Gödel's Completeness Theorem

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Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
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Troy, New York 12180 USA

Intro to Logic
11/25/2019

Only *steeples* of rationalism!

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel's "God Theorem"
- Could a Machine Match Gödel's Genius?

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Some Timeline Points

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1906 Brünn, Austria-Hungary

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1929 Doctoral Dissertation: Proof of Completeness Theorem

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Preliminaries: Propositional Calculus & First-Order Logic

...

Actually …

$$\mathscr{L}_0 < \mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3 \ldots$$

Actually ...

Second-order logic.

Third-order logic, which Gödel used for his "God Theorem."

$$\mathscr{L}_0 < \mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3 \ldots$$

Zero-order logic; subsumes the propositional calculus.

First-order logic; this is whaat the Completeness Theorem is about.

R&W's Axiomatization of the Propositional Calculus

A1	$(\phi \vee \phi) \rightarrow \phi$
A2	$\phi \rightarrow (\phi \vee \psi)$
A3	$(\phi \vee \psi) \rightarrow (\psi \vee \phi)$
A4	$(\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow (\phi \vee \chi))$

R&W's Axiomatization of the Propositional Calculus

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A4 $(\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow (\phi \vee \chi))$

All instances of these schemata are true no matter what the input (true or false). (Agreed?) And indeed every single formula in the propositional calculus that is true no matter what the permutation (as shown in a truth table), can be proved (somehow) from these four axioms (using the rules of inference given earlier in our semester). This, Gödel knew, and could use.

Exercise 1:

Verify that these are true-no-matter what in a truth table; then prove using our rules for the prop. calc.

$$(\phi \wedge \psi) \to (\psi \vee \chi)$$

$$\phi \to (\psi \to \phi)$$

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Truth Table showing this formula true no matter what the inputs.

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Truth Table showing this formula true no matter what the inputs.

Proof:

Exercise 1:

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$$(\phi \wedge \psi) \rightarrow (\psi \vee \chi)$$

√ **Truth Table** showing this formula true no matter what the inputs.

Proof:

1	$\phi \wedge \psi$	Supposition
2	ψ	1, Simplification
3	$\psi \vee \chi$	2, Addition
4	$(\phi \vee \psi) \rightarrow (\psi \vee \chi)$	1–3, Conditional Intro

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

EXAMPLE 3 State which rule of inference is the basis of the following argument: "It is below freezing
Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is ra

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$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

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From Language-Learning Slides: The Grammar of the Pure Predicate Calculus

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\begin{array}{lcl}
\mathit{Formula} & \Rightarrow & \mathit{AtomicFormula} \\
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Recall the Examples We Cited

Formula	$\Rightarrow$	*AtomicFormula*
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Lexicon

Sally likes Bill.

`(Likes sally bill)`

Sally likes Bill and Bill likes Sally.

Sally likes Bill's mother.

Sally likes Bill only if Bill's mother is tall.

Matilda is Bill's super-smart mother.

5 plus 5 equals the number 10.

...

Did you make sure you can simulate a machine that says "Yes that sentence is okay!" whenever it's conforms to this grammar?

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AtomicFormula	$\Rightarrow$	$(Predicate\ Term_1 \ldots Term_k)$
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Connective	$\Rightarrow$	$\wedge \mid \vee \mid \rightarrow \mid \leftrightarrow$
`Likes` *Predicate*	$\Rightarrow$	$P_1 \mid P_2 \mid P_3 \ \ldots$
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If Sally likes Bill then Sally likes Bill.

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Sally likes Bill's mother, or not.

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These are all true, yes; but can they be proved?!

Add the Final Addition/Deeper Challenge: Add Two Quantifiers to the Pure Predicate Calculus, Which Yields the First-order Logic = Predicate Calculus *simpliciter*

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there exists at least one thing *x* such that …

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$\exists x \ \ldots$ there exists at least one thing *x* such that …

$\forall x \ \ldots$ for all *x*, it's the case that …

$$\forall \epsilon(\epsilon > 0 \rightarrow \exists \delta(\delta > 0 \wedge \forall x(d(x,a) < \delta \rightarrow d(f(x),b) < \epsilon)))$$

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$\exists x \ \ldots$ there exists at least one thing *x* such that …

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Every natural number is greater than or equal to zero.

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$$\forall x (x \geq 0)$$

Every positive integer *x* is less-than-or-equal-to a positive integer *y*.

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$\forall x \ \ldots$ for all *x*, it's the case that …

Every natural number is greater than or equal to zero.

$$\forall x (x \geq 0)$$

Every positive integer *x* is less-than-or-equal-to a positive integer *y*.

$$\forall x \exists y (x \leq y) \qquad \forall x \exists y (\leq (x, y))$$

The Shoulders Available to Gödel for Standing Upon

. . .

Completeness Theorem for The Propositional Calculus

Let Γ be a set $\{\phi_1, \phi_2, \ldots\}$ of formulae in the the propositional calculus. Then either all of Γ are satisfiable, or the conjunction up to and including the point k (i.e. $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k$) of failure is refutable.

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Let Γ be a set $\{\phi_1, \phi_2, \ldots\}$ of formulae in the the propositional calculus. Then either all of Γ can be simultaneously true in some scenario, or the conjunction up to and including the point k (i.e. $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k$) of failure is **refutable** (i.e. $\vdash \neg(\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k)$).

What does the Completeness Theorem say?

. . .

Completeness Theorem as an Equation

In first-order logic: NECESSARY TRUTH = PROVABILITY.

Completeness Theorem, More Precisely Put

For every first-order statement ϕ: ϕ is a necessary or absolute truth (i.e. true in any scenario whatsoever) if, and only if, ϕ is provable.

And the version Gödel targeted, and proved:

For every first-order statement ϕ: Either ϕ is true in some scenarios, or ϕ is refutable (= it's negation $\neg\phi$ can be proved).

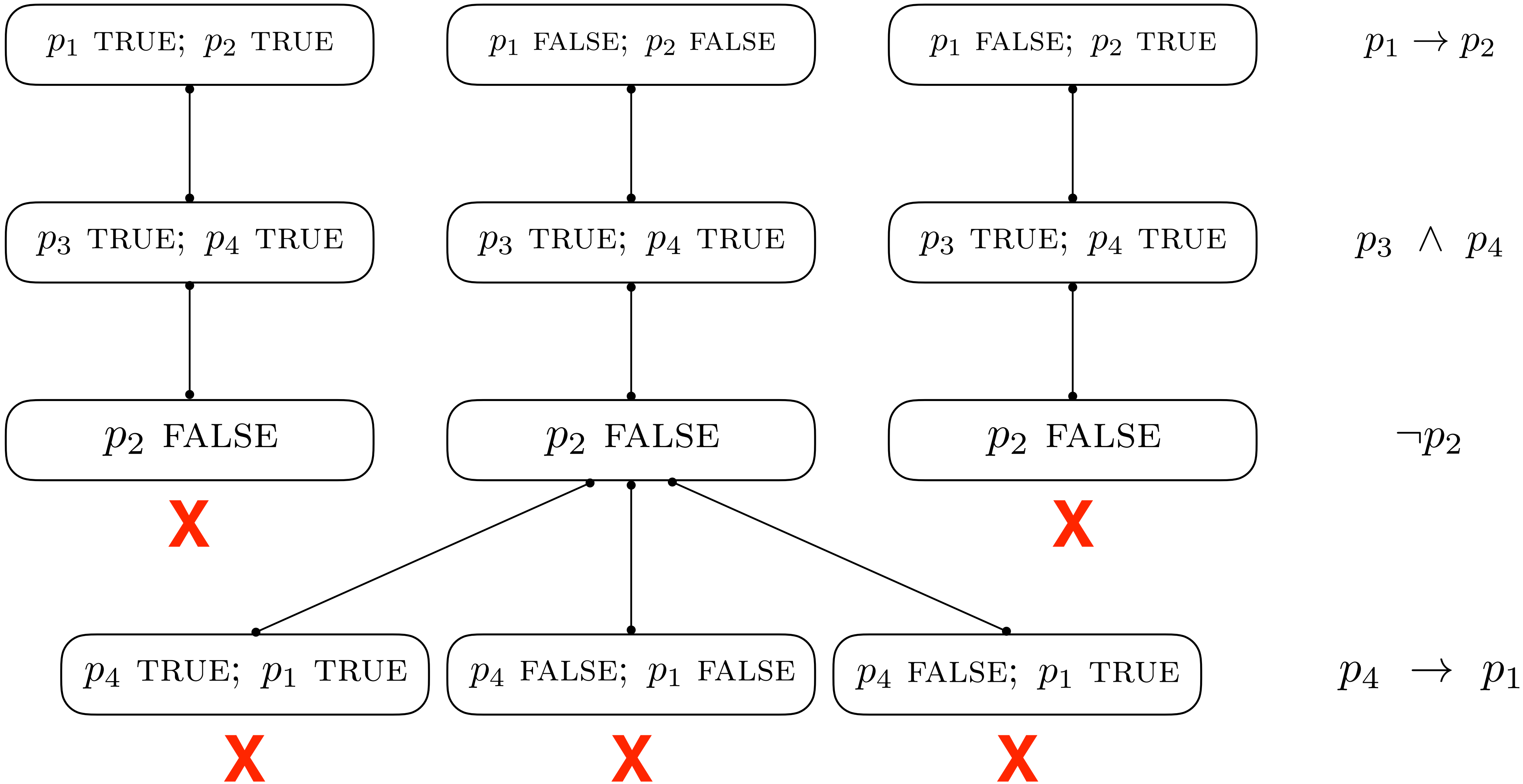
Comp$_G$

The Proof-Sketch

The Proof-Sketch

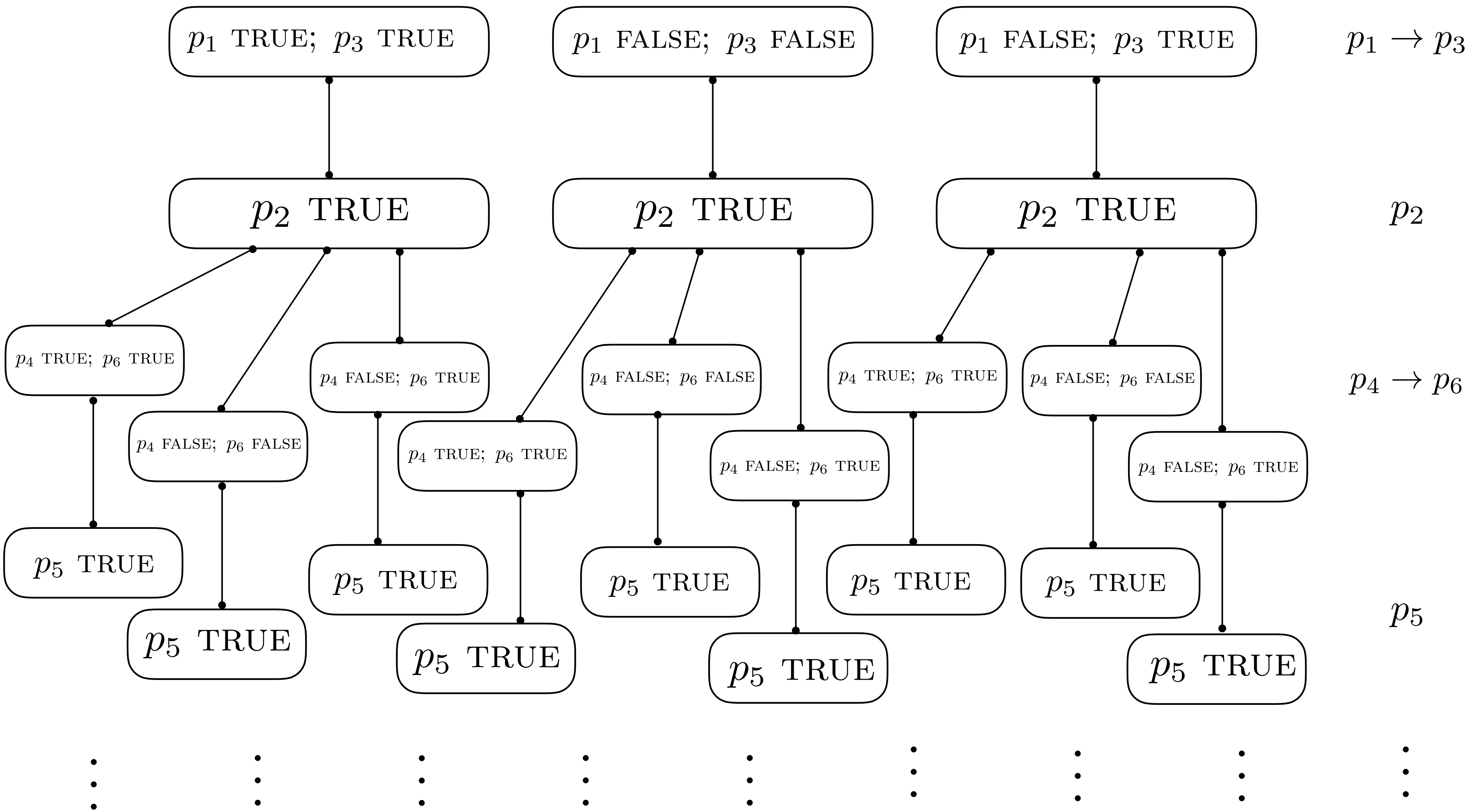
To prove the theorem in the case of first-order logic ($= \mathscr{L}_1$), we need to show that given any set Γ of formulae in first-order logic, either there's a scenario on which every member of this set is true; otherwise, there is a refutation of the set, i.e. a proof from the set to an outright contradiction $\phi \land \neg\phi$. We can accomplish this by finding a procedure $\mathscr{P}$ that first takes the set in question and goes hunting for a scenario that does the trick. If the scenario is found, we're done. But, if such a scenario *can't* be found, then our procedure moves on to find a proof of a contradiction from Γ!

$$\Gamma \coloneqq \{p_1 \to p_2,\ p_3 \wedge p_4,\ \neg p_2,\ p_4 \to p_1,\ \ldots\}$$



Therefore, there is no scenario in which all of the formulae are true!

$$\Gamma \coloneqq \{p_1 \to p_3,\ p_2,\ p_4 \to p_6,\ p_5,\ p_7 \to p_9,\ p_8,\ \ldots\}$$

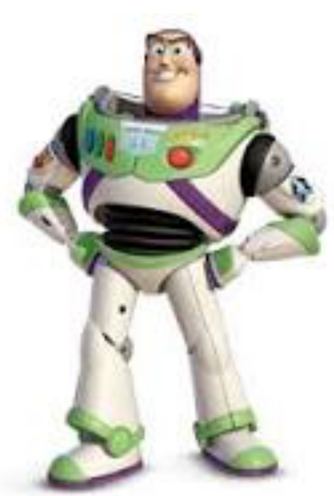


Therefore, since we can travel to infinity, there *is* a scenario in which all of the formulae are true: any infinite path down will do.

But the assumption that there *is* an infinite branch is based on König’s Lemma …

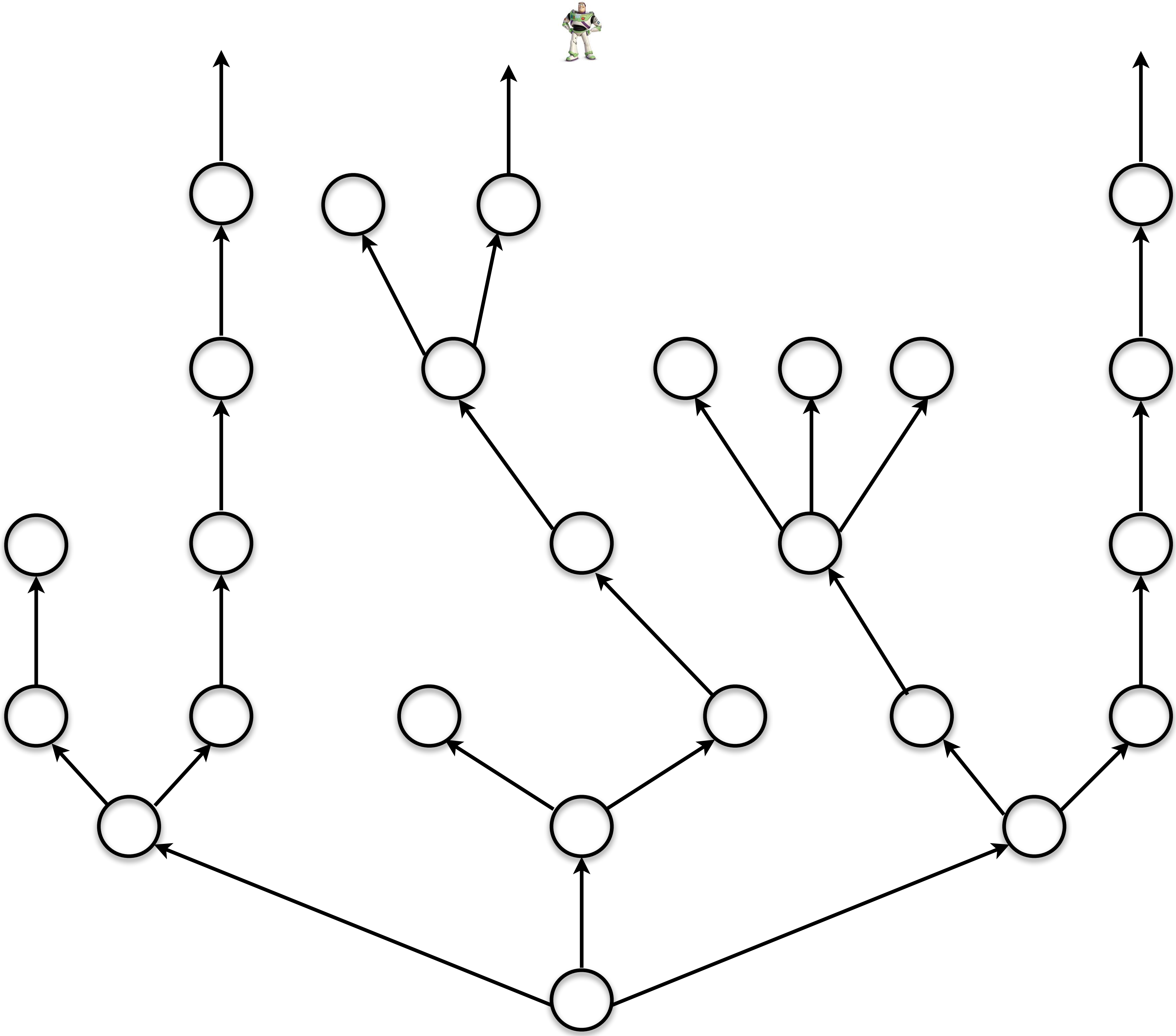
Toward König's Lemma as Train Travel

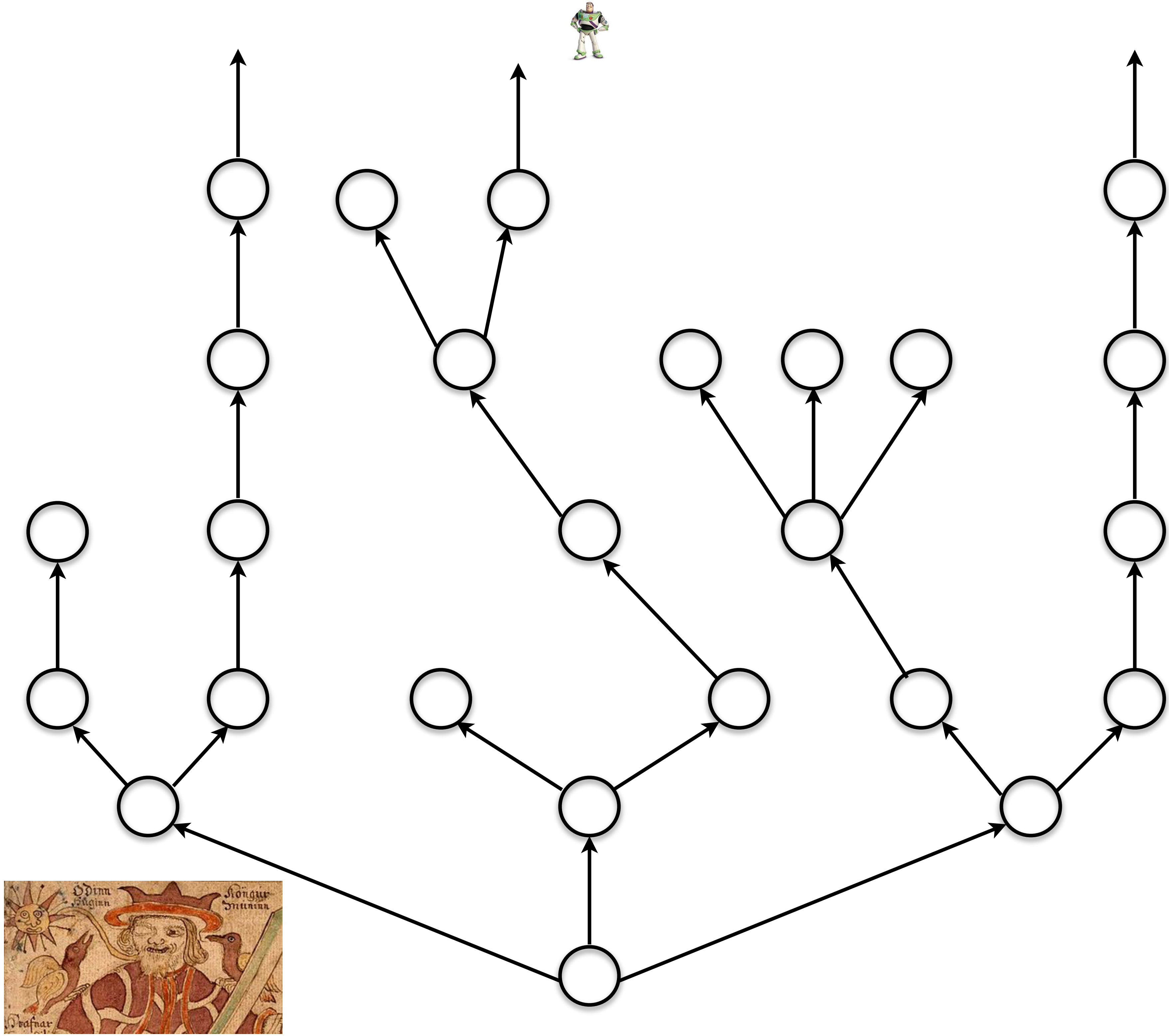
“To infinity and beyond!”

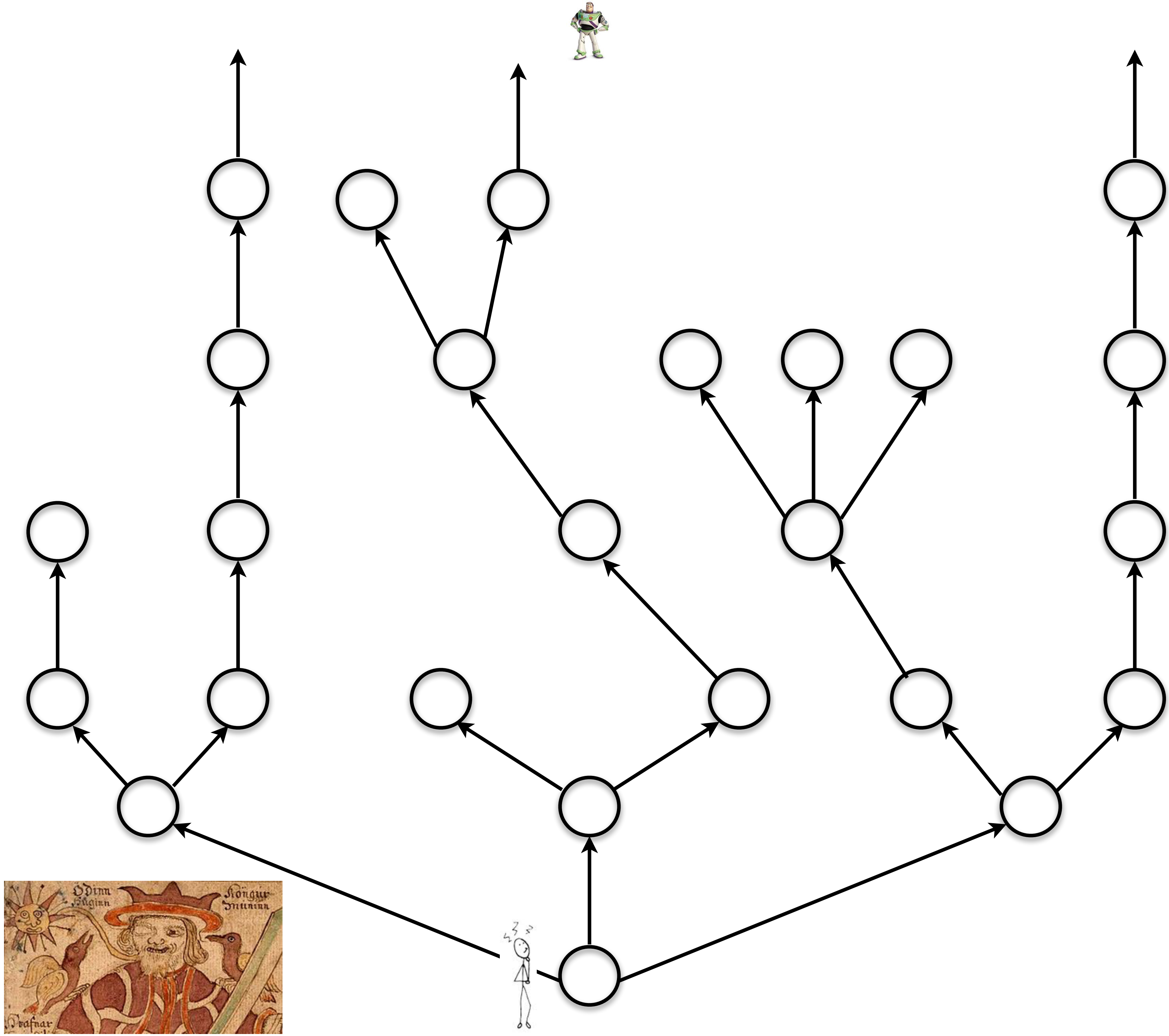


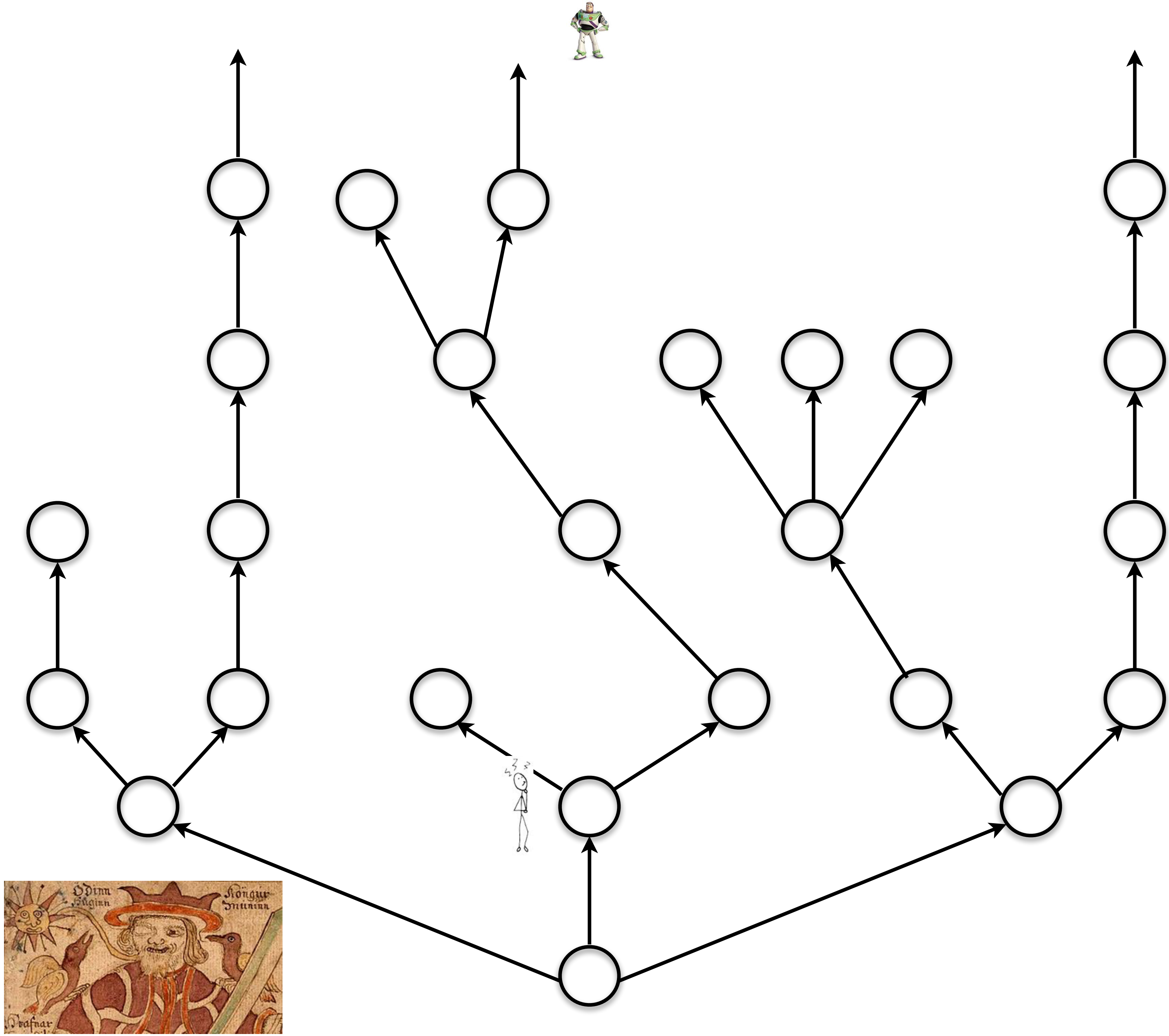
König’s Lemma (train-travel version)

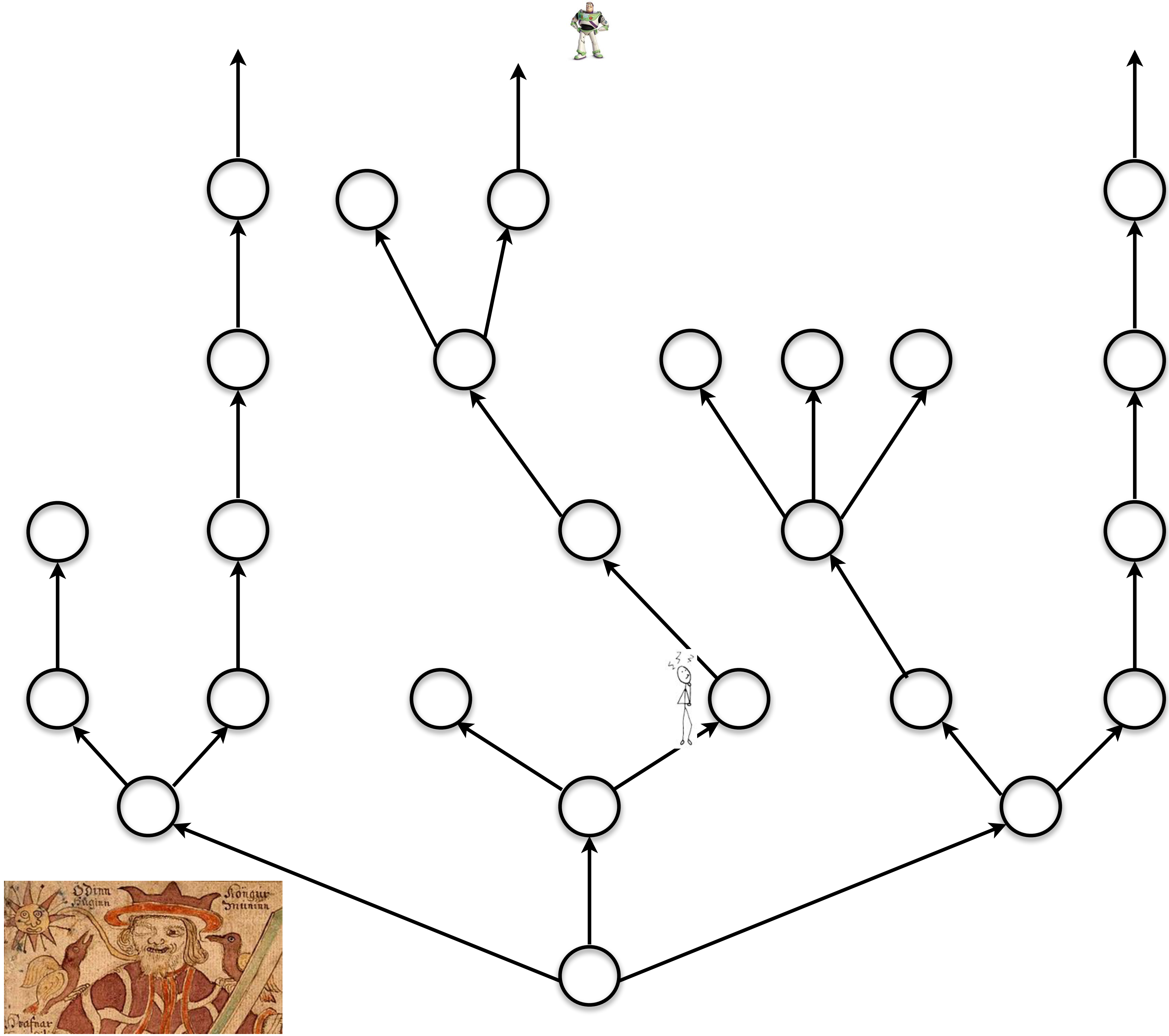
In a one-way train-travel map with finitely many options leading from each station, if there are partial paths forward of every finite length, there is an *infinite* path (= a path “to infinity”).







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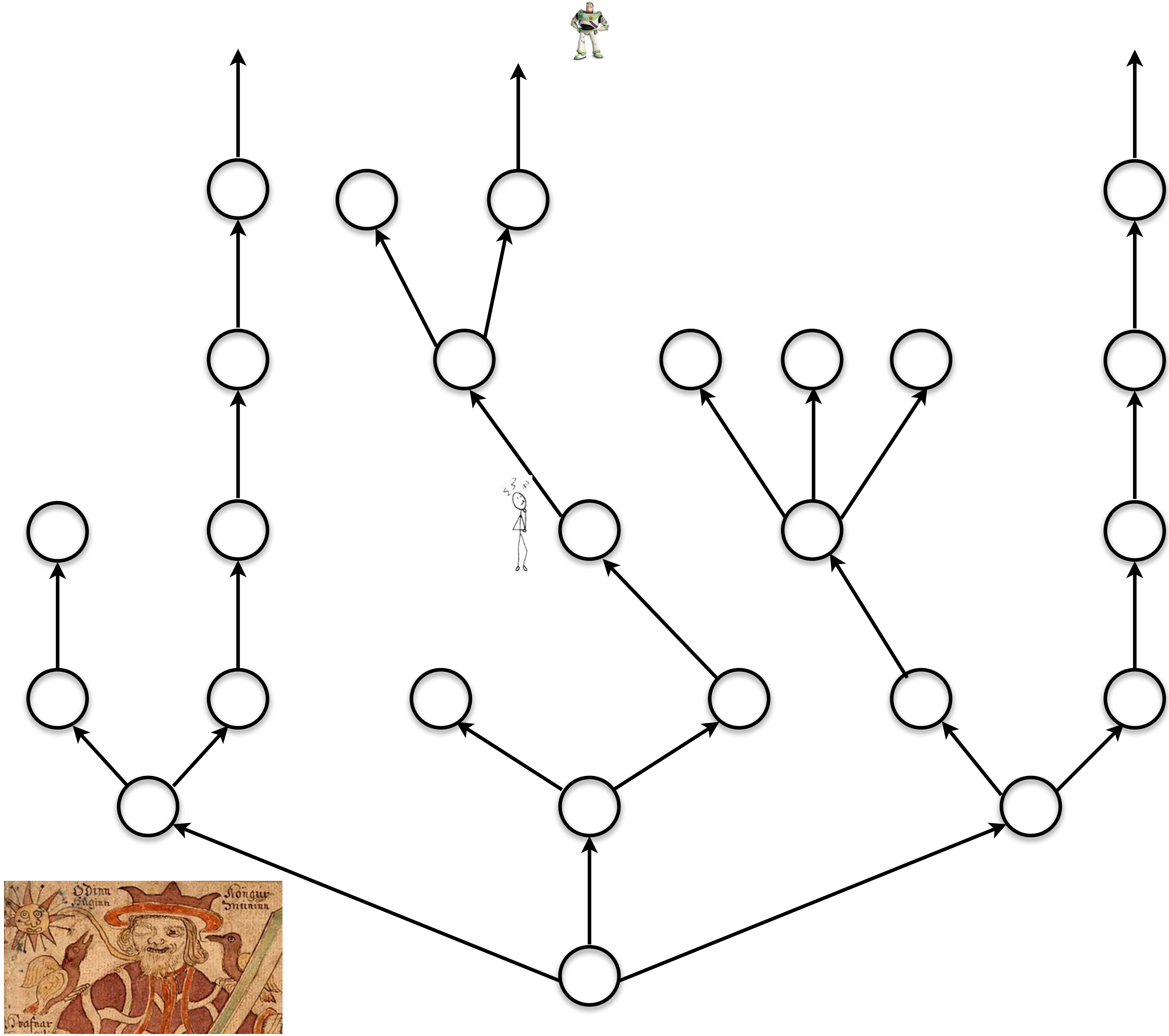

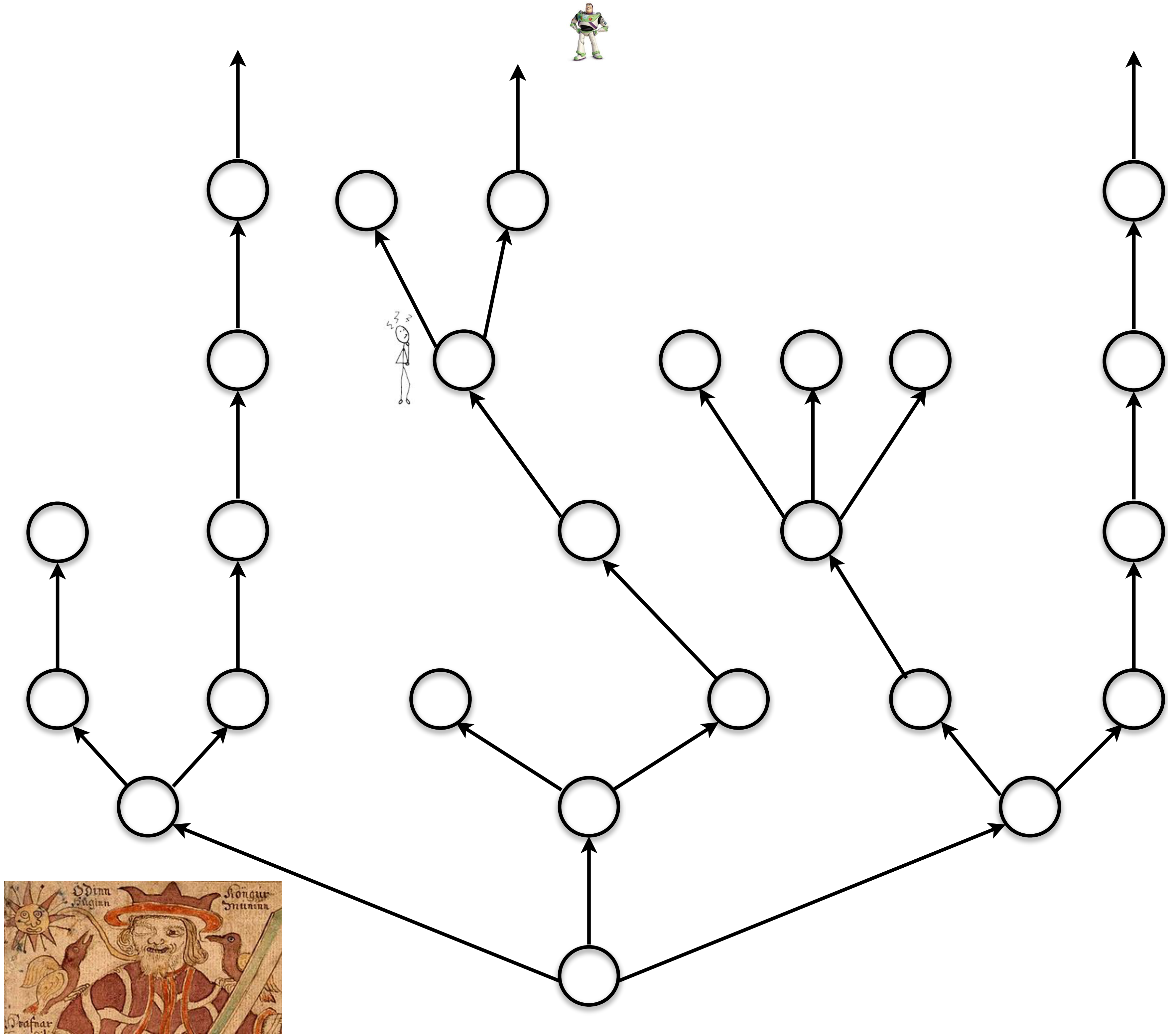

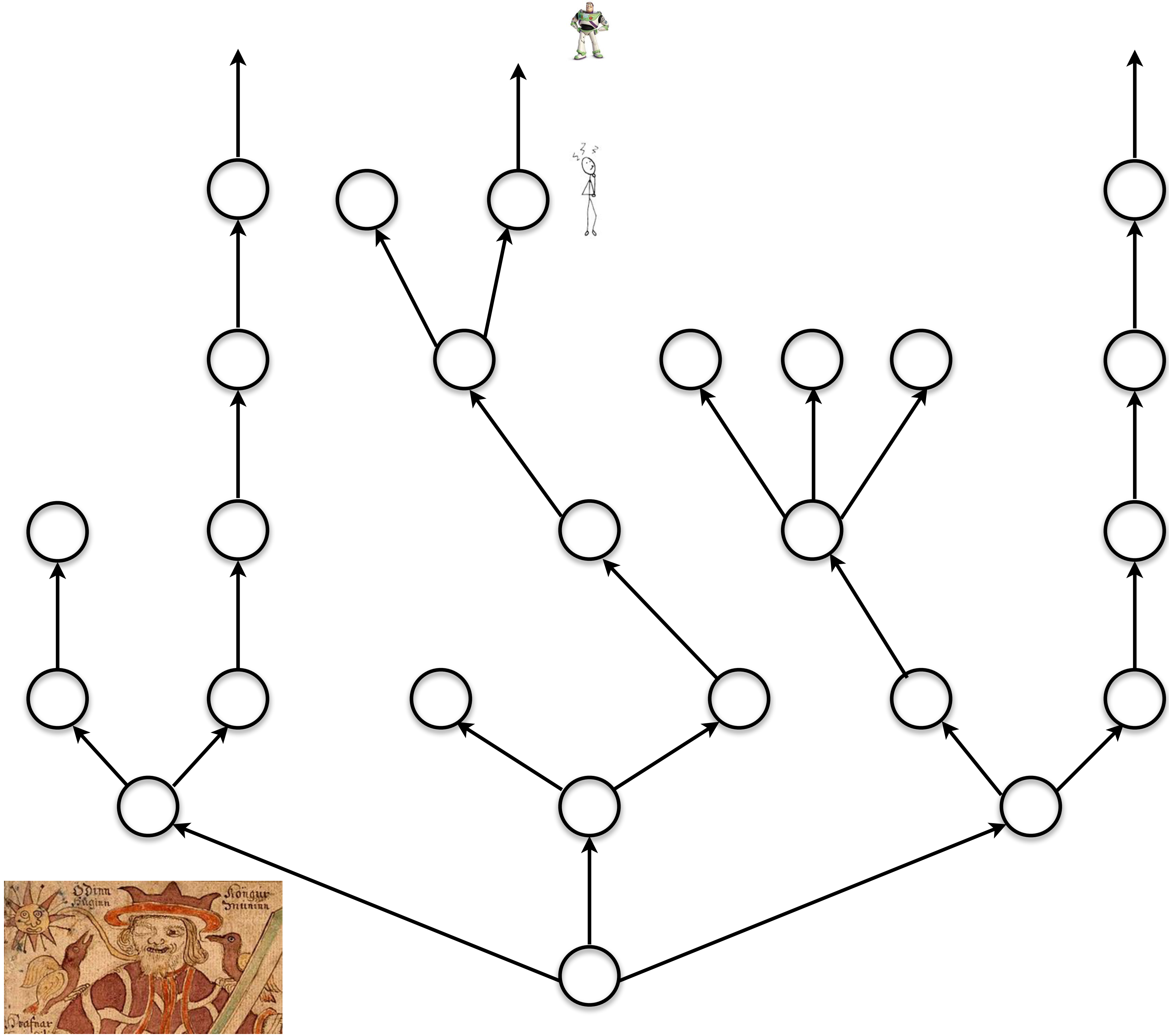

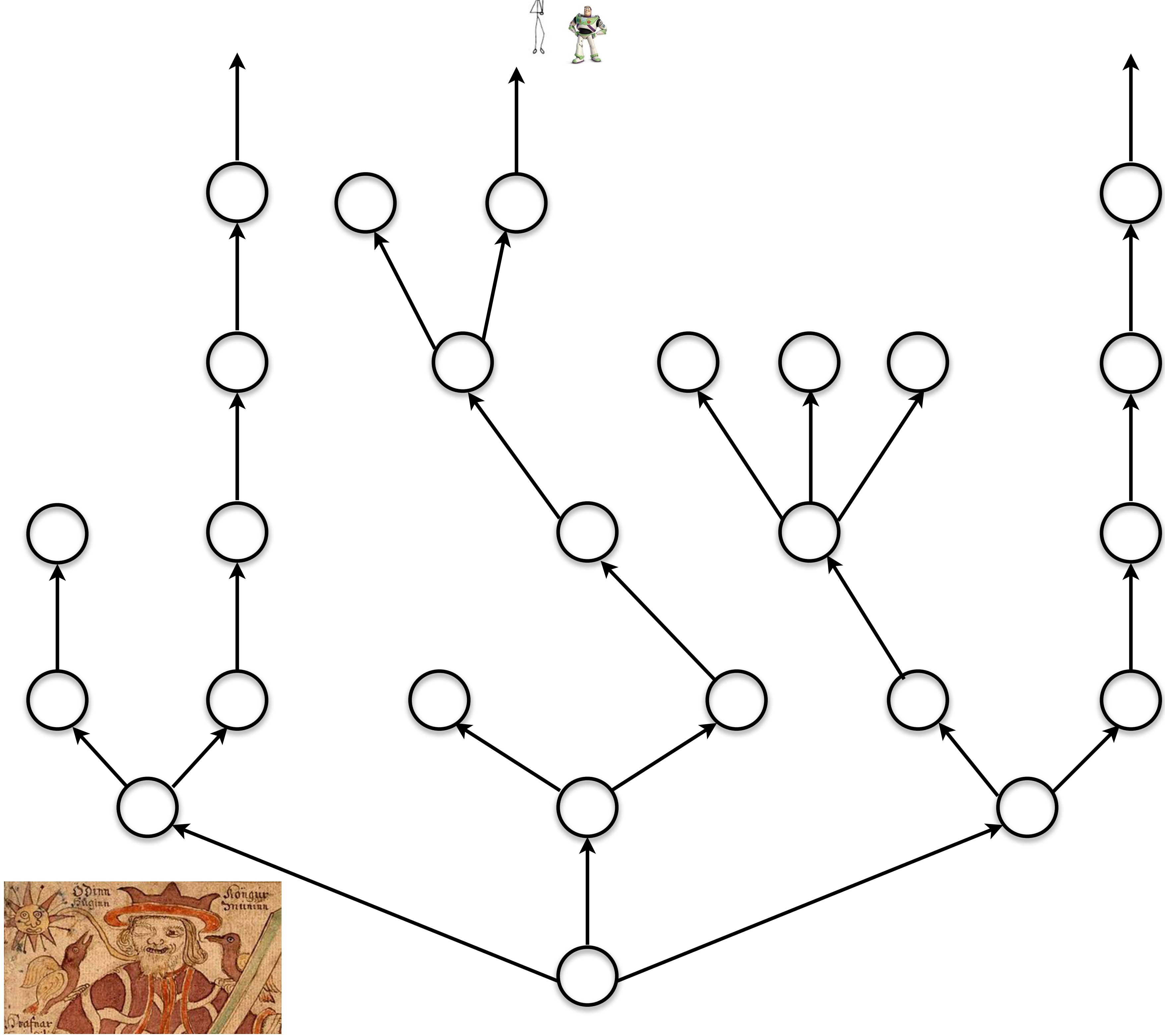

Exercise 2:
Is there an algorithm for traveling this way?

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No. This strategy for travel is beyond the reach of standard computation.

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No. This strategy for travel is beyond the reach of standard computation.

(Does it not then follow, assuming that humans can find and "use" a provably correct strategy for this travel, that humans can't be fundamentally computing machines?)

Proving the Lemma
(that there is an infinite branch)

Proof: We are seeking to prove that there is an infinite path (= that you can keep going forward forever = that the number of your stops forward are the size of $\mathbf{Z}^+$).

To begin, assume the antecedent of the theorem (i.e. that, (1), there are finitely many options leading from each station, and that, (2), in the map there are partial paths forward of every finite size).

Now, you are standing at Penn Station (S_1), facing *k* options. At least one of these options must lead to partial paths of arbitrary size (the size of any *m* in $\mathbf{Z}^+$). (**Sub-Proof**: Suppose otherwise for indirect proof. Then there is some positive integer *n* that places a ceiling on the size of partial paths that can be reached. But this violates (2) — contradiction.) Proceed to choose one of these options that lead to partial paths of arbitrary size. You are now standing at a new station (S_2), one stop after Penn Station. At least one of these options must lead to partial parts of arbitrary size (the size of any *m* in $\mathbf{Z}^+$). (**Sub-Proof**: Suppose otherwise for indirect proof …)

Since you can iterate this forever, you'll be on an infinite trip to infinity! Buzz will be happy.

For Further Reading

THE DISCOVERY OF MY COMPLETENESS PROOFS

LEON HENKIN

Dedicated to my teacher, Alonzo Church, in his 91st year.

§1. Introduction. This paper deals with aspects of my doctoral dissertation[1] which contributed to the early development of model theory. What was of use to later workers was less the results of my thesis, than the method by which I proved the completeness of first-order logic—a result established by Kurt Gödel in *his* doctoral thesis 18 years before.[2]

The ideas that fed my discovery of this proof were mostly those I found in the teachings and writings of Alonzo Church. This may seem curious, as his work in logic, and his teaching, gave great emphasis to the constructive character of mathematical logic, while the model theory to which I contributed is filled with theorems about very large classes of mathematical structures, whose proofs often by-pass constructive methods.

Another curious thing about my discovery of a new proof of Gödel's completeness theorem, is that it arrived in the midst of my efforts to prove an entirely different result. Such "accidental" discoveries arise in many parts of scientific work. Perhaps there are regularities in the conditions under which such "accidents" occur which would interest some historians, so I shall try to describe in some detail the accident which befell me.

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Small Steps Toward Hypercomputation via Infinitary Machine Proof Verification and Proof Generation

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Abstract. After setting a context based on two general points (that humans appear to reason in infinitary fashion, and two, that actual hypercomputers aren't currently available to directly model and replicate such infinitary reasoning), we set a humble engineering goal of taking initial steps toward a computing machine that can reason in infinitary fashion. The initial steps consist in our outline of automated proof-verification and proof-discovery techniques for theorems independent of PA that seem to require an understanding and use of infinitary concepts. We specifically focus on proof-discovery techniques that make use of a marriage of analogical and deductive reasoning (which we call *analogico-deductive reasoning*).

A Context: Infinitary Reasoning, Hypercomputation, and Humble Engineering

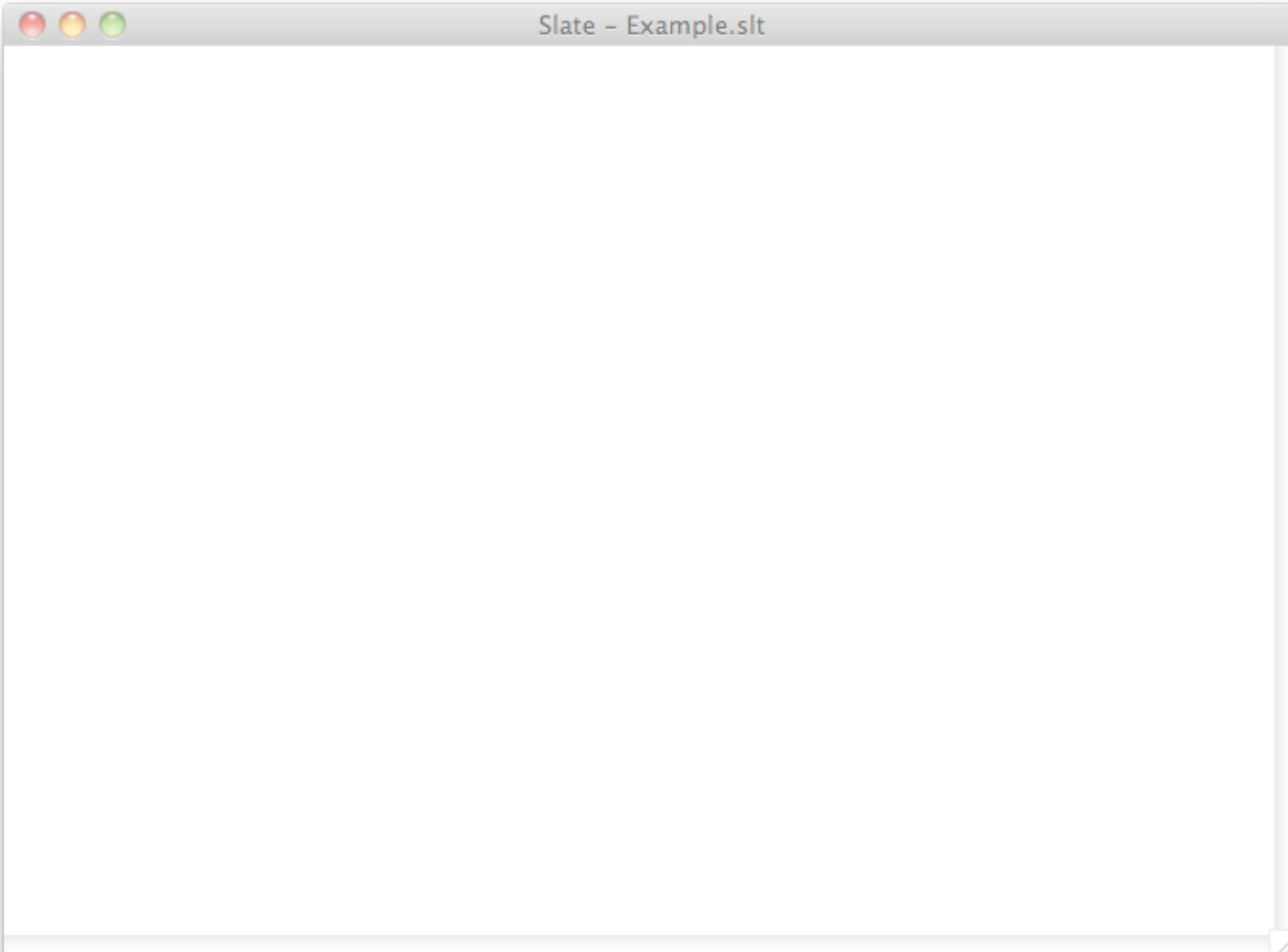
Bringsjord has repeatedly pointed out the obvious fact that the behavior of formal scientists, taken at face value, involve various infinitary structures and reasoning. (We say "at face value" to simply indicate we don't presuppose some view that denies the reality of infinite entities routinely involved in the formal sciences.) For example, in (Bringsjord & van Heuveln 2003), Bringsjord himself operates as such a scientist in presenting an infinitary paradox which to his knowledge has yet to be solved. And he has argued that apparently infinitary behavior constitutes a grave challenge to AI and the Church-Turing Thesis (e.g., see Bringsjord & Arkoudas 2006, Bringsjord & Zenzen 2003). More generally, Bringsjord conjectures that every human-produced proof of a theorem independent of Peano Arithmetic (PA) will make use of infinitary structures and reasoning, when these structures are taken at face value.[1] We have ourselves designed logico-computational logics for handling infinitary reasoning (e.g., see the treatment of the infinitized wise-man puzzle: Arkoudas & Bringsjord 2005), but this work simply falls back on the human ability to carry out induction on the natural numbers: it doesn't dissect and explain this ability. Finally, it must be admitted by all that there is simply no systematic, comprehensive model or framework anywhere in the formal/computational approach to understanding human knowledge and intelligence that provides a theory about how humans are able to engage with infinitary structures. This is revealed perhaps most clearly when one studies the fruit produced by the part of formal AI devoted to producing discovery systems: such fruit is embarrassingly finitary (e.g., see Shilliday 2009).

Given this context, we are interested in exploring how one might give a machine the ability to reason in infinitary fashion. We are not saying that we in fact have figured out how to give such ability to a computing machine. Our objective here is much more humble and limited: it is to push forward in the *attempt* to engineer a computing machine that has the ability to reason in infinitary fashion. Ultimately, if such an attempt is to succeed, the computing machine in question will presumably be capable of outright hypercomputation. But the fact is that from an engineering perspective, we don't know how to create and harness a hypercomputer. So what we must first try to do, as explained in (Bringsjord & Zenzen 2003), is pursue engineering that initiates the attempt to engineer a hypercomputer, and takes the first few steps. In the present paper, the engineering is aimed specifically at giving a computing machine the ability to, in a limited but well-defined sense, reason in infinitary fashion. Even more specifically, our engineering is aimed at building a machine capable of at least providing a strong case for a result which, in the human sphere, has hitherto required use of infinitary techniques.

[1] A weaker conjecture along the same line has been ventured by Isaacson, and is elegantly discussed by Smith (2007).

Govindarajulu, N.; Licato, J.; Bringsjord, S. 2013. Small Steps Toward Hypercomputation via Infinitary Machine Proof Verification and Proof Generation. In *Proceedings of UCNC 2013.* Pdf

Needs Understanding of Ordinal Numbers …



Needs Understanding of Ordinal Numbers …

Slate – Example.slt

$$19_1 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0 < \omega^{\omega^{\omega^{\omega^0}}} + \omega^{\omega^0} + \omega^0$$

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strictly decreasing

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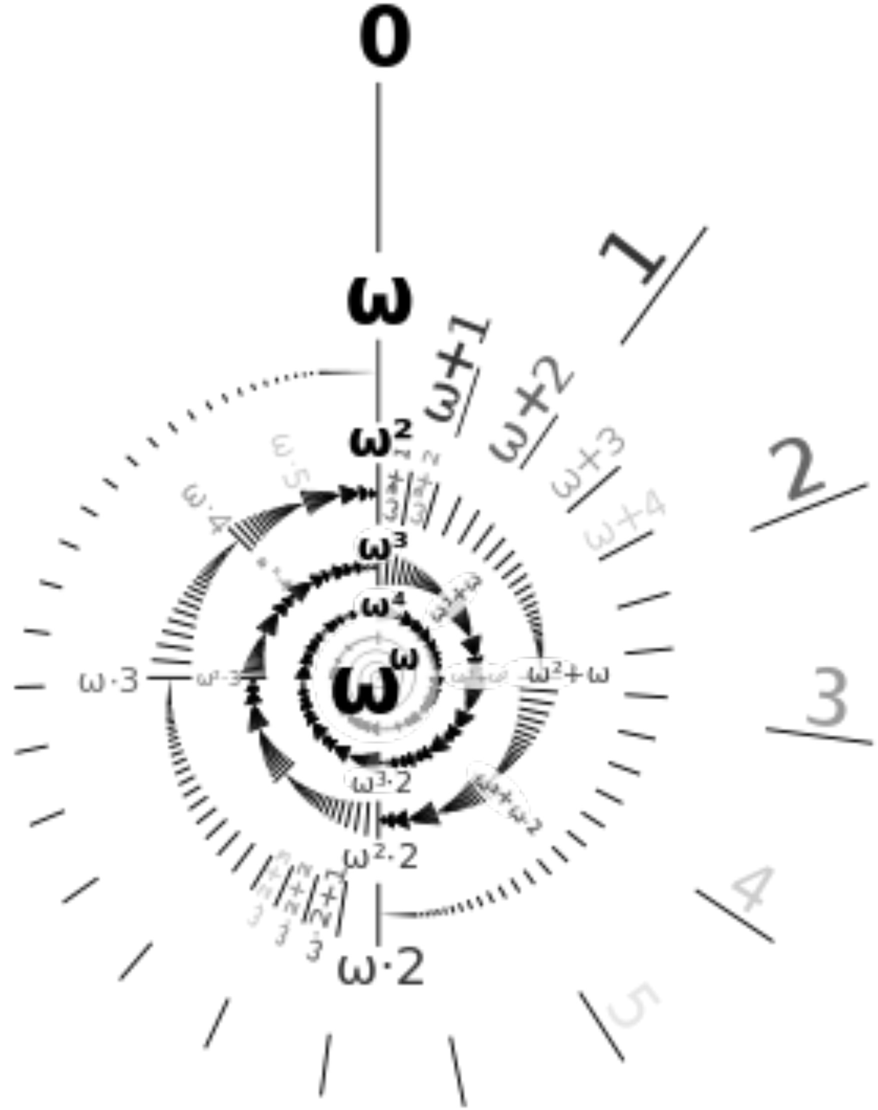

Yet, Conjecture (**C**)

(see "Isaacson's Conjecture")

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In order to produce a rationally compelling proof of any true sentence *S* formed from the symbol set of the language of arithmetic, but independent of **PA**, it's necessary to deploy concepts and structures of an irreducibly infinitary nature.

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In order to produce a rationally compelling proof of any true sentence *S* formed from the symbol set of the language of arithmetic, but independent of **PA**, it's necessary to deploy concepts and structures of an irreducibly infinitary nature.

If this is right, and computing machines can't use irreducibly infinitary techniques, they're in trouble — or: there won't be a Singularity.