# Quadrilaterals and Their Properties

**A 4-gon Hypothesis**
**Lesson 15-1  Kites and Triangle Midsegments**

**Learning Targets:**
- Develop properties of kites.
- Prove the Triangle Midsegment Theorem.

**SUGGESTED LEARNING STRATEGIES:** Discussion Groups, Shared Reading, Create Representations, Think-Pair-Share, Interactive Word Wall, Group Presentations

Mr. Cortez, the owner of a tile store, wants to create a database of all of the tiles he sells in his store. All of his tiles are quadrilaterals, but he needs to learn the properties of different quadrilaterals so he can correctly classify the tiles in his database.

Mr. Cortez begins by exploring convex quadrilaterals. The term *quadrilateral* can be abbreviated "quad."



1. Given quad *GEOM*.
   a. List all pairs of opposite sides.

   b. List all pairs of consecutive sides.

   c. List all pairs of opposite angles.

   d. List all pairs of consecutive angles.
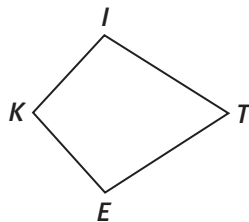
   e. Draw the diagonals, and list them.

**My Notes**

A *kite* is a quadrilateral with exactly two distinct pairs of congruent consecutive sides.



2. Given quad *KITE* with $\overline{KI} \cong \overline{KE}$ and $\overline{IT} \cong \overline{ET}$.
   a. One of the diagonals divides the kite into two congruent triangles. Draw that diagonal and list the two congruent triangles. Explain how you know the triangles are congruent.

   b. Draw the other diagonal. Explain how you know the diagonals are perpendicular.

   c. Complete the following list of properties of a kite. Think about the angles of a kite as well as the segments.
      **1.** Exactly two pairs of consecutive sides are congruent.
      **2.** One diagonal divides a kite into two congruent triangles.
      **3.** The diagonals of a kite are perpendicular.
      **4.**
      **5.**
      **6.**

3. **Critique the reasoning of others.** Mr. Cortez says that the diagonals of a kite bisect each other. Is Mr. Cortez correct? Support your answer with a valid argument.

**Check Your Understanding**

4. Why is a square not considered a kite?
5. Suppose $\overline{AC}$ and $\overline{BD}$ are the diagonals of a kite. What is a formula for the area of the kite in terms of the diagonals?

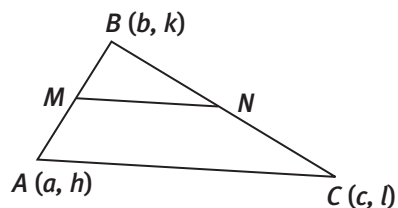**My Notes**

The segment whose endpoints are the midpoints of two sides of a triangle is called a *midsegment*.

**Triangle Midsegment Theorem** The midsegment of a triangle is parallel to the third side, and its length is one-half the length of the third side.

6. Use the figure and coordinates below to complete the coordinate proof for the Triangle Midsegment Theorem.



a. Complete the hypothesis and conclusion for the Triangle Midsegment Theorem.

**Hypothesis:** $M$ is the midpoint of _____.

$N$ is the midpoint of _____.

**Conclusion:** $\overline{MN} \parallel$ _____

$MN =$ _____

b. Find the coordinates of midpoints $M$ and $N$ in terms of $a$, $b$, $c$, $h$, $k$, and $l$.

c. Find the slope of $\overline{AC}$ and $\overline{MN}$.

d. Simplify your response to part c and explain how your answers to part c show $\overline{MN} \parallel \overline{AC}$.

**MATH TIP**

Given $A(x_1, y_1)$ and $B(x_2, y_2)$.

**Midpoint Formula:**

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

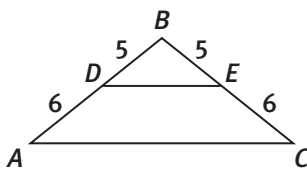Slope of $\overline{AB}$: $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$

**Distance Formula:**

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**e.** Find *AC* and *MN*.

**f.** Simplify your response to part e and explain how your answers to part e show that $MN = \frac{1}{2}AC$.

### Check Your Understanding

**7.** Are the midsegments of an isosceles triangle congruent? Explain.

**8.** Given $\overline{DE} \parallel \overline{AC}$. Is $\overline{DE}$ a midsegment of triangle *ABC*? Explain.
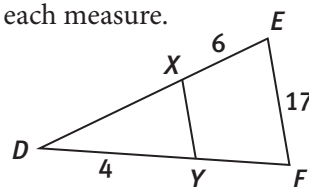


### LESSON 15-1 PRACTICE

**9.** $\overline{XY}$ is a midsegment of triangle *DEF*. Find each measure.

$XY = $ _____
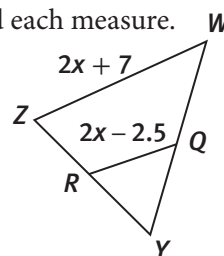
$DX = $ _____

$YF = $ _____



**10.** $\overline{QR}$ is a midsegment of triangle *WYZ*. Find each measure.

$x = $ _____

$WZ = $ _____

$QR = $ _____



**11. Make sense of problems.** Figure *ABCD* is a kite with diagonals $\overline{BD}$ and $\overline{AC}$. Complete each statement.
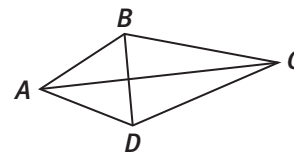
$\overline{BD} \perp$ _____

$\triangle ABC \cong \triangle$ _____

$\angle ABC \cong \angle$ _____

$\overline{AB} \cong$ _____

$\angle BAC \cong \angle$ _____

**Learning Targets:**
- Develop properties of trapezoids.
- Prove properties of trapezoids.

> **SUGGESTED LEARNING STRATEGIES:** Visualization, Shared Reading, Create Representations, Think-Pair-Share, Interactive Word Wall

A ***trapezoid*** is a quadrilateral with exactly one pair of parallel sides. The parallel sides of a trapezoid are called ***bases***, and the nonparallel sides are called ***legs***. The pairs of consecutive angles that include each of the bases are called ***base angles***.

1. Sketch a trapezoid and label the vertices *T*, *R*, *A*, and *P*. Identify the bases, legs, and both pairs of base angles.

The ***median of a trapezoid*** is the segment with endpoints at the midpoint of each leg of the trapezoid.

**Trapezoid Median Theorem** The median of a trapezoid is parallel to the bases and its length is the average of the lengths of the bases.



**Given:** Trapezoid *EFGH*
$\overline{MN}$ is a median.

**Prove:** $\overline{MN} \parallel \overline{FG}$ and $\overline{MN} \parallel \overline{EH}$
$MN = \frac{1}{2}(FG + EH)$

2. Draw one diagonal in trapezoid *EFGH*. Label the intersection of the diagonal with $\overline{MN}$ as *X* and explain below how the Triangle Midsegment Theorem can be used to justify the Trapezoid Median Theorem.

**CONNECT TO LANGUAGE**

The British use the term *trapezium* for a quadrilateral with exactly one pair of parallel sides and the term *trapezoid* for a quadrilateral with no parallel sides. They drive on a different side of the road, too.

3. Given trapezoid *EFGH* and $\overline{MN}$ is a median. Use the figure in Item 2, properties of trapezoids, and/or the Trapezoid Median Theorem for each of the following.
   a. If $m\angle GFE = 42°$, then $m\angle NME =$ _____ and $m\angle MEH =$ _____.

   b. Write an equation and solve for *x* if $FG = 4x + 4$, $EH = x + 5$, and $MN = 22$.

   c. Find *FG* if $MN = 19$ and $EH = 12$.

4. **Make use of structure.** What property or postulate allowed you to draw the auxiliary line in Item 2?

### Check Your Understanding

5. How does a trapezoid differ from a kite?
6. Can a trapezoid have bases that are congruent? Explain.

An ***isosceles trapezoid*** is a trapezoid with congruent legs.

7. Given $\triangle ABC$ is isosceles with $AB = CB$ and $AD = CE$.
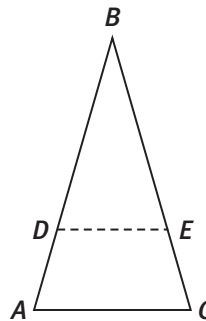


   a. $\angle A \cong$ _____. Explain.

   b. Explain why $\triangle BDE$ is isosceles.

   c. $\overline{AC} \parallel$ _____. Explain.

   d. Explain why quad *ADEC* is an isosceles trapezoid.

   e. $\angle ADE \cong$ _____. Explain.

**My Notes**

**f.** Complete the theorem.

The base angles of an isosceles trapezoid are _____.

**8.** On grid paper, plot quad *COLD* with coordinates $C(1, 0)$, $O(2, 2)$, $L(5, 3)$, and $D(7, 2)$.

**a.** Show that quad *COLD* is a trapezoid.

**b.** Show that quad *COLD* is isosceles.

**c.** Identify and find the length of each diagonal.

**d.** Based on the results in part c, complete the theorem.

The diagonals of an isosceles trapezoid are _____.

**9.** At this point, the theorem in Item 8 is simply a conjecture based on one example. Given the figure below, write the key steps for a proof of the theorem. Hint: You may want to use a pair of overlapping triangles and the theorem from Item 8 as part of your argument.
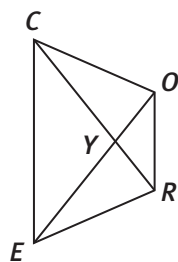


| **Hypothesis:** | *CORE* is a trapezoid. |
|---|---|
| | $\overline{CO} \cong \overline{ER}$ |
| **Conclusion:** | $\overline{CR} \cong \overline{EO}$ |

**My Notes**

## Check Your Understanding

**10.** Given quad *PLAN* is an isosceles trapezoid, use the diagram below and the properties of isosceles trapezoids to find each of the following.
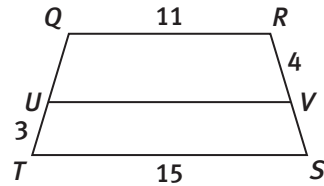


**a.** $\angle LPN \cong$ _____
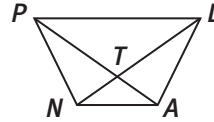
**b.** If $m\angle PLA = 70°$, then $m\angle LPN =$ _____ and $m\angle PNA =$ _____.

**c.** Write an equation and solve for $x$ if $AP = x$ and $NL = 3x - 8$.
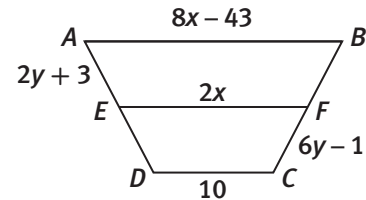
## LESSON 15-2 PRACTICE

**11.** $\overline{UV}$ is a midsegment of trapezoid *QRST*. Find each measure.

$QU =$ _____

$VS =$ _____

$UV =$ _____



**12. Reason abstractly.** $\overline{EF}$ is a midsegment of isosceles trapezoid *ABCD*. Find each measure.

$x =$ _____

$y =$ _____

$AE =$ _____

$ED =$ _____

$BF =$ _____

$FC =$ _____

$AB =$ _____

$EF =$ _____

**My Notes**

### Learning Targets:
- Develop properties of parallelograms.
- Prove properties of parallelograms.

> **SUGGESTED LEARNING STRATEGIES:** Visualization, Create Representations, Think-Pair-Share, Interactive Word Wall, Discussion Groups

A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel. For the sake of brevity, the symbol □ can be used for parallelogram.

1. Given □*KATY* as shown.
   a. Which angles are consecutive to ∠*K*?

   b. Use what you know about parallel lines to complete the theorem.
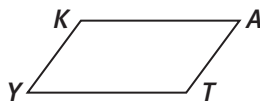
   Consecutive angles of a parallelogram are ⎯⎯⎯⎯⎯.

2. **Express regularity in repeated reasoning.** Use three index cards and draw three different parallelograms. Then cut out each parallelogram. For each parallelogram, draw a diagonal and cut along the diagonal to form two triangles. What do you notice about each pair of triangles?

3. Based upon the exploration in Item 2, complete the theorem.

   Each diagonal of a parallelogram divides that parallelogram into ⎯⎯⎯⎯⎯.
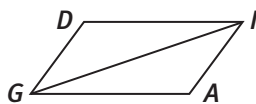
4. Given parallelogram *DIAG* as shown above. Complete the theorems.
   a. Opposite sides of a parallelogram are ⎯⎯⎯⎯⎯.

   b. Opposite angles of a parallelogram are ⎯⎯⎯⎯⎯.

   c. Prove the theorem you completed in part a. Use the figure in Item 3.

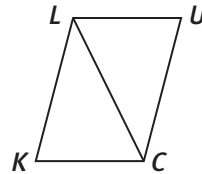   d. Prove the theorem you completed in part b. Use the figure in Item 3.

**MATH TERMS**

A **corollary** is a statement that results directly from a theorem.

5. Explain why the theorems in Item 4 can be considered as *corollaries* to the theorem in Item 3.

6. Given $\square LUCK$, use the figure and the theorems in Items 1, 3, and 4 to find the following.



a. $\triangle KCL \cong$ _____

b. Solve for $x$ if $m\angle KCU = 10x - 15$ and $m\angle K = 6x + 3$.

c. Solve for $x$ and $y$ if $KL = 2x + y$, $LU = 7$, $UC = 14$, and $KC = 5y - 4x$.

**Theorem:** The diagonals of a parallelogram bisect each other.

7. a. Rewrite the above theorem in "if-then" form.

   b. Draw a figure for the theorem, including the diagonals. Label the vertices and the point of intersection for the diagonals. Identify the information that is "given" and what is to be proved.

**Given:**

**Prove:**

   c. Write a two-column proof for the theorem.

**CONNECT TO AP**

Theorems are key to the development of many branches of mathematics. In calculus, two theorems that are frequently used are the Mean Value Theorem and the Fundamental Theorem of Calculus.

## Check Your Understanding

**8.** Why are trapezoids and kites not parallelograms?

**9.** The measure of one angle of a parallelogram is 68°. What are the measures of the other three angles of the parallelogram?

**10.** The lengths of two sides of a parallelogram are 12 in. and 18 in. What are the lengths of the other two sides?

## LESSON 15-3 PRACTICE

**11.** $\overline{AC}$ and $\overline{DB}$ are diagonals of parallelogram *ABCD*. Find each measure.

$AE =$ _____

$EC =$ _____

$DE =$ _____

$EB =$ _____



**12.** **Make sense of problems.** One of the floor tiles that Mr. Cortez sells is shaped like a parallelogram. Find each measure of the floor tile.

$m\angle W =$ _____

$m\angle X =$ _____

$m\angle Y =$ _____

$m\angle Z =$ _____

$WZ =$ _____

$XY =$ _____

**Learning Targets:**
- Develop properties of rectangles, rhombuses, and squares.
- Prove properties of rectangles, rhombuses, and squares.

> **SUGGESTED LEARNING STRATEGIES:** Visualization, Create Representations, Think-Pair-Share, Interactive Word Wall, Discussion Groups

A *rectangle* is a parallelogram with four right angles.

1. Given quad *RECT* is a rectangle. List all right triangles in the figure. Explain how you know the triangles are congruent.
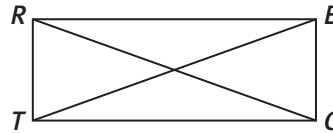
2. Complete the theorem.

   The diagonals of a rectangle are _____.

3. Explain how you know the theorem in Item 2 is true.

4. List all of the properties of a rectangle. Begin with the properties of a parallelogram.

5. Given quad *PINK* is a rectangle with coordinates $P(3,0)$, $I(0,6)$, and $N(8,10)$. Find the coordinates of point $K$.

6. Given quad *TGIF* is a rectangle. Use the properties of a rectangle and the figure at right to find the following.
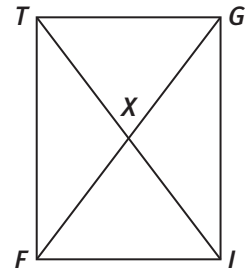
   a. If $TX = 13$, then $TI =$ _____ and $FG =$ _____.

   b. Solve for $x$ if $TX = 4x + 4$ and $FX = 7x - 23$.

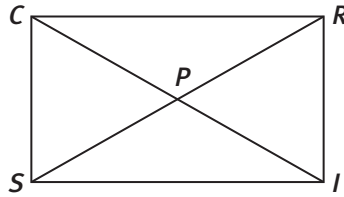   c. Solve for $x$ if $m\angle XFT = 6x - 4$ and $m\angle XTG = 10x - 2$.

*Indirect proofs* can be useful when the conclusion is a negative statement.

**Example of an Indirect Proof**

**Given:** $m\angle SCR \neq m\angle CSI$
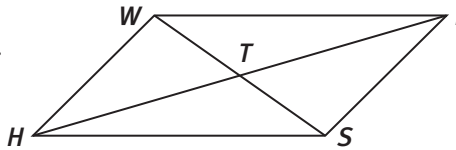**Prove:** $\square RISC$ is not a rectangle.

| Statements | Reasons |
|---|---|
| 1. $\square RISC$ is a rectangle. | 1. Assumption |
| 2. $m\angle SCR = m\angle CSI = 90°$ | 2. Definition of a rectangle |
| 3. $m\angle SCR \neq m\angle CSI$ | 3. Given |
| 4. $\square RISC$ is not a rectangle. | 4. The assumption led to a contradiction between statements 2 and 3. |

**7.** Complete the missing reasons in this indirect proof.

**Given:** $WT \neq TS$
**Prove:** Quad *WISH* is not a $\square$.

| Statements | Reasons |
|---|---|
| 1. $\square WISH$ | 1. |
| 2. $\overline{WS}$ and $\overline{HI}$ bisect each other. | 2. |
| 3. $WT = TS$ and $HT = TI$ | 3. |
| 4. $WT \neq TS$ | 4. |
| 5. Quad *WISH* is not a $\square$. | 5. |

**MATH TIP**

An indirect proof begins by assuming the opposite of the conclusion. The assumption is used as if it were given until a contradiction is reached. Once the assumption leads to a contradiction, the opposite of the assumption (the original conclusion) must be true.

**Check Your Understanding**

**8.** What do rectangles, trapezoids, and kites have in common? How do they differ?

**9.** Tell whether each of the following statements is true or false.
 **a.** All rectangles are parallelograms.
 **b.** Some rectangles are trapezoids.
 **c.** All parallelograms are rectangles.
 **d.** All rectangles are quadrilaterals.

**My Notes**

A ***rhombus*** is a parallelogram with four congruent sides.

**10.** Graph quad *USMC* with coordinates *U*(1, 1), *S*(4, 5), *M*(9, 5), and *C*(6, 1) on the grid below.
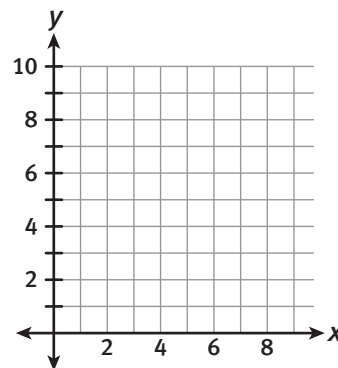


**a.** Verify that quad *USMC* is a parallelogram by finding the slope of each side.

**b.** Verify that ▱*USMC* is a rhombus by finding the length of each side.
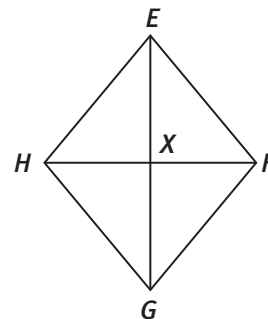
**c.** Find the slopes of the diagonals, $\overline{MU}$ and $\overline{SC}$.

**d.** Use the results in part c to complete the theorem.

The diagonals of a rhombus are _____.

**11.** Given quad *EFGH* is a rhombus.
**a.** List the three triangles that are congruent to △*HXE*.



**b.** Explain why ∠*EFX* ≅ ∠*GFX* and ∠*HGX* ≅ ∠*FGX*.

**c.** Complete the theorem.

Each diagonal of a rhombus _____.

A formal proof for the theorem in Item 11 is left as an exercise.

**12.** List all of the properties of a rhombus. Begin with the properties of a parallelogram.

**My Notes**

13. Given quad *UTAH* is a rhombus. Use the properties of a rhombus and the figure at right to find each of the following.
    a. Solve for $x$ if $m\angle UPT = 4x + 18$.

    b. Solve for $x$ and $y$ if
    $UT = 5x + 4$,
    $TA = 2x + y$,
    $HA = 2y - 8$, and
    $UH = 24$.

    c. Solve for $x$ if $m\angle PAH = 8x + 2$ and $m\angle PAT = 10x - 10$.

A **square** is a parallelogram with four right angles and four congruent sides.

14. Alternate definitions for a square.
    a. A square is a rectangle with

    _____.

    b. A square is a rhombus with

    _____.

15. List all of the properties of a square.

16. Match each region in the Venn diagram below with the correct term in the list.

kites                 isosceles trapezoids        parallelograms
polygons              quadrilaterals              rectangles
rhombi                squares                     trapezoids

**My Notes**

**17. Model with mathematics.** Mr. Cortez uses the table below to organize his findings before he enters information in the database. Place a check mark if the polygon has the given property.

| | 4 Sides | Opposite Sides Parallel | Opposite Sides Congruent | Opposite Angles Congruent | Diagonals Bisect Each Other | Consecutive Angles Supplementary | Diagonals Perpendicular | 4 Right Angles | 4 Congruent Sides | Exactly One Pair of Opposite Sides Parallel |
|---|---|---|---|---|---|---|---|---|---|---|
| Quadrilateral | | | | | | | | | | |
| Kite | | | | | | | | | | |
| Trapezoid | | | | | | | | | | |
| Parallelogram | | | | | | | | | | |
| Rectangle | | | | | | | | | | |
| Rhombus | | | | | | | | | | |
| Square | | | | | | | | | | |

### Check Your Understanding

**18.** Tell whether each statement is true or false.
   **a.** All squares are rectangles.
   **b.** All rhombuses are squares.
   **c.** All squares are parallelograms.
   **d.** Some squares are kites.
   **e.** No rhombuses are trapezoids.

**19.** What do all rectangles, squares, and rhombuses have in common?
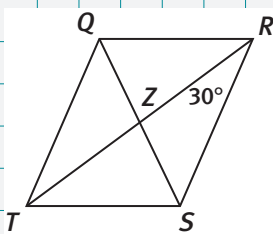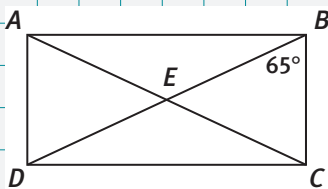
### LESSON 15-4 PRACTICE

**20.** $\overline{AC}$ and $\overline{DB}$ are diagonals of rectangle *ABCD*. Find each measure.

$m\angle DAB = $ _____          $m\angle AEB = $ _____

$m\angle ADC = $ _____          $m\angle BEC = $ _____

$m\angle BDC = $ _____          $m\angle BCE = $ _____

$m\angle BDA = $ _____

**21.** $\overline{QS}$ and $\overline{RT}$ are diagonals of rhombus *QRST*. Find each measure.

$m\angle QSR = $ _____          $m\angle QZR = $ _____

$m\angle QST = $ _____          $m\angle QTR = $ _____

$m\angle QTS = $ _____          $m\angle RZS = $ _____

**22. Make sense of problems.** A diagonal of a square tile is 10 mm. What is the area of the tile?

## ACTIVITY 15 PRACTICE

Write your answers on notebook paper.
Show your work.

### Lesson 15-1

1. Tell whether each statement about kites is *always*, *sometimes*, or *never* true.
   a. Exactly two pairs of consecutive sides are congruent.
   b. The diagonals divide the kite into four congruent triangles.
   c. The diagonals are perpendicular.
   d. A kite is a parallelogram.
   e. One diagonal bisects a pair of opposite angles.
   f. A kite is a rhombus.

### Lesson 15-2

2. Make a true statement by filling in each blank with *always*, *sometimes*, or *never*.
   a. A trapezoid is _____ isosceles.
   b. A trapezoid is _____ a quadrilateral.
   c. The length of the median of a trapezoid is _____ equal to the sum of the lengths of the bases.
   d. Trapezoids _____ have a pair of parallel sides.
   e. Trapezoids _____ have two pairs of supplementary consecutive angles.

3. Given quad *GHJK* is a trapezoid. $\overline{PQ}$ is the median.



   a. If $HJ = 40$ and $PQ = 28$, find $GK$.
   b. If $HJ = 5x$, $PQ = 5x - 9$, and $GK = 3x + 2$, then solve for $x$.
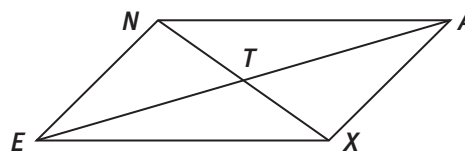
4. Given quad *JONE* is a trapezoid.
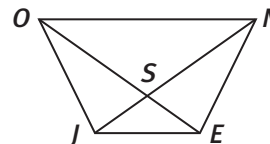


   a. $\angle ONJ \cong$ _____
   b. If $\overline{OJ} \cong \overline{NE}$, then $\overline{OE} \cong$ _____.
   c. If $\overline{OJ} \cong \overline{NE}$, then $\angle NEJ \cong$ _____.

### Lesson 15-3

5. Quadrilateral *XENA* is a parallelogram. *T* is the point of intersection of the diagonals. For each situation, write an equation and solve for $y$.



   a. $EN = 5y + 1$ and $AX = 8y - 5$
   b. $m\angle ANX = 3y - 1$ and $m\angle NXE = 2y + 1$
   c. $ET = y - 1$ and $EA = 3y - 10$
   d. $m\angle ANE = 7y - 5$ and $m\angle NEX = 3y + 5$

6. *M* is the fourth vertex of a parallelogram. The coordinates of the other vertices are (6, 4), (8, 1), and (2, 0). *M* can have any of the following coordinates except:
   **A.** (6, −2)  **B.** (12, 5)
   **C.** (4, −3)  **D.** (0, 3)

7. Given quad *QRST* with coordinates $Q(0, 0)$, $R(2, 6)$, $S(12, 6)$, and $T(12, 0)$.
   a. What is the best name for quad *QRST*? Explain.
   b. Find the coordinates of the midpoint for each side of quad *QRST* and label them *M, N, O,* and *P*. What is the best name for quad *MNOP*? Explain.

### Lesson 15-4

8. Given quad *WHAT* with vertices $W(2, 4)$, $H(5, 8)$, $A(9, 5)$, and $T(6, 1)$. What is the best name for this quadrilateral?
   **A.** parallelogram      **B.** rhombus
   **C.** rectangle          **D.** square

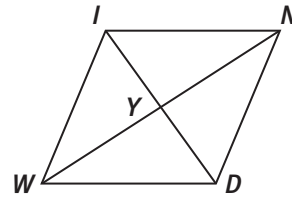9. Given quad *ABCD* is a rhombus and $m\angle ABD = 32°$. Find the measure of each numbered angle.
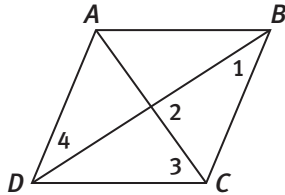


10. Given quad *RIGH* is a rectangle.
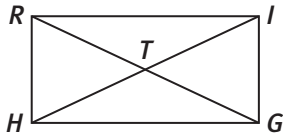


    **a.** If $RT = 18$, then $RG =$ _____.

    **b.** If $RG = 4x + 12$ and $HI = 10x - 15$, then $x =$ _____.

11. **Given:** Parallelogram *PQRS* with diagonal *PR*.
    **Prove:** $\triangle PQR \cong \triangle RSP$
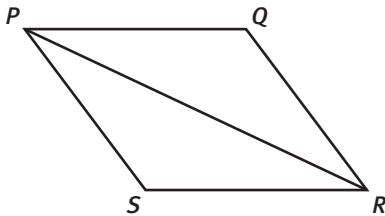


12. Write an indirect proof.
    **Given:** $\triangle WIN$ is not isosceles.
    **Prove:** Quad *WIND* is not a rhombus.



### MATHEMATICAL PRACTICES
**Reason Abstractly and Quantitatively**

13. Ginger noticed that no matter the height of the adjustable stand for her electric piano, the keyboard remains level and centered over the stand. What has to be true about the legs of the stand? Explain.