Game Theory and Nash Equilibrium

by
Jenny Duffy

A project submitted to the Department of
Mathematical Sciences in conformity with the requirements
for Math 4301 (Honours Seminar)

Lakehead University
Thunder Bay, Ontario, Canada

Abstract

This honours seminar project will focus on some of the primary knowledge that is associated with game theory. The research will be examining two-person zero-sum games and non-zero sum games, as well as N-person games. Included will be some inspection of basic matrix games in two-person zero-sum games. As well, both pure and mixed strategies will be studied within these games along with some basic algorithms to find said strategies. Another aspect of this thesis will be to look into Nash Equilibrium and the importance it has related to game theory. Additionally there will be some study done on the prisoner's dilemma in both two-person non-zero sum games and N-person games. Lastly, there will be some application to real world games throughout the paper.

Acknowledgements

I would like to thank and acknowledge everyone who has assisted me throughout the duration of writing this thesis. First, I would like to thank my supervisor, Dr. Yin Chen for his knowledge and guidance throughout this process. I would also like to recognize Dr. Razvan Anisca for his feedback and assistance. Next, I would like to thank my Mother for supporting me throughout my education and especially this past year while I have been writing this paper. A very special thank you to my Father for teaching me not to give up and to keep going no matter what life may throw my way and for being my very special guardian angel. Lastly I would like to express my appreciation for all of those who have read over or edited my thesis, as well as given me much encouragment along the way.

Contents

CHAPTER 1

Introduction

Historians give credit to John Von Neumann as the founder of Game Theory as a mathematical discipline. His first paper, published in 1928, laid out the fundamental theorem of two-person zero-sum games. John Von Neumann continued his research in collaboration with Oskar Morgenstern and their work was published into the book entitled *Theory of Games and Economic Behaviour*, which was published in 1944.

The aim of this project is to investigate and understand some of the basic knowledge of Game Theory. This includes understanding both pure and mixed strategies within games and how to apply some basic algorithms to find said strategies. In addition, this paper will be studying Nash Equilibrium and the important role that it plays within Game Theory. Game Theory is a branch of applied mathematics that analysis situations, both mathematically and logically, in order to create strategies that a player should take into action to ensure the best outcome for themself within a game.

Game Theory is primarily used within economics, political science and psychology. When it was first introduced, Game Theory focused soley on two-person zero-sum games, but has since evolved to encompass strategies and game play between more players. During the 1950s Game Theory was largely advanced by many scholars researching this area of mathematics. For example in 1950, John Nash wrote a dissertation on non-cooperative games which outlined what is now known as Nash Equlibrium. Nash equilibrium occurs in non-cooperative games when two players have optimal game strategies such that no matter how they change their strategy, or game play, they will not gain any benefit.

CHAPTER 2

Preliminaries

In this section, we will present some of the important definitions and terms, which will be used throughout this paper. The main sources for this chapter are [1] and [2]. First and foremost, we need to begin by defining what we mean when we are talking about a game.

DEFINITION 2.1. We define a **game**, to be a situation in which

- There are two or more players
- Each player has a number of strategies that they may choose
- The strategies chosen by each player will determine the outcome of the game
- Each possible outcome of the game has a payoff to each player

An important thing to note is that we can represent our games in different forms. The two forms that we will be looking at are **normal form** and **characteristic function form**, the latter being discussed in Chapter 5 more intently. Games that are described as in *Definition 2.1* are called **normal form** and we usually represent these types of games through the use of a *payoff matrix game*. The purpose of representing games through a matrix game, is that players can more easily calculate the strategies that they should play within a game, which we will be looking at within this paper. Additionally, it is assumed that the players of games are rational thinkers hence the importance of forming a strategy within the games.

To describe a payoff matrix game, assume we are playing a basic game and there are two players involved. The payoff matrix will represent two players, their choices and their payoffs. So, how does a matrix game represent these elements of the game? Well first we have player one, who is also known as the row player. This player is known as the row player because the rows are labelled with their choices/actions that they are able to play within the game. We will label these as $r_1...r_i$ in the matrix game that follows for clarification. Next, we have player two, who is known as the column player. Player two is called the column player because the columns of our game are labelled with the choices/actions that the second player may choose within a game. We will label these as $c_1...c_j$. Now the choices that each player makes will correlate with the payoffs each player receives. Since we have two players, one and two, with player one having i number of choices and player two having j number of choices, we will have $i(j)$ number of outcomes in the matrix. The elements in the matrix, a_{ij} represent the corresponding payoff for player one, if player one chooses action i and player two chooses action j. In zero sum

games the matrix will always represent the payoff to the row player, but in non-zero sum games both payoffs will be written in the corresponding a_{ij} spot. It is also important to note for later when we are talking about zero-sum games that player twos payoff for the game will be $(-a_{ij})$.

TABLE 1. Representing Normal Form Games in Matrix Games

		Player 2		
		c_1	...	c_j
Player 1	r_1	$a_{1,1}$	...	$a_{1,j}$
	$\vdots$	$\vdots$		$\vdots$
	r_i	$a_{i,1}$	...	$a_{i,j}$

Games in normal form, for example the game in table 1 of Chapter 3 is of this form, will include all of the strategies that can be played and the corresponding payoffs for these actions, for each player. We will learn how to find many of these strategies, including both pure and mixed strategies, within Chapter 3.

The next thing that we must define for the benefit of this paper is what we have in mind when we are discussing *strategies*.

DEFINITION 2.2. A players **strategy** is what actions or options a player can choose within a game, or situation, where the outcome is dependent on that option. The outcome is also dependent on the actions of the players opponent and their choice in the game. A players strategy helps determine what action they are going to play within the game.

For example, if a player is playing the game rock-paper-scissors they have three possible actions that they are able to play; rock, paper or scissors. If they are playing rock every time this is called a pure strategy. If they are playing a mixture of actions with different probabilities for each choice, this is known as a mixed strategy. So a player would play a probability x for rock, a probability y for paper and a probability $(1 - x - y)$ for scissors to play a mixed strategy.

Something to note, for later on in the paper is that every matrix game has a solution, either in pure strategies or in mixed strategies.

CHAPTER 3

Two Person Zero Sum Games

The first type of game that we will be looking at are those that involve only two players. This can be a game or situation that is between two humans, a human and a computer or human versus nature. Whoever is involved in the game will not take away from the main goal of the game, which is to play such that they are coming out with the best possible outcome no matter the circumstances. This is why it is important when playing a game, to play strategically. For the most part, players will be facing opponents who are rational in game play and hence they need to evaluate the actions they are able to take in a game. To create an optimal outcome, players need to look at their strategies. There are two types of strategies that we will be studying, these include **pure** and **mixed** strategies. It is important to understand the difference between these two types. In this chapter, the primary sources used were [1], [3] and [4].

1. Pure strategies

First we will be looking at *pure strategies*, which are the most basic of the two types and then we will continue on to mixed strategies.

DEFINITION 3.1. A **Pure Strategy** defines the specific action a player takes, no matter the situation of the game. Pure strategies are not random and the player does not change their actions during the game.

Consider the situation where you are playing the simple game of rock, paper, scissors. The rules are simple, each player has three choices to choose from in an attempt to defeat their opponent; rock, paper or scissors. The outcomes of the game are; paper beats rock, rock beats scissors and scissors beat paper. A player either wins, losses or ties. If a player were to play a pure strategy of rock everytime it would most likely not be beneficial for them in this type of game. As we have stated before, it is assumed that in a game we are playing against a rational player. If you are playing rock every single turn, most likely your rational opponent will catch on to this pure strategy and play accordingly, hence choosing paper. In situations or games such as rock, paper, scissors it is beneficial for a player to play a mixed strategy, which we will discuss in section 2 of this chapter.

There are different types of pure strategies that a player can use when playing a game. The first of these pure strategies is *dominant strategy*.

DEFINITION 3.2. A **dominant strategy** is a pure strategy such that: a strategy S dominates a strategy T if every outcome in S is at least as good as the corresponding outcome in T, and at least one outcome in S is strictly better than the corresponding outcome in T.

Lets look at a simple example of dominance. Assume that you are on a game show and you have the choice to spin one of two wheels, A or B, as seen in figure 1.
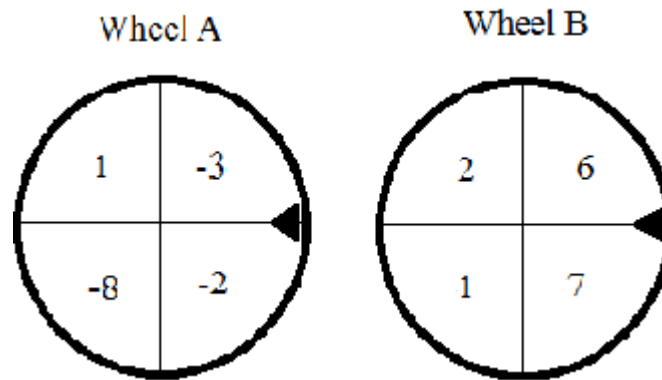

FIGURE 1. Dominance Diagram

It should be clear that every outcome in wheel B is better than, or at least equal to every outcome in wheel A. So we have that wheel B donimantes wheel A and when making a choice on which wheel to spin, a rational player would choose wheel B. This leads us to what is known as the *dominance principle.*

DEFINITION 3.3. The **Dominance principle** states: A rational player should never play a dominated strategy

The importance behind dominant strategies is that they are useful in ruling out other strategies, or actions to play within a game. It would not be logical to play a strategy that is inferior to another. One thing that readers should note is that there may not be a dominant strategy within a game. The next type of pure strategy that we will be looking at is called a saddle point strategy.

DEFINITION 3.4. An outcome of a matrix game (with payoffs to the row player) is called a **saddle point** if the entry at that outcome is both less than or equal to any entry in its row, and greater than or equal to any entry in its column.

This strategy has players looking at what their best response would be to what their opponent may play and also their own choices. If we have a payoff matrix game representing a normal form game, and we have found an element a_{ij}, where a_{ij} is the maximum of all of the minimums of all the rows and it is also the minimum of all of the maximums of the columns, then a_{ij} is a saddle point for the game.

It is an easy process to find a saddle point within a game, if it does exist. A player will begin by writing down the minimum entry for each row. From these minimum entries choose the maximum entry, as done in the table *Finding Saddle Points in a Game*. This is called the *maximin*. The next step is to look at the columns and write down the maximum entry for each column. Next choose the minimum entry of these column maximums. This is called the *minimax*. If the maximin of the rows and the minimax of the columns are equal, then they appear at the saddle point strategies for the players. If the maximin and minimax are not equal, then there is not a saddle point in the game. In the table *Finding Saddle Points in a game* the saddle point occurs are the outcome AA.

TABLE 1. Finding Saddle Points in a Game

		Player 2			
		A	B	C	Min
Player 1	A	2	4	3	**2**
	B	1	-10	5	-10
	C	-1	6	-8	-8
	Max	**2**	6	5	

When a game contains a saddle point, then along with it is an important principle called the *saddle point principle*. It is also worthy to note that the saddle point will be the same for both players, and hence why both players should play a strategy that involves the saddle point if it does exist.

DEFINITION 3.5. **Saddle Point Principle:** If a matrix game has a saddle point, both players should play a strategy which contains it.

If a saddle point exists within a payoff matrix game, then this saddle point strategy will find what is known as the *value of the game*.

1.1. Value of the Game, Pure strategy.

DEFINITION 3.6. For any matrix game if there is a number **v** such that player A has a strategy which guarantees that she will win at least **v**, and player B has a strategy which guarantees that player A will win no more than **v**, then **v** is called the **value of the game**.

If the value of the game exists, then a player can guarantee themselves an outcome of at least v. If a player does not play the saddle point strategy they are risking an outcome that is inferior to v and their opponent can possibly have an outcome with a value better than v. In a non-zero sum game a player would obviously want their opponents outcome to be at max v because if it is any higher than they are losing from their payoff of the game.

Note: It is possible for a game to have more than one saddle point, however the saddle points will have the same value. These points will also appear in the corners of the

rectangle that makes up the outcomes of the payoff matrix game. We will show this in the theorem 3.7.

THEOREM 3.7. *Any two saddle points in a matrix game have the same value. Furthermore, if player A and player B both play strategies containing a saddle point outcome, the result will always be a saddle point.*

PROOF. Suppose that we have a matrix game with corner entries a, b, c and d, where a and b are saddle points of the game as follows;

$$\begin{matrix} a & ... & c \\ \vdots & & \vdots \\ d & ... & b \end{matrix}$$

By our definition of a saddle point and the minmax theorem, we know that if a is a saddle point of our game, then it is the smallest entry in its row and similarily if b is a saddle point of our game then b is the largest entry in its column. This tells us that $a \leq c$ and $c \leq b$. So we know that $a \leq c \leq b$, we denote this as (1). Similarly b is the smallest entry is its row and a is the largest entry in its column. So we have $b \leq d$ and $d \leq a$, giving us $b \leq d \leq a$, we denote this as (2). Combing the inequalities (1) and (2) we get $a = b$, as well as $a = c = d = b$. This implies that c and d must also be the largest entries in their columns and the smallest entries in their rows, hence they too are saddle points and all of the saddle points are equal.   □

Again as a reminder, if we have that the *minimax* and *maximin* are not of the same value, when examining if there is a saddle point in a game, then we are able to conclude that there is no saddle point to the game.

2. Mixed Strategy

Until this point we have looked solely at pure strategies, but what if there is not a saddle point strategy or dominant strategy to a game? Then a player will want to create what is called a *mixed strategy*.

DEFINITION 3.8. A plan, which involves playing a mixture of strategies according to certain fixed probabilities is called a **mixed strategy**.

Instead of playing the same choice every turn, a player may want to play a mixture of choices that will optimize their overall outcome and overall payoff while playing a game. Now, let us suppose that we are looking at a basic 2 by 2 game, where there are no pure strategies within the game. Hence, we want to find a mixed strategy for the game.

In games where a player is playing a mixed strategy, they want to play a choice x percentage of the time, which they have calculated will have the best possible outcome of their game and another choice $x - 1$ percentage of the time (For a basic 2 by 2 matrix). How would we find this players mixed strategy, such that it is optimal in a payoff matrix game? One way is to calculate the following algorithms;

(2.1)
$$\mid \frac{(d-c)}{(d-c)+(a-b)} \mid A, \mid \frac{(a-b)}{(d-c)+(a-b)} \mid B \qquad\qquad [Player\ 1's\ Optimal\ strategy]$$

(2.2)
$$\mid \frac{(a-c)}{(a-c)+(b-d)} \mid A, \mid \frac{(b-d)}{(a-c)+(b-d)} \mid B \qquad\qquad [Player\ 2's\ Optimal\ strategy]$$

With the a, b, c and d values all corresponding to some payout in our basic matrix as follows.

TABLE 2. Optimal Mixed Strategy

		Player 2	
Player 1		A	B
	A	a	b
	B	c	d

These algorithms will give the player the percentages that they will want to play A and the percentage they will want to play B. Let us take the following game as an example of a matrix game we are playing;

TABLE 3. Optimal Mixed Strategy

		Player 2	
Player 1		A	B
	A	$a = 7$	$b = -2$
	B	$c = 3$	$d = 4$

The first step is to check that the game does not have a dominant or saddle point pure strategy, which this game does not. Next we want to use the above algorithms, (2.1), (2.2), to calculate the optimal playing strategies for player 1 and player 2.
Player 1 Optimal Strategy:

$$(2.3) \qquad A = \mid \frac{(4-3)}{(4-3)+(7-(-2))} \mid = \mid \frac{1}{10} \mid, \ B = \mid \frac{(7-(-2))}{(4-3)+(4-(-2))} \mid = \mid \frac{9}{10} \mid$$

Player 2 Optimal Strategy:

$$(2.4) \qquad A = \mid \frac{(7-3)}{(7-3)+(4-(-2))} \mid = \mid \frac{4}{10} \mid, \ B = \mid \frac{(-2-4)}{(7-3)+(4-(-2))} \mid = \mid \frac{6}{10} \mid$$

Subsitituting in our values for a, b, c and d we will get that player 1 should play $\frac{1}{10}A$, and $\frac{9}{10}B$, for their optimal strategy, while player 2 should play $\frac{4}{10}A$ and $\frac{6}{10}B$, for their optimal strategy. These probabilities that we have found for the players to play will help them to find what is known as their *expected value of the game.*

DEFINITION 3.9. The **expected value** of getting payoffs a_1, a_2, ..., a_k with respective probabilities p_1, p_2, ..., p_k is $a_1p_1 + a_2p_2 + ... + a_kp_k$. The expected value of a set of payoffs is just the weighted average of those payoffs, where the weights are the probabilities that each will occur.

Player one will have an expeted payoff of $\left[\frac{1}{10} \times 7\right] + \left[\frac{1}{10} \times (-2)\right] + \left[\frac{9}{10} \times 3\right] + \left[\frac{9}{10} \times 4\right] = \frac{68}{10}$ and player two will have a payoff of $\left[\frac{4}{10} \times (-7)\right] + \left[\frac{4}{10}(-3)\right] + \left[\frac{6}{10} \times (2)\right] + \left[\frac{6}{10} \times (-4)\right] = \frac{-52}{10}$.

To summarize, the expected value of a game is basically the average value a player should expect as their payoff, when playing their optimal mixed strategy over a period of time. Next, consider the case in which you were able to figure out your opponents mixed strategy. A rational player should be able to take the knowledge of their opponents strategy and play their own strategy that will give them the highest payoff, with the certain circumstances. This leads us to what is known as the *expected value principle*.

DEFINITION 3.10. **Expected value principle**: If you know that your opponent is playing a given mixed strategy, and will continue to play it regardless of what you do, you should play your strategy which has the largest expected value.

If a player is unable to predict or find out the mixed strategy of an opponent, then the expected value principle will not apply and the player will want to focus on creating a mixed strategy that will ensure themself the best payoff not matter the situation they are dealing with.

2.1. Value of the Game, Mixed Strategies.

THEOREM 3.11. **Minimax theorem**: *Every $m \times n$ matrix game has a* **solution**. *That is there is a unique number v, called the* **value of the game***, and there are optimal (pure or mixed) strategies for player A and player B such that;*

   (i) *if player A plays their optimal strategy, player A's expected payoff will be $\geq v$ no matter what player B does and*
  (ii) *if player B plays their optimal strategy, player A's expected payoff will be $\leq v$, no matter what player A does.*

The **minimax theorem**, tells us that there is always a solution to be found in some $k \times k$ subgame of an original $m \times n$ game. Another important thing to note for this theorem is that the method of equalizing expectations will fail if the solution of a 3×3 game has a 1×1 or a 2×2 subgame. When working with games that are larger than 2×2, you should always check for a saddle point strategy or dominance strategy before trying to equalize expectations. Additionally, one must be cautious about finding 2×2 subgames as there may be many of them in large $m \times n$ games. A graphing technique exists, which can be used to find the 2×2 matrix that will provide the solution of the game. For example, assume we are playing the following game;

TABLE 4. Game for Graphing Techniques

		A	B	Min
		Player 2		
Player 1	A	6	-3	-3
	B	2	-1	**-1**
	C	-5	4	-5
	Max	6	**4**	

First, we are able to deduce that there is no saddle point strategy to the game. Similarly, there are no dominant strategies. The next step when playing this game is to find a mixed strategy for the game. To do this we can begin by drawing a figure such as the one in figure 2, *Using upper and lower envelopes to find subgames.* On the left axis of our graph, a player will plot all of the outcomes for player 1 with the criteria that player 2 choose to play A (Player 1 chooses option A, player 2 chooses option A, we get the outcome AA which is 6, hence plot 6 on this axis). On the right axis of our graph, the player will then plot player 1's outcomes if player 2 played B. The next step is to then connect the outcomes where player 1 played the same choice for each side. For example, connect points AA and AB, where AA represents both players selecting A and AB represents player 1 selecting A and player 2 selecting B (Player one made the choice of A for each of these outcomes). The other connections are BA and BB, as well as CA and CB.
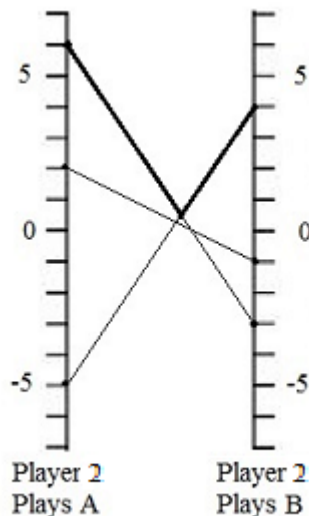

FIGURE 2. Using Upper and Lower Envelopes to Find Subgames

The bolded section of the graph is called the **upper envelope** and player 1 would want to play a strategy that plays this section to optimize their payoff. Also note the

lowest point of this upper envelope is at the intersection of Player 1 playing A and C. This tells us that in our mixed strategy for player 1, they would not want to play option B. With this information, we can change the 3×2 game, into a simpler 2×2 subgame and solve it with a mixed strategy.

TABLE 5. Game for Graphing Techniques

		A	B
		Player 2	
Player 1	A	6	-3
	C	-5	4

Through quick inspection, this game does not have a pure strategy, but the mixed strategy for player one would be to play $\frac{1}{2}A$ and $\frac{1}{2}B$.

3. Utility Theory

In this next section, we will be discussing games where the numerical payoffs in the games are not given. Rather than knowing the players outcome as a numerical quantity or unit, we will be considering their preference for that choice compared to the other options. This will be done by assigning numbers to the outcomes, such that they coordinate with the players preference to that outcome, compared to the other possible outcomes. We simply number the outcomes from 1 to n, where n is the number of outcomes. This is known as *utility theory*.

DEFINITION 3.12. **Utility theory** is the science of assigning numbers to outcomes in a way which reflects a player's preferences.

When dealing with utility theory we want to look at two cases. The first is when the game has a saddle point and the second is when we are using mixed strategy. Lets look at the first case. Since we have a saddle point, this tells us that the player will prefer this entry compared to the others in its coloumn. Similarily, the player will prefer any other entry in the row compared to the saddle point. We are **ordering** the numbers that the player prefers from most to least. Sometimes a player is unable to rank the payoffs, this is called **intransitive preference** and no ordering is possible. For example, someone may prefer the colour red to the colour green, and the colour green to the colour purple, but still prefer the colour purple to the colour red. If ordering is possible in a zero-sum matrix game, then the order of preferibility for player 1 will be the complete reverse for player 2. These scales that we are creating are called *ordinal scales*.

DEFINITION 3.13. A scale in which higher numbers represent more preferred outcomes in such a way that only the order of the numbers matter, not their absolute or relative magnitude, is called **ordinal scale**. Numbers determined from preferences in this way are called **ordinal utilities**.

Now we will look at the second case, mixed strategies. In this case we need to look at the **ratios of difference**, which in the case of the table 3 game would be $(d - c) : (a - b)$.

DEFINITION 3.14. A scale on which not only the order of numbers, but also the ratios of differences of the numbers is meaningful is called an **interval scale**. Numbers reflecting preferences on an interval scale are called **cardinal utilities**.

In these situations a player may not know the outcomes from their game play, but they have the distribution of probabilities for these outcomes. Let us then assume that we have a game with four outcomes, $a, b, c,$ and d and player 1 orders them as d, a, c and b. We want to find the players utilities on a cardinal scale. To do this we use lotteries. We will assign numbers to our d and b, the most and least prefered outcomes for the player. First assign 0 to b and 100 to d (Step 1, Figure 3). The next step is to then ask questions to the player, that break down the probabilities. So, we could ask player 1 if they prefer c, or to play $\frac{1}{2}d$ and $\frac{1}{2}b$? If they said they prefered c and due to the assigning of 0 to b and 100 to d we know that c is somewhere between 50 and 100 on the scale ($\frac{1}{2} \times 100 + \frac{1}{2}0 = 50$). If they did not prefer c, then we would know it lies somewhere on the scale between 0 and 50 (Step 2, Figure 3). Assuming the ladder is the case we could ask if the player prefers c or $\frac{1}{4}d$ and $\frac{3}{4}b$ ($\frac{1}{4} \times 100 + \frac{3}{4} \times 0 = 25$). If they choose c then c is between 25 and 50, but if they choose $\frac{1}{4}d$ and $\frac{3}{4}b$ then c is between 0 and 25 (Step 3, Figure 3). We can constantly change the lottery probabilities in order to get a better understanding of where a lies on the scale. For example, do you prefer c to $\frac{3}{4}d$ and $\frac{1}{4} b$, to see if c will now lie between 50 and 75 on the scale or 75 and 100. Once the utility for c was found, the same process can be used to find a on the scale. Using the players preference we are able to assign a unit or worth to each of the choices for the game. The scale that we created will represent the players cardinal utilities.
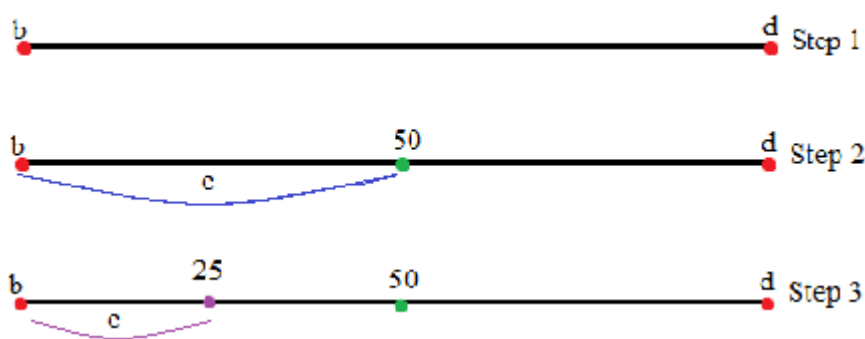

FIGURE 3. Dominance Diagram

Overall, a *cardinal utility* helps to preserve a players preference of the ordering of the choices that they are able to choose within a game. It can also be noted that cardinal utilies can be transformed by any positive linear function without changing the information they are showing. Consider the following game;

TABLE 6. Game A

Player 2		
	A	B
A	$(0\,,4)$	$(12,-2)$
B	$(14,-3)$	$(10,-1)$

Player 1 (row label, left side)

We can observe that by transforming game A by the function $g(x) = (\frac{x}{2} - 4)$, we will get a zero sum game. We will then be able to solve the new game B by zero sum game methods.

TABLE 7. Game B

Player 2		
	A	B
A	$(-4\,,4)$	$(2,-2)$
B	$(3,-3)$	$(1,-1)$

Player 1 (row label, left side)

It is important for players to determine if they are able to transform a game into a zero sum game from a non-zero sum game, as it tells us whether or not cooperation is available. Additionally, if a player knows they are strictly in opposition of their opponent they will then play one of our zero-sum game strategies.

4. Analysis of strategies

In the previous sections we have either known, assumed or calculated what an opponents probabilities will be for each of their choices. Now what if you are not sure about the assignment of probabilities and are unable to calculate them? This is actually a different type of zero sum game that is called games against nature. When playing these games a player may assume that all the actions 'nature' will play are equally likely. Well then, how will a player play optimally in this type of situation? There are four strategies that a player may choose from to play; *Laplace, Wald, Hurwicz and Savage.*

Strategy 1 This first strategy, created by Pierre-Simon de Laplace, has the player assume that all events are equally likely to occur and then apply the expected value principle, which we discussed in Section 3, Definition 10. Overall, Laplace's advice it to "Choose the row with the highest average entry. Equivalently, choose the row with the highest row sum." It is a very basic strategy, where a player is soley looking at their own payoffs and disregard their opponents.

Strategy 2 The second strategy is associated with Abraham Wald and this strategy states that players should write down the lowest entry from each row and select the row that has the largest minimum to play. This strategy is choosing the saddle point

strategy in a game if it contains a saddle point. However in a game without a saddle point this strategy is merely choosing a pure strategy, called the maximin strategy which will actually not be the mixed strategy games solution. This strategy is assuming that the worst situation will occur and attempts to create the best outcome in this type of situation, which is known as the *maximin principle.*

DEFINITION 3.15. The **maximin principle** assumes that the least favorable outcome will happen in the game. The opposite of this principle is the **maximax principle** that assumes the most favorable outcome will occur.

Strategy 3 The third strategy is named after Leonid Hurwicz. This strategy combines the first two strategies to create a new approach. In this strategy a player chooses what is called a "coefficient of optimism." This is usually denoted as α, where α, is between 0 and 1. Then for each row you are to compute the following;

$$(4.1) \qquad\qquad \alpha(row\,maximum) + (1 - \alpha)(row\,minimum)$$

After calculating this for each row, the player will chose the row which has the highest weighted average.

Strategy 4 The fourth strategy to choose from, when making decisions dealing with uncertain probabilities, comes from L. J. Savage. This strategy is designed to minimize the maximum regret that a player may feel from a decision by creating a regret matrix. "For the regret matrix, write down the largest entry in each row. Choose the row for which this largest entry is smallest." To create the regret matrix a player will take each entry and subtract it from the highest entry in its column. If the entry is the highest entry in a column then in the regret matrix the corresponding value will be 0. This strategy is attempting to make the player feel as good as possible no matter the circumstances of their game play.

Let us try and put these four strategies into practice. Begin by assuming we are playing the following game;

TABLE 8. Example Game

		Player 2			
		A	B	C	D
Player 1	A	1	1	0	2
	B	3	3	4	1
	C	0	0	2	3
	D	3	2	4	0

Applying each of the four strategies separately, we would get the following results. We would want to play the strategy that has the best or optimal outcome/payoff.

Laplace strategy: This is the easiet to compute of the four strategies. All we have to do is write down the sums of each row. Doing this, we will get row one sum = 1+1+0+2 = 4, row two sum= 3+3+4+1=11, row three sum=0+0+2+3=5 and row four sum = 3+2+4+0=9. According to laplace strategy, because row 2 has the highest sum we would want to play B.

Wald Strategy: This next strategy is also simple to compute. We begin by writing down the lowest entry of each row. In row one we have, 1, 1, 0 and 2, with zero clearly being the lowest. In row two we have 3, 3, 4, 1, with 1 begin the lowest. For row three we have 0, 0, 2 and 3, with 0 being the lowest. Lastly, for row four we have 3, 2, 4 and 0 with 0 being the smallest entry. Row one=0, row two=1, row three=0 and row four=0. The next step is we want to choose the highest of these minimum row entries. We see that row two has the highest minimum, thus the Wald strategy tells us to play B, the largest of the minimums.

Hurwicz Strategy: The first step in this strategy is to choose an α between 0 and 1, that is our coefficient of optimism. Suppose we choose our α to be 0.6. We can then plug this value into the Hurwicz equation,

$$(4.2) \qquad \alpha(row\,maximum) + (1-\alpha)(row\,minimum)$$

and we will get our equations for each row for this game, to calculate the weighted averages, according to our "coefficient of optimism" to be;

$$(4.3) \qquad 0.6(row\,maximum) + (1-0.6)(row\,minimum)$$

For row 1,

$$(4.4) \qquad (0.6)(2) + (0.4)(0) = 1.2$$

For row 2,

$$(4.5) \qquad (0.6)(4) + (0.4)(1) = 2.8 \leftarrow Highest$$

For row 3,

$$(4.6) \qquad (0.6)(3) + (0.4)(0) = 1.8$$

and lastly for row 4,

$$(4.7) \qquad (0.6)(4) + (0.4)(0) = 2.4$$

From Hurwicz strategy, the weighted averages for each row are; Row one = 1.2, Row two = 2.8, Row three = 1.8 and Row four = 2.4. We can clearly see that the second row has the highest weighted average and we would want to play option B in this game.

Savage strategy: For this strategy we need to create the least regret possible for the player. To do this we calculate the regret matrix, as seen in figure 9.

TABLE 9. Regret Matrix

		Player 2				
		A	B	C	D	Row Max
Player 1	A	2	2	4	1	4
	B	0	0	0	2	2
	C	3	3	2	0	3
	D	0	1	0	3	4

How did we find these values for the regret matrix? Well, lets begin with column one of our game matrix, we have the values, $a_{1,1} = 1$, $a_{2,1} = 3$, $a_{3,1} = 0$ and $a_{4,1} = 3$. Our elements $a_{2,1}$ and $a_{4,1}$ both have the highest values in this column. We ill take this value of 3 and subtract our a_{i1} values to get the first column of our regret matrix.

a_{21} - a_{11} = 3-1 = 2

a_{21} - a_{21} = 3-3 = 0

a_{21} - a_{31} = 3-0 = 3

a_{21} - a_{41} = 3-3 = 0

We do this procedure for each separate column to create the regret matrix. Once the the regret matrix has been created we take the maximum of each row within this matrix and choose the smallest value of these. Accordng to the Savage theory we would want to play row two because it has the smallest row maximum. So, the strategy suggests that the player plays B. They will never do more than 2 worse than the best they could have done, compared to say row one where we could do 4 worse than the best. Obviously, a player would have less regret doing only 2 worse compared to four.

Something to take into account is that when playing a game, a player may not want to constantly evaluate each strategy to determine what is the best for their game play. Additionally there are issues with each of these strategies depending on the criteria of the game. So, before choosing which of the strategies to play, a player must evaluate them according to six important axioms.

Axiom 1: *Symmetry.* Rearranging the rows or columns should not affect which strategy is recommended as best.

Axiom 2: *Strong Domination.* If every entry in row X is larger than the corresponding entry in row Y, a method should not recommend strategy Y.

Axiom 3: *Linearity.* The recommended strategy should not change if all entries in the matrix are multiplied by a positive constant, or if a constant is added to all entries.

Axiom 4: *Column duplication.* The recommended strategy should not change if we add to the matrix a new column which is a duplicate of a column already in the matrix.

Axiom 5: *Bonus Invariance.* The recommended strategy should not change if a constant is added to every entry in some column.

Axiom 6: *Row Adjunction.* Suppose a method chooses row X as the best strategy to follow in a game against nature, and then a new row Z is added to the matrix. The method should then choose either X or Z, but not some other row.

Note: Even though all four of the strategies will violate at least one of the six axioms, we can either try to find a new strategy to play in the game or for certain types of games against nature we can exclude one of the axioms. Obviously we must then evaluate the axiom against the game and if it is important in relation to the game, then we would not play one of the strategies that violate said axiom. For example the Savage strategy will violate axiom number 6.

CHAPTER 4

Two Person Non-Zero Sum Games

1. Two person Non-zero sum games

In this chapter we will be looking at two person non-zero sum games. The primary source used for research in this chapter was [1].

In zero sum games there was a straightforward set of outcomes, whatever one player won the other player lost. When dealing with non-zero sum games this is not the case and the payoffs must be written for both players in the payoff matrix to describe the game. In this section, we will also be looking at the types of assumptions that we can and cannot make when playing theses games. Players must ask themselves certain questions: *do players have no communication at all, can they talk to each other before choosing a strategy or do they agree on cooperative strategies?*

For example, assume two individuals both want to go to a movie. They decide they either want to go to a war movie, or a romantic comedy. They both also prefer to go to the movies together compared to going alone. The first individual prefers the war movie, while the second prefers the romantic comedy. However, the first individual would prefer going to the romcom, rather than go to the war movie by themself, and the second individual would prefer going to the war movie, rather than to the romcom by themself. We can put this into a game matrix.

TABLE 1. Movie Night: Non-Zero Sum

| Individual 1	Individual 2		
		War Movie	Romcom
	War movie	$(3,2)$	$(1,1)$
	Romcom	$(1,1)$	$(2,3)$

This game is clearly not a zero sum game and to know each players payoff we must write each within the game matrix. In zero-sum games, the preferences and interests of each player were opposed. In non-zero sum games, this is not always the case, as seen above. This can lead to cooperation. While looking at equilibrium in the next section, we will be focusing on non-cooperative solutions. Another thing to note, is that the dominance principle that we covered within zero-sum games also applies to non-zero sum games, as do saddle points as we are about to see.

2. Equilibrium

When playing a game, players want an outcome that will give them their largest payoff and they usually play for themselves while not taking into account their opponents decisions. Playing with a group of people that are always trying to guarantee themselves the best payoff may lead to a payoff that is not desirable, so we take into account equilibrium. For example, consider the following game;

FIGURE 1. Finding Equilibrium
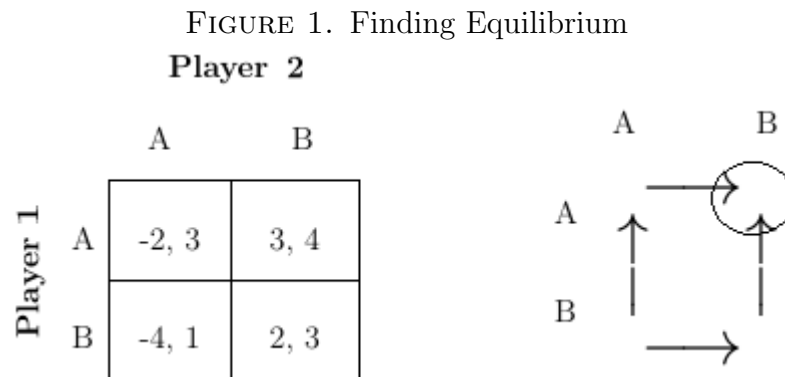
The equilibrium outcomes in a non-zero sum game actually correspond to saddle points in zero-sum games. To create an equilibrium chart we look at each players outcomes individually. For the arrows that are vertical the arrow will point to the payoff that player one prefers and for the horizontal arrows they point towards the payoff that player two prefers. **Note:** Not every non-zero sum game will have an equilibrium outcome, just like not every zero-sum game contains a saddle point. Similarily there can be more than one equilibrium outcome in a game such as in the following example;

FIGURE 2. Multiple Equilibrium
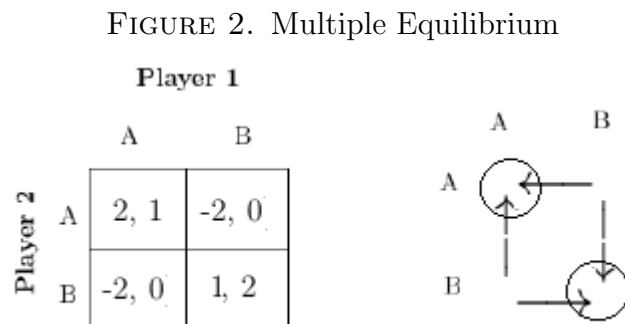
For this game we have two equilibrium outcomes, one at AA and one at BB. (A reminder when we say AA, this tells us that player one chooses option A and player two chooses option A). In this specific example, player one would prefer the outcome AA, while player two would prefer the outcome BB. This can cause problems within our game

play. Player one will most likely play option A, while player two would most likely play B. This gives us the outcome AB, giving player one a -2 outcome and player two a 0 outcome. Both players would prefer either of the two equilbrium, so communication and cooperation are needed in this game.

Another interesting point is that there may be a different payoff choice from the equilibrium outcome that is better than said equilibrium. If this is the case than our outcome is known as *non-pareto optimal*.

DEFINITION 4.1. An outcome of a game is **non-pareto-optimal** (or Pareto inferior or Pareto inefficient) if there is another outcome which would give both players higher payoffs, or would give one player the same payoff but the other player a higher payoff. An outcome is **Pareto-Optimal** if there is no such other outcome.

If a game does have an equilibrium outcome, that is not inferior to any other choices or options, then both players would not want to change their game play or strategy. If they did change their actions in the game, they would not be playing optimally. It is also important to note that if a single equilibrium is found within the game, this is called the solution of the game and is known as the *pareto principle*.

DEFINITION 4.2. **Pareto principle:** To be acceptable as a solution to a game, an outcome should be pareto optimal.

By this principle, one must always check that the equilibrium solution is not inferior to any other solution. An excellent example when the equilbrium is not the most acceptable solution is in a special case called the prisoners dilemma, which we will be looking at more in depth within chapter 6. Consider the following game;

TABLE 2. (Non)-Pareto Optimal

		Player 2	
		A	B
Player 1	A	$(3,3)$	$(5,-1)$
	B	$(-1,5)$	$(0,0)$

When creating a moving diagram, we find the equilibrium solution to be (0,0) at the point BB. It is clear however that this is inferior to the point AA, which is (3,3). This tells us that the equilibrium solution is non-pareto-optimal and both players would be better off playing A.

2.1. Nash Equilibrium. One of the more interesting and important developments of game theory is that of the *Nash Equilibrium*, founded by John Nash. He found that in every single two-person game, zero-sum, or non-zero sum, we can find at least one equilibrium (through pure or mixed strategies).

DEFINITION 4.3. **Nash equilibrium** is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of each of the other players, and no player has anything to gain by changing only his or her own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

In more basic terms, Nash Equilibrium tells us that player one makes the best decision that they can, while taking into consideration player two's decision, and player two makes their best decision, while taking into consideration player one's decision.

The equilibrium we have been considering to this point have been pure strategies. However just like in zero-sum games, we can create non-zero sum game mixed strategies. To do this a player will look at their opponents payoffs only within the game. From the following game, player one can use their opponents payoffs to create a mixed-strategy for themselve, which is similar to what we do in the zero-sum case.

FIGURE 3. Mixed Strategy in Non-zero Sum Games

A: $3x + 1(1 - x) = 2x + 1$

B: $2x + 3(1 - x) = -x + 2$

Set A=B and solve for x, $2x + 1 = -x + 2 \rightarrow x = 2/3$. So player one should play $\frac{2}{3} A \, and \, \frac{1}{3} B$. This is player one's equalizing strategy. Player two would do the same procedure, but look at player ones payoffs.

Next we will be looking at a player's *prudential strategy*. In this strategy a player looks at their own payoffs and ignores their opponents.

DEFINITION 4.4. In a non-zero-sum game, a player's optimal strategy in their game is called the player's **prudential strategy**. The value of the player's game is called the player's **security level**. If a player is playing their prudential strategy, they can assume themselves at least their security level.

Again lets look at the game in figure 3. Calculating players one mixed strategy we will find;

A: $-2x + 3(1 - x) = -5x + 3$

B: $4x + 2(1 - x) = 2x + 2$

Set A=B, we get $-5x + 3 = 2x + 2 \rightarrow x = \frac{1}{7}$. So player one would play A $\frac{1}{7}$ of the time and B $\frac{6}{7}$ of the time. This is player one's prudential strategy. Playing this strategy, they will have a security level of $\frac{16}{7}$.

DEFINITION 4.5. In non-zero-sum game, a player's **counter-prudential strategy** is their optimal response to the opponents prudential strategy.

Obviously a player would want to play an option that counter-acts their opponents prudential strategy as that would be rational play. As we should remember, the goal of game theory is to play as strategically as we can in order to get an optimal payoff.

DEFINITION 4.6. A two person game is **solvable in the strict sense** (or sss) if
i) there is at least one equilibrium outcome, which is pareto optimal, and
ii) if there is more than one pareto optimal equilibrium, all of them are equivalent and interchangeable.

Let us now look at an example where a game is solvable in the strict sense.

TABLE 3. Solvable in the Strict Sense

		Player 2		
		A	B	C
Player 1	A	$(3, 2)$	$(2, 0)$	$(1, -1)$
	B	$(1, 0)$	$(3, 1)$	$(2, -2)$
	C	$(0, 0)$	$(2, 3)$	$(1, 2)$

From our game matrix we can create a equilibrium chart and a payoff polygon.

FIGURE 4. SSS Equilibrium and Payoff Polygon
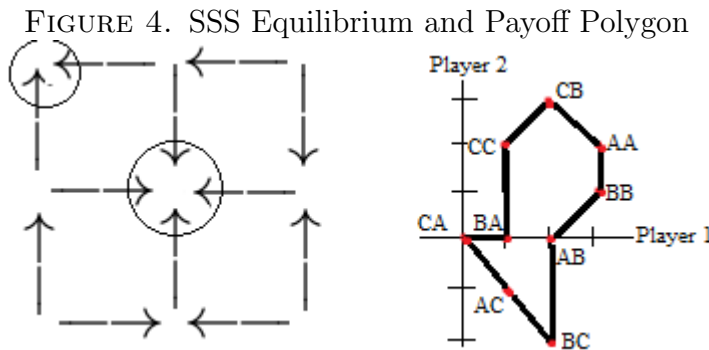
We can see that we get two equilibrium at AA and BB. It is clear however when we look at our payoff polygon on the right, when looking above or to the right of our point

BB, there are choices that are better than this point BB, in particular AA. We know that BB is not pareto optimal. On the other hand, our AA is pareto optimal because if we were to switch to say CB, player 1 would be getting a worse payoff, even though player 2 is getting a better value for this choice. Remembering our definition of pareto-optimal, this can not be the case. Hence this game has only one optimal equilibrium and is thus solvable in the strict sense. We would encourage player 1 to play A and player 2 to play A as well for an outcome of AA, (3,2).

The payoff polygon is just another tool that we are able to use to check that an option is pareto-optimal. It is also good to note that what the players are told to play from this above strategy may not give them there prudential strategy.

CHAPTER 5

N-Person Games

In the preceding chapters, we have been looking at games that involve only two players, but more real world games and applications of game theory will involve more than 2 opponents. These are called $n-person\, games$ where $n \geq 3$. These games become much more indepth, as we can introduce games that have coalitions, communication and side payments. The sources used for this chapter were [1] and [2].

1. Characteristic Function Form

When looking at games with more than two players we can form what are called coalitions. Coalitions occur when groups of players form together in a group, through cooperative behaviour, and split the payoff from that grouping with the members of the coalition. Coalitions that form may force other players to cooperate. These games are then a contest between coalitions. Players can form coalitions with two or more players, and all players can be in a coalition together.

DEFINITION 5.1. A game in **characteristic function form** is a set of N players, together with a function v, which for any subset $S \subseteq N$ gives a number $v(S)$. The number $v(S)$ called the **value of S** is to be interpreted as the amount that the players in S could win if they formed a coalition. The function v is the **characteristic function** of the game.

When we are looking at n-person games that allow cooperation, there are two questions players need to ask;

   (i) What coalitions can form within the game
   (ii) How should a coalition which forms divide it's winnings among its members.

Some other things to note are that $v(\emptyset)$ is the empty coalition. This means no players formed a coalition together and it will always equal 0. Players will not join a coalition if it does not benefit them in the game.

Any zero-sum game that is of normal form can be turned into a game that is in characteristic function form by taking $v(S)$ to be the security level of S. This means that the players are assuming that coalition S forms, and then they play optimally under the worst possible conditions. In this situation the worst possible conditions is that all other players that oppose $v(S)$ form a coalition, $N-S$ and are playing to try and minimize the payoff to S.

(1.1) $$v(S) + v(S - N) = 0$$

DEFINITION 5.2. A characteristic function form game (N, v) is called **superadditive** of $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalition S and T.

Let us look at a game that is in characteristic function form. **Note:** Characteristic function form games are not represented through payoff matrix games as normal form games have been.

Game in Characteristic function form: Assume we have three telephone companies that are putting up cellular towers around Ontario. Company one will be labelled as A, company two will be labelled as B and the third company will be labelled as C. The values of the coalitions can be calculated as; v(∅)=0, v(A)=4, v(B)=3, v(C)=2.1, v(AB)=9, v(AC)=7.5, v(BC)=9.2 and v(ABC)=13.2. We can see if this game is superadditive, remembering that superadditive means $v(S \cup T) \geq v(S) + v(T)$.

v(A)+v(B)=4+3=7 $\leq$ v(ABC),

v(A)+v(C)=4+2.1=6.1 $\leq$ v(ABC),

v(B)+v(C)=3+2.1=5.2 $\leq$ v(ABC),

v(A)+v(B)+v(C)=4+3+2.1=9.1 $\leq$ v(ABC),

v(A)+v(BC)=4+9.2=13.2 $=$ v(ABC)

v(B)+v(AC)=3+7.5=10.5 $\leq$ v(ABC),

v(C)+v(AB)=2.1+9=11.1 $\leq$ v(ABC).

Clearly we have that this game is superadditive because any coalitions that form will always be less than or equal to $v(ABC)$.

Note: if (N, v) arises from a normal form game by taking $v(S)$ to be the security level of S, then this characteristic function form game will always be superadditive.

2. Imputations, Domination, and Stable Sets

In this following section we will be looking at how to find solutions to games that are in characteristic function form. To do this we must remember those two questions from the previous section that every player must ask themselves, (i) what coalitions are able to form within the game, and (ii) how should a coalition which forms divide its winnings among the members of said coalition. In the forming of coalitions in games, it is important to keep the second question in the forefront of one's mind. Consider the example where a player A can be involved in two coalitions, $v(AD)$ or $v(ABC)$. In order for these coalitions to form player A must be willing to cooperate with the members involved in the coalitions. Lets say that $v(AD) = 10$ units and $v(ABC) = 12$ units. Although the second coalition has a larger payoff to split, player A must take into account that there are more players in $v(ABC)$. If players are dividing the payoffs equally, player A would be better off joing the coalition with player D and getting a payoff of 5 units, (10/2), rather than the second

coalition where they would get a payoff of 4 units, (12/3). Also, depending on the other coalitions that can form, some players may have more pull within their coalition. For example, perhaps D could form a separate coalition where they would get a payoff of at least 7 units. Then player D would want to split the 10 units in such a way that they get 7 units and player D would get 3 units, meaning player A would now be better off joining coalition ABC. This brings us to two important terms, *individual rationality and collective rationality.*

DEFINITION 5.3. **Individual rationality**: $x_i \geq v(i) \, \forall i$

Individual ratiionality basically tells us that if $v(i)$ is the amount player i can assure themself, without having cooperation with any one else in the game, they it would not be rational player i to accept a payoff that is less than this.

As we stated in our preliminaries, we are assuming that everyone involved in game play is a rational thinker. Being a rational thinker entails having ones best interest in mind and having optimal game play. In the example that we were just discussing, it would not be individually rational for player A to join the coalition AD and split the payoff 3 for A and 7 for D.

DEFINITION 5.4. **Collective rationality**: $\sum_{i \in N} x_i = v(N)$.

When dealing with collective rationality we are looking at games that are superaddi-tive. This tells use that $v(N)$ is the maximum amount that we can divide between the players and hence we would not have that for each member of our coalition would do better then joining the coalition, $\sum_{i \in N} x_i > v(N)$. This tells us that we would not be able to have $\sum_{i \in N} x_i < v(N)$ as each player would not have reason to join the coaltion as they can do better in the game if they were to play individually.

DEFINITION 5.5. A payoff n-tuple satisfying individual rationality and collective ra-tionality is called an **imputation** for the game (N, v).

So, now in order to solve our characteristic function form games, we must try to determine which imputations (or sets of imputations, depending on the coalitions that form) that will result in rational play in the game. This can be a very difficult task.

DEFINITION 5.6. A superadditive n-person game (N, v) is **inessential if** $\sum_{i \in N} v(i) = v(N)$. **Otherwise it is essential.**

In superadditive n-person games, if $\sum_{i \in N} v(i) = v(N)$ then any coalition can only assure each one of its members the value that they could get on their own, hence it would not make sense for them to form a coalition.

DEFINITION 5.7. Two games (N, v) and (N, w) are **strategically equivalent** if we can change v into w by;

(i) adding a constant c_i to the payoff of some player i (and hence to the value of any coalition containing player i) and

(ii) multiplying all payoffs by a positive constant.

This is basically tell us that two games (N, v) and (N, w) are strategically equivalent if we have contants $c_1, c_2, ..., c_n$ and a value x ≥ 0 such that;

$w(S) = x[v(S) + \sum_{i \in S} c_i]$ for all S $\subseteq$ N.

Lets take the following two games and see if they are equivalent;

game v: $v(A) = -3, \quad v(B) = -2, \quad v(C) = -4,$
$v(AB) = 4, \quad v(AC) = 2, \quad v(BC) = 3, \quad v(ABC) = 0,$
game w: $w(A) = w(B) = w(C) = 0,$
$w(AB) = 18, \quad w(AC) = 18, \quad w(BC) = 18, \quad w(ABC) = 18.$

Assume we are looking at gave v first. If we add a constant $c_1 = 3$ to player A, a constant $c_2 = 2$ to player B and a constant $c_3 = 4$ to player C and then have a value positive constant 2 as our x, we get that these to games are strategically equivalent and $w(S) = x[v(S) + \sum_{i \in S} c_i]$ for all S $\subseteq$ N. How? Well lets look at v(AB)=4 for example. If we add our $c_1 and c_2$ to this value and multiply it by our x=2, we get $2[v(AB) + c_1 + c_2] = 2[4 + 3 + 2] = 18 = w(AB)$

DEFINITION 5.8. An imputation $x = (x_1, x_2, ..., x_n)$ dominates an imputation $y = (y_1, y_2, ..., y_n)$ if there is some coalition S, such that;
(i) $x_1 \leq y_1$ for all i in S and
(ii) $\sum_{i \in s} x_i \leq v(s)$

If we have that an imputation x dominates an imputation y, then it is expected that a coaltion S would want to have x played rather than y. There are three properties of domination;

(1) If we are given two imputations, x and y, then it may be the case that neither of these two imputations dominates the other.
(2) It is possible to have cycles of domination. This means that we can have three imputations, x,y and z such that x dom y, y dom z and z dom x.
(3) It is possible that every imputation is dominated by some other imputation.

A great example for property number two and three would be the game *Divide the Dollar*. In this game we have three players who are dividing a dollar between themselves. Player 1 would begin by suggesting that the three players divide the dollar into three even thirds and then figuring out some other way to assign the last penny. However, player 2, wanting a better outcome, may suggest that player 1 and player 2 divide the dollar evenly and player 3 is left out of the coalition. Obviously player three will not appreciate this coalition and propose to player 1 that they split the dollar 60-40, where player 1 gets 60 cents and player 3 gets 40 cents. Player 2 will now counter offer and say player 1 and player 2 should spit the dollar, 70-30, where player 1 gets 70 cents and player 2 gets 30 cents, after saying this player 2 will realize they are getting less then the orginal plan of splitting the penny in equal thirds and will offer that player 2 and player 3 split the dollar 50-50. Obviously this argument will continue on and they will not reach an equal division of the dollar.

DEFINITION 5.9. A **stable set** for a game G is a set J of imputations such that;

    (i) J is internally stable: no imputation in J is dominated by any other imputation in J, and

    (ii) J is externally stable: every imputation not in J is dominated by some imputation in J.

A set is called **symmetric stable** if it treats all of its n players symmetrically.

3. The Core

In this next section we will be discussing briefly what the core is.

DEFINITION 5.10. The core of a game in characteristic function from is the set of all undominated imputations. i.e. the set of all imputations $(x_1, ..., x_n)$ such that for all $S \subseteq N, \sum_{i \in S} x_i \geq v(s)$.

It is important to note that when we have distributions, where a player is receiving less than they could obtain on their own, without the use of cooperation, it is not an acceptable way to play the game. They are not playing individually rational. This implies that imputations are efficient and individually rational.

An imputation will be in the core of a game if the members of any coalition S are getting, in total, at least as much as S could guarantee them. There is however the case that the core could be empty of imputations, and is then the empty set for any essential constant sum game.

The core cannot be improved on from any other subset of coalitions. So an imputation will have the property of being in the core if there is no other coalition that can improve the core. The basic property of the core is that there is cooperation between the players in a game that cannot be broken, if every coalition $S \subset N$ receives as least the value that they can get in the game, $v(S)$ (collective and individual rationality).

So how exactly do we describe the core? Well, in order to do so the game will have the following properties. (Note x_A, x_B, x_C are all imputations in the game.)

$x_A, x_B, x_C \geq 0$

$x_A + x_B \geq v(AB)$

$x_A + x_C \geq v(AC)$

$x_B + x_B C \geq v(BC)$

The last three inequalities must fold for the core. Let us look at an example of a game that has a core and how to find it.

EXAMPLE 5.11. $v(A) = v(B) = v(C) = 0, v(AB) = 1/2, v(AC) = 3/4.v(BC) = 1/2, v(ABC) = 1$ Our next step is to use the above properties of the core to find our x_A, x_B, x_C imputations of the game. Once we have found these imputations we will create is what is known as an imputation diagram which will give us our core.

Our first imputation: $x_A + x_B \geq v(AB) = 1/2$ this tells us that $x_C \leq 1/2$
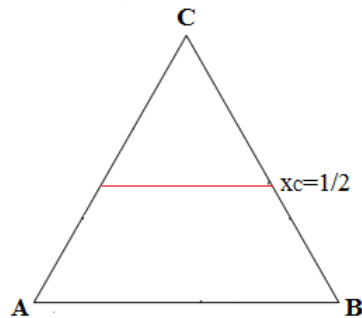

Because we had the above inequality, to create our imputation diagram we move half way from point A to point C and move half way from point B to point C and mark these points. Then we conntected those points and that is our imputation x_C.

Our second imputation: $x_B + x_C \geq v(BC) = 3/4$, this tells us that $x_A \leq 1/4$
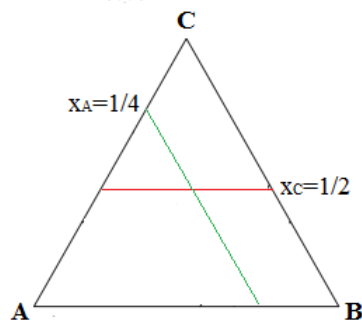
We followed the same reasoning as above to add our next imputation, $x_A = 1/4$ to our imputation diagram.

Our third imputation: $x_A + x_C \geq v(AC) = 1/2$, this tells us that $x_B \leq 1/2$. Again, we can now go half way from A to B and half way from C to B to graph our final imputation, x_B.
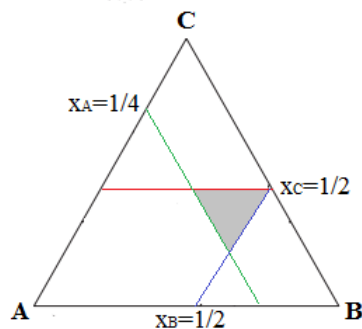
We can notice that when we graphed all of our imputations they have a non-empty intersection, which is filled with grey in the image. This is our core of the game.

CHAPTER 6

Prisoner's Dilemma

One of the most widely recognized and contemplated games is that of the prisoner's dilemma. We will be looking at two-person prisoner dilemma games, but it can also be applied to n-person games. The prisoner's dilemma shows us how we can have a unique equilibrium outcome within our game, but it does not satisfy the pareto-optimal criteria. It is also an example of where two rational players may not cooperate, although it is in their best interest. We will now give a brief description of the prisoner's dilemma game.

There are two individuals who are arrested for the exact same crime. Now, the authorities do not want the two criminals to communicate with one another so they are put into solitary confinement. Next, they are each offered the exact same options for what they want to do in this situation. They can either remain silent, or they can betray the other prisoner. From this we get the following results;

   i If prisoner 1 and 2 both betray each other they will each serve two years in jail

   ii If they both remain silent, they will serve only one year in jail

   iii If prisoner 1 betrays prisoner 2, and prisoner 2 remains silent (or the other way around) then the betrayer will be set free and the silent prisoner will serve 3 years

We can thus create the following game matrix with this information (S stands for silent, B stands for betray).

FIGURE 1. Mixed Strategy in Non-zero Sum Games

Prisoner 1		Prisoner 2	
		S	B
	S	$(-1, -1)$	$(-3, 0)$
	B	$(0, -3)$	$(-2, -2)$

When we create our equilibrium chart we see that there is an equilibrium at BB. This equilibrium is non-pareto-optimal since both of the prisoners would do better if they both did not say anything and we get the outcome SS. However since we are dealing with rational thinkers who have their own best interest in mind and they would rather spend no time in jail then one year, so each player believes they are better off to confess no matter what they believe their opponent to do (if they did not betray their partner This however leads both players to confessing, and we are back to the BB putcome of out game,

and they are both worse off had they both kept quiet. This is creating a conflict between individual rationailty, dominance and group rationality.

The general form of the prisoners dilemma is as follows;

TABLE 1. General form of Prisoners Dilemma

Player 2		
	C	D
C	(R,R)	(S,T)
D	(T,S)	(U,U)

(Player 1 labels the rows C and D.)

Labels- C: Cooperate, D: Defect, R: Reward for cooperation, S: Sucker Payoff, T: Temptation payoff, U: Uncooperative payoff

Conditions $T \leq R \leq U \leq S$ and $R \leq \frac{(S+T)}{2}$

This prisoner's dilemma is an interesting game, as it can be used to model many different situations that involve cooperation. The prisoners dilemma illustrates the conflict that occurs between individual rationality and group rationality and many people have tried to solve this game with arguments that will somehow lead players to playing the cooperative strategy even though it is not a dominating choice in the game. Additionally, the outcome of the prisoners dilemma and game play will depend on if the game is being played once or repeatedly and can be played with N players as well.

CHAPTER 7

Conclusion

In this paper, we examined two types of games that are involved within game theory, two-person and n-person games. In both of these games we studies zero-sum and non-zero sum games and some of the strategies that we may apply to games to have the best outcome in our play. We studied rational play between two players and concluded that players should play their pure or mixed strategy that will give them the value of the game. We then applied some of these strategies, such as the dominance and saddle point strategies, into basic games and were able to find the value of the games through the use of algorithms and basic steps. Next, we researched games where we were not playing games against a rational opponent, but rather games against nature. New strategies were introduced for these types of games and they included, Laplace, Hurwicz, Wald and Savage strategies.

Following this, we studied equilibrium and more importantly Nash Equilbrium, from which we can take away the ability to apply mathematics with the study of human interactions. Additionally we researched some important terms such as pareto optimal and non-pareto-optimal and in order for the equilibrium to be a solution to the game it must have been pareto-optimal.

Until the point of equilibrium, we had focused solely on two-person games. We then expanded our research into n-person games and characteristic function form. Within this part of our studying we looked at imputations, domination and stable sets. Lastly in n-person games we studied the solution concept of the core. This told us that if we can find an imputation within our core that does not have any imputation or coalition that dominates it then this is the solution to our game.

Bibliography

[1] Straffin, Phillip D. *Game Theory and Strategy* Washington: The Mathematical Associa tion of America, 1993

[2] Osborne, Martin J. *An Introduction to Game Theory* New York: Oxford University Press, 2004

[3] Fudenberg, D., Tirole, J. *Game Theory* Massachusetts: The MIT Press, 1991

[4] Binmore, Ken. *Game Theory: A very short introduction* New York: Oxford University Press, 2007