (1/2)b(0)$, *with base length* $b(0)$ *and top length* $L$. *For any* $\tilde{A} \in \Delta_0{}^l$, *the endograph is an isosceles triangle.*

*Define*
$f_1 : \mathcal{E}^l \to \mathcal{E}_0{}^l$ *by* $f_1(\tilde{A})^t = [a(t) - a(0), b(t) - b(0)]$,
$f_2 : \mathcal{E}_0{}^l \to \mathcal{T}_0{}^l$ *by* $f_2(\tilde{A})^t = [tM, b(0) - tM]$,
*where* $M = (1/2)b(0) - (1/8)(b(0) - a(1))$;
$f_3 : \mathcal{T}_0{}^l \to \mathcal{T}_0{}^l$ *by* $f_3(\tilde{A}) = g(b(0))\tilde{A}^t = [g(b(0))a(t), g(b(0))b(t)]$,
*where* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *is the function*

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1/2, \\ -2 + 2/x & \text{if } 1/2 \leq x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

*Also define* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $h(x) = xg(x)\ \forall x \in \mathbb{R}^+$. *Define* $f : \mathcal{E}^l \to \mathcal{E}^l$ *by* $f = f_3 \circ f_2 \circ f_1$. *Then* $f$ *is continuous with respect to the* $d_\infty$ *metric and maps* $\Delta_0{}^l$ *into itself. Now any* $\tilde{A} \in \Delta_0{}^l$ *is determined uniquely by its value* $b(0)$, *written* $b$ *from now on, and will be denoted by* $\tilde{A}_b$. *Then* $f(\tilde{A}) = \tilde{A}_{h(b)}$.

*Let* $n_1 = n_2 = 1$, $\mathcal{A} = \{\tilde{A}_b \in \Delta_0{}^l | 9/16 \leq b \leq 7/8\}$, *and* $\mathcal{B} = \{\tilde{A}_b \in \Delta_0{}^l | 3/4 \leq b \leq 7/8\}$. *Then Theorem 3.41 applies for* $f$. *To see this let note that* $\mathcal{B} \subseteq \mathcal{A}$;
$f(\mathcal{A}) = \{\tilde{A}_b \in \Delta_0{}^l | 1/4 \leq b \leq 7/8\}$ *so* $\mathcal{A} \subseteq f(\mathcal{A})$,
$f(\mathcal{B}) = \{\tilde{A}_b \in \Delta_0{}^l | 1/4 \leq b \leq 1/2\}$ *so* $f(\mathcal{B}) \cap \mathcal{A} = \emptyset$,
$f^2(\mathcal{B}) = \{\tilde{A}_b \in \Delta_0{}^l | 1/2 \leq b \leq 1\}$ *so* $\mathcal{A} \subseteq f^2(\mathcal{B})$.

*Moreover,* $f$ *is expanding on* $\mathcal{A}$ *since* $h$ *is expanding on* $[9/16, 7/8]$ *with*
$|h(x) - h(y)| = |(2 - 2x) - (2 - 2y)| = 2|x - y|$.
*there. Thus*
$d_\infty(f(\tilde{A}_x), f(\tilde{A}_y)) = 2d_\infty(\tilde{A}_x, \tilde{A}_y)$
*for any* $\tilde{A}_x, \tilde{A}_y \in \mathcal{A}$. *Finally,* $h^2$ *is one-to-one on* $[3/4, 7/8]$ *since*
$h^2(x) = 2(2 - 2x) = 4 - 4x$
*there. Thus* $f^2$ *is one-to-one on* $\mathcal{B}$. *The function* $f$ *is thus chaotic. Its chaotic action is most apparent in the compact subset* $\{\tilde{A}_b \in \Delta_0{}^l \mid 0 \leq b \leq 1\}$ *of* $\mathcal{E}^l$. *It can be shown that for any* $\tilde{A} \in \mathcal{E}^l$, *the successive iterates*

*$f^n(\tilde{A})$ asymptote towards this set. Their endographs become more and more triangular in shape, unless $b(0)-a(0)>1$, in which case they collapse onto a singleton fuzzy subset.*

## 3.9 Starshaped Fuzzy Subscts

The material in this section is taken for the most part from [30]. As we will see, convexity plays an important role in the theory and applications of fuzzy subsets. In this section, we examine a generalization of the concept of convexity. In particular, we examine the notion of starshaped sets and fuzzy starshaped sets. They extend convexity in important ways, e.g., unions of fuzzy starshaped sets with common kernel points are fuzzy starshaped. They also share similar metric properties with convex sets and convex fuzzy subsets, where a fuzzy subset $\tilde{A}$ of $\mathbb{R}^n$ is said to be *convex* if for any line segment $\overline{PQ}$ in $\mathbb{R}^n$ and any point $R$ in $\overline{PQ}$, $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$.

We restrict our attention to $\mathbb{R}^n$. We denote the Euclidean norm by $\|\ \|$, the inner product by $<\ ,\ >$, and the unit sphere by $S^{n-1}$. The Hausdorff distance between compact subsets $A$ and $B$ of $\mathbb{R}^n$ is given by

$\delta_\infty(A,B) = \vee\{\vee\{\wedge\{\|a-b\| b \in B\} \mid a \in A\}, \vee\{\wedge\{\|a-b\| a \in A\} \mid b \in B\}\}$.

Denote by $\mathcal{K}^n$ the space of all nonempty compact sets of $\mathbb{R}^n$ endowed with this metric.

A set $K \in \mathcal{K}^n$ is said to be *starshaped* with respect to a point $x \in K$ if for each $y \in K$, the line segment $\overline{xy}$ joining $x$ to $y$ is contained in $K$. The *kernel* of $K$, denoted by $ker(K)$, is the set of all points $x \in K$ such that $\overline{xy} \subset K$ for each $y \in K$ and co$(K)$ is the convex hull of $K$.

**Definition 3.39** *Let $\mathcal{K}^n_{ST} = \{K \in \mathcal{K}^n \mid K$ is starshaped with respect to the origin$\}$. Define $g_K : S^{n-1} \to \mathbb{R}$ by $\forall x \in S^{n-1}$,*

$g_K(x) = \wedge\{\lambda \mid x \in \lambda K\}$.

*Then $g_K$ is called the* gauge *of $K$.*

In Definition 3.39, the gauge of $K$ is a nonnegative lower semicontinuous function.

Suppose that $K$ is convex. Then $g_K$ is the support function of $K$. Also if $K, L \in \mathcal{K}^n_{ST}$, then

$\vee\{g_K(x) - g_L(x) \mid x \in S^{-1}\} = \delta_\infty(K,L)$.

For other properties of starshaped sets and gauge functions defined on all of $\mathbb{R}^n$, the reader is referred to [9]. The other $L_p$ metrics are defined in terms of gauge functions

$$\delta_p(K,L) = \left( \int_{S^{n-1}} |g_K(x) - g_L(x)|^p \mu(dx) \right)^{1/p}, 1 \leq p < \infty,$$

where $\mu(\cdot)$ is unit Lebesgue measure.

Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^n$. Recall that $\tilde{A}$ is said to be normal if $\exists x \in \mathbb{R}^n$ such that $\tilde{A}(x) = 1$. Let $\mathcal{U}^n$ denote the set of all normal, upper semicontinuous, fuzzy subsets of $\mathbb{R}^n$.

**Definition 3.40** *Let $\tilde{A} \in \mathcal{U}^n$. Then $\tilde{A}$ is said to be* fuzzy starshaped *with respect to $y \in \mathbb{R}^n$ if $\forall\ x \in \mathbb{R}^n$ and $\forall \lambda \in [0,1]$, $\tilde{A}(\lambda(x-y)) \geq \tilde{A}(x-y)$. Let $\mathcal{S}^n = \{\tilde{A} \in \mathcal{U}^n | \exists y \in \mathbb{R}^n$ such that $\tilde{A}$ is starshaped with respect to $y\}$ and let $\mathcal{S}_0^n = \{\tilde{A} \in \mathcal{U}^n |\ \tilde{A}$ is starshaped with respect to the origin$\}$.*

**Proposition 3.42** *Let $\tilde{A} \in \mathcal{U}^n$ and let $y \in \mathbb{R}^n$. Then $\tilde{A}$ is fuzzy starshaped with respect to $y$ if and only if $\forall t \in [0,1]$, $\tilde{A}^t$ is starshaped with respect to $y$.*

**Proof.** We may prove the result for $y = 0$ without loss of generality. Suppose that $\tilde{A}$ is fuzzy starshaped with respect to 0. Let $t, \lambda \in [0,1]$ and let $x \in \tilde{A}^t$. Then $\tilde{A}(\lambda(x-0)) \geq \tilde{A}(x-0) \geq t$. Hence $\lambda x \in \tilde{A}^t$. Thus $\overline{0x}$ is contained in $\tilde{A}^t$. Conversely suppose that $\forall t \in [0,1]$, $\tilde{A}^t$ is starshaped with respect to 0. Let $x \in \mathbb{R}^n$ and let $t = \tilde{A}(x)$. Then $\lambda x \in \tilde{A}^t$. Thus $\tilde{A}(\lambda x) \geq \tilde{A}(x)$ and so $\tilde{A}$ is fuzzy starshaped at 0. ■

**Definition 3.41** **(i)** *Let $\tilde{A} \in \mathcal{U}^n$. Define $ker(\tilde{A}) = \{y \in \mathbb{R}^n \mid \tilde{A}$ is fuzzy starshaped with respect to $y\}$.*

**(ii)** *Let $\tilde{A} \in \mathcal{S}^n$. Define the fuzzy subset $fker(\tilde{A})$ of $\mathbb{R}^n$ by $(fker(\tilde{A}))^t = ker(\tilde{A}^t)\ \forall t \in [0,1]$.*

**Proposition 3.43** **(i)** *Let $\tilde{A} \in \mathcal{U}^n$. Then $ker(\tilde{A})$ is a convex set in $\mathbb{R}^n$.*

**(ii)** *Let $\tilde{A} \in \mathcal{S}^n$. Then $fker(\tilde{A})$ is a convex fuzzy subset in $\mathcal{U}^n$.* ■

**Proposition 3.44** *$\mathcal{S}^n$ has the structure of a real cone.* ■

**Proposition 3.45** *Suppose that $\tilde{A}$ and $\tilde{B} \in \mathcal{S}^n$ and that $ker(\tilde{A}) \cap ker(\tilde{B}) \neq \emptyset$. Then $\tilde{A} \cap \tilde{B}$ and $\tilde{A} \cup \tilde{B}$ are fuzzy starshaped.* ■

We now discuss metric properties of $\mathcal{S}_0^n$. Let $\tilde{A}, \tilde{B} \in \mathcal{S}_0^n$. Then $\forall t \in [0,1]$, $\delta_\infty(\tilde{A}^t, \tilde{B}^t) = \vee\{|g_{\tilde{A}^t}(x) - g_{\tilde{B}^t}(x)| \mid x \in S^{n-1}\}$. For each $1 \leq p \leq \infty$, define

$d_p(\tilde{A}, \tilde{B}) = (\int_0^1 \delta_\infty(\tilde{A}^t, \tilde{B}^t)^p dt)^{1/p}$

and

$d_\infty(\tilde{A}, \tilde{B}) = \vee\{\delta_\infty(\tilde{A}^t, \tilde{B}^t) \mid 0 \leq t \leq 1\}$.

The properties of gauge functions guarantee that $d_p$ is defined for all $\tilde{A}$ and $\tilde{B} \in \mathcal{S}_0^n$, [11]. Also $d_p \leq d_q$ for $p \leq q$ and

$d_\infty(\tilde{A}, \tilde{B}) = \lim_{p \to \infty} d_p(\tilde{A}, \tilde{B})$.

The interested reader may refer to [32] and [78] for further properties of metrics on convex fuzzy subsets.

Another class of metrics may be defined directly from $L_p$ metrics on gauge functions as follows: For each $\tilde{A} \in \mathcal{S}_0^n$, write $\tilde{A}^*(t,x) = g_{\tilde{A}^t}(x)$, $0 \leq t \leq 1$ and $x \in S^{n-1}$. This definition is similar to that for support functions of convex fuzzy subsets. Further details may be found in [31]. For $1 \leq p < \infty$, set

$\rho_p(\tilde{A}, \tilde{B}) = \{\int_0^1 dt \int_{S^{n-1}} |\tilde{A}^*(t,x) - \tilde{B}^*(t,x)|^p \mu(dx)\}^{1/p}$.

Again, properties of gauge functions imply that $\rho_p$ is well defined on $\mathcal{S}_0^n$. Also, $\rho_p \leq \rho_q$ for $1 \leq p \leq q < \infty$ and $\rho_p \leq d_p \leq d_\infty$ for $1 \leq p < \infty$.

**Theorem 3.46** *$(\mathcal{S}_0^n, d_p)$ and $(\mathcal{S}_0^n, \rho_p)$ are metric spaces.* ■

The next result is an extension of a result in [140] to $\mathcal{K}_{ST}^n$. It is fundamental to the relation between metrics $d_p$ and $\rho_p$. For example, it is used in the proof of Theorem 3.48 in showing the topological equivalence stated there.

**Theorem 3.47** *Let $K, L \in \mathcal{K}_{ST}^n$ with $D = diam(K \cup L)$. Then for $1 \leq p < \infty$,*

$D^{(n-1)/p} C_p{}'(K,L) \cdot \delta_\infty(K,L)^{(n+p-1)/p} \leq \delta_p(K,L)$,

*where*

$$C_p'(K,L)^p = B(p+1, n-1)/B(1/2, 1/2(n-1))$$

*and $B(\cdot,\cdot)$ is the beta function.* ■

We say that $U \subseteq \mathcal{S}_0^n$ is *uniformly support bounded* if the support sets $\tilde{A}^0$ are bounded in $\mathbb{R}^n$, uniformly for $\tilde{A} \in U$. A family of support functions $U^* = \{\tilde{A}^* \mid \tilde{A} \in U\}$ is said to be *equileftcontinuous* if the family is equileftcontinuous in $t$ for all $\tilde{A}^* \in U^*$ and $x \in S^{n-1}$. $U$ is said to be a *Blaschke set* if it is uniformly bounded and $U^*$ is equileftcontinuous. In [31], it is shown that a closed set in the space of normal convex fuzzy subsets, endowed with the $d_\infty$ metric, is compact if and only if it is Blaschke.

**Theorem 3.48** *For each given $p$, $1 \leq p < \infty$, $d_p$ and $\rho_p$ induce equivalent topologies on $\mathcal{S}_0^n$, each of which is complete and separable, and in which closed Blaschke sets are compact.* ■

Theorem 3.48 shows that the properties of metric spaces are essentially unaffected by the generalization to the fuzzy starshaped case.

We now look at compactness in $(\mathcal{S}_0^n, d_p)$. That a closed set be Blaschke is sufficient in the $d_p$ metric topology, but is too strong to be necessary because $d_p \leq d_\infty$. An appropriate analogue for the weaker topology in $\mathcal{E}^n$ involved an $L_p$ space type definition [32]. This also holds for fuzzy starshaped sets and for completeness. In [30], the concept of $U$ being $p$-Blaschke is defined. It is noted in [30] that

1. Every Blaschke set in $\mathcal{S}_0^n$ is $p$-Blaschke, but not conversely.
2. A closed set $U$ in $(\mathcal{S}_0^n, d_p)$, $1 \leq p < \infty$, is compact if and only if it is $p$-Blaschke.
3. A set $U$ in $(\mathcal{S}_0^n, d_p)$, $1 \leq p < \infty$, is locally compact if and only if every uniformly support bounded and closed subset of $U$ is $p$-Blaschke.
4. $(\mathcal{S}_0^n, d_p)$, $1 \leq p < \infty$, is a locally compact space.

The reader is encouraged to examine [30] for further details and discussion.

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# 4
# FUZZY DIGITAL TOPOLOGY

## 4.1 Introduction

Digital picture processing deals with image compression or enhancement and image analysis. In image compression or enhancement, the output desired is a picture which is an approximation or improvement of the input picture. In image analysis, the desired output is a description of the input picture. In digital pictures, geometric considerations of parts of the picture are important in analysis and description processes.

Various kinds of segmentation processes are used to extract from a digital picture geometric properties and relationships defined on subsets of it. However it is often preferable to extract fuzzy subsets rather than crisp subsets. In this chapter, we develop the topological concepts of connectedness, surroundedness, and convexity for fuzzy subsets. Further discussion of these concepts for the crisp case can be found in [8, Chapters 8, 9] and [3].

## 4.2 Crisp Digital Topology

In this section, we briefly review some of the basic concepts of crisp digital topology.

Let $\Sigma$ be a rectangular array of integer-coordinate points or *lattice points*. Thus the point $P \equiv (x, y)$ of $\Sigma$ has two horizontal neighbors and two vertical neighbors, namely $(x \pm 1, y)$ and $(x, y \pm 1)$ respectively. These are known as the 4-neighbors of $P$. Similarly, it is possible to define 8-neighbors of $P$ by including the four diagonal neighbors as well, namely, $(x \pm 1, y \pm 1)$ and $(x \pm 1, y \mp 1)$. If $P$ is on the border of $\Sigma$, then some of these neighbors may not exist.

Let $P, Q$ be points of $\Sigma$. Then a *path* $\rho$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_i$ is adjacent to $P_{i-1}, 1 \leq i \leq n$. Note that this is two definitions in one ("4-path" and "8-path"), depending on whether "adjacent" means "4-adjacent" or "8-adjacent". The same is true for the definitions and results that follow, unless otherwise specified.

Let $S$ be any subset of $\Sigma$. We say that the points $P, Q$ of $\Sigma$ are *connected* in $S$ if there is a path from $P$ to $Q$ consisting entirely of points of $S$. Then this notion of connectedness in $S$ defines an equivalence relation on $\Sigma : P$ is connected to $P$ (by a path of length 0); if $P$ is connected to $Q$, then $Q$ is connected to $P$ (the reversal of a path is a path); and if $P$ is connected to $Q$ and $Q$ to $R$, then $P$ is connected to $R$ (the concatenation of two paths is a path). The equivalence classes induced by this equivalence relation are maximal subsets $S_i$ of $S$ such that every $P, Q$ belonging to a given $S_i$ are connected. These classes are called the (*connected*) *components* of $S$.

Recall that $S^c$ denotes the complement of $S$. Recall also that a border point is one which does not have all its neighbors. For simplicity, we assume that the border points of $\Sigma$ are all in $S^c$. Thus one component of $S^c$ always contains the border $B$ of $\Sigma$. The other components, if any, are called *holes* in $S$. If $S$ has no holes, it is called *simply-connected.* As pointed out in [8, Section 9.1], opposite types of connectedness (4– and 8–, or 8– and 4–) should be used for $S$ and for $S^c$ in order for various algorithms to work properly.

Let $S$ and $T$ be disjoint subsets of $\Sigma$. We say that $S$ *surrounds* $T$ if any path from $T$ to the border of $\Sigma$ meets $S$. If $S$ surrounds $T$, then $T$ does not surround $S$; and if $S$ surrounds $T$ and $T$ surrounds $W$, then $S$ surrounds $W$. More generally, for any subsets $U, V, W$ of $\Sigma$, we say that $V$ *separates* $U$ from $W$ if any path from $U$ to $W$ meets $V$. Thus $S$ surrounds $T$ if it separates $T$ from the border $B$ of $\Sigma$.

The results of this chapter are based primarily on [2], [6], and [7].

## 4.3 Fuzzy Connectedness

In this section, we extend the concepts of the previous section to fuzzy sets. Let $\widetilde{A}$ be a fuzzy subset of $\Sigma$. Recall that the fuzzy subset $\widetilde{A}^c$ of $\Sigma$ is defined by $\forall P \in \Sigma$, $\widetilde{A}^c(P) = 1 - \widetilde{A}(P)$ and is called the complement of $\widetilde{A}$. We assume for all points $P$ on the border $B$ of $\Sigma$ that $\widetilde{A}(P) = 0$.

**Definition 4.1** *Let $\widetilde{A}$ be a fuzzy subset of $\Sigma$ and let $\rho : P = P_0, P_1, \ldots, P_n = Q$ be any path between two points $P$ and $Q$ of $\Sigma$. Define the* strength $s_{\widetilde{A}}(\rho)$ *of $\rho$ (with respect to $\widetilde{A}$) to be $\wedge\{\widetilde{A}(P_i) \mid 0 \leq i \leq n\}$. Define the* degree of connectedness $c_{\tilde{A}}(P,Q)$ *of* $P$ *and* $Q$ *(with respect to* $\tilde{A}$*) to be* $\vee\{s_{\tilde{A}}(\rho)|\rho$ *is a path from* $P$ *to* $Q\}$.

**Proposition 4.1** *For all* $P,Q$ *in* $\Sigma$, *we have* $c_{\tilde{A}}(P,P) = \tilde{A}(P)$ *and* $c_{\tilde{A}}(P,Q) = c_{\tilde{A}}(Q,P)$.

**Proof.** Since $P$ is on any path $\rho$ from $P$ to $P$, we have for any such path that $s_{\tilde{A}}(\rho) = \wedge\{\tilde{A}(P_i) \mid 0 \leq i \leq n\} \leq \tilde{A}(P)$ and so $c_{\tilde{A}}(P,P) \leq \tilde{A}(P)$. On the other hand, $P$ itself is a path of length 0 from $P$ to $P$, for which $s_{\tilde{A}}(\rho) = \tilde{A}(P)$. Hence $c_{\tilde{A}}(P,P) \geq \tilde{A}(P)$. Thus $\tilde{A}(P) = c_{\tilde{A}}(P,P)$. That $c_{\tilde{A}}(P,P) = c_{\tilde{A}}(P,P)$ follows from the fact that the reversal of a path is a path, and reversal preserves path strength. ■

Suppose that $\tilde{A}$ maps $\Sigma$ into $\{0,1\}$. Let $S = \tilde{A}^{-1}(1) \equiv \{P|P \in \Sigma$ and $\tilde{A}(P) = 1\}$. Then $s_{\tilde{A}}(\rho) = 1$ if and only if $\rho$ consists entirely of points of $S$, and $c_{\tilde{A}}(P,Q) = 1$ if and only if $P$ and $Q$ are connected in $S$. Thus degree of connectedness generalizes the ordinary (nonfuzzy) concept of connectedness. Note that in general, $c_{\tilde{A}}(P,Q) = 1$ if and only if there exists a path from $P$ to $Q$ all of whose points are mapped into 1 by $\tilde{A}$ (and in particular, $\tilde{A}(P) = \tilde{A}(Q) = 1$). For any set $T \subseteq \Sigma$, we can define the *degree of connectedness* of $T$ (with respect to $\tilde{A}$) as $c_{\tilde{A}}(T) \equiv \wedge\{c_{\tilde{A}}(P,Q) \mid P,Q \in T\}$.

**Proposition 4.2** *Let* $P,Q \in \Sigma$. *Then* $c_{\tilde{A}}(P,Q) \leq \tilde{A}(P) \wedge \tilde{A}(Q)$.

**Proof.** For any path $\rho : P = P_0, P_1, \ldots, P_n = Q$, we have $s_{\tilde{A}}(\rho) = \wedge\{\tilde{A}(P_i)|0 \leq i \leq n\} \leq \tilde{A}(P_0) \wedge \tilde{A}(P_n) = \tilde{A}(P) \wedge \tilde{A}(Q)$. Hence $\vee\{s_{\tilde{A}}(\rho)|\rho$ is a path from $P$ to $Q\} \leq \tilde{A}(P) \wedge \tilde{A}(Q)$. ■

**Corollary 4.3** *Let* $T$ *be a subset of* $\Sigma$. *Then* $c_{\tilde{A}}(T) \leq \wedge\{\tilde{A}(P)|P \in T\}$.

**Proof.** $c_{\tilde{A}}(T) = \wedge\{c_{\tilde{A}}(P,Q)|P,Q \in T\} \leq \wedge\{\tilde{A}(P) \wedge \tilde{A}(Q) \mid P,Q \in T\} = \wedge\{\tilde{A}(P)|P \in T\}$. ■

It also follows from Proposition 4.2 that $c_{\tilde{A}}$, regarded as a fuzzy relation on $\Sigma$, i.e., as a fuzzy subset of $\Sigma \times \Sigma$, is a fuzzy relation on the fuzzy subset $\tilde{A}$ in the sense defined in Chapter 1.

**Definition 4.2** *Let* $P,Q \in \Sigma$ *and let* $\tilde{A}$ *be a fuzzy subset of* $\Sigma$. *Then* $P$ *and* $Q$ *are said to be* connected *in* $\tilde{A}$ *if* $c_{\tilde{A}}(P,Q) = \tilde{A}(P) \wedge \tilde{A}(Q)$.

We have that $c_{\tilde{A}}(P,Q)$ in Definition 4.2 takes on its maximum possible value.

**Proposition 4.4** *Let* $P,Q \in \Sigma$ *and let* $\tilde{A}$ *be a fuzzy subset of* $\Sigma$. *Then* $P$ *and* $Q$ *are connected in* $\tilde{A}$ *if and only if there exists a path* $\rho' : P = P_0, P_1, \ldots, P_n = Q$ *such that* $\tilde{A}(P_i) \geq \tilde{A}(P) \wedge \tilde{A}(Q), 0 \leq i < n$.

**Proof.** If there exists such a path $\rho'$, we have $c_{\tilde{A}}(P,Q) = \vee\{s_{\tilde{A}}(\rho)|\rho$ is a path from $P$ to $Q\} \geq s_{\tilde{A}}(\rho') = \wedge\{\tilde{A}(P_i)|0 \leq i < n\} \geq \tilde{A}(P) \wedge \tilde{A}(Q)$ so that $c_{\tilde{A}}(P,Q) = \tilde{A}(P) \wedge \tilde{A}(Q)$ by Proposition 4.2. Conversely, suppose that $P$ and $Q$ are connected in $\tilde{A}$. Then there is a path $\rho' : P = P_0, P_1, \ldots, P_n = Q$ such that $s_{\tilde{A}}(\rho') = \vee\{s_{\tilde{A}}(\rho)|\rho$ is a path from $P$ to $Q\} = c_{\tilde{A}}(P,Q) = \tilde{A}(P) \wedge \tilde{A}(Q)$. Thus for all $P_i$ on $\rho'$, we have $\tilde{A}(P_i) \geq \wedge\{\tilde{A}(P_i)|0 \leq i \leq n\} = s_{\tilde{A}}(\rho')$. ■

If $\tilde{A}(P) = \tilde{A}(Q) = 1$, $P$ and $Q$ are connected if and only if there exists a path from $P$ to $Q$ such that, for any point $P'$ of $\tilde{A}$, we have $\tilde{A}(P') = 1$. Thus if $\tilde{A}$ maps $\Sigma$ into $\{0,1\}$, and $S = \tilde{A}^{-1}(1)$, then two points $P, Q$ of $S$ are connected in $\tilde{A}$ if and only if they are connected in $S$. It is the case, however, that points can be connected in $\tilde{A}$ without being connected in $S$. In fact, if $\tilde{A}(P) = 0, P$ is connected in $\tilde{A}$ to any $Q$, with degree of connectedness zero. Thus "connected in $\tilde{A}$" is a generalization of "connected in $S$" only in some respects, but not in others. In fact, $C_{\tilde{A}} \equiv \{(P,Q)|P,Q$ are connected in $\tilde{A}\}$ is not in general, an equivalence relation, as we now see.

For all $P$ in $\Sigma$, $c_{\tilde{A}}(P,P) = \tilde{A}(P) = \tilde{A}(P) \wedge \tilde{A}(P)$ and so $C_{\tilde{A}}$ is reflexive. That $C_{\tilde{A}}$ is symmetric is clear since $c_{\tilde{A}}$ is symmetric and $\tilde{A}(P) \wedge \tilde{A}(Q) = \tilde{A}(Q) \wedge \tilde{A}(P)$. Let $\Sigma$ be the 1-by-3 array $P, Q, R$ and let $\tilde{A}(P) = \tilde{A}(R) = 1, \tilde{A}(Q) < 1$. Then $(P,Q)$ and $(Q,R)$ are connected in $\tilde{A}$, but $P$ and $R$ are not. That is, $(P,Q), (Q,R) \in C_{\tilde{A}}$, but $(P,R) \notin C_{\tilde{A}}$. Hence $C_{\tilde{A}}$ is not necessarily transitive.

Nevertheless, $C_{\tilde{A}}$ is a useful relation on $\Sigma$, as we show in the next section. For any set $T \subseteq \Sigma$, we call $T$ *connected* with respect to $\tilde{A}$ if all $P, Q$ in $T$ are connected in $\tilde{A}$.

## 4.4 Fuzzy Components

Although $C_{\tilde{A}}$ is not an equivalence relation, we can still define a notion of "connected component" with respect to $\tilde{A}$. Our definition is based on the concept of a "plateau" in $\tilde{A}$. We will see that this definition has many properties in common with the standard one, even though the components do not constitute a partition. For example, let $\Sigma$ be the 1-by-3 array $P, Q, R$ and let $\tilde{A}(P) = \tilde{A}(R) = 1, \tilde{A}(Q) < 1$. Then $\{P,Q\}$ and $\{Q,R\}$ are components, but $\{P,Q\} \cap \{Q,R\} \neq \emptyset$. However it is worth noticing that $\{P\} \cap \{Q,R\} = \emptyset$ and $\{P,Q\} \cap \{R\} = \emptyset$.

**Definition 4.3** *Let $\tilde{A}$ be a fuzzy subset of $\Sigma$. A subset $\Pi$ of $\Sigma$ is called a* plateau *in $\tilde{A}$ if the following conditions hold:*

(i) $\Pi$ *is connected;*

**(ii)** *$\tilde{A}(P) = \tilde{A}(Q)$ for all $P, Q$ in $\Pi$;*

**(iii)** *$\tilde{A}(P) \neq \tilde{A}(Q)$ for all pairs of neighboring points $P \in \Pi, Q \notin \Pi$.*

We see that $\Pi$ in Definition 4.3 is a plateau of $\tilde{A}$ if and only if it is a maximal connected subset of $\Sigma$ on which $\tilde{A}$ has constant value. Clearly any $P \in \Sigma$ belongs to exactly one plateau.

**Definition 4.4** *Let $\tilde{A}$ be a fuzzy subset of $\Sigma$ and let $\Pi$ be a plateau of $\tilde{A}$. Then $\Pi$ is called a* top *if $\tilde{A}(P) > \tilde{A}(Q)$ for all pairs of neighboring points $P \in \Pi, Q \notin \Pi$; and $\Pi$ is called a* bottom *if $\tilde{A}(P) < \tilde{A}(Q)$ for all pairs of neighboring points $P \in \Pi, Q \notin \Pi$.*

We see that $\Pi$, in Definition 4.4 is a top if its $\tilde{A}$ value is a local maximum. Similarly, $\Pi$ is a bottom if its $\tilde{A}$ value is a local minimum.

**Example 4.1** *Consider the fuzzy subset $\tilde{A}$ of $\Sigma$, with all non-zero values as shown below.*

| | | | | | |
|---|---|---|---|---|---|
| 0.4 | 0.5 | 0.6 | 1.0 | 0.6 | 0.5 |
| 0.3 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
| 0.2 | 0.7 | 0.9 | 0.9 | 0.6 | 0.2 |
| 0.6 | 0.7 | 0.9 | 0.9 | 0.5 | 0.7 |
| 0.5 | 0.5 | 0.8 | 0.5 | 0.7 | 0.8 |
| 0.8 | 0.6 | 0.7 | 0.6 | 0.8 | 0.4 |

*For the sake of our discussion, let us assume we use the "4-neighbor" convention. Then we have six different tops as shown below.*

| | | | | | |
|---|---|---|---|---|---|
| | | | 1 | | |
| | 2 | | | | |
| | | 3 | 3 | | |
| | | 3 | 3 | | |
| | | | | | 4 |
| 5 | | | | 6 | |

*Note that two tops are not adjacent to each other. In particular, note that the top labeled 2 and the top labeled 3 are not adjacent.*

**Proposition 4.5** *Let $\tilde{A}$ be a fuzzy subset of $\Sigma$. Then $\Pi$ is a plateau in $\tilde{A}$ if and only if it is a plateau in $\tilde{A}^c$. $\Pi$ is a bottom in $\tilde{A}$ if and only if it is a top in $\tilde{A}^c$, and vice versa.* ■

In the crisp case, the plateaus are just the connected components of $S$ and of $S^c$. In fact, if $S \neq \emptyset$, the tops are just the components of $S$, and the bottoms are the components of $S^c$. Also, every plateau is either a top or a bottom. Thus we can regard tops and bottoms as generalizations of connected components. In the remainder of this section, $\Pi$ is a top, and we assume that the points $P \in \Pi$ have $\tilde{A}(P) > 0$,otherwise $\Pi = \Sigma$.

With any top $\Pi$ we associate the following three sets of points.

$A_\Pi = \{P \in \Sigma| \exists$ path $P = P_0, P_1, ..., P_n = Q \in \Pi$ such that $\tilde{A}(P_{i-1}) \leq \tilde{A}(P_i), 1 \leq i \leq n\}$.

$B_\Pi = \{P \in \Sigma| \exists$ path $P = P_0, P_1, ..., P_n = Q \in \Pi$ such that $\tilde{A}(P) \leq \tilde{A}(P_i) \leq \tilde{A}(Q), 1 \leq i \leq n\}$.

$C_\Pi = \{P \in \Sigma| \exists$ path $P = P_0, P_1, ..., P_n = Q \in \Pi$ such that $\tilde{A}(P) \leq \tilde{A}(P_i), 1 \leq i \leq n\}$.

The next result follows immediately from the definitions.

**Proposition 4.6** $\Pi \subseteq A_\Pi \subseteq B_\Pi \subseteq C_\Pi$. ■

We call a path $\rho : P_0, P_1, \ldots, P_n$ *monotonic* in $\tilde{A}$ if either $\tilde{A}(P_i) \geq \tilde{A}(P_{i-1})$ for $i = 1, 2, \ldots, n$ or $\tilde{A}(P_i) \leq \tilde{A}(P_{i-1})$ for $i = 1, 2, \ldots, n$. Now $P \in A_\Pi$ if and only if there is a monotonically nondecreasing path from $P$ to $\Pi$. Thus there cannot be a local minimum between $P$ and $\Pi$. Similarly, if $P$ is in $B_\Pi$, there cannot be a peak higher than $\Pi$ between $P$ and $\Pi$. The sets $B_\Pi$ and $C_\Pi$ need not be connected in the ordinary sense (though $C_\Pi$ is connected in the $\tilde{A}$ sense; see Theorem 4.8). However $A_\Pi$ is connected in the ordinary sense as we now show. Let $P, P' \in A_\Pi$. Then there exists paths $\rho : P = P_0, P_1, ..., P_n = Q \in \Pi$ and $\rho' : P' = P'_0, P'_1, ..., P'_m = Q' \in \Pi$ such that $\tilde{A}(P_{i-1}) \leq \tilde{A}(P_i), i = 1, 2, \ldots, n$ and $\tilde{A}(P'_{j-1}) \leq \tilde{A}(P'_j), j = 1, 2, \ldots, m$. Hence by definition of $A_\Pi, P_i, P'_j \in A_\Pi$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Now there exists a path $\sigma$ in $\Pi$ connecting $Q$ and $Q'$. Hence $\rho\sigma\rho'^{-1}$ is a path in $A_\Pi$ connecting $P$ and $P'$.

Note also that all points whose $\tilde{A}$ values are sufficiently low will be in $C_\Pi$, e.g., if $\tilde{A}(P) = 0, P$ is in $C_\Pi$ for all $\Pi$. On the other hand, as we show in Proposition 4.7, points whose values are higher than $\tilde{A}(\Pi)$ (the common $\tilde{A}$ value of the points in $\Pi$) cannot be in $C_\Pi$; indeed, if $P$ is in $C_\Pi$ and $\tilde{A}(P) \geq \tilde{A}(\Pi)$, then we have $P \in \Pi$.

The points adjacent to a top $\Pi$ are evidently in $A_\Pi$. Also two tops can never be adjacent to one another, for if they have the same height, they belong to the same top; if they have different heights, the shorter one cannot be a top.

**Example 4.2** . *Let $\Sigma$ be the 1-by-4 array $P, Q, R, S$ and let $\tilde{A}(P) = 1$, $\tilde{A}(Q) = 1/2, \tilde{A}(R) = 3/4$, and $\tilde{A}(S) = 1/4$. Then $\Pi = \{P\}$ is a top, as is $\{R\}$. $A_\Pi = \{P, Q\}$ and $B_\Pi = \{P, Q, S\} = C_\Pi$.*

**Example 4.3** *Now let $\Sigma$ be the 1-by-4 array $P, Q, R, S$ and let $\tilde{A}(P) = 1/2$, $\tilde{A}(Q) = 1, \tilde{A}(R) = 1/2$, and $\tilde{A}(S) = 3/4$. Then $\Pi = \{S\}$ is a top. $A_\Pi = \{R, S\} = B_\Pi$ and $C_\Pi = \{P, R, S, \}$.*

**Example 4.4** *Let $\Sigma$ and $\tilde{A}$ be as in Example 4.1. Let $\Pi$ denote the top labeled 3. Then $A_\Pi, B_\Pi$ and $C_\Pi$ are from left to right as follows:*

| | | | | | |
|---|---|---|---|---|---|
| * | * | * | | | |
| * | | * | * | * | * |
| * | * | * | * | * | * |
| * | * | * | * | * | |
| * | * | * | * | | |
| | * | * | * | | |

| | | | | | |
|---|---|---|---|---|---|
| * | * | * | | | * |
| * | | * | * | * | * |
| * | * | * | * | * | * |
| * | * | * | * | * | |
| * | * | * | * | | |
| | * | * | * | | * |

| | | | | | |
|---|---|---|---|---|---|
| * | * | * | | * | * |
| * | | * | * | * | * |
| * | * | * | * | * | * |
| * | * | * | * | * | |
| * | * | * | * | | |
| | * | * | * | | * |

*We see that $A_\Pi \neq B_\Pi \neq C_\Pi \neq A_\Pi$.*

**Proposition 4.7** *If $P \in C_\Pi$ and $P \notin \Pi$, then $\tilde{A}(P) < \tilde{A}(\Pi)$.*

**Proof.** Suppose that $\tilde{A}(P) \geq \tilde{A}(\Pi)$. Then $P \in C_\Pi$ and so there exists a path $\rho$ from $P$ to $\Pi$ such that for all $P_i$ on $\rho$, $\tilde{A}(P_i) \geq \tilde{A}(P) \geq \tilde{A}(\Pi)$. However, if $P \notin \Pi$, $\rho$ must pass through a point $Q$ that is adjacent to $\Pi$, but not in $\Pi$. For any such $Q$, we have $\tilde{A}(Q) < \tilde{A}(\Pi)$ since $\Pi$ is a top, a contradiction. ∎

**Theorem 4.8** *$C_\Pi$ is the set of all points of $\Sigma$ that are connected to points of $\Pi$.*

**Proof.** Let $Q \in \Pi$ and let $P$ be connected to $Q$. Then by Proposition 4.4, there exists a path $\rho$ from $P$ to $Q$ such that for all $P_i$ on $\rho$, $\tilde{A}(P_i) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. Suppose that $\tilde{A}(P) > \tilde{A}(Q)$. Then $P \notin \Pi$ and $\tilde{A}(P_i) \geq \tilde{A}(Q)$ for all $P_i$ on $\rho$. However, by the proof of Proposition 4.7, this is impossible since $\rho$ must pass through a point $Q'$ adjacent to $\Pi$, but not in $\Pi$, and for such a point we must have $\tilde{A}(Q') < \tilde{A}(\Pi)$. Hence $\tilde{A}(P) \leq \tilde{A}(Q)$ and $\tilde{A}(P_i) \geq \tilde{A}(P)$ for all $P_i$ on $\rho$ so that $P \in C_\Pi$.

Conversely, suppose that $P \in C_\Pi$. Then $\tilde{A}(P) \leq \tilde{A}(\Pi)$ by Proposition 4.7. Hence there exists a path $\rho$ from $P$ to a point $Q$ of $\Pi$ such that for all $P_i$ on $\rho$, $\tilde{A}(P_i) \geq \tilde{A}(P) = \tilde{A}(P) \wedge \tilde{A}(Q)$. Thus $P$ connected to $Q$ by Proposition 4.4. ∎

Since $A_\Pi \subseteq B_\Pi \subseteq C_\Pi$, it follows that every point in $A_\Pi(B_\Pi)$ is connected to points of $\Pi$.

**Theorem 4.9** *For any $P \in \Sigma$, there exists a top $\Pi$ such that $P \in A_\Pi$.*

**Proof.** Let $P$ be in the plateau $\Pi_0$. If $\Pi_0$ is a top, we have $P \in \Pi_0 \subseteq A_{\Pi_0}$ and we are done. If $\Pi_0$ is not a top, let $P_1$ be a neighbor of $\Pi_0$ such that $\tilde{A}(P_1) > \tilde{A}(P_0)$, where $P_0 = P$. Then we have a monotonically nondecreasing path from $P_0$ to $P_1$ (going through $\Pi_0$ up to a neighbor of $P_1$). Repeat this argument with $P_1$ replacing $P$, and continue in this way to obtain $P_2, P_3, \ldots$. This process must terminate, say at $P_n$, since $\Sigma$ is finite.

Then $\Pi_n$.the plateau containing $P_n$, is a top, and we have a monotonic nondecreasing path from $P$ to $P_n$. Hence $P \in A_{\Pi_n}$. ■

We see from Theorem 4.9 that if $\Pi$ is a unique top, then $A_\Pi = \Sigma$.

**Theorem 4.10** *For any two distinct tops* $\Pi, \Pi'$, *we have* $\Pi' \cap C_\Pi = \emptyset$.

**Proof.** Suppose that $P \in \Pi' \cap C_\Pi$. Then there exists a path $\rho$ from $P$ to $\Pi$ such that for any point $P_i$ on $\rho$, $\tilde{A}(P) \leq \tilde{A}(P_i)$. However for a point $P_i$ adjacent to $\Pi'$, but not in $\Pi$, we must have $\tilde{A}(P_i) < \tilde{A}(\Pi') = \tilde{A}(P)$, a contradiction. ■

Theorems 4.9 and 4.10 show that the tops $\Pi'$ and their connected "components" $C_\Pi$ have partition-like properties: Every point belongs to some top in a strong sense ($\Sigma = \cup A_\Pi$, where the union is taken over all tops), and a fortiori in a weak sense ($A_\Pi \subseteq C_\Pi$); but no top can belong to another top even in a weak sense ($\Pi' \cap C_\Pi = \emptyset$). These remarks are further supported by the following.

**Theorem 4.11** *Let* $P, Q \in \Sigma$. *Then* $P$ *and* $Q$ *are connected if and only if there exists a top* $\Pi$ *such that* $P$ *and* $Q \in C_\Pi$.

**Proof.** Suppose that $P, Q$ are in $C_\Pi$. Then there are paths $\rho_1, \rho_2$ from $P$ and $Q$, respectively, to $\Pi$ such that for all $P_i$ on $\rho_1$, $\tilde{A}(P_i) \geq \tilde{A}(P)$ and for all $Q_i$ on $\rho_2$, $\tilde{A}(Q_i) \geq \tilde{A}(Q)$. Thus $\rho_1 {\rho_2}^{-1}$ is a path from $P$ to $Q$ such that for all points $R$ on this path, $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. Hence $P$ is connected to $Q$ by Proposition 4.4.

Conversely, suppose that $P$ and $Q$ be connected. Let $\tilde{A}(P) \leq \tilde{A}(Q)$ (say). By Theorem 4.9, there exists a monotonic nondecreasing path $\rho'$ from $Q$ to some top $\Pi$. Then $Q \in A_\Pi \subseteq C_\Pi$. On the other hand, by Proposition 4.7, there is a path $\rho$ from $P$ to $Q$ such that for all $P_i$ on $\rho$, $\tilde{A}(P_i) \geq \tilde{A}(P) \wedge \tilde{A}(Q) = \tilde{A}(P)$. Now for all $Q_i$ on $\rho'$, we have $\tilde{A}(Q_i) \geq \tilde{A}(Q) \geq \tilde{A}(P)$. Thus the path $\rho\rho'$ from $P$ to $\Pi$ guarantees that $P \in C_\Pi$. ■

The following result is an immediate consequence of Theorems 4.10 and 4.11.

**Corollary 4.12** $\Sigma$ *is connected with respect to* $\tilde{A}$ *if and only if there exists a unique top in* $\tilde{A}$. ■

Since bottoms are tops with respect to $\tilde{A}^c$, results for bottoms hold which are analogous to those for tops. In particular, the connected component of points having $\tilde{A} = 0$ that contains the border of $\Sigma$ is a bottom, which we can think of as the "background component" of $\tilde{A}^c$; while all other bottoms can be regarded as "holes in $\tilde{A}$". If $\tilde{A}$ has no holes, we call it *simply-connected.*

For any top $\Pi$, we can define a fuzzy subset $\tilde{A}_\Pi$ of $\Sigma$ defined by

$$\tilde{A}_\Pi(P) = \begin{cases} \tilde{A}(P)/\tilde{A}(\Pi), & \text{if } P \in C_\Pi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by Proposition 4.7, $\widetilde{A}_{\Pi}(P) = 1$ if and only if $P \in \Pi$.

An alternative method of defining membership in a component can be found in [6].

## 4.5 Fuzzy Surroundedness

In this section, we deal with the concept of fuzzy surroundedness. Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy subsets of $\Sigma$. We say that $\widetilde{B}$ *separates* $\widetilde{A}$ from $\widetilde{C}$ if for all points $P, R$ in $\Sigma$, and all paths $\rho$ from $P$ to $R$, there exists a point $Q$ on $\rho$ such that $\widetilde{B}(Q) \geq \widetilde{A}(P) \wedge \widetilde{C}(R)$. In particular, we say that $\widetilde{B}$ *surrounds* $\widetilde{A}$ if it separates $\widetilde{A}$ from the border of $\Sigma$. Since the border $\tilde{C}$ of $\Sigma$ is a nonfuzzy subset, we have

$$\widetilde{C}(R) = \begin{cases} 1 & \text{if } R \text{ is in the border of } \Sigma \\ 0 & \text{if } R \text{ is not in the border of } \Sigma \end{cases}$$

Thus the definition of surroundedness reduces to the following statement. For all $P \in \Sigma$ and all paths $\rho$ from $P$ to the border, there exists a point $Q$ on $\rho$ such that $\widetilde{B}(Q) \geq \widetilde{A}(P)$ since $R$ is in the border of $\Sigma$.

If $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are ordinary subsets these definitions reduce to the ordinary ones given in Section 4.2. Indeed, we need only consider the case where $P \in \tilde{A}$ and $R \in \tilde{C}$, since otherwise the minimum is zero. The definition of separatedness thus reduces to: $\tilde{B}$ separates $\tilde{A}$ from $\tilde{C}$ if for all $P \in \tilde{A}$ and $R \in \tilde{C}$ and all paths $\rho$ from $P$ to $R$, there exists a point $Q$ on $\rho$ such that $Q \in \tilde{B}$.

In Section 4.2 we defined "surrounds" only for disjoint sets and pointed out that it is antisymmetric and transitive. For nondisjoint sets, the situation is more complicated since two sets can surround one another without being the same. We illustrate this in the following example.

**Example 4.5** *Consider the* $4 \times 4$ *array given below.*

$$\begin{matrix} a & a & a & a \\ a & b & b & a \\ a & c & c & a \\ a & a & a & a \end{matrix}$$

*Let* $b \in S, c \in T$, *and* $a \in S \cap T$. *Then* $S$ *and* $T$ *surround each other. However, it can be shown that if* $S$ *and* $T$ *surround each other, then* $S \cap T$ *must surround both of them which is impossible for disjoint nonempty sets, since the empty set can only surround itself.*

Analogously, in the fuzzy case we can prove the next result.

**Theorem 4.13** *"Surrounds" is a weak partial order relation. That is, for all fuzzy subsets* $\widetilde{A}, \widetilde{B}, \widetilde{C}$ *of* $\Sigma$ *the following properties hold.*

(i) *Reflexivity:* $\tilde{A}$ *surrounds* $\tilde{A}$.

(ii) *Antisymmetry: If* $\tilde{A}$ *and* $\tilde{B}$ *surround each other, then* $\tilde{A} \cap \tilde{B}$ *surrounds both of them.*

(iii) *Transitivity: If* $\tilde{A}$ *surrounds* $\tilde{B}$ *and* $\tilde{B}$ *surrounds* $\tilde{C}$, *then* $\tilde{A}$ *surrounds* $\tilde{C}$.

**Proof.** (i) Take $Q = P$.

(ii) Let $\rho$ be any path from $P$ to the border and let $Q$ be the last point on $\rho$ such that $\tilde{B}(Q) \geq \tilde{A}(P)$. Since $\tilde{A}$ surrounds $\tilde{B}$, there is a point $Q'$ on $\rho$ beyond $Q$ (or equal to $Q$) such that $\tilde{A}(Q') \geq \tilde{B}(Q)$. Since $\tilde{B}$ surrounds $\tilde{A}$, there is a point $Q''$ on $\rho$ beyond (or equal to) $Q'$ such that $\tilde{B}(Q'') \geq \tilde{A}(Q') \geq \tilde{A}(P)$. By our choice of $Q$, this implies that $Q = Q' = Q''$ so that $\tilde{A}(Q) \cap \tilde{B}(Q) \geq \tilde{A}(P)$. Since $P$ was arbitrary, we have thus proved that $\tilde{A} \cap \tilde{B}$ surrounds $\tilde{A}$. Similarly, $\tilde{A} \cap \tilde{B}$ surrounds $\tilde{B}$.

(iii) Given any $P \in \Sigma$ and any path $\rho$ from $P$ to the border $B$, there is a point $Q$ on $\rho$ such that $\tilde{B}(Q) \geq \tilde{C}(P)$ since $\tilde{B}$ surrounds $\tilde{C}$. Moreover, on the part of $\rho$ between $Q$ and the border there is a point $R$ such that $\tilde{A}(R) \geq \tilde{B}(Q)$ since $\tilde{A}$ surrounds $\tilde{B}$. ■

Recall that for any fuzzy subset $\tilde{A}$ of $\Sigma$ and any $0 \leq t \leq 1$, the level set $\tilde{A}^t = \{P \in \Sigma | \tilde{A}(P) \geq t\}$.

**Proposition 4.14** *If* $\tilde{A}$ *surrounds* $\tilde{B}$, *then for any* $t, \tilde{A}^t$ *surrounds* $\tilde{B}^t$. ■

## 4.6 Components, Holes, and Surroundedness

In ordinary digital topology, if a component of $S$ and a component of $S^c$ are adjacent, then one of them surrounds the other. This is not true about the tops and bottoms of a fuzzy subset as illustrated in the following example.

**Example 4.6** *Consider the following* $4 \times 4$ *array.*

| | | | |
|---|---|---|---|
| 0.4 | 0.5 | 1.0 | 0.7 |
| 1.0 | 0.0 | 0.0 | 0.8 |
| 0.4 | 0.0 | 0.0 | 1.0 |
| 0.6 | 1.0 | 0.4 | 0.6 |

*In the above array of membership values, the* 1*'s are all adjacent to the* 0*'s, but the* 0*'s are not surrounded by any one of these components.*

Nevertheless, we can establish some relationships between surroundedness for tops or bottoms and surroundedness for the corresponding components.

**Theorem 4.15** *Let $\Pi$ be a top, $\Pi'$ a bottom, and let $\widetilde{A}_\Pi$ surround $\widetilde{A}_{\Pi'}$. Then $\Pi$ surrounds $A_{\Pi'} \supseteq \Pi'$, while outside $\Pi$ we have $C_\Pi \cap C_{\Pi'} = \emptyset$.*

**Proof.** By Proposition 4.14, if $\widetilde{A}_\Pi$ surrounds $\widetilde{A}_{\Pi'}$, then $\Pi$ must surround $\Pi'$ since $\Pi = (\widetilde{A}_\Pi)^1$ is just the set of points for which $\widetilde{A}_\Pi$ has value 1, and similarly for $\Pi'$. Moreover, $\Pi$ must even surround $A_{\Pi'}$ since we cannot have a monotonic path from a point outside $\Pi$ to a point (of $\Pi'$) inside $\Pi$ (the path must go both up and down when it enters and leaves $\Pi$). On the other hand, suppose that $\Pi$ is a top and $\Pi'$ is a bottom (or vice versa), that $P$ is outside $\Pi$, and that $P$ is in both $C_\Pi$ and $C_{\Pi'}$. Then we have $\widetilde{A}(\Pi') < \widetilde{A}(P) < \widetilde{A}(\Pi)$ and there is a path from $P$ to $\Pi'$ that has membership $\tilde{A}$ values below $\widetilde{A}(P)$. However this is impossible since the path must cross $\Pi$. ■

**Corollary 4.16** *If $\Pi$ is simply-connected, $\widetilde{A}_\Pi$ cannot surround any $\widetilde{A}_{\Pi'}$.* ■

We have seen in the proof of Theorem 4.15 that if $\Pi$ and $\Pi'$ are tops, and $\Pi$ surrounds $\Pi'$, it also surrounds $A_{\Pi'}$.

Assume that $\Pi$ and $\Pi'$ are tops and $A_\Pi$ surrounds $\Pi'$. Suppose $P \in A_{\Pi'}$ is not surrounded by $A_\Pi$ so that $P \notin A_\Pi$. Let $\rho$ be a monotonic path from $P$ to $\Pi'$. Then $\rho$ meets $A_\Pi$ since otherwise we could get from $\Pi'$ to $B$ (first using $\rho^{-1}$ to get to $P$) without crossing $A_\Pi$. Let $\rho$ meet $A_\Pi$ at the point $Q$. Then there is a monotonic path from $P$ to $\Pi$ (use $\rho$ up to $Q$, then take a monotonic path from $Q$ to $\Pi$) so that $P \in A_\Pi$, a contradiction. Thus we have the following result.

**Theorem 4.17** *If $\Pi$ and $\Pi'$ are tops and $A_\Pi$ surrounds $\Pi'$, then it also surrounds $A_{\Pi'}$.* ■

**Theorem 4.18** *If a point $P$ is surrounded by a union $\bigcup \Pi_i$ of tops, it is surrounded by one of them.*

**Proof.** If $P$ is in one of the $\Pi_i$, then that $\Pi_i$ surrounds it. Hence we may assume that $P$ is not in any of the $\Pi_i$. Each $\Pi_i$ is a connected set and $P$ is contained in its complement $\Pi_i^c$. This complement consists of a background component (containing the border $B$ of $\Sigma$) and possibly other components which are holes in $\Pi_i$. If $P$ is contained in a hole, then $\Pi_i$ surrrounds it and we are done. Otherwise, $P$ is in the background component of $\Pi_i^c$. If a path $\rho$ from $P$ to $B$ meets $\Pi_i$, we can divert $\rho$ to pass through points adjacent to $\Pi_i$; and none of these points can be in any other $\Pi_j$ by the remarks following Proposition 4.6. Hence points in $\Pi_i$ can be eliminated from $\rho$, and this is true for any $i$ so that we can find a $\rho$ that does not meet any of the $\Pi_i$ s, contradicting the assumption that $\bigcup \Pi_i$ surrounds $P$. ■

**Theorem 4.19** *If a connected set is surrounded by a union of tops, it is surrounded by one of them.*

**Proof.** Suppose that a connected set $C$ is surrounded by $\bigcup \Pi_i$. where each $\Pi_i$ is a top. We can assume that without loss of generality that no two $\Pi_i$'s surround one another. Suppose $C$ meets more than one of the $\Pi_i$'s, say $\Pi_j$ and $\Pi_k$. Since $\Pi_k$ is in the background component of $\Pi_j^c$, and $C$ is connected, there must exist a point $Q \in C$ adjacent to $\Pi_j$ and in the background component of $\Pi_j^c$. $Q$ is not in any $\Pi_i$ since tops cannot be adjacent. Moreover, since no $\Pi_i$, distinct from $\Pi_j$, surrounds $\Pi_j$, there is a path from $Q$ to $B$ (through $\Pi_j$) that does not meet any $\Pi_i \neq \Pi_j$. Thus no $\Pi_i$ surrounds $Q$, and neither does $\Pi_j$. It follows by Theorem 4.18 that $\bigcup \Pi_i$ does not surround $Q$, contradicting the fact that $Q \in C$. Thus $C$ can meet at most one of the $\Pi_i$'s say $\Pi_j$; and by the argument just given, no point of $C$ can be in the background component of $\Pi_j^c$ (since there would then be such a point $Q$ adjacent to $\Pi_j$, which would lead to the same contradiction). Hence $C$ is contained in the union of $\Pi_j$ and its holes so that $\Pi_j$ surrounds it. ■

In particular, if a top or bottom is surrounded by a union of tops (or bottoms), it is surrounded by one of them. On the other hand, a union of tops and bottoms can nontrivially surround a point (without it being surrounded by any one of them) since tops and bottoms can be mutually adjacent.

## *Component Counting; The Genus*

Define the *number of components* of $\tilde{A}$ as the number of its tops. It is possible to design a "one-pass" algorithm that counts these tops. The central idea is to scan $\Sigma$ row by row and assign distinct labels to each plateau $\Pi$. We also note whether or not the neighbors of each plateau have higher or lower $\tilde{A}$ values. Once the scan is completed, we determine all the equivalence classes of neighbors that were found to belong to the same plateau. If all the labels in a given class had only neighbors with lower $\tilde{A}$ values, the corresponding plateau is a top; and similarly for bottoms.

The *genus* of $\tilde{A}$ is defined as the number of its tops minus the number of its bottoms, excluding the border of $\Sigma$. Since the tops and bottoms can be computed in a single pass, by counting both the tops and the bottoms, it is possible to compute the genus also in one pass.

## *Application to Digital Image Processing*

We now discuss the practicality of the results developed in this chapter to digital picture segmentation. The discussion is taken from [6]. Let $f$ be a digital picture defined on the array $\Sigma$. First, normalize the grayscale of

$f$ to the interval $[0,1]$. Thus $f$ defines a fuzzy subset $\tilde{A}_f$ of $\Sigma$, where the membership of a point $P \in \Sigma$ in $\tilde{A}_f$ is given by $f(P)$. If $f$ contains dark objects on a light background, or vice versa, it is reasonable to segment it by thresholding due to the fact that the objects become connected components of above-threshold points. However, if we want to be more flexible in terms of thresholding, we can try to segment peaks in $\tilde{A}_f$. Observe that for any top $\Pi$, there exists a threshold namely, $\tilde{A}_f(\Pi)$, which yields exactly $\Pi$ as a connected component of above-threshold points. Moreover, $P$ is in $C_\Pi$ if and only if thresholding at $\tilde{A}_f(P)$ puts $P$ into the same connected component as $\Pi$. Thus the theory of fuzzy components is a generalized theory of "thresholdable connected objects" in digital pictures that does not require choosing a specific threshold.

If the objects in $f$ have smooth profiles, so that each object contains only one top, we can count objects by simply counting tops, as described in Section 4.5. If $f$ is noisy, there will be many "local tops" that do not correspond to significant objects; but such tops would presumably be "dominated by" other tops (e.g., we might say that $\Pi$ dominates $\Pi'$ if $A_\Pi$ surrounds $A_{\Pi'}$; see Theorem 4.17), or would be small and could be discarded on grounds of size.

## 4.7 Convexity

Let $E$ be the Euclidean plane and let $\tilde{A}$ be a fuzzy subset of $E$. We say that $\tilde{A}$ is *convex*, if for all $P, Q$ in $E$ and all $R$ on the line segment $\overline{PQ}$, we have $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. Note that if $\tilde{A}$ maps $E$ into $\{0,1\}$, the condition $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$ requires that any point on the segment $\overline{PQ}$ also be in $\tilde{A}$, which is the standard definition of convexity.

A real-valued function $f$ defined on $\mathbb{R}$ is called *min-free* if, for all points $A \leq B \leq C$ in $\mathbb{R}$, we have $f(B) \geq f(A) \wedge f(C)$. Then a fuzzy subset $\tilde{A}$ is convex if and only if all its cross-sections are min-free functions, where a cross section of $\tilde{A}$ by a line $l$ is the restriction of $\tilde{A}$ to $l$. In Section 4.9, we determine when the projections of convex fuzzy subsets are min-free functions. Note that a fuzzy subset of the real line is convex if and only if, regarded as a real-valued function, it is min-free.

**Proposition 4.20** *$\tilde{A}$ is convex if and only if its level sets are all convex.*

**Proof.** Suppose that $\tilde{A}$ is convex. Let $t \in [0,1]$ and $P, Q \in \tilde{A}^t$. Then $\forall R \in \overline{PQ}$, $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q) \geq t$ and so $R \in \tilde{A}^t$. Thus $\tilde{A}^t$ convex. Conversely, suppose $\tilde{A}^t$ is convex $\forall t \in [0,1]$. Let $P, Q \in E$ and $t = \tilde{A}(P) \wedge \tilde{A}(Q)$. Then $P, Q \in \tilde{A}^t$ and $\forall R \in \overline{PQ}$, $R \in \tilde{A}^t$. Hence $\tilde{A}(R) \geq t = \tilde{A}(P) \wedge \tilde{A}(Q)$. Thus $\tilde{A}$ is convex. ∎

A similar argument shows that Proposition 4.20 is also true if we define "level set" using $>$ rather than $\geq$.

**Example 4.7** *Consider the fuzzy subsets $\tilde{A}, \tilde{B}$ and $\tilde{C}$ of $E$ defined as follows: $\forall x, y \in E, \tilde{A}(x,y) = 1$ if $|x| + |y| \leq 1, \tilde{A}(x,y) = t$ if $|x| + |y| = 1/t$, where $t \in (0,1]$; $\tilde{B}(x,y) = 1$ if $x^2 + y^2 \leq 1, \tilde{B}(x,y) = t$ if $x^2 + y^2 = 1/t$, where $t \in (0,1]$; $\tilde{C} = 1 \wedge |x| \wedge |y|$. It may be noted that $\tilde{A}$ and $\tilde{B}$ are convex. To verify that $\tilde{A}$ is convex, we show that $\tilde{A}^t$ is convex $\forall t \in [0,1]$. Now $\tilde{A}^1 = \{(x,y) \mid |x| + |y| \leq 1\}$, $\tilde{A}^t = \{(x,y) \mid |x| + |y| \leq 1/t\}, t \in (0,1)$, and $\tilde{A}^0 = E$. Clearly, then $\tilde{A}^t$ is convex. In a similar fashion, $\tilde{B}$ can be shown to be convex. However, $\tilde{C}$ is not convex.*

## 4.8 The Sup Projection

For any line $l$ and any point $P \in l$, let $l_P$ be the line perpendicular to $l$ at $P$. By the *sup projection* of a fuzzy subset $\tilde{A}$ of $E$ on $l$ we mean the function $\tilde{A}_l$ such that $\forall P \in l, \tilde{A}_l(P) = \vee\{\tilde{A}(R) | R \in l_P\}$. Evidently $\tilde{A}_l$ is a fuzzy subset of $l$ since $0 \leq \tilde{A}_l(P) \leq 1, P \in l$. Also, for all $l$, $\tilde{A}_l$ can be considered as fuzzy subset of $E$ by defining $\tilde{A}_l(P) = 0$ for $P \in l^c$. It is easily seen that if $\tilde{A}$, considered as a crisp subset of $E$, is connected, then $\tilde{A}_l$ is an interval. Indeed, let $\tilde{A}_l(P) = \tilde{A}_l(Q) = 1$. Then there exist points on $l_P$ and $l_Q$ for which $\tilde{A} = 1$. Since $\tilde{A}$ is connected, these points are joined by a path consisting of points for which $\tilde{A} = 1$, and the projection of this path on $l$ contains the interval $\overline{PQ}$.

**Proposition 4.21** *If $\tilde{A}$ is convex, so is $\tilde{A}_l$.*

**Proof.** Let $A, B, C$ (in that order) be points of $l$. Let $\epsilon > 0$. Then $\exists$points $A'$ and $C'$ on $l_A$ and $l_C$, respectively, such that $\tilde{A}_l(A) < \tilde{A}(A') + \epsilon$ and $\tilde{A}_l(C) < \tilde{A}(C') + \epsilon$. Let $B'$ be the intersection of segment $\overline{A'C'}$ with $l_B$. Since $\tilde{A}$ is convex and $B' \in \overline{A'C'}$, we have $\tilde{A}(B') \geq \tilde{A}(A') \wedge \tilde{A}(C') > [\tilde{A}_l(A) - \epsilon] \wedge [\tilde{A}_l(C) - \epsilon] = [\tilde{A}_l(A) \wedge \tilde{A}_l(C)] - \epsilon$. But $\tilde{A}(B') \leq \tilde{A}(B)$ by definition of the sup projection. Hence $\tilde{A}_l(B) \geq [\tilde{A}_l(A) \wedge \tilde{A}_l(C)] - \epsilon$, and since $\epsilon$ is arbitrary, we have $\tilde{A}_l(B) \geq \tilde{A}_l(A) \wedge \tilde{A}_l(C)$. Thus $\tilde{A}_l$ is convex. ■

The converse of Proposition 4.21 is false; even if all the sup projections of $\tilde{A}$ are convex, $\tilde{A}$ need not be convex. To see this, let $\tilde{A}$ be an ordinary set and suppose that $\tilde{A}$ is connected. By the remarks preceding Proposition 4.21, the sup projection of $\tilde{A}$ on any $l$ is an interval and hence is convex, but $\tilde{A}$ itself need not be convex.

## 4.9 The Integral Projection

By the *integral projection* of $\tilde{A}$ on $l$ we mean the function $\overline{\tilde{A}_l}$ that maps each point $P \in l$ into $\int_{l_P} \tilde{A}$, the integral of $\tilde{A}$ over the line $l_P$ perpendicular to $l$ at $P$. If $\tilde{A}$ is an ordinary convex set, $l_P$ meets $\tilde{A}$ in an interval and

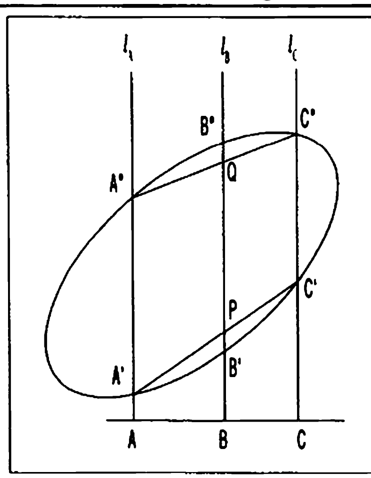
$\int_{l_{l'}} \tilde{A}$ is just the length of this interval. We assume here that this integral always exists. Note that we no longer have $0 \leq \overline{\tilde{A}_l} \leq 1$ as we did in the case of the sup projection.

**Example 4.8** *Consider the convex fuzzy subset $\tilde{B}$ defined in Example 4.7. Let $l$ be the $x$-axis. Then* $\overline{\tilde{B}_l}(x,0) = \int_{-\infty}^{\infty} \tilde{B}(x,y)dy$

$$= 2\left(\int_0^1 \tilde{B}(x,y)dy + \int_1^{\infty} \tilde{B}(x,y)dy\right)$$
$$= 2\left(\int_0^1 1dy + \int_1^{\infty}(x^2+y^2)^{-1}dy\right)$$
$$= 2\left(1 + \left[\frac{\tan^{-1}(\frac{y}{x})}{x}\right]_{y=1}^{y=\infty}\right)$$
$$= 2\left(1 + \frac{\pi}{2x} - \frac{\tan^{-1}(\frac{1}{x})}{x}\right).$$

**Proposition 4.22** *If $\tilde{A}$, considered as a crisp subset of $E$, is convex, then $\overline{\tilde{A}_l}$ is a min-free function.*

**FIGURE 4.1** A convex set and intersecting lines[1] .



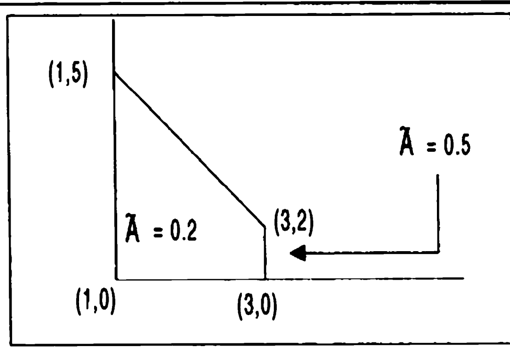

**Proof.** Let $A, B, C$ be points of $l$ with $B$ on the line segment $\overline{AC}$. Each of the lines $l_A, l_B, l_C$ meets the convex set $\tilde{A}$ in an interval (possibly degenerate or empty). Let the end-points of these intervals be $A'$, $A''$, $B'$, $B''$, and $C'$, $C''$, respectively (see Figure 4.1). Since $\tilde{A}$ is convex, the segments $\overline{A'C'}$ and $\overline{A''C''}$ are subsets of $\tilde{A}$. Hence the points $P, Q$, where these segments meet $l_B$ are in $\tilde{A}$ and lie between $B'$ and $B''$. Now $|A'A''| \wedge |C'C''| \leq |\overline{PQ}| \leq$

[1]Reprinted from: Pattern Recognition, 15, no 5, 379 382, 1982, Some Results on Fuzzy (Digital) Convexity, L. Janos and A. Rosenfeld, with permission from Elsevier Science.

$|A'A''| \vee |C'C''|$, where vertical bars denote the length of an interval. But $|A'A''| = \bar{\tilde{A}}_l(A)$ and $|C'C''| = \bar{\tilde{A}}_l(C)$, as pointed out in the paragraph preceding Example 4.8. Hence $\bar{\tilde{A}}_l(B) = |B'B''| \geq |\overline{PQ}| \geq |A'A''| \wedge |C'C''| = \bar{\tilde{A}}_l(A) \wedge \bar{\tilde{A}}_l(C)$. Therefore $\bar{\tilde{A}}_l$ is min-free. ■

It is interesting to note that Proposition 4.22 is false if $\tilde{A}$ is only assumed to be a convex fuzzy subset. To see this, let $\tilde{A}$ be defined as follows: $\tilde{A} = 0.2$ in the quadrilateral whose vertices are $(1,0), (1,5), (3,2)$ and $(3,0)$; except that $\tilde{A} = 0.5$ on the line segment $\overline{(3,0),(3,2)}$. Since the level sets of $\tilde{A}$ are convex, $\tilde{A}$ is convex by Proposition 4.20. But for the integral projection of $\tilde{A}$ on the $x$-axis, we have $\bar{\tilde{A}}_l(1,0) = \int_0^5 .2\ dx = 1$, $\bar{\tilde{A}}_l(2,0) = \int_0^{3.5} .2\ dx = 0.7$ and $\bar{\tilde{A}}_l(3,0) = \int_0^2 .5\ dx = 1$. Hence $\bar{\tilde{A}}_l$ is not a min-free function. (See Figure 4.2.)

**FIGURE 4.2** An example to illustrate: $\bar{\tilde{A}}_l$ is not min-free.[1]




The converse of Proposition 4.22 is also false; even if all the integral projections of $\tilde{A}$ are min-free functions, $\tilde{A}$ is not necessarily convex. In fact, consider the L-shaped polygon $\Psi$ whose vertices are (0,0), (0,10), (10,0), (5,10), (5,5), and (10,5) (Figure 4.3) and project $\Psi$ onto an arbitrary line $l$ (Figure 4.4). Then the value of this projection $\tilde{A}$ has no strict local minimum (see Figure 4.4) (it strictly increases from $P_1$ to $P_2$, remains constant from $P_2$ to $P_3$, strictly decreases from $P_3$ to $P_4$, remains constant from $P_4$ to $P_5$, and strictly decreases from $P_5$ to $P_6$), and hence is a min-free function, but $\Psi$ is not convex.

[1]Reprinted from: Pattern Recognition, 15, no. 5, 379 382, 1982, Some Results on Fuzzy (Digital) Convexity, L. Janos and A. Rosenfeld, with permission from Elsevier Science.

**FIGURE 4.3** A counter example: Converse of Proposition 4.22 is false. [1]

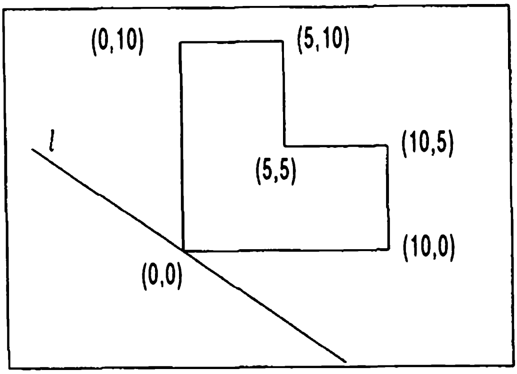
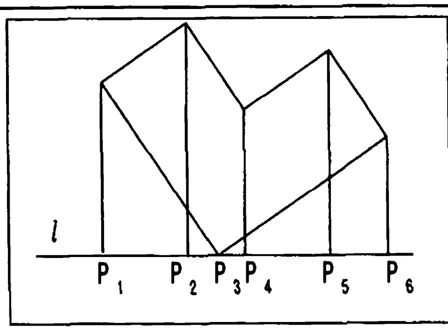



**FIGURE 4.4** Figure 4.3 rotated to make the line $l$ horizontal.[1]




## 4.10 Fuzzy Digital Convexity

### *Digital Convexity*

Let $R$ be a subset of the plane such that $\overline{(R^\circ)} = R$ ($R$ is the closure of its interior); we call such an $R$ *regular.* We regard each lattice point $P$ as the center of an open unit square $P^*$. We call such a square a *cell.* The set $I(R) \equiv \{P | R \cap P^* \neq \emptyset\}$ is called the *digital image* of $R$. Note that the digital image is defined only for regular sets.

By the definition of $I(R), R$ meets $Q^*$ if and only if $Q \in I(R)$. If $R$ meets any $\overline{Q^*}$ on its boundary, it meets the interior of at least one of the cells that share that boundary. Hence we have the following result.

**Proposition 4.23** *$R \subseteq \bigcup\{\overline{P^*} | P \in I(R)\}$ and $I(R)$ is the smallest set of lattice points for which this is true.* ■

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A set $S$ of lattice points is called *digitally convex* if it is the digital image of a convex regular set $R$.

We show that the digital image $S$ of any arcwise connected regular set $R$ is 4-connected. For all $P, Q \in S$, $R$ meets $P^*$ and $Q^*$, say in the points $(x, y)$ and $(u, v)$, and there is a path in $R$ from $(x, y)$ to $(u, v)$. Hence this path meets a sequence of interiors of 4-adjacent cells which thus yield a 4-path in $S$ from $P$ to $Q$. The following result follows from this argument.

**Proposition 4.24** *A digitally convex set is 4-connected.* ∎

The proofs of the following two theorems can be found in [3]

**Theorem 4.25** *The following properties of a 4-connected set $S$ are equivalent:*

**(i)** *for all $P, Q$ in $S$, no point not in $S$ lies on the line segment $\overline{PQ}$,*

**(ii)** *for all $P, Q$ in $S$ and all $(u, v) \in \overline{PQ}$, there exists a point $(x, y) \in S$ such that $|x - u| \vee |y - v| < 1$.* ∎

We call $S$ *regular* if every $P \in S$ has at least two (horizontal or vertical) neighbors in $S$.

**Theorem 4.26** *Any digitally convex set has the properties of Theorem 4.25. A regular set $S$ is digitally convex if and only if it has the properties of Theorem 4.25.* ∎

If $S$ is not regular, it may satisfy the properties of Theorem 4.25, but not have a convex pre-image.

## *Fuzzy Digital Convexity*

We discuss the possibility of generalizing Theorems 4.25 and 4.26 to convex fuzzy subsets. Given a fuzzy subset $\tilde{A}$ of the plane, we define a fuzzy subset $\tilde{A}'$ of the lattice points by

$\tilde{A}'(P) \equiv \vee\{\tilde{A}(x, y) | (x, y) \in P^*\}$.

If $\tilde{A}$ is a fuzzy subset of the plane and $t \in [0, 1]$, we let $\tilde{A}_t = \{(x, y) \mid \tilde{A}(x, y) > t\}$.

**Proposition 4.27** *If $\tilde{A}_t$ is regular, $\tilde{A}'_t$ is its digital image.*

**Proof.** $P \in I(\tilde{A}_t)$ if and only if $\tilde{A}_t \cap P^* \neq \emptyset$ if and only if $\vee\{\tilde{A}(x, y) | (x, y) \in P^*\} > t$ if and only if $\tilde{A}'(P) > t$ if and only if $P \in \tilde{A}'_t$. ∎

The corresponding result is not true if we replace $>$ with $\geq$. Indeed, if such a level set $\tilde{A}^t$ meets $P^*$, we have $\vee\{\tilde{A}(x, y) | (x, y) \in P^*\} \geq t$, so that $\tilde{A}'(P) \geq t$ and $P \in \tilde{A}'^t$; but conversely, if the supremum $\geq t$, $\tilde{A}^t$ may only meet $P^*$ (though it does have to meet the interior of some cell that shares its border with $P^*$, if $\tilde{A}^t$ is regular). Thus we know only that if $\tilde{A}^t$ is regular $\tilde{A}'^t$ contains its digital image.

**Corollary 4.28** *If $\tilde{A}$ is an ordinary regular set, then $\tilde{A}'$ is its digital image.*

**Proof.** Since $\tilde{A}$ is an ordinary set, $\tilde{A} = \tilde{A}_0$ and so is regular. Thus $\tilde{A}'_0 = \tilde{A}'$ is its digital image. ■

We call $\tilde{A}$ *fuzzily regular* if all $\tilde{A}_t$ are regular, $0 \leq t < 1$. If $\tilde{A}$ is fuzzily regular, we call $\tilde{A}'$ its *digital image.*

We call $\tilde{A}'$ *fuzzily digitally convex (FDC)* if it is the digital image of a fuzzily regular, convex $\tilde{A}$. Analogous to Proposition 4.20, we then have the following result.

**Proposition 4.29** *If $\tilde{A}'$ is FDC, then $\tilde{A}_t$ is digitally convex for all $t \in [0,1]$.*

**Proof.** Every $\tilde{A}'_t$ is the digital image of $\tilde{A}_t$ by Proposition 4.27 and $\tilde{A}_t$ is convex. ■

Analogous to Condition (i) in Theorem 4.25, we have the next result.

**Proposition 4.30** *If $\tilde{A}'$ is FDC, then for all collinear triples of lattice points $A, B, C$, with $B$ between $A$ and $C$, we have $\tilde{A}'(B) \geq \tilde{A}'(A) \wedge \tilde{A}'(C)$.*

**Proof.** Let $\epsilon > 0$. By the definition of $\tilde{A}'$, there exists points $A'$ and $C'$ of the cell interiors $A^*$and $C^*$ such that $\tilde{A}'(A) \leq \tilde{A}(A') + \epsilon$ and $\tilde{A}'(C) \leq \tilde{A}(C')$ $+\epsilon$. Now, $\overline{A'C'}$ meets the cell interior $B^*$. Let $B'$ be a point of $B^* \cap \overline{A'C'}$. Since $\tilde{A}$ is convex, we have $\tilde{A}(B\,) \geq \tilde{A}(A') \wedge \tilde{A}(C') > (\tilde{A}'(A) - \epsilon) \wedge (\tilde{A}'(C) - \epsilon) = (\tilde{A}'(A) \wedge \tilde{A}'(C)) - \epsilon$. Since $\tilde{A}'(B) = \vee\{\tilde{A}(x,y) | (x,y) \in B^*\} \geq \tilde{A}(B')$, we thus have $\tilde{A}'(B) > \tilde{A}'(A) \wedge \tilde{A}'(C) - \epsilon$; and since $\epsilon$ is arbitrary, it follows that $\tilde{A}'(B) \geq \tilde{A}'(A) \wedge \tilde{A}'(C)$. ■

## 4.11 On Connectivity Properties of Grayscale Pictures

The purpose of this section is to present some additional results on connectivity properties of gray scale pictures.

In studying topological properties in the case where points take on only the values 0 and 1, it is customary to use opposite types of connectedness for the two types of points, regarding diagonal neighbors as adjacent for the 1's but not for the 0's, or vice versa. For multi-valued pictures, e.g., those whose points have values from $[0,1]$, the situation is more complicated. To avoid these complications, we deal primarily with pictures defined on a hexagonal grid, where a point has only one kind of neighbor.

Let $\Sigma$ be a bounded regular grid of points in the plane. We will assume, in most of this section, that the grid is hexagonal rather than Cartesian, so that each point of $\Sigma$ has six neighbors. Let $S$ be any subset of $\Sigma$, which we assume does not meet the border of $\Sigma$. Two points $P, Q$ of $S$ are *connected* in $S$ if there exists a path $\rho : P = P_0, P_1, \ldots, P_n = Q$ of points of $S$ such that $P_i$ is a neighbor of $P_{i-1}, 1 \leq i \leq n$. The notion of connectedness is an equivalence relation and its equivalence classes are called the connected components of $S$. If there is only one component, we call $S$ connected. The complement of $S$ also consists of connected components. One of them, called the *background,* contains the border of $\Sigma$. The others, if any, are called *holes* of $S$.

Let $\tilde{A}$ be a fuzzy subset of $\Sigma$. Since $\Sigma$ is finite, $\tilde{A}$ takes on only finitely many values on $\Sigma$. We are only interested in the relative size of these values and can thus assume them to be rational numbers. Hence if we let $\alpha = 1/\lambda$, where $\lambda$ is the least common multiple of the denominators of these values, then these values are integer multiples of $\alpha$. For the remainder of the chapter, we will assume $\tilde{A}$ takes on integer values (dividing the original rational values by $\alpha$), so that $\tilde{A}$ defines a digital picture on $\Sigma$ whose gray level at $P$ is $\tilde{A}(P)/\alpha \equiv g(P)$, where $0 \leq g(P) \leq M$ (say). We assume that $\tilde{A}$ has value 0 on the border of $\Sigma$.

A digital picture $\tilde{A}$ can be decomposed into connected components $C$ of constant gray levels, i.e., for some gray level $l, C$ is a connected component of the set $\tilde{A}_l$ of points having gray level $l$. $C$ is called a *top* if the components adjacent to $C$ all have lower gray levels than $C$; a *bottom* is defined analogously. Hence, for any point $P$, there is a monotonically non-descending (non-ascending) path to a top (bottom). The gray level of a component $C$ will be denoted by $l(C)$.

Recall that $\tilde{A}$ is connected if, for all $P, Q$ in $\tilde{A}$, there exists a path $P = P_0, P_1, \ldots, P_n = Q$ such that each $\tilde{A}(P_i) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. It was shown in [5] that $\tilde{A}$ is connected if and only if $\tilde{A}$ has a unique top.

## *Equivalent Characterizations of Connectedness*

Let $0 \leq l \leq M$. Then the set of points that have gray level $l$ will be denoted by $\tilde{A}_l$ and the set of points that have gray levels $> l$ will be denoted by $\tilde{A}_{l^+}$. For the sake of brevity, a connected component of $\tilde{A}_l$ will be called an *$l$-component* and a connected component of $\tilde{A}_{l^+}$ will be called an *$l^+$-component.*

Now for any non-empty $l^+$-component and for any $P$ in the component there is a monotonically non-descending path to a top, and this path lies in the component. Thus we have the following result.

**Proposition 4.31** *Any non-empty $l^+$-component contains a top.* ■

**Theorem 4.32** *The following properties of $\tilde{A}$ are all equivalent:*

**(i)** *$\tilde{A}$ has a unique top.*

**(ii)** *For all $l$, $\tilde{A}_{l^+}$, is connected.*

**(iii)** *Every $l$-component is adjacent to, at most, one $l^+$-component.*

**Proof.** (i) $\Rightarrow$ (ii): If some $\tilde{A}_{l^+}$ had two components, each of them would contain a top by Proposition 4.31, and these tops must be distinct.

(ii) $\Rightarrow$ (iii): Immediate.

(iii) $\Rightarrow$ (i): Suppose that $\tilde{A}$ had two tops $U, V$ and consider a sequence of components $U = C_0, C_1, \dots, C_n = V$ such that $C_i$ is adjacent to $C_{i-1}, 1 \leq i \leq n$. Of all such sequences, select one for which $\wedge\{l(C_i) \mid i = 1, 2, \dots, n\} = l$ is as large as possible, and of all these, pick one for which the value $l$ is taken on as few times as possible. Let $l(C_j) = l$, then evidently $0 < j < n$. If $C_{j-1}$ and $C_{j+1}$ were in the same $l^+$-component, the sequence could be diverted to avoid $C_j$ by passing through a succession of components of values $> l$. The diverted sequence would thus have fewer terms of value $l$, a contradiction. Hence, $C_j$ is adjacent to two $l^+$-components., ■

Note that an $l$-component is adjacent to no $l^+$-components if and only if it is a top.

We now conclude this chapter with some comments concerning coherence. The interested reader should see [7] for more details and also for a discussion concerning the genus.

We call $\tilde{A}$ *coherent* if, for any component $C$, exactly one component is adjacent to $C$ along each of its borders.

**Proposition 4.33** *Let $\tilde{A}$ have the property that, along any of its borders, any $C$ meets components that are either all higher or all lower than it in value. Then $\tilde{A}$ is coherent.* ■

**Proposition 4.34** *If $\tilde{A}$ is coherent, the conditions of Theorem 4.32 are also equivalent to the following condition.*

**(iv)** *Every $l$-component is adjacent to, at most, one $l'$-component such that $l' > l$.* ■

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# 5

# FUZZY GEOMETRY

## 5.1 Introduction

In this chapter, we concentrate on fuzzy geometry. Fuzzy geometry has been studied from different perspectives. The theory presented in this chapter is applicable to pattern recognition, computer graphics and image processing and follows closely the theory as developed by Rosenfeld, [37,47–51]. Buckley and Eslami, [7,8], are developing a fuzzy plane geometry which is quite different, but has the potential for applications in various fields of computer science.

In pattern recognition one often wants to measure geometric properties of regions in images. Such regions are not always two–valued. It is some times more appropriate to regard them as fuzzy subsets of the image. It is not obvious how to measure geometric properties of fuzzy subsets. In this chapter, we deal with different geometric concepts of a fuzzy subset of the plane and show that they reduce to the usual ones if the fuzzy subset is an ordinary subset.

## 5.2 The Area and Perimeter of a Fuzzy Subset

Let $A$ and $B$ be subsets of the Euclidean plane $\mathbb{R}^2$ such that $B \supseteq A$. Then the area of $A$ is less than or equal to the area of $B$, assuming their areas exist. Similarly, the perimeter of $A$ is less than or equal to the perimeter of $B$, provided their perimeters are defined and both $A$ and $B$ are convex.

Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. If $\tilde{A}$ is integrable, we define its *area* as $\iint \tilde{A}\, dx\, dy$, where the integration is performed over the entire plane, $\mathbb{R}^2$. Thus, if $\tilde{B}$ is fuzzy subset of $\mathbb{R}^2$ containing $\tilde{A}$, we have $Area(\tilde{B}) \geq Area(\tilde{A})$, if both are defined. Defining the perimeter is not as simple as the area.

We first define perimeter for a simple class of 'piecewise constant' fuzzy subsets. This class includes the class of digital pictures as a special case. We show that this definition reduces to the ordinary one when the fuzzy subset is crisp. We also define the perimeter for 'smooth' fuzzy subsets and outline an argument showing that the smooth and piecewise constant definitions agree 'in the limit'. We then point out that a unified definition, including both the smooth and piecewise constant cases, can be given in terms of generalized functions.

We consider convex fuzzy subsets and show that if $\tilde{B} \supseteq \tilde{A}$, the perimeter of $\tilde{B}$ is greater than or equal to that of $\tilde{A}$. This generalizes a theorem about crisp convex sets to piecewise constant convex fuzzy subsets. We conclude the section with a study of fuzzy disks.

A set of points $\Pi = \{x_0, x_1, \ldots, x_n\}$ satisfying the inequality $a = x_0 < x_1 < \ldots < x_n = b$ is called a *partition* of $[a, b]$. Let $f$ be a real-valued function defined on $[a, b]$. If there exists a positive integer $M$ such that $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M$ for all partitions of $[a, b]$, then $f$ is said to be a *bounded variation* on $[a, b]$.

**Example 5.1** *Let $f(x) = x\cos(\pi/2x)$ for all $x \in (0, 1]$ and $f(0) = 0$. Let $\Pi = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$. Then $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ which cannot be bounded for all $n$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Thus $f$ is not of bounded variation over $[0, 1]$ even though the derivative $f'$ exists in $(0, 1)$.*

Let $x = x(t), y = y(t)$ be a parameterization of a continuous curve $C$ in $\mathbb{R}^2, a \leq t \leq b$. A partition $\Pi$ of the interval $[a, b]$ determines an inscribed polygon $\Pi(C)$ formed by joining the points corresponding to parameter values. The length $L(\Pi(C))$ of such a polygon is defined the usual way. The *length* $l(C)$ of the curve is defined to be the least upper bound of $L(\Pi(C))$ for all partitions $\Pi$. If $l(C)$ is finite, the curve $C$ is said to be *rectifiable*. It is a well known fact that $C$ is rectifiable if and only if $x(t)$ and $y(t)$ representing $C$ are of bounded variation.

We now examine the piecewise constant case. Let $\sum = \{S_1, ..., S_n\}$ be a partition of $\mathbb{R}^2$ such that all but one of the $S_i$'s (say $S_n$) are bounded. The set $S_{ij} = \bar{S}_i \cap \bar{S}_j$ (where the overbar denotes closure with respect to the Euclidean metric) is called the *common boundary* of $S_i$ and $S_j$. We call $\Sigma$ a *segmentation* of $\mathbb{R}^2$ if for all $1 \leq i, j \leq n$, $S_{ij}$ is a finite union of rectifiable arcs $A_{ijk}$ of finite lengths, where $1 \leq k \leq n_{ij}$ and thus $S_{ij} = \bigcup_{k=1}^{n_{ij}} A_{ijk}$.

**Definition 5.1** *The total* perimeter *of a segmentation $\sum = \{S_1, ..., S_n\}$ is defined as*

$$\sum_{\substack{i,j=1\\i<j}}^{n}\sum_{k=1}^{n_{ij}} l(A_{ijk}),$$

*where $l$ denotes arc length.*

**Definition 5.2** *A fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ is called* piecewise constant *if there exists a segmentation $\sum = \{S_1, ..., S_n\}$ of $\mathbb{R}^2$ such that $\tilde{A}$ has constant value $a_i$ on each $S_i$ and $\tilde{A} \equiv 0$ on $S_n$ (i.e., $a_n = 0$). If $\tilde{A}$ is piecewise constant, we define the* perimeter *of $\tilde{A}$ as*

$$p(\tilde{A}) \equiv \sum_{\substack{i,j=1\\i<j}}^{n}\sum_{k=1}^{n_{ij}} |a_i - a_j| l(A_{ijk}). \tag{5.2.1}$$

Let $\sum = \{S_1, ..., S_n\}$ be a segmentation of $\mathbb{R}^2$. Let $A$ be a subset of $\mathbb{R}^2$. Then the only pairs $(S_i, S_j)$ that make nonzero contributions to $p(\chi_A)$ are those such that $\chi_A = 1$ on $S_i$ and $\chi_A = 0$ on $S_j$ (or vice versa), and for such pairs we have $|a_i - a_j| = 1$, so that $p(\chi_A) = \sum\sum l(A_{ijk})$. Thus, if $A$ is a union of connected regions $S_i$ whose borders are the rectifiable closed curves $A_{ik}$, then $p(\chi_A) = \sum\sum l(A_{ik})$, which is just the total perimeter of $A$ as ordinarily defined.

The digital pictures with gray levels in $[0, 1]$ are a special case of the piecewise constant fuzzy subsets. Here we partition $\mathbb{R}^2$ into an $m \times m$ array of half-open unit squares and $\tilde{A}$ is constant on each of these squares, with $\tilde{A} = 0$ on the rest of $\mathbb{R}^2$.

We now define the perimeter for 'smooth' fuzzy subsets.

**Definition 5.3** *A fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ is called* smooth *if it has first partial derivatives everywhere. We define the perimeter of such an $\tilde{A}$ as*

$$\iint |\nabla\tilde{A}| dxdy, \tag{5.2.2}$$

*provided this integral exists, where*

$$|\nabla\tilde{A}| \equiv \sqrt{\left(\frac{\partial\tilde{A}}{\partial x}\right)^2 + \left(\frac{\partial\tilde{A}}{\partial y}\right)^2}$$

*is the magnitude of the gradient of $\tilde{A}$.*

To see the connection between Definitions 5.2 and 5.3, we consider the case where $\tilde{A}$ is a digital picture. Let $\tilde{A}^*$ be the same as $\tilde{A}$, except in strips of width $\varepsilon$ centered on the common borders of the unit squares, where it changes linearly from one constant value to another. (At the corners where four squares meet, $\tilde{A}^*$ has to be defined in a more complicated way to make it smooth, but this only involves strip intersections whose areas are $O(\varepsilon^2)$ and so does not affect our argument.) Thus $|\nabla\tilde{A}^*| = 0$, except on the strips;

while on the strip $\sigma_{ij}$ centered on $A_{ij}$ (say), where the value of $\tilde{A}^*$ changes from $a_i$ to $a_j$, we have $|\nabla \tilde{A}^*| = |a_i - a_j|/\varepsilon$ (except near the ends of $\sigma_{ij}$). Since the area of $\sigma_{ij}$ is $l(A_{ij})\varepsilon$, we thus have

$$\iint |\nabla \tilde{A}^*| dxdy = \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\iint_{\sigma_{ij}} |\nabla \tilde{A}^*| dxdy\right]$$

$$\doteq \sum_{\substack{i,j=1\\ i\neq j}}^{n} (l(A_{ij})\varepsilon)(|a_i - a_j|/\varepsilon)$$

$$= \sum_{\substack{i,j=1\\ i\neq j}}^{n} l(A_{ij})|a_i - a_j| = p(\tilde{A}),$$

where $\doteq$ denotes approximately equal to.

This argument shows that the two definitions of perimeter agree 'in the limit'. If $\tilde{A}$ is piecewise constant, we can approximate it by a smooth $\tilde{A}^*$ such that

$$\iint |\nabla \tilde{A}^*| dxdy \doteq p(\tilde{A}).$$

We now show, using the theory of generalized functions, how we can formulate a definition of perimeter that has both the piecewise constant and smooth definitions as special cases. We first review some pertinent results from analysis.

Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$. Then $f$ is called *ordinary* if it is integrable in each finite interval $[a, b]$. (By integrable, one usually means Lebesgue integrable, but for our purpose Riemann integrability suffices.) Thus continuous functions are ordinary. The set $E$ of all continuous functions form a vector space over $\mathbb{R}$.

Now certain ordinary functions including some that are continuous have no derivatives at all, for example, the Weierstrass function [60]. Certain other ordinary functions have a derivative, but the derivative is not ordinary, for example, $f(x) = 1/\sqrt{|x|}$. It is also possible for the derivative of an ordinary function to exist and that the derivative is ordinary, but such that when integrated the original function is not recovered, for example, the Cantor function [55]. We now examine this situation a little more closely.

Let $f$ be a function from a subset $S$ of $\mathbb{R}$ into $\mathbb{R}$. Then $f$ is said to be *absolutely continuous* on $S$ if $\forall \in > 0, \exists \delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \in$ for every finite collection of nonoverlapping intervals $\{(x_i', x_i) | x_i', x_i \in S, i = 1, \ldots, n\}$ such that $\sum_{i=1}^{n}(x_i - x_i') < \delta, n \in \mathbb{N}$.

It is known that a real-valued function from $[a, b]$ possesses an ordinary derivative $f'(x)$ such that $f(x) = f(a) + \int_a^x f'(u)du$ if and only if $f$ is absolutely continuous [55]. It is possible for a function $f$ to be uniformly continuous on $[a, b]$, but not absolutely continuous and not of bounded variation [59]. However if $f$ is absolutely continuous on $[a, b]$,then $f$ is of bounded variation on $[a, b]$.

In the theory of distributions or generalized functions, [20], a function $f$ is represented via its action on 'test' functions $\varphi$ through the integral

$$(f, \varphi) \equiv \int f(x)\varphi(x)dx, \tag{5.2.3}$$

where integration is over the whole space.

We assume for test functions that all derivatives exist and are continuous and to either be identically zero outside some fixed interval or else to go to zero rapidly, together with all their derivatives, as $x$ goes to infinity.

These concepts may be used for functions of one or several variables. For uniformity of notation, and also because the formulas introduced may be of interest for sets in 3-space as well as in the plane, we shall follow [20] and have $x$ denote a point $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ and $dx$ denote $dx_1\ dx_2 \ldots dx_n$. Thus

$$(f, \varphi) \equiv \int_{\mathbb{R}^n} f(x_1, x_2, ..., x_n) \cdot \varphi(x_1, x_2, ..., x_n)dx_1dx_2...dx_n.$$

By a 'test' function $\varphi$ being zero at infinity, we mean either that $\varphi(x)$ is identically zero for all points $x$ outside some fixed sphere centered at the origin, or else $\varphi$ and all its partial derivatives of all orders go to zero with some specified rapidity as the Euclidean norm of $x$, $(x_1^2 + x_2^2 + ... + x_n^2)^{1/2}$, approaches infinity.

When $n = 1$ the 'distributional derivative' $f'$ is introduced via the formula

$$(f', \varphi) = \int_{-\infty}^{\infty} f'(x)\varphi(x)dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx = -(f, \varphi'). \tag{5.2.4}$$

When the function $f$ is actually differentiable this is merely the formula for integration by parts; recall that $\varphi$, as a test function, is zero at $\pm\infty$. Note that when $f$ is not actually differentiable, then equation (5.2.4) defines a new generalized function $f'$. Thus we see that if $f$ is not differentiable, but we need the value of the integral of $f'$ multiplied by a test function $\varphi$ which has a bounded derivative, we could obtain one as if $f'(x)$ existed by taking $(f', \varphi)$ in (5.2.4). This technique is used in certain problems in mathematical physics.

For example, if $f$ is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

then

$$(f', \varphi) = \int_{-\infty}^{\infty} f'(x)\varphi(x)dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx = -\int_{0}^{\infty} \varphi'(x)dx = \varphi(0). \tag{5.2.5}$$

Thus $f'$ is the 'Dirac delta' function.

Generalized functions are not necessarily functions in the ordinary sense. They are rather defined only in terms of their actions on test functions. In equation (5.2.5), the notation $f'(x)$ is used for its convenience in manipulation of formulas. It does not stand for a specific real number for each real value of $x$.

For $n > 1$, partial derivatives are defined by the formula

$$\left(\frac{\partial f}{\partial x_i}, \varphi\right) \equiv -\left(f, \frac{\partial \varphi}{\partial x_i}\right). \tag{5.2.6}$$

The gradient vector $\nabla f$ is a generalized vector function that acts on test vector functions

$$\varphi(x) = (\varphi_1(x_1, x_2, ..., x_n), \varphi_2(x_1, x_2, ..., x_n), ..., \varphi_n(x_1, x_2, ..., x_n))$$

by the formula

$$(\nabla f, \varphi) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}, \varphi_i\right) = -\sum_{i=1}^{n} \left(f, \frac{\partial \varphi_i}{\partial x_i}\right).$$

We also use the notation

$$\int_{\mathbb{R}^n} \nabla f(x) \cdot \varphi(x) dx$$

for $(\nabla f, \varphi)$.

Suppose that $G$ is a crisp subset of $\mathbb{R}^n$, having a smooth or piecewise smooth boundary, and $\chi_G$ is its characteristic function. Then

$$(\nabla \chi_G, \varphi) = -\sum_{i=1}^{n} \left(\chi_G, \frac{\partial \varphi_i}{\partial x_i}\right) = -\sum_{i=1}^{n} \int_G \frac{\partial \varphi_i}{\partial x_i} dx.$$

Integrating by parts, we can show that the last integral is equal to

$$\int_\Gamma \varphi_i \cos \nu_i d\sigma,$$

where $\Gamma$ is the boundary of $G$, $d\sigma$ is the 'area' element on $\Gamma$ (when $n = 2$ and $\Gamma$ is a curve, $d\sigma$ is the element of arc length, usually written $ds$) and $\nu_i$ is, at each point of $\Gamma$, the angle between the positive $x_i$ direction and the unit outward pointing vector normal to $\Gamma$ at that point. Thus we have

$$\int_{\mathbb{R}^n} \nabla \chi_G \cdot \varphi dx = (\nabla \chi_G, \varphi) = \int_\Gamma \left(\sum_{i=1}^{n} \varphi_i \cos \nu_i\right) d\sigma.$$

If we now take for $\varphi$ any vector test function that is equal on $\Gamma$ to the unit outward pointing normal vector $N$, whose components are cos $\nu_1$, cos $\nu_2, ...,$ cos $\nu_{n_0}$, and denote this $\varphi$ simply by $N$, we have

$$\int_{\mathbb{R}^n} \nabla \chi_G \cdot N dx = \int_\Gamma \left( \sum_{i=1}^n (\cos \nu_i)^2 \right) d\sigma = \int_\Gamma d\sigma, \tag{5.2.7}$$

which is the area of $\Gamma$ or the surface area (in two dimensions the perimeter) of $G$. (When the normal to $\Gamma$ is not an entirely smooth function of position on $\Gamma$, (5.2.7) may be obtained as a limit using smooth $\varphi$'s which more and more closely approximate $N$ on $\Gamma$.)

We now extend (5.2.7) to generally define the 'surface area' of a fuzzy subset $\tilde{A}$ in $\mathbb{R}^n$ as

$$\int_{\mathbb{R}^n} \nabla \tilde{A} \cdot N, \tag{5.2.8}$$

where $N$ is, at each point, the unit vector pointing in the direction in which $\tilde{A}$ increases most rapidly. Where $\tilde{A}$ is smooth, $N$ is the unit vector in the direction of the gradient of $\tilde{A}$, so that $\nabla \tilde{A} \cdot N = |\nabla \tilde{A}|$. On surfaces $\Gamma$ (curves when $n = 2$) where $\tilde{A}$ has a finite jump, $N$ is a unit normal to the surface. If, in a neighborhood of $\Gamma$, $\nabla \tilde{A}$ is either zero or else the direction of $\nabla \tilde{A}$ approaches that of the normal to $\Gamma$ as any point of $\Gamma$ is approached, then (5.2.8) can be used to calculate the contribution of $\Gamma$ to the 'perimeter' of $\tilde{A}$ in the manner in which (5.2.7) was derived. This includes, in particular, the case of $\tilde{A}$ being a step-function, so that (5.2.8) includes both our previous definitions (5.2.1) and (5.2.2).

We now consider the convex case. Recall that $\tilde{A}$ is called convex if for all triples of collinear points $P, Q, R$ in $\mathbb{R}^2$, with $Q$ between $P$ and $R$, we have $\tilde{A}(Q) \geq \tilde{A}(P) \wedge \tilde{A}(R)$. Then by Proposition 4.20, we have that $\tilde{A}$ is convex if and only if the level set $\tilde{A}^t = \{P \in \mathbb{R}^2 | \tilde{A}(P) \geq t\}$ is a convex subset of $\mathbb{R}^2$ for all $t \in [0, 1]$.

Let $\tilde{A}$ be convex and piecewise constant, say with values $0 = a_0 < a_1 < \ldots < a_n$. Then we have

$$\tilde{A}^{a_n} \subset \tilde{A}^{a_{n-1}} \subset \ldots \subset \tilde{A}^{a_1} \subset \tilde{A}^{a_0} = \mathbb{R}^2.$$

Since each $\tilde{A}^{a_i}$ is convex, it is simply connected and its border (for $i \geq 1$) is a simple closed curve, call it $C_i$. The perimeter of $\tilde{A}$ in this case is thus

$$\sum_{i=1}^n (a_i - a_{i-1}) l(C_i) = a_n \sum_{i=1}^n \frac{(a_i - a_{i-1})}{a_n} l(C_i).$$

Since

$$\sum_{i=1}^n (a_i - a_{i-1}) = a_n - a_0 = a_n,$$

this last sum can be regarded as a weighted average of the $l(C_i)$'s since the coefficients

$$\frac{(a_i - a_{i-1})}{a_n}$$

sum to 1.

Let $A$ and $B$ be convex subsets of $\mathbb{R}^2$ whose perimeters are well defined. Then it can be shown that $A \subseteq B$ implies $p(A) \leq p(B)$. (This is false for non-convex sets. For example, let $B$ be a disk and $A$ a subset of $B$ with a very 'wiggly' border.) We now show that this property generalizes to fuzzy perimeter.

**Proposition 5.1** *Let $\tilde{A}$ and $\tilde{B}$ be piecewise constant convex fuzzy subsets of $\mathbb{R}^2$ such that $\tilde{A} \subseteq \tilde{B}$. Then $p(\tilde{A}) \leq p(\tilde{B})$.*

**Proof.** Let the values of $\tilde{A}$ and $\tilde{B}$ be $a_1 < \ldots < a_r$ and $b_1 < \ldots < b_s$ respectively. Then

$$\tilde{A}^{a_r} \subset \ldots \subset \tilde{A}^{a_1} \text{ and } \tilde{B}^{b_s} \subset \ldots \subset \tilde{B}^{b_1}.$$

Let the outer borders of these level sets be denoted by $C_r, \ldots, C_1$ and $D_s, \ldots, D_1$. Let $\tilde{A}_i = \tilde{A}^{a_i}$ for $i = 1, \ldots, r$ and $\tilde{B}_j = \tilde{B}^{b_j}$ for $j = 1, \ldots, s$. Since the $\tilde{A}_i$ and $\tilde{B}_j$ are all convex, we have $l(C_r) < \ldots < l(C_1)$ and $l(D_s) < \ldots < l(D_1)$. Moreover, since $\tilde{A} \subseteq \tilde{B}$, the value of $\tilde{B}$ on each $\tilde{A}_i$ is at least $a_i$. Hence for all $b_j \leq a_i$, we have $\tilde{B}_j \supseteq \tilde{A}_i$ and so $l(C_i) \leq l(D_j)$.

Since $\tilde{B} \supseteq \tilde{A}$, we have $\tilde{B} > 0$ on $\tilde{A}_1$. Thus $\tilde{A}_1 \subseteq \tilde{B}_1$. Let $\tilde{B}_1 \supseteq \tilde{B}_2 \supseteq \ldots \supseteq \tilde{B}_{j_1} \supseteq \tilde{A}_1$, but $\tilde{B}_{j_1+1} \not\supseteq \tilde{A}_1$, where $j_1 \geq 1$. Let $a_{i_1} \leq b_{j_1} < a_{i_1+1}$, where $r \geq i_1 \geq 1$. Thus we have the following equation:

$$\sum_{i=1}^{i_1} (a_i - a_{i-1}) l(C_i) \leq l(C_1) \sum_{i=1}^{i_1} (a_i - a_{i-1}) = a_{i_1} l(C_1) \leq b_{j_1} l(C_1)$$
$$= \sum_{j=1}^{j_1} (b_j - b_{j-1}) l(C_1) \leq \sum_{j=1}^{j_1} (b_j - b_{j-1}) l(D_j).$$

If $i_1 = r$, we are done since the left-hand side of the above equation is $p(\tilde{A})$ and the right-side is less than or equal to $p(\tilde{B})$. Otherwise, since $\tilde{B} \supseteq \tilde{A}$, we have $\tilde{B} \geq a_{i_1+1}$ on $\tilde{A}_{i_1+1}$. Thus there must exist $j_1' > j_1$ such that $\tilde{B}_{j'_1} \supseteq \tilde{A}_{i_1+1}$. Let $\tilde{B}_{j'_1} \supseteq \tilde{B}_{j'_1+1} \supseteq \ldots \supseteq \tilde{B}_{j_2} \supset \tilde{A}_{i_1+1}$, but $\tilde{B}_{j_2+1} \not\supseteq \tilde{A}_{i_1+1}$, where $j_2 \geq j'_1$. Let $a_{i_2} \leq b_{j_2} < a_{i_2+1}$, where evidently $i_2 \geq i_1 + 1$. Then

$$\sum_{i=i_1+1}^{i_2} (a_i - a_{i-1}) l(C_i) \geq l(C_{i_1+1}) \sum_{i=i_1+1}^{i_2} (a_i - a_{i-1}) = l(C_{i_1+1})(a_{i_2} - a_{i_1})$$
$$\leq l(C_{i_1+1})(a_{i_2} - b_{j_1}) + l(C_{i_1+1})(b_{j_1} - a_{i_1}).$$

The first term, of the right-hand side of the above equation is

$$\leq l(C_{i_1+1})(b_{j_2} - b_{j_1}) = l(C_{i_1+1}) \sum_{j=j_1+1}^{j_2} (b_j - b_{j-1}) \leq \sum_{j=j_1+1}^{j_2} (b_j - b_{j-1}) l(D_j).$$

which again is part of $p(\tilde{B})$; while the second term $\leq (b_{j_1} - a_{i_1}) l(C_1)$.

Continuing in the same way, we can show that $p(\tilde{A}) \leq p(\tilde{B})$. ■

We now introduce a class of fuzzy subsets of $\mathbb{R}^2$ which we use to show that fuzzy subsets differ from crisp subsets with respect to their perimeters.

A fuzzy subset $\tilde{A}$ in the plane is called a *fuzzy disk* if there is a point $Q$ such that $\tilde{A}(P)$ depends only on the distance from $P$ to $Q$. $Q$ is called the *center* of the fuzzy disk.

In dealing with fuzzy disks, we shall take the center of the disk as the origin of a polar coordinate system $(r, \theta)$ in the plane. Then $\tilde{A}$ is a function of $\sqrt{x^2 + y^2}$ and so the substitution $x = r\cos\theta$ and $y = r\sin\theta$ yields $\tilde{A}$ as a function of $r$ alone. The area of the fuzzy disk is given by

$$A(\tilde{A}) = \iint \tilde{A}dxdy = \int_0^{2\pi} \int_0^{\infty} r\tilde{A}(r)drd\theta = 2\pi \int_0^{\infty} r\tilde{A}(r)dr. \quad (5.2.9)$$

Since $\tilde{A}$ is a function of $r$ alone and $r = \sqrt{x^2 + y^2}$ , it follows that $\frac{\partial \tilde{A}}{\partial x} = \tilde{A}'(r)\left(x/\sqrt{x^2+y^2}\right)$ and $\frac{\partial \tilde{A}}{\partial y} = \tilde{A}'(r)\left(y/\sqrt{x^2+y^2}\right)$. Thus $\sqrt{\left(\frac{\partial \tilde{A}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{A}}{\partial y}\right)^2} = \sqrt{\left(\tilde{A}'(r)\right)^2}$Hence $|\nabla \tilde{A}| = |\tilde{A}'(r)|$. We thus obtain

$$p(\tilde{A}) = 2\pi \int_0^{\infty} r|\tilde{A}'(r)|dr. \quad (5.2.10)$$

We consider only those fuzzy disks for which $\tilde{A}$ is piecewise smooth, with at most finite jumps between intervals of smoothness. Let $\tilde{A}$ have jumps $j_1, j_2, \ldots, j_n$ at $r = r_1, r_2, \ldots, r_n$, respectively. Each jump then contributes a delta-function $|j_i| \cdot \delta(r - r_i)$ to $|\tilde{A}'(r)|$, and (5.2.10) must be interpreted as the sum of

$|j_1|2\pi r_1 + |j_2|2\pi r_2 + \ldots + |j_n|2\pi r_n$

and the integrals of $2\pi r|\tilde{A}'(r)|$ over the intervals of smoothness of $\tilde{A}$.

It is clear from (5.2.10) that we may make a fuzzy disk have arbitrarily large perimeter by having $\tilde{A}$ oscillate rapidly, while its area is small.

A fuzzy disk for which $\tilde{A}$ oscillates is not convex. In fact, a fuzzy disk is convex if and only if $\tilde{A}(r)$ is nonincreasing in $r$. For a convex fuzzy disk $\tilde{A}$ that is smooth, we have $|\tilde{A}'(r)| = -\tilde{A}'(r)$ and we may integrate by parts in (5.2.10). If we further assume that $r\tilde{A}(r) \to 0$ as $r \to \infty$ (which is certainly true in the most interesting case, i.e., when $\tilde{A}$ is zero outside some bounded region of the plane), we obtain

$$p(\tilde{A}) = 2\pi \int_0^{\infty} \tilde{A}(r)dr. \quad (5.2.11)$$

If $\tilde{A}$ has jumps as described above, then we temporarily introduce the auxiliary function

$\tilde{A}_*(r) = \tilde{A}(r) - \sum_{i=1}^{n} j_i(H(r - r_i) - 1),$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Formula (5.2.10) becomes
$p(\tilde{A}) = -\sum_{i=1}^{n} 2\pi r_i j_i - 2\pi \int_0^{\infty} r\tilde{A}_*'(r)dr.$
Integrating by parts, we obtain

$$p(\tilde{A}) = -\sum_{i=1}^{n} 2\pi r_i j_i + 2\pi \int_0^{\infty} \tilde{A}_*(r)dr = 2\pi \int_0^{\infty} \tilde{A}(r)dr,$$

where the second equality holds since $\int_0^{\infty}(H(r - r_i) - 1)dr = -r_i$ for $i = 1, \ldots, n$. This verifies (5.2.11) in the more general case.

Comparing (5.2.11) and (5.2.9), we see that it can happen that if $\tilde{A}(r)$ is small, but not zero for very large values of $r$, $A(\tilde{A})$ may be large while $p(\tilde{A})$ is small. For example, if

$$\tilde{A}(r) = \begin{cases} \frac{\varepsilon}{1+r^2}, & r \leq R_0 \\ 0, & r \geq R_0 + 1 \end{cases}$$

and $\tilde{A}$ drops smoothly from $\varepsilon/(1+r^2)$ to 0 in the interval $[R_0, R_0+1]$, then $p(\tilde{A}) = 2\pi\varepsilon \arctan R_0 + \alpha < \pi^2\varepsilon + \alpha$, where $\alpha$ is the contribution to $p$ of the interval $[R_0, R_0 + 1]$, and it is clear that

$$0 < \alpha < \frac{2\pi\varepsilon}{(1 + R_0^2)}$$

since $\tilde{A}(r) \leq \varepsilon/(1 + R_0^2)$ on $[R_0, R_0 + 1]$. Also

$$A(\tilde{A}) = \pi\varepsilon \log(1 + R_0) + \beta,$$

where $\beta$ is the contribution of $[R_0, R_0 + 1]$ :

$$0 < \beta < \frac{2\pi R_0}{(1 + {R_0}^2)}.$$

By choosing $\varepsilon$, we can make $p(\tilde{A})$ arbitrarily small and we can at the same time, by choosing $R_0$, make $A(\tilde{A})$ arbitrarily large.

In the case of crisp sets, the isoperimetric inequality relates the area and perimeter by

$$p^2 \geq 4\pi A. \tag{5.2.12}$$

Hence $p$ cannot be small while $A$ is large. Crisp disks are important in connection with (5.2.12). They are the only sets for which equality is achieved.

It is interesting to note that for convex fuzzy disks we get a reversed inequality. If $\tilde{A}$ is piecewise smooth convex fuzzy disk such that $r\tilde{A}(r) \to 0$ as $r \to \infty$, then

$$p^2(\tilde{A}) \leq 4\pi A(\tilde{A}). \tag{5.2.13}$$

We refer the reader to [51] for a discussion of (5.2.13).

## 5.3 The Height, Width and Diameter of a Fuzzy Subset

In this section, we introduce the definitions of the height, width, extrinsic diameter, and intrinsic diameter of a fuzzy subset. These definitions reduce to the usual ones if the fuzzy subset is an ordinary subset. We also establish some properties of these definitions, particularly for convex fuzzy subsets, and we show how they relate to area and perimeter.

Recall that if $\tilde{A}$ is integrable, we defined its area as $\iint \tilde{A}\, dx\, dy$, where the integration is performed over the entire plane. We assume in this section that $\tilde{A}$ is integrable and has bounded support, i.e., $\tilde{A} = 0$ outside a bounded region $R$ of $\mathbb{R}^2$. Hence

$\iint \tilde{A}\, dxdy = \iint\limits_R \tilde{A}\, dxdy \leq \iint\limits_R dxdy \leq$ area of $R$.

**Definition 5.4** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. The* height *of $\tilde{A}$ is defined as*

$h(\tilde{A}) = \int \vee\{\tilde{A}(x,y)|x \in \mathbb{R}\}\, dy,$

*and its* width *as*

$w(\tilde{A}) = \int \vee\{\ \tilde{A}(x,y)|y \in \mathbb{R}\}\, dx.$

It may be noted that for digital pictures, $x$ and $y$ take on only discrete values, and since $\tilde{A} = 0$ outside a bounded region, the supremums are over finite sets. In the Euclidean plane, the integrals are all finite since $\tilde{A}$ has bounded support.

Let $\tilde{A}$ be crisp. Then $\vee\{\tilde{A}(x,y_0) \mid x \in \mathbb{R}\} = 1$ if $\tilde{A}$ meets the horizontal line $y = y_0$, and $= 0$ otherwise. Thus $h(\tilde{A}) = \int \vee\{\tilde{A}(x,y)|x \in \mathbb{R}\}dy$ is the measure of the set of $y_0$'s such that $\tilde{A}$ meets $y = y_0$, which is a reasonable way of defining the height of $\tilde{A}$. Each connected component of $\tilde{A}$ gives rise to an interval of $y$'s. In particular, if $\tilde{A}$ is connected, there is only one interval, and $h(\tilde{A})$ is just the length of this interval. For example, suppose $\tilde{A}(x,y) = 1$ if $x^2 + y^2 \leq r^2$ and $\tilde{A}(x,y) = 0$ otherwise. Then $h(\tilde{A}) = 2r$ which is the length of the interval $[-r,r]$. Similar remarks apply to our definition of width. Of course, in the digital case the height (width) of a crisp set $\tilde{A}$ is just the number of rows (columns) that $\tilde{A}$ meets.

**Proposition 5.2** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. Then $A(\tilde{A}^2) = \iint \tilde{A}^2\, dx\, dy \leq h(\tilde{A})w(\tilde{A})$.*

**Proof.** Since
$\tilde{A}^2(x,y) \leq \vee\{\tilde{A}(x,y)|x \in \mathbb{R}\} \cdot \vee\{\tilde{A}(x,y)|y \in \mathbb{R}\}$,
it follows that
$\iint \tilde{A}^2 \leq \iint \vee\{\tilde{A}(x,y)|x \in \mathbb{R}\} \cdot \vee\{\tilde{A}(x,y)|y \in \mathbb{R}\}dx\, dy$
$= \int \vee\{\tilde{A}(x,y)|x \in \mathbb{R}\}dy \int \vee\{\tilde{A}(x,y)|y \in \mathbb{R}\}dx = h(\tilde{A})w(\tilde{A})$. ∎

Note that we cannot replace $\tilde{A}^2$ by $\tilde{A}$ in Proposition 5.2. For example, let $\tilde{A} = \alpha$ on an $H \times W$ upright rectangle and $\tilde{A} = 0$ elsewhere, where $\alpha < 1$. Then $h(\tilde{A}) = H\alpha$, $w(\tilde{A}) = W\alpha$, but $A(\tilde{A}) = HW\alpha > HW\alpha^2$. As this simple example shows, the height and width of a fuzzy subset depend not only on its geometrical measurements, but also on its membership values.

More generally, we define the *extrinsic diameter* of $\tilde{A}$ as
$E(\tilde{A}) = \bigvee_u \int \vee\{\tilde{A}(u,v)|v \in \mathbb{R}\}du$,
where $u, v$ denote any pair of orthogonal directions. Evidently, if $\tilde{A}$ is crisp, the $u$ giving the maximum is the direction of the line on which $\tilde{A}$'s projection has the largest measure. Clearly, $h(\tilde{A}) \leq E(\tilde{A})$ and $w(\tilde{A}) \leq E(\tilde{A})$, so that $A(\tilde{A}^2) \leq E^2(\tilde{A})$. As an example, suppose $\tilde{A}(x,y) = 1$ if $|x| \leq s$ and $|y| \leq s$, $\tilde{A}(x,y) = 0$ otherwise. Then $h(\tilde{A}) = 2s$. However, if we take for the coordinate axes, the diagonals of the square, we see that $E(\tilde{A}) = 2\sqrt{2}s$.

Let $\tilde{A}$ be a crisp, connected set, and let $P, Q$ be any points of $\tilde{A}$. Let $\rho_{PQ}$ be any rectifiable path from $P$ to $Q$ that lies entirely in $\tilde{A}$; such paths exist since $\tilde{A}$ is connected. (We assume here and in what follows that $\tilde{A}$ is not a pathological set, so that there is no problem with the rectifiability of the paths.) The *intrinsic diameter* of $\tilde{A}$ is defined as

$$I(\tilde{A}) \equiv \bigvee_{P,Q} \{\wedge_{\rho_{PQ}} |\rho_{PQ}|\} \tag{5.3.1}$$

where $|\rho_{PQ}|$ denotes the length of $\rho_{PQ}$.

**Proposition 5.3** *If $\tilde{A}$ is crisp and connected, then $E(\tilde{A}) \leq I(\tilde{A})$.*

**Proof.** Since $\tilde{A}$ is connected, the line on which $\tilde{A}$'s projection is the longest is an interval. The length of this interval is $E(\tilde{A})$. Let $P, Q$ be points of $\tilde{A}$ that project into (or within $\epsilon$ of) the endpoints of this interval. Then the shortest path in $\tilde{A}$ from $P$ to $Q$ is at least as long as the straight line segment $\overline{PQ}$ joining them, and this segment is at least as long as the interval since the interval is a projection of it. ∎

**Proposition 5.4** *If $\tilde{A}$ is crisp and convex, then $E(\tilde{A}) = I(\tilde{A})$.*

**Proof.** Let $P, Q$ be the endpoints of the path giving the maximum in the definition of $I(\tilde{A})$. Since $\tilde{A}$ is convex, the shortest path in $\tilde{A}$ from $P$ to $Q$

is just the line segment $\overline{PQ}$. The projection of $\tilde{A}$ on the line $PQ$ thus has length at least $I(\tilde{A})$. Thus $E(\tilde{A}) \geq I(\tilde{A})$. ■

We define intrinsic diameter only for connected fuzzy subsets. A fuzzy subset $\tilde{A}$ is connected if and only if its level sets $\tilde{A}^t = \{P|\tilde{A}(P) \geq t\}$ are connected for all $t$. Let $P, Q$ be any points and let $\rho_{PQ}$ be any rectifiable path from $P$ to $Q$ such that for any point $R$ on $\rho_{PQ}$, $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. Since $\tilde{A}$ is connected, such a path always exists. We now define the *intrinsic diameter* of $\tilde{A}$ as

$$I(\tilde{A}) \equiv \vee_{P,Q}\{\wedge_{\rho_{PQ}} \int_{\rho_{PQ}} \tilde{A}\}. \tag{5.3.2}$$

If $\tilde{A}$ is crisp and $P$ or $Q$ (or both) is not in $\tilde{A}$, then $\rho_{PQ}$ can be any path, and the minimum is taken on for a path that has the smallest possible intersection with $\tilde{A}$. Thus we get the same maximum over $P, Q$ if we restrict ourselves to $P, Q$ that both lie in $\tilde{A}$. For such $P, Q$ we have $\tilde{A} = 1$ on $\rho_{PQ}$, so that $\int \rho_{PQ}\ \tilde{A}$ is just the length of $\rho_{PQ}$, which proves that (5.3.2) is the same as (5.3.1) in the crisp case.

We now give an example in order to show how $I(\tilde{A})$ depends on both the shapes and the membership values of the level sets of $\tilde{A}$. Define the fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ by $\forall(x, y) \in \mathbb{R}^2$,

$$\tilde{A}(x,y) = \begin{cases} a & \text{if } x^2 + y^2 \leq r^2, \\ b & \text{if } r^2 < x^2 + y^2 \leq s^2, \\ 0 & \text{otherwise.} \end{cases}$$

Here, if $s$ is not much larger than $r$ and $a$ is much larger than $b$, we may get the maximum when $P$ and $Q$ are both in the inner disk (diametrically opposite), so that $I(\tilde{A}) = 2ra$. To see this, note that when $P$ and $Q$ are diametrically opposite in the outer disk, the path joining them that gives the minimum may be a diameter of the outer disk, yielding $2ra + 2(s-r)b$; but it may also be a path that goes around the inner disk (e.g., between the points where the tangents from $P$ and $Q$ touch the inner disk), yielding $b[2(s^2 - r^2)^{1/2} + r(\pi - 2\cos^{-1}(r/s))]$, and this can be smaller than $2ra$.

This example shows that we can have $E(\tilde{A}) > I(\tilde{A})$ in the fuzzy case since $E(\tilde{A})$ for our fuzzy disk equals $2ra + 2(s-r)b$ in all cases, no matter what the relative values of $a, b, r$, and $s$. On the other hand, if $\tilde{A}$ is convex, we have $\tilde{A}(r) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$ for all $R$ on $\overline{PQ}$ so that

$\int_{\overline{PQ}} \tilde{A} \geq \int_{\rho_{PQ}} \tilde{A}$

for the path $\rho_{PQ}$ giving the minimum. If we project $\tilde{A}$ on the line $\overline{PQ}$, say in direction $u$, we have

$\int(\vee_v \tilde{A})du \geq \int_{\overline{PQ}} \tilde{A}du.$

We thus have

$E(\tilde{A}) = \vee_u \int \vee\{\tilde{A}|v \in \mathbb{R}\}du \geq \int_{\overline{PQ}} \tilde{A} \geq \int_{\rho_{PQ}} \tilde{A} = I(\tilde{A}),$

so that we have the following result.

**Proposition 5.5** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. If $\tilde{A}$ is convex, then $E(\tilde{A}) \geq I(\tilde{A})$.* ■

Note that $E(\tilde{A})$ can be strictly greater than $I(\tilde{A})$ even when $\tilde{A}$ is convex. For example, the fuzzy disk is convex when $a \geq b$.

In [43] it is shown, in the digital case, that the intrinsic diameter of a set is at most half of the set's total perimeter. A discussion concerning the relationship of the intrinsic diameter and the perimeter of a fuzzy subset can be found in [47].

**Theorem 5.6** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. If $\tilde{A}$ is piecewise constant and convex, then $I(\tilde{A}) \leq \frac{1}{2}p(\tilde{A})$.*

**Proof.** Since $\tilde{A}$ is convex, for any points $P, Q$ and any point $R$ on the line segment $\overline{PQ}$, we have $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. Thus $\overline{PQ}$ is one of the paths $\rho_{PQ}$ in the definition of $I(\tilde{A})$ and so

$I(\tilde{A}) = \underset{P,Q}{\vee} \{ \underset{\rho_{PQ}}{\wedge} \int_{\rho_{PQ}} \tilde{A}\} \leq \underset{P,Q}{\vee} \int_{\overline{PQ}} \tilde{A}$.

Let $\text{Im}(\tilde{A}) = \{0, a_1, ..., a_n\}$ where $0 < a_1 < ... < a_n$. Let $M_i = \tilde{A}^{a_i} \setminus \tilde{A}^{a_{i+1}}$ for $i = 1, \ldots, n-1$. Now $\tilde{A}(x) = 0$ if $x \in \mathbb{R}^2 \setminus \tilde{A}^{a_1}$, $\tilde{A}(x) = a_i$ if $x \in M_i, i = 1, \ldots, n-1$, $\tilde{A}(x) = a_n$ if $x \in \tilde{A}^{a_n}$. Also the $\tilde{A}^{a_i}$ are convex. Let $c_i$ be the perimeter of $\tilde{A}^{a_i}$ (= the outer perimeter of $M_i$), $i = 1, \ldots, n$. Thus, a segment $\overline{PQ}$ which yields $\underset{P,Q}{\vee} \int_{\overline{PQ}} \tilde{A}$ may be taken to be a chord of $\tilde{A}^{a_1}$, i.e., with $P$ and $Q$ on the border of $\tilde{A}^{a_1}$. [If $P$ or $Q$ were interior to $\tilde{A}^{a_1}$ we could extend $PQ$ (until it hits the border) and increase $\int_{\overline{PQ}} \tilde{A}$, a contradiction; and we need not extend $\overline{PQ}$ past the border since $\tilde{A} = 0$ outside $\tilde{A}^{a_1}$ and such an extension would not increase $\int_{\overline{PQ}} \tilde{A}$.] Let this $PQ$ intersect the $M_i$'s in the sequence of segments $m_1, m_2, ..., m_{k-1}, m_k, m'_{k-1}, ..., m'_2, m'_1$, where $M_k$ is the innermost of the $M$'s that $\overline{PQ}$ intersects. Hence, the concatenation of $m_i, m_{i+1}, ..., m_k, ..., m'_{i+1}, m'_i$ is a chord of $\tilde{A}^{a_i}$. Thus by the nonfuzzy relationship between perimeter and diameter we have $m_i + m_{i+1} + ... + m_k + ... + m'_{i+1} + m'_i \leq \frac{1}{2}c_i$. We thus have

$\int_{\overline{PQ}} \tilde{A} = m_1a_1 + m_2a_2 + ... + m_ka_k + ... + m'_2a_2 + m'_1a_1$
$= m_ka_k + (m_{k-1} + m'_{k-1})a_{k-1} + (m_{k-2} + m'_{k-2})a_{k-2}$
$+ ... + (m_2 + m'_2)a_2 + (m_1 + m'_1)a_1$
$= m_k(a_k - a_{k-1}) + (m_{k-1} + m_k + m'_{k-1})(a_{k-1} - a_{k-2})$
$+ (m_{k-2} + m_{k-1} + m_k + m'_{k-1} + m'_k)(a_{k-2} - a_{k-3})$
$+ ... + (m_2 + m_3 + ... + m'_k + ... + m'_3 + m'_2)(a_2 - a_1)$
$+ (m_1 + m_2 + ... + m_k + ... + m'_2 + m'_1)(a_1 - 0) \leq \frac{1}{2}[c_k(a_k - a_{k-1}) + c_{k-1}(a_{k-1} - a_{k-2}) + ... + c_2(a_2 - a_1) + c_1(a_1 - 0)]$

in which the last expression is just the fuzzy perimeter of $\tilde{A}$. ■

Although our presentation so far is based on the work of Rosenfeld, we find it worthwhile to include some results and ideas appearing [4]. In [4], Bogomolny modifies some of the definitions presented in the first two sections of this chapter and defines a projection operator that should be used along with the multiplicative operator in the process of fuzzy reasoning. Bogomolony shows that his definitions of perimeter, diameter, and height fit comfortably with the notion of area. For example, the area of a set is less than or equal to its height times its width. Also the isoperimetric inequality holds for a large class of fuzzy subsets.

For any two orthonormal vectors $\bar{a}$ and $\bar{b}$ in $\mathbb{R}^2$, the projection $Pr_{\bar{a}}\tilde{A}$ of a fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ onto the direction of $\bar{a}$ parallel to the direction $\bar{b}$ is a fuzzy subset of $\mathbb{R}$ defined by

$$(Pr_{\bar{a}}\tilde{A})(r) = \vee\{ \tilde{A}^{1/2}(r\bar{a}+s\bar{b}) | s \in \mathbb{R}\} \ \forall r \in \mathbb{R}.$$

For a unit vector $\bar{a}$ the $\bar{a}$-width of a fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ is defined by

$$w_{\bar{a}}(\tilde{A}) = \int_{\mathbb{R}}(Pr_{\bar{a}}\tilde{A})(r)dr.$$

The height $h(\tilde{A})$ and the width $w(\tilde{A})$ are defined respectively by

$$h(\tilde{A}) = w_{\bar{e}_2}(\tilde{A}) \text{ and } w(\tilde{A}) = w_{\bar{e}_1}(\tilde{A}),$$

where $\bar{e}_1 = (1,0)$ and $\bar{e}_2 = (0,1)$.

**Proposition 5.7** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. Then $Area(\tilde{A}) \leq h(\tilde{A})w(\tilde{A})$.*

**Proof.** We have

$$0 \leq \tilde{A}(x_1, x_2) = \tilde{A}^{1/2}(x_1, x_2) \cdot \tilde{A}^{1/2}(x_1, x_2)$$
$$\leq \vee\{\tilde{A}^{1/2}(x_1, x_2) \mid x_1 \in \mathbb{R}\} \cdot \vee\{\tilde{A}^{1/2}(x_1, x_2) \mid x_2 \in \mathbb{R}\}.$$

Thus

$$Area(\tilde{A}) = \int \int_{\mathbb{R}^2} \tilde{A}(x_1, x_2)dx_1dx_2$$
$$\leq \int_{\mathbb{R}} \vee\{\tilde{A}^{1/2}(x_1, x_2) \mid x_1 \in \mathbb{R}\}dx_2 \cdot \int_{\mathbb{R}} \vee\{\tilde{A}^{1/2}(x_1, x_2) \mid x_2 \in \mathbb{R}\}dx_1$$
$$= w_{\bar{e}_2}(\tilde{A}) \cdot w_{\bar{e}_1}(\tilde{A}) = h(\tilde{A})w(\tilde{A}). \blacksquare$$

We now consider the perimeter of a fuzzy subset $\tilde{A}$ of $\mathbb{R}^2$ which is a step function. An explicit definition is given below. We let $\bar{S}$ denote the closure of a subset $S$ of $\mathbb{R}^2$ in the Euclidean topology.

**Definition 5.5** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. Then $\tilde{A}$ is called a* fuzzy step function *if the following conditions hold:*

**(i)** *there exist (crisp) open sets $S_1, ..., S_{n+1}$, of which all but one (say $S_{n+1}$) are bounded;*

**(ii)** $S_i \cap S_j = \emptyset, i \neq j$;

**(iii)** $\bigcup_{i=1}^{n+1} \bar{S}_i = \mathbb{R}^2$;

**(iv)** *if $i \neq j$, $\bar{S}_i \cap \bar{S}_j = \bigcup_{k=1}^{n_{ij}} A_{ijk}$, where each $A_{ijk}$ is a rectifiable Jordan arc of length $l(A_{ijk})$;*

**(v)** $\tilde{A} \equiv a_i$ *on* $S_i$, $i = 1, ..., n+1$, $a_{n+1} = 0$.

The the *perimeter* of a fuzzy step function $\tilde{A}$ is defined to be

$$p(\tilde{A}) = \sum_{\substack{i,j=1\\ i<j}}^{n+1} \sum_{k=1}^{n_{ij}} |a_i^{1/2} - a_j^{1/2}| l(A_{ijk}).$$

**Theorem 5.8** *Let $\tilde{A}$ be a fuzzy step function. Then $(p(\tilde{A}))^2 \geq 4\pi Area(\tilde{A})$, i. e., the isoperimetric inequality holds.* ■

## 5.4 Distances Between Fuzzy Subsets

Let $S$ be a metric space with metric $d$. The distance between two subsets $U$ and $V$ is normally defined as

$$\wedge\{d(P,Q) \mid P \in U, Q \in V\}. \tag{5.4.1}$$

This definition yields the shortest distance between $U$ and $V$. In [10,11] two methods of defining the distance between two fuzzy subsets $\tilde{A}, \tilde{B}$ of $S$ are presented. The first method yields a 'distance' which is a fuzzy subset of $\mathbb{R}^+$, the nonnegative reals. It is defined for all $r \in \mathbb{R}^+$ by

$$\tilde{d}_{\tilde{A},\tilde{B}}(r) = \vee\{\tilde{A}(P) \wedge \tilde{B}(Q) \mid P, Q \in S, d(P,Q) = r\}. \tag{5.4.2}$$

Suppose that $\tilde{A}$ and $\tilde{B}$ are crisp. Then $\tilde{d}_{\tilde{A},\tilde{B}}(r) = 1$ if there exist $P \in \tilde{A}, Q \in \tilde{B}$ such that $d(P,Q) = r$ and $\tilde{d}_{\tilde{A},\tilde{B}}(r) = 0$ otherwise. Thus $\tilde{d}_{\tilde{A},\tilde{B}}$ generalizes the set of distances between two crisp sets, but not the shortest distance between two crisp sets. We propose a definition very similar to (5.4.2) that does generalize the concept of shortest distance.

The second definition discussed in [15,16] generalizes the Hausdorff distance between crisp sets, namely

$$T^\lambda \equiv \{P \in S \mid \exists Q \in T, d(P,Q) \leq \lambda\}, \tag{5.4.3}$$

where $T \subseteq S$ and $\lambda \in \mathbb{R}^+$. We can think of $T^\lambda$ as being the result of 'expanding' $T$ by the radius $\lambda$. Define the function $L$ of $\wp(S) \times \wp(S)$ into $\mathbb{R}^+$ by $\forall u, v \in \wp(S)$,

$$L(U,V) = \wedge\{\lambda \in \mathbb{R}^+ \mid U^\lambda \supseteq V\}.$$

Then the Hausdorff distance between $U$ and $V$ is the function $L^*$ of $\wp(S) \times \wp(S)$ into $\mathbb{R}^+$defined by

$$L^*(U,V) = L(U,V) \vee L(V,U). \tag{5.4.4}$$

We see that, $L^*(U,V)$ is the smallest amount that one of $U$ or $V$ must be expanded in order to contain the other one. This definition has an immediate generalization to fuzzy subsets of $S$. For any such fuzzy subset $\tilde{T}$ and any $\lambda \in \mathbb{R}^+$, we define the function $\tilde{T}^\lambda$ of $\wp(S)$ into $\mathbb{R}^+$ by $\forall P \in S$,

$$\tilde{T}^\lambda(P) = \vee\{\tilde{T}(Q) \mid d(P,Q) \leq \lambda\}. \tag{5.4.5}$$

Here, we can think of $\tilde{T}^\lambda$ as the result of applying to $\tilde{T}$ a 'local maximum' operation with radius $\lambda$. It clearly generalizes (5.4.3). We then define the functions $L$ and $L^*$ of $\mathfrak{F}\wp(S) \times \mathfrak{F}\wp(S)$ into $\mathbb{R}^+$ by $\forall \tilde{A}, \tilde{B} \in \mathfrak{F}\wp(S)$

$$L(\tilde{A},\tilde{B}) = \wedge\{\lambda \in \mathbb{R}^+ \mid \tilde{A}^\lambda \geq \tilde{B}\},$$

and

$$L^*(\tilde{A},\tilde{B}) = L(\tilde{A},\tilde{B}) \vee L(\tilde{B},\tilde{A}).$$

Later, we shall indicate how our definition is related to this approach.

Let $\tilde{A}, \tilde{B}$ be fuzzy subsets of $S$. Define the fuzzy subset $\Delta_{\tilde{A},\tilde{B}}$ of $\mathbb{R}^+$ by for all $r \in \mathbb{R}^+$,

$$\Delta_{\tilde{A},\tilde{B}}(r) \equiv \vee\{\tilde{A}(P) \wedge \tilde{B}(Q) \mid P,Q \in S, d(P,Q) \leq r\}. \tag{5.4.6}$$

This definition is almost identical to (5.4.2) except that $=$ is replaced by $\leq$ . If $\tilde{A}$ and $\tilde{B}$ are crisp, we have $\Delta_{\tilde{A},\tilde{B}}(r) = 0$ for all $r < d(\tilde{A},\tilde{B})$, and $\Delta_{\tilde{A},\tilde{B}}(r) = 1$ for all $r \geq d(\tilde{A},\tilde{B})$.

**Proposition 5.9** *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of $S$. Then $\Delta_{\tilde{A},\tilde{B}}$ is a monotonically nondecreasing function of $r$.* ■

**Proposition 5.10** *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of $S$. Then $\Delta_{\tilde{A},\tilde{B}}(0) = \vee\{\tilde{A}(P) \wedge \tilde{B}(P) \mid P \in S\}$. In particular, $\Delta_{\tilde{A},\tilde{B}}(0) = 0$ if and only if $(\tilde{A} \cap \tilde{B}) = \chi_\emptyset$.*

**Proof.** $\Delta_{\tilde{A},\tilde{B}}(0) = \vee\{\tilde{A}(P) \wedge \tilde{B}(P) \mid P \in S\}$ since $d(P,Q) \leq 0$ if and only if $P = Q$. Thus $\Delta_{\tilde{A},\tilde{B}}(0) = 0$ if and only if $(\tilde{A} \cap \tilde{B})(P) = 0$ $\forall P \in S$. ■

**Proposition 5.11** *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of $S$. Then $\lim_{r\to\infty} \Delta_{\tilde{A},\tilde{B}}(r) = (\vee\{\tilde{A}(P) \mid P \in S\}) \wedge (\vee\{\tilde{B}(Q) \mid Q \in S\})$. In particular, $\Delta_{\tilde{A},\tilde{B}} \equiv 0$ if and only if $\tilde{A} = \chi_\emptyset$ or $\tilde{B} = \chi_\emptyset$.*

**Proof.** $\lim_{r\to\infty} \Delta_{\tilde{A},\tilde{B}}(r) = \vee\{\tilde{A}(P) \wedge \tilde{B}(Q) \mid P,Q, \in S\} = (\vee\{\tilde{A}(P) \mid P \in S\}) \wedge (\vee\{\tilde{B}(Q) | Q \in S\})$.

Now $\Delta_{\tilde{A},\tilde{B}}(r) = 0$ $\forall r \in \mathbb{R}^+$ if and only if $\tilde{A}(P) \wedge \tilde{B}(Q) = 0$ $\forall P,Q \in S$ if and only if either $\tilde{A} \equiv 0$ or $\tilde{B} \equiv 0$. ■

**Proposition 5.12** *If $\tilde{A}' \subseteq \tilde{A}$ and $\tilde{B}' \subseteq \tilde{B}$, then $\Delta_{\tilde{A}',\tilde{B}'} \subseteq \Delta_{\tilde{A},\tilde{B}}$.* ■

If $\tilde{A}$ and $\tilde{B}$ are crisp, then $\Delta_{\tilde{A},\tilde{B}}$ has a step from 0 to 1 at $r = d(\tilde{A}, \tilde{B})$ and is constant everywhere else. (This is not strictly true if $\tilde{A}$ and $\tilde{B}$ are nondisjoint since then $\Delta_{\tilde{A},\tilde{B}} \equiv 1$ for all $r \in \mathbb{R}^+$; but if we extend the definition of $\Delta_{\tilde{A},\tilde{B}}$ to the entire real line by defining $\Delta_{\tilde{A},\tilde{B}}(r) \equiv 0$ for all $r < 0$, we now have a step at $r = 0$ in the nondisjoint case.) Let $\tilde{A}$ and $\tilde{B}$ be discrete valued, say taking on the values $0 \leq t_0 < t_1 < \dots < t_n \leq 1$. By (5.4.6), we have

$$\Delta_{\tilde{A},\tilde{B}}(r) = \vee\{t_i | P \in \tilde{A}^{t_i}, Q \in \tilde{B}^{t_i}, d(P,Q) \leq r\}.$$

Let $d_i \equiv d(\tilde{A}^{t_i}, \tilde{B}^{t_i})$. Note that $0 = d_0 \leq d_1 \leq \dots \leq d_n$. Suppose that $0 = d_0 < d_{i_1} < \dots < d_{i_k}$ are the strictly increasing $d$'s. This means that $\tilde{A}^{t_{i_j}}$ and $\tilde{B}^{t_{i_j}}$ are strictly farther apart than $\tilde{A}^{t_{i_{j-1}}}$ and $\tilde{B}^{t_{i_{j-1}}}$. Thus $\Delta_{\tilde{A},\tilde{B}}$ has a step of height $t_{i_j} - t_{i_{j-1}}$ at each $d_{i_j}$, and it has no other steps (except for a step of height

$$t_{i_0} \equiv \Delta_{\tilde{A},\tilde{B}}(0) = \vee\{t_i | P \in \tilde{A}^{t_i} \cap \tilde{B}^{t_i}\}$$

at $r = 0$ if this supremum is nonzero).

In the discrete valued case, we could obtain a very nice generalization of the shortest distance between two sets by defining a discrete fuzzy subset of $\mathbb{R}^+$ having membership $t_{i_j} - t_{i_{j-1}}$ at each $d_{i_j}$ (and $t_{i_0}$ at 0), and zero elsewhere. Note that when $\tilde{A}$ and $\tilde{B}$ are crisp, this fuzzy subset is just the crisp point $\{d(\tilde{A}, \tilde{B})\}$. However, it is not clear how we could extend this definition to the situation where $\tilde{A}$ or $\tilde{B}$ is not discrete valued.

The reader may wish to consult [48] for a discussion of another possible approach.

In the following examples, we show that there is no simple relationship between the differentiability of $\Delta_{\tilde{A},\tilde{B}}$ and the continuity or differentiability of $\tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ is a fuzzy subset of $\mathbb{R}$. Let $d$ denote the Euclidean metric on $\mathbb{R}$. Then $d(x, y) = |x - y|$ $\forall x, y \in \mathbb{R}$. Let $\tilde{A}$ is the fuzzy subset of $\mathbb{R}$ defined by $\tilde{A}(x) = 0$ $\forall x \in \mathbb{R}, x \neq 0$, and $\tilde{A}(0) = 1$.

**Example 5.2** *If $\tilde{B}$ is continuous, $\Delta_{\tilde{A},\tilde{B}}$ is continuous, except possibly at 0: If $\tilde{B}(0) = b > 0$, the value of $\Delta_{\tilde{A},\tilde{B}}$ jumps from 0 to $b$ at $r = 0$. Suppose $\Delta_{\tilde{A},\tilde{B}}$ had a discontinuity at $r = a > 0$, say a jump from $c$ to $c + h$. Then we must have $\tilde{B}(x) \leq c$ for $|x| < a$, and $\tilde{B}(x) = c + h$ either at $x = a$, at $x = -a$, or at a sequence of $x$'s having $|x| > a$ and $x$ arbitrarily close to $a$ or $-a$. In either case, this makes $\tilde{B}$ discontinuous at $a$ or $-a$.*

**Example 5.3** *If $\tilde{B}$ is differentiable, $\Delta_{\tilde{A},\tilde{B}}$ need not be differentiable: Let $\tilde{B}$ be any differentiable fuzzy subset that satisfies the following conditions: $\tilde{B}(0) = 0$; $\tilde{B}$ is strictly monotonically increasing as we move away from 0 in either direction; $\tilde{B}(a) = \tilde{B}(-a) = b$ (where $a, b > 0$); and $\tilde{B}'(a) > -\tilde{B}'(-a)$. Since $\tilde{B}(x) < b$ for $|x| < a$ and $\tilde{B}(x) > b$ for $|x| > a$, we have $\Delta_{\tilde{A},\tilde{B}}(r) < b$ for $r < a, \Delta_{\tilde{A},\tilde{B}}(r) > b$ for $r > a$, and $\Delta_{\tilde{A},\tilde{B}}(r) = b$ for $r = a$. In a sufficiently small neighborhood, the values of $\tilde{B}(x)$ for $|x| < a$ are greater for $x$ near $-a$ than for $x$ near $+a$, because $\tilde{B}'(a) > -\tilde{B}'(-a)$;*

*while the values for $|x| > a$ are greater for $x$ near $+a$ than for $x$ near $-a$. Hence in this neighborhood the slope of $\Delta_{\tilde{A},\tilde{B}}(r)$ is (approximately) equal to $-\tilde{B}'(-a)$ for $r < a$; to $\tilde{B}'(a)$ for $r > a$; and the slope has a discontinuity at $r = a$.*

**Example 5.4** *$\Delta_{\tilde{A},\tilde{B}}$ can be differentiable even if $\tilde{B}$ is not continuous: Let $\tilde{B}(x)$ be differentiable and strictly monotonically increasing for $x \geq 0$, and let $\tilde{B}$ be arbitrary for $x < 0$, subject only to the restriction that $\tilde{B}(-x) < \tilde{B}(x)$ for all $x > 0$. Evidently for all $r \geq 0$ we have $\Delta_{\tilde{A},\tilde{B}}(r) = \tilde{B}(r)$, so that $\Delta$ is differentiable.*

Let $P \in S$ and let $\tilde{B}$ be a fuzzy subset of $S$. The *distance distribution* from $P$ to $\tilde{B}$, denoted $\Delta_{P,\tilde{B}}$ is defined as $\Delta_{\tilde{A},\tilde{B}}$, where $\tilde{A}(P) = 1, \tilde{A}(Q) = 0$ for all $Q \in S, Q \neq P$. Hence
$\Delta_{P,\tilde{B}}(r) = \vee\{\ \tilde{B}(Q) \mid d(P,Q) \leq r\}$.
The effect on $\Delta$ of applying a local maximum operation to $\tilde{B}$ (or $\tilde{A}$) is easy to describe. We recall that $\tilde{B}^{\lambda}(P) \equiv \vee\{\tilde{B}(Q)|d(P,Q) \leq \lambda\}$. Then the following result holds readily.

**Example 5.5** $\Delta_{\tilde{A},\tilde{B}^{\lambda}}(r) = \Delta_{\tilde{A},\tilde{B}}(r + \lambda) = \Delta_{\tilde{A}^{\lambda},\tilde{B}}(r)$.

For a discussion concerning the application of the operation $()^{\lambda}$ to $\tilde{A}$ or $\tilde{B}$, the reader is referred to [48]. For a discussion on local minimum and maximum operations and their effects on distance, see [28]. Other pertinent results can be found in [52], [39], and [48].

## 5.5 Fuzzy Rectangles

In this section, we define the notion of a separable fuzzy subset of the plane $\mathbb{R}^2$ For such a fuzzy subset, connectedness, convexity, and orthoconvexity are all equivalent. We call a fuzzy subset with these properties a fuzzy rectangle. We also define a fuzzy convex polygon in terms of an infimum of fuzzy halfplanes and show that a fuzzy rectangle is a fuzzy convex polygon.

**Definition 5.6** *Let $\tilde{A}$ be a fuzzy subset of the plane. Then $\tilde{A}$ is called* separable *if there exists a coordinate system $(x, y)$ and fuzzy subsets $\tilde{B}$ and $\tilde{C}$ of $\mathbb{R}$ such that*

$$\tilde{A}(x, y) = \tilde{B}(x) \wedge \tilde{C}(y) \text{ for all } x, y \in \mathbb{R}.$$

Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. Then $\tilde{A}$ is *connected* if for all points $P, Q$ there exists an arc $\rho$ from $P$ to $Q$ such that $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$ $\forall R \in \rho$. $\tilde{A}$ is *convex* if $\forall P, Q$ and $\forall R$ on the line segment $\overline{PQ}$, we have $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. $\tilde{A}$ is *orthoconvex* if the same is true whenever $P$ and $Q$ have the same $x$ coordinate or the same $y$ coordinate, i. e., whenever the line segment $\overline{PQ}$ is vertical or horizontal.

**Theorem 5.13** *Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}^2$. Then the following conditions are equivalent:*

(i) *$\tilde{A}$ is connected,*

(ii) *$\tilde{A}$ is convex,*

(iii) *$\tilde{A}$ is orthoconvex.*

**Proof.** There exists fuzzy subsets $\tilde{B}$ and $\tilde{C}$ of $\mathbb{R}$ such that $\tilde{A}(x,y) = \tilde{B}(x) \wedge \tilde{C}(x)$ $\forall x, y \in \mathbb{R}$.

(i) $\Leftrightarrow$ (ii): Suppose that (i) holds. Let $P = (a,b), Q = (c,d)$, and let $a \le r \le c$ (or vice versa). There exists an arc $\rho$ from $P$ to $Q$ such that $\tilde{A}(R) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$ $\forall R \in \rho$. Now $\rho$ must cross the vertical line with abscissa $r$. Thus for some $s_0$, we have $(r, s_0)$ on $\rho$ so that $\tilde{A}(r, s_0) \ge \tilde{A}(a,b) \wedge \tilde{A}(c,d)$. Hence $\tilde{B}(r) \wedge \tilde{C}(s_0) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$ so that $\tilde{B}(r) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$. The analogous argument shows that if $b \le s \le d$ (or vice versa) we must have $\tilde{C}(s) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$. In particular, in and on the (possibly degenerate) rectangle that has $P$ and $Q$ as opposite corners, we have both

$\tilde{B}(r) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$ and $\tilde{C}(s) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$

so that

$\tilde{A}(r,s) = \tilde{B}(r) \wedge \tilde{C}(s) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$;

but the line segment $\overline{PQ}$ is a diagonal of this rectangle. That (ii) implies (i) is immediate.

(ii) $\Leftrightarrow$ (iii): Clearly (ii) implies (iii). Thus it remains only to show that (iii) implies (i). Let $P = (a,b), Q = (c,d)$, and $R = (a,d)$. Then

$\tilde{A}(R) = \tilde{B}(a) \wedge \tilde{C}(d)\} \ge (\tilde{B}(a) \wedge \tilde{C}(b)) \wedge (\tilde{B}(c) \wedge \tilde{C}(d)) = \tilde{A}(P) \wedge \tilde{A}(Q)$.

Since $\tilde{A}$ is orthoconvex, we have $\tilde{A}(S) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$ for all $S$ on the vertical line segment $\overline{PR}$. Hence

$\tilde{A}(S) \ge \tilde{A}(P) \wedge (\tilde{A}(P) \wedge \tilde{A}(Q)) = \tilde{A}(P) \wedge \tilde{A}(Q)$.

Similarly,

$\tilde{A}(S) \ge \tilde{A}(Q) \wedge \tilde{A}(R) \ge \tilde{A}(Q) \wedge (\tilde{A}(P) \wedge \tilde{A}(Q)) = \tilde{A}(P) \wedge \tilde{A}(Q)$

for all $S$ on the horizontal line segment $\overline{QR}$. Thus for all $S$ on the $L$-shaped path from $P$ to $Q$ composed of these two line segments, we have $\tilde{A}(S) \ge \tilde{A}(P) \wedge \tilde{A}(Q)$, which proves that $\tilde{A}$ is fuzzy connected. ■

If $\tilde{A}$ is separable and satisfies (i), (ii), and (iii) of Theorem 5.13, we call $\tilde{A}$ a *fuzzy rectangle*.

**Theorem 5.14** *$\tilde{A}$ is a fuzzy rectangle if and only if there exist two convex fuzzy subsets $\tilde{B}, \tilde{C}$ of $\mathbb{R}$ such that*

$\tilde{A}(x,y) = \tilde{B}(x) \wedge \tilde{C}(y)$

*for all $x, y \in \mathbb{R}$.*

**Proof.** If $\tilde{B}$ and $\tilde{C}$ exist, then it is immediate that $\tilde{A}$ is a fuzzy rectangle. Conversely, suppose that $\tilde{A}$ is a fuzzy rectangle. Then $\tilde{A}$ is separable and so there exist fuzzy subsets $\tilde{B}$ and $\tilde{C}$ of $\mathbb{R}$ such that $\tilde{A}(x,y) = \tilde{B}(x) \wedge \tilde{C}(y)$

$\forall x, y \in \mathbb{R}$. Suppose that $\tilde{B}$ is not convex. Then there exist $x_1 \leq x_2 \leq x_3$ such that $\tilde{B}(x_2) < \tilde{B}(x_1) \wedge \tilde{B}(x_3)$. Suppose for some $y$ that $\tilde{C}(y) > \tilde{B}(x_2)$. Then

$\tilde{A}(x_1, y) = \tilde{B}(x_1) \wedge \tilde{C}(y) > \tilde{B}(x_2)$,

$\tilde{A}(x_2, y) = \tilde{B}(x_2) \wedge \tilde{C}(y) = \tilde{B}(x_2)$,

and

$\tilde{A}(x_3, y) = \tilde{B}(x_3) \wedge \tilde{C}(y) > \tilde{B}(x_2)$,

which contradicts the convexity of $\tilde{A}$. Thus $\tilde{C}(y) \leq \tilde{B}(x_2)$ for all $y$ so that

$\tilde{A}(x, y) = \tilde{B}(x) \wedge \tilde{C}(y) = \tilde{B}(x) \wedge (\tilde{C}(y) \wedge \tilde{B}(x_2)) = (\tilde{B}(x) \wedge \tilde{B}(x_2)) \wedge \tilde{C}(y)$.

Let $\tilde{B}(x_2)$ be the smallest value of $\tilde{B}$ that violates convexity. Then $\tilde{B}(x) \wedge \tilde{B}(x_2)$ must be convex. (If $\tilde{B}(x) \wedge \tilde{B}(x_2)$ had a triple of values that violated convexity, say $x_1' < x_2' < x_3'$, then $\tilde{B}(x_2') \wedge \tilde{B}(x_2) < \tilde{B}(x_1') \wedge \tilde{B}(x_2) \wedge \tilde{B}(x_3') \wedge \tilde{B}(x_2)$ and so $\tilde{B}(x_2') < \tilde{B}(x_1') \wedge \tilde{B}(x_3')$ and $\tilde{B}(x_2') < \tilde{B}(x_2)$. The same triple of points would also violate the convexity of $\tilde{B}$, contrary to the minimality of $\tilde{B}(x_2)$.) Thus if $\tilde{B}$ is not convex, we can replace it by $\tilde{B}(x) \wedge \tilde{B}(x_2)$, which is convex, and we still have $\tilde{A}(x, y) = (\tilde{B}(x) \wedge \tilde{B}(x_2)) \wedge \tilde{C}(y)$; and similarly for $\tilde{C}$. ∎

**Proposition 5.15** *$\tilde{A}$ is a fuzzy rectangle if and only if $\tilde{A}^t$ is a rectangle for all $t \in [0, 1]$. (The rectangle may be degenerate: it may have zero width or height (or both), or it may be infinite in $x$ or $y$ or both, in one or both directions.)*

**Proof.** Suppose that $\tilde{A}$ is a fuzzy rectangle. Then there exists fuzzy subsets $\tilde{B}, \tilde{C}$ of $\mathbb{R}$ such that $\tilde{A}(x, y) = \tilde{B}(x) \wedge \tilde{C}(y)$. Thus $\forall t \in [0, 1]$, $\tilde{A}(x, y) \geq t$ if and only if $\tilde{B}(x) \geq t$ and $\tilde{C}(y) \geq t$. Hence

$\tilde{A}^t = \{x | \tilde{B}(x) \geq t\} \times \{y | \tilde{C}(y) \geq t\} = \tilde{B}^t \times \tilde{C}^t$.

By Theorem 5.14, if $\tilde{A}$ is a fuzzy rectangle, we can assume that $\tilde{B}$ and $\tilde{C}$ are convex. Thus for any $t$, $\tilde{B}^t$ and $\tilde{C}^t$ are convex, i. e., are intervals (possibly degenerate) so that their direct product is a rectangle. Conversely, if $\tilde{A}^t$ is a rectangle for all $t$, $\tilde{B}^t$ and $\tilde{C}^t$ must be intervals for all $t$, hence are convex. Thus $\tilde{B}$ and $\tilde{C}$ are convex, which makes $\tilde{A}$ a fuzzy rectangle by Theorem 5.14. ∎

A fuzzy subset $\tilde{A}$ of the plane is called a *fuzzy halfplane* if there exists a direction $x$ and a fuzzy subset $\tilde{B}$ of $\mathbb{R}$ such that

(a) $\tilde{A}(x, y) = \tilde{B}(x)$ for all $x, y \in \mathbb{R}$;

(b) $\tilde{B}$ is monotonically nonincreasing, i. e., $x_1 > x_2$ implies $\tilde{B}(x_1) \leq \tilde{B}(x_2)$.

Therefore, $\tilde{A}$ is a fuzzy halfplane if and only if the $\tilde{A}^t, 0 \leq t \leq 1$, are halfplanes (possibly degenerate).

**Proposition 5.16** *Let $\tilde{A}$ be a fuzzy halfplane. Then $\tilde{A}$ is convex.*

**Proof.** Let $\tilde{B}$ be a fuzzy subset of $\mathbb{R}$ such that $\tilde{A}(x, y) = \tilde{B}(x)$ $\forall x, y \in \mathbb{R}$. For any points $P, Q, R$ such that $R$ is on the line segment $\overline{PQ}$, let $u, v, w$ be the $x$-coordinates of $P, Q, R$ respectively. Then $u \leq w \leq v$ (or vice versa). Thus $\tilde{B}(u) \geq \tilde{B}(w) \geq \tilde{B}(v)$ (or vice versa) so that $\tilde{A}(P) \geq \tilde{A}(R) \geq \tilde{A}(Q)$ (or vice versa). Hence $\tilde{A}(R) \geq \tilde{A}(P) \wedge \tilde{A}(Q)$. ∎

Let $\tilde{A}_1, \ldots, \tilde{A}_k$ be fuzzy halfplanes whose associated directions $x_1, \ldots, x_k$ are in cyclic order (modulo $2\pi$). If every pair of successive directions (modulo $k$) differs by less than $\pi$, we call $\tilde{A}_1 \cap \ldots \cap \tilde{A}_k$ a *fuzzy convex polygon*. Since an infimum of convex fuzzy subsets is convex, it follows from Proposition 5.16 that a fuzzy convex polygon is a convex fuzzy subset. Thus, $(\tilde{A}_1 \cap \ldots \cap \tilde{A}_k)$ is a fuzzy convex polygon if and only if $\{(\tilde{A}_1 \cap \ldots \cap \tilde{A}_k)^t | 0 \leq t \leq 1\}$ is a set of convex polygons.

**Proposition 5.17** *A fuzzy subset $\tilde{B}$ of $\mathbb{R}$ is convex if and only if there exists fuzzy subsets $\tilde{B}_1, \tilde{B}_2$ of $\mathbb{R}$ such that $\tilde{B} = \tilde{B}_1 \cap \tilde{B}_2$, where $\tilde{B}_1$ is monotonically nonincreasing and $\tilde{B}_2$ is monotonically nondecreasing.*

**Proof.** As in the proof on Proposition 5.16, a nonincreasing or nondecreasing fuzzy subset of the line is convex. Hence the infimum of two such fuzzy subsets is convex. Conversely, let $\tilde{B}$ be convex. If $\tilde{B}$ ever decreases, it can never increase afterwards. Thus either $\tilde{B}$ is nondecreasing or nonincreasing, or it is first nondecreasing, say up to $x_0$, and then nonincreasing. Let
$\vee\{\tilde{B}(x) | x \in \mathbb{R}\} = M$.
Then we must have $\tilde{B}(x_0) = M$. Thus in the first two cases, we can take $\tilde{B}_1 \equiv \tilde{B}$ and $\tilde{B}_2 = M$ or vice versa, while in the last case, we can define $\tilde{B}_1 = M$ for $x \leq x_0, \tilde{B}_1 = \tilde{B}$ for $x \geq x_0$, and $\tilde{B}_2 = \tilde{B}$ for $x \leq x_0$, $\tilde{B}_2 = M$ for $x \geq x_0$. ∎

**Theorem 5.18** *A fuzzy rectangle is a fuzzy convex polygon.*

**Proof.** By Theorem 5.14, if $\tilde{A}$ is a fuzzy rectangle, we have $\tilde{A}(x, y) = \tilde{B}(x) \wedge \tilde{C}(y)$ , where $\tilde{B}$ and $\tilde{C}$ are convex. By Proposition 5.17, we have $\tilde{B} = \tilde{B}_1 \cap \tilde{B}_2$ and $\tilde{C} = \tilde{C}_1 \cap \tilde{C}_2$, where the $\tilde{B}$'s and $\tilde{C}$'s are monotonic. Hence $\tilde{A}$ is the infimum of four fuzzy halfplanes whose associated directions are $\pm x$ and $\pm y$. ∎

## 5.6 A Fuzzy Medial Axis Transformation Based on Fuzzy Disks

Let $S$ be a subset of a metric space. Let $P$ be any point of $S$ and let $D_P^S$ be the maximal disk centered at $P$ and contained in $S$. Then $S$ is the union of the $D_P^S$'s. Let $S'$ be any subset of $S$ such that for all $P \in S$, there exists $Q \in S'$ such that $D_P^S \subseteq D_Q^S$; we call $S'$ a *sufficient subset* of $S$. Clearly, for any such $S'$, $S$ is the union of the $D_Q^S$'s.

In particular, let $S$ be a set of nodes in a graph. Then the graph can be regarded as a metric space under the metric defined by path length. Let

$S^*$ be the set of nodes $Q$ of $S$ at which $D_Q^S$ is a local maximum (i. e., for any neighbor $P$ of $Q$, $D_P^S \subseteq D_Q^S$). Then $S^*$ is a sufficient subset of $S$. $S^*$ is called the *medial axis* of $S$ and the set of $D_Q^S$'s for $Q \in S^*$ is called the *medial axis transformation (MAT)* of $S$. When the graph is the set of pixels in a digital image under any of the standard definitions of 'neighbor', this reduces to a standard definition of the MAT [52].

In this section, we generalize the definition of the MAT to fuzzy subsets of a metric space. Our definition is based on the concept of a fuzzy disk, which is a fuzzy subset in which membership depends only on distance from a given point. Unfortunately, specifying the MAT may require, at times, more storage space than specifying $\tilde{A}$ itself.

There have been several generalizations of the MAT to grayscale images, e. g., using a 'gray-weighted' definition of distance [27,61], or using the methods of gray-weighted 'mathematical morphology' [39,56]; a related idea is the concept of thinning or skeletonization [18,29]. If the gray levels are scaled to lie in the range $[0, 1]$, we can regard the gray level of a pixel as its degree of membership in the set of high-valued ('bright') pixels. Thus a gray scale image can be regarded as a fuzzy subset.

The general definition of the MAT given above generalizes straightforwardly to fuzzy subsets of a metric space. We recall from Section 5.2 that for any metric, a fuzzy disk centered at the point $P$ is a fuzzy subset in which membership depends only on distance from $P$.

Let $D$ be a metric space with metric $d$ and let $\tilde{A}$ be a fuzzy subset of $D$. For each $P \in D$, let $\tilde{B}_P^{\tilde{A}}$ be the fuzzy subset of $D$ defined by for all $Q \in D$,

$$\tilde{B}_P^{\tilde{A}}(Q) = \wedge\{\tilde{A}(R) \mid d(P,R) = d(P,Q),\ R \in D\}.$$

Then, $\tilde{B}_P^{\tilde{A}}$ is a fuzzy disk and $\tilde{B}_P^{\tilde{A}} \subseteq \tilde{A}$. In fact, $\tilde{B}_P^{\tilde{A}}$ is the maximal disk centered at $P$ such that $\tilde{B}_P^{\tilde{A}} \subseteq \tilde{A}$. Let $Q \in D$. Then $\tilde{B}_Q^{\tilde{A}}(Q) \leq \left(\bigcup_{P \in D} \tilde{B}_P^{\tilde{A}}\right)(Q) = \vee\left\{\tilde{B}_P^{\tilde{A}}(Q) \mid P \in D\right\} \leq \tilde{A}(Q) = \tilde{B}_Q^{\tilde{A}}(Q)$.

Hence $\bigcup_{P \in D} \tilde{B}_P^{\tilde{A}} = \tilde{A}$.

Let $D'$ be any subset of $D$ such that

$$\forall P \in D, \exists Q \in D' \text{ such that } \tilde{B}_P^{\tilde{A}} \subseteq \tilde{B}_Q^{\tilde{A}}.$$

We call such a $D'$ an *$\tilde{A}$-sufficient subset* of $D$. For any such $D'$, we have

$$\tilde{A} \supseteq \bigcup_{Q \in D'} \tilde{B}_Q^{\tilde{A}} \supseteq \bigcup_{P \in D} \tilde{B}_P^{\tilde{A}} = \tilde{A} \text{ and so } \tilde{A} = \bigcup_{Q \in D'} \tilde{B}_Q^{\tilde{A}}.$$

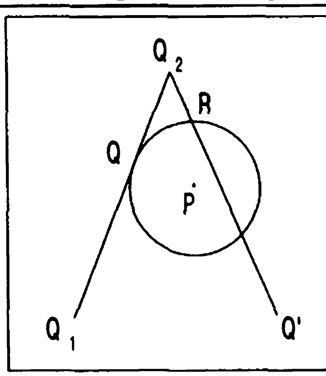
In particular, let $D$ be the set of nodes of a graph under the path metric. We say that $P \in D$ is a *(nonstrict) local maximum* of $\tilde{A}$ if $P$ has no neighbor $Q$ such that $\tilde{B}_P^{\tilde{A}} \subset \tilde{B}_Q^{\tilde{A}}$. Evidently, the set $D_{\tilde{A}}$ of such local maxima of $\tilde{A}$ is an $\tilde{A}$-sufficient subset of $D$ so that $\tilde{A}$ is such that $\tilde{A}(Q) = \vee\{\tilde{B}_P^{\tilde{A}}(Q) \mid P \in D_{\tilde{A}}\}$ for all $Q \in D$. We call $D_{\tilde{A}}$ the *fuzzy medial axis* of $\tilde{A}$, and we call $\{\tilde{B}_P^{\tilde{A}} \mid P \in D_{\tilde{A}}\}$ the *fuzzy medial axis transformation* of $\tilde{A}$. It is easily seen that if $\tilde{A}$ is a crisp subset of $D$, these definitions reduce to those given at the beginning of the section.

It should be pointed out that although crisp disks are convex, fuzzy disks are not necessarily convex. However, we can show that if $D$ is the plane under the Euclidean metric and $\tilde{A}$ is convex, then the $\tilde{B}_P^{\tilde{A}}$'s are all convex. Hence, a fuzzy disk centered at $P$ in the Euclidean plane is convex if and only if its membership function is a monotonically nonincreasing function of distance to $P$.

**Proposition 5.19** *If $\tilde{A}$ is convex, then the $\tilde{B}_P^{\tilde{A}}$ are convex for all $P$.*

**Proof.** Assume that there exists collinear points $Q_1, Q, Q_2$ with $Q$ on the line segment $\overline{Q_1Q_2}$ such that $\tilde{B}_P^{\tilde{A}}(Q) < \tilde{B}_P^{\tilde{A}}(Q_1) \wedge \tilde{B}_P^{\tilde{A}}(Q_2)$. Then there exists a point $R$ such that $\tilde{A}(R) < \tilde{B}_P^{\tilde{A}}(Q_1) \wedge \tilde{B}_P^{\tilde{A}}(Q_2)$ and $d(P,Q) = d(P,R)$. Now there exists a point $Q'$ such that $d(P,Q_1) = d(P,Q')$ and $Q', R$ and $Q_2$ are collinear with $R$ on the line segment $\overline{Q'Q_2}$. Since $d(P,Q_1) = d(P,Q'), \tilde{B}_P^{\tilde{A}}(Q_1) = \tilde{B}_P^{\tilde{A}}(Q')$. Thus $\tilde{A}(R) < \tilde{B}_P^{\tilde{A}}(Q') \wedge \tilde{B}_P^{\tilde{A}}(Q_2)$ and so $\tilde{A}$ is convex. This gives the desired result by contrapositive. ■

**FIGURE 5.1** Illustration of the proof of Proposition 5.19.




The converse of Proposition 5.19 is false. If $\tilde{A} = 1$ at two points and $\tilde{A} = 0$ elsewhere, then every $\tilde{B}_P^{\tilde{A}}$ is convex, but $\tilde{A}$ is not. In fact, $\tilde{A}$ is not even connected.

Since crisp disks are convex, we also have a generalization of the crisp MAT if we define the FMAT using the maximal convex fuzzy disk centered at (every) $P$ and not exceeding $\tilde{A}$. We shall consider both the general definition and this 'convex' definition in our examples.

The following discussion is from [37]. Let $D$ be the digital plane (the integer-coordinate lattice points) under the chessboard metric, i.e., $\max\{|x-u|, |y-v|\}$ for points $(x, y)$ and $(u, v)$. A 'disk' in this metric is an upright square of odd side length. Let $\tilde{A}$ be zero except on an $n \times n$ array of lattice points so that $\tilde{A}$ represents an $n \times n$ digital image with gray levels in the range $[0, 1]$. For any pixel $P$, we have $\tilde{B}_P^{\tilde{A}}(r) = 0$ for any $r \geq$ the chessboard distance $d_P$ from $P$ to the border of the image. Thus to specify $\tilde{B}_P^{\tilde{A}}$, we need only list its values for the (integer) chessboard distances up to $d_P$. In an $n \times n$ digital image, the average number of values will be on the order of $n$, and if the number of pixels that belong to the fuzzy medial axis is also $O(n)$, the total number of values needed to specify the FMAT is $O(n^2)$.

As an example, consider the $5 \times 5$ digital image shown in Table 5.1(a). For the pixels on the border of the image, $\tilde{B}_P^{\tilde{A}}$ is specified by the single value $\tilde{A}(P)$. For the pixels having value 0.2, $\tilde{B}_P^{\tilde{A}}$ is defined by the pair of values (0.2, 0.1) except for the lower left-hand 0.2, where $\tilde{B}_P^{\tilde{A}}$ is defined by (0.2, 0).Finally, for the center pixel, $\tilde{B}_P^{\tilde{A}}$ is defined by the triple of values (0.3, 0.2, 0). This implies that, as shown in Table 5.1(b), all the pixels having value 0.2, except the lower left-hand one, belong to the fuzzy medial axis (since the center of pixel's $\tilde{B}_P^{\tilde{A}}$ is not $\geq$ their $\tilde{B}_P^{\tilde{A}}$'s); thus 8 of the 25 pixels belong to the fuzzy medial axis. Thus specifying the FMAT requires 17 membership values (one disk requires three values, and seven require two each). This result remains true if we define the FMAT using only convex fuzzy disks since the disks in this example are all convex. Note that specifying the image itself requires only 25 values.

**TABLE 5.1** (a) $5 \times 5$ digital image. (b) $X$'s denote pixels belonging to the fuzzy medial axis of (a). [1]

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | | · | · | · | · | · |
| 0.1 | 0.2 | 0.2 | 0.2 | 0.1 | | · | $X$ | $X$ | $X$ | · |
| 0.1 | 0.2 | 0.3 | 0.2 | 0.1 | | · | $X$ | $X$ | $X$ | · |
| 0.1 | 0.2 | 0.2 | 0.2 | 0.1 | | · | · | $X$ | $X$ | · |
| 0.0 | 0.1 | 0.1 | 0.1 | 0.1 | | · | · | · | · | · |
| | | (a) | | | | | | (b) | | |

For real images, the situation can even be worse. In [37, Figures 3(a) and 4(a), p.588 and p.589] show, respectively, a $16 \times 41$ chromosome image and a $36 \times 60$ image of an '$S$'; each of them has 32 possible membership values. In the first case, all but 189 of the 656 pixels belong to the fuzzy medial axis [37, Figure 3(b), p.588], and in the second case all but 411 of the 2160 pixels belong to it [37, Figure 4(b), p.589]. The results are similar if we allow only convex disks in the FMAT. In the first case, we still need all

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but 211 of the pixels [37, Figure 3(c), p.588], and in the second case all but 730 [37, Figure 4(c), p.589]. (The fact that we need fewer maximal convex fuzzy disks, even though their values are smaller than those of the maximal fuzzy disks (the $\tilde{B}_P^{\tilde{A}}$'s), is apparently because the convex fuzzy disks have fewer nonstrict local maxima.)

Somewhat better results can be obtained by defining the FMAT using disks of bounded radius (i. e., disks whose memberships are 0 beyond a given radius $r$); evidently, for any $r \geq 0$ the image is still the supremum of these disks. (Of course, for $r = 0$ the FMAT is just the entire set of pixels in the image.) As we decrease $r$, the number of disks needed will increase, but since small disks are specified by fewer values, the total number of values needed may decrease. Unfortunately, for the images in [37, Figures 3 and 4] it turns out that for every value of $r$, the number of values needed to specify the FMAT is at least as great as the number of pixels in the images, as shown in Tables 5.2 and 5.3.

The FMAT, defined using either fuzzy disks or convex fuzzy disks, is a natural generalization of the MAT. Unfortunately, since $O(n)$ membership values are required to specify a fuzzy disk in an $n \times n$ digital image, the FMAT is a compact representation of the image only if it involves a relatively small number of fuzzy disks.

The compactness of the representation can be improved in two ways:

(a) The FMAT, like the MAT, is redundant. $\tilde{B}_P^{\tilde{A}}$ is not used if there exists a $Q$ such that $\tilde{B}_P^{\tilde{A}} \subseteq \tilde{B}_Q^{\tilde{A}}$, but we could use also eliminate many other $\tilde{B}_P^{\tilde{A}}$'s for which there exist sets $\mathcal{Q}$ of $Q$'s such that $\tilde{B}_P^{\tilde{A}} \subseteq \bigcup_{Q \in \mathcal{Q}} \tilde{B}_Q^{\tilde{A}}$. On this approach to reducing the redundancy of the MAT see [2].

(b) The MAT and FMAT completely determine the original image. For many purposes it would suffice to use a representation from which an approximation to the image could be constructed. A MAT-based approach of this type is described in [1].

It would be of interest to generalize both these approaches to the FMAT.

**TABLE 5.2** Number of disks and number of values needed for the chromosome image ([37, Figure 3], 656 pixels) when we use disks of radii $\leq 7, 5, 3, 2, 1$, or 0. [1]

| Radius | Disks | Values |
|---|---|---|
| 7 | 304 | 1237 |
| 5 | 310 | 1193 |
| 3 | 311 | 980 |
| 2 | 319 | 828 |
| 1 | 345 | 656 |
| 0 | 656 | 656 |

**TABLE 5.3** Number of disks and number of values needed for the $S$ image ([37, Figure 4], 2160 pixels) when we use disks of radii $\leq 17, 13, 10, 7, 5, 3, 1$ or 0.[1]

| Radius | Disks | Values |
|---|---|---|
| 17 | 1744 | 15,278 |
| 13 | 1749 | 14,678 |
| 10 | 1752 | 13,411 |
| 7 | 1764 | 11,251 |
| 5 | 1817 | 9,440 |
| 3 | 1903 | 7,074 |
| 1 | 2059 | 4,031 |
| 0 | 2160 | 2,160 |

## 5.7 Fuzzy Triangles

In this section, we introduce the notion of a fuzzy triangle in the plane. We define the notions of area, perimeter, and side lengths. We show that side lengths are related to the vertex angles by the Law of Sines. The material is based on [49].

For any direction $\theta$ in the plane, let $(x_\theta, y_\theta)$ be Cartesian coordinates with $x_\theta$ measured along $\theta$ and $y_\theta$ measured perpendicular to $\theta$. A fuzzy subset $\tilde{A}$ of the plane is called a *fuzzy halfplane* in direction $\theta$ (Section 5.5) if $\tilde{A}(x_\theta, y_\theta)$ depends only on $x_\theta$ and is a monotonically nondecreasing function of $x_\theta$. Hence, a level set of a fuzzy halfplane in direction $\theta$ is either the entire plane, or a halfplane bounded by a line perpendicular to $\theta$, or empty.

For example, let $\theta$ be a direction in the plane. Define the fuzzy subset $\tilde{A}$ of the plane by

$$\tilde{A}(x_\theta, y_\theta) = \begin{cases} 0 & \text{if } x_\theta < 1, \\ \frac{x_\theta - 1}{x_\theta} & \text{if } x_\theta \geq 1. \end{cases}$$

Then $1 \leq x_\theta < x'_\theta$ implies $\frac{x_\theta - 1}{x_\theta} < \frac{x'_\theta - 1}{x'_\theta}$ and so $\tilde{A}$ is a monotonically nondecreasing function of $x_\theta$. It follows that $\tilde{A}$ is a fuzzy halfplane in the direction of $\theta$.

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**Proposition 5.20** *If $\tilde{A}$ is a fuzzy halfplane, then $\tilde{A}$ is convex.*

**Proof.** Let $P, Q$, and $R$ be points such that $Q$ is on the line segment $\overline{PR}$. For any direction $\theta$, the $x_\theta$-coordinates $P_\theta, Q_\theta, R_\theta$ of such a collinear triple $P, Q, R$ must satisfy either $P_\theta \le Q_\theta \le R_\theta$ or $P_\theta \ge Q_\theta \ge R_\theta$. Hence $\tilde{A}(P) \wedge \tilde{A}(Q) \wedge \tilde{A}(R)$ must equal either $\tilde{A}(P)$ or $\tilde{A}(R)$. ∎

Fuzzy convex polygons of various types can be defined as infimums of fuzzy halfplanes (Section 5.5). Note that such polygons must be convex fuzzy subsets since an infimum of convex fuzzy subsets is convex. We will be primarily concerned with fuzzy triangles, with emphasis on the case where the membership functions are discrete-valued.

Let $\alpha, \beta, \gamma$ be three directions in the plane which are not all contained in a halfplane. Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy halfplanes in directions $\alpha, \beta, \gamma$, respectively. To avoid degenerate cases, we will assume that $\tilde{A}, \tilde{B}$,and $\tilde{C}$ are all nonconstant and all take on the value 0. Then $\tilde{A} \cap \tilde{B} \cap \tilde{C}$ is called a *fuzzy triangle.*

**Proposition 5.21** *Let $\tilde{A}, \tilde{B}, \tilde{C}$ be as described above. Any nonempty level set of $\tilde{A} \cap \tilde{B} \cap \tilde{C}$ is a triangle with its sides perpendicular to $\alpha, \beta, \gamma$.*

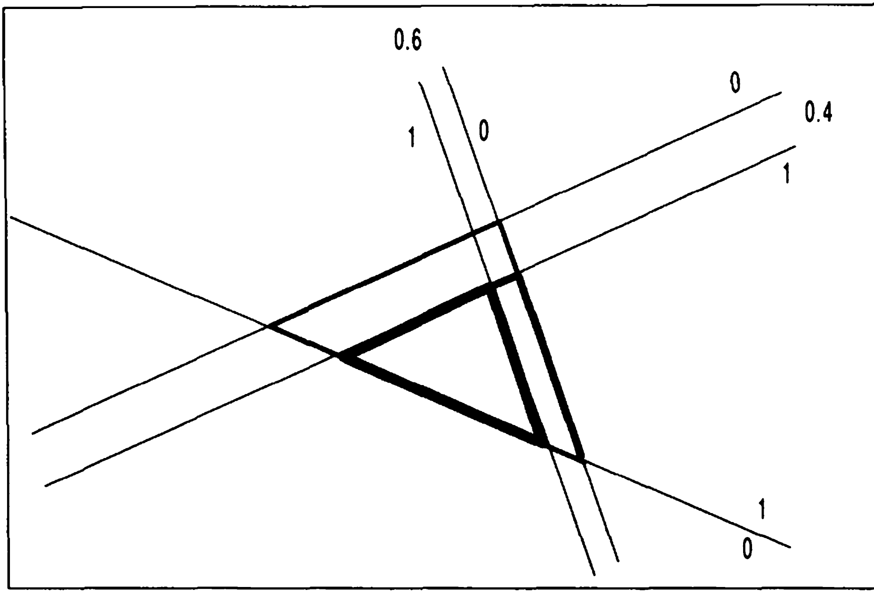
**Proof.** The nonempty level sets of $\tilde{A}$ are halfplanes bounded by lines perpendicular to $\alpha$ and they lie on the sides of these lines in the direction of $\alpha$ (i. e., the direction of nondecreasing $\tilde{A}$); and similarly for the level sets of $\tilde{B}$ and $\tilde{C}$. Now $\forall t \in [0,1]$, $\left(\tilde{A} \cap \tilde{B} \cap \tilde{C}\right)^t = \tilde{A}^t \cap \tilde{B}^t \cap \tilde{C}^t$. Since $\alpha, \beta$,and $\gamma$ are not all contained in a halfplane, an intersection of level sets of $\tilde{A}, \tilde{B}$, and $\tilde{C}$ is either empty or a triangle. ∎

Let $\tilde{A}, \tilde{B}$,and $\tilde{C}$ be discrete-valued and suppose that $\tilde{A} \cap \tilde{B} \cap \tilde{C}$ takes on the values $0 < t_1 < \ldots < t_n \le 1$. Then we can specify a fuzzy triangle $\tilde{T}$ by defining a nest of triangles $T_i$ each of which has its sides perpendicular to $\alpha, \beta$, and $\gamma$. On the innermost nonempty triangle $T_n$, $\tilde{T}$ has value $t_n$; on the remaining part of the triangle $T_{n-1}$ immediately surrounding $T_n$, $\tilde{T}$ takes on value $t_{n-1}, \ldots$; on the remaining part of the outermost triangle $T_1$, $\tilde{T}$ takes on value $t_1$; and its value on the rest of the plane is zero. Note that the $T_i$'s can be irregularly placed, as long as they are parallel-sided and nested; and note that the $T_i$'s must all be similar. A simple example of a fuzzy triangle, involving only the membership values $0, .4, .6$ and $1$, is shown in Fig. 5.2. This fuzzy triangle is defined by fuzzy halfplanes whose images in $[0,1]$ are $\{0,1\}, \{0, 0.6, 1\}$ and $\{0, 0.4, 0.6, 1\}$. As this example shows, some of the sides of the $T_i$'s may coincide.

Recall that the sup projection of a fuzzy subset $\tilde{A}$ onto a line $L$ is a fuzzy subset of $L$ whose value at $P \in L$ is the supremum of the values of $\tilde{A}$ on the line perpendicular to $L$ at $P$.

**Proposition 5.22** *A fuzzy triangle is completely determined by its sup projections on lines perpendicular to any two of the directions $\alpha, \beta, \gamma$.* ∎

**FIGURE 5.2** A fuzzy triangle.




Since these lines are parallel to the sides of the $T_i$'s, we can think of them as defining the directions of the sides of $\tilde{T}$.

Let the areas of $T_1, ..., T_n$ be $S_1, ..., S_n$, let their perimeters be $P_1, ..., P_n$, and let $\delta_i = t_i - t_{i-1}$ (where $t_0 = 0$). Then the *area* of $\tilde{T}$ is $S = \sum_{i=1}^{n} \delta_i S_i$.

We see that this sum counts the area $S_1$ of $T_1$ with weight $t_1$ and counts the area $S_i$ of each successive inner $T_i$ with additional weight $\delta_i$. The *perimeter* of $\tilde{T}$ is

$$P = \sum_{i=1}^{n} \delta_i P_i.$$

Let the side lengths of $T_i$ perpendicular to $\alpha, \beta$, and $\gamma$ be $a_i, b_i$, and $c_i$, respectively. Then we can define the *side lengths* of $\tilde{T}$ as

$$a = \sum_{i=1}^{n} \delta_i a_i, b = \sum_{i=1}^{n} \delta_i b_i, c = \sum_{i=1}^{n} \delta_i c_i.$$

Thus, we have $a + b + c = P$. Note that since the $T_i$'s are parallel-sided, they all have the same vertex angles, say $A, B, C$. We can regard these as the *vertex angles* of $\tilde{T}$. Note that by the Law of Sines, we have for each $T_i$

$$\frac{a_i}{\sin A} = \frac{b_i}{\sin B} = \frac{c_i}{\sin C}.$$

If we multiply by $\delta_i$ and sum over $i$, this gives the following result.

**Proposition 5.23** $a/\sin A = b/\sin B = c/\sin C$. ■

**Corollary 5.24** *If two vertex angles of $\tilde{T}$ are equal, their opposite side lengths must be equal, and conversely.* ■

Many properties of ordinary triangles do not generalize to arbitrary fuzzy triangles. For example, let the side lengths of a fuzzy right triangle be

$$a = \sum_{i=1}^{n} \delta_i a_i, b = \sum_{i=1}^{n} \delta_i b_i, c = \sum_{i=1}^{n} \delta_i c_i.$$

Since the $T_i$'s are all right triangles, we have $a_i{}^2 + b_i{}^2 = c_i{}^2$ for each $i$. However, we cannot conclude that $a^2 + b^2 = c^2$, in general. Some other generalization failures are described in [50].

## 5.8 Degree of Adjacency or Surroundedness

In this section, we propose definitions of the degree of adjacency of two regions in the plane and the degree of surroundedness of one region by another. Our results are from [54]. We show that some of these concepts have natural generalizations to fuzzy subsets of the plane. Applications of the proposed measures to digital polygons are given and algorithms for computing these measures are presented.

In describing a picture, one often needs to specify geometric relations among the regions of which the picture is composed. A review of such relations and their measurement in digital pictures can be found in [53].

The concept of adjacency is an important relation between regions. In a digital picture, sets $S$ and $T$ are adjacent if some border pixel of $S$ is a neighbor of some border pixel of $T$. In the Euclidean plane, regions $S$ and $T$ are adjacent if their borders intersect. However, this relation is not quantitative since $S$ and $T$ are not considered adjacent even if they are very close to one another. It also doesn't matter whether they are adjacent at one point or at many points. We propose a quantitative definition of adjacency which takes these factors into account.

Quantitative definitions of adjacency have been used in defining criteria for region merging in segmentation. For example, merge merit can be based on the length of common border of two regions relative to their total border lengths, [64]. However, this assumes that the regions are exactly adjacent. Parts of the borders that lie very close to one another do not contribute to the length of common border. The definition presented here takes near-misses into account, and can even be extended to define degree of adjacency for fuzzy subsets.

The concept of surroundedness is also an important region relation. All pictures are assumed to be of finite size. The region of the plane outside a

**FIGURE 5.3** Examples of near-adjacency (a, b) and non-adjancency (c).

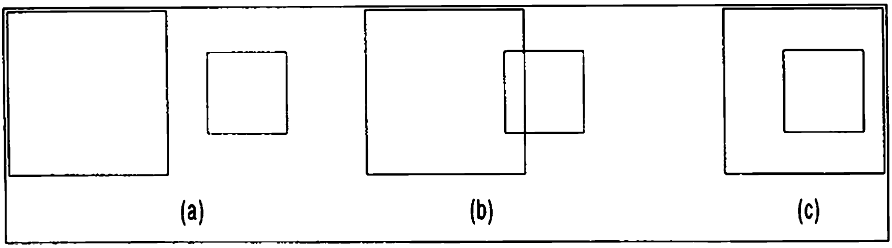
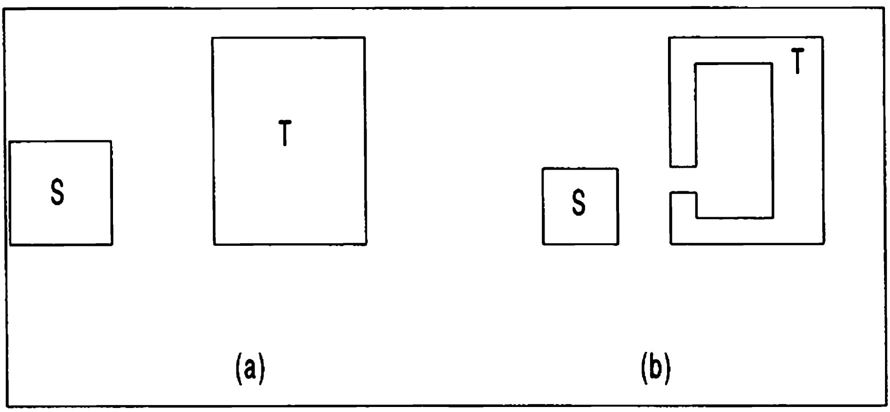



**FIGURE 5.4** The line of sight requirement in measuring adjancency.




picture is called the *background.* $S$ is said to *surround* $T$ if any path from $T$ to the background must intersect $S$. This definition is also non-quantitative. We propose two ways of defining the degree to which $S$ surrounds $T$.

We first consider quantitative adjacency in Euclidean regions.

Let $C_0$ be a rectifiable simple closed curve in the plane and let $C_1,\ldots, C_n$ be rectifiable simple closed curves not crossing one another and contained in the interior $\widehat{C}_0$ of $C_0$. According to the orientation of $C_0$, the closed set $C_0 \cup \widehat{C}_0$ is either a bounded set or the infinite plane except for a bounded set (infinite case). For $C_1,\ldots,C_n$, we assume that $\widehat{C}_1,\ldots,\widehat{C}_n$ are bounded sets. Then $(C_0 \cup \widehat{C}_0)\backslash(\widehat{C}_1 \cup \ldots \cup \widehat{C}_n)$ is called a *region.* $C_0$ is called its *outer border* and $C_1,\ldots,C_n$ are called its *hole borders.* In the infinite case, there is no unique distinction between the outer border and the hole borders because the outer border may be considered to be a hole border itself. The *perimeter* of the region is defined to be the sum $\sum_{i=0}^{n}|C_i|$ of the lengths of its borders.

It is intuitive to consider two regions $S$ and $T$ to be "somewhat" adjacent if some border of $S$ "nearly" touches some border of $T$. The degree of adjacency depends on how nearly they touch and along how much of their lengths they touch. The borders nearly touch if they are close to one another. That is illustrated in Figures 5.3(a) and (b). Note that $S$ and $T$ are allowed to overlap. However, not all cases in which borders are close to each other imply near-adjacency. This is shown in Figure 5.3(c). The difference is that in Figures 5.3(a) and (b), the shortest paths between the close borders lie outside both regions or inside both of them, while in Figure 5.3(c) these paths lie inside one region and outside the other. It also seems reasonable that only line of sight paths should be counted in defining adjacency. In Figure 5.4(a), the left-hand edge of $T$ should contribute to its degree of adjacency to $S$, but its other edges should not, and similarly in Figure 5.4(b), the parts of the border of the concavity in $T$ from which $S$ is not visible should not contribute. Finally, note that as can be seen from Figures 5.5(a) and (b), quantitative adjacency is not symmetric. In Figure 5.5(a), $S$ is highly adjacent to $T$ since much (or all) of its border nearly coincides with the border of $T$, but $T$ is not as highly adjacent to $S$ since only a small part of its border coincides with that of $S$.

Due to above line of reasoning, the degree of adjacency of $S$ to $T$ is defined as follows: Let $P, Q$ be any border points of $S$ and $T$, respectively, If $P \neq Q$, we say that the line segment $\overline{PQ}$ is *admissible*, with respect to $(S,T)$ understood, if its interior lies entirely outside both $S$ and $T$ or entirely inside both of $S$ and $T$. If $P = Q$, we call $\overline{PQ}$ *admissible* if the (signed) normals to the borders of $S$ and $T$ at $P$ do not point in the same direction. Let $d_P$ be the length of the shortest admissible line segment $\overline{PQ}$ having $P$ as an endpoint; if no such segment exists, let $d_P = \infty$. Then we define the adjancency of $S$ and $T$ as follows:

$a(S,T) = \int_{\partial S} 1/(d_P + 1)dP,$

where the integration is over the border $\partial S$ of $S$. Suppose for example that $S$ and $T$ are two squares of size $a \times a$ with distance $b$ between them. Then

$a(S,T) = a(T,S) = a/(1+b).$

As another example suppose that $S$ is a square of size $a \times a$ located at the center of a square $T$ of size $b \times b$, defining an infinite region $\overline{T}$. Then

$a(S,\overline{T}) = 4a/(1 + (b-a)/2).$

Hence

$a(S,\overline{T}) \to 4a$ as $a \to b.$

Consequently, a border point $P$ of $S$ contributes maximally to $a(S,T)$ if it also lies on the border of $T$ (and the conditions for the case $P = Q$ are met), since in this case $d_P = 0$ and

$1/(d_P + 1) = 1.$

It does not contribute at all if no admissible segment $\overline{PQ}$ exists, e.g., if the border of $T$ is not visible from $P$ since in this case $d_P = \infty$ and

$1/(d_P + 1) = 0.$

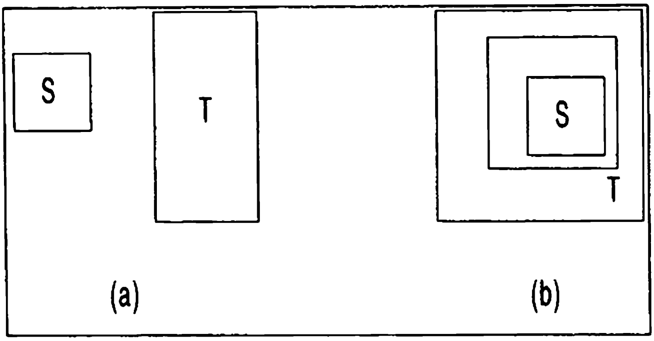
Since $d_P \geq 0$ in any case, we have

$1/(d_P + 1) \leq 1.$

Thus

$a(S,T) \leq \int_{\partial S} 1dP = p(S),$

**FIGURE 5.5** Degree of adjacency is not symmetric.




the perimeter of $S$. Now $a(S,T)$ can be normalized by dividing it by $p(S)$. It then lies between 0 (not at all adjacent) and 1 (maximally adjacent). For example, it follows from Proposition 5.26 that a hole in a region is maximally adjacent to that region. A different function $f(d_P)$ could have been used in place of $1/(d_P+1)$ in defining $a(S,T)$. The essential requirements are that $f$ be a monotonically decreasing function of $d_P$ and that $f(0)=1, f(\infty)=0$.

**Proposition 5.25** *Let $S$ and $T$ be regions. Then $a(S,T)=0$ if and only if $S \subseteq T$.*

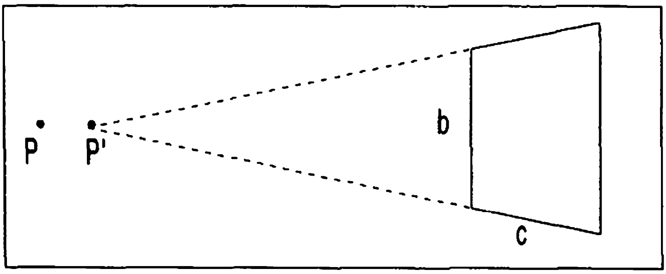
**Proof.** Suppose that $S \subseteq T$. Then there are no admissible segments $\overline{PQ}$. Note that where the borders touch, the signed normals of $S$ and $T$ point in the same direction. Conversely, suppose $S \not\subseteq T$. Then by the definition of a region, there must exist a border arc of $S$ at every point of which there is an admissible segment. Hence, $a(S,T) \neq 0$. ■

**Proposition 5.26** *Let $S$ and $T$ be regions. Then $a(S,T)=p(S)$ if and only if either $S$ is bounded, lies inside a hole in $T$ and its border is identical to the border of that hole or $S$ is unbounded, $T$ lies inside a hole in $S$ and the border of $S$ is identical to the outer border of $T$.*

**Proof.** Suppose the conditions hold. Then $d_P=0$ at every border point of $S$. Conversely, under no other circumstances can the entire border of $S$ coincide with a part of the border of $T$. Note that since a region cannot consist of several isolated parts, it must be contained in one hole only. ■

The definition of $a(S,T)$ can be extended to types of sets other than regions. As an example, consider a single point $P$. Define $a(P,T)$ to be 0 if $P \in \widehat{T}$ and $1/(d(P,T)+1)$ if $P \notin \widehat{T}$, where $d(P,T)$ is the distance from $P$ to $T$. (We note that this definition is not exactly analogous to the one for regions; a single point has a zero border length, so that integrating over it should always give zero. The analogy would be better if, in the region

**FIGURE 5.6** The degree of adjacency of a region $T$ to a point $P$ is not necessarily a monotonically decreasing function of $d(P,T)$ and is not necessarily a continuous function of the position of $P$.




definition, we normalized $a(S,T)$ by dividing by $p(S)$. We also note that it follows from this definition that $a(P,T) = 1$ when $P$ is on the border of $T$.) Conversely, for a single point $Q$ we can define $a(S,Q)$ using the original definition for sets $S$ and $\{Q\}$. Here too there are no admissible segments if $Q \in \widehat{S}$, but that otherwise $a(S,Q)$ is obtained by integration over the part of the border of $S$ visible from $Q$. Similarly, we can define $a(S,T)$ if $S$ or $T$ is an arc, although the details are not provided here. If the $a(S_i,T_j)$ are defined for $S_1,...,S_m$ and $T_1,...,T_n$, it is also possible to define $a(\cup_{i=1}^m S_i, \cup_{j=1}^n T_j)$.

However, we omit the details.

**Proposition 5.27** *If $T' \subseteq T$ and $S \cap T = \emptyset$, then $a(S,T') \leq a(S,T)$.*

**Proof.** Let $\overline{PQ}$ be any admissible segment in the definition of $a(S,T)$. Let $R$ be the first point in which $\overline{PQ}$ meets $T$. Then $\overline{PR}$ or a shorter line segment is admissible in the definition of $a(S,T)$. (Note that $R \neq P$ since $S \cap T = \emptyset$.) If $d_P, d'_P$ are the lengths of the shortest admissible segments in the definitions of $a(S,T)$ and $a(S,T')$, respectively, then $d_P \leq d'_P$. (Note that for some $P'$s there may be admissible segments with respect to $T$, but not with respect to $T'$.) Thus

$\int_{\partial S} 1/(d'_P + 1)dP < \int_{\partial S} 1/(d_P + 1)dP.$ ■

**Proposition 5.28** *If $P, P' \notin T$ and $d(P',T) \leq d(P,T)$, then $a(P',T) \geq a(P,T)$, but it is not necessarily the case that $a(T,P') \geq a(T,P)$.*

**Proof.** We have that

$a(P',T) = 1/(d(P',T) + 1) \geq 1/(d(P,T) + 1).$

On the other hand, let $P, P'$ and $T$ be as shown in Figure 5.6. Then for $P$ sufficiently close to $P'$, the contributions of side $b$ to $a(T,P)$ and $a(T,P')$ are approximately equal, but sides $a$ and $c$ do not contribute to $a(T,P')$. Hence its total contribution is smaller. (See also the example above.) ■

**Proposition 5.29** *Suppose that $P \notin T$. Then $a(P,T)$ is a continuous function of the position of $P$, but $a(T,P)$ is not.*

**Proof.** The proof follows from Proposition 5.28 and the example given there. ■

The definition of adjacency can be generalized to the case of bounded fuzzy subsets of the plane, i. e., fuzzy subsets which are equal to zero outside a bounded region $\overline{B}$. Let $\widetilde{A}$ and $\widetilde{B}$ be such fuzzy subsets. The desirability of defining geometric concepts in the fuzzy case, so that they can be measured without having to first crisply segment a picture, is discussed in [46].

We assume in the following that $\widetilde{A}$ and $\widetilde{B}$ are 'piecewise constant' in the following sense. Partition $\overline{B}$ into a finite number of regions whose interiors are disjoint and such that the border of each region is contained in the union of the borders of the other regions. Let $U$ be the union of all borders of these regions. In the interior of each region, $\widetilde{A}$ has constant value and at each point of a border, it has one of the neighboring interior values. Another case of interest is that in which $\widetilde{A}$ and $\widetilde{B}$ are 'smooth', that is, everywhere differentiable. Note that we can approximate a piecewise constant $\widetilde{A}$ by a smooth $\widetilde{B}$ which is constant except near the borders, where it changes rapidly from one constant value to another. 'Smooth' versions of the definitions in this section could be given, using derivatives in place of differences.

If $P \neq Q$, we call the segment $\overline{PQ}$ *admissible with respect to* $(\widetilde{A}, \widetilde{B})$ (or simply *admissible*) if

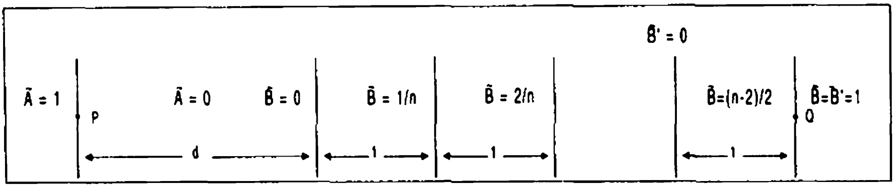
(a) $P$ is on a border of $U$ and $Q$ is on a border of $U$. We assume that only two of the constant regions of $\widetilde{A}(\widetilde{B})$ meet at $P(Q)$. (More than two regions meet only at a finite number of points and these can be ignored in defining degree of adjacency.)

(b) Let $R$ be a point of $\overline{PQ}$ such that $R \neq P$ and $\widetilde{A}$ changes value at $R$ as we move from $P$ to $Q$ and let $\Delta_R$ be this change in value. Note that there can be only a finite number of such $R$'s. Let the values of $\widetilde{A}$ at the two regions that meet at $P$ be $a$ and $b$, where $b$ is the value on $\overline{PQ}$ near $P$, and let $\Delta_P = a - b$. Assume that $\Delta_P$ and all the $\Delta_R$'s have the same sign and that $|\Delta_P| > |\Delta_R|$ for all $R$.

(c) Let $\nabla_R$ and $\nabla_Q$ be defined analogously to $\Delta_R$ and $\Delta_P$ in (b), with $\widetilde{B}$ replacing $\widetilde{A}$ and the roles of $P$ and $Q$ reversed. Assume that $\nabla_Q$ and all the $\nabla_R$'s have the same sign. Assume also that this is the same sign as in (b) and that $|\nabla_Q| > |\nabla_R|$ for all $R$.

It follows from conditions (b) and (c) that the changes in $\widetilde{A}$ as we move from $P$ to $Q$ are all in the same direction and that the change 'at $P$' is the largest of them. The changes in $\widetilde{B}$ as we move from $Q$ to $P$ are analogous. What this means is that the border points $P$ and $Q$ of $U$ are facing either toward or away from each other since the changes have the same signs in both cases. This also means that no 'stronger' border points

**FIGURE 5.7** Counterexample to the fuzzy generalization of Proposition 5.27




(at which larger changes occur) lie between them, so that they are within 'line of sight'. Clearly if $\tilde{A}$ and $\tilde{B}$ are crisp, these conditions reduce to the definition of admissibility given previously for $P \neq Q$.

When $P = Q$, we call $\overline{PQ}$ *admissible* if $P$ is a border point of $U$ and the changes in $\tilde{A}$ and $\tilde{B}$ at $P$ in a fixed direction from one region of the partition of $\overline{B}$ to the other touching it at $P$ have opposite sign. In this case, let $\Delta_P$ and $\nabla_Q$, respectively, be the changes in the values of $\tilde{A}$ and $\tilde{B}$ at $P$, defined as in the preceding paragraph, where $\Delta_P \nabla_Q \geq 0$ is assumed.

For each $P$, let

$g(P) = \vee\{\Delta_P \nabla_Q/(d(P,Q)+1) \mid \overline{PQ} \text{ is admissible}\}$,

where $d(P,Q)$ is the distance from $P$ to $Q$. The numerator is always positive since the changes in $\tilde{A}$ and $\tilde{B}$ at $P$ and $Q$ have the same sign. Since $|\Delta_P| \leq 1$ and $|\nabla_Q| \leq 1$, the numerator is in the interval $(0,1]$. Hence, in the crisp case the numerator must be 1, and the sup is achieved when the denominator is as small as possible. Hence $g(P)$ is the same as $1/(d_P+1)$ mentioned above. It is understood that $g(P) = 0$ if no admissible $\overline{PQ}$ exists.

The definition of $g(P)$ involves a trade-off between the border strengths (= sizes of changes) at $P$ and $Q$ and the distance $d(P,Q)$: the supremum may arise from weak changes that are close together (or even coincide) or from stronger changes that are farther apart. The nature of the trade-off can be manipulated by using some other monotonic function of $d(P,Q)$ in place of $1/(d_P+1)$. It is of interest to compare this with the previous remark about $f(P)$.

The adjacency between $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$a(\tilde{A},\tilde{B}) = \int_U g(P)dP$,

where the integration is along all borders $U$ of the partition of $\overline{B}$. It may be of interest to extend this definition by defining $a(P,\tilde{B})$ and $a(\tilde{A},Q)$ where $P$ and $Q$ are points and to investigate the possibility of fuzzy generalizations of the propositions above. We prove here only the following result.

**Proposition 5.30** *Let $p(\tilde{A})$ denote the perimeter of $\tilde{A}$. Then $a(\tilde{A},\tilde{B}) \leq p(\tilde{A})$.*

**Proof.** By Definition 5.2, $p(\tilde{A})$ is just the sum of the lengths of the border arcs of $U$ at which pairs of regions of $\tilde{A}$ meet, each multiplied by the absolute difference in value between that pair of regions. This difference at a given border point $P$ is $\triangle_P$. Hence

$p(\widetilde{A}) = \int_{U} |\triangle_P| dP \geq \int_{U} g(P) dP$

for all $\widetilde{A}$ since

$g(P) = \triangle_P \nabla_Q / (d(P,Q) + 1)$

for a certain point $Q$ with $|\nabla_Q| \leq 1$ and

$1/(d(P,Q) + 1) \leq 1$,

while $\triangle_P \nabla_Q = |\triangle_P| \cdot |\nabla_Q|$ since they have the same sign. ■

The fuzzy generalization of Proposition 5.27 does not hold. This follows since $\widetilde{A} \cap \widetilde{B} = \chi_0$ and $\widetilde{B}' \subseteq \widetilde{B}$ do not imply $a(\widetilde{A}, \widetilde{B}') \leq a(\widetilde{A}, \widetilde{B})$. In Figure 5.7, $\overline{PQ}$ is admissible for $\widetilde{B}$ and

$g(P) = 1/(d + n - 1)$.

With respect to $\widetilde{B}$, however, the steps in value are all $1/n$, except for the last, which is $2/n$. The only possible maximal values of $g(P)$ are thus

$(1/n)/(d + 1)$ and $(2/n)/(d + n - 1)$.

If $n > 2$ and $d + n - 1 < (d + 1)n$, these values are both smaller than that for $\widetilde{B}'$. This counterexample would fail if a different definition of $g(P)$ were used such as

$[\triangle_P/(d(P,Q) + 1) + \sum_{R \in \overline{PQ}} (\triangle_R/(d(R,Q) + 1)] \times [\nabla_Q/(d(Q,P) + 1) + \sum_{R \in \overline{QP}} (\nabla_R/(d(R,P) + 1)$.

However, this alternative approach is not pursued here.

In the example in Figure 5.6, $T$ is a polygon. If $T$ has a smooth boundary, then $a(T, P)$ is continuous, but whether or not it is monotonic may depend on the function of distance that is used in defining adjacency. (We use $1/(d + 1)$ here.) For example, if $T$ is a disk, as $P$ moves away from $T$ the amount of $T$'s border visible from $P$ increases and this may compensate for the fact that the border is farther away from $P$.

We now consider digital polygons.

Subsets of digital pictures may be considered from different points of view. For example, they may be considered as sets of grid points, as sets of cells, or as digital polygons. For the purpose of defining quantitative adjacency in the digital (crisp) case, it is convenient to deal with digital polygons.

In a digital simple polygon $S = \langle P_0, P_1, ..., P_n \rangle$ for $k = 0, 1, ..., n$, the $P_k$ are all grid points with integer coordinates. Points $P_k$ and $P_{k+1}$ are 8-neighbors, where $P_{n+1} = P_0$. In relation to the interior of $S$, the sequence $P_0, P_1, ..., P_n$ has clockwise orientation; the border of $S$ is non-crossing, i.e. $S$ is a simple polygon in the usual sense. Since the orientation is fixed, finite and infinite digital simple polygons can both be defined in this way. For $S = \langle P_0, P_1, ..., P_n \rangle$, the complementary polygon $S$ is given by $\langle P_n, P_{n-1}, ..., P_0 \rangle$.

Since digital simple polygons are regions as defined above, the degree of adjacency $a(S, T)$ is defined for digital polygons $S$ and $T$. However, for the needs of picture processing or computer graphics, a more specifically digital approach is used. With this in mind, the set of border points $BP(S)$ of a digital polygon $S$ are restricted to grid points on the (real) borders of $S$, i.e., $BP(S) = \{P_0, P_1, ..., P_n\}$ for $S = \langle P_0, P_1, ..., P_n\rangle$. Admissible line segments are defined as above for $P \in BP(S)$ and $Q \in BP(T)$, where $S$ and $T$ are digital polygons. (At a vertex of a digital polygon, the (signed) normal is defined to be the bisector of the vertex angle.) Let $AL(S,T)$ denote the set of all admissible line segments from border points of $S$ to border points of $T$. Then $AL(S,T) \neq AL(T,S)$ for almost all digital polygons $S$ and $T$ and $AL(S,\overline{S}) = BP(S)$, and $AL(S,S) = \emptyset$. For digital polygons $S$ and $T$, define the digital degree of adjacency as follows:

$a_{dig}(S,T) = \sum_{P\in BP(S)} 1/(1+d_P)$ if $AL(S,T) \neq \emptyset$ and 0 otherwise.

It follows that all the properties given above for the function $a$ are true for $a_{dig}$ also. The property $a(S,T) \leq p(S)$, the perimeter of $S$, is replaced by $a_{dig}(S,T) \leq \text{card}(BP(S))$, which may be considered to be the digital perimeter of $\overline{S}$.

**Proposition 5.31** *Let $S$ and $T$ be digital polygons. Then $a_{dig}(S,T) = card(BP(S))$ if and only if $T = \overline{S}$.*

**Proof.** Since $AL(S,\overline{S}) = BP(S)$, $d_p = 0$ for all $P \in BP(S)$. Thus, $a_{dig}(S,\overline{S}) = \text{card } BP(S)$. Conversely, if $T \neq \overline{S}$, then there exists at least one point $P \in BP(S)$ with $d_p > 0$. ■

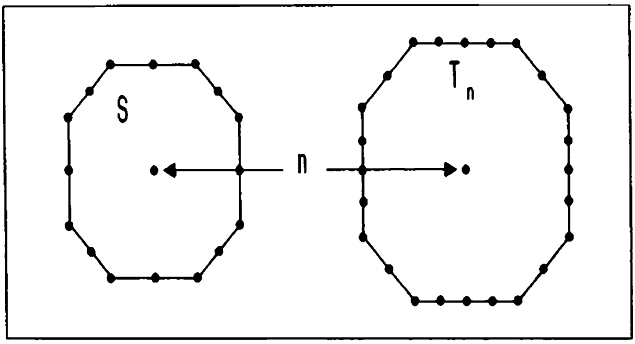
The *normalized degree of adjacency* is defined as follows:

$a^*_{dig}(S,T) = a_{dig}(S,T)/\text{card}(BP(S))$,

where $S$ and $T$ are digital polygons. The following examples illustrate the behavior of this concept of degree of adjacency.

**Example 5.6** *Let $S$ and $T_n$ be convex digital polygons with distance $n$ between them, as shown in Figure 5.8. For different values of $n$, the polygon $T_n$ changes its position in relation to $S$. For example, for $n = 0, S$ is in a centralized position within $T_0$ and for $n = 7$ and $n = -7$, $S$ and $T_n$ are in touching positions. By symmetry, $a_{dig}(S,T_n) = a_{dig}(S,T_{-n})$ and $a_{dig}(T_n,S) = a_{dig}(T_{-n},S)$ for $n = 0,1,2,....$ For the normalized degrees of adjacency, we use card$(BP(S)) = 16$ and card$(BP(T_n)) = 24$. For $n = 0,1,2,...,10$, the values of $a^*_{dig}(S,T_n)$ and $a^*_{dig}(T_n,S)$ are given in [54, Table 1, p. 175]. These values are graphically illustrated in Figure 5.9. As seen in this figure, there is a somewhat unbalanced behavior of the proposed measure $a^*_{dig}$ for intersecting positions ($-6 \leq n \leq 6$) and non-intersecting positions ($|n| \geq 8$) of the two polygons. Even for the 'most adjacent' positions ($n = 7$) we don't have the maximum value, which occurs when $|n| = 2$ This behavior is due to the influence of border points for which $P = Q, P \in BP(S)$ and $Q \in BP(T_n)$. The definition of $a^*_{dig}$ may be changed by requiring $d_P = 0$ if and only if there are border segments $PP'$ and $QQ'$ of $S$ and $T_n$, respectively, such that $P \neq P'$, $PP' = QQ'$ and the signed normals of $S$ and $T_n$ on this common border segment point in exactly inverse directions. In all other cases of $P = Q, P \in BP(S)$ and*

**FIGURE 5.8** Two convex digital polygons $n$ units apart.




*$Q \in BP(T_n)$, we let $d_P = \infty$. The resulting modified function $a^*_{dig}$ is denoted by $a^+_{dig}$. Results for $a^+_{dig}$ can be found in [54, Table 1, p. 175] for* $-10 \leq n \leq 10$.

Convex digital polygons are restricted to be octagons at most, where 'convexity' is understood as in the real plane. Then, for two convex digital polygons $S$ and $T$ with $n =$ card$(BP(S))+$ card$(BP(T))$, there exists an $O(n)$ worst case time algorithm for computing $a_{dig}(S,T)$ or $a^*_{dig}(S,T)$. The basic ideas of the algorithm are as follows.

(i) Determine the upper and lower tangents on $S \cup T$ by dividing the polygonal border of $S$ (or $T$) into a connected part, where no admissible line segments to $T$ (or to $S$) are possible, and a second connected part containing all points which may contribute to $a_{dig(S,T)}$ (or to $a_{dig}(T,S)$).

(ii) Perform two search procedures, one top-down and one bottom-up, to compute candidate values for $d_P$ for all points $P$ in the interesting part of the border of $S$, where only points in the interesting part of the border of $T$ need be considered. In both search procedures the connection line between two points $P \in BP(S)$ and $Q \in BP(T)$ which are under consideration for computing $d_P$ moves monotonically down (or up) in the interesting parts of the borders of $S$ and $T$. During these search procedures, at most two crossings of the borders of $S$ and $T$ are possible.

(iii) For all points $P$ in the interesting part of the border of $S$, take the minimum of both candidate values found in (ii) to compute $d_P$.

(iv) Determine $a_{dig}(S,T)$ by using the values of $d_P$.

Using small examples, it can already be seen that two search procedures are necessary in step (ii), which cooperate to give the final result in step (iv).

**Example 5.7** *Here, we consider a moving point in relation to a fixed digital polygon $T$; see Proposition 5.28. $P_n$ denotes the point at distance $n$ from $T$, as illustrated in Figure 5.10. In this case, it follows that $a_{dig}(P_n,T) =$*

*$1/(n+1)$ for $n \geq 0$ and $a_{dig}(T, P_0) = 1, a_{dig}(T, P_1) = 1.94646, a_{dig}(T, P_2) = 1.47377, a_{dig}(T, P_3) = 1.99526$, etc.*

Functions $a^*_{dig}(S, T_n)$ and $a^*_{dig}(T_n, S)$ are described pictorially in [54, Figure 7, p. 174] for the polygon in Figure 5.8, for $-10 \leq n \leq +10$. Results for Example 5.6 can be found in [54, Table 1, p. 175].

We now consider quantitative surroundedness. We present two quantitative definitions of surroundedness in the Euclidean plane. We show how each of them generalizes to the fuzzy case. We then discuss quantitative surroundedness in digital pictures. We first consider visual surroundedness.

Let $P$ be a point and $T$ a bounded set. Let $r_\theta(P,T) = 1$ if the ray emanating from $P$ in direction $\theta$ meets $T$ and $r_\theta(P,T) = 0$ otherwise. Define the *degree of visual surroundedness* of $P$ by $T$ as follows:

$v(P,T) = 1/2\pi \int_0^{2\pi} r_\theta(P,T)d\theta.$

We note that this integral might not be defined for certain sets $T$, but it is defined for various types of well-behaved sets such as regions and arcs.

If $S$ is a (well-behaved) set, define $v(S,T)$ as $\wedge\{v(P,T) \mid P \in S\}$. (Another possibility would be to take the 'average' value of $v(P,T)$ for all $P \in S$). It can be shown that $v(S,T)$ is defined by a border point of $S$, i.e., $v(S,T) = \wedge\{v(P,T) \mid P \in \partial S\}$.

For the sets given in Figure 5.10, $v(S,T) = v(P,T) = \tan^{-1}(5/6)/\pi = 0.2211$ and $v(T,S) = v(Q,S) = \tan^{-1}(1/2)/\pi = 0.1476$.

If $\widetilde{A}$ and $\widetilde{B}$ are fuzzy subsets, define $r_\theta(P, \widetilde{A}, \widetilde{B}) = 1$ if $\widetilde{B}(R) \geq \widetilde{A}(P)$ at some point $R$ on the ray emanating from $P$ in direction $\theta$. (Recall that $\widetilde{B}$ surrounds $\widetilde{A}$ if for any point $P$ and any path $\pi$ from $P$ to $B$, there exists $R \in \pi$ such that $\widetilde{B}(R) \geq \widetilde{A}(P)$.) Then define $v(P, \widetilde{A}, \widetilde{B})$ to be

$1/2\pi \int_0^{2\pi} r_\theta(P, \widetilde{A}, \widetilde{B})dP$

and define $v(\widetilde{A}, \widetilde{B})$ by taking the minimum over all $P$ in the plane. (In the case of taking the 'average' value of $v(P, \widetilde{A}, \widetilde{B})$, the denominator for the average is

$\int_{\widetilde{B}} \widetilde{A} dP$.

It follows that this generalizes the crisp definition.

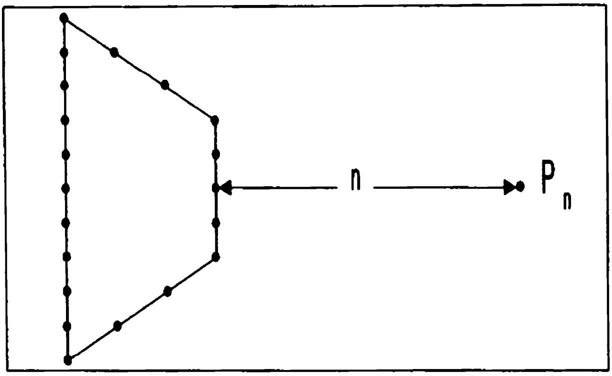
**Proposition 5.32** *Suppose that $T \supseteq T'$. Then $v(P,T) \geq v(P,T')$ for any $P$ and $v(S,T) \geq v(S,T')$ for any $S$.*

**Proof.** The result is immediate from the fact that $r_0(P,T) \geq r_0(P,T')$ for any $P$. ■

Analogously, in the fuzzy case, if $\widetilde{B} \supseteq \widetilde{B}'$, then $v(P, \widetilde{A}, \widetilde{B}) \geq v(P, \widetilde{A}, \widetilde{B}')$ for any $P$ and $\widetilde{A}$.

Clearly, $v(P,T)$ is a continuous function of the position of $P$. On the other hand, $v(P,T)$ need not increase as $P$ moves closer to $T$, even if $T$ is convex. This is illustrated in Figure 5.11.

**FIGURE 5.9** Point $P_n$ at distance $n$ from polygon $T$, as used in Example 5.7.




If $T$ subtends angle $\alpha$ from $P$, it follows that $v(P,T) = \alpha/2\pi$. Consequently, if $T$ is convex, as $P$ approaches $T$, $v(P,T)$ approaches $1/2$ since $\alpha$ approaches $\pi$. If $P \in T$, then $v(P,T) = 1$.

Let $T$ be non-convex and let $H(T)$ be its convex hull. It follows easily that if $P \notin H(T)$, then any ray from $P$ that meets $H(T)$ must also meet $T$. Hence, $v(P, H(T)) = v(P,T)$. In order for $v(P.T)$ to exceed $1/2$, $P$ must lie in $H(T)$.

We now consider topological surroundedness. Even if $v(P,T) = 1, T$ may not surround $P$ in the usual sense since there may be a curved path from $P$ to $B$ (the 'background' region, outside the picture) that does not intersect $T$. This is illustrated in Figure 5.12. We now introduce an alternative definition of quantitative surroundedness that is more closely related to the usual topological definition.

The degree to which $T$ topologically surrounds $P$ is intuitively related to how much a path from $P$ must change direction in order to reach $B$ without intersecting $T$. For example, if $T$ is a spiral and $P$ is 'surrounded' by a very large number of turns of $T$, a path from $P$ that does not intersect $T$ must turn through a very large multiple of $2\pi$ before it can reach $B$.

Let $\pi_\theta$ be any rectifiable path from $P'$ to $B$ that starts at $P'$ in direction $\theta$ and that does not intersect $T$. (If no such path exists, define $t(P',T) = \infty$.) Let

$C_{\pi_\theta}(P',T) = \int_{\pi_\theta} \mid c_{\pi_\theta}(P) \mid dP$,

where $c_{\pi_\theta}(P)$ denotes the absolute curvature of $\pi_\theta$ at a point $P$ on $\pi_\theta$. Let

$C_{\pi_\theta}(P',T) = \wedge\{C_{\pi_\theta}(P',T) \mid \pi_\theta\}$

if $T$ is a 'well behaved' set (e.g. a region) and $P' \notin T$ and $P'$ is not inside a hole of $T$. Then $C_\theta(P',T)$ is finite. Finally, let

$t(P',T) = 1/2\pi \int_0^{2\pi} C_\pi(P'.T)d\theta$,

the average of $C_\theta(P'.T)$ over $\theta$. We could have used $\wedge\{C_\theta(P',T)$ for our definition of topological surroundednss, but using the average allows our

**FIGURE 5.10** Example sets for illustrating surroundedness.

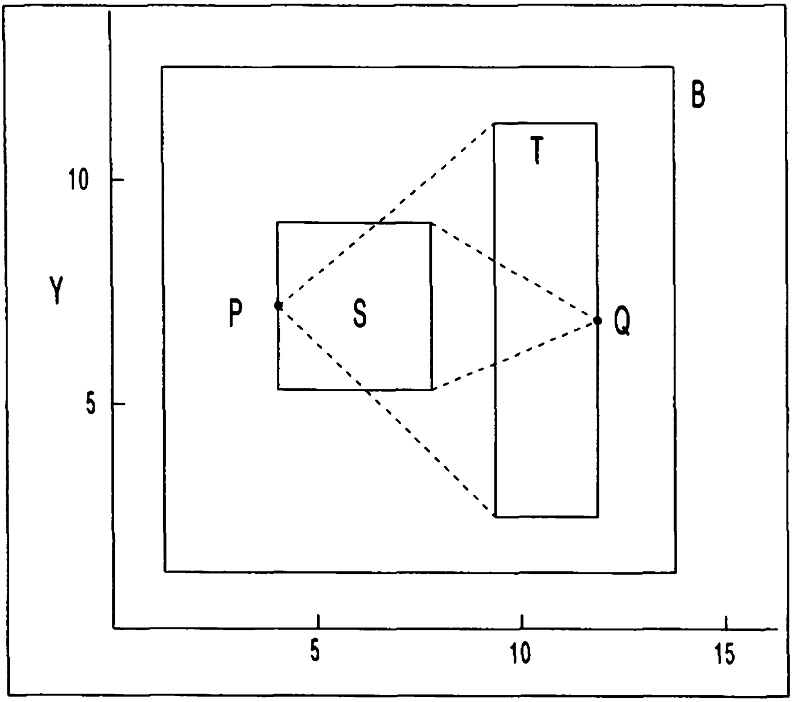
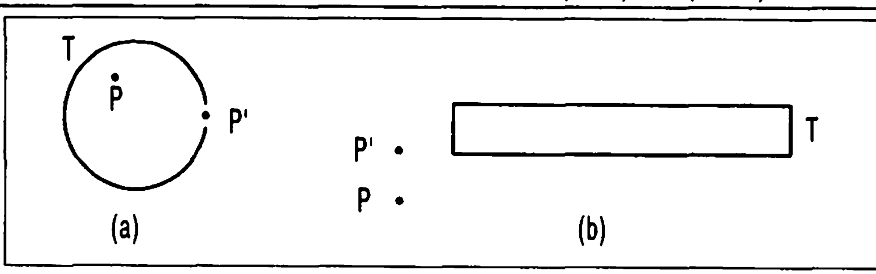



**FIGURE 5.11** $P'$ is closer to $T$ than $P$, but $v(P,T) > v(P',T)$.



definition to be sensitive to 'partial' surroundedness of $P'$ by $T$. For example, if $T$ is a circle with a small gap and $P'$ is at its center, there exists a direction in which $\pi_\theta$ does not have to turn at all, so that the definition gives 0, as if $T$ were not there at all. On the other hand, the averaging definition reflects the fact that some paths may have to turn by as much as $\pi$ before they can get out of $T$. In fact, the average is approximately $\pi/2$, but it gets smaller as the gap in the circle gets wider.

If $S$ is a (well behaved) set, define $t(S,T)$ as $\wedge\{t(P,T) \mid P \in \partial S\}$. If $\tilde{A}$ and $\tilde{B}$ are fuzzy subsets, use analogous definitions, except that $\pi_\theta$ is a path from $P$ to $B$ such that $\tilde{B}(R) < \tilde{A}(P)$ for all $R$ on $\pi_\theta$. This is the fuzzy version of 'does not intersect $T$'. In the fuzzy case, $t(\tilde{A},\tilde{B})$ would be $(\int\int t(P,\tilde{A},\tilde{B})dxdy)/\int\int \tilde{A}dxdy$ if we use the averaging definition.

**Proposition 5.33** *Suppose that $T \supseteq T'$. Then $t(P,T) \geq t(P,T')$ for all $P$ and $t(S,T) \geq t(S,T')$ for all $S$.*

**Proof.** Any path (from any $P$) that meets $T'$ also meets $T$. Hence the desired result follows immediately. ■

Analogously, in the fuzzy case, if $\tilde{B} \supseteq \tilde{B}'$, then $t(\tilde{A},\tilde{B}) \geq t(\tilde{A},\tilde{B}')$ for any $\tilde{A}$.

It follows easily that $t(P,T)$ is a continuous function of the position of $P$. However, $t(P,T)$ need not increase as $P$ moves closer to $T$, even if $T$ is convex. This can be seen from the examples in Figure 5.11.

Let $T$ be convex and subtend angle $\alpha$ at $P$. Clearly, for all $\theta$ outside that angular sector, paths from $P$ to $B$ exist that do not turn at all and do not meet $T$. However, if $\theta$ is inside the sector, say $\beta$ away from the nearer boundary of the sector, a path from $P$ in direction $\theta$ must turn by at least $\beta$ in order to reach $B$ without meeting $T$. Furthermore, such paths exist that do not turn by more than $\beta$. It follows that $t(P,T)$ is just the average value of $\beta$ for all directions $\theta$ in the sector; this is evidently just $\alpha/2$. In particular, as $P$ approaches $T, t(P,T)$ approaches $\pi/2$ since $\alpha$ approaches $\pi$. It follows that if $P \in T$, then $t(P,T) = \infty$. For nonconvex $T$, remarks similar to those above apply.

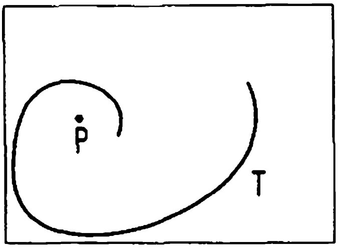
## *Surroundedness for Digital Polygons*

For subsets of digital pictures, approaches to quantitative surroundedness must be 'digitized'. We define visual surroundedness as follows:

$v_{dig}(S,T) = \wedge\{v_{dig}(P,T) \mid P \in BP(S)\}$

for digital polygons $S$ and $T$, where $v_{dig}(P,T) = v(P,T) = \alpha/2\pi$ if $T$ subtends angle $\alpha$ from $P$. The rectangles $S$ and $T$ in Fig. 5.10 may be considered to be digital polygons, for example. Then the values of $v_{dig}(S,T)$ and $v_{dig}(T,S)$ remain the same as given above, by $v(S,T)$ and $v(T,S)$ respectively. The straightforward approach to computing $v_{dig}(S,T)$ would be as follows:

**FIGURE 5.12** Visual surroundedness does not imply surroundedness.




angle = + $\infty$,
compute the convex hulls $S', T'$ of $S, T$ using any desired linear time algorithm
**for** all points $P$ in $BP(S')$ **do**
compute the two tangents from $P$ to $T'$
the angle $\alpha$ between the two tangents from $P$ to $T'$
if angle > $\alpha$, then angle = $\alpha$
**return** angle/$2\pi$.

Since when $P$ moves around $S'$ the released tangential points $Q_1, Q_2 \in BP(T')$ move around $T'$ monotonically, this algorithm leads to an $O(n)$ time algorithm $n$ = card($BP(S)$)+ card($BP(T)$) by using two points to the actual tangential points in $BP(T')$.

In the case of topological surroundedness for a digital polygon $S$, besides the restriction of $\partial S$ to $BP(S)$, the set of possible directions $\theta$ for paths from $BP(S)$ to the background $B$ must be digitized. Assume that $\theta$ is restricted to the set

$\text{ang}_m = \{n2\pi/m \mid n = 0, 1, 2, ..., m-1\}$,

for $m \geq 1$. Then $C_\theta(P, T)$ denotes the minimal angle that a path $\pi_\theta$ in direction $\theta$ starting at $P$ may take around $T$ to $B$, as defined previously and $t^m_{dig}(P, T)$ is defined by

$1/m \sum_{\theta \in ang_m} C_\theta(P, T)$

for a digital polygon $T$ and a point $P$. Finally, we have

$t^m_{dig}(S.T) = \wedge\{t^m_{dig}(P.T) \mid P \in BP(S)\}$.

Clearly, the computational requirements for computing $t^m_{dig}(S, T)$ exceed those for computing the visual surroundedness measure $v^m_{dig}(S, T)$, but nevertheless $t^m_{dig}$ seems to be a practically useful function. For example, in the situation of Figure 5.10, we have $t^8_{dig}(P, T) = 1/8(\tan^{-1}(5/6) + 0 + 0 + 0 + 0 + 0 + 0 + 0) = 0.0276 \cdot \pi = 0.0868$. It follows that $t^8_{dig}(S, T) \leq t^8_{dig}(P, T) = 0.0868$. Analogously, $t^8_{dig}(T, S) \leq t^8_{dig}(Q, S) = 0.0184 \cdot \pi = 0.0579$. Nearly the same algorithm for computing $v_{dig}(S, T)$ can be used for computing $t^m_{dig}(S, T)$ with some extensions. After computing the two tangents from $P$ to $T'$, we determine $\alpha = t^m_{dig}(P, T')$ by using the minimal angular differences to these tangents if $\theta$ is between these tangents; otherwise $C_\theta(P.T') = 0$.

Thus $t^m_{dig}(S,T)$ with $n$ = card($BP(S)$)+ card($BP(T)$) may be computed within $O(mn)$ time in the worst case sense.

Some algorithms for computing quantitative adjacency and surroundedness have been presented for the digital case. Fast algorithms for the adjacency measure in the general case (arbitrary polygons) need development.

It is noted in [54] that the proposed measures should be of interest in the study of stochastic geometry in the real plane and that these measures can be used to characterize relationships between objects in a segmented digital picture or to compare objects in two different pictures.

## 5.9 Image Enhancement and Thresholding Using Fuzzy Compactness

The results of this section are from [36]. Algorithms based on minimization of compactness and of fuzziness are developed so that it is possible to obtain both fuzzy and nonfuzzy thresholded versions of an ill-defined image. By incorporating fuzziness in the spatial domain, i.e., in describing the geometry of regions, it becomes possible to provide more meaningful results than by considering fuzziness in grey level alone. The effectiveness of the algorithms is shown for different bandwidths of the membership function using a blurred chromosome image having a bimodal histogram and a noisy tank image having a unimodal histogram as input.

The problem of grey level thresholding is important in image processing and recognition. For example, in enhancing contrast in an image, proper threshold levels must be selected so that some suitable non-linear transformation can highlight a desirable set of pixel intensities compared to others. Similarly, in image segmentation it is necessary to have proper histogram thresholding whose objective is to establish boundaries in order to partition the image space (crisply) into meaningful regions.

When the regions in an image are ill-defined, it is natural and also appropriate to avoid committing to a specific segmentation by allowing the segments to be fuzzy subsets of the image. Fuzzy geometric properties which generalize those for ordinary regions as defined in Sections 5.1 – 5.7 are helpful in such an analysis.

The above mentioned task is performed automatically with the help of a compactness measure [51] which takes into account fuzziness in the spatial domain, i.e., in the geometry of the image regions. In addition to this measure, the ambiguity in grey level through the concepts of index of fuzziness [26], entropy [14] and index of nonfuzziness (crispness) are considered, [31]. These concepts were found in [30, 32–35] to provide objective measures for image enhancement, threshold selection, feature evaluation and seed point extraction.

The algorithms described in this section extract the fuzzy segmented version of an ill-defined image by minimizing the ambiguity in both the intensity and spatial domain. In order to making a nonfuzzy decision, one may consider the cross-over point of the corresponding $S$ function [63] as the threshold level. The nonfuzzy decisions corresponding to various algorithms are compared here when a blurred chromosome image and a noisy tank image are used as input.

We now consider various measures of fuzziness in an image developed as in [30, 31, 33, 34].

We consider an image $X$ of size $M \times N$ and $L$ levels of brightness as an array of fuzzy singletons, each having a value of membership denoting its degree of brightness relative to some brightness level $l$, $l = 0, 1, 2, ..., L-1$. Let $\tilde{A}_X(x_{mn}) = \tilde{A}_{mn}$, where $\tilde{A}_{mn} \in [0,1]$ for $m = 1, ..., M; n = 1, ..., N$. The values $\tilde{A}_{mn}$ denote the grade of possessing some brightness property by the $(m,n)$-th pixel $x_{mn}$. This brightness property is defined below.

The index of fuzziness reflects the average amount of ambiguity (fuzziness) present in an image $X$ by measuring the distance ('linear' and 'quadratic' corresponding to linear index of fuzziness and quadratic index of fuzziness) between its fuzzy property $\tilde{A}_X$ and the nearest two-level property $\tilde{A}_X$, i.e., the distance between the gray tone image and its nearest two-tone version. The term 'entropy' uses Shannon's function but its meaning is quite different from classical entropy because no probabilistic concept is needed to define it. The index of nonfuzziness measures the amount of nonfuzziness (crispness) in $\tilde{A}_X$ by computing its distance from its complement version. These indices are defined as follows.

(a) Linear index of fuzziness:

$$\begin{aligned}\nu_1(X) &= 2/MN \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} |\tilde{A}_X(x_{mn}) - \tilde{A}_{\underline{X}}(x_{mn})| \\ &= 2/MN \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{A}_{X \cap \overline{X}}(x_{mn}) \\ &= 2/MN \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{A}_X(x_{mn}) \wedge (1 - \tilde{A}_X(x_{mn}),\end{aligned}$$

where $\tilde{A}_{\underline{X}}(x_{mn})$ denotes the nearest two-level version of $X$ such that

$$\tilde{A}_X(x_{mn}) = \begin{cases} 0 & \text{if } \tilde{A}_X(x_{mn}) \leq 0.5 \\ 1 & \text{otherwise.} \end{cases} \tag{5.9.1}$$

(b) Quadratic index of fuzziness:

$$\tilde{B}_q(X) = 2/\sqrt[2]{MN}[\textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} (\tilde{A}_X(x_{mn}) - \tilde{A}_{\underline{X}}(x_{mn}))]^{1/2},$$

(c) Entropy

$$H(X) = 1/(MN \ln 2) \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} S_n(\tilde{A}_X(x_{mn}))$$

where

$$S_n(\tilde{A}_X(x_{mn})) = -\tilde{A}_X(x_{mn}) \ln \tilde{A}_X(x_{mn}) - (1 - \tilde{A}_X(x_{mn}) \ln(1 - \tilde{A}_X(x_{mn})),$$

$m = 1, 2, ..., M; n = 1, 2, ..., N$.

(d) Index of nonfuzziness (crispness):

$$\eta(X) = 1/MN \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} |\tilde{A}_X(x_{mn}) - \tilde{A}_{\overline{X}}(x_{mn})|.$$

The above measures lie in $[0,1]$ and have the following properties for $m = 1, ..., M$ and $n = 1, ..., N$:

$I(X) = 0$ (min) for $\tilde{A}_X(x_{mn}) = 0$ or 1,
$I(X) = 1$ (max) for $\tilde{A}_X(x_{mn}) = 0.5$,
$I(X) \geq I(X^*)$,
$I(X) = I(\overline{X})$,
where $I$ stands for $\nu(X), H(X)$ and $1 - \eta(X)$, and where $X^*$ is the 'sharpened' or 'intensified' version of $X$ such that

$$\tilde{A}_{X^*}(x_{mn}) \begin{cases} \geq \tilde{A}_X(x_{mn}) & \text{if } \tilde{A}_X(x_{mn}) \geq 0.5 \\ \leq \tilde{A}_X(x_{mn}) & \text{otherwise} \end{cases}$$

### *Fuzzy Geometry of Image Subsets*

In Sections 5.1 – 5.7, the concepts of digital picture geometry were extended to fuzzy subsets and some of the standard geometric properties of and relationships among regions were generalized to fuzzy subsets. We only consider here the area, perimeter and compactness of a fuzzy image subset, characterized by $\tilde{A}_X(x_{mn})$. These extensions will be used in the following for developing threshold selection algorithms. For simplicity, we replace $\tilde{A}_X(x_{mn})$ by $\tilde{A}$ in defining these parameters.

The area of $\tilde{A}$, written $a(\tilde{A})$, is defined as follows:

$$a(\tilde{A}) = \int \tilde{A},$$

where the integral is taken over any region outside which $\tilde{A} = 0$.

If $\tilde{A}$ is piecewise constant, the case in a digital image, $a(\tilde{A})$ is the weighted sum of the areas of the regions on which $\tilde{A}$ has constant values, weighted by these values.

For the piecewise constant case, the perimeter of $\tilde{A}$, written $p(\tilde{A})$, is defined to be

$$p(\tilde{A}) = \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \sum_{k=1}^{r_{ij}} |\tilde{A}_i - \tilde{A}_j||A_{ijk}|.$$

This is the weighted sum of the length of the arcs $A_{ijk}$ along which the $i$-th and $j$-th regions meet and have constant $\tilde{A}$ values $\tilde{A}_i$ and $\tilde{A}_j$ respectively, weighted by the absolute difference of these values.

The compactness of $\tilde{A}$, written comp$(\tilde{A})$, is defined to be

$$\text{comp}(\tilde{A}) = a(\tilde{A})/p^2(\tilde{A}).$$

For crisp sets, this is largest for a disk, where it is equal to $1/4\pi$. For a fuzzy disk where $\tilde{A}$ depends only on the distance from the origin (center), it can be shown that

$$a(\tilde{A})/p^2(\tilde{A}) \geq 1/4\pi.$$

That is, of all possible fuzzy disks, the compactness is smallest for its crisp version. Consequently, we use minimization rather than maximization of fuzzy compactness as a criterion for image enhancement and threshold selection.

### *Threshold Selection*

We now consider minimizing fuzziness. Consider for example, the minimization of $\nu_1(X)$. It follows from equation (5.9.1) that the nearest ordinary plane $\tilde{A}_X$ (which represents the closest two-tone version of the grey tone image $X$) is dependent on the position of the cross-over point, i.e., the 0.5 value of $\tilde{A}_X$. Consequently, a proper selection of the cross-over point may be made which results in a minimum value of $\nu(X)$ only when the cross-over point corresponds to the appropriate boundary between regions (clusters) in $X$.

This can be explained further as follows. Consider the standard $S$-function in Figure 1 of [63].

$$\tilde{A}_X(x_{mn}) = S(x_{mn}; a, b, c) = \begin{cases} 0 & \text{if } x_{mn} \leq a, \\ 2[(x_{mn} - a)/(c - a)]^2 & \text{if } a \leq x_{mn} \leq b, \\ 1 - 2[(x_{mn} - c)/(c - a)]^2 & \text{if } b \leq x_{mn} \leq c, \\ 1 & \text{if } x_{mn} \geq c, \end{cases} \tag{5.9.2}$$

with cross-over point $b = (a + c)/2$ and bandwidth
$\Delta b = b - a = c - b$
for obtaining $\tilde{A}_X(x_{mn})$ or $u_{mn}$ (representing the degree of brightness of each pixel) from the given $x_{mn}$ of the image $X$. Then for a cross-over point selected at, say, $b = l_i$, $\tilde{A}_X(l_i) = 0.5$ and $u_{mn}$ would take on values $> 0.5$ and $< 0.5$ corresponding to $x_{mn} > l_i$ and $< l_i$. This implies allocation of the grey levels into two ranges. The term $\nu(X)$ then measures the average ambiguity in $X$ by computing $\tilde{A}_{X \cap \overline{X}}(x_{mn})$ in such a way that the contribution of the levels towards $\nu(X)$ comes mostly from those near $l_i$ and decreases as we move away from $l_i$.

Hence, modification of the cross-over point results in different segmented images with varying $\nu(X)$. If $b$ corresponds to the appropriate boundary (threshold) between two regions, then there is a minimum number of pixel intensities in $X$ having $u_{mn} \simeq 0.5$ (resulting in $\nu \simeq 1$) and a maximum number of pixel intensities having $u_{mn} \simeq 0$ or 1 (resulting in $\nu \simeq 0$) thus contributing least towards $\nu(X)$. This optimum (minimum) value of fuzziness would be greater for any other selection of the cross-over point.

We now consider some algorithms.

**Algorithm 1**

**Input:** A $M \times N$ image with minimum and maximum grey levels $l_{\min}$ and $l_{\max}$ respectively..

**Step 1.** Construct the 'bright image' of membership $\tilde{A}_X$, where
$\tilde{A}_X(l) = S(l; a, l_i, c)$, $l_{\min} \leq l, l_i \leq l_{\max}$,
using equation (5.9.2) with cross-over point $b = l_i$ and a particular bandwidth $\Delta b = c - l = l_i - a$.

**Step 2.** Compute the amount of fuzziness in $\tilde{A}_X$ corresponding to $b = l_i$ with

$$\begin{aligned}\nu(X)|l_i &= 2/MN\sum_i \min\{S(l;a,l_i,c), 1 - S(l;a,l_i,c)\}h(l) \\ &= 2/MN\sum_i T_i(l)h(l) \qquad (5.9.3)\end{aligned}$$

where $T_i(l) = \min\{S(l;a,l_i,c), 1-S(1;a,l_i,c)\}$ and $h(l)$ denotes the number of occurrences of the level $l$.

**Step 3.** Vary $l_i$ from $l_{\min}$ to $l_{\max}$ and select $l_i = l_c$, say, for which $\nu(X)$ is a minimum.

$l_c$ is thus the cross-over point of $\tilde{A}_X(x_{mn})$ having minimum ambiguity, i.e., for which $\tilde{A}_X$ has minimum distance from its closest two-tone version. Now $\tilde{A}_{mn}$ can be regarded as a *fuzzy segmented version* of the image, with $\tilde{A}_{mn} < 0.5$ and $> 0.5$ corresponding to regions $[l_{\min}, l_c - l]$ and $[l_c, l_{\max}]$.

For the purpose of nonfuzzy segmentation, the level $l_c$ can be considered as the threshold between background and object, or the boundary of the object region. This can further be verified from equation (5.9.3) which shows that the minimum value of $\nu(X)$ would always correspond to the valley region of the histogram having minimum number of occurrences.

We now consider variation of bandwidth $(\triangle b)$.

Call $T_i(l)$ (in equation 5.9.3) a Triangular Window function centered at $l_i$ with bandwidth $\triangle b$. As $\nabla b$ decreases, $\tilde{A}_X$ has more intensified contrast around the cross-over point resulting in decrease of ambiguity in $\tilde{A}_X$. Therefore, the possibility of detecting some undesirable thresholds (spurious minima in the histogram) increases due to the smaller width of the $T_i(l)$ function.

On the other hand, an increase of $\triangle b$ results in a higher value of fuzziness and thus leads to the possibility of losing some of the weak minima.

The application of this technique to both bimodal and multimodal images with various $T_i$ functions based on $\nu_1(X)$, $\nu_q(X)$, $H(X)$ and $\tilde{A}(X)$ is demonstrated in [33,34].

We consider next minimizing compactness.

In the previous discussion of threshold selection, fuzziness in the grey levels of an image was considered. Fuzziness in the spatial domain is now taken into consideration by using the compactness measure for selecting nonfuzzy thresholds.

It follows that both the perimeter and area of a fuzzy segmented image depend on the membership value, denoting the degree of brightness, say, of each region. Furthermore, the compactness of a fuzzy region decreases as its $\tilde{A}$ value increases and it is smallest for a crisp one. We now present two algorithms to show how the above mentioned concept can be utilized for selecting a threshold between two regions (say, the background and a single object) in a bimodal image $X$.

As in Algorithm 1, we construct $u_{mn}$ with different $S$ functions having constant $\triangle b$ value and select the cross-over point of the $\tilde{A}_X$ as the boundary of the object for which comp($\tilde{A}$) is a minimum.

**Algorithm 2**

**Input:** Given an $M \times N$ image with minimum and maximum grey levels $l_{\min}$ and $l_{\max}$

**Step 1.** Construct 'bright' image $\tilde{A}_X$ as in Step 1 of Algorithm 1.

**Step 2.** Compute the area and perimeter of $\tilde{A}_X$ corresponding to be $b = l_i$ with

$a(\tilde{A})|l_i = \sum_{m=1}^{M} \sum_{n=1}^{N} u_{mn} = \sum_i S(l; a, l_i, c)h(l)$,

$l_{\min} \leq l, l_i \leq l_{\max}$

and

$p(\tilde{A})|l_i = \sum_{m=1}^{M} \sum_{n=1}^{N-1} |u_{mn} - u_{m,n+1}| + \sum_{n=1}^{N} \sum_{m=1}^{M-1} |u_{mn} - u_{m+1,n}|$

excluding the frame of the image.

For example, consider the $4 \times 4$ $\tilde{A}_{mn}$ array

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & \delta & 0 & 0 \end{matrix} \quad \text{where } 1 \geq \alpha, \beta, \gamma, \delta > 0.$$

Here, $a(\tilde{A}) = \alpha + 2\beta + \gamma + \delta$ and

$p(\tilde{A}) = [\alpha + |\beta - \alpha| + \beta + \beta + |\gamma - \beta| + \delta + \delta] + [\alpha + \alpha + \delta + \beta + 0 + \beta + \gamma + \gamma]$.

**Step 3.** Compute the compactness of $\tilde{A}_X$ corresponding to $b = l_i$ with

$$\text{comp}\left(\tilde{A}\right)|l_i = (a(\tilde{A})|l_i)/(p^2(\tilde{A})|l_i). \tag{5.9.4}$$

**Step 4.** Vary $l_i$ from $l_{\min}$ to $l_{\max}$ and select that $l_i = l_c$ for which comp$\left(\tilde{A}\right)$ is minimum.

Consequently, the level $l_c$ denotes the cross-over point of the fuzzy image plane $u_{mn}$ which is the least compact (or most crisp). These $u_{mn}$ can therefore be viewed as a fuzzy segmented version of the image $X$.

The level $l_c$ can be considered as the threshold for making a nonfuzzy decision on classifying/segmenting the image into regions.

Approximate the definitions of area and compactness of $\tilde{A}_X$ by considering that $\tilde{A}_X$ has only two values corresponding to the background and object regions. The $\tilde{A}$-value for the background is assumed to be zero, whereas the $\tilde{A}$-value of the object region is monotonically increasing with increase in threshold level. Consequently, by varying the threshold, one can have different segmented versions of the object region. Each segmented version thresholded at $l_i$ has its area and perimeter computed as follows:

$a(\tilde{A}_t) = a \cdot \tilde{A}_t = \tilde{A}_t \sum_i h(l)$, $l_t \leq l \leq l_{\max}$,

where $a$ denotes the area of the region on which $\tilde{A} = \tilde{A}_t$ (constant), i.e., the number of pixels having grey level greater than or equal to $l_t$ and

$p(\tilde{A}_t) = \tilde{A}_t \cdot p$

where $p$ denotes the length of the arcs along which the regions having $\tilde{A} = \tilde{A}_t$ and $\tilde{A} = 0$ meet, or, in other words, the perimeter of the region on which $\tilde{A} = \tilde{A}_t$ (constant).

For the example considered in Algorithm 2, the values of $a(\tilde{A}_t)$ and $p(\tilde{A}_t)$ for $\alpha = \beta = \gamma = \delta = \tilde{A}_t$ are $5\tilde{A}_t$ and $12\tilde{A}_t$ respectively.

The algorithm for selecting the boundary of a single-object region from an $M \times N$ dimensional image may therefore be stated as follows:

**Algorithm 3**

**Input:** Given an $M \times N$ image with minimum and maximum grey levels $l_{\text{min}}$ and $l_{\text{max}}$

**Step1.** Construct the 'bright' image $\tilde{A}_X$ using
$\tilde{A}_X(l) = S(l; a, b, c)$
with $a = l_{\text{min}}, c = l_{\text{max}}$ and $b = (a + c)/2$.

**Step 2.** Generate a segmented version putting
$\tilde{A} = 0$ if $\tilde{A} < \tilde{A}_t$ else $\tilde{A} = \tilde{A}_t$,
where $\tilde{A}_t$ is the value of $\tilde{A}_x(l_t)$ obtained in Step 1.

**Step 3.** Compute the compactness of the segmented version thresholded at $l_t$ :
$\text{comp}(\tilde{A}_t) = a \cdot \tilde{A}_y/p^2 \cdot \tilde{A}_t^2 = a/p^2\tilde{A}_t.$

$$\text{comp}(\tilde{A}_t) = a \cdot \tilde{A}_y/p^2 \cdot \tilde{A}_t^2 = a/p^2\tilde{A}_t. \tag{5.9.5}$$

**Step 4.** Vary $l_t$ in $(l_{\text{min}}, l_{\text{max}})$ and hence $\tilde{A}_t$ in $(0, 1)$ and select the level as boundary of the object for which equation (5.9.5) attains its minimum.

Note that after approximation of the area and perimeter of $\tilde{A}_{mn}$, the compactness measure (equation (5.9.5)) reduces to $1/\tilde{A}_t$ times the crisp compactness of the object region. Unlike Algorithms 1 and 2, here $\tilde{A}_X$ is kept fixed throughout the process and the output of the algorithm is a nonfuzzy segmented version of $X$ determined by $l_t$.

**Algorithm 4**

Algorithms 1-3 minimize either the amount of fuzziness or the compactness of an image $X$. We combine these measures and compute the product of fuzziness and compactness, and determine the level for which the product becomes a minimum. Compute using equations (5.9.3) and (5.9.4),

$$\theta_t = \nu(X)|l_i \cdot \text{comp}(\tilde{A})|l_i \tag{5.9.6}$$

or we compute using equations (5.9.3) and (5.9.5),

$$\theta_t = \nu(X)|l_t \cdot \text{comp}(\tilde{A}_t) \tag{5.9.7}$$

at each value of $l_i$ (or $l_t$), $l_{\text{min}} < l_i, l_t < l_{\text{max}}$, and select $l_i = l_c$, say as threshold for which equation (5.9.6) (or (5.9.7)) is a minimum. The corresponding $u_{mn}$ represents the fuzzy segmented version of the image as far as minimization of its fuzziness in grey level and the spatial domain is concerned.

Note that although the linear index of fuzziness in Algorithms 1 and 4 is considered, the other measures, namely $\nu_q(X), H(X)$ and $\tilde{A}(X)$ can be considered for computing the total amount of fuzziness in $u_{mn}$.

Figure 2a in [36, p. 82] shows a $64 \times 64, 64$ level image of a blurred chromosome with $l_{\min} = 12$ and $l_{\max} = 59$. Its bimodal histogram is shown in [36, Fig. 2b, p.82]

The different minima obtained using Algorithms 1-4 for $\Delta b = 2, 4, 8, 16$ are given in Table 1 of [36, p.83]. The enhanced version of the chromosome corresponding to these thresholds (minima) are shown in [36, Figures 3-8] only for $\Delta b = 4, 8$ and 16. In each of Figures 3-5 of [36], (a), (b) and (c) correspond to Algorithm 1, Algorithm 2 and equation (5.9.6) of Algorithm 4. Similarly, Figures 6-7 in [36], (a), (b) and (c) correspond to Algorithm 1, Algorithm 3 and equation (5.9.7) of Algorithm 4.

The interested reader is strongly encouraged to see [36] for a detailed discussion of the algorithms concerning implementation and results.

The compactness measure usually results in more minima as compared to index of fuzziness. The index of fuzziness (Algorithm 1) sharpens the histogram and it detects a single threshold in the valley region of the histogram for $\Delta b = 4, 8$ and 16. At $\Delta b = 2$, the algorithm as expected results in some undesirable thresholds corresponding to weak minima of the histogram. This conforms to the earlier investigation [33]. Algorithms 2 and 3 based on the compactness measure detect a higher-valued threshold (global minimum) which results in better segmentation (or enhancement) of the chromosome as far as its shape is concerned.

The advantage of the compactness measures over the index value is that they take fuzziness in the spatial domain (i.e., the geometry of the object) into consideration in extracting thresholds. The index value, on the other hand, incorporates fuzziness only in grey level. In addition, for Algorithm 2 as $\Delta b$ increases, the number of and the separation between minima also decrease.

Multiplying $\nu(X)$ by comp$(\tilde{A}_t)$, in equation (5.9.7), produces at least as many thresholds as are generated by the individual measures. However this is not the case for equation (5.9.6) where the number of thresholds is (except for $\Delta b = 2$) equal to or less than the numbers for the individual measures.

We now explain the observations made above. As mentioned before, $\nu(X)$ basically sharpens the histogram. Hence as $l_i$ increases, it first increases until it reaches a maximum, and then decreases until a minimum (threshold) is attained. Then it follows the same pattern for the other mode of the histogram. The compactness measure, on the other hand, first starts decreasing until it reaches a minimum, then increases for awhile, and then starts decreasing again.

It can also be seen that the variation of compactness in Algorithm 3 plays a more dominant role than the variation of index value in Algorithm 1 in detecting minima. The case is reversed for the combination of Algorithm 1 and Algorithm 2, where the product is influenced more by the index value. Consequently, the threshold obtained by equation (5.9.6) is found to be within the range of threshold values obtained by the individual measures. Equation (5.9.7), on the other hand, is able to create a higher-valued (or at least equal) threshold which results in better object enhancement than those of the individual measures.

Figures 9(a) and 9(b) in [36, p.84] show a noisy image of a tank and its unimodal histogram, having $l_{\min} = 14$, $l_{\max} = 50$. The minima obtained by the different algorithms for $\Delta b = 2, 4, 8$ and 16 are given in Table 2 in [36, p.85]. The corresponding enhanced versions for $\Delta b = 4, 8$ and 16 are shown in Figures 10-12 in [36, p.85] for various combinations of algorithms.

As expected, the index of fuzziness alone was not able to detect a threshold for the tank image because of its unimodal histogram. However the compactness measure does give good thresholds. As in the case of the chromosome image, equation (5.9.7) yields at least as many thresholds as are generated by the compactness measure. Except for $\Delta b = 2$, equation (5.9.6) yields at most as many thresholds as the compactness measure.

The following conclusions are drawn in [36]. Algorithms based on compactness measures of fuzzy sets are developed and used to determine thresholds (both fuzzy and nonfuzzy) of an ill-defined image (or the enhanced version of a fuzzy object region) without referring to its histogram. The enhanced chromosome images obtained from the global minima of the measures are found to be better than those obtained on the basis of minimizing fuzziness in grey level, as far as the shape of the chromosome is concerned. Consideration of fuzziness in the spatial domain, i.e., in the geometry of the object region, provides more information by making it possible to extract more than a single thresholded version of an object. Similarly in the case of the unimodal (noisy) tank image, the compactness measure is able to determine some suitable thresholds but the index parameter is not. Furthermore, optimization of both compactness and fuzziness usually allows better selection of thresholded enhanced versions.

## 5.10 Fuzzy Plane Geometry: Points and Lines

In this section, we present a version of fuzzy plane geometry different from that of the previous sections. This version was initiated by Buckley and Eslami, [7,8]. In the previous sections, the concepts of the area, height, width, diameter, and perimeter of fuzzy subsets are real numbers. The approach used here is one which will lead to these measures being fuzzy real numbers. This approach will have as an application, the superimposing of objects from fuzzy geometry onto databases to obtain a fuzzy landscape over the data base. A soft query could be a fuzzy probe into the landscape with the system's response the number data points in a level set of the interaction of the fuzzy probe and the fuzzy landscape.

**Definition 5.7** *Let $\tilde{N}$ be a fuzzy subset of $\mathbb{R}$. Then $\tilde{N}$ is called a (real)* fuzzy number *if the following conditions hold:*

(i) *$\tilde{N}$ is upper semi-continuous,*

(ii) *there exist $c, d \in \mathbb{R}$ with $c \leq d$ such that $\forall x \notin [c, d]$, $\tilde{N}(x) = 0$,*

(iii) *there exist $a, b \in \mathbb{R}$ such that $c \leq a \leq b \leq d$ and $\tilde{N}$ is increasing on $[c, a]$, $\tilde{N}$ is decreasing on $[b, d]$, and $\tilde{N}(x) = 1 \forall x \in [a, b]$.*

It follows $\forall t \in [0, 1]$ that if $\tilde{N}$ is a fuzzy number, then $\tilde{N}^t$ is a bounded closed interval.

Suppose that $\tilde{N}$ is a fuzzy subset of $\mathbb{R}$ satisfying (ii) of Definition 5.7 with $a = b$ such that $\tilde{N}(a) = 1$ and the graph of $\tilde{N}$ is a straight line segment from $c$ to $a$ and a straight line segment from $a$ to $b$. Then $\tilde{N}$ is a fuzzy number and is called a *triangular fuzzy number.*

A natural way to define a fuzzy point in the plane would be as an ordered pair of real fuzzy numbers. However this definition does not give good results for fuzzy lines. Also pictures of fuzzy points under this definition cannot be constructed. Hence the following definition of a fuzzy point is used.

**Definition 5.8** *Let $(a, b) \in \mathbb{R}^2$ and let $\tilde{P}$ be a fuzzy subset of $\mathbb{R}^2$. Then $\tilde{P}$ is called a* fuzzy point *at $(a, b)$ if the following conditions hold:*

(i) *$\tilde{P}$ is upper semi-continuous;*

(ii) *$\forall (x, y) \in \mathbb{R}^2$, $\tilde{P}(x, y) = 1$ if and only if $(x, y) = (a, b)$;*

(iii) *$\forall t \in [0, 1]$, $\tilde{P}^t$ is a compact, convex subset of $\mathbb{R}^2$. (If $\tilde{P}$ is a fuzzy point at $(a, b)$, we sometimes write $\tilde{P}_{(a,b)}$ for $\tilde{P}$.)*

The concept of fuzzy point is based on the idea of a fuzzy vector in $\mathbb{R}^n$, [5, 8].

Let $(a, b) \in \mathbb{R}^2$ and let $\tilde{P}$ be a fuzzy point at $(a, b)$. Then we can visualize $\tilde{P}$ as a surface in $\mathbb{R}^3$ through the graph of the equation $z = \tilde{P}(x, y)$, $(x, y) \in \mathbb{R}^2$.

**Example 5.8** *Let $\tilde{X}$ and $\tilde{Y}$ be real fuzzy numbers, where $\tilde{X}(x) = 1$ if and only if $x = a$ and $\tilde{Y}(y) = 1$ if and only if $y = b$. Then the fuzzy subset $\tilde{P}$ of $\mathbb{R}^2$ defined by $\tilde{P}(x, y) = \tilde{X}(x) \wedge \tilde{Y}(y)$ $\forall (x, y) \in \mathbb{R}^2$ is a fuzzy point at $(a, b)$.*

In the following, we let $d$ denote the usual Euclidean distance metric on $\mathbb{R}^2$.

We now define the fuzzy distance between two fuzzy points.

**Definition 5.9** *Let $\tilde{P}_1$ and $\tilde{P}_2$ be two fuzzy points. $\forall t \in [0,1]$, let $\Omega(t) = \{d(u,v) | u \in (\tilde{P}_1)^t$ and $v \in (\tilde{P}_2)^t\}$. Define the fuzzy subset $\tilde{D}(\tilde{P}_1, \tilde{P}_2)$ of $\mathbb{R}$ by $\tilde{D}(\tilde{P}_1, \tilde{P}_2)(a) = \vee\{t | a \in \Omega(t)\}$ $\forall a \in \mathbb{R}$.*

Let $t \in [0,1]$. We note that in Definition 5.9, $\Omega(t)$ is defined in terms of a pair of fuzzy points, say $\tilde{P}_1, \tilde{P}_2$. Then $\Omega(t) = \{r \in \mathbb{R} | \exists u \in (\tilde{P}_1)^t, \exists v \in (\tilde{P}_2)^t$ such that $r = d(u,v)\}$.

**Theorem 5.34** *Let $\tilde{P}_1$ and $\tilde{P}_2$ be two fuzzy points. Then $\forall t \in [0,1]$, the level set $\tilde{D}(\tilde{P}_1, \tilde{P}_2)^t = \Omega(t)$. Further, $\tilde{D}(\tilde{P}_1, \tilde{P}_2)$ is a fuzzy number.*

**Proof.** We first show that $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)^t = \Omega(t)$, $0 \leqslant t \leqslant 1$. Let $d \in \Omega(t)$. Then $\widetilde{D}(d) \geqslant t$ and $\Omega(t) \subseteq \widetilde{D}^t$. We now show that $\widetilde{D}^t$ is a subset of $\Omega(t)$. Let $d \in \widetilde{D}^t$. Then $\widetilde{D}(t) \geqslant t$. Set $\widetilde{D}(d) = s$. We consider the cases $s > t$ and $s = t$. Suppose that $s > t$. Then there is an $r, t < r \leqslant s$, with $d \in \Omega(r)$. Since $\Omega(r) \subseteq \Omega(t)$, we have $d \in \Omega(t)$. Hence $\widetilde{D}^t \subseteq \Omega(t)$. Assume that $s = t$. Let $K = \{w \mid d \in \Omega(w)\}$. Then $\vee K = s = t = \widetilde{D}(d)$. There is a sequence $r_n$ in $K$ such that $r_n \uparrow t$. Given $\varepsilon > 0$ there is a positive integer $N$ such that $t - \varepsilon < r_n$, for all $n \geqslant N$. Now $d$ in $\Omega(r_n)$ for all $n$ implies that $d$ is in $\Omega(t-\varepsilon)$ for all $\varepsilon > 0$. Thus $d = d(u,v)$ for some $u \in \widetilde{P}(a_1, b_1)^{t-\varepsilon}$ and $v \in \widetilde{P}(a_2, b_2)^{t-\varepsilon}$. Hence $\widetilde{P}(a_1, b_1)(u) \geqslant t - \varepsilon$ and $\widetilde{P}(a_2, b_2)(v) \geqslant t - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have that $\widetilde{P}(a_1, b_1)(u) \geqslant t$ and $\widetilde{P}(a_2, b_2)(v) \geqslant t$. Therefore, $d \in \Omega(t)$. Thus $\widetilde{D}^t \subseteq \Omega(t)$. Hence $\widetilde{D}^t = \Omega(t)$ for $0 < t \leqslant 1$. It follows easily that $\widetilde{D}^0 = \Omega(0)$.

We now show that $\widetilde{D}$ is a fuzzy number.

Since the $t$-cuts of $P(a_1, b_1)$ and $P(a_2, b_2)$ are compact it, follows that $\Omega(t)$ is a bounded closed interval for all $t$. Let $\Omega(t) = [l(t), r(t)], 0 \leqslant t \leqslant 1$. It is also is known that if the $t$-cuts of a fuzzy number are closed sets, then its membership function is upper semi-continuous [4]. But $\widetilde{D}^t = \Omega(t)$ is a closed interval for all $t$. Hence, $\widetilde{D}$ is upper semi-continuous.

Let $\Omega(0) = [c, d]$. Then $\widetilde{D}(d) = 0$ outside $[c, d]$.

Let $\Omega(1) = a$, where $a = d((a_1, b_1), (a_2, b_2))$. Now since $\widetilde{D}^t = [l(t), r(t)]$ for all $t$ with $l(t)$ is increasing from $c$ to $a$ and $r(t)$ decreasing from $d$ to $a$ we obtain $\widetilde{D}$ is increasing on $[c, a]$ and decreasing on $[a, d]$ with $\widetilde{D}(d) = 1$ at $d = a$.

Consequently, $\widetilde{D}$ is a fuzzy number ■

Let $\tilde{P}$ and $\tilde{Q}$ be fuzzy points at $(a,b)$ and $(c,d)$, respectively. Suppose that $\tilde{D}(\tilde{P}, \tilde{Q})(r) = 0$ $\forall r > 0$. Then $0 = \vee\{t | \exists u \in (\tilde{P})^t, \exists v \in (\tilde{Q})^t$ such that $r = d(u,v)\} \forall r > 0$. Thus there do not exist $u \in (\tilde{P})^t$ and $v \in (\tilde{Q})^t$ such that $r = d(u,v)$ for any $r > 0$ and for any $t \in (0,1]$. Hence $(a,b) = (c,d)$ else for $r = d((a,b),(c,d))$, $\tilde{D}(\tilde{P}, \tilde{Q})(r) = 1$ and $r > 0$. Now suppose that $\tilde{D}(\tilde{P}, \tilde{Q})(r) = 1$ $\forall r > 0$, where $(a,b) = (c,d)$. Suppose that $\tilde{Q}$ is not crisp. Then $\exists v \in (\tilde{Q})^{t_0}, v \neq (a,b)$, for some $t_0$ such that $0 < t_0 < 1$. Now $d((a,b),v) > 0$. Hence $\tilde{D}(\tilde{P}, \tilde{Q})(d((a,b),v)) = 1$ by assumption and also

$\tilde{D}(\tilde{P},\tilde{Q})(d((a,b),(a,b))) = 1$. However this is impossible since $\tilde{D}(\tilde{P},\tilde{Q})$ is a fuzzy number and thus attains the value 1 uniquely.

Now suppose that $\tilde{P}$ and $\tilde{Q}$ are (crisp) fuzzy points at $(a,b)$ and $(c,d)$, respectively. Let $r > 0$. Then $(\tilde{P})^t = \{(a,b)\}$ and $\tilde{Q}^t = \{(c,d)\}$ $\forall t \in (0,1]$. Hence $\tilde{D}(\tilde{P}.\tilde{Q})(r) = \vee\{t | \exists u \in \{(a,b)\}, \exists v \in \{(c,d)\}$ such that $r = d(u,v)\}$. Thus

$$\tilde{D}(\tilde{P},\tilde{Q})(r) = \begin{cases} 1 & \text{if } r = d((a,b),(c,d)) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\tilde{D}$ reduces to $d$ if $\tilde{P}$ and $\tilde{Q}$ are crisp.

**Definition 5.10** *Define the fuzzy subset $\tilde{0}$ of $\mathbb{R}$ by $\tilde{0}(x) = 0$ if $x < 0$ and $\tilde{0}(x) = 1$ if and only if $x = 0$ and $\tilde{0}$ is decreasing in some interval $(0,d)$ for some $d > 0$, and $\tilde{0}(x) = 0$ for $x \geq d$.*

**Definition 5.11** *Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers and set $\tilde{A}^t = [a_1(t), a_2(t)]$, $\tilde{B}^t = [b_1(t), b_2(t)]$ $\forall t \in [0,1]$. We write $\tilde{A} \leq_s \tilde{B}$ if and only if $a_1(t) \leq b_1(t)$ and $a_2(t) \leq b_2(t)$ $\forall t \in [0,1]$. We write $\tilde{A} \leq_w \tilde{B}$ if and only if $a_2(t) \leq b_2(t)$ $\forall t \in [0,1]$. We call $\leq_s$* a strong ordering *and $\leq_w$* a weak ordering.

**Theorem 5.35** *The relation $\leq_s$ is a partial order (reflexive, transitive, antisymmetric) on the set of fuzzy numbers. The relation $\leq_w$ is reflexive and transitive.* ■

The addition of fuzzy numbers in the following definition is done by interval arithmetic. That is, we just add $\tilde{A}^t$ and $\tilde{B}^t$ $\forall t \in [0,1]$ to determine $\tilde{A} + \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers.

**Definition 5.12** *A fuzzy metric $\tilde{M}$ is a function from the set of all pairs of fuzzy points $(\tilde{P}_1, \tilde{P}_2)$ into the set of all fuzzy numbers such that the following conditions hold:*

**(i)** $\tilde{M}(\tilde{P}_1, \tilde{P}_2) = \tilde{0}$ *if and only if $\tilde{P}_1$ and $\tilde{P}_2$ are both fuzzy points at some $(a,b)$;*

**(ii)** $\tilde{M}(\tilde{P}_1, \tilde{P}_2) = \tilde{M}(\tilde{P}_2, \tilde{P}_1)$;

**(iii)** $\tilde{M}(\tilde{P}_1.\tilde{P}_2) \leq \tilde{M}(\tilde{P}_1, \tilde{P}_3) + \tilde{M}(\tilde{P}_3, \tilde{P}_2)$ *for all fuzzy points $\tilde{P}_1.\tilde{P}_2$, and $\tilde{P}_3$.*

*If $\leq$ is $\leq_s$ $(\leq_w)$, we call $\tilde{M}$ a strong (weak) fuzzy metric.*

**Theorem 5.36** *$\tilde{D}$ is a weak fuzzy metric.*

**Proof.** Clearly, $\tilde{D}(\tilde{P}_1, \tilde{P}_2) = \tilde{D}(\tilde{P}_2, \tilde{P}_1)$. Let $\tilde{P}_1 = \tilde{P}(a_1, b_1)$, $\tilde{P}_2 = \tilde{P}(a_2, b_2)$. Suppose that $\tilde{D}(\tilde{P}_1, \tilde{P}_2) = 0$. Then 0 is in $\tilde{D}(\tilde{P}_1, \tilde{P}_2)^1$. However, $\tilde{D}(\tilde{P}_1, \tilde{P}_2)^1$ is the set of all $d(u.v)$, where $u \in (\tilde{P}_1)^1 = \{(a_1, b_1)\}$, $v \in (\tilde{P}_2)^1 = \{(a_2, b_2)\}$.

Hence $d((a_1,b_1),(a_2,b_2)) = 0$ implies $(a_1,b_1) = (a_2,b_2)$. Now suppose that $\widetilde{P}_1$ and $\widetilde{P}_2$ are fuzzy points at $(a,b)$. It follows that $\widetilde{D}(\widetilde{P}_1,\widetilde{P}_2)^1 = \{0\}$ and $\widetilde{D}(\widetilde{P}_1,\widetilde{P}_2)$ has the correct shape to be called an $\widetilde{O}$.

Let $\widetilde{P}_3 = \widetilde{P}(a_3,b_3)$, $\widetilde{A} = \widetilde{D}(\widetilde{P}_1,\widetilde{P}_2)$, $\widetilde{B} = \widetilde{D}(\widetilde{P}_1,\widetilde{P}_3)$, $\widetilde{C} = \widetilde{D}(\widetilde{P}_3,\widetilde{P}_2)$, $\widetilde{A}^t = [a_1(t),a_2(t)]$, $\widetilde{B}^t = [b_1(t),b_2(t)]$, $\widetilde{C}^t = [c_1(t),c_2(t)]$. We now show that $a_2(t) \leqslant b_2(t) + c_2(t)$ for all $t$.

We have from Theorem 5.34 that

$a_2(t) = \vee\{d(u,v) \mid u \in (\widetilde{P}_1)^t, v \in (\widetilde{P}_2)^t\}$,

$b_2(t) = \vee\{d(u,v) \mid u \in (\widetilde{P}_1)^t, v \in (\widetilde{P}_3)^t\}$,

$c_2(t) = \vee\{d(u,v) \mid u \in (\widetilde{P}_3)^t, v \in (\widetilde{P}_2)^t\}$.

Hence,

$a_2(t) \leqslant \vee_{(u,v)}\{d(u,w) + d(w,v) \mid u \in (\widetilde{P}_1)^t, w \in (\widetilde{P}_3)^t, v \in (\widetilde{P}_2)^t\}$

$\leqslant \vee_u\{d(u,w) \mid u \in (\widetilde{P}_1)^t, w \in (P_3)^t\} + \vee_v\{d(w,v) \mid w \in (P_3)^t, v \in (P_2)Ht\}$

$\leqslant b_2(t) + c_2(t)$. ■

The following example shows that $\widetilde{D}$ is not a strong fuzzy metric.

**Example 5.9** *Let $\tilde{P}_1, \tilde{P}_2$, and $\tilde{P}_3$ be fuzzy points at* $(1,0), (3,0)$,*and* $(2,0)$, *respectively. The shape of each $\tilde{P}_i$ is a right circular cone. For example, $\tilde{P}_1$ is a right circular cone with base $(x-1)^2+y^2 \leq 1/4$ and vertex $(1,0)$. The base of $\tilde{P}_2$ is $(x-3)^2+y^2 \leq 1/4$ and the base of $\tilde{P}_3$ is $(x-2)^2+y^2 \leq 1/4$. Then $\tilde{D}(\tilde{P}_1,\tilde{P}_2)(0) = [1,3]$, $\tilde{D}(\tilde{P}_1,\tilde{P}_3)(0) = \tilde{D}(\tilde{P}_3,\tilde{P}_2)(0) = [0,2]$ so that $\tilde{D}(\tilde{P}_1,\tilde{P}_3)(0) + \tilde{D}(\tilde{P}_3,\tilde{P}_2)(0) = [0,4]$ and $[1,3]$ is not $\leq_s [0,4]$.*

There several possible ways to define a fuzzy line. First, we might define a fuzzy line to be the set of all pairs of fuzzy numbers $(\tilde{X},\tilde{Y})$ which are solutions to

$\tilde{A}\tilde{X} + \tilde{B}\tilde{Y} = \tilde{C}$

for given fuzzy numbers $\tilde{A},\tilde{B},\tilde{C}$. However this equation often has no solution for $\tilde{X}$ and $\tilde{Y}$ using standard fuzzy arithmetic.

Another possible method is to define a fuzzy line to be the set of all pairs of fuzzy numbers $(\tilde{X},\tilde{Y})$ which are solutions to

$\tilde{Y} = \tilde{M}\tilde{X} + \tilde{B}$ for given fuzzy numbers $\tilde{M},\tilde{B}$. With this method, one cannot construct pictures of this type of fuzzy line.

A third possible method is to use the following approach. Let $\tilde{A},\tilde{B},\tilde{C}$ be fuzzy numbers. If $\tilde{A}(1) = \{a\}$ and $\tilde{B}(1) = \{b\}$, we assume that $a$ and $b$ are not both zero. Let

$\Omega_{11}(t) = \{(x,y) | ax + by = c, a \in \tilde{A}^t, b \in \tilde{B}^t, c \in \tilde{C}^t\}$ $\forall t \in [0,1]$. Then we let $\tilde{L}_{11}$ denote the fuzzy subset of $\mathbb{R}^2$ defined by $\forall(x,y) \in \mathbb{R}^2$,

$\tilde{L}_{11}(x,y) = \vee\{t | (x,y) \in \Omega_{11}\}$.

If $\tilde{A}(1) = 0$ and $\tilde{B}(1) = 0$, then $\Omega_{11}(1)$ can be empty since we then have the equation $0x + 0y = c, c \in \tilde{C}(1)$, which will have no solution when $c \neq 0$.

Another possible method to define a fuzzy line is with the equation $y = mx + b$. Let $\tilde{M}$ and $\tilde{B}$ be fuzzy numbers. Let

$\Omega_{12}(t) = \{(x, y)|y = mx + b, m \in \tilde{M}^t, b \in \tilde{B}^t\}\ \forall t \in [0,1]$.
Then we let $\tilde{L}_{12}$ denote the fuzzy subset of $\mathbb{R}^2$ defined by $\forall(x, y) \in \mathbb{R}^2$,
$\tilde{L}_{12}(x, y) = \vee\{t|(x, y) \in \Omega_{12}\}$.
Still another possible method is by using a point-slope form. Let $\tilde{K}$ be a fuzzy point in $\mathbb{R}^2$ and let $\tilde{M}$ be a fuzzy number. Let
$\Omega_2(t) = \{(x, y)|y - v = m(x - u), (u, v) \in \tilde{K}^t, m \in \tilde{M}^t\}\ \forall t \in [0,1]$.
Then we let $\tilde{L}_2$ denote the fuzzy subset of $\mathbb{R}^2$ defined by $\forall(x, y) \in \mathbb{R}^2$,
$\tilde{L}_2(x, y) = \vee\{t|(x, y) \in \Omega_2\}$.
Finally, another possibility is the two-point method. Let $\tilde{P}_1$ and $\tilde{P}_2$ be two fuzzy points in the plane. Let
$\Omega_3(t) = \{(x, y)|(y - v_1)/(x - u_1) = (v_2 - v_1)/(u_2 - u_1), (u_1, v_1) \in (\tilde{P}_1)^t, (u_2, v_2) \in (\tilde{P}_1)^t\}\ \forall t \in [0,1]$.
Then we let $\tilde{L}_3$ denote the fuzzy subset of $\mathbb{R}^2$ defined by $\forall(x, y) \in \mathbb{R}^2$,
$\tilde{L}_3(x, y) = \vee\{t|(x, y) \in \Omega_3\}$.
We consider $\tilde{L}_{11}, \tilde{L}_{12}, \tilde{L}_2$, and $\tilde{L}_3$ to be four different types of fuzzy lines.

**Theorem 5.37** *$(\tilde{L}_{11})^t = \Omega_{11}(t)$, $(\tilde{L}_{12})^t = \Omega_{12}(t)$, $(\tilde{L}_2)^t = \Omega_2(t)$, and $(\tilde{L}_3)^t = \Omega_3(t)\ \forall t \in [0,1]$.* ■

We now give some examples of fuzzy lines.

**Example 5.10** *Let $\tilde{A} = (-1,0,1)$, $\tilde{B} = (-1,1,2)$, and $\tilde{C} = (0,1,2)$ be triangular fuzzy numbers. Then the support of $\tilde{L}_{11}$, $cl(\bigcup_{0<t\leq 1} \tilde{L}_{11}^t)$, is all of $\mathbb{R}^2$. Also the level set $(\tilde{L}_{11})^1$ is the crisp line $y = 1$.*

**Example 5.11** *Let $\tilde{L}_{12}$ be defined by $y = 2x + \tilde{B}$, where $\tilde{B} = (0,1,2)$ is a fuzzy triangular number. Here $\tilde{M}$ is the crisp number 2. The graph of $z = \tilde{L}_{12}(x, y)$ is generated by $\tilde{B}$ on the y-axis, base on the interval $[0,2]$, and "running" the triangle along the crisp line $y = 2x + 1$.*

**Example 5.12** *Let $\tilde{M}$ denote the crisp real number 1 and let $\tilde{K}$ be a fuzzy point at $(1,1)$. Then $(\tilde{L}_2)^t$ will be all lines, slope 1, through a point in $\tilde{K}^t$. $(\tilde{L}_2)^1$ is the crisp line $y = x$. $\tilde{L}_2$ is "thin" when $cl(\bigcup_{0<t\leq 1} \tilde{K}^t)$ is "small".*

**Example 5.13** *Let $\tilde{P}_1(0,0)$ and $\tilde{P}_2(1,1)$ be two fuzzy points whose graph is a right circular cone. The base of $\tilde{P}_1(0,0)$ is $B_1 = \{(x, y)|x^2 + y^2 \leq (1/3)^2\}$ and vertex $(0,0)$. The base of $\tilde{P}_2(1,1)$ is $B_2 = \{(x, y)|(x - 1)^2 + (y - 1)^2 \leq (1/3)^2\}$ and vertex $(1,1)$. Then the support of $\tilde{L}_3$, is all lines through a point in $B_1$ and $B_2$. $(\tilde{L}_3)^1$ is the line $y = x$. $\tilde{L}_3$ is thin between $B_1$ and $B_2$, but gets wider and wider as we move along $y = x$ for $x > 1$ or for $x < 0$.*

A fuzzy line $\tilde{L}$ is said to contain a fuzzy point $\tilde{P}$ if and only if $\tilde{P} \subseteq \tilde{L}$.

Clearly $\tilde{L}_2$ contains $\tilde{K}$. If $\tilde{P}_{(c,d)}$ is a fuzzy point at $(c, d)$ and $\tilde{L}_2$ contains $\tilde{P}_{(c,d)}$, then $(c, d) \in \Omega_2(1)$. Let the level set $\tilde{M}^1 = [m_1, m_2]$ be an interval. We have that $\Omega_2(1)$ is all lines through $(a, b)$ with slope $m, m_1 \leq m \leq m_2$.

If $\tilde{M}$ is a triangular number, $\tilde{M}^1 = \{m\}$, then $\Omega_2(1)$ is the crisp line $y-b = m(x-a)$.

Let $\tilde{P}_1$ and $\tilde{P}_2$ be fuzzy points at $(a_1, b_1)$ and $(a_2, b_2)$, respectively, which define $\tilde{L}_3$. Then $\tilde{L}_3$ contains both $\tilde{P}_1$ and $\tilde{P}_2$. Also $(\tilde{L}_3)^1 = \Omega_3(1)$ will always be the crisp line through $(a_1, b_1)$ and $(a_2, b_2)$. If $\tilde{L}_3$ contains some other point $\tilde{Q}$, then $\tilde{Q}^1$ must be on the line which is $\Omega_3(1)$.

We now consider some relationships. We show how, under certain conditions, $\widetilde{L}_{11}$ is an $\widetilde{L}_{12}$, $\widetilde{L}_3$ is an $\widetilde{L}_2$, $\widetilde{L}_2$ is an $\widetilde{L}_{12}$, and $\widetilde{L}_{12}$ is an $\widetilde{L}_2$. We first consider $\widetilde{L}_{11}$ and $\widetilde{L}_{12}$.

Assume that zero does not belong to $\widetilde{B}^0$. Define

$\Omega_m(t) = \{-a/b \mid a \in \widetilde{A}^t, b \in \widetilde{B}^t\}, \qquad 0 \leqslant t \leqslant 1,$

and define $\widetilde{M}$ by $\forall x \in \mathbb{R}^2$,

$\widetilde{M}(x) = \vee\{t \mid x \in \Omega_m(t)\}.$

Next set

$\Omega_b(t) = \{c/d : b \in \widetilde{B}^t, c \in \widetilde{C}^t\}, \qquad 0 \leqslant t \leqslant 1,$

and define $\widetilde{B}_0$ by $\forall x \in \mathbb{R}^2$,

$\widetilde{B}_0(x) = \vee\{t \mid x \in \Omega_b(t)\}.$

In the above definitions, $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are the fuzzy numbers in the definition of $\widetilde{L}_{11}$. It follows that $\widetilde{M}$ and $\widetilde{B}_0$ are also fuzzy numbers and that $\widetilde{M}^t = \Omega_m(t)$ and $(\widetilde{B}_0)^t = \Omega_b(t)$ for all $t$. Thus let $\widetilde{M}$ and $\widetilde{B}_0$ be the fuzzy numbers in the definition of $\widetilde{L}_{12}$.

**Theorem 5.38** $\widetilde{L}_{11} = \widetilde{L}_{12}$.

**Proof.** We show that $(\widetilde{L}_{11})^t = \Omega_{11}(t)$ is the same as $(\widetilde{L}_{12})^t = \Omega_{12}(t)$ for all $t$.

If $(x, y) \in \Omega_{11}(t)$, then $ax + by = c$ for some $a \in \widetilde{A}^t, b \in \widetilde{B}^t, c \in \widetilde{C}^t$. Then $y = mx + b_0$ for $m = -a/b, b_0 = c/b$. However $m \in \widetilde{M}^t, b_0 \in (\widetilde{B}_0)^t$. Thus $(x, y) \in \Omega_{12}(t)$. Hence, $\Omega_{11}(t)$ is a subset of $\Omega_{12}(t)$.

Similarly, it follows that $\Omega_{12}(t)$ is a subset of $\Omega_{11}(t)$. ∎

We now consider $\widetilde{L}_3$ and $\widetilde{L}_2$.

Let $\widetilde{P}_1 = \widetilde{P}(a_1, b_1), \widetilde{P}_2 = \widetilde{P}(a_2, b_2)$ be two fuzzy points which define $\widetilde{L}_3$. Define $\mathrm{Proj}_x(\mathrm{Proj}_y)$ to be the projection of a subset of the plane onto the $x$-axis ($y$-axis). Assume that $\mathrm{Proj}_x(\widetilde{P}_1)^0 \cap \mathrm{Proj}_x(\widetilde{P}_2)^0 = \emptyset$. Hence if $(u_1, v_1) \in (\widetilde{P}_1)^0$ and $(u_2, v_2) \in (\widetilde{P}_2)^0$, then $u_1 - u_2$ will never be zero. Define

$\Omega_m(t) = \{m \mid m = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in (\widetilde{P}_1)^t, (u_2, v_2) \in (\widetilde{P}_2)^t\}$ for $0 \leqslant t \leqslant 1$,

and set

$\widetilde{M}(x) = \vee\{t \mid x \in \Omega_m(t)\}.$

It follows that $\widetilde{M}$ is a fuzzy number and $\widetilde{M}^t = \Omega_m(t)$ for all $t$. Now let $\widetilde{M}$ and $\widetilde{P}_1$ be fuzzy subsets in the definition of $\widetilde{L}_2$.

**Theorem 5.39** $\widetilde{L}_3 = \widetilde{L}_2$.

**Proof.** We show $\Omega_3(t) = \Omega_2(t)$ for all $t$. Let $(x, y) \in (\widetilde{L}_3)^t$. Then $y - v_1 = m(x - u_1)$ for $m \in \widetilde{M}^t$, $(u_1, v_1) \in (\widetilde{P}_1)^t$. But then $(x, y) \in \Omega_2(t)$ and $\Omega_3(t)$ is a subset of $\Omega_2(t)$. Similarly, we can show that $\Omega_2(t)$ is a subset of $\Omega_3(t)$. ■

We consider $\widetilde{L}_{12}$ and $\widetilde{L}_2$ next.

We first show that given an $\widetilde{L}_{12}$, we can define an $\widetilde{L}_2$ so that $\widetilde{L}_2 = \widetilde{L}_{12}$. Let $\widetilde{M}$, $\widetilde{B}$ be fuzzy numbers that define $\widetilde{L}_{12}$. We use the same $\widetilde{M}$ for $\widetilde{L}_{12}$. [7, Figure. 1. p. 186] shows a typical $t$-cut of an $\widetilde{L}_{12}$. Recall that $(\widetilde{L}_{12})^t = \Omega_{12}(t)$. We need to specify a fuzzy point $\widetilde{K}$ to completely define $\widetilde{L}_2$. Let $\widetilde{M}^t = [m_1, m_2]$ and $\widetilde{B}^t = [b_1, b_2]$. Assume that $\widetilde{B}(x) = 1$ if and only if $x = b^*$ where $b_1 < b^* < b_2$. That is, $\widetilde{B}$ is normalized at only one point. Define $\widetilde{K}$ to be a fuzzy point at $(0, b^*)$ so that its $t$-cut lies in region $\widetilde{R}^t$ in [7, Figure 1, p. 186]. Then $\Omega_2(t) = \Omega_{12}(t)$ and $\widetilde{L}_2$ will be the same as $\widetilde{L}_{12}$.

Conversely, let $\widetilde{M}$ and $\widetilde{K}$ be the fuzzy subsets in the definition of $\widetilde{L}_2$. Then $\widetilde{K}$ must be a fuzzy point on the $y$-axis. Let $\widetilde{K}$ be a fuzzy point at $(0, b^*)$ so that its $t$-cuts fit inside region $\widetilde{R}^t$ of [7, Figure 1, p.186]. $\mathrm{Proj}_y \widetilde{K}^t$ will be $[b_1, b_2]$ in [7, Figure 1, p. 186]. Define the fuzzy number $\widetilde{B}$ so that $\widetilde{B}^t = \mathrm{Proj}_y \widetilde{K}^t$ for all $t$. The same $\widetilde{M}$ is used for $\widetilde{L}_{12}$. Then $\Omega_{12}(t) = \Omega_2(t)$ for all $t$ so that $\widetilde{L}_{12} = \widetilde{L}_2$.

We consider some general properties.

A fuzzy line $\widetilde{L}$ will be an $\widetilde{L}_{11}$, $\widetilde{L}_{12}, \widetilde{L}_2$, or an $\widetilde{L}_3$. Then the following properties hold:

(1) $t$-cuts of $\widetilde{L}$ are closed, connected and arcwise connected, but not necessarily convex;

(2) $\widetilde{L}(x, y)$ is upper semi-continuous since $t$-cuts are closed; and

(3) $\widetilde{L}$ is normalized, or there is always at least one crisp line in $\widetilde{L}^1$.

**Definition 5.13** *Let $\widetilde{L}_a, \widetilde{L}_b$ be two fuzzy lines. A measure of parallelness $(\rho)$ of $\widetilde{L}_a$ and $\widetilde{L}_b$ is defined to be $1 - \lambda$ where $\lambda = \vee\{\widetilde{L}_a(x, y) \wedge \widetilde{L}_b(x, y) \mid (x, y) \in \mathbb{R}^2\}$.*

In Definition 5.13, $\lambda$ is just the height of the intersection of $\widetilde{L}_a$ and $\widetilde{L}_b$. Hence, if $\widetilde{L}_a \cap \widetilde{L}_b$ is the empty set (completely parallel), then $\lambda = 0$ and $\rho = 1$. Let $l_a$ and $l_b$ be a crisp lines in $(\widetilde{L}_a)^1$ and $(\widetilde{L}_b)^1$, respectively. If $l_a$ and $l_b$ intersect, then $\lambda = 1$ and $\rho = 0$. Thus $\rho$ has some properties we would expect for a measure of parallelness.

Suppose $\widetilde{L}_a$ and $\widetilde{L}_b$ are both crisp lines. Then $\rho = 1$ if and only if $\widetilde{L}_a$ and $\widetilde{L}_b$ are parallel.

Now let $\widetilde{L}_a$ be a crisp line and $\widetilde{L}_b$ a fuzzy line. For example, let $\widetilde{L}_b = \widetilde{L}_{12}$ or $\widetilde{L}_2$. If $\widetilde{M}^1 = [m_1, m_2]$, $m_1 < m_2$, then $\widetilde{L}_a$ intersects a crisp line in $(\widetilde{L}_b)^1$ and $\rho = 0$. Hence $\rho = 0$ for a crisp line and a fuzzy line.

**Definition 5.14** *Let $\widetilde{L}_a$ and $\widetilde{L}_b$ be two fuzzy lines. Assume $\rho \leq 1$. The fuzzy region $\widetilde{R}$ of intersection of $\widetilde{L}_a$ and $\widetilde{L}_b$ is $\widetilde{R} = \widetilde{L}_a \cap \widetilde{L}_b$. Then $\widetilde{R}(x, y) = \widetilde{L}_a(x, y) \wedge \widetilde{L}_b(x, y)$.*

Clearly, if $\rho = 1$, then the fuzzy region of intersection $\widetilde{R}$ is the empty set.

## 5.11 Fuzzy Plane Geometry: Circles and Polygons

This section continues the development of fuzzy plane geometry of the previous section. The material is from [8]. We investigate fuzzy circles and fuzzy polygons. We show that the fuzzy area of a fuzzy circle, or a fuzzy polygon, is a fuzzy number. We also show that the fuzzy perimeter of a fuzzy circle, or a fuzzy polygon, is a fuzzy number.

We define a fuzzy circle and show how to get $t$-cuts of a fuzzy circle. We also consider some examples of fuzzy circles. We define the fuzzy area of a fuzzy circle and show that it is a fuzzy number. We then define the fuzzy circumference of a fuzzy circle and prove that it is also a fuzzy number. We look at some examples of the fuzzy area and the circumference of fuzzy circles.

The equation $x^2 + ax + y^2 + by = \mathrm{c}$ defines a circle when $4\mathrm{c} > a^2 + b^2$. Our first method is to fuzzify this procedure.

Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy numbers. A fuzzy circle $\mathfrak{C}$ is all pairs of fuzzy numbers $(\widetilde{X}, \widetilde{Y})$ which are solutions to

$(\widetilde{X})^2 + \widetilde{A}\widetilde{X} + (\widetilde{Y})^2 + \widetilde{B}\widetilde{Y} = \widetilde{C}$,

where $4\widetilde{C} > (\widetilde{A})^2 + (\widetilde{B})^2$.

However, the above equation usually has no solution (using standard fuzzy arithmetic) for $\widetilde{X}$ and $\widetilde{Y}$, [9]. Therefore, we do not use this method in defining a fuzzy circle.

Another possible method in specifying a circle is to use the standard equation for a circle: $(x - a)^2 + (y - b)^2 = \mathrm{c}^2$. This leads us to our second method of defining a fuzzy circle.

Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy numbers. A fuzzy circle $\mathfrak{C}$ is all pairs of fuzzy numbers $(\widetilde{X}, \widetilde{Y})$ which are solutions to

$(\widetilde{X} - \widetilde{A})^2 + (\widetilde{Y} - \widetilde{B})^2 = (\widetilde{C})^2$.

Unfortunately, this equation also has few, if any, solutions for $\widetilde{X}$ and $\widetilde{Y}$, [9]. This leads to the next approach in defining fuzzy circles [6,10,11].

Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy numbers. Let

$\Omega(t) = \{(x, y) \mid (x - a)^2 + (y - b)^2 = \mathrm{c}^2, a \in \widetilde{A}^t, b \in \widetilde{B}^t, \mathrm{c} \in \widetilde{C}^t\}$,

for $0 \leqslant t \leqslant 1$. A fuzzy circle is defined as follows: $\forall (x, y) \in \mathbb{R}^2$,

$\mathfrak{C}(x, y) = \vee\{t \mid (x, y) \in \Omega(t)\}$.

We will adopt this method to define a fuzzy circle. This type of fuzzy circle will be seen to have desirable properties including its fuzzy area and circumference being fuzzy numbers (or real numbers as a special case of fuzzy numbers).

As the following theorem, shows it is not too difficult to obtain $t$-cuts of fuzzy circles.

**Theorem 5.40** $\mathfrak{C}^t = \Omega(t), 0 \leqslant t \leqslant 1$.

**Proof.** We first show that the $t$-cuts are the same for $0 < t \leqslant 1$. Let $v \in \Omega(t)$. Then $\mathfrak{C}(v) \geqslant t$ and $\Omega(t)$ is a subset of $\mathfrak{C}^t$.

We now show that $\mathfrak{C}^t$ is a subset of $\Omega(t)$. Let $v \in \mathfrak{C}^t$. Then $\mathfrak{C}(v) \geqslant t$. Set $\mathfrak{C}(v) = s$. We consider two cases: (a) $s > t$; and (b) $s = t$.

(a) Suppose that $s > t$. There is an $r, t < r \leqslant s$ with $v$ in $\Omega(r)$. Since $\Omega(r)$ is a subset of $\Omega(t)$, we have $v$ in $\Omega(t)$.

(b) Suppose that $s = t$. Let $K = \{w \mid v \in \Omega(w)\}$. Then $\vee K = s = t = \mathfrak{C}(v)$. There is a sequence $r_n$ in $K$ such that $r_n \uparrow t$. Given $\varepsilon > 0$ there is a positive integer $N$ so that $t - \varepsilon < r_n, n \geqslant N$. Now $v \in \Omega(r_n)$ for all $n$ implies that $v$ is also in $\Omega(t-\varepsilon)$ for all $\varepsilon > 0$. If $v = (x, y)$, then $(x-a)^2 + (y-b)^2 = c^2$ for some $a$ in $\widetilde{A}^{t-\varepsilon}$, $b$ in $\widetilde{B}^{t-\varepsilon}$, $c$ in $\widetilde{C}^{t-\varepsilon}$. Hence, $\widetilde{A}(a) \geqslant t-\varepsilon$, $\widetilde{B}(b) \geqslant t-\varepsilon$, $\widetilde{C}(c) \geqslant t-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\widetilde{A}(a) \geqslant t$, $\widetilde{B}(b) \geqslant t$, $\widetilde{C}(c) \geqslant t$ and $a \in \widetilde{A}^t, b \in \widetilde{B}^t, c \in \widetilde{C}^t$. Hence $v = (x, y)$ is in $\Omega(t)$. Thus $\mathfrak{C}^t$ is a subset of $\Omega(t)$.

It follows that $\mathfrak{C}^0 = \Omega(0)$ since $\mathfrak{C}^t = \Omega(t), 0 < t \leqslant 1$. ∎

**Example 5.14** *The fuzzy circle defined here is called "thick" or "fat" in [8]. Let $\widetilde{A} = (0/1/2) = \widetilde{B} = \widetilde{C}$ be triangular fuzzy numbers. The support of $\mathfrak{C}$, $\mathfrak{C}^0 = \Omega(0)$ is the rectangle $[-2, 4] \times [-2, 4]$ with rounded edges. In fact, all t-cuts of $\mathfrak{C}, 0 \leqslant t \leqslant 1$, are rounded corner rectangles with $\mathfrak{C}^1$ the crisp circle $(x-1)^2 + (y-1)^2 = 1$. The graph of the membership function of $\mathfrak{C}$ is a four-sided pyramid, with rounded edges and vertex at $(1, 1)$.*

**Example 5.15** *By a "regular" fuzzy circle, we mean that t-cuts, $0 \leqslant t < 1$, will be disks. Let $\widetilde{A} = \widetilde{B} = 1$ (real number one) and $\widetilde{C} = (1/2/3)$. All t-cuts, $0 \leqslant t < 1$, of $\mathfrak{C}$ are disks with $\mathfrak{C}^0 = \{(x, y) \mid 1 \leqslant (x-1)^2 + (y-1)^2 \leqslant 9\}$. $\mathfrak{C}^1$ is the crisp circle $(x-1)^2 + (y-1)^2 = 1$.*

**Definition 5.15** *Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy numbers in the definition of the fuzzy circle $\mathfrak{C}$. Set*

$\Omega_a(t) = \{\theta \mid \theta$ *is the area of* $(x-a)^2 + (y-b)^2 = c^2, a \in \widetilde{A}^t, b \in \widetilde{B}^t, c \in \widetilde{C}^t\}$,

$0 \leqslant t \leqslant 1$. *The area $\widetilde{\Theta}$ of $\mathfrak{C}$ is defined as follows:*

$\widetilde{\Theta}(\theta) = \vee\{t \mid \theta \in \Omega_a(t)\}$.

There are some degenerate cases which must be considered where $\widetilde{\Theta}$ becomes a (crisp) real number (a special case of a fuzzy number). If $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are all real numbers, then $\mathfrak{C}$ is a crisp circle and $\widetilde{\Theta}$ is a real number. Thus assume that at least one of the $\widetilde{A}, \widetilde{B}, \widetilde{C}$ is a fuzzy number which is not a real number. The only case where $\widetilde{\Theta}$ can degenerate into a crisp number is when $\widetilde{C}$ is real. Let $\widetilde{C} = c, c > 0$, a real number. Then $\widetilde{\Theta} = \pi c^2$ a real number. Thus, in the next theorem, it is assumed that $\widetilde{C}$ is not a real number.

**Theorem 5.41** $\widetilde{\Theta}^t = \Omega_a(t)$ *for all $t$ and $\widetilde{\Theta}$ is a fuzzy number.*

**Proof.** The proof that $\widetilde{\Theta}^t = \Omega_a(t)$ for all $t$ is similar to the proof that $\mathfrak{C}^t = \Omega(t)$ for all $t$ in Theorem 5.40.

We now show that $\widetilde{\Theta}$ is a fuzzy number. Since the $t$-cuts of $\widetilde{A}, \widetilde{B}, \widetilde{C}$ are compact intervals, it follows that $\Omega(t)$ is a bounded closed interval for all $t$. Let $\Omega_a(t) = [l(t), r(t)], 0 \leqslant t \leqslant 1$. If the $t$-cuts are closed intervals, then the membership function is upper semi-continuous [11]. However, $\widetilde{\Theta}^t = \Omega(t) =$ a closed interval for all $t$. Hence, $\widetilde{\Theta}(\theta)$ is upper semi-continuous. Let $\Omega(0) = [c, d]$. Clearly, $\widetilde{\Theta}(\theta) = 0$ outside $[c, d]$. Let $\Omega(1) = [a, b]$. Then, $\widetilde{\Theta}(\theta) = 1$ on $[a, b]$. Now since $\Omega(t) = [l(t), r(t)] = \widetilde{\Theta}^t$ for all $t$ with $l(t)$ increasing from $c$ to $a$ and $r(t)$ decreasing from $d$ to $b$, it follows that $\widetilde{\Theta}(\theta)$ is increasing on $[c, a]$, decreasing on $[b, d]$, and it equals 1 on $[a, b]$. Thus $\widetilde{\Theta}$ is a fuzzy number. ■

**Definition 5.16** *Let $\widetilde{A}, \widetilde{B}, \widetilde{C}$ be fuzzy numbers in the definition of $\mathfrak{C}$. Define*

$\Omega_c(t) = \{\Delta \mid \Delta$ *is the circumference of* $(x-a)^2 + (y-b)^2 = c^2, a \in \widetilde{A}^t, b \in \widetilde{B}^t, c \in \widetilde{C}^t\}$,

$0 \leqslant t \leqslant 1$. *Define the circumference $\delta$ as*

$\delta(\Delta) = \vee\{t \mid \Delta \in \Omega_c(t)\}$.

In the following theorem, assume that $\widetilde{C}$ is not a real number. If $\widetilde{C}$ is real, then $\delta$ is a real number.

**Theorem 5.42** $\delta^t = \Omega_c(t)$ *for all $t$ and $\delta$ is a fuzzy number.*

**Proof.** The proof is similar to the proof of Theorem 5.41. ■

**Example 5.16** *Consider the "regular" fuzzy circle of Example 5.15. Then $\widetilde{A} = \widetilde{B} = 1$ and $\widetilde{C} = (1/2/3)$ is a triangular fuzzy number. The level sets of $\widetilde{C}$ are $[t+1, 3-t], 0 \leqslant t \leqslant 1$. Thus $\widetilde{\Theta}^t = [\pi(t+1)^2, \pi(3-t)^2], \delta^t = [2\pi(t+1), 2\pi(3-t)], 0 \leqslant t \leqslant 1$. Hence, the fuzzy area $\widetilde{\Theta}$ is a* triangular-shaped *fuzzy number with support $[\pi, 9\pi]$ and vertex at $(4\pi, 0)$. The fuzzy circumference $\delta$ is a triangular fuzzy number $(2\pi/4\pi/6\pi)$.*

**Example 5.17** *Consider Example 5.14, where $\widetilde{A} = \widetilde{B} = \widetilde{C} = (0/1/2)$. The level sets of $\widetilde{C}$ are $[t, 2-t], 0 \leqslant t \leqslant 1$. It follows that $\widetilde{\Theta}$ is a* triangular-shaped *fuzzy number with t-cuts $[\pi t^2, \pi(2-t)^2], 0 \leqslant t \leqslant 1$. Also, $\delta$ is a triangular fuzzy number $(0/\pi/4\pi)$.*

We next discuss what we mean by a regular $n$-sided polygon, define (regular) $n$-sided polygons, and compute $t$-cuts of fuzzy polygons. We define the fuzzy area and perimeter of a fuzzy polygon and show that they are fuzzy numbers. We also look at two special cases as fuzzy rectangles and triangles.

A polygon is a rectilinear figure with $n$ sides, $n \geqslant 3$. We allow $n = 3$ and $n = 4$ so that triangles and rectangles can be considered polygons. A regular polygon will have a convex interior.

**Definition 5.17** *Distinct points $v_1, ..., v_n$ in $\mathbb{R}^2$ are said to be* convex independent *if and only if any $v_i$ does not belong to the convex hull (convex closure) of the rest of the $v_j, 1 \leqslant j \leqslant n, j \neq i$.*

Let $v_1, ..., v_n$ be convex independent. Assume they are numbered counterclockwise, i.e., as we travel from $v_1$ to $v_2, ..., v_{n-1}$ to $v_n$ we continually travel in a counterclockwise direction. We connect adjacent $v_i$ with line segments in the following manner. We draw a line segment $l_{12}$ from $v_1$ to $v_2, ..., l_{n1}$ from $v_n$ to $v_1$. The $v_i$, together with the line segments, define a regular $n$-sided polygon. We call such a polygon an $n$-gon. The interior of an $n$-gon is convex. When $n = 3$ we have a triangle.

Let $v_1, ..., v_n$ be $n$ distinct points in the plane that define an $n$-gon (a regular $n$-sided polygon). Next let $\widetilde{P}_i$ be a fuzzy point at $v_i, 1 \leqslant i \leqslant n$. We now need the definition of a fuzzy line segment.

**Definition 5.18** *Let $\widetilde{P}$ and $\widetilde{Q}$ be distinct fuzzy points. Let $\Omega_l(t)$ denote the set of all line segments from a point in $\widetilde{P}^t$ to a point in $\widetilde{Q}^t\}$. The fuzzy line segment $\widetilde{L}_{pq}$, from $\widetilde{P}$ to $\widetilde{Q}$ is defined as follows:*

$\widetilde{L}_{pq}(x, y) = \vee\{t \mid (x, y) \in \Omega_l(t)\}$.

As in the proof of Theorem 5.40, we can show that $(\widetilde{L}_{pq})^t = \Omega_l(t)$ for all $t$.

**Definition 5.19** *Let $\widetilde{L}_1, ..., \widetilde{L}_n$ be fuzzy line segments from $\widetilde{P}_1$ to $\widetilde{P}_2, ..., \widetilde{P}_n$ to $\widetilde{P}_1$, respectively. Then a (regular) $n-$sided fuzzy polygon $\mathfrak{P}$ is defined to be $\mathfrak{P} = \cup_{i=1}^n \widetilde{L}_i$.*

It follows that

$$\mathfrak{P}(x, y) = \vee\{\widetilde{L}_i(x, y) \mid i = 1, ..., n\} \tag{5.11.1}$$

for all $(x, y) \in \mathbb{R}^2$.

If the two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ are such that $(\widetilde{P}_2)^0$ is a subset of $(\widetilde{P}_1)^0$, then we obtain a degenerate fuzzy polygon. In a degenerate fuzzy polygon the support, $\mathfrak{P}^0$, does not show all the (fuzzy) $n$ vertices.

**Example 5.18** *Let $n = 3$ and the fuzzy points be all right circular cones. $\widetilde{P}_1$ has base $B_1 = \{(x, y) \mid x^2 + y^2 \leqslant (0.1)^2\}$ and vertex at (0,0), $\widetilde{P}_2$ has base $B_2 = \{(x, y) \mid (x-1)^2 + y^2 \leqslant (0.1)^2\}$ with vertex at (1,0), and $\widetilde{P}_3$ has base $B_3 = \{(x, y) \mid (x-1)^2 + (y-0.5)^2 \leqslant 1\}$ having vertex at (1,0.5). Clearly, $(\widetilde{P}_2)^0 \subset (\widetilde{P}_3)^0$ and $(\widetilde{P}_1)^0$, $(\widetilde{P}_3)^0$ are disjoint. Thus, $(\widetilde{L}_2)^0 \subset (\widetilde{P}_3)^0$ and $\mathfrak{P}$ is degenerate. Also, $L_1(0) \subset L_3(0)$. Hence support of $\mathfrak{P}$ is $(\widetilde{L}_3)^0$.*

**Definition 5.20** *We call a regular $n$-sided fuzzy polygon a fuzzy $n$-gon. We say a fuzzy $n$-gon is non-degenerate if $(\widetilde{P}_i)^0$ is not a subset of $(\widetilde{P}_j)^0$, $j \neq i$, for all $i = 1, \ldots, n$. We say the fuzzy $n$-gon is strongly non-degenerate if the $(\widetilde{P}_i)^0 i \leqslant n$, are pairwise disjoint.*

**Example 5.19** *Let $n = 4$ and let $\widetilde{P}_1$ be a fuzzy point at $(0,0)$, $\widetilde{P}_2$ a fuzzy point at $(1,0)$, $\widetilde{P}_3$ a fuzzy point at $(1,1)$, and $\widetilde{P}_4$ a fuzzy point at $(0,1)$. Let $\widetilde{P}_1$ be a right circular cone with base $B_1 = \{(x,y) \mid x^2 + y^2 \leqslant (0.1)^2\}$ and vertex at $(0,0)$. The rest of the $\widetilde{P}_i, i > 1$, will just be rigid translations of $\widetilde{P}_1$ to their position in the plane. Then $\mathfrak{P}$ is strongly non-degenerate; in fact, we would call $\mathfrak{P}$ a fuzzy rectangle.*

**Theorem 5.43** $\mathfrak{P}^t = \cup_{i=1}^{n}(\widetilde{L}_i)^t$ *for all $t$.*

**Proof.** We have that $(\widetilde{L}_i)^t = \Omega_i(t)$, where $\Omega_i(t)$ is the set of all line segments from $(P_i)^t$to $(P_{i+1})^t, 0 \leqslant t \leqslant 1$. (We replace $i+1$ by 1 when $i = n$). Thus, we must show

$$\mathfrak{P}^t = \cup_{i=1}^{n}\Omega_i(t), 0 \leqslant t \leqslant 1. \tag{5.11.2}$$

We first show that Eq.(5.11.2) is true for $0 < t \leqslant 1$.

Let $v$ belong to the union of the $\Omega_i(t)$. Then there is some value of $i$, say $i = 1$, such that $v \in \Omega_1(t)$. Then $\widetilde{L}_1(v) \geqslant t$. Hence by Eq. (5.11.1) $\mathfrak{P}(v) \geqslant t$ and $v$ belongs to $\mathfrak{P}^t$. Thus the union of the $\Omega_i(t)$ is a subset of $\mathfrak{P}^t$.

Let $v$ belong to $\mathfrak{P}^t$ and let $\mathfrak{P}(v) = s$. Then $s > t$ or $s = t$.

Suppose $s = t$. By (5.11.1), there is an $i$, say $i = 1$, such that $\widetilde{L}_1(v) = t$. Hence $v \in \Omega_1(t)$ and $v$ belongs to the union of the $\Omega_i(t)$.

Suppose $s > t$. By (5.11.1), there is an $i$, say $i = 1$, such that $\widetilde{L}_1 = s$. Then $v \in \Omega_1(s)$. However since $s > t$ we also have $v$ in $\Omega_1(t)$. Hence $v$ belongs to the union of the $\Omega_i(t)$. Thus $\mathfrak{P}^t$ is a subset of the union of the $\Omega_i(t)$.

Since Eq.(5.11.2) holds for $0 < t \leqslant 1$, it holds for $t = 0$. ■

We now consider the fuzzy area of a fuzzy $n$-gon.

**Definition 5.21** *Let $\mathfrak{P}$ be a strongly non-degenerate fuzzy $n$-gon defined by fuzzy points $\widetilde{P}_i, 1 \leqslant i \leqslant n$. Define $\Omega_a(t)$ to be the set of all areas of $n$-gons such that $v_i \in (\widetilde{P}_i)^t$, $1 \leqslant i \leqslant n, 0 \leqslant t \leqslant 1$. The area $\widetilde{\Theta}$ of $\mathfrak{P}$ is defined to be*
$\widetilde{\Theta}(\theta) = \vee\{t \mid \theta \in \Omega_a(t)\}$.

In a similar way, we can define the fuzzy perimeter of a fuzzy $n$-gon.

**Definition 5.22** *Let $\mathfrak{P}$ be a strongly non-degenerate fuzzy $n$-gon defined by fuzzy points $\widetilde{P}_i, i \leqslant i \leqslant n$. Set $\Omega_p(t)$ equal to the set of all perimeters of $n$-gons defined by $v_i$ in $(\widetilde{P}_i)^t, 1 \leqslant i \leqslant n$. $0 \leqslant t \leqslant 1$. Then the fuzzy*

*perimeter* $\widetilde{\delta}$ *is*
$\widetilde{\delta}(\triangle) = \vee\{\alpha \mid \triangle \in \Omega_p(t)\}$.

**Theorem 5.44** $\widetilde{\Theta}^t = \Omega_a(t)$ *and* $\widetilde{\delta}^t = \Omega_p(t)$ *for all* $t$.

**Proof.** The proof is similar to those of Theorems 5.40 and 5.41. ■

If we allow $\mathfrak{P}$ to be degenerate, we still obtain $\widetilde{\Theta}^t = \Omega_a(t)$, $\widetilde{\delta}^t = \Omega_p(t)$, for all $t$, but $\widetilde{\Theta}$ and $\widetilde{\delta}$ may not be fuzzy numbers.

**Example 5.20** *Consider the degenerate fuzzy polygon in Example 5.18. Since some* $t$*-cuts of* $\widetilde{P}_2$ *are inside the corresponding* $t$*-cuts of* $\widetilde{P}_3$, $\Omega_a(t)$ *is the interval* $(0, r(t)], 0 \leqslant t \leqslant t^*$ *for some* $0 < t^* < 1$. *However* $\widetilde{\Theta}^t = \Omega_a(t)$ *so that* $\widetilde{\Theta}^t$ *is not always be a closed interval. Hence,* $\widetilde{\Theta}(\theta)$ *is not be upper semi-continuous. Similar results hold for the fuzzy perimeter. Here neither* $\widetilde{\Theta}$ *nor* $\widetilde{\delta}$ *are fuzzy numbers.*

If $n = 3$ a fuzzy $n$-gon, it is a fuzzy triangle. If it is strongly non-degenerate, its fuzzy area and perimeter are fuzzy numbers. Now, since we have a fuzzy triangle we may investigate the beginnings of fuzzy trigonometry (see also [51] and Section 5.2). For the remainder of the section, we assume that the fuzzy triangle is strongly non-degenerate.

**Definition 5.23** *Let* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ *be the fuzzy points that define the fuzzy triangle* $\mathfrak{T}$. *Let* $\widetilde{L}_{12}, \widetilde{L}_{23}, \widetilde{L}_{31}$ *be the fuzzy line segments connecting* $\widetilde{P}_1$ *to* $\widetilde{P}_2$, $\widetilde{P}_2$ *to* $\widetilde{P}_2$ *to* $\widetilde{P}_3$, *and* $\widetilde{P}_3$ *to* $\widetilde{P}_1$, *respectively. Define* $\Omega_\pi(t)$ *denote the set of all angles (in radians) between* $l_{12}$ *and* $l_{31}$ *such that* $l_{12}$ *is a line segment from a point in* $(\widetilde{P}_1)^t$ *to a point in* $(\widetilde{P}_2)^t$, $l_{31}$ *is a line segment from* $(\widetilde{P}_3)^t$ *to* $(\widetilde{P}_1)^t$, *and* $l_{12}$, $l_{31}$ *initiate from the same point in* $(\widetilde{P}_1)^t, 0 \leqslant t \leqslant 1$. *We define* $\widetilde{\Pi}$, *the fuzzy angle between* $\widetilde{L}_{12}$ *and* $\widetilde{L}_{31}$, *as follows:*
$\widetilde{\Pi}(\pi) = \vee\{t \mid \pi \in \Omega_\pi(t)\}$.

**Theorem 5.45** $\widetilde{\Pi}^t = \Omega_\pi(t)$ *for all* $t$ *and* $\widetilde{\Pi}$ *is a fuzzy number.*

**Proof.** The proof is similar to that of Theorems 5.40 and 5.41. ■

In the definition of $\widetilde{\Pi}$, assume that $\widetilde{P}_i$ is a fuzzy point at $v_i$ in $\mathbb{R}^2$, $i = 1, 2, 3$. $\widetilde{\Pi}$ depends not only on the $v_i$, but also on the "size" of the fuzzy points. That is, if we substitute fuzzy point $\widetilde{P}'_2$ at $v_2$ for $\widetilde{P}_2$ and $(\widetilde{P}_2)^t \subset (\widetilde{P}'_2)^t$ for all $t$ and $\widetilde{\Pi}'$ is the resulting fuzzy angle, then we can get $\widetilde{\Pi} \subset \widetilde{\Pi}'$. Thus we write $\widetilde{\Pi} = \widetilde{\Pi}(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3)$ to show the dependence of $\widetilde{\Pi}$ on the fuzzy points.

Consider elementary fuzzy right triangle trigonometry. Consider fuzzy points $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ at $v_1 = (0, 0)$, $v_2 = (a, 0)$, $v_3 = (a, b)$, $a > 0$, $b > 0$, respectively. The $\widetilde{P}_i$, $1 \leqslant i \leqslant 3$, form a right triangle. $\widetilde{\Pi}$ is the fuzzy angle between $\widetilde{L}_{12}$ and $\widetilde{L}_{31}$. Define
$\tan(\widetilde{\Pi}) = \widetilde{D}(\widetilde{P}_2, \widetilde{P}_3)/\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$,

where $\widetilde{D}(\widetilde{P}, \widetilde{Q})$ is the fuzzy distance from fuzzy point $\widetilde{P}$ to $\widetilde{Q}$. Now $\tan(\widetilde{\Pi})$ is well-defined and is a fuzzy number, since (a) zero does not belong to the support of $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$ and (b) $\widetilde{D}(\widetilde{P}_2, \widetilde{P}_3)$ and $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$ are both fuzzy numbers. We can define $\sin(\widetilde{\Pi})$ and $\cos(\widetilde{\Pi})$ in a similar manner and both will be fuzzy numbers.

However, many properties of triangles and crisp trigonometry identities such as $\cos^2(\theta) + \sin^2(\theta) = 1$ do not generalize to fuzzy triangles (see also [51] and Section 5.2). The following example shows that the Pythagorean theorem may not hold for fuzzy right triangles.

**Example 5.21** *Let $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ be fuzzy points at $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$, respectively. Each $\widetilde{P}_i$ is a right circular cone, base a circle of radius 0.1 centered at $v_i$, with vertex at $v_i$, $i = 1,2,3$. Let $\widetilde{A} = \widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$, $\widetilde{B} = \widetilde{D}(\widetilde{P}_2, \widetilde{P}_3), \widetilde{C} = \widetilde{D}(\widetilde{P}_2, \widetilde{P}_3)$. Now $\widetilde{L}_{12}$ and $\widetilde{L}_{23}$ are the two sides of the fuzzy right angle $\widetilde{\mathfrak{T}}$ and $\widetilde{L}_{31}$ is the hypotenuse. The lengths of $\widetilde{L}_{12}, \widetilde{L}_{23}, \widetilde{L}_{31}$ are $\widetilde{A}, \widetilde{B}, \widetilde{C}$, respectively. But $(\widetilde{A})^2 + (\widetilde{B})^2 \neq (\widetilde{C})^2$ because $\widetilde{A}^0 = \widetilde{B}^0 = [0.8, 1.2]$, $\widetilde{C}^0 = [\sqrt{2} - 0.2, \sqrt{2} + 0.2]$ so that the support of $(\widetilde{A})^2 + (\widetilde{B})^2$ is $[1.28, 2.88]$ but the support of $(\widetilde{C})^2$ is $[1.474, 2.606]$.*

We now consider fuzzy rectangles. Let $v_1 = (x,y), v_2 = (x+r,y), v_3 = (x+r, y+s)$, and $v_4 = (x, y+s)$, $r > 0$, $s > 0$ be four points in the plane. Next let $\widetilde{P}_i$ be a fuzzy point at $v_i, i = 1,2,3,4$. The $\widetilde{P}_i$ define a fuzzy 4-gon. We assume it is strongly non-degenerate. We call this fuzzy 4-gon a *fuzzy rectangle.* We have seen above that we can define the area and perimeter of a fuzzy rectangle and they will be fuzzy numbers.

However, the fuzzy area of the fuzzy rectangle may not be the product of the side lengths and the fuzzy perimeter may not equal the sum of the side lengths. Let $\widetilde{A} = \widetilde{D}(\widetilde{P}_1, \widetilde{P}_2)$, $\widetilde{B} = \widetilde{D}(\widetilde{P}_2, \widetilde{P}_3)$, $\widetilde{C} = \widetilde{D}(\widetilde{P}_3, \widetilde{P}_4)$ and $\widetilde{D} = \widetilde{D}(\widetilde{P}_4, \widetilde{P}_1)$. Since the $t$-cuts of the fuzzy points may have different sizes, we may get $\widetilde{A} \neq \widetilde{C}$ and $\widetilde{B} \neq \widetilde{D}$. Thus assume the fuzzy points are all the "same", just centered at different points in the plane. The following example shows that still the area may not be the product of the side lengths and the perimeter may not equal to the sum of the side lengths.

**Example 5.22** *Let $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4$ be fuzzy points at $v_1 = (0,0), v_2 = (0,1)$, $v_3 = (1,1), v_4 = (1,0)$, respectively. Then we have a fuzzy square. Assume each $\widetilde{P}_i$ is a right circular cone with base a circle of radius 0.1 centered at $v_i$, and vertex at $v_i$, $1 \leqslant i \leqslant 4$.*

*Let $\widetilde{A} = \widetilde{D}(\widetilde{P}_1, \widetilde{P}_2), \widetilde{B} = \widetilde{D}(\widetilde{P}_2, \widetilde{P}_3), \widetilde{C} = \widetilde{D}(\widetilde{P}_3, \widetilde{P}_4), \ldots, \widetilde{D} = \widetilde{D}(\widetilde{P}_4, \widetilde{P}_1)$. We see that $\widetilde{A}^0 = \widetilde{B}^0 = \widetilde{C}^0 = \widetilde{D}^0 = [0.8, 1.2]$. Thus, $(\widetilde{AB})^0 = [0.64, 1.44]$ and $(\widetilde{A} + \widetilde{B} + \widetilde{C} + \widetilde{D})^0 = [3.2, 4.8]$. Let $\Omega_a(t)$ denote the set of all areas of a rectangle with vertices in $\widetilde{P}_i(\alpha), 1 \leqslant i \leqslant 4, 0 \leqslant t \leqslant 1$, then the fuzzy area $\widetilde{\Theta}$ is such that $\widetilde{\Theta}(\theta) = \vee\{t \mid \theta \in \Omega_a(t)\}$.*

*Similarly, we define the fuzzy perimeter $\tilde{\delta}$. We easily see that the left end point of $\tilde{\Theta}^0$ is greater than 0.64 and the left end point of $\tilde{\delta}^0$ is more than 3.2. Hence,* $\tilde{\Theta} \neq \tilde{A}\tilde{B}$, $\tilde{\delta} \neq \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D}$.

The authors of [8] plan to extend above results to $\mathbb{R}^n, n \geqslant 3$. They may define and study fuzzy points and lines in $\mathbb{R}^n, n \geqslant 3$. Then introduce fuzzy planes in $\mathbb{R}^3$ and fuzzy hyperplanes in $\mathbb{R}^n, n \geqslant 3$. They will look at fuzzy distance in $\mathbb{R}^n, n \geqslant 3$, and the intersection of fuzzy lines and fuzzy hyperplanes, etc. The authors of [8] also plan to apply their results to fuzzy data bases.

## 5.12 Fuzzy Plane Projective Geometry

In this section, we introduce some concepts of fuzzy projective geometry as initiated by Gupta and Ray, [23]. The approach used here is different than that used in the previous sections. For example in the previous section, fuzzy numbers were central to the development, while in this section fuzzy singletons are central. Also, an axiomatic approach is used in this section.

Let $S$ be a nonempty set. A collection $\Pi$ of fuzzy points (singletons) of $S$ is called a complete set of fuzzy points if given $x \in S$, there exists $t \in (0, 1]$ such that $x_t \in \Pi$. If $x_t \in \Pi$ and $t > 0$, then $x_t$ is called a *fuzzy vertical point*. It is possible for $x_t, x_s \in \Pi$ with $t \neq s$. If $x_t, y_s \in \Pi$ with $x \neq y$, then $x_t$ and $y_s$ are called *fuzzy distinct*. A nonzero fuzzy subset $\tilde{L}$ of $S$ is called a *fuzzy line* through $\Pi$ if $\forall x \in S$, $\tilde{L}(x) > 0$ implies that $x_t \in \Pi$, where $t = \tilde{L}(x)$. A fuzzy line $\tilde{L}$ is said to *contain* or *pass* through the fuzzy point $x_t$ of $\Pi$ if $\tilde{L}(x) = t$. In this case, we say that $x_t$ lies on $\tilde{L}$. This gives a symmetric incidence relation $I$ such that $\tilde{L}Ix_t$ or $x_tI\tilde{L}$ means that $\tilde{L}$ contains the fuzzy point $x_t$. A fuzzy plane projective geometry ($FPPG$) is an axiomatic theory with the triple $(\Pi, \Lambda, I)$ as its fundamental notions and $F1, F2$, and $F3$ as its axioms (listed below), where $\Pi$ is complete set of fuzzy points of a nonempty set $S$ and $\Lambda$ is a collection of fuzzy lines through $\Pi$.

F1a. Given two fuzzy distinct points in $\Pi$, there is a least one fuzzy line in $\Lambda$ with which both are incident.

F1b. Given two fuzzy distinct points in $\Pi$, there is at most one fuzzy line in $\Lambda$ with which both are incident.

F2. Given two distinct fuzzy lines in $\Lambda$, there is at least on fuzzy point in $\Pi$ with which both are incident.

F3. $\Pi$ contains at least four fuzzy distinct points such that no three of them are incident with one and the same fuzzy line in $\Lambda$.

We now give three examples of fuzzy plane projective geometries.

**Example 5.23** *(The Straight Line Model) Let* $S = \mathbb{R} \cup \{ri | r \neq 0, r \in \mathbb{R}\} \cup \{\infty\}$, *where $i$ is the imaginary number and $\infty$ is an object which* *is not a complex number. Let* $\Pi = \{x_t | x_t, x \in \mathbb{R}, 0 < t < 1\} \cup \{ri_s | s = (1/\pi)\cot^{-1}(r), r \in \mathbb{R}, r \neq 0\} \cup \{\infty_{1/2}\}$. *The fuzzy lines are defined as follows: For* $0 \neq m \in \mathbb{R}, c \in \mathbb{R}$, *define the fuzzy line* $[m, c]$ *through* $\Pi$ *by*

$$[m,c](x) = \begin{cases} (1/\pi)\cot^{-1}(mx + c) & \text{if } x \in \mathbb{R}, \\ (1/\pi)\cot^{-1}(m) & \text{if } x = mi, \\ 0 & \text{otherwise.} \end{cases}$$

*For* $d \in \mathbb{R}$, *define the fuzzy line* $[d]$ *through* $\Pi$ *by*

$$[d](x) = \begin{cases} (1/\pi)\cot^{-1}(d) & \text{if } x \in \mathbb{R}, \\ 1/2 & \text{if } x = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

*Define the unique fuzzy line* $\omega$ *through* $\Pi$ *by* $\omega(xi) = (1/\pi)\cot^{-1}(x)$, $0 \neq r \in \mathbb{R}, \omega(\infty) = 1/2, \omega(x) = 0$ *if* $x \in \mathbb{R}$. *It follows easily that this model satisfies F1 , F2, and F3.*

**Example 5.24** *(Model M) In this model, S and* $\Pi$ *are as defined in the previous model. The fuzzy lines are defined as follows: For* $0 \neq m \in \mathbb{R}, c \in \mathbb{R}$, *define the fuzzy line* $[m, c]$ *through* $\Pi$ *by*

$$[m,c](x) = \begin{cases} (1/\pi)\cot^{-1}(mx + c) & \text{if } m < 0, x \in \mathbb{R}, \text{ or } m > 0, x \leq 0, \\ (1/\pi)\cot^{-1}(2mx + c) & \text{if } m > 0, x > 0, \\ (1/\pi)\cot^{-1}(m) & \text{if } x = mi, \\ 0 & \text{otherwise.} \end{cases}$$

*For* $d \in \mathbb{R}$, *define the fuzzy line* $[d]$ *through* $\Pi$ *by*

$$[d](x) = \begin{cases} (1/\pi)\cot^{-1}(d) & \text{if } x \in \mathbb{R}, \\ 1/2 & \text{if } x = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

*Define the unique fuzzy line* $\omega$ *through* $\Pi$ *by* $\omega(xi) = (1/\pi)\cot^{-1}(x)$, $0 \neq x \in \mathbb{R}, \omega(\infty) = 1/2, \omega(x) = 0$ *if* $x \in \mathbb{R}$. *It follows easily that this model satisfies F1 , F2, and F3.*

**Example 5.25** *(The Spherical Geodesic Model) Let* $S = (0, \pi]$ *and* $\Pi = \{x_y | 0 < x \leq \pi, 0 < y < 1\}$. *The set* $\Lambda$ *consists of fuzzy lines* $[a, b]$, $a, b \in \mathbb{R}$, *where* $[a, b]$ *denotes the function*

$$y = (1/\pi)\cot^{-1}(a\cos(x) + b\sin(x)), 0 \leq x \leq \pi.$$

*If we do not insist that the values assumed by* $[a, b]$ *should lie in* $[0, 1]$, *we may take* $\Pi = \{x_y | 0 < x \leq \pi, 0 < y < \pi\}$ *and by* $[a, b]$ *we mean the function*

$$y = \cot^{-1}(a\cos(x) + b\sin(x)), 0 \leq x \leq \pi.$$

*These are functions from $S$ into $(0, \pi)$. This revised model is called the Expanded Spherical Model (ESG) Model. All the geometric properties, viz., cut points, formation of triangles, configurations for Desargues's theorem, etc. are invariant for both the SG and the ESG Models. It can be shown that these models are fuzzy plane projective geometries, [23, Theorem 3.5, p. 193].*

**Theorem 5.46** *Let $(\Pi, \Lambda, I)$ be a fuzzy plane geometry. Then the following assertions hold.*

**(i)** *Given two distinct fuzzy lines, there is at most one fuzzy point with which both are incident.*

**(ii)** *$\Lambda$ contains at least four distinct fuzzy lines such that no three of them pass through one and the same fuzzy point in $\Pi$.*

**(iii)** *Every fuzzy point is incident with at least three distinct fuzzy lines.*

**(iv)** *Every fuzzy line is incident with at least three fuzzy distinct points.* ■

The point of intersection of two distinct fuzzy lines $\tilde{L}$ and $\tilde{M}$ is the unique fuzzy point with which both are incident. It is denoted by $\tilde{L} \cap \tilde{M}$. If $\tilde{A}$ and $\tilde{B}$ are two fuzzy distinct points, then we let $\tilde{A}\tilde{B}$ denote the unique fuzzy line with which both $\tilde{A}$ and $\tilde{B}$ are incident. Two or more fuzzy points $\tilde{A}_i$, $i = 1, 2, \ldots, n$, are said to be *fuzzy collinear* if there is a fuzzy line with which each of them is incident. Clearly no two of them are fuzzy vertical. The fuzzy lines $\tilde{L}_i$, $i = 1, \ldots, n$, are said to be *fuzzy concurrent* if there is a fuzzy point with which each of them is incident.

**Definition 5.24** *A* fuzzy triangle *in an FPPG is a set of three fuzzy distinct points $\tilde{A}_1, \tilde{A}_2$, and $\tilde{A}_3$ and a set of three fuzzy lines $\tilde{L}_1, \tilde{L}_2$, and $\tilde{L}_3$ such that $\tilde{A}_i I \tilde{L}_j$ for $i \neq j$ and it is not the case that $\tilde{A}_i I \tilde{L}_i$ $(i, j = 1, 2, 3)$. The fuzzy points $\tilde{A}_i$ are called the* vertices *and the fuzzy lines $\tilde{L}_i$ are called the* sides *of the triangle. The triangle is denoted by $\tilde{A}_1\tilde{A}_2\tilde{A}_3$.*

**Fuzzy Desargues' proposition** $(fD_{11})$ : Let $\tilde{A}_1\tilde{A}_2\tilde{A}_3$ and $\tilde{B}_1\tilde{B}_2\tilde{B}_3$ be two fuzzy triangles. Let $\tilde{A}_i$ and $\tilde{B}_i$ be corresponding vertices. Let $\tilde{L}_i$ and $\tilde{M}_i$ be corresponding sides. If every two corresponding vertices are fuzzy distinct and every two corresponding sides are distinct and the fuzzy lines connecting corresponding vertices are incident with a fuzzy point $\tilde{O}$, then the corresponding sides intersect in three fuzzy points which are either fuzzy collinear or fuzzy vertical.

**Theorem 5.47** *The fuzzy Desargues' proposition is not valid in the Model $M$.* ■

**Theorem 5.48** *The fuzzy Desargues' proposition is valid in the Spherical Geodesic Model.* ■

**Theorem 5.49** *The fuzzy Desargues' proposition is independent of F1, F2, and F3.* ■

**Fuzzy small Desargues' proposition** ($fD_{10}$) : Let $\tilde{A}_1\tilde{A}_2\tilde{A}_3$ and $\tilde{B}_1\tilde{B}_2\tilde{B}_3$ be two fuzzy triangles such that the corresponding vertices are fuzzy distinct and the corresponding sides are distinct. Let $\tilde{C}_i = \tilde{L}_i \cap \tilde{M}_i$, where $\tilde{L}_i$ and $\tilde{M}_i$ are the sides of $\tilde{A}_1\tilde{A}_2\tilde{A}_3$ and $\tilde{B}_1\tilde{B}_2\tilde{B}_3$, respectively, $i = 1, 2, 3$. The lines connecting corresponding vertices are incident with a fuzzy point $\tilde{O}$. There is an extra incidence $\tilde{A}_1 I \tilde{M}_1$. Then $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ are either fuzzy collinear or fuzzy vertical.

**Theorem 5.50** *The fuzzy small Desargues' proposition is independent of F1, F2, and F3.* ■

## 5.13 A Modified Hausdorff Distance Between Fuzzy Subsets

The results of this section are from [13]. The concept of distance is of importance in science and engineering. It is often desirable that the distance be a metric. Let $S$ be a set. A function $d$ from $S \times S$ into $\mathbb{R}$ is called a *metric* if (1) $\forall P, Q \in S$, $d(P,Q) \geqslant 0$, (2) $\forall P, Q \in S$, $d(P,Q) = 0$ if and only if $P = Q$, (3) $\forall P, Q \in S$, $d(P,Q) = d(Q,P)$, and (4) $\forall P, Q, R \in S$, $d(P,R) \leqslant d(P,Q) + d(Q,R)$. If $d$ satisfies (1) and (2), it is called *positive definite.* If $d$ satisfies (2) it is called *symmetric.* If $d$ satisfies (4), it is said to satisfy the triangle inequality.

Let $\mathbb{R}^p$ denote $p$-dimensional Euclidean space. Minkowski defined a class of metric distances on $\mathbb{R}^p$ as follows:

$$d_n(P,Q) = \{\sum_{i=1}^{p} |x_i - y_i|^n\}^{1/n},$$

where $P, Q$ are two points in $\mathbb{R}^p$ and $x_i$ and $y_i$ are the $i$-th coordinate values of $P$ and $Q$, respectively. Of these distances, the one most used in pattern recognition and image applications are the city block, chessboard and Euclidean distances, i. e., those corresponding to $n = 1, \infty$ and 2, respectively.

The concept of distance has been extended to subsets of a metric space. One popular set distance is the Hausdorff distance. It is a metric. Let $W^r$ denote the operation of dilating the set $W$ by radius $r$ (i.e., $W^r$ is the set of all points within distance $r$ of $W$). For any two non-empty, compact (closed and bounded) subsets $U$ and $V$ of $\mathbb{R}^p$, let

$L(U,V) = \wedge\{r \in \mathbb{R}^+ \mid U^r \supseteq V\}$.

Then the *Hausdorff distance* $H(U,V)$ is defined to be

$\vee\{L(U,V), L(V,U)\}$.

We first review methods of defining the distance between two fuzzy subsets, including methods of generalizing the Hausdorff distance to fuzzy subsets, and discuss their shortcomings. We then propose a modified Hausdorff distance for fuzzy subsets and establish its metric properties. We provide some examples and compare our definition to other recently proposed definitions. We conclude the section with a discussion concerning applications and we give an example showing that our definition is relatively robust to noise.

## *Distances Between Fuzzy Subsets*

Two methods of defining the distance between two fuzzy subsets $\widetilde{A}$ and $\widetilde{B}$ of $\mathbb{R}^p$ have been proposed in [16]. One of these was later modified in [48].

In one of the methods in [16], a distance which is a fuzzy subset of $\mathbb{R}^+$ was defined, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. For $r \in \mathbb{R}^+$, the distance is defined as

$d_{\widetilde{A},\widetilde{B}}(r) = \vee\{\widetilde{A}(P) \wedge \widetilde{B}(Q)) \mid d(P,Q) = r; P, Q \in \mathbb{R}^p\}$.

For two non-fuzzy sets $U$ and $V$, this definition leads to $d_{U,V}(r) = 1$ if there exist $P \in U, Q \in V$ such that $d(P,Q) = r$; otherwise $d_{U,V}(r) = 0$. The metric properties of this distance function are discussed in [16]. The function $d_{U,V}$ is not a metric in the usual sense of the term.

The definition was modified and renamed in [48] as follows:

$\Delta_{\widetilde{A},\widetilde{B}}(r) \equiv \vee\{\widetilde{A}(P) \wedge \widetilde{B}(Q)) \mid d(P,Q) \leq r. P, Q \in \mathbb{R}^p\}$.

It follows that $\Delta_{\widetilde{A},\widetilde{B}}$ is a monotonically non-decreasing function of $r$ and if $\widetilde{A}' \subseteq \widetilde{A}$ and $\widetilde{B}' \subseteq \widetilde{B}$ for fuzzy subsets $\widetilde{A}', \widetilde{B}'$of $\mathbb{R}^p$, then $\Delta_{\widetilde{A}',\widetilde{B}'} \leqslant \Delta_{\widetilde{A},\widetilde{B}}$. The distance $\Delta_{\widetilde{A},\widetilde{B}}$ satisfies some other desirable properties.

The mean distance between two fuzzy subsets $\widetilde{A}$ and $\widetilde{B}$ ($\widetilde{A} \neq \chi_\emptyset \neq \widetilde{B}$ was also defined in [48] as follows:

$\bar{d}_{\widetilde{A},\widetilde{B}} = \frac{\sum\sum_{P,Q\in S} d(P,Q)[\widetilde{A}(P)\wedge\widetilde{B}(Q)]}{\sum\sum_{P,Q\in S}[\widetilde{A}(P)\wedge\widetilde{B}(Q)]}$.

In [16], the Hausdorff distance was generalized to fuzzy subsets as follows. For a fuzzy subset $\widetilde{W}$ for all $r \in \mathbb{R}^+$, let

$\widetilde{W^r}(P) \equiv \vee\{\widetilde{W}(Q) \mid d(P,Q) \leqslant r\}$.

Here $\widetilde{W^r}$, the expansion of $\widetilde{W}$ by $r$, is the result of applying to all points of $\widetilde{W}$ a local max operation within a region of radius $r$. Now let

$L(\widetilde{A},\widetilde{B}) \equiv \wedge\{r \in \mathbb{R}^+ \mid \widetilde{A}^r \supseteq \widetilde{B}\}$.

The fuzzy Hausdorff distance $H_f(\widetilde{A},\widetilde{B})$ is then defined as

$H_f(\widetilde{A},\widetilde{B}) \equiv L(\widetilde{A},\widetilde{B}) \vee L(\widetilde{B},\widetilde{A})]$.

It follows that if the supremum of $\widetilde{A}$ does not equal the supremun of $\widetilde{B}$, then either $L(\widetilde{A},\widetilde{B})$ or $L(\widetilde{B},\widetilde{A})$ does not exist and so $H_f(\widetilde{A},)$ cannot be defined. Thus, two fuzzy subsets of $S$ must have the same supremum for the distance $H_f$ between them to exist. This is a serious drawback of this definition.

Another definition, proposed in [42], compared the level sets of the fuzzy subsets. This definition is also limited to fuzzy subsets that have equal maximum membership values.

In order to handle fuzzy subsets with unequal maximum memberships, an expression for Hausdorff distance was proposed in [12]. Here the fuzzy subsets $\widetilde{A}$ and $\widetilde{B}$ are modified to $\widetilde{A}'$ and $\widetilde{B}'$ that have maximum membership value equal to 1. For sets with finite and countable support, the distance is defined as

$$H_f(\widetilde{A}', \widetilde{B}') = \frac{\sum_{i=1}^{m} t_i H(A_{t_i}, B_{t_i})}{\sum_{i=1}^{m} t_i} + \epsilon \frac{\sum_{P \in S} \mid \widetilde{A}(P) - \widetilde{B}(P) \mid}{\text{card}(S)}, \quad (5.13.1)$$

where $t_i$, $i = 1, \ldots, m$, are the distinct membership values of $\widetilde{A}'$ and $\widetilde{B}'$ and where $A_{t_i}$ is the crisp set defined as

$A_{t_i} = \{P \mid \widetilde{A}'(P) \geqslant t_i\}$

(and similarly for $B_{t_i}$) and finally where $\epsilon$ is a small positive constant, and card$(S)$ is the cardinality of the finite countable support $S$.

If $S$ is an uncountable metric space, then this definition is modified to

$$H_f(\widetilde{A}', \widetilde{B}') = \int_0^1 tH(A_t, B_t)dt + \epsilon \frac{\int_s \left| \widetilde{A}'(s) - \widetilde{B}'(s) \right| ds}{\int_s ds}. \quad (5.13.2)$$

Both (5.13.1) and (5.13.2) have two terms. The first term defines geometric distance in the Hausdorff sense, while the second term represents dissimilarity. Combining two terms representing two unrelated notions is not very appealing.

To overcome this, a single-term expression is proposed in this section that represents geometric distance only. This expression is a metric. It is a generalization of the 'crisp' Hausdorff distance and is applicable to arbitrary fuzzy subsets. Other modifications of our definition are given in [5] and in [19]. These definitions will be compared with the one presented later.

## *A Proposed Modified Hausdorff Distance*

Let $\widetilde{A}$ and $\widetilde{B}$ be any two non-empty fuzzy subsets of a metric space $S$. The maximum membership of $\widetilde{A}$ is

$u^* = \vee\{\widetilde{A}(x) \mid x \in S\}$.

Let

$A_{\max} = \{x \mid \widetilde{A}(x) = u^*\}$.

Let $A_a$ be a crisp subset of $S$ such that

$A_a \supset A_{\max}$

and such that for any two fuzzy subsets $\widetilde{A}$ and $\widetilde{B}$

$A_a = B_a$ if and only if $A_{\max} = B_{\max}$.

For $t \in [0,1]$, let
$A_t = \{x \mid \widetilde{A}(x) \in [t, u^*]\}$ if $t \leqslant u^*$,
$= A_a$ if $t > u^*$.
It follows that $A_t = A_{\max}$ if $t = u^*$, and that the second case does not arise if $u^* = 1$.

For any subfuzzy set $\widetilde{A}$ such that $u^* > 0$, $A_t \neq \emptyset\ \forall t \in [0,1]$. If $\widetilde{A} = \widetilde{B}$, $A_t = B_t$ for all $t \in [0,1]$.

**Proposition 5.51** *If $\widetilde{A} \neq \widetilde{B}$, then there exists $t > 0$ such that $A_t \neq B_t$.*

**Proof.** We consider two cases, depending on whether or not $\widetilde{A}$ and $\widetilde{B}$ have the same maximum membership value.

Suppose that $u^* = v^*$. Since $\widetilde{A} \neq \widetilde{B}$, there exist $x \in S$ such that $\widetilde{A}(x) \neq \widetilde{B}(x)$. If $\widetilde{A}(x) > \widetilde{B}(x)$ then, by the definition of $A_t$ and $B_t$, we have $A_{\widetilde{A}(x)} \neq B_{\widetilde{B}(x)}$. This follows since $A_{\widetilde{A}(x)}$ contains the point $x$, but $B_{\widetilde{B}(x)}$ does not. Similarly, if $\widetilde{B}(x) > \widetilde{A}(x)$, we must have $A_{\widetilde{A}(x)} \neq B_{\widetilde{B}(x)}$. Thus, the desired result holds.

Suppose that $u^* \neq v^*$. Without loss of generality, let $u^* > v^*$. If $A_{\max} = B_{\max}$, then the proposition is true for $t = u^*$ since in that case $A_t = A_{\max}$, but $B_t = B_a = A_a$ which is a superset of $A_{\max}$.

Suppose that $A_{\max} \neq B_{\max}$. Then the proposition is true for $t = u^*$ provided $A_{\max} \neq B_a$ because then $A_t = A_{\max}$ while $B_t = B_a \neq A_{\max}$.

Now suppose $B_a$ is constructed so that $B_a = A_{\max}$. Since $B_a$ is a superset of $B_{\max}$, there exists $p \in B_a \backslash B_{\max}$. Clearly, at $t = v^*$ the proposition is true since then $B_t = B_{\max}$, but $A_t \supseteq A_{\max} \supseteq B_{\max} \cup \{p\} \supset B_t$. ■

Suppose first that the fuzzy subsets take on only a discrete set of membership values $t_1, t_2, \ldots, t_m$. Let $H(A_{t_i}, B_{t_i})$ be the crisp Hausdorff distance between $A_{t_i}$ and $B_{t_i}$. Then we define

$$H_f(\widetilde{A}, \widetilde{B}) = \frac{\sum_{i=1}^{m} t_i H(A_{t_i}, B_{t_i})}{\sum_{i=1}^{m} t_i} \tag{5.13.3}$$

as the fuzzy Hausdorff distance between $\widetilde{A}$ and $\widetilde{B}$. That $H_f$ is a metric follows from Proposition 5.51 and comments preceding it.

Now $H_f$ is a membership-weighted average of the crisp Hausdorff distances between the level sets of the two fuzzy sets, where some of the level sets are modified, if necessary, to preserve the metric properties. In some sense, this average can be regarded as an expected value of the Hausdorff distance.

More generally, the membership values $t$ in the numerator and denominator of $H_f$ could be raised to some power. But in computing the Hausdorff distance between two fuzzy subsets, one can work directly with a power of $t$ rather than with $t$. Therefore an exponent is not used in the expression for $H_f$.

If $\widetilde{A}$ and $\widetilde{B}$ are continuous-valued, in analogy with (5.13.3), we have the integral expression

$$H_f(\widetilde{A},\widetilde{B}) = \int_0^1 tH(A_t, B_t)dt / \int_0^1 tdt = 2\int_0^1 tH(A_t, B_t)dt. \qquad (5.13.4)$$

It can be shown that both definitions reduce to the conventional Hausdorff distance in the crisp case. This follows since for a crisp set, $H(A_t, B_t)$ is constant for all $t > 0$ and this constant can be taken out of the integral sign.

The definition here depends on $A_a$, which can be chosen in many ways. If a fuzzy subset $\widetilde{A}$ has maximum membership $u^* = 1$, $A_a$ need not be defined since $A_t$ does not use it. If $u^* < 1$, define $A_a$ by adjoining a single point $x_{\widetilde{A}}$ to $A_{\max}$, i.e., $A_a = A_{\max} \cup \{x_{\widetilde{A}}\}$. This single point can have a negligible effect on the distance, because it can be given negligible area. Consider the more specific case of a discrete space of pixels defined by a square tessellation, as is encountered in digital image processing. Let $U_{\max}$ be the set of pixels at which $u$ attains its maximum. Then $x_{\widetilde{A}}$ can be a pixel adjacent to any border pixel of $A_{\max}$. Any of the adjacent pixels can be chosen as $x_{\widetilde{A}}$, provided that $x_{\widetilde{A}} = x_{\widetilde{B}}$ if $A_{\max} = B_{\max}$. Note that the same singleton should be used for all fuzzy subsets that have the same maximum-value level set. A convention of where to place the singleton can be defined provided that this convention is maintained, the results will be consistent.

In [13, Figure 1, p. 165], an example in the digital domain is presented, using Euclidean distance between (centers of) pixels. The reader is encouraged to see [13] for example. In [13, Figure 2, p. 167], another example is presented in which one of the two fuzzy subsets, $\widetilde{A}$, does not have any pixels with membership value 1. Thus its 1-level set is empty. The two sets at membership value 1 cannot be compared directly. To apply our definition to this example, a pixel to the level set of the highest membership value of $\widetilde{A}$ is appended and this set along with the 1-level set of $\widetilde{B}$ is used to compute the 1-level Hausdorff distance.

In [5], a definition was proposed that modifies those given by (5.13.1) and (5.13.2). It was assumed that there exists a non-empty set $S'$ disjoint from the set $S$ on which the fuzzy subsets are defined. The membership of any point of $S'$ in any fuzzy subset was defined to be zero and every fuzzy subset was modified by appending the zero membership region. Hence, every fuzzy subset was forced to have a non-empty set of zero membership values. However, this method gives more weight to distances between low-membership regions than to distances between high-membership regions. This is counterintuitive since the higher the membership of a region, the stronger is its degree of belongingness to the fuzzy subset. Also, the definition in [5] does not reduce to the conventional Hausdorff distance in the crisp case. To see this suppose, for example, that $S$ is a disk of diameter $D$.

Let $A$ and $B$ be two disks of diameter $d \leqslant D$ whose centers are located at distance $(D-d)/2$ from the center of $S$ in opposite directions. Consider a fuzzy set $\widetilde{A}$ whose membership is 1 on $A$ and 0 elsewhere, and a fuzzy subset $\widetilde{B}$ whose membership is 1 on $B$ and 0 elsewhere. It can be shown that, according to the definition in [5], the distance between $\widetilde{A}$ and $\widetilde{B}$ is $d/6$, which is not the crisp Hausdorff distance $(D-d)$ between $A$ and $B$. The definition used here does not suffer from this drawback.

Another modification of our previous definition was given in [19]. There it was assumed that the underlying metric space $S$ is a compact subset of Euclidean space. Thus, $D = \vee h(U, V)$ exists for all non-empty compact proper subsets of $S$. $h$ is then extended to all compact subsets of $S$ by defining $h(\emptyset, \emptyset) = 0$ and $h(\emptyset, W) = h(W, \emptyset) = D$ for all non-empty compact subsets $W$ of $S$. The extended $h$ can then be generalized to fuzzy subsets of $S$ using expressions analogous to (5.13.1) and (5.13.2), provided their level sets are all compact. However, this $h$ is biased on the diameter $D$ of $S$. This drawback is demonstrated by an example given in [19].

We close this section with the summary given in [13]. The proposed fuzzy Hausdorff distance can be used in a wide variety of practical applications. For example, consider the problem of matching two gray-tone images. Working with gray-tone images may be advantageous over their two-tone thresholded versions since it is not necessary to commit to any specific thresholding method. Instead, the gray-tone images can be considered as fuzzy subsets (for example, by rescaling the gray-tone values to the range [0,1]) and the proposed measure can be used to define their degree of match. The measure presented here can also be used to match sets of feature points (for example, edge or corner points) extracted from two images, where these points are characterized by their locations and by their strengths. These strengths, scaled to the range [0,1], can be regarded as fuzzy membership values. They do not have to be thresholded.

Since the definition of crisp Hausdorff distance makes use of maximum and minimum functions, the presence or absence of a single stray datum in the set $A$ can drastically change the value of its distance from another set $B$. This is the well-known robustness problem of the Hausdorff distance. It limits its practical applications. To deal with this problem, several modifications to Hausdorff distance have been proposed. In [25], a ranked distance was used. In [17], several modifications were considered, one of which performs well under Gaussian noise. In [38], a concept called censored Hausdorff distance was used. These modifications can handle noise to various extents, but they are no longer metrics. Since the fuzzy Hausdorff distance proposed here is the membership-weighted average of the crisp Hausdorff distances of level sets, any of these modifications in (5.12.3) and (5.12.4) can also be used. However, the metric property of the fuzzy Hausdorff distance is then lost.

A different approach to handling noise is proposed here. It allows the metric property to be preserved. Any binary digital image can be regarded as a fuzzy subset, where the white pixels have zero membership values, and the membership values of the black pixels are determined by examining their neighborhoods. (The 3-by-3 neighborhood was used here, but larger neighborhoods could also be used.) If a black pixel $p$ has $k$ black neighbors, its membership value is taken to be $k/8$. Thus $p$ has membership 1 if and only if all its 8 neighbors are black. Black pixels having no black neighbors have zero membership value, as do all white pixels. Consequently, black pixels that are due to noise will have small or zero membership values.

Any two binary images can be converted into fuzzy subsets in this way and their fuzzy Hausdorff distance can be computed using (5.13.3). It was verified experimentally in [13] that the noise has less effect on this distance compared to its effect on the classical crisp Hausdorff distance.

The experiment used binary images of numerals. The original and noisy images are shown in [13, Figure 3, p. 169]. Noise was added to the images by converting 5% of the white pixels randomly into black ones. For each pair of images, the minimum Hausdorff distance was found by translating and rotating one image with respect to the other. It can be proved as follows that the minimum of any metric under translation and rotation is also a metric: Let $D$ be a metric and let $A, B, C$ be sets. The minimum distance $D'$ between any two sets can be achieved by keeping one of them fixed and translating and rotating only the other. Let $B'$ be the translation and rotation of $B$ such that $D(A, B')$ is a minimum, and let $C'$ be defined analogously. Let $C^*$ be the translation and rotation of $C$ such that $D(B', C^*)$ is a minimum. Then $D'(A,C) = D(A,C') \leqslant D(A,C^*) \leqslant D(A,B') + D(B',C^*) = D'(A,B) + D'(B,C)$ since $D$ is a metric. This shows that the minimum of $D$ under translation and rotation satisfies the triangle inequality. Clearly, it is positive definite and symmetric.

The classical Hausdorff distances between all the pairs of numerals were computed for both the noise-free and noisy images. The results are presented in [13, Tables 1 and 2, pp. 169, 170]. The images were also converted into fuzzy subsets as described above and the fuzzy Hausdorff distances between them were then computed. The results are presented in [13, Tables 3 and 4, p.170]. The diagonal entries in [13, Tables 2 and 4] are not zeros since the noisy images are matched with their noiseless ideal prototypes.

It was demonstrated in [13] that the classical Hausdorff distance was drastically changed by noise. The fuzzy Hausdorff distance was also changed, but the change was much smaller. In fact, [13, Tables 3 and 4] show that the fuzzy distance can be used for pattern classification in the presence of noise since the intra-class distances are much smaller than the interclass distances.

## 5.14 References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# 6 FUZZY ABSTRACT ALGEBRA

In 1971, Azriel Rosenfeld wrote his seminal paper on fuzzy subgroups, [52]. This paper led to a new area in abstract algebra as well as a new area in fuzzy mathematics. Hundreds of papers examining various fuzzy substructures of algebraic structures have since appeared in the literature. In this chapter, we examine those fuzzy substructures which have applications or strong potential for applications outside of mathematics, namely computer science or engineering. We emphasize that the nature of fuzzy abstract algebra differs from that of the algebra of fuzzy numbers such as triangular fuzzy numbers and others.

## 6.1 Crisp Algebraic Structures

### *Semigroups*

In this section, we review some basic results of those crisp algebraic structures needed in this chapter. The first algebraic structure we examine is a semigroup. Applications of semigroups can be found in automata theory, codes, algebraic linguistics, and combinatorics.

**Definition 6.1** *The* $(n+1)$*-tuple* $(S, *_1, ..., *_n)$ *is called an* algebraic system *if* $S$ *is a nonempty set and* $*_i$ *is a binary operation on* $S$, *that is,* $*_i : S \times S \to S$, $i = 1, 2, ..., n$. *If* $(S, *)$ *is an algebraic system and* $\forall$ $a, b \in S$, $a * b = b * a$, *we say that* $*$ *or* $(S, *)$ *is* commutative.

**Definition 6.2** *An algebraic system* $(S, *)$ *is called a* semigroup *if* $*$ *is associative, that is,* $\forall a, b, c \in S$, $a * (b * c) = (a * b) * c$. *A semigroup* $(S, *)$ *is called a* monoid *if it has an* identity, *that is,* $\exists e \in S$ *such that* $\forall a \in S, e * a = a = a * e$.

If $(S, *)$ is a semigroup, we sometimes merely refer to $S$ as a semigroup when the binary operation $*$ is understood.

Let $(S, *)$ be a monoid and let $e, f$ be identities of $S$. By treating $f$ and then $e$ as an identity in the equation, $e = e * f = f$, we see that the identity of a monoid is unique.

Let $(S, *)$ be a semigroup and let $a \in S$. Define $a^1 = a$. Suppose $a^n$ is defined for $n \in \mathbb{N}$. Define $a^{n+1} = a^n * a$. If $(S, *)$ is a monoid with identity $e$, we define $a^0 = e$. Since $*$ is associative, we have that $a^i * a^j = a^{i+j}$ $\forall i, j \in \mathbb{N}$.

**Example 6.1** *The mathematical systems* $(\mathbb{N} \cup \{0\}, +)$, $(\mathbb{N}, \cdot)$, *and* $(\mathbb{Z}, \cdot)$ *are monoids, where* $+$ *and* $\cdot$ *are the usual operations of addition and multiplication, respectively. The mathematical system* $(\mathbb{N}, +)$ *is a semigroup which is not a monoid.*

**Definition 6.3** *Let* $(S, *)$ *be a semigroup (monoid) and let* $X$ *be a subset of* $S$. *Then* $(X, *)$ *is a* subsemigroup (submonoid) *of* $(S, *)$ *if* $(X, *)$ *is a semigroup (monoid with the same identity as* $S$*). (The operation* $*$ *on* $X$ *is really* $*|_{X \times X}$, *the restriction of* $*$ *to* $X \times X$.)

We see that if $(S, *)$ is a semigroup and $X$ is a nonempty subset of $S$, then $X$ is a subsemigroup of $S$ if and only if $\forall a, b \in X$, $a * b \in X$. Similarly we have that if $(S, *)$ is a monoid and $X$ is a nonempty subset of $S$, then $X$ is a submonoid of $S$ if and only if $\forall a, b \in X, a * b \in X$ and $e \in X$, where $e$ is the identity of $S$. In Example 6.1, $(\mathbb{N}, \cdot)$ is a submonoid of $(\mathbb{Z}, \cdot)$ and $(\mathbb{N}, +)$ is a subsemigroup of $(\mathbb{N} \cup \{0\}, +)$.

**Theorem 6.1** *Let* $S$ *be a semigroup (monoid) and let* $\{S_i \mid i \in I\}$ *be a collection of subsemigroups (submonoids) of* $S$, *where* $I$ *is a nonempty index set. Then* $\bigcap_{i \in I} S_i$ *is a subsemigroup (submonoid) of* $S$. ■

**Definition 6.4** *Let* $S$ *be a semigroup (monoid) and let* $X$ *be a subset of* $S$. *Let* $\langle X \rangle$ *denote the intersection of all subsemigroups (submonoids) which contain* $X$. *Then* $\langle X \rangle$ *is called the subsemigroup (submonoid)* generated *by* $X$.

Let $S$ be any semigroup. Let $e$ be an element not in $S$ and let $S^* = S \cup \{e\}$. Extend the binary operation of $S$ to $S^*$ by defining $e * e = e$ and $e * a = a * e = a$ $\forall a \in S$. Then $S^*$ is a monoid with $e$ as the identity and with $S$ as a subsemigroup. If $S$ is a monoid, it is not a submonoid of $S^*$ since $e \notin S$. If $S$ is commutative, then so is $S^*$.

**Definition 6.5** *Let* $(S, *)$ *and* $(T, \cdot)$ *be semigroups and let* $f$ *be a function of* $S$ *into* $T$. *Then* $f$ *is called a* homomorphism *of* $S$ *into* $T$ *if* $\forall a, b \in S$, $f(a * b) = f(a) \cdot f(b)$. *If* $f$ *is a one-to-one homomorphism of* $S$ *onto* $T$, *then* $f$ *is referred to as an* isomorphism *of* $S$ *onto* $T$. $S$ *and* $T$ *are said to be* isomorphic, *and we write* $S \simeq T$.

Let $X$ be a set. By a *free* semigroup on the set $X$, we mean a semigroup $F$ together with a function $f : X \to F$ such that for every function $g$ of $X$ into a semigroup $S$, there exists a unique homomorphism $h : F \to S$ such that $h \circ f = g$. We sometimes write $(F, f)$ for a free semigroup with function $f$. The proofs of the following results on free semigroups can be found in [14].

**Theorem 6.2** *If a semigroup* $F$ *together with a function* $f : X \to F$ *is a free semigroup on the set* $X$, *then* $f$ *is one-to-one and its image* $f(X)$ *generates* $F$. ■

**Theorem 6.3** *If* $(F, f)$ *and* $(F', f')$ *are free semigroups on the same set* $X$, *then there exists a unique isomorphism* $j$ *of* $F$ *onto* $F'$ *such that* $j \circ f = f'$. ■

Let $X$ be a set, let $X^*$ denote the set of all strings of finite length of elements of $X$ including the empty string, $\lambda$, and let $X^+ = X^* \setminus \{\lambda\}$. Define a binary operation on $X^+$ by $\forall x_1 ... x_m, y_1 ... y_n \in X^+$,

$(x_1 ... x_m)(y_1 ... y_n) = x_1 ... x_m y_1 ... y_n$.

This operation is associative and is sometimes called *concatenation*. $X^+$ with this binary operation is a free semigroup. The proof of this fact, normally appears in the proof of the next theorem. As a consequence, we have that $X^*$ with this binary operation is a free monoid with identity $\lambda$.

**Theorem 6.4** *For any set* $X$, *there exists a free semigroup on* $X$. ■

**Theorem 6.5** *If* $X$ *is a set of generators of a semigroup* $S$, *that is,* $\langle X \rangle = S$, *then* $S = \{x_1 * ... * x_n \mid x_i \in X, n \in \mathbb{N}\}$, *i. e., every element of* $S$ *is a finite product of elements of* $X$. ■

Let $X$ be a set. Then the strings of elements of $X$, say $x_1 x_2 ... x_m$ and $y_1 y_2 ... y_n$, are equal if and only if $m = n$ and $x_i = y_i, i = 1, 2, ..., m$. Hence every element of $X^+$ admits a unique factorization of elements of $X$.

Every set $X$ determines an essentially unique free semigroup $(F, f)$. Since the function $f : X \to F$ is one-to-one, we may identify $X$ with its image $f(X)$ in $F$. This having been done, $X$ becomes a subset of $F$ which generates $F$. Every function $g : X \to S$ from $X$ into an arbitrary semigroup $S$ extends to a unique homomorphism $h : F \to S$. We refer to the semigroup $F$ as the *free semigroup generated* by $X$.

Consider the monoid $F^* = F \cup \{e\}$, where $F$ is a free semigroup generated by a set $X$. Then every function $g : X \to S$, where $S$ is a monoid, extends to a unique homomorphism $h : F^* \to X$. This property characterizes the monoid $F^*$ by means of the discussion analogous to the one given above for semigroups. This monoid $F^*$ is called the *free monoid generated* by $X$.

## *Groups*

We now review the notion of a group. The theory of groups is one of the oldest branches of abstract algebra. The concept of a group is so universal that it has appeared in many different branches of mathematics and science. Group theory can be used to describe symmetries and permutations in nature and mathematics. For example, groups can be used to classify all the forms chemical crystals can take. Groups also can be used to count the number of nonequivalent objects under permutations or symmetries; for example, the number of different switching functions of $n$ variables when permutations of the inputs are allowed. Besides crystallography and combinatorics, groups have applications in quantum mechanics, particle physics, and coding theory, to name just a few.

**Definition 6.6** *A monoid $(G, *)$ is called a* group *if $\forall a \in G$, $\exists b \in G$ such that $a * b = e = b * a$, where $e$ is the identity of $G$. The element $b$ is called an inverse of $a$.*

It is also easily shown that in a group $G$, the inverse of an element $a$ is unique. For example, if $a'$ and $a''$ are inverses of an element $a$ in a group $G$ with identity $e$, then $a' = a' * e = a' * (a * a'') = (a' * a) * a'' = e * a'' = a''$. We denote the inverse of $a \in G$ by $a^{-1}$. If $n \in \mathbb{N}$, we define $a^{-n} = (a^{-1})^n$ and $a^0 = e$. If we use additive notation for the binary operation of the group, then we write $na$ for $a^n$ $\forall n \in \mathbb{Z}$.

**Example 6.2** *The mathematical systems*
$(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +)$, *and* $(\mathbb{C}, +)$
*are commutative groups as are*
$(\mathbb{Q} \setminus \{0\}, \cdot), (\mathbb{R} \setminus \{0\}, \cdot)$, *and* $(\mathbb{C} \setminus \{0\}, \cdot)$,
*where $+$ is the usual operation of addition and $\cdot$ is the usual operation of multiplication.*

Another important example of a group is $(S, \circ)$, where $S$ is the set of all one-to-one functions of a set $X$ onto itself and $\circ$ denotes the composition of functions. If $X$ is finite, then $S$ is known as the *symmetric group* on $X$.

**Definition 6.7** *Let $(G, *)$ be a group. Let $H$ be a subset of $G$. Then $H$ is a* subgroup *of $G$ if $(H, *)$ is a group. (The operation $*$ on $H$ is really $*|_{H \times H}$, the restriction of $*$ to $H \times H$.)*

**Example 6.3** *Each group in Example 6.2 is a subgroup of the group appearing to its right.*

We supply the proof of the next result because of its fundamental importance.

**Theorem 6.6** *Let $(G, *)$ be a group and let $H$ be a nonempty subset of $G$. Then $H$ is a subgroup of $G$ if and only if $\forall a, b \in H$, $a * b^{-1} \in H$.*

**Proof.** Suppose that $H$ is a subgroup of $G$. Then $\forall a, b \in H$, $a, b^{-1} \in H$ and so $a * b^{-1} \in H$. Conversely, suppose that $\forall a, b \in H$, $a * b^{-1} \in H$. Since $H \neq \emptyset$, $\exists a \in H$. Thus $e = a * a^{-1} \in H$. Hence $\forall b \in H$, $b^{-1} = e * b^{-1} \in H$. Thus $\forall a, b \in H$, $a, b^{-1} \in H$ and so $a * b = a * (b^{-1})^{-1} \in H$. The associative law holds for $H$ since it holds in the larger set $G$. Therefore $H$ is a subgroup of $G$. ■

**Theorem 6.7** *Let $(G, *)$ be a group and let $\{H_i \mid i \in I\}$ be a collection of subgroups of $G$, where $I$ is a nonempty index set. Then $\bigcap_{i \in I} H_i$ is a subgroup of $G$.*

**Proof.** Since $e \in H_i \forall i \in I$, $e \in \bigcap_{i \in I} H_i$ and so $\bigcap_{i \in I} H_i \neq \emptyset$. Let $a, b \in \bigcap_{i \in I} H_i$. Then $a, b \in H_i \forall i \in I$. Hence $a * b^{-1} \in H_i \forall i \in I$ and so $a * b^{-1} \in \bigcap_{i \in I} H_i$ . Thus $\bigcap_{i \in I} H_i$ is a subgroup of $G$ by Theorem 6.6. ■

**Definition 6.8** *Let $(G, *)$ be a group and let $X$ be a subset of $G$. Let $\langle X \rangle$ denote the intersection of a all subgroups of $G$ which contain $X$. Then $\langle X \rangle$ is called the subgroup of $G$* generated *by $X$.*

In Definition 6.8, it follows that $\langle X \rangle$ is the smallest subgroup of $G$ which contains $X$.

Let $x_1, ..., x_n \in X$. Then since $\langle X \rangle \supseteq X$ and since $\langle X \rangle$ is a subgroup of $G$ by Theorem 6.7, products of the form $x_1^{e_1} * ... * x_n^{e_n}$ must be in $\langle X \rangle$, where $x_i \in X, e_i = \pm 1$, $i = 1, 2, ..., n$; $n \in \mathbb{N}$. The next result says that $\langle X \rangle$ is exactly the set of such elements.

**Theorem 6.8** *Let $(G, *)$ be a group and let $X$ be a nonempty subset of $G$. Then $\langle X \rangle = \{x_1^{e_1} * ... * x_n^{e_n} \mid x_i \in X, e_i = \pm 1, i = 1, 2, ..., n; n \in \mathbb{N}\}$.*

**Proof.** Let $A = \{x_1^{e_1} * ... * x_n^{e_n} \mid x_i \in X, e_i = \pm 1, i = 1, 2, ..., n; n \in \mathbb{N}\}$. We have just argued above that $\langle X \rangle \supseteq A$. One shows that $\langle X \rangle \subseteq A$ by showing that $A$ is a subgroup of $G$ containing $X$ and then applying the fact that $\langle X \rangle$ is the smallest subgroup of $G$ containing $X$. ■

**Corollary 6.9** *Let $(G, *)$ be a group and let $a \in G$. Then $\langle \{a\} \rangle = \{a^m \mid m \in \mathbb{Z}\}$.*

**Proof.** $\langle \{a\} \rangle = \{a^{e_1} ... a^{e_n} \mid e_i = \pm 1, i = 1, 2, ..., n; n \in \mathbb{N}\} = \{a^{e_1 + ... + e_n} \mid e_i = \pm 1, i = 1, 2, ..., n; n \in \mathbb{N}\} = \{a^m \mid m \in \mathbb{Z}\}$. ■

If $(G, *)$ is a group such that there exists $a \in G$ such that $G = \langle \{a\} \rangle$, then $G$ is called a *cyclic group*. We often simply write $\langle a \rangle$ for $\langle \{a\} \rangle$.

The *order* of a group is defined to be the number of elements in a group. If $G$ is a finite group (a group with a finite number of elements) of $n$ elements, then $G$ has order $n$. If $G$ is an infinite group, then $G$ is said to have *infinite order*. Let $G$ be a group and let $a \in G$. If there exists a positive integer $n$ such that $a^n = e$, then the smallest such positive integer is called the *order* of $a$. If no such positive integer exists, we say that $a$ has *infinite order*.

It can be shown that cyclic groups fall into one of two types. One type consists of infinite cyclic groups. Here the $a^m$ in $\{a^m \mid m \in \mathbb{Z}\}$ are distinct. The second type consists of finite cyclic groups. Here the $a^m$ in $\{a^m \mid m \in \mathbb{Z}\}$ are not distinct. If $< a >$ is a finite cyclic group of order $n$, then $a$ has order $n$, $\langle a \rangle = \{e, a, a^2 ..., a^{n-1}\}$, and $a^i * a^j = a^k$, where $k = i + j$ modulo $n$.

In the following example, we give a group which is not cyclic. The group is known as the Klein four-group.

**Example 6.4** *Let $G = \{e, a, b, c\}$. Define the binary operation $*$ on $G$ by the following operation table.*

| $*$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

*We have*
$\langle e \rangle = \{e\} \neq G, < a > = \{e, a\} \neq G, < b > = \{e, b\} \neq G$, *and* $< c > = \{e, c\} \neq G$.
*Thus $G$ is not cyclic.*

## *Rings and Ideals*

The importance of commutative algebra and algebraic geometry to the computer scientist and engineer has been enhanced by the development of computers fast enough to run new algorithms for manipulating polynomial equations. We now review some concepts from these two areas. We apply these concepts to polynomial equations.

**Definition 6.9** *The mathematical system $(R, +, \cdot)$ is called a* ring *if $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and the distributive properties hold, that is, $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.*

Let $(R, +, \cdot)$ be a ring. If $(R, \cdot)$ is a commutative semigroup, then we call $R$ a *commutative ring*. If $(R, \cdot)$ is a monoid, we let 1 denote its identity and we say that the ring $R$ is with identity. We let 0 denote the identity of $(R, +)$. Let $S$ be a subset of $R$. Then $S$ is called a *subring* of $R$ if $(S, +, \cdot)$ is a ring.

**Definition 6.10** *The mathematical system $(F, +, \cdot)$ is called a* field *if it a commutative ring with identity such that $\forall a \in F$, $a \neq 0$, there exists $a^{-1} \in F$ such that $aa^{-1} = 1$.*

We see that the mathematical system $(F, +, \cdot)$ is a field if and only if $(F, +)$ and $(F \setminus \{0\}, \cdot)$ are commutative groups such that the distributive laws hold.

**Example 6.5** *The mathematical systems $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot)$, and $(\mathbb{C}, +, \cdot)$ are commutative rings with identity. In fact, the latter three are fields. The mathematical system $(\{qn \mid q \in \mathbb{Z}\}, +, \cdot)$ is a commutative ring without identity, where $n \in \mathbb{N} \setminus \{1\}$.*

In what follows, we are only interested in commutative rings with identity. Some of the concepts and results we review from ideal theory hold for more general types of rings.

We now give a nonrigorous definition of a polynomial over a ring.

**Definition 6.11** *Let $R$ be a commutative ring with identity and let $x$ be an indeterminate over $R$. A polynomial in $x$ with coefficients in $R$ or a polynomial in $x$ over $R$ is an expression of the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 x^0$, where $a_i \in R$, $i = 0, 1, \ldots, n$; $n \in \mathbb{N} \cup \{0\}$ and $x^0 = 1$. (By an indeterminate $x$ over $R$, we mean that $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 x^0 = 0$ if and only if $a_i = 0$, $i = 0, 1, \ldots, n$.)*

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial over a commutative ring $R$. If we write $\sum_{i=0}^{n} a_i x^i$ for $p(x)$ with $n > m$, then we mean $a_i = 0$ for $i = m+1, \ldots, n$. By Definition 6.11, we have that two polynomials $\sum_{i=0}^{m} a_i x^i$ and $\sum_{j=0}^{n} b_j x^j$ are equal if and only if $a_i = b_i$ for $i = 0, 1, \ldots, m \vee n$.

**Definition 6.12** *Let $R$ be a commutative ring with identity and let $x$ be an indeterminate over $R$. Let $R[x]$ denote the set of all polynomials in $x$ over $R$. Define $+$ and $\cdot$ on $R[x]$ as follows: $\forall$ $\sum_{i=0}^{m} a_i x^i$ and $\sum_{j=0}^{n} b_j x^j$ $\in R[x]$, $\sum_{i=0}^{m} a_i x^i + \sum_{j=0}^{n} b_j x^j = \sum_{k=0}^{q} (a_k + b_k) x^k$, where $q = m \vee n$ and $(\sum_{i=0}^{m} a_i x^i) \cdot (\sum_{j=0}^{n} b_j x^j) = \sum_{k=0}^{m+n} c_k x^k$, where $c_k = \sum_{h=0}^{k} a_h b_{k-h}$, $k = 0, 1, \ldots, m+n$.*

Let $R[x]$ be the set of all polynomials over the commutative ring $R$ with identity. If we associate $a$ with $ax^0$, then we can consider $R$ to be a subset of $R[x]$.

**Theorem 6.10** *Let $R$ be a commutative ring with identity. The mathematical system $(R[x], +, \cdot)$ is a commutative ring with identity. In fact, the identities of $R$ and $R[x]$ coincide. Furthermore, $R$ is a subring of $R[x]$.* ■

We assume that the reader is familiar with the more basic properties of polynomials.

We now extend the definition of a polynomial ring from one indeterminate to several indeterminates. Let $R$ be a commutative ring with identity. We define recursively

$R[x_1, x_2, ..., x_n] = R[x_1, x_2, ..., x_{n-1}][x_n]$, where $x_1$ is an indeterminate over $R$ and $x_n$ is an indeterminate over $R[x_1, x_2, ..., x_{n-1}]$. $R[x_1, x_2, ..., x_n]$ is called a polynomial ring in $n$ indeterminates.

We now describe a polynomial ring in three indeterminates $x, y, z$ over a commutative ring $R$ with identity.

$R[x, y, z] = \{\sum_{k=0}^{q} \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ijk}x^i y^j z^k \mid a_{ijk} \in R,\ i = 0, 1, ..., m;\ j = 0, 1, ..., n;\ k = 0, 1, ..., q;\ m, n, q \in \mathbb{N} \cup \{0\}\}$.

**Definition 6.13** *Let $(R, +, \cdot)$ be a commutative ring with identity and let $I$ be a subset of $R$. Then $I$ is said to be an* ideal *of $R$ if $(I, +)$ is a subgroup of $(R, +)$ and $\forall r \in R$ and $\forall a \in I$, $ra \in I$.*

**Theorem 6.11** *Let $(R, +, \cdot)$ be a commutative ring with identity and let $I$ be a nonempty subset of $R$. Then $I$ is an ideal of $R$ if and only if $\forall a, b \in I$, $a - b \in I$ and $\forall r \in R$ and $\forall a \in I$, $ra \in I$.* ■

**Example 6.6** *Consider the ring $(\mathbb{Z}, +, \cdot)$. Let $n \in \mathbb{N}$. Then $I = \{qn \mid q \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$ since $\forall qn, q'n \in I$ and $\forall r \in \mathbb{Z}$, $qn - q'n = (q - q')n \in I$ and $r(qn) = (rq)n \in I$. However even though $\mathbb{Z}$ is a subring of $\mathbb{Q}$, $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$ since $(1/2) \cdot 1 \notin \mathbb{Z}$ and $1 \in \mathbb{Z}$.*

**Theorem 6.12** *Let $R$ be a commutative ring with identity and let $\{I_j \mid j \in J\}$ be a nonempty collection of ideals of $R$. Then $\bigcap_{j \in J} I_j$ is an ideal of $R$.* ■

**Definition 6.14** *Let $R$ be a commutative ring with identity and let $X$ be a subset of $R$. Define $\langle X \rangle$ to be the intersection of all ideals of $R$ which contain $X$. Then $\langle X \rangle$ is called the ideal of $R$* generated *by $X$.*

In Definition 6.14, $\langle X \rangle$ is the smallest ideal of $R$ which contains $X$. Suppose that $x_i \in X$ and $r_i \in R$ for $i = 1, 2, ..., n$. Then from the definition of an ideal, $r_i x_i \in \langle X \rangle$ for $i = 1, 2, ..., n$. Since an ideal is closed under addition, $\sum_{i=1}^{n} r_i x_i \in \langle X \rangle$. The next theorem states that $\langle X \rangle$ is precisely the set of all such sums. This follows by showing that $\{\sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X,\ i = 1, 2, ..., n; n \in \mathbb{N}\}$ is an ideal of $R$ containing $X$ and then using the fact that $\langle X \rangle$ is the smallest ideal of $R$ containing $X$.

**Theorem 6.13** *Let $R$ be a commutative ring with identity and let $X$ be a nonempty subset of $R$. Then $\langle X \rangle = \{\sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X,\ i = 1, 2, ..., n; n \in \mathbb{N}\}$.* ■

**Corollary 6.14** *Let $R$ be a commutative ring with identity and let $x \in R$. Then $< \{x\} > = \{rx \mid r \in R\}$.* ■

If $R$ is a commutative ring with identity and $x \in R$, we often write $\langle x \rangle$ for $< \{x\} >$.

Corollary 6.14 gives us a method to construct examples of ideals. This can be seen from the following example.

**Example 6.7** *Consider any commutative ring $R$ with identity. Let $x$ be any (fixed) element of $R$. Then $\{rx \mid r \in R\}$ is an ideal of $R$ by Corollary 6.14.*

**Example 6.8** *Let $R[x]$ be a polynomial ring in the indeterminate $x$, where $R$ is a commutative ring with identity. Then $\langle x \rangle = \{r(x)x | r(x) \in R[x]\}$ is an ideal of $R[x]$. $\langle x \rangle$ is the set of all polynomials with zero constant term.*

**Example 6.9** *Let $R[x,y,z]$ be a polynomial ring in three indeterminates over $R$. Then $< y-x^2, x^2z > = \{r(x,y,z)(y-x^2)+s(x,y,z)x^2z \mid r(x,y,z), s(x,y,z) \in R[x,y,z]\}$ by Theorem 6.13.*

If $I$ and $J$ are ideals of $R$, we define the product of $I$ and $J$, written $I \cdot J$, to be the set $I \cdot J = \{\sum_{k=1}^{n} i_k j_k \mid i_k \in I, j_k \in J, k = 1, \ldots, n; n \in \mathbb{N}\}$. It follows that $I \cdot J$ is an ideal of $R$.

The concept introduced in the next definition is quite useful in the study of nonlinear systems of equations.

**Definition 6.15** *Let $R$ be a commutative ring with identity. Then $R$ is said to satisfy the* ascending chain condition *for ideals or to be* Noetherian *if for every ascending chain of ideals*

$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots,$

there exists a positive integer $m$ such that $\forall n \geq m$, $I_n = I_m$.

**Example 6.10** *The ring $\mathbb{Z}$ of integers is Noetherian. We can see this from the following argument. It can be shown that for every ideal $I$ of $\mathbb{Z}$, there exists $n \in \mathbb{N} \cup \{0\}$ such that $I = < n >$. Let $< k_1 > \subseteq \ldots \subseteq < k_n > \subseteq \ldots$ be an ascending chain of ideals of $\mathbb{Z}$ for $k_n \in \mathbb{N}$, $n = 1, 2, \ldots$. Then $k_n \in < k_{n+1} >$ and so $\exists r_n \in \mathbb{Z}$ such that $k_n = r_n k_{n+1}$. Thus $k_n \geq k_{n+1} \geq 0$ for $n = 1, 2, \ldots$. Hence $\exists m \in \mathbb{N}$ such that $\forall n \geq m$, $k_n = k_m$.*

**Example 6.11** *The polynomial ring $F[x]$ over a field $F$ is Noetherian by an argument similar to the one used in the previous example.*

Any field $F$ is Noetherian since $F$ has only the ideals $\{0\}$ and $F$.

**Theorem 6.15** *If $R$ is a Noetherian ring, then a polynomial ring $R[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$ over $R$ is Noetherian.* ■

**Theorem 6.16** *Let $R$ be a commutative ring with identity. Then $R$ is Noetherian if and only if every ideal has a finite generating set.* ■

From Theorem 6.16, we have immediately that every ideal in the polynomial ring $F[x_1, ..., x_n]$ over the field $F$ is finitely generated.

The notion of a prime ideal introduced in the next definition is an extension of the notion of a prime integer in the ring $\mathbb{Z}$. It will allow us to get a type of Fundamental Theorem of Arithmetic which we can apply to the study of nonlinear systems of equations. In fact, the study of nonlinear systems of equations motivates the presentation of the material in the remainder of this section.

**Definition 6.16** *Let $R$ be a commutative ring with identity and let $P$ be an ideal of $R$. Then $P$ is said to be a* prime ideal *of $R$ if $\forall a, b \in R$, $ab \in P$ and $a \notin P$ implies $b \in P$.*

**Example 6.12** *Consider the ring of integers $\mathbb{Z}$ and let $p$ be a prime element of $\mathbb{Z}$. Then $< p >$ is a prime ideal of $\mathbb{Z}$. We can see this from the following reasoning. Let $ab \in < p >$. By Corollary 6.14, $\exists r \in \mathbb{Z}$ such that $ab = rp$. Hence either $a$ or $b$ is a multiple of $p$ since $p$ is prime and so either $a \in < p >$ or $b \in < p >$, respectively.*

**Example 6.13** *Consider the polynomial ring $F[x]$, where $F$ is a field. Let $p(x)$ be an irreducible polynomial in $F[x]$. Then $< p(x) >$ is a prime ideal of $F[x]$ by a similar argument as used in the previous example.*

**Definition 6.17** *Let $R$ be a commutative ring with identity and let $Q$ be an ideal of $R$. Then $Q$ is called a* primary ideal *of $R$ if $\forall a, b \in R$, $ab \in Q$ and $a \notin Q$ implies there exists $n \in \mathbb{N}$ such that $b^n \in Q$.*

It is clear from the definitions that a prime ideal in a commutative ring with identity is also a primary ideal.

**Example 6.14** *Consider the ring of integers $\mathbb{Z}$ and let $p$ be a prime element of $\mathbb{Z}$. Then $< p^n >$ is a primary ideal of $\mathbb{Z}$, where $n \in \mathbb{N}$.*

**Example 6.15** *Consider the polynomial ring $F[x]$, where $F$ is a field. Let $p(x)$ be an irreducible polynomial in $F[x]$. Then $< p(x)^n >$ is a primary ideal of $F[x]$, where $n \in \mathbb{N}$.*

**Definition 6.18** *Let $R$ be a commutative ring with identity and let $I$ be an ideal of $R$. Then the* radical *of $I$, denoted $\sqrt{I}$, is defined to be the set*
$$\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

**Theorem 6.17** *Let $Q$ be an ideal of a commutative ring with identity $R$. Then*

**(i)** *$\sqrt{Q}$ is an ideal of $R$ and $\sqrt{Q} \supseteq Q$;*

**(ii)** *if $\sqrt{Q}$ is a primary ideal, then $\sqrt{Q}$ is a prime ideal.* ■

**Example 6.16** *Consider the ring of integers $\mathbb{Z}$. Let $p \in \mathbb{Z}$ be a prime and let $n$ be a positive integer. Then $< p^n >$ is a primary ideal whose radical is the prime ideal $< p >$.*

**Example 6.17** *Consider the polynomial ring $F[x]$ in the indeterminate $x$ over the field $F$. Let $p(x)$ be an irreducible polynomial in $F[x]$ and $n$ a positive integer. Then $< p(x)^n >$ is a primary ideal whose radical is the prime ideal $< p(x) >$.*

Let $R$ be a commutative ring with identity and $Q$ a primary ideal of $R$. Then the radical $P = \sqrt{Q}$ of $Q$ is called the *associated prime ideal* of $Q$ and $Q$ is called a *primary ideal* belonging to (or *primary* for) the prime ideal $P$.

If $I$ and $J$ are ideals of a commutative ring $R$ with identity, then one can show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

**Definition 6.19** *Let $R$ be a commutative ring with identity, $I$ be an ideal of $R$, and $Q_1, ..., Q_n$ be primary ideals of $R$. If $I$ has a representation,*

$$I = Q_1 \cap ... \cap Q_n,$$

*then this representation is called a* primary representation *of $I$. It is called* redundant *or* reduced *if no $Q_i$, $i = 1, ..., n$, contains the intersection of the other $Q_j$'s and the $Q_i$'s have distinct radicals.*

**Theorem 6.18** *Every ideal $I$ in a Noetherian ring $R$ has a reduced primary representation.* ■

Theorems 6.15, 6.18 and Examples 6.10, 6.11 provide us with examples of rings in which every ideal has a reduced primary representation.

**Example 6.18** *Consider the ring $\mathbb{Z}$ of integers. Then every ideal of $R$ has a reduced primary representation since $\mathbb{Z}$ is Noetherian. For example, let $n$ be any positive integer and let*

$$n = p_1^{e_1} ... p_k^{e_k}$$

*be the prime factorization of $n$, where $p_i$ is a prime and $e_i$ is a positive integer, $i = 1, ..., k$. Then*

$$< n > = < p_1{}^{e_1} > \cap ... \cap < p_k{}^{e_k} >$$

*is a primary representation of $< n >$. We also have that*

$$\sqrt{< n >} = \sqrt{< p_1^{e_1} >} \cap ... \cap \sqrt{< p_k^{e_k} >}$$

*and that*

$$\sqrt{< p_i^{e_i} >} = < p_i >, \ i = 1..., k.$$

From Example 6.18, we can see the connection of the concepts just presented to the Fundamental Theorem of Arithmetic.

## *Varieties*

The remainder of this section is concerned with the geometry dealing with affine varieties. An affine variety is defined by polynomial equations. These

polynomial equations may define for example curves and surfaces. Throughout the rest of this section, we let $F[x_1, ..., x_n]$ denote a polynomial ring in the indeterminates $x_1, ..., x_n$ over the field $F$. Let
$f = f(x_1, ..., x_n) = \sum_{i_1=0}^{q_1} \cdots \sum_{i_n=0}^{q_n} a_{i_1 ... i_n} {x_1}^{i_1} ... {x_n}^{i_n} \in F[x_1, ..., x_n]$. We sometimes write $\sum_{\mathbf{i}} a_{\mathbf{i}} {x_1}^{i_1} ... {x_n}^{i_1}$ for $f(x_1, ..., x_n)$, where $\mathbf{i} = (i_1, ..., i_n)$.

**Definition 6.20** *Let $n$ be a positive integer. The set* $F^n = \{(a_1, ..., a_n) \mid a_i \in F, i = 1, ..., n\}$ *is called the* affine space *over $F$.*

For $f \in F[x_1, ..., x_n]$, we can interpret $f$ as a function from $F^n$ into $F$ as follows: For all $(a_1, ..., a_n) \in F^n$,
$f((a_1, ..., a_n)) = \sum_{\mathbf{i}} a_{\mathbf{i}} a_1^{i_1} ... a_n^{i_1}$.

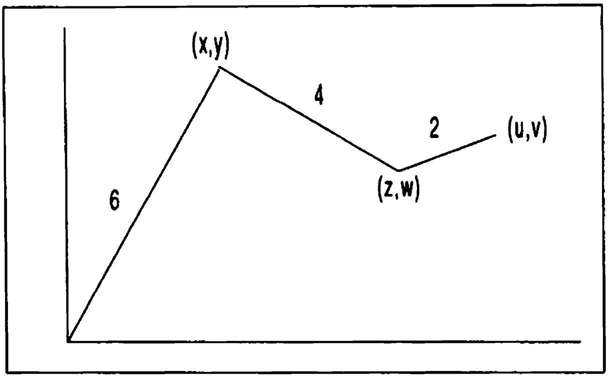
**Definition 6.21** *Let $f_1, ..., f_m \in F[x_1, ..., x_n]$. The set* $V(f_1, ..., f_m) = \{(a_1, ..., a_n) \in F^n \mid f_i(a_1, ..., a_n) = 0, i = 1, ..., m\}$ *is called the* affine variety *defined by $f_1, ..., f_m$.*

Consider, for example, the following linear system of equations
$x + 2y + z = 2$
$x + y - z = 1$.
We replace the second equation by the second equation minus the first equation to obtain
$x + 2y + z = 2$
$-y - 2z = -1$.
We then replace the first equation by the first equation plus twice the second equation. This gives us
$x - 3z = 0$
$-y - 2z = -1$.
Thus
$V(x + 2y + z - 2, x + y - z - 1) = \{(3t, 1 - 2t, t) \mid t \in F\}$.
For an application of polynomial equations, we can turn to robotics. We consider the motion of a robot's arm in the plane. We assume that we have three linked rods of length 6, 4, 2, respectively.
The positions or states of the arm are determined by the solution in $\mathbb{R}^6$ to the following polynomial equations.
$x^2 + y^2 = 36$
$(z - x)^2 + (w - y)^2 = 16$
$(u - z)^2 + (v - w)^2 = 4$
Other applications can be found in computer graphics and geometric theorem proving, [8].

**Lemma 6.19** *Let $V, W \subseteq F^n$ be affine varieties. Then $V \cap W$ and $V \cup W$ are affine varieties.*

**Proof.** Let $V = V(f_1, ..., f_m)$ and $W = V(g_1, ..., g_q)$ for some $f_1, ..., f_m, g_1, ..., g_q \in F[x_1, ..., x_n]$. Now $(a_1, ..., a_n) \in V \cap W$ if and only if $f_i(a_1, ..., a_n)$

FIGURE 6.1 Robotic arm




$= 0$ and $g_j(a_1, ..., a_n) = 0$ for $i = 1, ..., m$ and $j = 1, ..., q$ if and only if $(a_1, ..., a_n) \in V(f_1, ..., f_m, g_1, ..., g_q)$. Thus $V \cap W = V(f_1, ..., f_m, g_1, ..., g_q)$.

Let $(a_1, ..., a_n) \in V$. Then $f_i(a_1, ..., a_n)g_j(a_1, ..., a_n) = 0$ for $i = 1, ..., m$ and $j = 1, ..., q$. Hence $V \subseteq V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\})$. Similarly, $W \subseteq V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\})$. Thus $V \cup W \subseteq V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\})$. Let $(a_1, ..., a_n) \in V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\})$. Suppose there exists $i$ such that $f_i(a_1, ..., a_n) \neq 0$. Since $f_i(a_1, ..., a_n)g_j(a_1, ..., a_n) = 0$ for $i = 1, ..., m$ and $j = 1, ..., q$, we have $g_j(a_1, ..., a_n) = 0$ for $j = 1, ..., q$. Therefore $(a_1, ..., a_n) \in W$. Suppose $f_i(a_1, ..., a_n) = 0$ for $i = 1, ..., m$. Then $(a_1, ..., a_n) \in V$. Thus $V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\}) \subseteq V \cup W$. Consequently, $V(\{f_i g_j \mid i = 1, ..., m$ and $j = 1, ..., q\}) = V \cup W$. ■

**Definition 6.22** *Let $I$ be an ideal of the polynomial ring $F[x_1, ..., x_n]$ in indeterminates $x_1, ..., x_n$ over a field $F$. Define $V(I)$ to be the set*
$V(I) = \{(a_1, ..., a_n) \in F^n \mid f(a_1, ..., a_n) = 0, f \in I\}$.

**Proposition 6.20** *Let $I$ be an ideal of the polynomial ring $F[x_1, ..., x_n]$ in indeterminates $x_1, ..., x_n$ over a field $F$. Then $V(I)$ is an affine variety. In fact, if $I = < f_1, ..., f_m >$, then $V(I) = V(f_1, ..., f_m)$.*

**Proof.** By Theorem 6.16, there exist $f_1, ..., f_m \in F[x_1, ..., x_n]$ such that $I = < f_1, ..., f_m >$ . Since $\{f_1, ..., f_m\} \subseteq I$, $V(I) \subseteq V(f_1, ..., f_m)$. For any $g \in I$, we have by Theorem 6.13 that $g = \sum_{i=1}^{m} r_i f_i$ for some $r_i \in F[x_1, ..., x_n], i = 1, ..., m$. Thus if $(a_1, ..., a_n) \in V(f_1, ..., f_m)$, $f_i(a_1, ..., a_n) = 0$ for $i = 1 , ..., m$. Hence $g(a_1, ..., a_n) = 0$. Therefore $(a_1, ..., a_n) \in V(I)$. Consequently, $V(f_1, ..., f_m) \subseteq V(I)$. ■

**Definition 6.23** *Let $V \subseteq F^n$ be an affine variety. Let $I(V) = \{f \in F[x_1, ..., x_n] \mid f(a_1, ..., a_n) = 0 \ \forall (a_1, ..., a_n) \in V\}$.*

**Lemma 6.21** *Let $V \subseteq F^n$ be an affine variety. Then $I(V)$ is an ideal of $F[x_1, ..., x_n]$.*

**Proof.** Clearly the zero polynomial is in $I(V)$ since $0(a_1, ..., a_n) = 0 \; \forall \; (a_1, ..., a_n) \in F^n$. Let $f, g \in I(V)$. Then $(f - g)(a_1, ..., a_n) = f(a_1, ..., a_n) - g(a_1, ..., a_n) = 0 - 0$. Thus $f - g \in I(V)$. Let $h \in F[x_1, ..., x_n]$. Then $(hf)(a_1, ..., a_n) = h(a_1, ..., a_n)f(a_1, ..., a_n) = h(a_1, ..., a_n)0 = 0$. Therefore $hf \in I(V)$. Hence $I(V)$ is an ideal of $F[x_1, ..., x_n]$. ■

**Example 6.19** *Let $V = \{(0,0)\}$ in $F^2$. Then $I(V) = < x, y >$.*

**Example 6.20** *Let $V = F^n$. Then $f \in I(F^n)$ if and only if $f(a_1, ..., a_n) = 0 \; \forall (a_1, ..., a_n) \in F^n$. Hence if $F$ is infinite, then $f$ is the 0 polynomial. Thus $I(F^n) = \{0\}$ if $F$ is infinite.*

**Proposition 6.22** *Let $V$ and $W$ be affine varieties in $F^n$. Then*

**(i)** *$V \subseteq W$ if and only if $I(V) \supseteq I(W)$,*

**(ii)** *$V = W$ if and only if $I(V) = I(W)$.* ■

**Definition 6.24** *An affine variety $V \subseteq F^n$ is* irreducible *if whenever $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are affine varieties, then either $V_1 = V$ or $V_2 = V$.*

**Proposition 6.23** *Let $V \subseteq F^n$ be an affine variety. Then $V$ is irreducible if and only if $I(V)$ is a prime ideal of $F[x_1, \ldots, x_n]$.* ■

**Proposition 6.24** *Let $V_1 \supseteq V_2 \supseteq ... \supseteq V_n \supseteq ...$ be a descending chain of varieties in $F^n$. Then there exists a positive integer $m$ such that $\forall n \geq m$, $V_n = V_m$.* ■

**Example 6.21** *Consider the variety $V(xz, yz)$ in $\mathbb{R}^3$. If $z = 0$, then $x$ and $y$ are arbitrary and if $z \neq 0$, then $x = y = 0$. Thus $V(xz, yz)$ is the union of the $xy$-plane and the $z$-axis.*

**Example 6.22** *Consider the variety $V = V(xz - y^2, x^3 - yz)$. It follows nontrivially that $V$ is the union of the two irreducible varieties $V(x, y)$ and $V(xz - y^2, x^3 - yz, x^2y - z^2)$. The details may be found in [8].*

**Definition 6.25** *Let $V \subseteq F^n$ be an affine variety. A decomposition $V = V_1 \cup ... \cup V_m$, where each $V_i$ is an irreducible variety, is called a* minimal *(or* irredundant*) decomposition if $V_i \not\subseteq V_j$ for $i \neq j$.*

**Theorem 6.25** *Let $V \subseteq F^n$ be an affine variety. Then $V$ has a minimal decomposition $V = V_1 \cup ... \cup V_m$. Furthermore, this minimal decomposition is unique up to the order in which $V_1, ..., V_m$ are written.* ■

**Example 6.23** *Consider the variety $V(xz - y^2, x^3 - yz)$ of Example 6.22 and the ideal $I =< xz - y^2, x^3 - yz >$ . Then from the decomposition $V(xz - y^2, x^3 - yz) = V(x, y) \cup V(xz - y^2, x^3 - yz, x^2y - z^2)$, we deduce that $I =< x, y > \cap < xz - y^2, x^3 - yz, x^2y - z^2 >$ . It can be shown that $< x, y >$ and $< xz - y^2, x^3 - yz, x^2y - z^2 >$ are prime ideals. The details can be found in [8].*

We close this section by applying the ideas presented here on rings to nonlinear systems of equations. Let $\mathbb{R}[x, y, z]$ denote a polynomial ring in the indeterminates $x, y, z$ over the field of real numbers $\mathbb{R}$. Consider the nonlinear system of equations

$x^2 - y = 0$

$x^2z = 0$.

We know by Theorem 6.15 that $\mathbb{R}[x, y, z]$ is Noetherian and hence from Theorem 6.18 that the ideal $< x^2 - y, x^2z >$ has a reduced primary representation. This representation is

$< x^2 - y, x^2z > = < x^2 - y, z > \cap < x^2, y >$ .

The radicals of the primary ideals $< x^2 - y, z >$ and $< x^2, y >$ are the prime ideals $< x^2 - y, z >$ and $< x, y >$, respectively. The corresponding irreducible affine varieties are $\{(x, x^2, 0) \mid x \in \mathbb{R}\}$ and $\{(0, 0, z) \mid z \in \mathbb{R}\}$, respectively. It is clear that the union of these affine varieties is the solution set to the given nonlinear system.

## 6.2 Fuzzy Substructures of Algebraic Structures

**Definition 6.26** *Let $(S, *)$ be a semigroup. Let $\tilde{A}$ be a fuzzy subset of $S$. Then $\tilde{A}$ is called a fuzzy subsemigroup of $S$ if $\forall a, b \in S$, $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b)$.*

The definition of a fuzzy subsemigroup $\tilde{A}$ of a semigroup $S$ is motivated by the following reasoning. If $\tilde{A}(a) = 1$ and $\tilde{A}(b) = 1$, then $\tilde{A}(a * b) = 1$, i. e., in the crisp sense if $a \in \tilde{A}$ and $b \in \tilde{A}$, then $a * b \in \tilde{A}$. We see that Definition 6.26 extends Definition 6.3.

**Definition 6.27** *Let $(S, *)$ be a monoid and let $\tilde{A}$ be a fuzzy subset of $S$. Then $\tilde{A}$ is called a fuzzy monoid of $S$ if $\forall a, b \in S$, $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b)$ and $\forall a \in S, \tilde{A}(e) \geq \tilde{A}(a)$, where $e$ is the identity of $S$.*

If we note in Definition 6.27 that if $\tilde{A}(a) = 1$, then $\tilde{A}(e) = 1$ and so we see that Definition 6.27 extends Definition 6.3.

**Proposition 6.26** *Let $S$ semigroup (monoid) and let $\tilde{A}$ be a fuzzy subset of $S$.*

**(i)** *Then $\tilde{A}$ fuzzy subsemigroup (submonoid) of $S$ if and only if $\tilde{A}^t$ is a subsemigroup (submonoid) of $S$ $\forall t \in Im(\tilde{A})$.*

(ii) *If $\tilde{A}$ is a fuzzy subsemigroup (submonoid) of $S$ and supp$(\tilde{A}) \neq \emptyset$, then supp$(\tilde{A})$ is a subsemigroup (submonoid) of $S$.*

**Proof.** (i) Suppose that $\tilde{A}$ is a fuzzy subsemigroup (submonoid) of $S$. Let $t \in \text{Im}(\tilde{A})$. Let $a, b \in \tilde{A}^t$. Then $\tilde{A}(a) \geq t$ and $\tilde{A}(b) \geq t$. Since $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b) \geq t$, $a * b \in \tilde{A}^t$. (If $\tilde{A}$ is a fuzzy submonoid, then $e \in \tilde{A}^t$ $\forall t \in \text{Im}(\tilde{A})$ since $\tilde{A}(e) \geq \tilde{A}(a)$ $\forall a \in S$.) Conversely, suppose that $\tilde{A}^t$ is a subsemigroup (submonoid) of $S$ $\forall t \in \text{Im}(\tilde{A})$. Let $a, b \in S$. Let $\tilde{A}(a) = t$ and $\tilde{A}(b) = s$ with $t \geq s$, say. Then $a, b \in \tilde{A}^s$ and so $a * b \in \tilde{A}^s$. Hence $\tilde{A}(a * b) \geq s = \tilde{A}(a) \wedge \tilde{A}(b)$. (If each $\tilde{A}^t$ is a monoid, then $e \in \tilde{A}^t$ for each $t \in \text{Im}(\tilde{A})$ and so $\tilde{A}(e) \geq \tilde{A}(a)$ $\forall a \in S$.)

(ii) Let $a, b \in \text{supp}(\tilde{A})$. Then $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b) > 0$ and so $a * b \in \text{supp}(\tilde{A})$. (If $\tilde{A}$ is a fuzzy monoid, then $e \in \text{supp}(\tilde{A})$ since $\tilde{A}(e) \geq \tilde{A}(a) \forall a \in S$.) ■

**Definition 6.28** *Let $(G, *)$ be a group and let $\tilde{A}$ be a fuzzy subset of $G$. Then $\tilde{A}$ is called a* fuzzy subgroup *of $G$ if $\forall a, b \in G$, $\tilde{A}(a * b^{-1}) \geq \tilde{A}(a) \wedge \tilde{A}(b)$.*

The definition of a fuzzy subgroup of a group is motivated by Theorem 6.6. We see that if $\tilde{A}(a) = 1$ and $\tilde{A}(b) = 1$ in Definition 6.28, then $\tilde{A}(a * b^{-1}) = 1$ and Theorem 6.6 is thus extended.

**Proposition 6.27** *Let $(G, *)$ be a group and let $\tilde{A}$ be a fuzzy subset of $G$. Then $\tilde{A}$ is a fuzzy subgroup of $G$ if and only if $\forall a, b \in G$, $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b)$ and $\tilde{A}(a^{-1}) \geq \tilde{A}(a)$.*

**Proof.** Suppose that $\tilde{A}$ is a fuzzy subgroup of $G$. Then $\forall a \in G$, $\tilde{A}(e) = \tilde{A}(a * a^{-1}) \geq \tilde{A}(a) \wedge \tilde{A}(a) = \tilde{A}(a)$. Hence $\forall a \in G$, $\tilde{A}(a^{-1}) = \tilde{A}(e * a^{-1}) \geq \tilde{A}(e) \wedge \tilde{A}(a) = \tilde{A}(a)$. Thus $\tilde{A}(a * b) \geq \tilde{A}(a) \wedge \tilde{A}(b^{-1}) \geq \tilde{A}(a) \wedge \tilde{A}(b) \forall a, b \in G$. For the converse, let $a, b \in G$. Then $\tilde{A}(a * b^{-1}) \geq \tilde{A}(a) \wedge \tilde{A}(b^{-1}) \geq \tilde{A}(a) \wedge \tilde{A}(b)$. ■

The proof of the following result follows in a similar manner as that of Proposition 6.26.

**Proposition 6.28** *Let $G$ be a group and let $\tilde{A}$ be a fuzzy subset of $G$.*

(i) *Then $\tilde{A}$ is a fuzzy subgroup of $G$ if and only if $\tilde{A}^t$ is a subgroup of $G$ $\forall t \in Im(\tilde{A})$.*

(ii) *If $\tilde{A}$ is a fuzzy subgroup of $G$ and supp$(\tilde{A}) \neq \emptyset$, then supp$(\tilde{A})$ is a subgroup of $G$.* ■

We now review some results from fuzzy ideal theory. These results are taken primarily from [24-26,28,29,38,40,48,49,56].

**Definition 6.29** *Let $(R, +, \cdot)$ be a commutative ring with identity and let $\tilde{I}$ be a fuzzy subset of $R$. Then $\tilde{I}$ is called a fuzzy ideal of $R$ if $\forall a, b \in R$, $\tilde{I}(a-b) \geq \tilde{I}(a) \wedge \tilde{I}(b)$ and $\tilde{I}(a \cdot b) \geq \tilde{I}(a) \vee \tilde{I}(b)$.*

If $\tilde{A}$ is a fuzzy ideal of $R$, then $\tilde{A}(0) \geq \tilde{A}(x)$ for every $x \in R$. We let $\tilde{A}_*$ $= \{x \in R \mid \tilde{A}(x) = \tilde{A}(0)\}$. Then by the following result, $\tilde{A}_*$ is an ideal of $R$.

**Proposition 6.29** *Let $R$ be a commutative ring with identity and let $\tilde{I}$ be a fuzzy subset of $R$.*

**(i)** *Then $\tilde{I}$ is a fuzzy ideal of $R$ if and only if $\tilde{I}^t$ is an ideal of $R$ $\forall t \in Im(\tilde{I})$.*

**(ii)** *If $\tilde{I}$ is a fuzzy ideal of $R$ and $supp(\tilde{A}) \neq \emptyset$, then $supp(\tilde{I})$ is an ideal of $R$.* ■

**Proposition 6.30** *Let $R$ be a commutative ring with identity. Then the intersection of any collection of fuzzy ideals of $R$ is a fuzzy ideal of $R$.* ■

**Definition 6.30** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ and $\tilde{B}$ be fuzzy ideals of $R$. Define the fuzzy subset $\tilde{A}\tilde{B}$ of $R$ by $\forall x \in R$,*

$\tilde{A}\tilde{B}(x) = \vee\{\wedge\{\tilde{A}(a_i) \wedge \tilde{B}(b_i) \mid i = 1, \ldots, n\} \mid x = \sum_{i=1}^{n} a_i b_i, a_i, b_i \in R,$ $i = 1, \ldots, n;\ n \in \mathbb{N}\}$.

**Proposition 6.31** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ and $\tilde{B}$ be fuzzy ideals of $R$. Then $\tilde{A}\tilde{B}$ is a fuzzy ideal of $R$.* ■

**Definition 6.31** *Let $R$ be a commutative ring with identity and let $\tilde{P}$ be a fuzzy ideal of $R$. Then $\tilde{P}$ is called a* prime fuzzy ideal *of $R$ if for all fuzzy ideals $\tilde{A}$ and $\tilde{B}$ of $R$, $\tilde{A}\tilde{B} \subseteq \tilde{P}$ and $\tilde{A} \not\subseteq \tilde{P}$ implies $\tilde{B} \subseteq \tilde{P}$.*

**Theorem 6.32** *Let $R$ be a commutative ring with identity and let $\tilde{P}$ be a nonconstant fuzzy ideal of $R$. Then $\tilde{P}$ is a prime fuzzy ideal of $R$ if and only if $Im(\tilde{P}) = \{1, t\}$, where $0 \leq t < 1$, and the level ideal $\tilde{P}^1$ is a prime ideal of $R$.* ■

**Definition 6.32** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ be a fuzzy ideal of $R$. The radical of $\tilde{A}$, denoted by $\sqrt{\tilde{A}}$, is defined by $\sqrt{\tilde{A}} = \bigcap \tilde{P}$, the intersection being taken over those prime fuzzy ideals $\tilde{P}$ such that $\tilde{P} \supseteq \tilde{A}$ and $\tilde{P}_* \supseteq \tilde{A}_*$.*

**Theorem 6.33** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ be a fuzzy ideal of $R$. Then $\sqrt{\tilde{A}}$ is a fuzzy ideal of $R$ such that $\sqrt{\tilde{A}} \supseteq \tilde{A}$ and $\sqrt{\tilde{A}}(0) = 1$.* ■

**Theorem 6.34** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ and $\tilde{B}$ be fuzzy ideals of $R$ such that $\tilde{A}(0) = 1 = \tilde{B}(0)$. Then $\sqrt{\tilde{A} \cap \tilde{B}} = \sqrt{\tilde{A}} \cap \sqrt{\tilde{B}}$.* ■

**Definition 6.33** *Let $R$ be a commutative ring with identity and let $\tilde{Q}$ be a fuzzy ideal of $R$. Then $\tilde{Q}$ is called a primary fuzzy ideal of $R$ if either $\tilde{Q} = \chi_R$ or $\tilde{Q}$ is nonconstant and for all fuzzy ideals $\tilde{A}$ and $\tilde{B}$ of $R$, $\tilde{A}\tilde{B} \subseteq \tilde{Q}$ and $\tilde{A} \not\subseteq \tilde{Q}$ implies $\tilde{B} \subseteq \sqrt{\tilde{Q}}$.*

**Theorem 6.35** *Let $R$ be a commutative ring with identity and let $\tilde{Q}$ be a nonconstant fuzzy ideal of $R$. Then $\tilde{Q}$ is a primary fuzzy ideal of $R$ if and only if $Im(\tilde{Q}) = \{1, t\}$, where $0 \leq t < 1$, and the level ideal $\tilde{P}^1$ is a primary ideal of $R$.* ■

**Theorem 6.36** *Let $R$ be a commutative ring with identity and let $\tilde{Q}$ be a primary fuzzy ideal of $R$. Then $\sqrt{\tilde{Q}}$ is a prime fuzzy ideal of $R$.* ■

**Definition 6.34** *Let $R$ be a commutative ring with identity and let $\tilde{Q}$ be a primary fuzzy ideal of $R$. Then $\tilde{P} = \sqrt{\tilde{Q}}$ is called the* associated prime fuzzy ideal *of $\tilde{Q}$ and $\tilde{Q}$ is called a* primary ideal belonging *to $\tilde{P}$ or simply* primary *for $\tilde{P}$.*

In the remainder of this section, we let $R$ denote the polynomial ring $F[x_1, ..., x_n]$ in indeterminates $x_1, ..., x_n$ over a field $F$. Let $L$ be a field containing $F$, possibly an algebraic closure of $F$. We now give definitions for the fuzzy counterparts of the affine variety of a set of points in $L^n$ and the ideal in $R$ of an affine variety. Let $c$ be a strictly decreasing function of $[0, 1]$ into itself such that $c(0) = 1, c(1) = 0$, and for all $t \in [0, 1]$, $c(c(t)) = t$. The following approach has the advantage that $c$ may be changed to fit the application.

**Definition 6.35** *Let $\tilde{X}$ be a finite-valued fuzzy subset of $L^n$, say $Im(\tilde{X}) = \{t_0, t_1, ..., t_n\}$, where $t_0 < t_1 < ... < t_n$. Define the fuzzy subset $I(\tilde{X})$ of $R$ as follows:*

$$I(\tilde{X})(f) = \begin{cases} c(t_n) & \text{if } f \in R \setminus I(\tilde{X}_{t_n}), \\ c(t_i) & \text{if } f \in I(\tilde{X}_{t_{i+1}}) \setminus I(\tilde{X}_{t_i}), i = 1, ..., n-1, \\ c(t_0) & \text{if } f \in I(\tilde{X}_{t_1}). \end{cases}$$

*If $n = 0$, then we define $I(\tilde{X})(0) = 1$.*

In Definition 6.35, it possible for $I(\tilde{X}_{t_{i+1}}) = I(\tilde{X}_{t_i})$ or $R = I(\tilde{X}_{t_n})$. In this case, $c(t_i) \notin \text{Im}(I(\tilde{X}))$, $i = 1, ..., n$.

**Definition 6.36** *Let $\tilde{A}$ be a finite-valued fuzzy ideal of $R$, say $Im(\tilde{A}) = \{s_0, s_1, ..., s_m\}$, where $s_0 < s_1 < ... < s_m$. Define the fuzzy subset $V(\tilde{A})$ of $L^n$ as follows:*

$$V(\tilde{A})(b) = \begin{cases} c(s_m) & \text{if } b \in L^n \setminus V(\tilde{A}_{s_m}), \\ c(s_i) & \text{if } b \in V(\tilde{A}_{s_{i+1}}) \setminus V(\tilde{A}_{s_i}), i = 1, ..., m-1, \\ c(s_0) & \text{if } b \in V(\tilde{A}_{s_1}). \end{cases}$$

*$V(\tilde{A})$ is called a* fuzzy affine variety.

In Definition 6.36, it is possible for $c(s_i) \notin \mathrm{Im}(V(\tilde{A}))$ for some $i = 1, ..., m$.

**Proposition 6.37** *Let $\tilde{X}$ be a finite-valued fuzzy subset of $L^n$ and let $\tilde{A}$ be a finite-valued fuzzy ideal of $R$. Then*

**(i)** $\forall t \in [0,1]$, $V(I(\tilde{X}))^t = V(I(\tilde{X}^t))$,

**(ii)** $\forall s \in [0,1]$, $I(V(\tilde{A}))^s = I(V(\tilde{A}^s))$. ■

**Proposition 6.38** *Let $\tilde{X}$ be a finite-valued fuzzy subset of $L^n$ and let $\tilde{A}$ be a finite-valued fuzzy ideal of $R$. Then*

**(i)** $I(V(I(\tilde{X}))) = I(\tilde{X})$,

**(ii)** $V(I(V(\tilde{A}))) = V(\tilde{A})$. ■

**Proposition 6.39** *Let $\tilde{X}$ be a fuzzy subset of $L^n$. Then $\tilde{X}$ is a fuzzy affine variety if and only if $\tilde{X}$ is finite-valued and for all $t \in Im(\tilde{X})$, $\tilde{X}^t$ is an affine variety.* ■

**Proposition 6.40** *If $\tilde{A}$ is a nonconstant prime fuzzy ideal of $R$, then $\tilde{A} = I(V(\tilde{A}))$.* ■

**Proposition 6.41** *Let $\tilde{A}$ be a finite-valued fuzzy ideal of $R$. If $\tilde{A}(0) = 1$, then $V(\tilde{A}) = V(\sqrt{\tilde{A}})$.* ■

**Corollary 6.42** *If $\tilde{P}$ is a prime fuzzy ideal of $R$ belonging to the primary fuzzy ideal $\tilde{Q}$ of $R$, then $V(\tilde{Q}) = V(\tilde{P})$.* ■

**Theorem 6.43** *Let $\tilde{A}$ and $\tilde{B}$ be finite-valued fuzzy ideals of $R$ such that $\tilde{A}(0) = \tilde{B}(0) = 1$. Then $V(\tilde{A} \cap \tilde{B}) = V(\tilde{A}) \cup V(\tilde{B})$.* ■

**Theorem 6.44** *Let $\tilde{X}$ and $\tilde{Y}$ be fuzzy affine varieties. If $0 \in Im(\tilde{X}) \cap Im(\tilde{Y})$, then $I(\tilde{X} \cup \tilde{Y}) = I(\tilde{X}) \cap I(\tilde{Y})$.* ■

## 6.3 Fuzzy Submonoids and Automata Theory

In our first application, we consider strings of fuzzy singletons as input to a fuzzy finite state machine. The notion of fuzzy automata was introduced in [58]. There has been considerable growth in the area, [18]. In this section, we present a theory of free fuzzy monoids and apply the results to the area of (fuzzy) automata. In (fuzzy) automata, the set of strings of input symbols can be considered to be a free monoid. We introduce the notion of fuzzy strings of input symbols, where the fuzzy strings form free fuzzy submonoids of the free monoid of input strings. We show that (fuzzy) automata with fuzzy input are equivalent to (fuzzy) automata with crisp input. Hence the results of (fuzzy) automata theory can be immediately applied to those of (fuzzy) automata theory with fuzzy input. The results are taken from [7] and [34].

Let $f$ be a homomorphism of a semigroup $F$ into a semigroup $S$. Let $\tilde{A}$ be a fuzzy subset of $F$ and $\tilde{B}$ a fuzzy subset of $S$. We recall that the fuzzy subset $f(\tilde{A})$ of $S$ and the fuzzy subset $f^{-1}(\tilde{B})$ of $F$ are defined as follows:

$\forall y \in S$, $f(\tilde{A})(y) = \vee\{\tilde{A}(x) \mid f(x) = y, x \in F\}$ if $y \in f(F)$ and $f(\tilde{A})(y) = 0$ if $y \notin f(F)$;

$\forall x \in F$, $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$.

**Proposition 6.45** *Let $f$ be a homomorphism of a semigroup (monoid) $F$ into (onto) a semigroup (monoid) $S$.*

**(i)** *If $\tilde{A}$ is a fuzzy subsemigroup (submonoid) of $F$, then $f(\tilde{A})$ is a fuzzy subsemigroup (submonoid) of $S$.*

**(ii)** *If $\tilde{B}$ is a fuzzy subsemigroup (submonoid) of $S$, then $f^{-1}(\tilde{B})$ is a fuzzy subsemigroup (submonoid) of $F$.* ■

**Definition 6.37** *Let $S$ be a semigroup (monoid) and $\tilde{C}$ a fuzzy subset of $S$. Let $< \tilde{C} >$ ($<< \tilde{C} >>$) denote the intersection of all fuzzy subsemigroups (submonoids) of $S$ which contain $\tilde{C}$. Then $< \tilde{C} >$ ($<< \tilde{C} >>$) is called the* fuzzy subsemigroup (submonoid) *generated by $\tilde{C}$.*

Clearly $< \tilde{C} >$ ($<< \tilde{C} >>$) in Definition 6.37 is the smallest fuzzy subsemigroup (submonoid) of $S$ which contains $\tilde{C}$. Let $t_{\tilde{C}}$ denote $\vee\{\tilde{C}(x) \mid x \in S\}$.

**Theorem 6.46** *Let $S$ be a semigroup (monoid) and $\tilde{C}$ a fuzzy subset of $S$. Define the fuzzy subset $\tilde{A}$ of $S$ by $\forall\ x \in S$,*

$\tilde{A}(x) = \vee\{((z_1)_{t_1} \dots (z_n)_{t_n})(x) \mid x = z_1 \dots z_n,\ z_i \in S,\ \tilde{C}(z_i) = t_i,\ i = 1, \dots, n; n \in \mathbb{N}\}$. *Then* $< \tilde{C} > = \tilde{A}$ ($<< \tilde{C} >> = \tilde{A} \vee e_{t_{\tilde{C}}}$). ■

**Definition 6.38** *Let $F$ be a free semigroup (monoid) on the set $X$ with respect to the function $f : X \to F$. Let $\tilde{Y}$ be a fuzzy subset of $X$. Let $\tilde{M}$ be a* *fuzzy subsemigroup (submonoid) of F. Then* $\tilde{M}$ *is said to be* free *with respect to* $\tilde{Y}$ *if* $f(\tilde{Y}) = \tilde{M}$ *on* $f(X)$, $< f(\tilde{Y}) > = \tilde{M}(<< f(\tilde{Y}) >> = \tilde{M})$ *and* $\forall$ *semigroups (monoids) S and* $\forall$ *fuzzy subsemigroups (submonoids)* $\tilde{A}$ *of S with* $g : X \to S$ *and* $g(\tilde{Y}) = \tilde{A}$ *on* $g(X)$, *there exists unique homomorphism h of F into S such that* $g = h \circ f$ *and* $h(\tilde{M}) \subseteq \tilde{A}$.

**Theorem 6.47** *Let F be a free semigroup (monoid) on the set X with respect to the function* $f : X \to F$. *Let* $\tilde{Y}$ *be a fuzzy subset of X. Then there exists a fuzzy subsemigroup (submonoid) of F which is free with respect to* $\tilde{Y}$. ■

**Proposition 6.48** *Let S be a semigroup (monoid) and* $\tilde{B}$ *a fuzzy subsemigroup (submonoid) of S. Then* $\exists$ *a free semigroup (monoid) F, a free fuzzy subsemigroup (submonoid)* $\tilde{M}$ *of F, and a homomorphism h of F onto S such that* $h(\tilde{M}) \subseteq \tilde{B}$. ■

**Proposition 6.49** *Let S be a semigroup (monoid) and* $\tilde{B}$ *a fuzzy subsemigroup (submonoid) of S. Then* $\exists$ *a free semigroup (monoid) F and a free fuzzy subsemigroup (submonoid)* $\tilde{M}$ *of F such that* $\tilde{B}$ *is weakly isomorphic [7,34,59] to a quotient semigroup (monoid) [7,34,59] of* $\tilde{M}$. ■

Before showing how the results on free fuzzy submonoids can be applied to the study of fuzzy finite state machines with fuzzy input, we give some basic definitions. A fuzzy finite state machine (ffsm) $M$ is a triple $(Q, X, \tilde{A})$ where $Q$ and $X$ are finite nonempty sets and $\tilde{A}$ is a function from $Q \times X \times Q$ into [0,1]. The elements of $Q$ are called *states* and the elements of $X$ are called *input symbols.* The function $\tilde{A}$ is the *fuzzy transition function.* Let $X^*$ denote the set of all strings of finite length of elements of $X$ including the empty string, $\lambda$. Then $X^*$ is a free monoid. Let $M = (Q, X, \tilde{A})$ be a ffsm. Define $\tilde{A}^* : Q \times X^* \times Q \to [0,1]$ by $\tilde{A}^*(q, \lambda, p) = 1$ if $q = p$ and $\tilde{A}^*(q, \lambda, p) = 0$ if $q \neq p$ and $\forall (q, x, p) \in Q \times X^* \times Q$, $\forall a \in X$, $\tilde{A}^*(q, xa, p) = \vee\{\tilde{A}^*(q, x, r) \wedge \tilde{A}^*(r, a, p) \mid r \in Q\}$.

Let $x_i \in X, i = 1, ..., n$. Let $\tilde{Y}$ be a fuzzy subset of $X$. Let $x = x_1...x_n$. Then $(x_1)_{\tilde{Y}(x_1)}...(x_n)_{\tilde{Y}(x_n)} = (x_1...x_n)_{\tilde{Y}(x_1)\wedge...\wedge\tilde{Y}(x_n)} = x_{\tilde{Y}^*(x)}$, where $\tilde{Y}^*(x) = \tilde{Y}(x_1)\wedge...\wedge\tilde{Y}(x_n)$. Thus inputting the string of fuzzy singletons $(x_1)_{\tilde{Y}(x_1)}$, ..., $(x_n)_{\tilde{Y}(x_n)}$ successively is the same as inputting $x_{\tilde{Y}^*(x)}$ where $x = x_1...x_n$. That is, there is a consistency between input and the semigroup operation, concatenation, of $X^*$. We also know that $<< \tilde{Y} >> = \tilde{Y}$ on $X$.

We can think of $\tilde{Y}^*$ as $<< \tilde{Y} >>$, the fuzzy subsemigroup of $X^*$ generated by $\tilde{Y}$.

We now apply our results to the theory of fuzzy automata. Let $M = (Q, X, \tilde{A})$ be a ffsm. Let $\tilde{Y}$ be a fuzzy subset of $X$ and let $\tilde{Y}^* = << \tilde{Y} >>$.

**Definition 6.39** *Let* $M = (Q, X, \tilde{A})$ *be a ffsm. Let* $\tilde{Y}$ *be a fuzzy subset of X. Define* $\tilde{A}_{\tilde{Y}} : Q \times X \times Q \to [0,1]$ *and* $\tilde{A}_{\tilde{Y}^*} : Q \times X^* \times Q \to [0,1]$ *as follows:*

$\forall(q,a,p) \in Q \times X \times Q$, $\tilde{A}_{\tilde{Y}}(q,a,p) = \tilde{Y}(a) \wedge \tilde{A}(q,a,p)$;
$\forall(q,x,p) \in Q \times X^* \times Q$. $\tilde{A}_{\tilde{Y}^*}(q,x,p) = \tilde{Y}^*(x) \wedge \tilde{A}^*(q,x,p)$.

If $x \in X^*$, we define the *length*, $|x|$, of $x$ as follows: if $x = \lambda$, then $|x| = 0$ and if $x = x_1 \ldots x_n$, where $x_i \in X$, $i = 1, \ldots, n$, then $|x| = n$.

**Theorem 6.50** *Let $M = (Q, X, \tilde{A})$ be a ffsm. Let $\tilde{Y}$ be a fuzzy subset of $X$. Then $(\tilde{A}_{\tilde{Y}})^* = \tilde{A}_{\tilde{Y}^*}$.*

**Proof.** $\tilde{A}_{\tilde{Y}^*}(q,\lambda,p) = \tilde{Y}^*(\lambda) \wedge \tilde{A}^*(q,\lambda,p) = 1 \wedge 1$ if $q = p$ and $1 \wedge 0$ if $q \neq p$. Thus $\tilde{A}_{\tilde{Y}^*}(q,\lambda,p) = (\tilde{A}_{\tilde{Y}})^*(q,\lambda,p)$. Assume $\tilde{A}_{\tilde{Y}^*}(q,x,p) = (\tilde{A}_{\tilde{Y}})^*(q,x,p)$ for $|x| \geq 0$. Now $\tilde{A}_{\tilde{Y}^*}(q,xa,p) = \tilde{Y}^*(xa) \wedge \tilde{A}^*(q,xa,p) = \tilde{Y}^*(x) \wedge \tilde{Y}^*(a) \wedge \vee\{\tilde{A}^*(q,x,r) \wedge \tilde{A}(r,a,p) \mid r \in Q\} = \vee\{\tilde{Y}^*(x) \wedge \tilde{Y}^*(a) \wedge \tilde{A}^*(q,x,r) \wedge \tilde{A}(r,a,p) \mid r \in Q\} = \vee\{\tilde{Y}^*(x) \wedge \tilde{Y}(a) \wedge \tilde{A}^*(q,x,r) \wedge \tilde{A}(r,a,p) \mid r \in Q\} = \vee\{(\tilde{A}_{\tilde{Y}})^*(q,x,r) \wedge \tilde{A}_{\tilde{Y}}(r,a,p) \mid r \in Q\}$ (by the induction hypothesis) $= (\tilde{A}_{\tilde{Y}})^*(q,xa,p)$. ■

**Corollary 6.51** *Let $M = (Q, X, \tilde{A})$ be a ffsm. Let $\tilde{Y}$ be a fuzzy subset of $X$. Then $\forall q,p \in Q, \forall x,y \in X^*$, $\tilde{A}_{\tilde{Y}^*}(q,xy,p) = \vee\{\tilde{A}_{\tilde{Y}^*}(q,x,r) \wedge \tilde{A}_{\tilde{Y}^*}(r,y,p) \mid r \in Q\}$.* ■

We see from Theorem 6.50 that a ffsm with fuzzy input acts like a ffsm. Hence results for ffsm's can be immediately carried over to ffsm's with fuzzy input.

## 6.4 Fuzzy Subgroups, Pattern Recognition and Coding Theory

The material for the presentation in this section is from [3,4,7]. We show how the concept of fuzzy subgroups can be used to examine the faithfulness of a device which decodes messages transmitted a cross a noisy channel. Let $G$ denote a group. The notion of a fuzzy subgroup of $G$ was introduced by Rosenfeld [52]. Subsequently Anthony and Sherwood [3,4] introduced the notion of a fuzzy subgroup where an arbitrary $t$-norm replaced the $t$-norm "minimum" used by Rosenfeld and where $\tilde{A}(e) = 1$ was required, $e$ the identity of $G$. By a $t$-norm, we mean a function $T$ of $[0,1] \times [0,1] \to [0,1]$ such that $\forall x, y, z \in [0,1]$, $T(x,1) = x$, $T(x,y) \leq T(z,y)$ if $x \leq z$, $T(x,y) = T(y,x)$, and $T(x,T(y,z)) = T(T(x,y),z))$. Then according to [3], a fuzzy subset $\tilde{A}$ of $G$ is a fuzzy subgroup of $G$ if $\forall x, y \in G$, $\tilde{A}(xy^{-1}) \geq T(\tilde{A}(x), \tilde{A}(y))$ and $\tilde{A}(e) = 1$. We introduce two classes of fuzzy subgroups. Each fuzzy subgroup in these classes satisfy the definition of a fuzzy subgroup with the $t$-norm $T_m$ given by $T_m(s,t) = (s+t-1) \vee 0 \; \forall \; s, t \in [0, 1]$. Although these classes look different, each fuzzy subgroup in either is isomorphic to one in the other. We note that a fuzzy subgroup satisfies the definition of fuzzy subgroup with $T$ = "minimum" if and only if it is a fuzzy subgroup generated of a very special type. These notions are then applied to some abstract pattern recognition problems and coding theory problems.

In the following result, we can think that the value of the fuzzy subgroup at a particular point $x$ represents the probability that $x$ will be found in a randomly selected subgroup. This gives us a particular way of generating fuzzy subgroups. We first quickly review some basic definitions from probability. For a set $\Omega$, a set $\mathcal{A}$ of subsets of $\Omega$ is a *$\sigma$-algebra* if (1) $\emptyset \in \mathcal{A}$, (2) $\forall A \in \mathcal{A}$, $A^c \in \mathcal{A}$, and (3) if $\{A_i \mid i \in I\}$ is a countable collection of elements of $\mathcal{A}$, then $\bigcup_{i \in I} A_i \in \mathcal{A}$. We call $P : \mathcal{A} \to \mathbb{R}$ a probability measure if $P(A) > 0$ $\forall A \in \mathcal{A}$, $P(\Omega) = 1$, and $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for any denumerable union of disjoint sets $A_i, i = 1, 2, \ldots$. The triple $(\Omega, \mathcal{A}, P)$ is called a *probability space.*

**Theorem 6.52** *Let $G$ be a group and let $\mathcal{S}$ be the set of all subgroups of $G$. For each $x \in G$, let $S_x = \{S \in \mathcal{S} \mid x \in S\}$ and let $\mathcal{T} = \{S_x \mid x \in G\}$. Let $\mathcal{A}$ be any $\sigma$-algebra on $\mathcal{S}$ which contains the $\sigma$-algebra generated by $\mathcal{T}$ and let $m$ be a probability measure on $(\mathcal{S}, \mathcal{A})$. Then the fuzzy subset $\tilde{A}$ of $G$ defined by $\tilde{A}(x) = m(S_x)$ $\forall$ $x \in G$ is a fuzzy subgroup of $G$ with respect to $T_m$. A fuzzy subgroup obtained in this manner is called a* subgroup generated. ■

In the next result, we think of a point which travels in some random fashion through a group and we compute the probability of finding the point in a particular subgroup. We thus have another way of generating fuzzy subgroups.

**Theorem 6.53** *Let $(G, +)$ be a group and $H$ a subgroup of $G$. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(\mathcal{G}, \oplus)$ be a group of functions mapping $\Omega$ into $G$ with $\oplus$ defined by pointwise addition in the range space. Suppose that $\forall$ $f \in \mathcal{G}$, $G_f = \{\omega \in \Omega \mid f(\omega) \in H\}$ is an element of $\mathcal{A}$. Then the fuzzy subset $\tilde{B}$ of $\mathcal{G}$ defined by $\tilde{B}(f) = P(G_f)$ $\forall$ $f \in \mathcal{G}$ is a fuzzy subgroup of $\mathcal{G}$ with respect to $T_m$. A fuzzy subgroup obtained in this manner is called* function generated. ■

The next theorems establish a basic equivalence between the notions of subgroup generated and function generated.

**Theorem 6.54** *Every function generated fuzzy subgroup is subgroup generated.* ■

**Theorem 6.55** *Every subgroup generated fuzzy subgroup is isomorphic to a function generated fuzzy subgroup.* ■

**Theorem 6.56** *Every fuzzy subgroup with respect to* $\wedge$ *is subgroup generated.* ■

**Theorem 6.57** *Let* $\tilde{A}$ *be a fuzzy subgroup of* $G$ *with* $(S, \mathcal{A}, m)$ *and the sets* $S_x$ *for* $x \in G$ *as defined in Theorem 6.52. If there exists* $S^* \in \mathcal{A}$ *which is linearly ordered by set inclusion such that* $m(S^*) = 1$*, then* $\tilde{A}$ *is a fuzzy subgroup with respect to* $\wedge$. ■

**Theorem 6.58** *A fuzzy subgroup is a fuzzy subgroup with respect to* $\wedge$ *if and only if it is subgroup generated and the generating family possesses a subfamily of measure one which is linearly ordered by set inclusion.* ■

Suppose that $F$ is a device which receives a stream of discrete inputs and produces a stream of discrete outputs. We assume the following conditions.

(1) $F$ is deterministic and acts independently on each individual input. That is, a particular input will produce the same output each time that input is provided to $F$. However, the output which is produced by a specific input is not known.

(2) There is complete knowledge of the outputs. That is, the output stream is observable.

(3) The input stream is not observable. The possible inputs are known and estimates can be obtained of their relative frequencies in a large segment of the input stream.

(4) The outputs have an algebraic character in the sense that they can be identified with the objects in a group. Thus there is a method of combining the outputs which has the ordinary properties of a group operation.

Let $\mathcal{I}$ designate the collection of inputs and let $\mathcal{O}$ be the collection of outputs. If $T \in \mathcal{I}$, then $F(T) \in \mathcal{O}$. Hence $F$ is identified with a function from $\mathcal{I}$ into $\mathcal{O}$. Suppose that $f$ is a known function from $\mathcal{I}$ into $\mathcal{O}$. Moreover, suppose that some particular character of $F$ which we shall call "faithfulness" is associated with solvability for $x$ of an equation in the output "group" of the form $x + f(T) = F(T)$, where $+$ is the group operation. If for some $T \in \mathcal{I}$ a solution for $x$ can be found in a given subgroup $H$, then the output $F(T)$ will be called *$H$-$f$ faithful* to the input $T$. For a sufficiently large finite segment of the output stream and for a given function $f$ and subgroup $H$, we examine the problem of approximating the proportion of the outputs which are $H$-$f$ faithful to their respective inputs.

To translate this problem into the setting of fuzzy subgroups, certain identifications are necessary. The outputs have already been identified with a group $(G, +)$. The inputs may be identified with a probability space $(\Omega, \mathcal{A}, P)$ where $\Omega = \mathcal{I}$, $\mathcal{A}$ is the power set of $\Omega$, and $P(T)$ is the known estimate of the relative frequency of $T$ in the input stream for each $T \in \Omega$. If $(\mathcal{G}, \oplus)$ is the set of all functions from $\Omega$ into $G$ with $\oplus$ defined by pointwise addition in the range space, then both $F$ and $f$ may be identified with elements of $\mathcal{G}$. The function $f$ is known while $F$ is not known. $H$ is a fixed subgroup of $G$. By Theorem 6.53, the fuzzy subset $\tilde{B}$ of $\mathcal{G}$ defined by $\tilde{B}(g) = P\{T \in \Omega \mid g(T) \in H\}$ is a function generated fuzzy subgroup of $\mathcal{G}$ with respect to $T_m$. Now $\tilde{B}(F)$ can be estimated by observing the output stream over some finite segment and computing the percentage of those outputs which are in $H$. Note that $\tilde{B}(\ominus f) = \tilde{B}(f)$ is known since $f$ is known. Now an output, $F(T)$, is $H$-$f$ faithful to $T$ if and only if $x + f(T) = F(T)$ has a solution for $x$ in $H$. This happens if and only if $x = F(T) - f(T) = (F \ominus f)(T) \in H$. Therefore $\tilde{B}(F \ominus f)$ is the probability that $F(T)$ is $H$-$f$ faithful to $T$. The solution to the original problem may now be identified with $\tilde{B}(F \ominus f)$. This may be estimated using $\tilde{B}(f)$, an estimate of $\tilde{B}(F)$, and the properties of the fuzzy subgroup, $\tilde{B}$, in the following way: Since $T_m(\tilde{B}(F), \tilde{B}(\ominus f)) = T_m(\tilde{B}(F), \tilde{B}(f)) = (\tilde{B}(F) + \tilde{B}(f) - 1) \vee 0 \geq \tilde{B}(F) + \tilde{B}(f) - 1$, we have

$$\tilde{B}(F \ominus f) \geq T_m(\tilde{B}(F), \tilde{B}(\ominus f)) = T_m(\tilde{B}(F), \tilde{B}(f)) \geq \tilde{B}(F) + \tilde{B}(f) - 1. \tag{6.4.1}$$

Similarly, since $T_m(\tilde{B}(F \ominus f), \tilde{B}(f)) = (\tilde{B}(F \ominus f) + \tilde{B}(f) - 1) \vee 0 \geq \tilde{B}(F \ominus f) + \tilde{B}(f) - 1$,we have

$$\tilde{B}(F) = \tilde{B}(F \ominus f + f) \geq T_m(\tilde{B}(F \ominus f), \tilde{B}(f)) \geq \tilde{B}(F \ominus f) + \tilde{B}(f) - 1. \tag{6.4.2}$$

Further, since $T_m(\tilde{B}(f \ominus F), \tilde{B}(F)) = T_m(\tilde{B}(F \ominus f), \tilde{B}(F)) = (\tilde{B}(F \ominus f) + \tilde{B}(F) - 1) \vee 0 \geq \tilde{B}(F \ominus f) + \tilde{B}(F) - 1$,

$$\tilde{B}(f) = \tilde{B}(f \ominus F \oplus F) \geq T_m(\tilde{B}(f \ominus F), \tilde{B}(F)) \geq \tilde{B}(F \ominus f) + \tilde{B}(F) - 1. \tag{6.4.3}$$

From (6.4.1) and (6.4.2), we obtain

$$\tilde{B}(F) - (1 - \tilde{B}(f)) \leq \tilde{B}(F \ominus f) \leq \tilde{B}(F) + (1 - \tilde{B}(f)). \tag{6.4.4}$$

From (6.4.1) and (6.4.3), we obtain

$$\tilde{B}(f) - (1 - \tilde{B}(F)) \leq \tilde{B}(F \ominus f) \leq \tilde{B}(f) + (1 - \tilde{B}(F)) \tag{6.4.5}$$

Thus we obtain the following estimate for the solution $\tilde{B}(F \ominus f)$ :

$$\mid \tilde{B}(F \ominus f) - (\tilde{B}(f) \wedge \tilde{B}(F)) \mid \leq 1 - (\tilde{B}(f) \vee \tilde{B}(F)). \tag{6.4.6}$$

This estimate is close only when $\tilde{B}(f)$ or $\tilde{B}(F)$ is close to 1. However, if $\tilde{B}$ can be shown to be a fuzzy subgroup with respect to $\wedge$, the situation changes considerably.

Suppose $\tilde{B}$ is a fuzzy subgroup of $G$ with respect to $\wedge$. Then (6.4.1), (6.4.2), and (6.4.3) become, respectively,

$$\tilde{B}(F \ominus f) \geq \tilde{B}(F) \wedge \tilde{B}(f) \tag{6.4.7}$$

$$\tilde{B}(F) \geq \tilde{B}(F \ominus f) \wedge \tilde{B}(f) \tag{6.4.8}$$

$$\tilde{B}(f) \geq \tilde{B}(F \ominus f) \wedge \tilde{B}(F) \tag{6.4.9}$$

Now if $\tilde{B}(f) > \tilde{B}(F)$, then from (6.4.7) we get $\tilde{B}(F \ominus f) \geq \tilde{B}(F)$ and from (6.4.8) we get $\tilde{B}(F) \geq \tilde{B}(F \ominus f)$. Thus $\tilde{B}(F \ominus f) = \tilde{B}(F)$. Similarly, if $\tilde{B}(f) < \tilde{B}(F)$, then from (6.4.7) we conclude that $\tilde{B}(F \ominus f) \geq \tilde{B}(f)$. From (6.4.9), we obtain $\tilde{B}(f) \geq \tilde{B}(F \ominus f)$ so that $\tilde{B}(F \ominus f) = \tilde{B}(f)$. Therefore if $\tilde{B}(f) \neq \tilde{B}(F)$, then $\tilde{B}(F \ominus f) = \tilde{B}(f) \wedge \tilde{B}(F)$ and we know the solution exactly. Finally, if $\tilde{B}(F) = \tilde{B}(f)$ the best one can say is $\tilde{B}(F) = \tilde{B}(f) \leq \tilde{B}$ $(F \ominus f) \leq 1$.

Let $n \in \mathbb{N}$. An $n \times n$ array of the integers $1, 2, 3, ..., n^2$ is called a *pattern*. Suppose that $F$ is a machine which accepts input patterns and produces output patterns. Each pattern may be identified with a transformation in $S_{n^2}$, the symmetric group on $n^2$ elements, in the following way.

$$\begin{matrix} k_1 & k_2 & \dots & k_n \\ k_{n+1} & k_{n+1} & \dots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n^2-n} & k_{n^2-n+1} & \dots & k_{n^2} \end{matrix} \quad \Leftrightarrow F,$$

where $F(i) = k_i$ for $i = 1, 2, \ldots, n^2$. Thus $F$ is identified with a function from $S_{n^2}$ into $S_{n^2}$. An output pattern will be called *recognizable* if it is a composition of translations and rotations of the input. There is a subgroup $H$ of $S_{n^2}$ such that an output pattern $F(T)$ is recognizable if and only if there exists a permutation $T^*$ in $H$ such that $T^* \circ T = F(T)$. Suppose that estimates of the relative frequency of patterns in the input stream can be obtained. Let $\Omega = S_{n^2}$, $\mathcal{A}$ be the power set of $S_{n^2}$, and $P$ be a probability measure on $S_{n^2}$ obtained from the estimates of the relative frequency of input patterns. Now $(\Omega, \mathcal{A}, P) = (S_{n^2}, \mathcal{A}, P)$ and $(G, +) = (S_{n^2}, \circ)$ with $(\mathcal{G}, \oplus)$ and $H$ defined appropriately. Let $f_1 : S_{n^2} \to S_{n^2}$ be defined by $f_1(T) = T$ for every $T \in S_{n^2}$. Then the output pattern $F(T)$ is recognizable if and only if the equation $x \circ f_1(T) = F(T)$ has a solution for $x$ in $H$. This is the definition of $F(T)$ being $H$-$f_1$ faithful to $T$. Note that

$\tilde{B}(f_1) = P\{T \in S_{n^2} \mid f_1(T) = T \in H\} = P(H)$.

From the discussion of the generalized recognition problem, the probability the output is recognizable ($h$-$f_1$ faithful) is $\tilde{B}(F \ominus f_1)$ which may be estimated using the inequality

$$\left|\tilde{B}(F \ominus f_1) - (P(H) \wedge \tilde{B}(F))\right| \leq 1 - (P(H) \vee \tilde{B}(F)).$$

Once again, in the event $\tilde{B}$ can be shown to be a fuzzy subgroup with respect to minimum, we obtain

$\tilde{B}(F \ominus f) = P(H) \wedge \tilde{B}(F)$
if $P(H) \neq \tilde{B}(F)$; otherwise $P(H) = \tilde{B}(F) \leq \tilde{B}(F \ominus f) \leq 1$. It should be remembered that $P(H)$ is known and $\tilde{B}(F)$ can be estimated by the percentage of outputs which are in $H$.

We now consider the standard problem concerning the transmission of strings of 0's and 1's across a symmetric binary channel with noise. Let $B = \{0, 1\}$ and $B^n$ denote the set of all binary $n$-tuples, $n \geq 2$. Then $B^n$ is a group under componentwise addition modulo 2. Let $C \subseteq B^n$ denote the set of all codewords. Then $C$ is a subgroup of $B^n$. We make the following identifications: $C = H = \mathcal{I} = \Omega$ and $B^n = G = \mathcal{O}$ where $H$, $G$, $\mathcal{I}$, $\Omega$ and $\mathcal{O}$ are as described above. In this situation, $f$ is known, $f(T)$ is unknown, $F$ is unknown, and $F(T)$ is known where $T \in H$. We let $f$ be the identity map since in the ideal situation their is no noise and so the output equals the input. We recall that $F(T)$ is observable and $\tilde{B}(F)$ can be estimated. Since $f$ is the identity map, $\tilde{B}(f) = 1$. Thus, $\tilde{B}(F \ominus f) = \tilde{B}(F)$ by inequality (6.4.1). Thus $F(T)$ is $H$-$f$ faithful to $T$.

We now consider a more general situation. Assume that $1 \geq \tilde{B}(f) > \tilde{B}(F)$. This is a reasonable assumption since $f$ represents the ideal situation while $F$ represents the real world situation. Then by inequality (6.4.1), $|\tilde{B}(F \ominus f) - (\tilde{B}(f) \wedge \tilde{B}(F))| \leq 1 - (\tilde{B}(f) \vee \tilde{B}(F))$ and so $|\tilde{B}(F \ominus f) - \tilde{B}(F)| \leq 1 - \tilde{B}(f)$. Now assume that $\forall\, s, t \in \operatorname{Im}(\tilde{B})$, $s \neq t$, $1 - \tilde{B}(f) < |s - t|$. Then $|\tilde{B}(F \ominus f) - \tilde{B}(F)| = 0$. Hence $\tilde{B}(F \ominus f) = \tilde{B}(F)$. Once again, $F(T)$ is $H$-$f$ faithful. Consider the fuzzy coset $\tilde{B}_f$ where $\tilde{B}_f(g) = \tilde{B}(g \ominus f)$ $\forall\, g \in \mathcal{G}$. Then $\tilde{B}_f(F) = \tilde{B}(F)$. Also, $g \in (\tilde{B}_f)^t \Leftrightarrow \tilde{B}_f(g) \geq t \Leftrightarrow \tilde{B}(g \ominus f) \geq t \Leftrightarrow g \ominus f \in \tilde{B}^t \Leftrightarrow g \in f \oplus \tilde{B}^t$.

We now note a structure result for the group $(\mathcal{G}, \oplus)$ and the fuzzy subgroup $\tilde{B}$ of $\mathcal{G}$. $\forall\, f \in \mathcal{G}, (g \oplus g)(T) = g(T) \oplus g(T) = 0\ \forall\, T \in \mathcal{I}$. Thus, $2\mathcal{G} = \{\theta\}$ where $\theta(T) = 0\ \forall\, T \in \mathcal{I}$. Thus $\mathcal{G} = \bigoplus_{g \in \mathcal{G}} < g >$. For all $g \in \mathcal{G}$, define the fuzzy subset $\tilde{B}^{(g)}$ of $\mathcal{G}$ as follows: $\tilde{B}^{(g)}(\theta) = \tilde{B}(\theta)$ which equals 1, $\tilde{B}^{(g)}(g) = \tilde{B}(g)$ if $g \in \operatorname{supp}(\tilde{B})$ and $\tilde{B}^{(g)}(h) = 0$ if $h \in \mathcal{G} \setminus \{\theta, g\}$. Then $\tilde{B}^{(g)}$ is a fuzzy subgroup of $\mathcal{G}$. Hence $\tilde{B} = \bigoplus_{g \in S} \tilde{B}^{(g)}$ for some subset $S \subseteq \operatorname{supp}(\tilde{B})$ where $|S| = [\operatorname{supp}(\tilde{B}) : \mathbb{Z}_2]$ by [37, Theorem 2.3, p. 96].

## 6.5 Free Fuzzy Monoids and Coding Theory

Semigroup theory has found a wide range of applications. It is used, for example, in the study of combinatorics, algebraic linguistics, and automata theory. In this section, we describe the use of free monoids and fuzzy free monoids in the construction of codes. The material is taken from [21] and [13].

Let $A$ be a set, $M$ be a monoid, and $f : A \to M$ be an injection of $A$ onto a set of generators of $M$. We recall from Section 6.1 that $M$ is free on

$f(A)$ if and only if for every monoid $M'$ and for every mapping $g : A \to M'$ there exists a homomorphism $h : M \to M'$ such that $g = h \circ f$.

For subsets $A$ and $B$ of a semigroup, we let $AB = \{ab \mid a \in A. b \in B\}$. Then for all $n \in \mathbb{N}$, $A^n = \{a_1 \ldots a_n \mid a_i \in A, i = 1, \ldots, n\}$.

**Proposition 6.59** *Let $M$ be monoid and let $S = M \setminus \{e\}$, where $e$ is the identity of $M$. Then $M$ is free if and only if every element of $S$ has a unique factorization as a product of elements of $S \setminus S^2$.*

**Proof.** Suppose that $M$ is free. Then $M = A^*$, where $A = S \setminus S^2$. Thus the unique factorization property holds by definition of $A^*$. Conversely, suppose that every element of $M \setminus \{e\}$ has a unique factorization as a product of elements of $A = S \setminus S^2$. Then there is a homomorphism $h$ of $A^*$ onto $M$ such that $h(a_1a_2...a_n) = a_1a_2...a_n$. Since every element of $S$ has a unique factorization as a product of elements of $A$, $h$ is an isomorphism. Thus $M$ is free. ■

**Definition 6.40** *A monoid $M$ is called* equidivisible *if for every $a, b, c, d \in M$, $ab = cd$ implies either $a = cu, ub = d$ for some $u \in M$ or $av = c, b = vd$ for some $v \in M$.*

**Proposition 6.60** *Let $M$ be a monoid and let $S = M \setminus \{e\}$. The $M$ is free if and only if $M$ is equidivisible and $\bigcap_{n \in \mathbb{N}} S^n = \emptyset$.* ■

**Corollary 6.61** *Let $M$ be a monoid. Then $M$ is free if and only if $M$ is equidivisible and there exists a homomorphism $h$ from $M$ into the monoid $(\mathbb{N} \cup \{0\}, +)$ such that $h^{-1}(0)$ is the identity of $M$.* ■

Let $M$ be a monoid and let $U = \{x \in M \mid \exists x' \in M$ such that $xx' = e = x'x\}$. Then $U$ is a group, called the *group of units* of $M$. $U$ is called *trivial* if $U = \{e\}$. $M$ is called *cancellative* if for all $a, b, c \in M$, $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$.

**Corollary 6.62** *Let $M$ be a monoid. Then $M$ is free if and only if $M$ is cancellative, equidivisible, has a trivial group of units and every $m \in M, m \neq e$, has a finite number of nontrivial left factors (factors $\neq e$).* ■

**Example 6.24** *Let $X$ be a nonempty set and $\mathbb{R}^+$ be the set of all nonnegative real numbers. Let $F(\mathbb{R}^+, X) = \{(f, r) \mid f : (0, r] \to X, r \in \mathbb{R}^+\}$. Define a binary operation $*$ on $F(\mathbb{R}^+, E)$ as follows: $(f, r) * (g, s) = (h, r + s)$, where $h(x) = f(x)$ if $x \in (0, r]$ and $h(x) = g(x - r)$ if $x \in (r, r + s]$. Then it can be shown that $F(\mathbb{R}^+, E)$ is a monoid under $*$ which is cancellative, equidivisible, and has a trivial group of units. Let $(h, t) \in F(\mathbb{R}^+, E)$ and $r \in (0, t)$. Then $(h, t) = (f, r) * (g, t)$, where $f(x) = h(x)$ if $x \in (0, r]$ and $g(x) = h(r + x)$ if $x \in (0, t - r]$. Then $(h, t)$ has infinitely many left factors. By Corollary 6.62, $F(\mathbb{R}^+, E)$ is not free.*

We sometimes call a set $A$ an *alphabet* and its elements *letters*. The elements of $A^*$ are called *words*.

**Proposition 6.63** *Let $M$ be a submonoid of a free monoid $A^*$ on the alphabet $A$. Then $M$ has a unique minimal set of generators $C = M^+ \setminus (M^+)^2$, where $M^+ = M \setminus \{e\}$. $C$ is called the* base *of $M$.*

**Proof.** It follows from Proposition 6.60 that $\bigcap_{n \in \mathbb{N}} (M^+)^n = \emptyset$. Thus for all $m \in M^+$, there exists $k \in \mathbb{N}$ such that $m \in (M^+)^k \setminus (M^+)^{k+1}$. Hence $m$ can be written as a product $c_1 c_2 ... c_k$ with $c_i \in M^+ \setminus (M^+)^2$ for $i = 1, ..., k$. Therefore $C$ is a set of generators of $M$. Let $C' \subseteq M^+$ be another set of generators of $M$. Then every $c \in C$ can be written $c = c'_1 c'_2 ... c'_n$ with $c'_i \in C' \subseteq M^+$. Since $c \in M^+ \setminus (M^+)^2$, $n = 1$. Hence $c \in C'$. Thus $C \subseteq C'$ and so $C$ is a minimal set of generators and is in fact unique with respect to this property. ■

**Example 6.25** *Let $A = \{a, b\}$. Then $M = \{a^i \mid i \in \mathbb{N} \cup \{0\}, i \neq 1\}$ is a submonoid of $A^*$ and $C = \{a^2, a^3\}$ is the base of $M$. Now $M$ is not free over $C$ since $a^6$ has a factorization as a product of elements of $C$ in two different ways, namely $a^2 a^2 a^2$ and $a^3 a^3$. By Proposition 6.59, $M$ is not free.*

**Proposition 6.64** *Let $M$ be a submonoid of a free monoid $A^*$. Then the following conditions are equivalent.*

**(i)** *$M$ is free.*

**(ii)** *$\forall w \in A^*$, $Mw \cap M \neq \emptyset$ and $wM \cap M \neq \emptyset$ imply $w \in M$.*

**(iii)** *$\forall w \in A^*, Mw \cap M \cap wM \neq \emptyset$ imply $w \in M$.* ■

**Definition 6.41** *Let $C$ be a subset of a free monoid $A^*$. Then $C$ is called a* code *over $A$ if $C$ is the base of a free submonoid $M$ of $A^*$. We write $M = C^*$ in this case.*

Let $C$ be any subset of $A^*$ and $B$ a set such that there exists a bijection $f : B \to C$. Then $f$ can be extended to a homomorphism $f^*$ of $B^*$ onto the submonoid $M$ of $A^*$ generated by $C$. Now $C$ is a code if and only if $f^*$ is an isomorphism. In this case, $f$ is called an *encoding* of $B$ in the alphabet $A$. The next example shows that it is not always easy to detect whether or not a subset of $A^*$ is a code.

**Example 6.26** *Let $A = \{a, b\}$ and $C = \{a^4, b, ba^2, ab, aba^2\}$. Then $C = \{a^4\} \cup \{a^i b a^j \mid i = 0, 1$ and $j = 0, 2\}$. Now $a^m \in C^*$ if and only if $m \equiv_4 0$. If a word $w$ contains occurrences of $b$ we write $w = a^{i_1} b a^{i_2} b ... a^{i_k} b a^{i_{k+1}}$ with $i_1, i_2, ..., i_{k+1} \geq 0$. If $w \in C^*$, then $i_1 \equiv_4 0$ or $1$ and $i_{k+1} \equiv_4 0$ or $2$. Conversely, if $i_1 \equiv_4 0$ or $1$ and $i_{k+1} \equiv_4 0$ or $2$, then $w \in C^*$. We now* *consider the uniqueness of the expressions.* $\forall j, 1 \leq j \leq k$, *the equivalence class of* $i_j$ *modulo 4 can be uniquely decomposed as a sum* $i_j = 4k_j + l_j + m_j$, *where* $l_j \in \{0, 1\}$, $m_j \in \{0, 2\}$. *So* $w = \ldots(a^{l_{j-1}}ba^{m_j})(a^4)^{k_j}(a^{l_j}ba^{m_{j+1}})\ldots$. *Since this decomposition of* $w \in C^*$ *is unique, we have that* $C$ *is a code.*

We note that the equivalence (ii) and (iii) in Proposition 6.64 is true for a submonoid of any monoid $N$ even if $N$ is not free. We call a submonoid $M$ of a monoid $N$ *weakly unitary* if it satisfies (ii) or (iii) of Proposition 6.64. For the proof of (iii) implies (ii), we have that $m_1 w \in M$ and $wm_2 \in M$ implies that $w(m_2 m_1 w) = (wm_2 m_1)w \in M$ and so $w \in M$ by (iii).

The next result provides a simple method for constructing examples of codes.

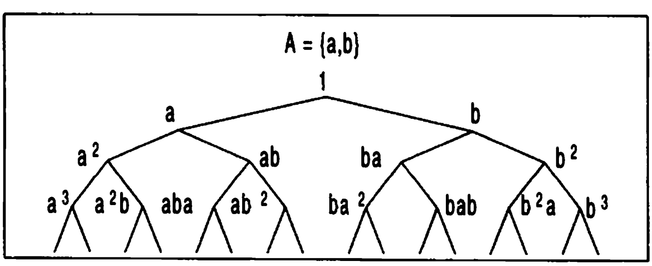
**Proposition 6.65** *Let* $M$ *be a submonoid of a free monoid* $A^*$ *and* $C$ *its base. Then the following conditions are equivalent.*

**(i)** $\forall w \in A^*, Mw \cap M \neq \emptyset$ *implies* $w \in M$.

**(ii)** $CA^+ \cap C = \emptyset$. ■

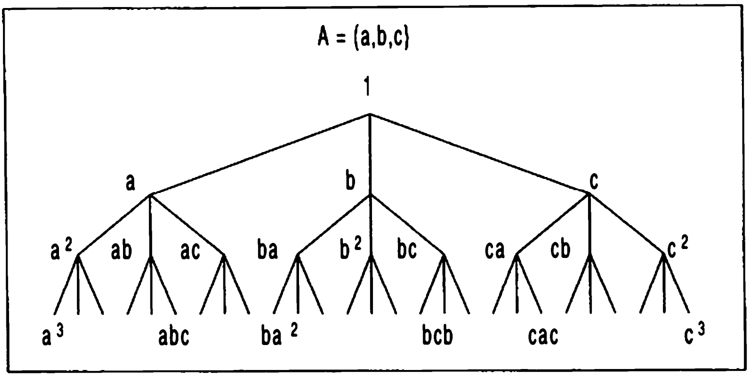
**Definition 6.42** *A code* $C$ *over the alphabet* $A$ *is called a* prefix (suffix) *code if it satisfies* $CA^+ \cap C = \emptyset$ $(A^+C \cap C = \emptyset)$. $C$ *is called a* biprefix *code if it is a prefix and a suffix code. A submonoid* $M$ *of any monoid* $N$ *satisfying (ii) of Proposition 6.65 is called* left unitary *in* $N$. $M$ *is called* right unitary (unitary) *in* $N$ *if it satisfies the dual of (i), namely* $A^+C \cap C = \emptyset$, *(both (i) and its dual).*

By condition (ii) in Proposition 6.65, no word of $C$ is a proper left factor of another word of $C$. Define the relation $\leq_l$ on $A^*$ by $\forall u, v \in A^*$, $u \leq_l v$ if $v$ is a left factor of $u$. Then $\leq_l$ is a partial ordering of $A^*$. In the diagrams to follow, we display the top part of $A^*$ partially ordered by $\leq_l$ when $A = \{a, b\}$ and when $A = \{a, b, c\}$.



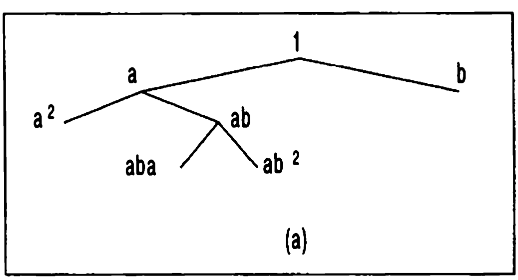

$A^*$ partially ordered by $\leq_l$ when[1] $A = \{a, b\}$

[1]Semigroups and Combinatorial Applications, G. Lallement, Copyright 1979. Reprinted by permission of John Wiley & Sons, Inc




$A^*$ partially ordered by $\leq_l$ when[1] $A = \{a, b, c\}$

A necessary and sufficient condition for a subset $C$ of $A^*$ to be a prefix code is that for every $c \in C$, $w \in A^*, w \leq_l c$ and $w \neq c$ implies $w \notin C$. Thus to obtain examples of prefix codes, it suffices to select subsets $C$ of $A^*$ that will be end points for $\leq_l$ . For example, the trees displayed below give the prefix codes $C_1 = \{a^2, aba, ab^2, b\}$ over $\{a, b\}$ and $C_2 = \{a^2, ab, ac, ba, b^2, cb, c^2\}$ over $\{a, b, c\}$. The set $B = \{a^n b \mid n \in \mathbb{N}\}$ is an example of an infinite prefix code over $\{a, b\}$ and is represented by an infinite falling tree with end points $a^n b$, $n \in \mathbb{N}$.




Prefix codes over[1] $\{a, b\}$

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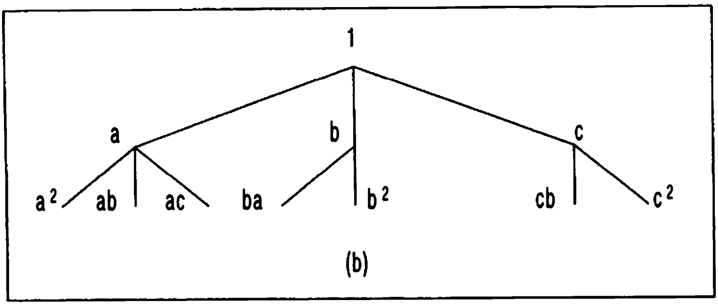


Prefix codes over[1] $\{a, b, c\}$

There is no simple characterization of general codes analogous to condition (ii) of Proposition 6.65 for prefix codes. However, we do have the following result.

**Proposition 6.66** *Let $A^*$ be a free monoid and $C$ be a subset of $A^*$. Define the sequence of subsets $D_i$ of $A^*$ recursively by $D_0 = C$ and $D_i = \{w \in A^+ \mid D_{i-1}w \cap C \neq \emptyset$ or $Cw \cap D_{i-1} \neq \emptyset\}$, $i = 1, 2, \ldots$. Then $C$ is a code over $A$ if and only if $C \cap D_i = \emptyset$ for $i = 1, 2, \ldots$.* ■

Suppose that $C$ is finite. Then the length of the words in $D_i$ for every $i$ is bounded by the maximal length of words in $C$. Hence, there is only a finite number of distinct $D_i$ and Proposition 6.66 gives an algorithm for deciding whether or not $C$ is a code.

**Example 6.27** **(i)** *For $C = \{a, a^3b, aba, \}$, we have $D_0 = C, D_1 = \{a^2b, ba\}, D_2 = \{ab\}$, and $D_3 = \{a, b\}$. Since $C \cap D_3 \neq \emptyset$, $C$ is not a code.*

**(ii)** *For $C = \{a, a^2b, bab, b^2\}$, we have $D_0 = C, D_1 = \{ab\}, D_2 = \{b\}$, and $D_i = \{ab, b\}$ for $i = 3, 4, \ldots$. Since $C \cap D_i = \emptyset$ for $i = 1, 2, \ldots$, $C$ is a code.*

We now consider the construction of codes using fuzzy subsemigroups. The material is taken from Gerla [13]. Let $L$ be a partially ordered set. Then $L$ is called a $\wedge$*-semilattice* ($\vee$*-semilattice*) if $\forall x, y \in L$, $x$ and $y$ have a greatest lower bound (least upper bound), say $x \wedge y$ ($x \vee y$). A $\wedge$-semilattice ($\vee$-semilattice) is called *complete* if for every subset of $L$ has greatest lower bound (least upper bound) in $L$. Let $B^+$ denote a free semigroup and $L$ be a complete semilattice. Then $\{\tilde{A} \mid \tilde{A} : B^+ \to L\}$ is a semilattice whose elements are $L$-subsets of the free semigroup $B^+$. An $L$-subset $\tilde{A}$ of $B^+$ is an $L$-subsemigroup of $B^+$ if for $t \in L$, the level set $\tilde{A}^t = \{x \in B^+ \mid \tilde{A}(x) \geq t\}$

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is a subsemigroup of $B^+$. Then $\tilde{A}$ is a $L$-subsemigroup of $B^+$ if $\forall x, y \in B^+$, $\tilde{A}(xy) \geq \tilde{A}(x) \wedge \tilde{A}(y)$.

The search for suitable codes for communication theory is known. It was proposed by Gerla that $L$-semigroup theory be used. To this end, free, pure, very pure, left unitary, right unitary, unitary $L$-subsemigroups of a free semigroup are defined. To every such $L$-subsemigroup there is a family of codes associated with it. An $L$-subsemigroup of a free semigroup is *free, pure, very pure, left unitary, right unitary, unitary* if and only if its level sets are free, pure, very pure, left unitary, right unitary, unitary, respectively. Thus any method used to construct an $L$-subsemigroup of a free semigroup of one of these types yields a family of semigroups of the same type, namely the level sets of the $L$-subsemigroup.

**Proposition 6.67** *Let $\tilde{A}$ be an $L$-subsemigroup of $B^+$. Then $\tilde{A}$ is free, pure, very pure, left unitary, right unitary, or unitary if and only if $\forall x, y \in B^+$*

**(i)** $\tilde{A}(x) \geq \tilde{A}(yx) \wedge \tilde{A}(xy) \wedge \tilde{A}(y)$,

**(ii)** $\tilde{A}(x) = \tilde{A}(x^n)\ \forall n \in \mathbb{N}$,

**(iii)** $\tilde{A}(x) \geq \tilde{A}(yx) \wedge \tilde{A}(xy)$,

**(iv)** $\tilde{A}(x) \geq \tilde{A}(yx) \wedge \tilde{A}(y)$,

**(v)** $\tilde{A}(x) \geq \tilde{A}(xy) \wedge \tilde{A}(y)$,

**(vi)** $\tilde{A}(x) \geq \tilde{A}(yx) \wedge \tilde{A}(xy)$ *and* $\tilde{A}(x) \geq \tilde{A}(xy) \wedge \tilde{A}(y)$,

*respectively.* ■

**Proposition 6.68** *If $L$ is a $\wedge$-complete semilattice, then the class of free (pure, very pure, left unitary, right unitary, or unitary) $L$-subsemigroups of $B^+$ is closed with respect to finite and infinite intersections.* ■

A closure system for a semigroup $S$ is an $\bigcap$-complete class $C$ of subsemigroups of $S$ such that $S \in C$. For any subset $X$ of $S$, let $\langle X \rangle = \bigcap \{M \in C \mid M \supseteq X\}$. Every closure system $C$ is a semilattice with respect to the operation $\vee$ defined by setting $X \vee Y = < X \cup Y >\ \forall X, Y \in C$. It is important to note that the order relation associated with $\vee$ is the dual of the ordinary inclusion.

**Proposition 6.69** *Let $S$ be a semigroup, $C$ a closure system for $S$, and $g$ a homomorphism of $B^+$ into $S$. Then the function $\tilde{A} : B^+ \to C$ defined by $\forall x \in B^+, \tilde{A}(x) = < g(x) >$ is a $C$-subsemigroup of $B^+$. If $\forall x, y \in B^+$,*

**(i)** $x \in < yx, xy, y >$,

**(ii)** $x \in < x^n > \forall n \in \mathbb{N}$,

**(iii)** $x \in < yx, xy >$,

**(iv)** $x \in < yx, y >$,

**(v)** $x \in < xy, y >$,

**(vi)** $x \in < yx, y > \cap < xy, y >$,

*then $\tilde{A}$ is free, pure, very pure, left unitary, right unitary, or unitary, respectively. Moreover, $\tilde{A}^X = g^{-1}(X)$ for every $X \in C$.* ■

**Corollary 6.70** *There exists an L-subsemigroup $\tilde{A}$ of $B^+$ such that $\{\tilde{A}^X \mid X \in L\}$ is the family of all free (pure, very pure, left unitary, right unitary, or unitary) subsemigroups of $B^+$.* ■

**Corollary 6.71** *Let $S$ be a semigroup, $C$ a class of subsemigroups of $S$, and $\bar{C}$ the closure system generated by $C$. Then if $g : B^+ \to S$ is any homomorphism, the function $\tilde{A}$ defined by $\tilde{A}(x) = < g(x) > \forall x \in B^+$ is a unitary $\bar{C}$-subsemigroup of $B^+$.* ■

Methods to construct examples can be found in [13].

## 6.6 Formal Power Series, Regular Fuzzy Languages, and Fuzzy Automata

We now describe an approach for the construction of fuzzy automata using formal power series representations. The formal power series approach yields a minimal fuzzy automata. The results of this section are from [51, 53].

**Definition 6.43** *A mathematical system $(A, +, \cdot)$ is called a* semiring *if (i) $(A, +)$ is a commutative monoid, (ii) $(A, \cdot)$ is a monoid, (iii) $\forall a, b, c \in A$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$, and $a \cdot 0 = 0 = 0 \cdot a$ $\forall a \in A$. (We let 0 denote the "additive" identity of $(A, +)$ and 1 denote the "multiplicative" identity of $(A, \cdot), 0 \neq 1$.)*

If $(A, +, \cdot)$ is a semiring, we sometimes write $ab$ for $a \cdot b$, where $a, b \in A$.

**Definition 6.44** *A commutative monoid $(D, +)$ is called an $A$-*semimodule *over the semiring $(A, +, \cdot)$ with respect to $\cdot : A \times D \to D$ if the following conditions hold:*

**(i)** *$\forall a, b \in A$ and $\forall d \in D$, $(ab) \cdot d = a \cdot (b \cdot d)$.*

(ii) $\forall a, b \in A$ *and* $\forall d_1, d_2 \in D$,

$$a \cdot (d_1 + d_2) = (a \cdot d_1) + (a \cdot d_2) \text{ and } (a + b) \cdot d_1 = (a \cdot d_1) + (b \cdot d_1)$$

(iii) $\forall d \in D$, $1 \cdot d = d$ *and* $0 \cdot d = 0$.

**Definition 6.45** *Let* $(D, +)$ *be an* $A$*-semimodule over the semiring* $(A, +, \cdot)$. *Let* $U$ *be a nonempty subset of* $D$. *Then* $(U, +)$ *is called a* subsemimodule *of* $D$ *if* $(U, +)$ *is an* $A$*-semimodule.*

**Proposition 6.72** *Let* $(D, +)$ *be an* $A$*-semimodule over the semiring* $(A, +, \cdot)$. *Let* $U$ *be a nonempty subset of* $D$. *Then* $(U, +)$ *is a subsemimodule of* $D$ *if and only if* $\forall u_1, u_2 \in U$ *and* $\forall a, b \in A, au_1 + bu_2 \in U$. ■

Let $(D, +)$ be an $A$-semimodule over the semiring $(A, +, \cdot)$. Let $U$ be a nonempty subset of $D$. Then the intersection of all subsemimodules of $D$ which contain $U$ is a subsemimodule of $D$ and is called the subsemimodule *generated* by $U$. We let $< U >$ denote this submodule. It is the smallest subsemimodule of $D$ containing $U$. It can be shown that $< U >= \{\sum_{i=1}^{n} a_i u_i \mid a_i \in A, u_i \in U, i = 1, ..., n; n \in \mathbb{N}\}$.

Let $A^{m \times m}$ be the set of all $m \times m$ matrices with elements from a semiring $A$. Then $A^{m \times m}$ is a semiring under the usual definitions of addition and multiplication. Let $M$ be a monoid. Then the multiplicative homomorphism $\mu : M \rightarrow A^{m \times m}$ is called a *representation* if $\forall w_1, w_2 \in M, \mu(w_1 w_2) = \mu(w_1)\mu(w_2)$.

**Definition 6.46** *Let* $M$ *be a monoid and* $A$ *be a semiring. A function* $r$ *of* $M$ *into* $A$ *is called* formal power series, *and* $r$ *is written as a formal sum*

$$r = \sum_{w \in M} (r, w) w. \tag{6.6.1}$$

*The values of* $(r, w) \in A$ *are also referred to as the* coefficients *of the series.*

We will consider here the free monoid $V_T{}^*$ generated by the words over an alphabet $V_T$ and $r$ will be a series with noncommuting variables in $V_T$.

Let $M$ be a monoid. The collection of all formal power series $r$, as defined above, is denoted by $A[[M]]$. For $r \in A[[M]]$, the set $\{w \in M \mid (r, w) \neq 0\}$ is called the *support* of $r$ and is denoted by supp$(r)$. A subset of $V_T{}^*$ is called a *language*. A language may be uniquely associated with a formal power series $r$ belonging to $A[[V_T{}^*]]$. The elements of the support of $r$, $r \in A[[V_T{}^*]]$, are the words $w \in V_T{}^*$ such that $(r, w) \neq 0$, and hence supp$(r)$ may be considered as a language over the alphabet $V_T{}^*$.

**Definition 6.47** *The elements of* $A[[M]]$ *consisting of all series with finite support are referred to as* polynomials. *We let* $A[M]$ *denote the set of all polynomials.*

We let 0 denote the series all of whose coefficients equal 0. We let $\lambda$ denote the identity of $V_T{}^*$. Then $\forall a \in A$, $a\lambda = a$. If $w \in V_T{}^*$, then $aw$ denotes the series whose coefficient of $w$ is $a$ and the remaining coefficients are 0. Then $aw \in A[V_T{}^*]$ and $aw$ is called a *monomial.*

The support of a series in $A[[V_T{}^*]]$ is a language over the alphabet $V_T{}^*$. A series $r \in A[[V_T{}^*]]$, where every coefficient equals 0 or 1, is called the *characteristic series* of its support $L$, and written $r = \text{char}(L)$.

For the purpose of inference of regular grammars, we require only rational series and recognizable series.

Define the function $d : A[[M]] \times A[[M]] \to \mathbb{R}$ by $\forall r, r' \in A[[M]]$,

$$d(r, r') = \begin{cases} 0 & \text{if } r = r' \\ 2^{-l} & \text{if } r \neq r' \end{cases}$$

where $l = \wedge\{lg(w) \mid (r, w) \neq (r', w), w \in M\}$ and $lg : M \to \mathbb{N}$. Then $d$ is a metric on $A[[M]]$. We can thus discuss convergence of sequences of elements of $A[[M]]$ with respect to $d$.

We now illustrate some of the concepts introduced up to now. Let $M = \{x^i \mid i = 0, 1, 2, ...\}$. Define the binary operation $\cdot$ on $M$ by $\forall x^i, x^j \in M$, $x^i \cdot x^j = x^{i+j}$. Then $M$ is a monoid under $\cdot$ with identity $x^0 = 1$. Then, using more familiar notation, $\sum_{w \in M}(r, w)w$ becomes $\sum_{i=0}^{\infty}(r, x^i)x^i$. Define $lg : M \to \mathbb{N}$ by $lg(x^i) = i$ $\forall x^i \in M$. If

$r = \sum_{i=0}^{\infty}(r, x^i)x^i$ and $r' = \sum_{i=0}^{\infty}(r', x^i)x^i$

with $(r, x^i) = (r', x^i)$ for $i = 0, 1, ..., k$ and $(r, x^{k+1}) \neq (r', x^{k+1})$, then $l = k + 1$.

**Definition 6.48** *An element $r$ of $A[[M]]$ is called* quasi-regular *if $(r, \lambda) = 0$, where $\lambda \in M$ is the null string. The quasi-regular series has the property that the sequence $r, r^2, ..., r^n, ...$ converges to 0 and*

$\lim_{n \to \infty} \sum_{k=1}^{n} r^k$

*exists. If $r$ is quasi-regular, then the series*

$r^+ = \sum_{k \geq 1} r^k$

*is called a* quasi-inverse *of $r$.*

**Definition 6.49** *A subsemiring $Q(A[[M]])$ is called* rationally closed *if it contains the quasi-inverse of every quasi-regular element.*

The family of $A$-rational series over $M$, denoted by $A^{rat}[[M]]$, is the smallest rationally closed subset of $A[[M]]$ containing all polynomials, i. e., $A^{rat}[[M]] \supseteq A[M]$. A series of $A[[M]]$ is termed $A$-recognizable, i.e., $r \in A^{rec}[[M]]$ if

$r = (r, \lambda)\lambda + \sum_{w \neq \lambda} p(\mu w)w,$

where $\mu : M \to A^{m \times m}$, $m \neq 1$, is a representation. The value $p(a_{ij})$ can be expressed as a linear combination of the entries $a_{ij}$ with coefficients in $A$, i. e.,

$$p(a_{ij}) = \sum_{i,j} a_{ij}p_{ij}.p_{ij} \in A. \tag{6.6.2}$$

**Example 6.28** *Consider* $\mathbb{N}[[X^*]]$, *where* $X = \{x, \bar{x}\}$. *The series*
$r = \sum_{n=1}^{\infty} 2^n (x\bar{x})^n x + 3x = (2x\bar{x})^+ x + 3x$
*is* $\mathbb{N}$*-rational.* $Supp(r)$ *is denoted by the regular expression*
$(x\bar{x})^+ x \cup x$.
*Consider the representation* $\mu$ *defined by*
$\mu(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\mu(\bar{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
*Let* $p$ *be the function defined by*
$p(a_{ij}) = a_{11} + a_{12}$.
*Then the* $\mathbb{N}$*-recognizable series*
$r' = \sum_{w \in X^*} p(\mu w) w$
*can be written in the form*
$r' = \sum_{n=0}^{\infty} a_n x^n$,
*where the sequence* $a_0, a_1, ..., a_n, ...$ *constitutes the Fibonacci sequence. To see this, let*
$\mu(x^n) = \begin{bmatrix} a_{n-1} & a_n \\ a_n & a_{n+1} \end{bmatrix}$
*where* $a_0 = 0, a_1 = 1, a_2 = 1$, *and* $a_k = a_{k-1} + a_{k-2}$ *for* $k = 2, 3, ....$ *Then*
$p(\mu(x^n)) = a_{n-1} + a_n$ *for* $n = 1, 2, ....$
*(We recall that the Fibonacci sequence is defined by* $a_1 = a_2 = 1$ *and* $a_n = a_{n-2} + a_{n-1}$ *for* $n = 3, 4, ....$*)*

**Example 6.29** *Consider* $\mathbb{N}[[X^*]]$, *where* $X = \{x, \bar{x}\}$, *and two sequences of polynomials* ${r_1}^{(i)}$ *and* ${r_2}^{(i)}$, $i = 0, 1, 2, ...$, *defined as follows:*
${r_1}^{(0)} = {r_2}^{(0)} = 0$,
${r_1}^{(i+1)} = {r_2}^{(i)} + {r_1}^{(i)}\,{r_2}^{(i)}$,
${r_2}^{(i+1)} = x{r_1}^{(i)}\bar{x} + \lambda$, *for all* $i \geq 0$.
*Then both of the sequences* ${r_1}^{(i)}$ *and* ${r_2}^{(i)}$ *converge and the limit of the former sequence is the characteristic series of the Dyck language (on 2 letters) over* $X$. *We recall that the Dyck language is given by the following grammar:* $(\{S\}, \{x, \bar{x}\}, S, \{S \to SS, S \to \lambda, S \to xS\bar{x}\})$, *where* $\{S\}$ *is the set of nonterminals,* $\{x, \bar{x}\}$ *is the set of terminals,* $S$ *is the initial symbol, and* $\{S \to SS, S \to \lambda, S \to xS\bar{x}\}$ *is the set of productions.*

The following theorem provides a convenient characterization of recognizable power series.

**Theorem 6.73** *(Schutzenberger) [54] If* $r \in A^{rec}[[M]]$, *then there exists a row vector* $\alpha$, *a representation* $\mu$, *and a column vector* $\beta$ *such that*

$$r = \sum_{w \in M} (\alpha(\mu w)\beta) w. \tag{6.6.3}$$

*Conversely, any series of the form*
$\sum_{w\in M}(\alpha(\mu w)\beta)w$
*belongs to* $A^{rec}[[M]]$. ■

**Example 6.30** *Consider* $\mathbb{N}[[X^*]]$, *where* $X = \{x,\bar{x}\}$. *Let the representation* $\mu$ *be defined as in Example 6.28. Let* $\alpha = (1,0)$ *and* $\beta = (1,1)^T$, *the transpose of* $(1,1)$. *Then* $\alpha\mu(x^n)\beta = (\alpha\mu(x^n))\beta = (a_{n-1}, a_n)(1,1)^T = (a_{n-1} + a_n)$. *This gives us once again the series* $r' = \sum_{n=0}^{\infty} a_n x^n$, *where the sequence* $a_0, a_1, \ldots, a_n, \ldots$ *constitutes the Fibonacci sequence.*

**Theorem 6.74** *(Kleene-Schutzenberger) [54] For the free monoid* $V_T{}^*$, *the sets* $A^{rec}[[V_T{}^*]]$ *and* $A^{rat}[[V_T{}^*]]$ *coincide.* ■

We now define Hankel matrices. They can be used to characterize rational power series.

**Definition 6.50** *The Hankel matrix of* $r \in A[[V_T{}^*]]$ *is a doubly infinite matrix* $H(r)$ *whose rows and columns are indexed by the words* $V_T{}^*$ *and whose elements with the indices* $u$ *(row index) and* $v$ *(column index) are equal to* $(r, uv)$.

A formal power series $r \in A[[V_T{}^*]]$ is a function from $V_T{}^*$ to $A$. We denote the set of all functions from $V_T{}^*$ to $A$ by $A^{V_T{}^*}$. The set $A^{V_T{}^*}$ also provides a convenient way to visualize the columns of $H(r)$ as elements in $A^{V_T{}^*}$.

We note that with the column $H(r)$ corresponding to the word $v \in V_T{}^*$ (the $v$-*th* column of $H(r)$), we may associate the function $F_v \in A^{V_T{}^*}$ as follows:

$$F_v(u) = (r, uv), \forall u \in V_T{}^*. \tag{6.6.4}$$

Then $F_v(u)$ is essentially equivalent to the $(u,v)$-*th* entry of $H(r)$.

**Example 6.31** *Let* $V_T{}^* = \{x^n \mid n \in \mathbb{N}\}$ *and* $A = \mathbb{Z}$. *Consider the formal power series* $r = \sum_{i=1}^{\infty} nx^n$. *Then the Hankel matrix* $H(r)$ *is the doubly infinite matrix whose* $(x^i, x^j)$*-th entry is* $i + j$ *for* $i,j = 0,1,\ldots$. *Define* $f : V_T{}^* \to A$ *by* $\forall x^n \in V_T{}^*$, $f(x^n) = n$. *Now* $f$ *is the set of ordered pairs* $f = \{(x^n, n) \mid n \in \mathbb{N}\}$. *If* $g : V_T{}^* \to A$ *is another such function, then* $g = \{(x^n, a_n) \mid a_n \in \mathbb{Z}, n \in \mathbb{N}\}$. *Now* $f = g$ *if and only if* $n = a_n$ $\forall n \in \mathbb{N}$. *Thus we see that we can uniquely associate* $f$ *with the power series* $\sum_{i=1}^{\infty} nx^n$. *Also* $F_{x^j}(x^i) = (i + j, x^i x^j)$ *which is the* $(x^i, x^j) - th$ *entry of* $H(r)$.

If we appropriately define addition of functions in $A^{V_T{}^*}$ and multiplication of functions in $A^{V_T{}^*}$ by an element $a \in A$, then $A^{V_T{}^*}$ becomes an $A$-semimodule. We define addition on $A^{V_T{}^*}$ by $\forall\ f_1, f_2 \in A^{V_T{}^*}$

$$(f_1 + f_2)(u) = f_1(u) + f_2(u) \forall u \in V_T{}^*. \tag{6.6.5}$$

Now $f_i(u) \in A$ for $i = 1, 2$. Hence $f_1(u) + f_2(u)$ corresponds to addition of elements of $A$.

Thus $f_1 + f_2 \in A^{V_T{}^*}$ and $A^{V_T{}^*}$ is a commutative monoid with respect to the addition of functions as defined here. For all $a \in A$ and $f \in A[[V_T{}^*]]$ define $af$ by

$$(af)(u) = a \cdot f(u) \forall u \in V_T{}^*. \tag{6.6.6}$$

Then $A^{V_T{}^*}$ becomes an $A$-semimodule.

We next introduce a new operation, where for $w \in V_T{}^*$ and $F \in A^{V_T{}^*}$, the function $wF \in A^{V_T{}^*}$ is defined as follows:

$$wF(v) = F(vw), \forall v \in V_T{}^*. \tag{6.6.7}$$

Let $F, G \in A^{V_T{}^*}$ and $w \in V_T{}^*$. Then $\forall u \in V_T{}^*$, $(w(F + G))(u) = (F + G)(uw) = F(uw) + G(uw) = wF(u) + wG(u) = (wF + wG)(u)$. Thus $w(F + G) = wF + wG$. Let $a \in A$. Then $\forall u \in V_T{}^*$, $(a(wF))(u) = a(wF)(u) = aF(uw) = (aF)(uw) = w(aF)(u)$. Hence $a(wF)) = w(aF)$. Now define $\Phi : A^{V_T{}^*} \to A^{V_T{}^*}$ by $\Phi(F) = wF$ $\forall F \in A^{V_T{}^*}$ and $w \in V_T{}^*$. Then $\Phi(F + G) = w(F + G) = wF + wG = \Phi(F) + \Phi(G)$ and $\Phi(aF) = w(aF) = a(wF) = a(\Phi(F))$. That is, $\Phi$ is linear.

If we consider the function $F_v$ corresponding to the $v$-th column of $H(r)$, then from Eqs. (6.6.4) and (6.6.7), we have

$$(wF_v)(u) = F_v(uw) = (r, uwv) \forall u \in V_T{}^*. \tag{6.6.8}$$

This results in

$$(wF_v)(u) = F_{wv}(u) \forall u \in V_T{}^*. \tag{6.6.9}$$

Thus the operation of premultiplication of $F_v$ by $w$ results in a new function $F_{wv}$ that corresponds to the $wv$-th column of $H(r)$.

We now define a stable subsemimodule.

**Definition 6.51** *A subsemimodule $S$ of $A^{V_T{}^*}$ is called* stable *if $w \in V_T{}^*$ and $F \in S$ imply that $wF \in S$.*

The following result determines whether a given formal power series is a rational series.

**Theorem 6.75** *Let $A$ be a commutative semiring and $r \in A\,[[V_T{}^*]]$. Then the following conditions are equivalent:*

**(i)** $r \in A^{rat}\,[[V_T{}^*]]$.

(ii) *The subsemimodules of* $A^{V_T{}^*}$ *generated by the columns of* $H(r)$ *are contained in a finitely generated stable subsemimodule of* $A^{V_T{}^*}$. ■

We will now be concerned only with a fuzzy semiring $A$ in our goal to construct the minimal fuzzy automaton that accepts sentences in $R^+$ of a fuzzy language.

A fuzzy language over an alphabet $V_T{}^*$ is defined to be a fuzzy subset $\tilde{A}$ of $V_T{}^*$ and a string $x$ in $V_T{}^*$ has a membership grade $\tilde{A}(x)$, $0 < \tilde{A}(x) \leq 1$, denoting its grade of membership in the fuzzy language.

A regular fuzzy language is a set of sentences generated by a regular fuzzy grammar whose finite set of productions are of the form

$$A \rightarrow \theta a B \text{ or } A \rightarrow \theta a, \tag{6.6.10}$$

where $0 < \theta \leq 1$, $A, B \in V_N$, $a \in V_T$.

A finite fuzzy automata over $V_T$ that accepts the strings generated by a regular fuzzy grammar is a 4-tuple

$$M = (Q, \pi, F, \eta), \tag{6.6.11}$$

where $Q$ is a nonempty finite set of internal states, $\pi$ is an $n$-dimensional fuzzy row vector called the *initial state designator,* $\eta$ is a column vector called the *final state designator,* and $F$ is a fuzzy transition matrix.

To construct a fuzzy automaton from a set of sentences belonging to a positive sample set of a fuzzy language, the Hankel matrix is formed using all possible factorizations of each of the strings $w_i$. As observed above, the Hankel matrix is formed by the words of $V_T{}^*$ with each element equal to $(r, uw)$, where $u$ and $v$ in $V_T{}^*$ correspond to the row and column indices of $H(r)$. We recall that $A^{V_T{}^*}$ becomes an $A$-semimodule if the functions in $A^{V_T{}^*}$ are suitably operated.

We now establish that the interval $[0, 1]$ becomes a commutative semiring with respect to the maximum and minimum operations, $\vee$ and $\wedge$, respectively. To see this, we first note that $[0, 1]$ is a commutative monoid with respect to $\vee$ : $\forall a, b, c \in [0, 1]$,

(i) $a \vee (b \vee c) = (a \vee b) \vee c$,

(ii) $a \vee 0 = 0 \vee a = a$,

(iii) $a \vee b = b \vee a$.

Secondly, we note that $[0, 1]$ is a commutative monoid with respect to $\wedge$ : $\forall a, b, c \in [0, 1]$,

(i) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(ii) $a \wedge 1 = 1 \wedge a = a$,

**(iii)** $a \wedge b = b \wedge a$.

Also, $\forall a, b, c \in [0,1]$,
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
and $\forall a \in [0,1]$,
$a \wedge 0 = 0 \wedge a = 0$.

These equations can be verified by examining the various cases:
$a \geq b \geq c$, $a \geq c \geq b$, $b \geq a \geq c$, $b \geq c \geq a$, $c \geq a \geq b$, and $c \geq b \geq a$.

The interval $[0,1]$ with respect to $\vee$ and $\wedge$ thus forms a semiring. In the remainder of the section, we call $([0,1], \vee, \wedge)$ a *fuzzy semiring.*

Let $\delta \in [0,1]$ and let $\widehat{h} = (h_1, ..., h_i, ...)^T$ and $\widehat{k} = (k_1, ..., k_i, ...)^T$ be fuzzy column vectors whose components are from $A$, where $A$ is a fuzzy semiring. By $\delta \wedge \widehat{h}$, we mean the fuzzy column vector $(\delta \wedge h_1, ...\delta \wedge h_i, ...)^T$. By $\widehat{h} \vee \widehat{k}$, we mean the fuzzy column vector $(h_1 \vee k_1, ..., h_i \vee k_i, ...)^T$. Now given the fuzzy column vectors $\widehat{h}_1, ..., \widehat{h}_n$, a fuzzy column vector $\widehat{h}$ belongs to the fuzzy $A$-subsemimodule generated by $\{\widehat{h}_1, ..., \widehat{h}_n\}$ if there exist $\delta_1, ..., \delta_n \in [0,1]$ such that

$$\widehat{h} = (\delta_1 \wedge \widehat{h}_1) \vee ... \vee (\delta_n \wedge \widehat{h}_n). \tag{6.6.12}$$

In this case, we say that $\widehat{h}$ is *dependent* on $\{\widehat{h}_1, ..., \widehat{h}_n\}$. If no such $\delta_i$ exists, $\widehat{h}$ is said to be *independent* of $\{\widehat{h}_1, ..., \widehat{h}_n\}$. A set $\mathcal{H}$ of fuzzy column vectors is said to be independent if $\forall \widehat{h} \in \mathcal{H}$, $\widehat{h}$ is not dependent on $\mathcal{H} \setminus \{\widehat{h}\}$. Since the interval $[0,1]$ is not a field with respect to the operations $\vee$ and $\wedge$, techniques of vector spaces to determine a basis are not directly applicable here. At the end of the section, we present an algorithm for identifying a set of independent columns of $H(r)$.

Suppose $H(r)$ has finitely many independent columns say, $F_1, ..., F_n$. We first show that the subsemimodule $S$ generated by $\{F_1, ..., F_n\}$ of $H(r)$ is stable. Let $F_v$ be a column of $H(r)$ indexed by $v \in V_T{}^*$. Then $F_v$ is dependent on $\{F_1, ..., F_n\}$ and hence $F_v \in S$. Now from (6.6.9), for $w \in V_T{}^*$, $w \cdot F_v = F_{wv}$, where $F_{wv}$ is the column of $H(r)$ indexed by $wv \in V_T{}^*$. Hence $F_{wv}$ is dependent on $\{F_1, ..., F_n\}$ and consequently $F_{wv} \in S$. Therefore from Definition 6.51, $S$ is stable. Thus from conditions (i) and (ii) of Theorem 6.75, $r \in A^{rat}[[V_T{}^*]]$. Conversely, suppose there do not exist finitely many independent columns of $H(r)$. Then from the definition of independence of a column vector of the fuzzy Hankel matrix, the subsemimodule generated by the columns of $H(r)$ also does not have a finite set of generators. Thus from Theorem 6.75, $r$ cannot be a rational power series. We have thus proved the following corollary to Theorem 6.75.

**Corollary 6.76** *If $A$ is a fuzzy semiring, then $r \in A^{rat}[[V_T{}^*]]$ if and only if there are finitely many independent columns of $H(r)$.* ■

Suppose that $H(r)$ has finitely many independent columns and $r \in A^{rat}[[V_T{}^*]]$. Then from Theorems 6.73 and 6.74, $r$ may be expressed as

$$r = \sum_{w \in V_T{}^*} (\alpha(\mu w)\beta)w, \tag{6.6.13}$$

where $\alpha$ is a row vector, $\beta$ is a column vector, $\mu$ is a representation, and $w \in V_T{}^*$.

Since $H(r)$ has finitely many independent columns, there exists a maximal set, $\{F_{v_1}, ..., F_{v_m}\}$, of independent columns of $H(r)$ associated with $\{v_1, ..., v_m\}$, where $v_i \in V_T{}^*$ are strings associated with these columns, $i = 1, ..., m$. Thus for $x \in V_T$, $xF_{v_i}$ must be dependent on $\{F_{v_1}, ..., F_{v_m}\}$, where $xF_{v_i}$ should be interpreted as in Eq. (6.6.9). Hence $xF_{v_i}$ may be represented as

$$xF_{v_i} = \sum_{j=1}^{m} (\mu x)_{ji} F_{v_i} \tag{6.6.14}$$

for $x \in V_T$, where $\mu : V_T{}^* \to A^{m \times m}$.

We must now establish that $\mu$ is a representation. Assuming that the above equation holds for $x = w_1$ and $x = w_2$. Since

$w_1 w_2 F_{v_i}(v) = F_{v_i}(v w_1 w_2) = \sum_{j=1}^{m} (\mu w_2)_{ji} (F_{v_j})(v w_1)$
$= \sum_{j=1}^{m} (\mu w_2)_{ji} \sum_{k=1}^{m} (\mu w_1)_{kj} F_{v_k}(v)$
$= \sum_{k=1}^{m} (\sum_{j=1}^{m} (\mu w_2)_{ji} (\mu w_1)_{kj}) F_{v_k}(v)$
$= \sum_{k=1}^{m} (\mu w_1 \mu w_2)_{ki} F_{v_k}(v)$,

$$(w_1 w_2) F_{v_i}(v) = \sum_{k=1}^{m} (\mu w_1 \mu w_2)_{ki} F_{v_k}(v). \tag{6.6.15}$$

This equation holds for $x = w_1 w_2$. Since it holds for $x \in V_T$, it also holds for any $x \in V_T{}^*$. Thus to construct $\mu$, we need to consider the dependencies of $xF_i$ for $i = 1, ..., m$ and $x \in V_T$ on $\{F_{v_1}, ..., F_{v_m}\}$. Once $\mu$ is constructed, $\alpha$ and $\beta$ can be constructed in the following manner.. Since $r$ belongs to a finitely generated subsemimodule of $A[[V_T{}^*]]$, there exist elements $\beta_1, ..., \beta_m \in A$ such that $r = \sum_{i=1}^{m} \beta_i F_{v_i}$, where $F_{v_i}$ is now treated as a function in $A^{V_T{}^*}$. Then

$(r, w) = \sum_{i=1}^{m} \beta_i F_{v_i}(w) = \sum_{i=1}^{m} \beta_i (w F_{v_i}(\lambda)) = \sum_{i=1}^{m} \beta_i \sum_{j=1}^{m} (\mu w)_{ji} F_{v_i}(\lambda)$
$= (\beta_1, ..., \beta_m)(\mu w)^T (F_{v_1}(\lambda), F_{v_2}(\lambda), ..., F_{v_n}(\lambda))^T$
$= (F_{v_1}(\lambda), F_{v_2}(\lambda), ..., F_{v_n}(\lambda))(\mu w)(\beta_1, ..., \beta_m)^T$

and so

$$(r, w) = (F_{v_1}(\lambda), F_{v_2}(\lambda), ..., F_{v_m}(\lambda))(\mu w)(\beta_1, ..., \beta_m)^T. \tag{6.6.16}$$

Considering $\alpha = (F_{v_1}(\lambda), F_{v_2}(\lambda), ..., F_{v_m}(\lambda))$ and $\beta = (\beta_1, ..., \beta_m)^T$, $(r, w) = \sum_\alpha (\mu w)\beta$. Here $\alpha$ corresponds to the entries in $F_{v_1}, ..., F_{v_m}$ for the row in $H(r)$ labeled by $\lambda \in V_T{}^*$. Also $\beta_i$ corresponds to the coefficients of $F_{v_i}$ in the expansion of $F_\lambda$ in terms of $F_{v_1}, ..., F_{v_m}$.

Now $\alpha$ and $\beta$ correspond to the initial and final states, respectively, of the fuzzy automaton $M$. Once $\alpha, \beta$, and $\mu$ are determined, the desired fuzzy automaton that recognizes the strings in $R^+$ can be constructed. The fuzzy automaton

$M = (\mu, \{q_1, ..., q_m\}, F, \eta)$

can now be defined as

$\pi = \alpha, \eta = \beta$, and $f(q_i, x, q_k) = [\mu(x)]_{ki}$, $x \in V_T$.

Thus the steps required to construct the fuzzy automaton that accepts only the strings in $R^+$($a$ positive sample set of strings) of a fuzzy language are as follows:

(1) Construct the fuzzy Hankel matrix $H(r)$.

(2) Identify a complete set of independent columns of $H(r)$.

(3) Obtain the fuzzy vectors $\alpha$ and $\beta$ and the fuzzy matrices $\mu(x_i)$, $\forall x_i \in V_T$.

(4) Construct the fuzzy automaton.

We note that while inferring a grammar from a positive set $R^+$ of samples of finite length, any column corresponding to a word $v$, $v \in V_T{}^*$, that is not a factorization of any string $w_i \in R^+$ will be identically zero. The same situation arises in the case of the rows of $H(r)$ corresponding to a word $u$ that is not a factorization of $w_i$. Thus the Hankel matrix essentially reduces to the form

$$H(r) = \begin{bmatrix} \widehat{H}(r) & 0 \\ 0 & 0 \end{bmatrix}$$

where the zeros are infinite matrices and $\widehat{H}(r)$ is a submatrix of $H(r)$. In the case of recursive production of strings with cycles, the inference procedure deals with a Hankel matrix of the form

$H(r) = [H_1(r), H_2(r), 0]$,

where $H_1(r)$ is a finite submatrix and contains all the relevant information [5].

The problem of identification of a set of independent columns of $H(r)$ thus reduces to identifying the set of independent columns of $H(r)$, which will be henceforth designated as $H(r)$ only.

We now show how a set of independent column vectors can be identified from the finite fuzzy Hankel matrix.

Previously, we defined the dependence of a column vector $\widehat{h}$ on a set of generators $\{\widehat{h}_1, ..., \widehat{h}_n\}$ of the finite fuzzy Hankel matrix $H(r)$. Here we present an algorithm that checks whether $\widehat{h}$ belongs to the subsemimodule $F$ generated by this set of generators and also identifies its coefficients $\delta_j$. The $j$-th element of the vector $\widehat{h}_k$ will be denoted by $h_{jk}$ and the $i$-th element of $\widehat{h}$ by $h_i$.

Given the set of fuzzy column vectors $\widehat{h}_1, ..., \widehat{h}_n$ of dimension $m$, a set of row vectors $S(i)$ are formed for $i = 1, ..., m$.

$S(i) = \{j \mid j \in \{1, ..., n\}$ such that $h_i \leq h_{ji}\}$.

In the following procedure, in order to identify $\delta_j, j = 1, ..., n$, we examine the dependencies of $h_i$ on $\{h_{1i}, h_{2i}, ..., h_{ni}\}$ for $i = 1, ..., m$.

When $h_i$ can be expressed in terms of $h_{ki}$, $k = 1, ..., n$, the coefficients of $h_{ji}$ will be denoted by $\delta_{ji}$, i. e.,

$$h_i = \sum_{j=1}^{n} \delta_{ji} h_{ji}. \tag{6.6.17}$$

Each such equation identifies a range of admissible values of $\delta_{ji}$.

To identify such constraints on $\delta_{ji}$, note that for $k \in S(i)$,

$h_i > h_{ji}$ implies $\delta_{ki} \in [0, 1]$ (no restriction on $\delta_{ki}$),

and for $k \in S(i)$,

$$h_i \leq h_{ji} \Rightarrow \delta_{ki} \in [0, h_i]. \tag{6.6.18}$$

If now $\mathrm{card}(S(i)) = 1$, say $S(i) = \{j\}$,then $\delta_{ji}$ has a single value, i. e., $\delta_{ji} = h_i$.

On the other hand, if $\mathrm{card}(S(i)) > 1$, for any $i \in \{1, ..., m\}$ and $j \in S(i)$, then the maximum value that $\delta_{ji}$ can have is $\delta_{ji}\mid_{max} = h_i$. Let $\delta_{ji_1}$ and $\delta_{ji_m}$ denote the minimum and maximum admissible values of $\delta_{ji}$ as dictated by Eq. (6.6.18). Let

$$\delta_{jl} = \vee\{\delta_{ji_1} \mid i = 1, ..., m\} \text{ and } \delta_{ju} = \vee\{\delta_{ji_m} \mid i = 1, ..., m\}. \tag{6.6.19}$$

If Eq. (6.6.12) is satisfied, then $\delta_j$ must belong to $\{\delta_{jl}, \delta_{ju}\}$.

Let $\delta_j \in \{\delta_{jl}, \delta_{ju}\}$ and suppose $R_j = \{i \mid i \in \{1, ..., m\}$ is such that $h_i = \delta_j \wedge h_{ji}$. We now present the condition of dependence of $\widehat{h}$ on the set of fuzzy column vectors in the following theorem.

**Theorem 6.77** *A fuzzy column vector $\widehat{h}$ is dependent on a set $\{\widehat{h}_1, ..., \widehat{h}_n\}$ of fuzzy column vectors if and only if*

$$R_1 \cup R_2 \cup ... \cup R_n = \{1, ..., m\}. \tag{6.6.20}$$

**Proof.** The necessity is obvious. Suppose $R_1 \cup R_2 \cup ... \cup Rn = \{1, ..., m\}$. Then there is at least one $i \in \{1, .., m\}$ such that $h_i = (\delta_i \wedge h_{1i}) \vee \ldots \vee(\delta_n \wedge h_{ni})$. Since not all $R_i$ are empty, $i = 1, ..., n$, let $R_{k1}, .., R_{kr} \neq \emptyset$, where $k_i \in \{1, ..., n\}$ are such that $R_{k1} \cup ... \cup R_{kn} = \{1, ..., m\}$. Then from the definition of the $R_i$, we can express the fuzzy vector $\widehat{h}$ as

$$\widehat{h} = (\delta_i \wedge \widehat{h}_1) \vee \ldots \vee (\delta_n \wedge \widehat{h}_n), \tag{6.6.21}$$

where $\delta_j = 0$ if $j \notin \{k_1, ..., k_n\}$ and $\delta_{ki} = h_k$ for $k \in R_{ki}$. ■

**Corollary 6.78** *A fuzzy column vector* $\widehat{h} \in A^n$ *is independent of a set of fuzzy column vectors* $\{\widehat{h}_1, ..., \widehat{h}_n\}$ *if* $S(i) = \emptyset$ *for any* $i \in \{1, ..., m\}$. ■

We give the algorithm for checking if a non-null column vector $x_k$ in the subsemimodule $F$ is linearly dependent on a set of fuzzy vectors at the end of this section.

If a set of column vectors $\widehat{g}_i, i = 1, ..., n$, is given, a complete set of independent fuzzy vectors $\widehat{f}_i, i = 1, ..., l$, can be selected such that the subsemimodule generated by $\{\widehat{f}_1, ..., \widehat{f}_m\}$ contains the $\widehat{g}_i$'s. The procedure is shown in the form of a flow chart given in [51].

We describe an application of the inference of fuzzy grammar in character recognition. Each of the 45 classes of Bengali alphabetic characters has been coded in the form of a string over $V_T = \{a, b, c, d\}$. Linguistic analysis is carried out for only a small zone of the pattern where the structural dissimilarity of the training patterns representing different pattern classes is maximum. For structural analysis these zones are represented by strings of pattern primitives. All the strings of a particular pattern class are next associated with a generative grammar that is not known a priori. The grammar corresponding to each class of patterns is next constructed using the inference procedure described [51].

It may be noted at this point that the positive sample $R^+(L(G))$ ($L(G)$ is the language corresponding to a pattern class whose grammar is $G$) must be structurally complete with respect to $G$. Otherwise, if a new string not hitherto included in $R$ is accepted by the automaton, the set $R$ is enhanced to include it and the fuzzy Hankel matrix is modified accordingly. The repetition of this procedure continues until the sample set $R^+$ is complete.

We now consider a positive sample set

$R^+ = 0.8ab, 0.8aabb, 0.3ab, 0.2bc, 0.9abbc.$

The finite submatrix of the fuzzy Hankel matrix $H(r)$ is shown in Table 6.1.

Using the algorithm DEPENDENCE, the independent columns of the fuzzy Hankel matrix have been indicated as $F_1, F_2, F_5, F_6$ and $F_7$.

The algorithm DEPENDENCE also identifies if any column vector $\widehat{h}(j)$ is dependent on the set of generators $HU(m) = \{\widehat{f}_1, ..., \widehat{f}_m\}$ of the Hankel matrix as constructed in Table 6.1. It also identifies the coefficients $\delta_i$, using the procedure ARRANG$(S(i), N, CARD(i))$ and the procedure COMPARE$(SO(K), SO(K-1))$.

Once the independent set of column vectors are extracted, the next step is to find out the matrices $\mu(x)$, $x \in V_T$.

In order to determine the matrices $\mu(x)$, $x \in V$, initially the expression $xF$ has to be computed for $x = a, b, c$ and $i = 1, ..., 7$. The matrices $\mu(a)$, $\mu(b)$, and $\mu(c)$ are given in Table 6.1.

The $\alpha_i$'s can be computed from the relationship $\alpha = (F_1(\lambda), ..., F_m(\lambda))$, where the vector corresponds to the entries in the set of independent columns $F_1, ..., F_m$ for the row in $H(r)$ labeled by $\lambda$. Thus

**TABLE 6.1** The finite submatrix of the fuzzy Hankel matrix $H(r)$ [2]

$$\mu(a) = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & 0 & 1 & 0 & .8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mu(b) = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mu(c) = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ \lambda \\ a \\ ab \\ abc \\ bc \\ abb \\ abbc \\ aa \\ aab \\ aabb \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} \lambda & ab & b & abc & bc & c & abbc & bbc & aabb & abb & bb \end{array} \\ \begin{bmatrix} 0 & .8 & 0 & .3 & .2 & 0 & .9 & 0 & .8 & 0 & 0 \\ 0 & 0 & .8 & 0 & .3 & 0 & 0 & .9 & 0 & .8 & 0 \\ .8 & 0 & 0 & 0 & .9 & .3 & 0 & 0 & 0 & 0 & 0 \\ .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .9 & 0 & 0 & 0 & 0 & 0 \\ .9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccccccccc} F_1 & & F_7 & & F_3 & F_4 & F_2 & F_5 & & & F_6 \end{array} \end{array}$$

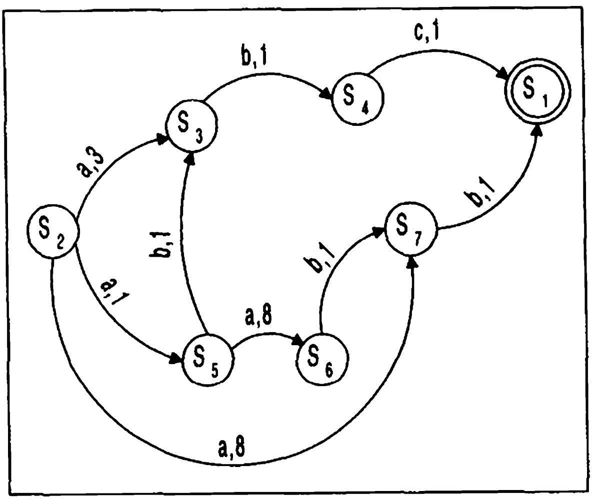
$\alpha = (0\ 0.9\ 0.2\ 0\ 0\ 0\ 0)$.

The vector

$\beta = (1\ 0\ 0\ 0\ 0\ 0\ 0)$,

because column $F_1$ is an independent column. Once $\alpha, \beta$, and $\mu(a), \mu(b)$,and $\mu(c)$ have been determined, the fuzzy automaton can be constructed by the method already described. The fuzzy automaton that accepts [1] the strings is shown in Figure 6.2.

**FIGURE 6.2** Inferred fuzzy automata.[2]




PROCEDURE DEPENDENCE

Step 1. $i = 1$; Form $S(i)$ such that

$$S(i) = \{v_j \mid h_{ji} \geq h_i\}.$$

Find card$(S(i))$.

Step 2. If card$(S(i)) = 0$, go to step 12. Else do

Step 3. If card$(S(i)) = 1$ and $S(i) = \{j_k\}$,

$$\delta_{jl} = \delta_{ju} = h_i \text{ for } j = j_k.$$

For any other $j \neq j_k$, $\delta_{jl} = 0, \delta_{ju} = 1$, go to step 5. Else do

Step 4. If card$(S(i)) > 1$,

$$\delta_{jl} = 0 \text{ and } \delta_{ju} = h_i \text{ for all } j = j_k.$$

[1] Reprinted from Information Sciences 55, A.K. Ray, B. Chatterjee, and A.K. Majumdar, A Formal Power Series Approach to the Construction of Minimal Fuzzy Automata, 189 207,1991 with permission from Elsevier Science

For any other $j \neq j_k, \delta_{j1} = 0, \delta_{ju} = 1$.

Step 5. $i = i + 1$. Repeat the procedure until $i = m$.

Step 6. $j = 1$. Find $\vee\{\delta_{jkl}\}$ and $\wedge\{\delta_{jkl}\}$, $k = 1, ..., m$. If $\vee\{\delta_{jkl}\} > \wedge\{\delta_{jku}\}$, go to step 12. Else do

Step 7. Select a $\delta_j$ such that

$$\delta_{jk1} \mid_{max} < \delta_j < \delta_{jku} \mid_{min}$$

and set $R_j = \emptyset$.

Step 8. Form $R_j = R_j \cup i$ ($i \in \{1, ..., n\}$ such that $h_i = \delta_j \wedge h_{ji}$).

Step 9. $j = j + 1$. If $j < n$, go to step 6.

Step 10. Check if $R_j$ covers all $i \in \{1, ..., m\}$. If $R_j = \{1, ..., m\}$, go to step 11.

Else go to step 12.

Step 11. $\widehat{h}$ is dependent, print the values of $\delta_j$'s.

Step 12. $\widehat{h}$ is independent.

## 6.7 Nonlinear Systems of Equations of Fuzzy Singletons

Recall that if $S$ is a set, $x \in S$, and $t \in [0,1]$, then the fuzzy subset $x_t$ of $S$ is called a *fuzzy singleton* if $\forall s \in S$, $x_t(s) = t$ if $s = x$ and $x_t(s) = 0$ if $s \neq x$. A fuzzy subset $\tilde{A}$ of $S$ is said to have the *sup property* if every subset of $\tilde{A}(S)$ has a maximal element.

In this section, we examine nonlinear systems of equations of fuzzy singletons, i. e., each equation is of the form
$\sum_{i_1=0}^{q_1} \cdots \sum_{i_n=0}^{q_n} a_{i_1 \ldots i_n}((x_1)_{t_1})^{i_1} \ldots ((x_n)_{t_n})^{i_n} = 0_t$,
where $\sum_{i_1=0}^{q_1} \cdots \sum_{i_n=0}^{q_n} a_{i_1 \ldots i_n} {x_1}^{i_1} \ldots {x_n}^{i_n} \in F[x_1, ..., x_n]$, the polynomial ring in $n$ indeterminates over the field $F$, and $t_1, ..., t_n, t \in (0,1]$.

**Example 6.32** *Consider the polynomial ring* $\mathbb{R}[x,y,z]$ *in indeterminates* $x, y, z$ *over* $\mathbb{R}$. *An example of a nonlinear system of equations of fuzzy singletons is*
$(x_s)^2 - y_t = 0_{1/4}$
$(x_s)^2 z_u = 0_{1/2}$,
*where* $x_s, y_t$, *and* $z_u$ *are fuzzy singletons.*

**Definition 6.52** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ be a fuzzy ideal of $R$. A representation of $\tilde{A}$ as a finite intersection $\tilde{A} = \tilde{Q}_1 \cap \ldots \cap \tilde{Q}_m$ of fuzzy primary ideals of $R$ is called a* fuzzy primary representation *(or* decomposition*) of $\tilde{A}$. It is called* irredundant *or* reduced *if no $\tilde{Q}_i$ contains $\bigcap_{\substack{j=1 \\ j \neq i}} \tilde{Q}_j$ and the $\tilde{Q}_i$ have distinct radicals.*

**Theorem 6.79** *Let $R$ be a commutative ring with identity. Every fuzzy ideal $\tilde{A}$ of $R$ such that $\tilde{A}(0) = 1$ and $\tilde{A}$ is finite-valued has a fuzzy primary representation if and only if every ideal of $R$ has a primary representation.* ■

**Corollary 6.80** *Suppose that $R$ is a Noetherian ring. Then every fuzzy ideal $\tilde{A}$ of $R$ such that $\tilde{A}(0) = 1$ and $\tilde{A}$ is finite-valued has a fuzzy primary representation.* ■

A commutative ring with identity is called *Artinian* if every descending sequence of ideals is finite.

**Theorem 6.81** *Let $R$ be a commutative ring with identity. Then every fuzzy ideal of $R$ such that $\tilde{A}$ such that $\tilde{A}(0) = 1$ has a fuzzy primary representation if and only if $R$ is Artinian.* ■

**Theorem 6.82** *Let $R$ be a commutative ring with identity and let $\tilde{A}$ be a fuzzy ideal of $R$. If $\tilde{A}$ has a primary representation, then $\tilde{A}$ has a reduced primary representation.* ■

**Example 6.33** *Consider the polynomial ring $R = \mathbb{R}[x, y, z]$ in indeterminates $x, y, z$ over $\mathbb{R}$. Let $\tilde{A}$ be the fuzzy ideal of $R$ generated by $(x_{1/4})^2 - y_{1/4}$ and $(x_{1/2})^2 z_{1/2}$ i. e., $\tilde{A} = <(x_{1/4})^2 - y_{1/4}, (x_{1/2})^2 z_{1/2}>$ . Then*

$\tilde{A} = \tilde{Q}_1 \cap \tilde{Q}_2 \cap \tilde{Q}_3 \cap \tilde{Q}_4 \cap \tilde{Q}_5$

*is a reduced fuzzy primary representation of $R$, where the primary fuzzy ideals $\tilde{Q}_i$, $i = 1, 2, 3, 4, 5$, of $R$ are defined as follows: $\forall u \in R$,*

$\tilde{Q}_1(u) = 1$ *if* $u \in <x^2, y>$, $\tilde{Q}_1(u) = 0$ *otherwise,*

$\tilde{Q}_2(u) = 1$ *if* $u \in <x^2 - y, z>$, $\tilde{Q}_1(u) = 0$ *otherwise,*

$\tilde{Q}_3(u) = 1$ *if* $u \in <x^2>$, $\tilde{Q}_3(u) = 1/4$ *otherwise,*

$\tilde{Q}_4(u) = 1$ *if* $u \in <z>$, $\tilde{Q}_4(u) = 1/4$ *otherwise,*

$\tilde{Q}_5(u) = 1$ *if* $u \in <0>$, $\tilde{Q}_5(u) = 1/2$ *otherwise.*

*The radicals of the $\tilde{Q}_i$, $i = 1, 2, 3, 4, 5$ are as follows:*

$\sqrt{\tilde{Q}_i} = \tilde{Q}_i$ *for* $i = 2, 4, 5,$

*and* $\forall u \in R$,

$\sqrt{\tilde{Q}_1}\ (u) = 1$ *if* $u \in <x, y>$ *and* $\sqrt{\tilde{Q}_1}\ (u) = 0$ *otherwise,*

$\sqrt{\tilde{Q}_3}\ (u) = 1$ *if* $u \in \langle x \rangle$ *and* $\sqrt{\tilde{Q}_3}\ (u) = 1/4$ *otherwise.*

*We now illustrate how the solution to the nonlinear system of equations of fuzzy singletons*

$(x_s)^2 - y_t = 0_{1/4}$

$(x_s)^2 z_u = 0_{1/2}$

*is related to the reduced fuzzy primary representation of $\tilde{A}$. Clearly the solution is given by*

$\{(x, y, z) \mid y = x^2, z = 0; x, y \in \mathbb{R}\} \cup \{(x, y, z) \mid x = 0 = y; z \in \mathbb{R}\}$ *and*

$s \wedge t = 1/4$ *and* $s \wedge u = 1/2$.

*Hence* $t = 1/4$ *and* $s \wedge u = 1/2$. *The radicals* $\sqrt{\tilde{Q}_i}, i = 1, 2$, *display the crisp part of the solution, namely* $\{(0, 0, z) \mid z \in \mathbb{R}\}$ *and* $\{(x, x^2, 0) \mid x \in \mathbb{R}\}$ *respectively, while the radicals* $\sqrt{\tilde{Q}_i}$ , $i = 3, 4, 5$ *display the fuzzy part. This will be better seen once we have developed the notion of a fuzzy affine variety of a fuzzy ideal, Definition 6.36.*

**Definition 6.53** *Let* $\tilde{X}$ *be a fuzzy affine variety. Then* $\tilde{X}$ *is called* irreducible *if for all fuzzy affine varieties* $\tilde{X}'$ *and* $\tilde{X}''$ *such that* $\tilde{X} = \tilde{X}' \cup \tilde{X}''$ *either* $\tilde{X} = \tilde{X}'$ *or* $\tilde{X} = \tilde{X}''$; *otherwise* $\tilde{X}$ *is called* reducible.

**Theorem 6.83** *Let* $\tilde{X}$ *be a fuzzy affine variety. Then* $\tilde{X}$ *is irreducible and nonconstant if and only if* $Im(\tilde{X}) = \{0, t\}, 0 < t$, *and* $\tilde{X}^t$ *is irreducible.* ■

**Theorem 6.84** *Let* $\tilde{A}$ *be a finite-valued fuzzy ideal of* $R$. *Then* $I(V(\tilde{A}))$ *is prime if and only if* $V(\tilde{A})$ *is irreducible.* ■

**Theorem 6.85** *Let* $\tilde{A}$ *be a finite-valued fuzzy ideal of* $R$ *with* $\tilde{A}(0) = 1$. *Then* $I(V(\tilde{A})) = \sqrt{\tilde{A}}$. ■

**Theorem 6.86** *There exists a one-to-one correspondence between fuzzy affine varieties* $\tilde{X}$ *with* $0 \in Im(\tilde{X})$ *and fuzzy radical ideals.* ■

**Theorem 6.87** *Every fuzzy affine variety* $\tilde{X}$ *with* $0 \in Im(\tilde{X})$ *can be uniquely expressed as the union of a finite number of irreducible algebraic varieties no one of which is contained in the union of the others.* ■

**Example 6.34** *Let* $R = F[x, y, z]$, *where* $F$ *is the field of complex numbers and* $x, y, z$ *are algebraically independent indeterminates over* $F$. *Define the fuzzy subset* $\tilde{A}$ *of* $R$ *by* $\tilde{A}(0) = 1, \tilde{A}(f) = 1/2$ *if* $f \in < x^2z > \setminus < 0 >$, $\tilde{A}(f) = 1/4$ *if* $f \in < x^2 + y^2 - 1, x^2z > \setminus < x^2z >$, *and* $\tilde{A}(f) = 0$ *if* $f \in R \setminus < x^2 + y^2 - 1, x^2z >$ . *Then* $\tilde{A}$ *is a fuzzy ideal of* $R$. *Now* $\sqrt{\tilde{A}}$ *is such that* $\sqrt{\tilde{A}}\,(0) = 1$, $\sqrt{\tilde{A}}(f) = 1/2$ *if* $f \in < xz > \setminus < 0 >$, $\sqrt{\tilde{A}}(f) = 1/4$ *if* $f \in < x^2 + y^2 - 1, xz > \setminus < xz >$, *and* $\sqrt{\tilde{A}}\,(f) = 0$ *if* $f \in R \setminus < x^2 + y^2 - 1, xz >$ .
*Hence*

$$\tilde{A}^0 = R, \left(\sqrt{\tilde{A}}\right)^0 = R$$

$$\tilde{A}^{1/4} = < x^2 + y^2 - 1, x^2z >, \left(\sqrt{\tilde{A}}\right)^{1/4} = < x^2 + y^2 - 1, xz >$$

$$\tilde{A}^{1/2} = < x^2z >, \left(\sqrt{\tilde{A}}\right)^{1/2} = < xz >$$

$$\tilde{A}^1 = < 0 >, \left(\sqrt{\tilde{A}}\right)^1 = < 0 > .$$

*Since* $F^3 = V(< 0 >)$,

$$V(\tilde{A})(b) = \begin{cases} c(1/2) & \text{if } b \in V(<0>) \setminus V(\langle xz \rangle), \\ c(1/4) & \text{if } b \in V(<xz>) \setminus V(<x^2+y^2-1, xz>), \\ c(0) = 1 & \text{if } b \in V(<x^2+y^2-1, xz>). \end{cases}$$

*Define the fuzzy subsets* $\tilde{Q}^{(i)}$ *of* $R$, $i = 1, \dots, 6$ *as follows:*
$\tilde{Q}^{(1)}(f) = 1$ *if* $f \in <x^2, y \quad 1>$, $\tilde{Q}^{(1)}(f) = 0$ *otherwise;*
$\tilde{Q}^{(2)}(f) = 1$ *if* $f \in <x^2, y+1>$, $\tilde{Q}^{(2)}(f) = 0$ *otherwise;*
$\tilde{Q}^{(3)}(f) = 1$ *if* $f \in <x^2+y^2-1, z>$, $\tilde{Q}^{(3)}(f) = 0$ *otherwise;*
$\tilde{Q}^{(4)}(f) = 1$ *if* $f \in <x^2>$, $\tilde{Q}^{(4)}(f) = 1/4$ *otherwise;*
$\tilde{Q}^{(5)}(f) = 1$ *if* $f \in <z>$, $\tilde{Q}^{(5)}(f) = 1/4$ *otherwise;*
$\tilde{Q}^{(6)}(f) = 1$ *if* $f \in <0>$, $\tilde{Q}^{(6)}(f) = 1/2$ *otherwise.*

*Then* $\tilde{Q}^{(i)}$ *is a fuzzy ideal of* $R$, $i = 1, \dots, 6$ *and* $\tilde{A} = \bigcap_{i=1}^{6} \tilde{Q}^{(i)}$. *In fact,* $\bigcap_{i=1}^{6} \tilde{Q}^{(i)}$ *an irredundant fuzzy primary representation of* $\tilde{A}$. *Now*
$\sqrt{\tilde{Q}^{(1)}}(f) = 1$ *if* $f \in <x, y-1>$, $\sqrt{\tilde{Q}^{(1)}}(f) = 0$ *otherwise;*
$\sqrt{\tilde{Q}^{(2)}}(f) = 1$ *if* $f \in <x, y+1>$, $\sqrt{\tilde{Q}^{(2)}}(f) = 0$ *otherwise;*
$\sqrt{\tilde{Q}^{(3)}}(f) = 1$ *if* $f \in <x^2+y^2-1, z>$, $\sqrt{\tilde{Q}^{(3)}}(f) = 0$ *otherwise;*
$\sqrt{\tilde{Q}^{(4)}}(f) = 1$ *if* $f \in \langle x \rangle$, $\sqrt{\tilde{Q}^{(4)}}(f) = 1/4$ *otherwise;*
$\sqrt{\tilde{Q}^{(5)}}(f) = 1$ *if* $f \in <z>$, $\sqrt{\tilde{Q}^{(5)}}(f) = 1/4$ *otherwise;*
$\sqrt{\tilde{Q}^{(6)}}(f) = 1$ *if* $f \in <0>$, $\sqrt{\tilde{Q}^{(6)}}(f) = 1/2$ *otherwise.*

*We see that* $\sqrt{\tilde{Q}^{(i)}} = \tilde{Q}^{(i)}$ *for* $i \in \{3, 5, 6\}$. *Also* $\sqrt{\tilde{A}} = \bigcap_{i=1}^{6} \tilde{P}^{(i)}$, *where* $\tilde{P}^{(i)} = \sqrt{\tilde{Q}^{(i)}}$ *is prime fuzzy ideal of* $R$, $i = 1, \dots, 6$.
*We have the following fuzzy affine varieties:*
$V(\tilde{P}^{(1)})(b) = 1$ *if* $b \in V(<x, y-1>)$, $V(\tilde{P}^{(1)})(b) = 0$ *otherwise;*
$V(\tilde{P}^{(2)})(b) = 1$, *if* $b \in V(<x, y+1>)$, $V(\tilde{P}^{(2)})(b) = 0$ *otherwise;*
$V(\tilde{P}^{(3)})(b) = 1$, *if* $b \in V(<x^2+y^2-1, z>)$, $V(\tilde{P}^{(3)})(b) = 0$ *otherwise;*
$V(\tilde{P}^{(4)})(b) = c(1/4)$, *if* $b \in V(\langle x \rangle)$, $V(\tilde{P}^{(4)})(b) = 0$ *otherwise;*
$V(\tilde{P}^{(5)})(b) = c(1/4)$, *if* $b \in V(<z>)$, $V(\tilde{P}^{(5)})(b) = 0$ *otherwise;*
$V(\tilde{P}^{(6)})(b) = c(1/2)$, *if* $b \in F^3$.

*Then* $V(\tilde{A}) = \bigcup_{i=1}^{6} V(\tilde{P}^{(i)})$ *and in fact* $V(\tilde{P}^{(i)})$ *is irreducible and no* $V(\tilde{P}^{(i)})$ *is contained in the union of the others,* $i = 1, \dots, 6$.

*Consider the nonlinear system of equations of fuzzy singletons:*
$(x_s)^2 + (y_t)^2 - 1_{1/4} = 0_{1/4}$,
$(x_s)^2 z_u = 0_{1/2}$.

*Then a solution is given by* $t \geq 1/4$ *and* $s \wedge u = 1/2$ *and the line* $x = 0, y = 1$; *the line* $x = 0, y = -1$; *and the circle* $z = 0, x^2 + y^2 = 1$.

*Note also that* $<\tilde{A}> = <(x^2+y^2-1)_{1/4}, (x^2z)_{1/2}>$ . *If we let* $c(0) = 1, c(1/4) = 1/2, c(1/2) = 1/4$, *and* $c(1) = 0$, *then the above representation of* $V(\tilde{A})$ *seems to better represent the solution of the above nonlinear system of equations of fuzzy singletons. The* $V(\tilde{P}^{(i)})$ *for* $i = 1, 2, 3$ *represent the crisp part of the solution while the* $V(\tilde{P}^{(i)})$ *for* $i = 4, 5, 6$ *yield the fuzzy part.*

In Example 6.34 it was shown how a solution to a system of fuzzy intersection equations could be displayed by a primary representation of the fuzzy ideal generated by the defining polynomials of the intersection equations. We now show this holds in general. The proofs of the results are in [41]. If $\tilde{A}$ is a fuzzy ideal of $R$, we let $\tilde{A}_* = \{x \in R \mid \tilde{A}(x) = \tilde{A}(0)\}$. Then $\tilde{A}_*$ is an ideal of $R$.

**Theorem 6.88** *Let* $\tilde{A} = \langle (f_1)_{t_1}, \ldots, (f_q)_{t_q} \rangle \cup 0_1$ *where* $f_1, \ldots, f_q \in R$, $1 \geq t_1 \geq \ldots \geq t_q > 0$ *and* $t \neq t_q$. *Suppose that* $\langle f_1, \ldots, f_q \rangle \neq R$. *Let* $\{t_{i_1}, \ldots, t_{i_m}\} = \{t_1, \ldots, t_q\}$ *be such that* $t_{i_1} > \ldots > t_{i_m}$. *Let*

$$\mathcal{F}_{t_{m-u-1}} = \{f_k | t_k > t_{i_{m-u}}\},$$

$u = 0, 1, \ldots, m-1$, *and let* $\mathcal{F}_{t_{i_m}} = \{f_1, .., f_q\}$. *Define the fuzzy subsets* $\widetilde{W}$, $\widetilde{W}_1, \ldots, \widetilde{W}_m$ *of* $R$ *as follows:*

$$\widetilde{W}(r) = \begin{cases} 1 & \text{if } r \in \langle \mathcal{F}_{t_{i_m}} \rangle \\ 0 & \text{if } r \notin \langle \mathcal{F}_{t_{i_m}} \rangle \end{cases}$$

$$\widetilde{W}_u(r) = \begin{cases} 1 & \text{if } r \in \left\langle \mathcal{F}_{t_{i_{m-u}}} \right\rangle \\ t_{i_{m-u+1}} & \text{if } r \notin \left\langle \mathcal{F}_{t_{i_{m-u}}} \right\rangle \end{cases}$$

$u = 1, \ldots, m-1$.

$$\widetilde{W}_m(r) = \begin{cases} 1 & \text{if } r \in \langle \emptyset \rangle \\ t_{i_1} & \text{if } r \notin \langle \emptyset \rangle \end{cases}$$

*Then* $\widetilde{W}, \widetilde{W}_1, \ldots, \widetilde{W}_m$ *are fuzzy ideals of* $R$ *and* $\tilde{A} = \widetilde{W} \cap \widetilde{W}_1 \cap \ldots \cap \widetilde{W}_m$. ■

**Theorem 6.89** *Let* $\tilde{A} = \langle (f_1)_{t_1}, \ldots, (f_q)_{t_q} \rangle \cup 0_1$ *where* $f_1, \ldots, f_q \in R$, $1 \geq t_1 \geq \ldots \geq t_q > 0$ *and* $t_1 \neq t_q$. *Suppose that* $\langle f_1, \ldots, f_q \rangle \neq R$. *Let* $\widetilde{W}$, $\widetilde{W}_1, \ldots, \widetilde{W}_m$ *be defined as in Theorem 6.88. Let*

$$\chi_{\widetilde{W}_*} = Q_{01} \cap \ldots \cap Q_{0k_0}$$

*and*

$$\chi_{(\widetilde{W}_u)_*} = Q_{u1} \cap \ldots \cap Q_{uk_u}$$

*be fuzzy primary representations of* $\chi_{\widetilde{W}_{\bullet}}$ *and* $\chi_{(\widetilde{W}_u)_{\bullet}}$, *respectively,* $u = 1, \ldots, m$. *For each* $u = 0.1, \ldots, m$, *define the fuzzy subsets* $\widetilde{A}_{u1}, \ldots, \widetilde{A}_{uk_u}$ *of* $R$ *as follows:* $\forall\, r \in R$.

$$\widetilde{A}_{0j}(r) = \begin{cases} 1 & \textit{if } r \in Q_{0j} \\ 0 & \textit{if } r \notin Q_{0j} \end{cases}$$

$j = 1, \ldots, k_0$.

$$\widetilde{A}_{uj}(r) = \begin{cases} 1 & \textit{if } r \in Q_{uj} \\ 0 & \textit{if } r \notin Q_{uj} \end{cases}$$

$j = 1, \ldots, k_u$; $u = 1, \ldots, m$. *Then the following assertions holds:*

**(i)** $\widetilde{A}_{u1}, \ldots, \widetilde{A}_{uk_u}$ *are fuzzy ideals of* $R$, $u = 0, 1, \ldots, m$.

**(ii)** $\widetilde{W}_u = \widetilde{A}_{u1} \cap \ldots \cap \widetilde{A}_{uk_u}$, $u = 0, 1, \ldots, m$.

**(iii)** $\widetilde{A} = (\widetilde{A}_{01} \cap \ldots \cap \widetilde{A}_{0k_0}) \cap (\widetilde{A}_{11} \cap \ldots \cap \widetilde{A}_{1k_1}) \cap \ldots \cap (\widetilde{A}_{m1} \cap \ldots \cap \widetilde{A}_{mk_m})$ *is a fuzzy primary fuzzy representation of* $\widetilde{A}$. ■

Let $R$ denote the polynomial ring $F[x_1, \ldots, x_n]$ in $n$ indeterminates over the field $F$. Then every ideal of $R$ has a primary representation. Let

$$\sum_{i_1=1}^{k_{1j}} \cdots \sum_{i_n=1}^{k_{nj}} (r_{i_1 \ldots i_n j})_1 ((x_1)_{s_{1j}})^{i_1} \ldots ((x_n)_{s_{nj}})^{i_n} = (r_j)_{t_j}, j = 1, \ldots, q, \tag{6.7.1}$$

denote $q$ nonlinear equations in the fuzzy singletons $(x_1)_{s_1}, \ldots, (x_n)_{s_n}$ where $s_{ij} = s_i$ if $x_i$ appears in equation $j$ and 1 otherwise, $i = 1, \ldots, n$; $j = 1, \ldots, q$ and where the $(r_{i_1 \ldots i_n j})_1$ and the $(r_j)_{t_j}$ are fuzzy singletons and the $r_j$ and the $r_{i_1 \ldots i_n}$ are in $F$. Let

$$f_j = \sum_{i_1=1}^{k_{1j}} \cdots \sum_{i_n=1}^{k_{nj}} r_{i_1 \ldots i_n j} (x_1)^{i_1} \ldots (x_n)^{i_n}, j = 1, \ldots, q.$$

Then the system of equations (6.7.1) is equivalent to the following two systems of equations:

$$f_j = r_j,\ j = 1, \ldots, q \tag{6.7.2}$$

and

$$s_{1j} \wedge \ldots \wedge s_{nj} = t_j, j = 1, \ldots, q. \tag{6.7.3}$$

Let $\widetilde{A} = \langle (f_1)_{t_1}, \ldots, (f_q)_{t_q} \rangle \cup 0_1$. It is clear that in (iii) of Theorem 6.89,

$$\sqrt{\widetilde{A}_{01}} \cap \ldots \cap \sqrt{\widetilde{A}_{0k_0}}$$

gives, via unions of the corresponding irreducible algebraic fuzzy varieties, the crisp part, (6.7.2), of the solution to the fuzzy intersection equations, (6.7.1), while

$$(\sqrt{\widetilde{A}_{11}} \cap \ldots \cap \sqrt{\widetilde{A}_{1k_1}}) \cap \ldots \cap (\sqrt{\widetilde{A}_{m1}} \cap \ldots \cap \sqrt{\widetilde{A}_{mk_m}})$$

gives the fuzzy part, (6.7.3).

## 6.8 Localized Fuzzy Subrings

The notion of algebraic fuzzy varieties was introduced in order to use primary representation theory of fuzzy ideals to examine the solution of fuzzy intersection equations. The concepts of quasi-local fuzzy subrings and complete local fuzzy subrings were developed in [27] and [11, 39], respectively, in order to lay the ground work for the examination of fuzzy intersection equations locally. In this section, we characterize local rings in terms of certain fuzzy ideals. We also characterize rings of fractions at a prime ideal in terms of fuzzy ideals. We apply our results to fuzzy intersection equations. In particular, we show that the fuzzy ideal which represents a system of fuzzy intersection equations in a polynomial ring is such that its extension in a ring of fractions represents the same system of fuzzy intersection equations.

Throughout this section $R$ denotes a commutative ring with identity. Let $\widetilde{A}_{\#} = \{x \in R | \widetilde{A}(x) > \widetilde{A}(1)\}$. If $\widetilde{A}$ is a fuzzy ideal of $R$, then $\widetilde{A}_{\#}$ is an ideal of $R$. Let $\mathcal{S}$ be a set of fuzzy singletons of $R$ such that if $x_t$, $x_s \in \mathcal{S}$, then $t = s > 0$. Let foot$(\mathcal{S}) = \{x | x_t \in \mathcal{S}\}$. If $\widetilde{A}$ is a fuzzy ideal of $R$ such that $\widetilde{A} = \langle \mathcal{S} \rangle \cup 0_{\widetilde{A}(0)}$ for some $\mathcal{S}$, then $\mathcal{S}$ is called a *generating set* for $\widetilde{A}$. If $\mathcal{S}$ is a generating set for $\widetilde{A}$, and $\langle \mathcal{S} \backslash \{x_t\} \rangle \cup 0_{\widetilde{A}(0)} \subset \widetilde{A}\ \forall x_t \in \mathcal{S}$, then $\mathcal{S}$ is called a *minimal generating set* for $\widetilde{A}$. If $S$ is a subset $R$, we let $\langle S \rangle$ denote the ideal of $R$ generated by $S$.

A commutative ring with identity, but not necessarily Noetherian, is said to be local if it has a unique maximal ideal. (Such a ring is called quasi-local in [27]). In [27] the definition of a quasi-local fuzzy subring of $R$ was given when $R$ was assumed to be local. That is, a fuzzy subring $\widetilde{A}$ of a local ring $R$ was called quasi-local if $\widetilde{A}(x) = \widetilde{A}(x^{-1})$ for all units $x$ of $R$. If $\widetilde{A}$ is a fuzzy ideal of $R$, then $\widetilde{A}(x) = \widetilde{A}(1)$ for all units $x$ of $R$. Hence if $R$ is a local ring and $\widetilde{A}$ is a fuzzy ideal of $R$, then $\widetilde{A}$ is a quasi-local fuzzy subring of $R$. We also know that if $\widetilde{A}$ is a fuzzy ideal of $R$, then $\widetilde{A}(y) \geq \widetilde{A}(1)\ \forall y \in R$. If $\widetilde{A}$ is a nonconstant fuzzy ideal of $R$, then $\widetilde{A}(0) > \widetilde{A}(1)$.

**Definition 6.54** *A fuzzy ideal $\widetilde{A}$ of $R$ is called* local *if* $\forall x \in R$, $\widetilde{A}(x) = \widetilde{A}(1)$ *is equivalent to $x$ being a unit in $R$.*

Note that if $\widetilde{A}$ is a fuzzy ideal of $R$ which is local, then $\mu$ is not constant since 0 is not a unit of $R$. Let $R$ denote the polynomial ring $F[x]$ over the field $F$. Define the fuzzy subring $\widetilde{A}$ of $R$ by $\widetilde{A}(z) = 1$ if $z = 0$. $\widetilde{A}(z) = \frac{1}{2}$ if $z \in F\backslash\{0\}$, and $\widetilde{A}(z) = \frac{1}{4}$ if $z \in R\backslash F$. Then $\widetilde{A}$ is a fuzzy subring of $R$. Also, $\widetilde{A}(z) = \widetilde{A}(1)$ if and only if $z$ is a unit. However $\widetilde{A}$ is not a fuzzy ideal of $R$. We also note that $R$ is not a local ring.

**Lemma 6.90** *Let $\widetilde{A}$ be a nonconstant fuzzy ideal of $R$. Then $\widetilde{A}$ is local if and only if $\widetilde{A}_{\#}$ is the unique maximal ideal of $R$.* ■

Recall that a fuzzy ideal $\widetilde{A}$ of $R$ is a generalized maximal fuzzy ideal if $\widetilde{A}$ is not constant and for any fuzzy ideal $\widetilde{B}$ of $R$, if $\widetilde{A} \subseteq \widetilde{B}$, then either $\widetilde{A}_* = \widetilde{B}_*$ or $\widetilde{B} = 1_R$. Then a fuzzy ideal $\widetilde{A}$ of $R$ is maximal if and only if $|\mathrm{Im}(\widetilde{A})| = 2$, $\widetilde{A}(0) = 1$, and $\widetilde{A}_*$ is a maximal ideal of $R$.

Let $\widetilde{A}$ and $\widetilde{C}$ be fuzzy ideals of $R$. Then $\widetilde{A}$ and $\widetilde{C}$ are said to be equivalent if

$$\{\widetilde{A}_t | t \in \mathrm{Im}(\widetilde{A})\} = \{\widetilde{C}_t | t \in \mathrm{Im}(\widetilde{C})\}.$$

**Theorem 6.91** *The following conditions are equivalent:*

**(i)** *$R$ is local;*

**(ii)** *$R$ has a fuzzy ideal which is local;*

**(iii)** *all generalized maximal fuzzy ideals of $R$ are local;*

**(iv)** *all generalized maximal fuzzy ideals of $R$ are equivalent.* ■

If $R$ is Artinian, we say that a fuzzy ideal $\widetilde{A}$ of $R$ is of *maximal chain* if the level ideals of $\widetilde{A}$ form a composition series.

**Theorem 6.92** *Let $R$ be Artinian. Then $R$ is local if and only if every fuzzy ideal of $R$ of maximal chain is local.* ■

A fuzzy ideal $\widetilde{A}$ of $R$ is called *normalized* if $\widetilde{A}(0) = 1$.

**Theorem 6.93** *$R$ is a field if and only if the set of all normalized L-ideals of $R$ which are local coincides with the set of all generalized maximal fuzzy ideals of $R$.* ■

Throughout the remainder of the section, $S$ denotes a closed multiplicative system in $R$ such that $0 \notin S$ and which is *saturated*, i.e., $\forall x, y \in R, xy \in S$ implies $x, y \in S$, [6]. Let $RS^{-1}$ denote the corresponding ring of fractions. Then $RS^{-1} = \{\phi(r)/\phi(w) \mid r \in R,\ w \in S\}$, where $\phi$ is a homomorphism of $R$ into $RS^{-1}$ such that $\mathrm{Ker}\ \phi = \{x \in R |\ xw = 0$ for some $w$ in $S\}$ and the elements of $\phi(S)$ are units in $RS^{-1}$, [61, p. 222]. If $I$ is an ideal of $R$, we use the notation $IS^{-1}$ for the ideal of $RS^{-1}$ generated by $\phi(I)$.

**Definition 6.55** *Assume $\widetilde{A}$ and $\widetilde{A}'$ are fuzzy ideals of $R$ and $RS^{-1}$, respectively. Then $\widetilde{A}'$ is called the L-subring of $\widetilde{A}$ in $RS^{-1}$ if $Im(\widetilde{A}) = Im(\widetilde{A}')$ and $\widetilde{A}'_t = \widetilde{A}_t S^{-1}$ $\forall t \in Im(\widetilde{A})$.*

In the following example, we show that not every fuzzy ideal of $R$ has a localized fuzzy subring in $RS^{-1}$.

We say that the ring of fractions $RS^{-1}$ is a localized ring of $R$ at a prime ideal, if there exists a prime ideal $P$ of $R$ such that $S = cP$, the complement of $P$ in $R$.

**Theorem 6.94** *The ring of fractions $RS^{-1}$ is a localized ring of $R$ at a prime ideal of $R$ if and only if there exists a fuzzy ideal $\widetilde{A}$ of $R$ which has a localized fuzzy subring $\widetilde{A}'$ in $RS^{-1}$ and $S \supseteq R \backslash \widetilde{A}_{\#}$. In such a case, $\widetilde{A}_{\#}$ is a prime ideal of $R$ and $RS^{-1}$ is a localized ring of $R$ at $\widetilde{A}_{\#}$.* ■

Let $\mathcal{S}$ be a set of fuzzy singletons. Define the fuzzy subset $\widetilde{C}_{\mathcal{S}}$ of $R$ by $\forall x \in R$,

$$\widetilde{C}_{\mathcal{S}}(x) = \vee\{t \mid x_t \in \mathcal{S}\}.$$

If $r \in R$ and $x_t$ is a fuzzy singleton, we let $rx_t$ denote the fuzzy singleton $(rx)_t$.

**Theorem 6.95** *Let $\mathcal{S}$ be a set of fuzzy singletons of $R$. Let $\widetilde{C}$ be the fuzzy subset of $R$ defined by $\forall x \in R$,*

$$\widetilde{C}(x) = \vee\{(\sum_{i=1}^{k} r_i(x_i)_{t_i})(x) \mid r_i \in R, x_{t_i} \in \mathcal{S}, i = 1, \ldots, k; k \in \mathbb{N}\}.$$

*Then $\widetilde{C} = \langle \mathcal{S} \rangle$, where $\langle \mathcal{S} \rangle = \left\langle \widetilde{C}_{\mathcal{S}} \right\rangle$.* ■

**Lemma 6.96** *Suppose that $\widetilde{A}$ is a fuzzy ideal of $R$ such that $\widetilde{A}$ has the sup property. Let $\mathcal{S} = \cup_{t \in [0,1]} \mathcal{S}_t$, where $\mathcal{S}_a \subseteq \{x_t \mid x \in R, \widetilde{A}(x) = t\}$ if $t \in Im(\widetilde{A})$ and $\mathcal{S}_t = \emptyset$ if $t \in [0,1] \backslash Im(\widetilde{A})$. Then $\widetilde{A} = \langle \mathcal{S} \rangle \cup 0_{\widetilde{A}(0)}$ if and only if $\widetilde{A}_t = (foot( \cup_{s \geq t} \mathcal{S}_s))$ $\forall t \in Im(\widetilde{A})$.* ■

**Proposition 6.97** *Suppose $\widetilde{A}$ is a fuzzy ideal of $R$ such that $\widetilde{A}$ has the sup property. Let*

$$\mathcal{S} = \cup_{t \in [0,1]} \mathcal{S}_t,$$

*where $\mathcal{S}_t \subseteq \{ x_t \mid x \in R, \widetilde{A}(x) = t\}$ if $t \in Im(\widetilde{A})$ and $\mathcal{S}_t = \emptyset$ if $t \in [0,1] \backslash Im(\widetilde{A})$. If $foot(\cup_{s \geq t}(\mathcal{S}_s)$ is a minimal generating set for $\widetilde{A}_t$ $\forall t \in Im(\widetilde{A})$, then $\mathcal{S}$ is a minimal generating set for $\widetilde{A}$.* ■

**Definition 6.56** *Let $\mathcal{S}$ denote a set of fuzzy singletons such that if $x_t$ and $x_s \in \mathcal{S}$, then $t = s > 0$. Let $\widetilde{A}$ be a fuzzy ideal of $R$. Then $\mathcal{S}$ is called an $\mathcal{S}$-minimal generating set for $\widetilde{A}$ if $\widetilde{A} = \langle \mathcal{S} \rangle \cup 0_{\widetilde{A}(0)}$ and $\forall x \in foot(\mathcal{S})$, there does not exist $w \in S$ such that $sw \in (foot(\mathcal{S}) \backslash \{x\})$.*

**Proposition 6.98** *Let $\mathcal{S}$ denote a set of fuzzy singletons such that if $x_t$ and $x_s \in \mathcal{S}$, then $t = s > 0$. Let $\widetilde{A}$ be a fuzzy ideal of $R$ such that $\widetilde{A}$ has the sup property. If $\mathcal{S}$ is an $\mathcal{S}$-minimal generating set for $\widetilde{A}$, then $\mathcal{S}$ is a minimal generating set for $\widetilde{A}$.* ■

If $x_t$ is a fuzzy singleton of $R$, then $\phi(x_t) = \phi(x)_t$. Let $\widetilde{A}$ and $\widetilde{A}'$ be fuzzy ideals of $R$ and $RS^{-1}$, respectively, such that $\widetilde{A}'$ is a localized fuzzy subring of $\widetilde{A}$ in $RS^{-1}$. If $\mathcal{S}$ is a set of fuzzy singletons which generate $\widetilde{A}$, then $\{\phi(x)_t \mid x_t \in \mathcal{S}\}$ generates $\widetilde{A}'$ and we say that $\widetilde{A}$ and $\widetilde{A}'$ have the same set of generators and we write $\phi(\mathcal{S})$ for $\{\phi(x)_t \mid x_t \in \mathcal{S}\}$.

**Theorem 6.99** *Let $\widetilde{C}, \widetilde{C}'$ be fuzzy ideals of $R$, $RS^{-1}$, respectively, such that $\widetilde{C}$ has the sup property. If $\widetilde{C}$ has an $\mathcal{S}$-minimal generating set and $\widetilde{C}'$ is a localized fuzzy subring of $\widetilde{C}$ in $RS^{-1}$, then $\widetilde{C}$ and $\widetilde{C}'$ have the same minimal generating sets and $Im(\widetilde{C}) = Im(\widetilde{C}')$. Conversely, if $RS^{-1}$ is a localized ring at a prime ideal of $R$, $\widetilde{C}$ and $\widetilde{C}'$ have the same minimal generating sets and $Im(\widetilde{C}) = Im(\widetilde{C}')$, then $\widetilde{C}'$ is a localized fuzzy subring of $\widetilde{C}$ in $RS^{-1}$.* ■

We now apply our results in the following example.

**Example 6.35** *Let $R$ denote the polynomial ring $\mathbb{R}[x, y, z]$ in the algebraically independent indeterminates $x, y, z$ over the field of $\mathbb{R}$ of real numbers. Then the ideal $\langle x^2 - y, x^2 z \rangle$ represents the nonlinear system of equations*

$$\begin{aligned} x^2 - y &= 0 \\ x^2 z &= 0 \end{aligned}$$

*and has the reduced primary representation*

$$\langle x^2 - y, x^2 z \rangle = \langle x^2 - y, z \rangle \cap \langle x^2, y \rangle .$$

*Hence*

$$\sqrt{\langle x^2 - y, x^2 z \rangle} = \langle x^2 - y, z \rangle \cap \langle x, y \rangle$$

*and the prime ideals $\langle x^2 - y, z \rangle$ and $\langle x, y \rangle$ display the solution of the nonlinear system of equations via their corresponding irreducible affine varieties.*

*Now consider the following nonlinear system of fuzzy intersection equations*

$$\begin{aligned} (x_s)^2 - y_t &= 0_{\frac{1}{4}} \\ (x_s)^2 z_u &= 0_{\frac{1}{2}}. \end{aligned}$$

*Then this system is represented by the fuzzy ideal* $\mu = \left\langle (x^2 - y)_{\frac{1}{1}}, (x^2 z)_{\frac{1}{2}} \right\rangle$ *and* $\mathcal{S} = \{(x^2 - y)_{\frac{1}{1}}, (x^2 z)_{\frac{1}{2}}\}$ *is a minimal generating set for* $\mu$. *In order to examine the system locally we consider either of the prime ideals* $\langle x^2 - y, z\rangle$ *and* $\langle x, y\rangle$, *say,* $P = \langle x, y\rangle$, *and we form the quotient ring* $R_P$. *Then in* $R_P$, *the extended ideal [61] of* $\langle x^2 - y, z\rangle$ *is*

$$\left\langle x^2 - y, x^2 z\right\rangle^e = \left\langle x^2, y\right\rangle^e.$$

*Hence the corresponding nonlinear system of fuzzy intersection equations is*

$$\begin{aligned} y_t &= 0_{\frac{1}{4}} \\ (x_s)^2 &= 0_{\frac{1}{2}}. \end{aligned}$$

*This system is represented by the fuzzy ideal* $\widetilde{B} = \left\langle y_{\frac{1}{4}}, (x^2)_{\frac{1}{2}} \right\rangle$ *in* $R$. *Now* $\mathcal{S} = \{y_{\frac{1}{4}}, (x^2)_{\frac{1}{2}}\}$ *is a minimal generating set for* $\widetilde{B}$. *By Theorem 6.99, we have that* $\mathcal{S}$ *is a minimal generating set for the fuzzy localized subring* $\widetilde{B}'$ *of* $\widetilde{B}$ *in* $R_P$. *Hence* $\widetilde{B}'$ *represents the same system of fuzzy intersection equations as* $\widetilde{B}$ *does.*

*If we consider the prime ideal* $N = \left\langle x^2 - y, z\right\rangle$, *then in* $R_N$

$$\left\langle x^2 - y, x^2 z\right\rangle^e = \left\langle x^2 - y, z\right\rangle^e.$$

*Hence the corresponding nonlinear system of fuzzy intersection equations is*

$$\begin{aligned} (x_b)^2 - y_a &= 0_{\frac{1}{4}} \\ z_u &= 0_{\frac{1}{2}}. \end{aligned}$$

*This system is represented by the fuzzy ideal* $\widetilde{C} = \left\langle (x^2 - y)_{\frac{1}{4}}, z_{\frac{1}{2}} \right\rangle$ *in* $R$. *We have that* $\{(x^2 - y)_{\frac{1}{4}}, z_{\frac{1}{2}}\}$ *is a minimal generating set for* $\widetilde{C}$ *and also for the fuzzy localized subring* $\widetilde{C}'$ *of* $\widetilde{C}$ *in* $R_N$.

## 6.9 Local Examination of Fuzzy Intersection Equations

In this section, $R$ denotes a commutative ring with identity. The notion of algebraic fuzzy varieties was introduced in order to use primary representation theory of fuzzy ideals to examine the solution of fuzzy intersection equations. Local concepts of subrings were developed in order to lay the ground work for the examination of fuzzy intersection equations locally. In this section, we carry out a local examination of fuzzy intersection equations. We show that a system of fuzzy intersection equations can be examined locally to obtain the general solution to the crisp part of the system. The details can be found in [2].

Let $\mathcal{M}$ be a multiplicative system in $R$ [61, p. 46]. Let $N = \{x \in R \mid mx = 0 \text{ for some } m \in \mathcal{M}\}$. Then $N$ is an ideal of $R$. If $N = \{0\}$, then $\mathcal{M}$ is said to be regular. Let $h$ be the natural homomorphism of $R$ onto $R/N \subseteq R_{\mathcal{M}}$, the quotient ring of $R$ with respect to $\mathcal{M}$. If $I$ is an ideal of $R$, then the ideal in $R_{\mathcal{M}}$ generated by $h(I)$ is called the *extended ideal* of $I$ in $R_{\mathcal{M}}$ and is denoted by $h(I)^e$. If $J$ is an ideal of $R_{\mathcal{M}}$, then $h^{-1}(J)$ is called the *contracted ideal* of $J$ in $R$.

Let $\widetilde{A}$ be a fuzzy ideal of $R$. Define the fuzzy subset $h(\widetilde{A})^e$ of $R_{\mathcal{M}}$ by $\forall y \in R_{\mathcal{M}}$, $h(\widetilde{A})^e(y) = \vee\{t \in [0,1] \mid y \in (h(\widetilde{A}))^t_{\mathcal{M}}\}$. Then $h(\widetilde{A})^e$ is a fuzzy ideal of $R_{\mathcal{M}}$. Let $t \in [0,1]$. Now $y \in h(\widetilde{A})^t \Leftrightarrow h(\widetilde{A})(y) \geq t \Leftrightarrow \vee\{\widetilde{A}(x) \mid h(x) = y\} \geq t \Leftarrow \exists x \in \widetilde{A}^t$ such that $h(x) = y \Leftrightarrow y \in h(\widetilde{A}^t)$, where the "$\Leftarrow$" becomes "$\Leftrightarrow$" if $\widetilde{A}$ has the sup property. Hence if $\widetilde{A}$ has the sup property, then $h(\widetilde{A})^t = h(\widetilde{A}^t)$ and so $(h(\widetilde{A}))^t_{\mathcal{M}} = h(\widetilde{A}^t)_{\mathcal{M}}$. We use the notation $\widetilde{A}^e$ for $h(\widetilde{A})^e$ at times. If $I$ is an ideal of $R$, we sometimes use the notation $I^e$ for $h(I)^e$. If $\widetilde{B}$ is a fuzzy ideal of $R_{\mathcal{M}}$, then we use the notation $\widetilde{B}^c$ for $h^{-1}(\widetilde{B})$ at times. If $J$ is an ideal of $R_{\mathcal{M}}$, we sometimes use the notation $J^c$ for $h^{-1}(J)$.

Suppose that $\widetilde{A}$ has the sup property. Then $(\widetilde{A}^e)(y) = t \Leftrightarrow h(\widetilde{A})^e(y) = t \Leftrightarrow \vee\{s \mid y \in h(\widetilde{A})^s_{\mathcal{M}} = h(\widetilde{A}^s)_{\mathcal{M}}\} = t \Leftrightarrow t$ is maximal in $[0,1]$ such that $y \in h(\widetilde{A})^t_{\mathcal{M}} = h(\widetilde{A}^t)_{\mathcal{M}} = h(\widetilde{A}^t)^e$ (since $\widetilde{A}$ has the sup property) $= (\widetilde{A}^t)^e$. Hence $(\widetilde{A}^e)^t = (\widetilde{A}^t)^e$ $\forall t \in [0,1]$.

**Theorem 6.100** *Let $\widetilde{B}$ be a primary fuzzy ideal of $R_{\mathcal{M}}$. Then $\sqrt{\widetilde{B}^c} = (\sqrt{\widetilde{B}}\,)^c$.* ■

**Theorem 6.101** *Let $\widetilde{A}$ be a primary fuzzy ideal of $R$ such that $\widetilde{A}_*$ is disjoint from $\mathcal{M}$.*

*(1) Then $\widetilde{A} = \widetilde{A}^{ec}$ and $\sqrt{\widetilde{A}} = (\sqrt{\widetilde{A}})^{ec}$.*

*(2) Then $\widetilde{A}^e$ is primary and $\sqrt{\widetilde{A}^e} = (\sqrt{\widetilde{A}})^e$.* ■

**Lemma 6.102** *Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy ideals of $R_{\mathcal{M}}$. Then $(\widetilde{A} \cap \widetilde{B})^c = \widetilde{A}^c \cap \widetilde{B}^c$.* ■

**Theorem 6.103** *Let $\widetilde{A}$ be a fuzzy ideal of $R$ such that $\widetilde{A}$ has a reduced primary representation $\widetilde{A} = \cap_{i=1}^n \widetilde{A}_i$. Suppose that for $1 \leq i \leq k$, $(\widetilde{A}_i)_* \cap \mathcal{M} = \emptyset$ and that for $k+1 \leq i \leq n$, $(\widetilde{A}_i)_* \cap \mathcal{M} \neq \emptyset$. Then $\widetilde{A}^e = \cap_{i=1}^k \widetilde{A}_i{}^e$ is a reduced primary representation. Furthermore, $\widetilde{A}^{ec} = \cap_{i=1}^k \widetilde{A}_i$.* ■

**Example 6.36** *Let $R$ denote the polynomial ring $\mathbb{R}[x,y,z]$ in algebraically independent indeterminates $x, y, z$ over the field $\mathbb{R}$ of real numbers. Then the ideal $\langle x^2 - y, x^2 z\rangle$ has the reduced primary representation*

$$\langle x^2 - y, x^2 z\rangle = \langle x^2 - y, z\rangle \cap \langle x^2, y\rangle .$$

*We also have*

$$\sqrt{\langle x^2 - y, x^2 z\rangle} = \langle x^2 - y, z\rangle \cap \langle x, y\rangle .$$

*Now consider the nonlinear system of fuzzy singletons*

$$\begin{array}{lcl} (x_s)^2 - y_t & = & 0_{\frac{1}{4}} \\ (x_s)^2 z_u & = & 0_{\frac{1}{2}}. \end{array} \qquad (6.9.1)$$

*The solution to system (6.9.1) is*

$$\{(0, 0, r) \mid r \in \mathbb{R}\} \cup \{(s, s^2, 0) | s \in \mathbb{R}\},$$

$t = \frac{1}{4}$, $s \wedge u = \frac{1}{2}$. *Let* $\widetilde{A}$ *denote the fuzzy ideal* $\left\langle (x^2 - y)_{\frac{1}{4}}, (x^2 z)_{\frac{1}{2}} \right\rangle \cup 0_1$. *Then*

$$\widetilde{A}(r) = \begin{cases} 1 & \text{if } r = 0 \\ \frac{1}{2} & \text{if } r \in \langle x^2 z\rangle \setminus \{0\} \\ \frac{1}{4} & \text{if } r \in \langle x^2 - y, x^2 z\rangle \setminus \langle x^2 z\rangle \\ 0 & \text{if } r \in R \setminus \langle x^2 - y, x^2 z\rangle . \end{cases}$$

*Define the fuzzy subset* $\widetilde{Q}_i$ *of* $R$, $i = 1, \ldots, 5$, *as in Example 6.3.3. Recall that* $\widetilde{Q}_i$ *is a primary fuzzy ideal of* $R$, $i = 1, \ldots, 5$ *and* $\widetilde{A} = \cap_{i=1}^{5} \widetilde{Q}_i$ *is a reduced primary representation of* $\widetilde{A}$.

*As before, we see that the crisp part of the solution to system (6.9.1) is displayed by* $\sqrt{\widetilde{Q}_1} \cap \sqrt{\widetilde{Q}_2}$ *while the fuzzy part is displayed by* $\sqrt{\widetilde{Q}_3} \cap \sqrt{\widetilde{Q}_4} \cap \sqrt{\widetilde{Q}_5}$. *(In order to see this more clearly, one should consider the irreducible fuzzy algebraic varieties corresponding to the* $\sqrt{\widetilde{Q}_i}$. *Then one would be concerned with* $c(\frac{1}{4}) = \frac{1}{2}$ *rather than* $\frac{1}{4}$ *and* $c(\frac{1}{2}) = \frac{1}{4}$ *rather than* $\frac{1}{2}$.*)*

*Consider the quotient ring* $R_P$, *where* $P$ *is the prime ideal* $\langle x, y\rangle$. *Since* $P \cap cP = \emptyset$, *we have in* $R_P$ *that* $\langle x^2 - y, x^2 z\rangle^e = \langle x^2, y\rangle^e$ *by [61, Theorem 17, p.225]. Now*

$$\widetilde{Q}_{1*} \cap cP = \langle x^2, y\rangle \cap cP = \emptyset,$$

$$\widetilde{Q}_{3*} \cap cP = \langle x^2\rangle \cap cP = \emptyset,$$

$$\widetilde{Q}_{5*} \cap cP = \langle 0\rangle \cap cP = \emptyset,$$

*while*

$$\widetilde{Q}_{2*} \cap cP = \langle x^2 - y, z\rangle \cap cP \neq \emptyset,$$

$$\widetilde{Q}_{4*} \cap cP = \langle z\rangle \cap cP \neq \emptyset.$$

*Thus by Theorem* 6.103, *we have in* $R_P$ *that* $\widetilde{A}^e = \widetilde{Q}_1{}^e \cap \widetilde{Q}_3{}^e \cap \widetilde{Q}_5{}^e$ *and so*

$$\widetilde{A}^e(r) = \begin{cases} 0 & \text{if } r \in R_P \backslash \left\langle x^2, y \right\rangle^e \\ \frac{1}{4} & \text{if } r \in \left\langle x^2, y \right\rangle^e \backslash \left\langle x^2 \right\rangle^e \\ \frac{1}{2} & \text{if } r \in \left\langle x^2 \right\rangle^e \backslash \{0\} \\ 1 & \text{if } r \in \{0\}. \end{cases}$$

*Hence by Theorem* 6.103,

$$\widetilde{A}^{ec}(r) = \widetilde{Q}_1 \cap \widetilde{Q}_3 \cap \widetilde{Q}_5(r) = \begin{cases} 0 & \text{if } r \in R \backslash \left\langle x^2, y \right\rangle \\ \frac{1}{4} & \text{if } r \in \left\langle x^2, y \right\rangle \backslash \left\langle x^2 \right\rangle \\ \frac{1}{2} & \text{if } r \in \left\langle x^2 \right\rangle \backslash \{0\} \\ 1 & \text{if } r \in \{0\}. \end{cases}$$

*Consider the nonlinear system of fuzzy singletons*

$$\begin{aligned} y_t &= 0_{\frac{1}{4}} \\ (x_s)^2 &= 0_{\frac{1}{2}}. \end{aligned} \qquad (6.9.2)$$

*Then* $\{(0,0,r) \mid r \in \mathbb{R}\}$, $t = \frac{1}{4}$, $s = \frac{1}{2}$ *is the solution to this system. It is represented by the fuzzy ideal* $\widetilde{B} = \left\langle (x_{\frac{1}{2}})^2, y_{\frac{1}{4}} \right\rangle \cup 0_1$. *Now* $\widetilde{B} = \widetilde{Q}_1 \cap \widetilde{Q}_3 \cap \widetilde{Q}_5$ *is a reduced primary representation of* $\widetilde{B}$. $\sqrt{\widetilde{Q}_1}$ *displays the crisp part of the solution while* $\sqrt{\widetilde{Q}_3} \cap \sqrt{\widetilde{Q}_5}$ *displays the fuzzy part. Now consider the prime ideal* $N = \left\langle x^2 - y, z \right\rangle$. *Since* $N \cap cN = \emptyset$, *we have in* $R_N$ *that* $\left\langle x^2 - y, x^2 z \right\rangle^e = \left\langle x^2 - y, z \right\rangle^e$. *Now*

$$\widetilde{Q}_{2*} \cap cN = \left\langle x^2 - y, x^2 z \right\rangle \cap cN = \emptyset,$$

$$\widetilde{Q}_{4*} \cap cN = \langle z \rangle \cap cN = \emptyset,$$

$$\widetilde{Q}_{5*} \cap cN = \langle 0 \rangle \cap cN = \emptyset,$$

*while*

$$\widetilde{Q}_{1*} \cap cN = \left\langle x^2, y \right\rangle \cap cN \neq \emptyset,$$

$$\widetilde{Q}_{3*} \cap cN = \left\langle x^2 \right\rangle \cap cN \neq \emptyset.$$

*Thus by Theorem* 6.103, *we have in* $R_N$ *that* $\widetilde{A}^e = \widetilde{Q}_2{}^e \cap \widetilde{Q}_4{}^e \cap \widetilde{Q}_5{}^e$ *and so*

$$\widetilde{A}(r) = \begin{cases} 0 & \text{if } r \in R_N \backslash \left\langle x^2 - y, z \right\rangle^e \\ \frac{1}{4} & \text{if } r \in \left\langle x^2 - y, z \right\rangle^e \backslash \langle z \rangle^e \\ \frac{1}{2} & \text{if } r \in \langle z \rangle^e \backslash \{0\} \\ 1 & \text{if } r \in \{0\}. \end{cases}$$

*Hence by Theorem* 6.103,

$$\widetilde{A}^{ec}(r) = \widetilde{Q}_2 \cap \widetilde{Q}_4 \cap \widetilde{Q}_5(r) = \begin{cases} 0 & \text{if } r \in R \setminus \langle x^2 - y, z \rangle \\ \frac{1}{4} & \text{if } r \in \langle x^2 - y, z \rangle \setminus \langle z \rangle \\ \frac{1}{2} & \text{if } r \in \langle z \rangle \setminus \{0\} \\ 1 & \text{if } r \in \{0\}. \end{cases}$$

*Consider the nonlinear system of fuzzy singletons*

$$\begin{aligned} (x_s)^2 - y_t &= 0_{\frac{1}{4}} \\ z_u &= 0_{\frac{1}{2}}. \end{aligned} \tag{6.9.3}$$

*Then* $\{(w, w^2, 0) | s \in \mathbb{R}\}$, $s \wedge t = \frac{1}{4}$, *and* $u = \frac{1}{2}$ *is the solution to this system. The system is represented by the fuzzy ideal* $\widetilde{C} = \left\langle (x^2 \quad y)_{\frac{1}{4}}, z_{\frac{1}{2}} \right\rangle \cup 0_1$. *Now* $\widetilde{C} = \widetilde{Q}_2 \cap \widetilde{Q}_4 \cap \mu \widetilde{Q}_5$ *is a reduced primary representation of* $\widetilde{C}$. $\sqrt{\widetilde{Q}_2}$ *displays the crisp part of the solution while* $\sqrt{\widetilde{Q}_4} \cap \sqrt{\widetilde{Q}_5}$ *displays the fuzzy part.*

*We have examined the system (6.9.1) locally. From the two examinations, we obtain for the crisp part of the solution* $\{(0, 0, r) \mid r \in \mathbb{R}\}$ *for (6.9.2) and* $\{(w, w^2, 0) \mid w \in \mathbb{R}\}$ *for (6.9.3). The union of these two gives us the crisp part of the solution to system (6.9.1). However the fuzzy solutions to (6.9.2) and (6.9.3) are* $t = \frac{1}{4}, s = \frac{1}{2}$ *and* $t \wedge s = \frac{1}{4}$, $u = \frac{1}{2}$, *respectively. The fuzzy part of the solution to (6.9.1) is* $t = \frac{1}{4}$ *and* $s \wedge u = \frac{1}{2}$. *The two "local" fuzzy solutions do not seem to give us the fuzzy part of the solution to (6.9.1), at least not immediately.*

*Consider all possible* $\wedge$*'s of the two fuzzy solutions above*

$$t \wedge s \wedge t = \frac{1}{4} \wedge \frac{1}{4}$$

$$t \wedge u = \frac{1}{4} \wedge \frac{1}{2}$$

$$s \wedge s \wedge t = \frac{1}{2} \wedge \frac{1}{4}$$

$$s \wedge u = \frac{1}{2} \wedge \frac{1}{2}.$$

*These equations reduce to*

$$s \wedge t = \frac{1}{4}$$

$$t \wedge u = \frac{1}{4}$$

$$s \wedge t = \frac{1}{4}$$

$$s \wedge u = \frac{1}{2}.$$

*Hence* $t = \frac{1}{4}$ *and* $s \wedge u = \frac{1}{2}$ *which is the solution to the original problem.*

It is an open problem to determine a general procedure to find the solution to the fuzzy part of the original problem from the local solutions.

An algorithm for solving fuzzy systems of intersection equations is given in [46] and an application to fuzzy graph theory is given in [47]. For a study if $L$-intersection equations for $L$ a complete distributive lattice, the reader is referred to [22].

The interested reader can consult [38, 40, 41, 46] for more results along these lines.

## 6.10 More on Coding Theory

In this section, we let $F$ denote the field of integers modulo 2. We define a fuzzy code as a fuzzy subset of $F^n$, where $F^n = \{(a_1, ..., a_n) \mid a_i \in F, i = 1, ..., n\}$ and $n$ is a fixed arbitrary positive integer. We recall that $F^n$ is a vector space over $F$. We give an analysis of the Hamming distance between two fuzzy codewords and the error-correcting capability of a code in terms of its corresponding fuzzy code. We assume that the channel is a binary symmetric channel so that an error in any one location is equally likely as an error in another. The results appearing in the first part of this section are from [17].

**Definition 6.57** *$\forall u = (u_1, ..., u_n) \in F^n$, define the fuzzy subset $\tilde{A}_u$ of $F^n$ by $\forall v = (v_1, ..., v_n) \in F^n, \tilde{A}_u(v) = p^{n-d}q^d$, where $d = \sum_{i=1}^n \mid u_i - v_i \mid$ and $p$ and $q$ are fixed positive real numbers such that $p + q = 1$.*

Define $\Phi : F^n \to \tilde{A}^n = \{\tilde{A}_u \mid u \in F^n\}$ by $\Phi(u) = \tilde{A}_u \ \forall u \in F^n$. Then $\Phi$ is a one-to-one function of $F^n$ onto $\tilde{A}^n$.

**Definition 6.58** *If $C \subseteq F^n$, then $\Phi(C)$ is called a* fuzzy code *corresponding to the code $C$. If $c \in C$, then $\tilde{A}_c$ is called a* fuzzy codeword.

We consider an example. Let $n = 3$ and $C = \{(0,0,0), (1,1,1)\}$. If $(0,0,0)$ is transmitted and $(0,1,0)$ is received, then assuming $q < 1/2$, there is a greater likelihood that $(0,0,0)$ was transmitted than $(1,1,1)$ (since we are assuming burst errors do not occur).

Let $u \in F^n$ and $v^n \in F^n$. Then $\sum_{i=1}^n |u_i - v_i|$ is the number of coordinate positions in which $u$ and $v$ differ. The number of errors required to transform $u$ into $v$ equals this number. We let $d(u, v)$ denote $\sum_{i=1}^n |u_i - v_i|$. $d(u, v)$ is called the *Hamming distance* of $u, v$.

**Definition 6.59** *Let $C \subseteq F^n$ be a code. The* minimum distance *of $C$ is defined to be $d_{min}(C) = \wedge\{d(a, b) \mid a, b \in C, a \neq b\}$.*

If $C$ is a subspace of $F^n$, then $d_{min}(C) = \wedge\{d(a, 0) \mid a \in C, a \neq 0\}$, where $0 = (0, ..., 0)$. Now $d(a, 0)$ is the number of nonzero entries in $a$ and is called the *weight* of $a$ and often is denoted by $|a|$. When a code $C$ is a subspace of $F^n$, we called it a *linear code.*

Let $[[\quad]]$ denote the greater integer function on the real numbers. For any code $C \subseteq F^n$,

$E_C = [[(d_{min}(C) - 1)/2]]$

is the maximum number of errors allowed in the channel for each $n$ bits transmitted for which received signals may be correctly decoded. One of the most important problems in coding theory is to define codes whose codewords are 'far apart' from each other as possible or whose value $E_C$ is maximized. It is also desirable to decode uniquely. For example, let $n = 3$ and $C = \{(0,0,0),(1,0,1)\}$. Then $d_{min}(C) = 2$. Suppose that a codeword is transmitted across the channel and $(0,0,1)$ is received. Then $(0,0,1)$ is of distance 1 from both the codewords $(0,0,0)$ and $(1,0,1)$. Hence $(0,0,1)$ cannot be decoded uniquely. Thus in order to always be able to correct a single error, we must have $d_{min}(C)$ at least equal to 3. If $C = \{(0,0,0),(1,1,1)\}$, then $(0,0,1)$ is decoded as $(0,0,0)$ since it is closer to $(0,0,0)$ than it is to $(1,1,1)$.

We now examine fuzzy codes. For any code $C \subseteq F^n$, we have seen that there is a corresponding fuzzy code $\Phi(C)$. If $u \in F^n$ is a received word and c is a codeword, i. e., $c \in C$, then $\tilde{A}_c(u)$ is the probability that c was transmitted. Fuzzy subsets appear to be a natural setting for the study of codes in that probability of error in the channel is included in the definition of the (fuzzy) code. In the following, we assume that $p \neq q$.

**Definition 6.60** *Let $C \subseteq F^n$ be a code. Define $\theta : F^n \to \{\tilde{A} \mid \tilde{A}$ is a fuzzy subset of $F^n\}$ by $\forall u \in F^n$, $\theta(u) = \{\tilde{A}_c \mid c \in C,\ \tilde{A}_c(u) \geq \tilde{A}_b(u)\ \forall b \in C\}$.*

A code for which $|\theta(u)| = 1\ \forall u \in F^n$ is uniquely decodable. In such a case, $u$ is decoded as $\Phi^{-1}(\theta(u))$.

As mentioned previously, an important criteria in designing good codes is spacing the codewords as far apart from each other as possible. The Hamming distance is the metric used in $F^n$ to measure distance. Analogously, the generalized Hamming distance between fuzzy subsets may be used as a metric in $\tilde{A}^n$. It is defined by $\forall \tilde{A}_u, \tilde{A}_v \in \tilde{A}^n$,

$d(\tilde{A}_u, \tilde{A}_v) = \sum_{w \in F^n} |\tilde{A}_u(w) - \tilde{A}_v(w)|$.

The theorem which follows shows that $d(\tilde{A}_u, \tilde{A}_v)$ is independent of $n$.

**Theorem 6.104** *Let $u, v \in F^n$ be such that $d(u,v) = d$. If $p \neq q$ and $p \neq 0,1$, then $d(\tilde{A}_u, \tilde{A}_v) = \sum_d$, where*

$\sum_d = \sum_{i=0}^{d} \binom{d}{i} |p^i q^{d-i} - p^{d-i} q^i|$. ■

**Lemma 6.105** *If $p \neq q$ and $p \neq 0,1$, then $\sum_0 < \sum_1 = \sum_2 < \sum_3 = \sum_4 < \sum_5 = \dots$.* ■

**Lemma 6.106** *Let $p \neq q$ and $p \neq 0,1$. If $d(a,b) < d(u,v)$, then $d(\tilde{A}_a, \tilde{A}_b) \leq d(\tilde{A}_u, \tilde{A}_v)$.* ■

For a fuzzy code $\Phi(C)$, we define its *minimum distance* by
$d_{min}(\Phi(C)) = \wedge \left\{ d(\tilde{A}_c, \tilde{A}_b) \mid \tilde{A}_c, \tilde{A}_b \in \Phi(C), \tilde{A}_c \neq \tilde{A}_b \right\}$

**Theorem 6.107** *Let $C_1$ and $C_2$ be two codes used in the same channel. If $p \neq q$ and $p \neq 0, 1$, then $E_{C_1} = E_{C_2}$ if and only if $d_{min}(\Phi(C_1)) = d_{min}(\Phi(C_2))$.*

**Proof.** Let $d_i = d_{min}(\Phi(C_i))$ for $i = 1, 2$. If $E_{C_1} = E_{C_2}$, then either $d_1 = d_2$ in which case the desired result holds or $d_2 = d_1 + 1$, where $d_1$ is an odd positive integer. From Lemma 6.105, we have $d_{min}(\Phi(C_1)) = d_{min}(\Phi(C_2))$.
Conversely, if $d_{min}(\Phi(C_1)) = d_{min}(\Phi(C_2))$, then it follows from Lemma 6.105 that either $d_1 = d_2$ and so $E_{C_1} = E_{C_2}$ or $d_1 = d_2 \pm 1$, where $d_1 \wedge d_2$ is odd and so $E_{C_1} = E_{C_2}$. ■

Let $M$ be a subset of $\mathbb{R}$. Let $C \subseteq F^n$ be a code. Suppose $c \in C$ is transmitted across the channel and that $u \in F^n$ is received. The signals that are transmitted are usually distorted by varying degrees. The electrical receiver may record the signals in one of two ways. The electrical waves representing the received word $u$ is measured bit by bit as real numbers. The signals then are either recorded as $n$-tuples over $M$ or each bit of $u$ is transformed into an element of $F$. $u$ is then decoded in $F^m$ for some positive integer $m < n$. For example, suppose that $(1, 0, 1)$ is encoded as $c = (1, 0, 1, 0)$ and transmitted electronically across the channel. Suppose the received waves are measured as $u = (1.02, 0.52, 0.98, 0.02)$ and recorded as $v = (1, 1, 1, 0)$. Then some information is lost in recording $u$ as $v$. Some possible directions for further study to overcome this loss are suggested in [15]. For example, let $x \in F^m$ and define the fuzzy subset $\tilde{A}_x$ of $M^n$ by $\forall y \in M^n$,
$\tilde{A}_x(y) = p(x, y)$,
where $p(x, y)$ is the probability that if $x$ is encoded and transmitted across the channel, $y$ is received. Then soft decoding may be studied via $\{\tilde{A}_x \mid x \in F^m\}$.

We have assumed that errors in the transmission of words across a noisy channel were symmetric in nature, i. e., that the probability of $1 \to 0$ and $0 \to 1$ crossover failures were equally likely. However errors in VSLI circuits and many computer memories are on a unidirectional nature, [8]. A unidirectional error model assumes that both $1 \to 0$ and $0 \to 1$ crossovers can occur, but only one type of error occurs in a particular data word. This has provided the basis for a new direction in coding theory and fault tolerance computing.

Also, the failure in the memory cells of some of the LSI single transistor-cell memories and NMOS memories are most likely caused by leakage of charge. If we represent the presence of charge in a cell by 1 and the absence of charge by 0, then the errors in those type of memories can be modeled as $1 \to 0$ type asymmetric errors, [8].

The results in the remainder of this section are from [15]. Once again $F$ denotes the field of integers modulo 2 and $F^n$ the vector space of $n$-tuples over $F$. We let $p$ denote the probability that there is no error in transmission, i. e., a transmitted 1 will be received as a 1 and a transmitted 0 will be received as a 0. Let $q = 1 - p$. Then $q$ is the probability that there is an error in transmission in an arbitrary bit.

In the following definition, we define a fuzzy word for a unidirectional error model.

**Definition 6.61** *Let $u = (u_1, ..., u_n) \in F^n$. Define the fuzzy subset $\tilde{A}_u$ of $F^n$ as follows:* $\forall v = (v_1, ..., v_n) \in F^n$,

$$\tilde{A}_u(v) = \begin{cases} 0 & \text{if } k_1 \wedge k_2 \neq 0, \\ p^{m-d}q^d & \text{otherwise,} \end{cases}$$

*where*
$k_1 = \sum_{i=1}^{n} 0 \vee (u_i - v_i)$, $k_2 = \sum_{i=1}^{n} 0 \vee (v_i - u_i)$,

$$d = \begin{cases} k_1 & \text{if } k_2 = 0, \\ k_2 & \text{if } k_1 = 0, \end{cases}$$

$$m = \begin{cases} \sum_{i=1}^{n} u_i & \text{if } k_2 = 0, \\ n - \sum_{i=1}^{n} u_i & \text{if } k_1 = 0, \\ (\sum_{i=1}^{n} u_i) \vee (n - \sum_{i=1}^{n} u_i) & \text{if } k_1 = k_2 = 0 \end{cases}$$

In Definition 6.61, $\tilde{A}_u(v)$ is zero if both one and zero transitions have occurred. It allows either, by themselves, to occur in a given received word. In the case that a received word is the same as that transmitted, one may choose either the one's or the zero's as possibly toggling. In our definition, we choose whichever there are more of. For example, if there are more ones, the definition would expect $1 \to 0$ transitions only. Hence $m$ equals the number of 1's and $d = 0$. In any event, we only allow $m$ to take the value of the number of bits which can transition in calculating our membership function. The bits that cannot change are not considered in the function.

In the following definition, we define a fuzzy word for an asymmetric error model in which only 1-errors may occur.

**Definition 6.62** *Let $u = (u_1, ..., u_n) \in F^n$. Define the fuzzy subset $\tilde{A}_u$ of $F^n$ as follows:* $\forall v = (v_1, ..., v_n) \in F^n$,

$$\tilde{A}_u(v) = \begin{cases} 0 & \text{if } d_h > d, \\ p^{m-d}q^d & \text{otherwise,} \end{cases}$$

*where*
$d = \sum_{i=1}^{n} 0 \vee (u_i - v_i)$, $d_h = \sum_{i=1}^{n} |u_i - v_i|$, *and* $m = \sum_{i=1}^{n} u_i$.

In Definition 6.62 only 1's are considered in the membership function calculation. They are the only bits that may transition under this asymmetric model. If both 1's and 0's transition, the membership is zero since this is not allowable by definition of asymmetric errors.

Let $\tilde{A}^n = \{ \tilde{A}_u \mid u \in F^n\}$. Define $\Psi : F^n \to \tilde{A}^n$ by $\Psi(u) = \tilde{A}_u$ for all $u \in F^n$. Then $\Psi$ is a one-to-one function of $F^n$ onto $\tilde{A}^n$. Let $C \subseteq F^n$ be a code. Then $\Psi(C)$ is called a *fuzzy code.* If $c \in C$, then $\tilde{A}_c$ is called a *fuzzy codeword.*

Previously in this section, the Hamming distance $d_H(u,v) = \sum_{i=1}^n |u_i - v_i|$ of $u, v \in F^n$ was given as was the generalized Hamming distance between fuzzy subsets of $\tilde{A}^n$. It is defined by $\forall \tilde{A}_u, \tilde{A}_v \in \tilde{A}^n$,

$d(\tilde{A}_u, \tilde{A}_v) = \sum_{w \in F^n} |\tilde{A}_u(w) - \tilde{A}_v(w)|$.

Since in this section we are using the asymmetric model, the asymmetric distance metric [8] may also provide a useful comparison in $F^n$.

**Definition 6.63** *Define the function $d_a : F^n \times F^n \to \mathbb{R}$ by $\forall u, v \in F^n$,*

$$d_a(u,v) = N(u,v) \vee N(v,u),$$

*where $N : F^n \times F^n \to \mathbb{R}$ is such that $N(u,v) = \sum_{i=1}^n 0 \vee (u_i - v_i)$, $u = (u_1, ..., u_n)$, and $v = (v_1, ..., v_n)$. Then $d_a$ is called the* asymmetric distance metric.

It was previously noted that the generalized Hamming distance for symmetric errors was independent of $n$. We now show that a similar result holds for asymmetric errors, but not for unidirectional errors.

The next example shows that for unidirectional errors $d(\tilde{A}_u, \tilde{A}_v)$ depends not only on $d(u,v)$, but on $n$ as well.

**Example 6.37** *Let $n = 2$. Let $u = (0,0)$ and $v = (1,1)$. Then*

$d(\tilde{A}_u, \tilde{A}_v) = |p^2q^0 - p^0q^2| + |p^1q^1 - p^1q^1| + |p^1q^1 - p^1q^1| + |p^0q^2 - p^2q^0|$
$= 2(p-q)$

*and*

$d_H(u,v) = 2.$

*Now let $n = 3$, $w = (1,1,1)$, and $x = (0,0,1)$ in $F^3$. Then*

$d(\tilde{A}_u, \tilde{A}_v) = |p^0q^3 - p^0q^1| + |p^1q^2 - p^2q^0| + |p^1q^2 - 0| + |p^2q^1 - p^1q^1| + |p^1q^2 - 0| + |p^2q^1 - p^1q^1| + |p^2q^1 - 0| + |p^3q^0 - p^0q^2|$
$= q - q^2 - q^3 + p^3 + p^2 + 2pq + pq^2 + pq^2 - p^2q \neq 2(p-q)$

*and*

$d_H(w,x) = 2.$

*We have the same Hamming distance between $u, v$ and $w, x$, respectively, but different distances between the corresponding fuzzy codewords.*

We now state a result which says that the distance between fuzzy codewords is independent of $n$ for ideal symmetric errors. The result holds when the Hamming distance or the asymmetric distance is used as the distance metric between two codewords.

**Theorem 6.108** *Let $u, v \in F^n$ and set $d_a(u, v) = d_a$. If $p \neq q$ and $0 \neq p \neq 1$, then $d_h(\tilde{A}_u, \tilde{A}_v) = \Gamma_{d_a}$, where $\Gamma_{d_a} = 2 - 2q^{d_a}$.* ■

**Lemma 6.109** $\Gamma_1 < \Gamma_2 < \ldots < \Gamma_n < 2.$ ■

If instead of using the Hamming distance to measure the distance between two fuzzy codewords, we used the asymmetric distance, so that
$D_a(\tilde{A}_u, \tilde{A}_v) = (\sum_{w \in F^n} (\tilde{A}_u(w) - \tilde{A}_v(w)) \vee (\sum_{w \in F^n} (\tilde{A}_v(w) - \tilde{A}_u(w))),$
then the following theorem holds.

**Theorem 6.110** *Let $u, v \in F^n$ and set $d_a(u, v) = d_a$. If $p \neq q$ and $0 \neq p \neq 1$, then $D_a(\tilde{A}_u, \tilde{A}_v) = \Phi_{d_a}$, where $\Phi_{d_a} = 1 - d^{d_a}$.* ■

**Lemma 6.111** $\Phi_1 < \Phi_2 < \ldots < \Phi_n < 1.$ ■

As the asymmetric distance between codewords on which fuzzy codes will be based becomes large, there is only a small increase in the measurable distance between codewords. For unidirectional errors, the case is that the space of the code will affect the distance between the fuzzy code words. These issues must be taken into account in designing fuzzy codes.

## 6.11 Other Applications

We now mention some other ways fuzzy abstract algebra has been applied. The paper [35] deals with the classification of knowledges when they are endowed with some fuzzy algebraic structure. By using the quotient group of symmetric knowledges, an algebraic method is given in [35] to classify them. Also the (anti) fuzzy subgroup construction is used to classify knowledges.

In the paper [20], fuzzy points are regarded as data and fuzzy objects are constructed from the set of given data on an arbitrary group. Using the method of least squares, optimal fuzzy subgroups are defined for the set of data and it is shown that one of them is obtained as a fuzzy subgroup by a set of some modified data.

In [55], a decomposition of an $L$-valued set ($L$ a lattice) gives a family of characteristic functions which can be considered as a binary block-code. Conditions are given under which an arbitrary block-code corresponds to an $L$-valued fuzzy set. An explicit description of the Hamming distance, as well as of any code distance is also given, all in lattice-theoretic terms. A necessary and sufficient condition is given for a linear code to correspond to an $L$-valued fuzzy set. In such a case, the lattice has to be Boolean.

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# LIST OF FIGURES

# LIST OF TABLES

# LIST OF SYMBOLS

$A, B, C$ are sets

$\tilde{A}, \tilde{B}, \tilde{C}$ are fuzzy subsets

| | |
|---|---|
| $A \cup B$ | union of $A$ and $B$, p. 1 |
| $A \cap B$ | intersection of $A$ and $B$, p. 1 |
| $B \setminus A$ | relative complement of $A$ in $B$, p. 1 |
| $A^c$ | the complement of $A$ in its universal set, p. 1 |
| $A \subseteq B$ | $A$ is contained in $B$, p. 1 |
| $A \supseteq B$ | $A$ contains $B$, p. 1 |
| $x \in A$ | $x$ is an element of $A$, p. 1 |
| $x \notin A$ | $x$ is not an element of $A$, p. 1 |
| $A \subset B$ | $A$ is strictly contained in $B$, p. 1 |
| $B \supset A$ | $B$ strictly contains $A$, p. 1 |
| $\|A\|$ | the cardinality of $A$, p. 1 |
| card$(A)$ | the cardinality of $A$, p. 1 |
| $\wp(A)$ | the power set of $A$, p. 1 |
| $\emptyset$ | the empty set, p. 1 |
| $\mathbb{N}$ | the positive integers, p. 1 |
| $\mathbb{Z}$ | the integers, p. 1 |
| $\mathbb{Q}$ | the reational numbers, p. 1 |
| $\mathbb{R}$ | the real numbers, p. 1 |
| $\mathbb{C}$ | the complex numbers, p. 1 |
| $(x, y)$ | ordered pair of $x$ and $y$, p. 1 |
| $X \times Y$ | Cartesian product of sets $X$ and $Y$, p. 2 |
| $X^n$ | set of ordered $n$-tuples p.2 |
| Dom$(R)$ | domain of the relation $R$, p. 2 |
| Im$(R)$ | image of the relation $R$, p. 2 |

| | |
|---|---|
| $[x]$ | equivalence class determined by $x$, p. 2 |
| $\tilde{A}$ | fuzzy subset, p. 3 |
| $\tilde{A}^t$ | level set or $t$-cut, p. 3 |
| $\mathrm{supp}(\tilde{A})$ | support of $A$, p. 3 |
| $\mathfrak{F}\wp(A)$ | fuzzy power set of $\tilde{A}$, p. 3 |
| $\chi_A$ | characteristic function of $A$, p. 3 |
| $\tilde{A} \subseteq \tilde{B}$ | $\tilde{A}$ is contained in $\tilde{B}$, p. 4 |
| $\tilde{A} \subset \tilde{B}$ | $\tilde{A}$ is strictly contained in $\tilde{B}$, p. 4 |
| $\wedge$ | infimum, p. 4 |
| $\vee$ | supremum. p. 4 |
| $\tilde{A} \cup \tilde{B}$ | union of $\tilde{A}$ and $\tilde{B}$, p. 4 |
| $\tilde{A} \cap \tilde{B}$ | intersection of $\tilde{A}$ and $\tilde{B}$, p. 4 |
| $\tilde{A}^c$ | complement of $\tilde{A}$, p. 4 |
| $\bigcap_{\tilde{C}\in S} \tilde{C}$ | intersection of those $\tilde{C}$ in the set $S$, p. 5 |
| $\bigcup_{\tilde{C}\in S} \tilde{C}$ | union of those $\tilde{C}$ in the set $S$, p. 5 |
| $\tilde{R} \circ \tilde{Q}$ | the composition of fuzzy relations $\tilde{R}$ and $\tilde{Q}$, p. 9 |
| $\tilde{R}^k$ | $\tilde{R}$ composed with itself $k$ times, $k > 1$, p. 9 |
| $\tilde{R}^\infty$ | p. 9 |
| $\tilde{R}^0$ | p. 9 |
| $\tilde{R}^{-1}$ | p. 9 |
| $F^{\tilde{R}}$ | p. 16 |
| $F_\epsilon{}^{\tilde{R}}$ | p. 16 |
| $\phi_{\tilde{R}}$ | p. 17 |
| $\phi_{\tilde{R}}^\epsilon$ | p. 18 |
| $(S, R)$ | graph, p. 21 |
| $(S, \tilde{A}, \tilde{R})$ | fuzzy graph, p. 21 |
| $(\tilde{A}, \tilde{R})$ | partial fuzzy subgraph, p. 21 |
| $< P >$ | the fuzzy subgraph induced by a subset $P$ of the set of vertices, p. 22 |
| $\rho$ | path, p. 23 |
| $dis(x, y)$ | distance between $x$ and $y$, p. 23 |
| $l(\rho)$ | $\tilde{R}$-length of $\rho$, p. 24 |
| $\delta(x, y)$ | $\tilde{R}-$distance, p. 24 |
| $d(x, y)$ | distance, p. 24 |
| $M_{\tilde{R}}$ | p. 30 |
| $(M_{\tilde{R}})_{ij} = \tilde{R}(v_i, v_j)$ | p. 30 |
| $M_{\tilde{R}^\infty}$ | p. 30 |
| $M_{\tilde{R}^\infty}^\epsilon$ | p. 30 |
| $C_G$ | p. 30 |

| | |
|---|---|
| $C_G^\epsilon$ | p. 30 |
| $MS\epsilon CS$ | p. 30 |
| $Z(N)$ | flexibility of a network. p. 32 |
| $B(N)$ | balancedness of a network, p.32 |
| $d(v)$ | degree of a vertex, p. 32 |
| $\delta(G)$ | minimum degree of a fuzzy graph $G$, p. 32 |
| $\Delta(G)$ | maximum degree of a fuzzy graph $G$, p. 32 |
| $G_1 \cup G_2$ | union of fuzzy graphs $G_1$ and $G_2$ , p. 32 |
| $\lambda(G)$ | edge connectivity of a fuzzy graph $G$, p. 33 |
| $h(e)$ | cohesiveness of an element $e$, p. 35 |
| $H_e$ | $h(e)$-edge component of $e$, p. 35 |
| $M_e$ | maximal connected subgraph of $G$ containing the element $e$, p. 35 |
| $Z = (C_1, C_2, ..., C_m)$ | slicing of $G$, p. 36 |
| $\Omega(G)$ | vertex connectivity of a fuzzy graph $G$, p. 38 |
| $G_1 \times G_2$ | Cartesian product of graphs $G_1$ and $G_2$, p. 45 |
| $\tilde{A}_1 \times \tilde{A}_2$ | p. 45 |
| $\tilde{E}_1 \tilde{E}_2$ | p. 45 |
| $G_1[G_2]$ | composition of graph $G_1$ with graph $G_2$, p. 47 |
| $\tilde{A}_1 \circ \tilde{A}_2$ | p. 47 |
| $\tilde{E}_1 \circ \tilde{E}_2$ | p. 47 |
| $G_1 \cup G_2$ | union of graphs $G_1$ and $G_2$, p. 49 |
| $\tilde{A}_1 \cup \tilde{A}_2$ | p. 49 |
| $\tilde{E}_1 \cup \tilde{E}_2$ | p. 49 |
| $G_1 + G_2$ | join of $G_1$ and graph $G_2$, p. 50 |
| $\tilde{A}_1 + \tilde{A}_2$ | p. 50 |
| $\tilde{E}_1 + \tilde{E}_2$ | p. 50 |
| $E\bar{x} = \bar{b}$ | p. 52 |
| $E^*$ | p. 53 |
| $DOM(A_i)$ | set of possible values for the attribute $A_i$, p. 58 |
| $\chi_{X \to Y}$ | p. 58 |
| $t_1[X]$ | p. 58 |
| $T_{\tilde{R}}(X, Y)$ | p. 59 |
| $T^+$ | the smallest fuzzy relation on $U^2$ which contains $T$, p. 60 |
| ${T_f}^+$ | p. 60 |
| $Z_\lambda$ | p. 60 |
| $(X, \mathcal{T})$ | topological space, p. 67 |
| $\mathcal{B}$ | base for a topology, p. 68 |
| $\mathcal{T}_A$ | relative topology on $A$, p. 68 |

| | |
|---|---|
| $A'$ | derived set, p. 68 |
| $\bar{A}$ | closure of $A$, p. 69 |
| $clA$ | closure of $A$, p. 69 |
| $A^\circ$ | interior of $A$, p. 70 |
| $\mathcal{C}$ | cover, p. 70 |
| $\mathcal{N}_x$ | neighborhood system of $x$, p. 72 |
| $d$ | metric, p. 74 |
| $(X, d)$ | metric space, p. 75 |
| $(X^*, d^*)$ | completion of a metric space $(X, d)$, p. 77 |
| $coA$ | convex hull of $A$, p. 77 |
| $\|\| \ \|\|$ | norm, p. 77 |
| $(X, \mathcal{FT})$ | fuzzy topological space, p. 80 |
| $\mathcal{N}$ | neighborhood system, p. 80 |
| $\tilde{A}^\circ$ | the interior of $\tilde{A}$, p. 81 |
| $\mathcal{H}(X)$ | set of nonempty compact subsets of $X$, where $(X, d)$ is a metric space, p. 85 |
| $h(d)$ | Hausdorff distance, p. 85 |
| $(\mathcal{H}(X), h)$ | space of fractals, p. 85 |
| $(X : w_n, n = 1, ..., N)$ | iterated function system (IFS), p.87 |
| $T_s$ | p. 88 |
| $\mathcal{F}^*(X)$ | p. 88 |
| $(X, w, \Phi)$ | iterated fuzzy subset system (IFZS), p. 88 |
| $\tilde{A}^+$ | closure of $\{ x \in X \| \tilde{A} > 0\}$, p. 89 |
| $D(A, B)$ | p. 89 |
| $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$ | p. 90 |
| $\overset{\approx}{A}$ | p. 91 |
| $T$ | p. 91 |
| $\mathcal{E}^n$ | p. 97 |
| ${\mathcal{E}_0}^l$ | p. 98 |
| ${\mathcal{T}_0}^l$ | p. 98 |
| ${\Delta_0}^l$ | p. 98 |
| $S^{n-1}$ | unit sphere in $\mathbb{R}^n$, p. 99 |
| $\delta_\infty$ | p. 99 |
| $\mathcal{K}^n$ | p. 99 |
| ${\mathcal{K}_{ST}}^n$ | p. 99 |
| $g_K$ | gauge function p. 99 |
| $\delta_p$ | p. 99 |
| $\mathcal{U}^n$ | set of all normal, upper semicontiuous, fuzzy subsets of $\mathbb{R}^n$, p. 100 |
| $\mathcal{S}^n$ | p. 100 |
| ${\mathcal{S}_0}^n$ | p. 100 |
| $ker(\tilde{A})$ | p. 100 |
| $fker(\tilde{A})$ | p. 100 |
| $d_p$ | p. 100 |

| | |
|---|---|
| $d_\infty$ | p. 100 |
| $\rho_p$ | p. 101 |
| $\rho : P_0, P_1, \ldots, P_n$ | path in a rectangular grid, p. 116 |
| $s_{\tilde{A}}(\rho)$ | strength of $\rho$ with respect to $\tilde{A}$,p. 116 |
| $c_{\tilde{A}}(P, Q)$ | degree of connectedness of $P$ and $Q$ with respect to $\tilde{A}$, p. 117 |
| $c_{\tilde{A}}(T)$ | degree of connectedness of a subset $T$ of a rectangular grid $\Sigma$ with respect to $\tilde{A}$, p. 117 |
| $C_{\tilde{A}}$ | p. 117 |
| $\Pi$ | plateau p. 118 top or bottom, p. 120 |
| $A_\Pi$ | p. 120 |
| $B_\Pi$ | p. 120 |
| $C_\Pi$ | p. 120 |
| $\tilde{A}_\Pi$ | p. 122 |
| $\tilde{A}_l$ | sup projection, p. 128 |
| $\bar{\tilde{A}}_l$ | integral projection, p. 128 |
| $I(R)$ | the digital image of a subset $R$ of the plane, p. 131 |
| $\tilde{A}_t$ | p. 132 |
| $\tilde{A}_l$ | the set points that have grey level $l$, p. 134 |
| $\tilde{A}_{l+}$ | the set points that have grey level $> l$,p. 134 |
| $\int\int \tilde{A} dx dy$ | area of $\tilde{A}$, p. 138 |
| $\Pi$ | partition, p. 138 |
| $l(C)$ | length of a curve, p. 138 |
| $l(A_{ijk})$ | arc length, p. 139 |
| $p(\tilde{A})$ | perimeter, p. 139, p. 183 |
| $\|\nabla \tilde{A}\|$ | magnitude of the gradient of $\tilde{A}$, p. 139 |
| $(f, \phi)$ | p. 141 |
| $h(\tilde{A})$ | height of $\tilde{A}$, p. 147 |
| $w(\tilde{A})$ | width of $\tilde{A}$, p. 147 |
| $E(\tilde{A})$ | extrinsic diameter of $\tilde{A}$, p. 148 |
| $\rho_{PQ}$ | rectifiable path from $P$ to $Q$, p. 148 |
| $I(\tilde{A})$ | intrinsic diameter of $\tilde{A}$, p. 148 |
| $\|\rho_{PQ}\|$ | length of the path $\rho_{PQ}$, p. 148 |
| $\tilde{d}_{\tilde{A},\tilde{B}}(r)$ | p. 152 |
| $T^\lambda$ | p. 152 |
| $L(U, V)$ | p. 152 |
| $L^*(U, V)$ | p. 152 |
| $L(\tilde{A}, \tilde{B})$ | p. 153 |
| $L^*(\tilde{A}, \tilde{B})$ | p. 153 |

| | |
|---|---|
| $\Delta_{\tilde{A},\tilde{B}}(r)$ | p. 153 |
| $\Delta_{P,\tilde{B}}$ | p. 155 |
| $D_P^S$ | p.158 |
| $\tilde{B}_P^{\tilde{A}}$ | p. 159 |
| $D_{\tilde{A}}$ | fuzzy medial axis of $\tilde{A}$, p. 159 |
| $P_\theta$ | p. 164 |
| $a(S,T)$ | adacency of $S$ and $T$, p. 168 |
| $d(P,T)$ | distance from $P$ to $T$, p. 169 |
| $\triangle_P$ | p.172 |
| $\nabla_Q$ | p. 172 |
| $BP(S)$ | set of border points of $S$, p. 173 |
| $a_{dig}(S,T)$ | digital degree of adjacency, p. 174 |
| $v(P,T)$ | degree of visual surroundedness, p. 176 |
| $C_{\pi_\theta}$ | p.177 |
| $t(P',T)$ | p. 177 |
| $\nu_1(X)$ | linear index of fuzziness, p. 182 |
| $\tilde{B}_q(X)$ | quadratic index of fuzziness, p. 182 |
| $H(X)$ | entropy, p. 182 |
| $\eta(X)$ | index of nonfuzziness, p. 182 |
| $\mathrm{comp}\left(\tilde{A}\right)$ | compactness of $\tilde{A}$, p. 186 |
| $\tilde{P}_{(a,b)}$ | fuzzy point at $(a,b)$, p. 190 |
| $\Omega(t)$ | p. 191 |
| $\tilde{D}(\tilde{P}_1,\tilde{P}_2)$ | p. 191 |
| $\leq_s$ | p. 192 |
| $\leq_w$ | p. 192 |
| $\Omega_{11}(t)$ | p. 193 |
| $\tilde{L}_{11}$ | p. 193 |
| $\Omega_{12}(t)$ | p. 194 |
| $\tilde{L}_{12}$ | p. 194 |
| $\Omega_2(t)$ | p. 194 |
| $\tilde{L}_2$ | p. 194 |
| $\Omega_3(t)$ | p. 194 |
| $\tilde{L}_3$ | p. 194 |
| $\mathfrak{C}$ | fuzzy circle, p. 197 |
| $\tilde{\Theta}$ | area of $\mathfrak{C}$, p. 198 |
| $\mathfrak{P}$ | fuzzy polygon, p. 200 |
| $\tilde{\Pi}$ | fuzzy angle, p. 202 |
| $(\Pi,\Lambda,I)$ | p. 204 |
| $[m,c]$ | p. 205 |
| $[d]$ | p. 205 |
| $H(U,V)$ | Hausdorff distance, p. 207 |
| $H_f(\widetilde{A},\widetilde{B})$ | fuzzy Hausdorff distance, p. 210 |
| $(S,*_1,\ldots,*_n)$ | algebraic system, p. 219 |

| | |
|---|---|
| $(S, *)$ | semigroup or monoid, p. 219 |
| $*\vert_{X \times X}$ | $*$ restricted to $X \times X$, p. 220 |
| $< X >$ | subsemigroup (submonoid) generated by $X$, p. 220 |
| $S^*$ | p. 220 |
| $\simeq$ | isomorphism p. 221 |
| $(F, f)$ | free semigroup $F$ with function $f$, p. 221 |
| $(G, *)$ | group, p. 222 |
| $< X >$ | subgroup generated by $X$, p. 223 |
| $(R, +, \cdot)$ | ring, p. 224 |
| $R[x]$ | polynomial ring over $R$, p. 225 |
| $R[x_1, ..., x_n]$ | polynomial ring in $n$ indeterminates over $R$, p. 226 |
| $< X >$ | ideal generated by $X$, p. 226 |
| $I \cdot J$ | product of two ideals, p. 227 |
| $\sqrt{I}$ | radical of an ideal $I$ p. 228 |
| $F^n$ | affine space over the field $F$, p. 230 |
| $V(f_1, ..., f_m)$ | affine variety defined by the polynomials $f_1, ..., f_m$, p. 230 |
| $V(I)$ | p. 231 |
| $I(V)$ | p. 231 |
| $\tilde{A}\tilde{B}$ | product of two fuzzy ideals, p. 235 |
| $\sqrt{\tilde{A}}$ | radical of a fuzzy ideal, p. 235 |
| $I(\tilde{X})$ | p. 236 |
| $V(\tilde{A})$ | p. 237 |
| $< \tilde{C} >$ | fuzzy subsemigroup generated by a fuzzy subset,p. 238 |
| $<< \tilde{C} >>$ | fuzzy monoid generated by a fuzzy subset, p. 238 |
| $X^*$ | set of all strings of elements of $X$, p. 239 |
| $\tilde{A}^*$ | p. 239 |
| $\tilde{A}_{\tilde{Y}}$ | p. 239 |
| $T_m$ | p. 240 |
| $P$ or $m$ | probability measure, p. 241 |
| $(\mathcal{S}, \mathcal{A})$ | $\mathcal{A}$ is a $\sigma$-algebra on the set $\mathcal{S}$, p. 241 |
| $(\Omega, \mathcal{A}, P)$ | probability space, p. 241 |
| $(\mathcal{G}, \oplus)$ | p. 241 |
| $G_f$ | p. 241 |
| $\ominus f$ | inverse of $f$ with rspect to $\oplus$, p. 243 |
| $S_{n^2}$ | symmetric group on $n^2$ elements, p. 244 |
| $F(\mathbb{R}^+, X)$ | p. 246 |
| $A^*$ | free monoid, p. 247 |
| $M^+$ | p. 247 |
| $Mw$ | p. 247 |

| | |
|---|---|
| $\leq_l$ | p. 248 |
| $A^{m\times m}$ | set of $m \times m$ matrices over $A$, p. 253 |
| $\mu : M \to A^{m\times m}$ | representation, p. 253 |
| $\sum_{w\in M} (r,w)w$ | power series, p. 253 |
| ${V_T}^*$ | free monoid over an alphabet $V_T$, p. 253 |
| $A[[M]]$ | set of power series over $A$, p. 253 |
| $A[M]$ | set of polynomials over $A$, p. 253 |
| $\sum_{i=0}^{\infty}(r,x^i)x^i$ | p. 254 |
| $A^{rat}[[M]]$ | p. 254 |
| $A^{rec}[[M]]$ | p. 254 |
| $H(r)$ | Hankel matrix. p. 256 |
| $A^{{V_T}^*}$ | p. 256 |
| $F_v$ | p. 256 |
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| $F^n$ | p. 281 |
| $\tilde{A}_u$ | p. 281 |
| $\tilde{A}^n$ | p. 281 |
| $d_{min}(C)$ | minimum distance of the code $C$, p. 281 |
| $[[\ ]]$ | greatest integer function, p. 282 |
| $E_C$ | p. 282 |
| $\sum_d$ | p. 282 |
| $d_{min}$ | p. 283 |
| $d_a$ | asymmetric distance metric, p. 285 |
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