Game Theory
Solutions & Answers to Exercise Set 1

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1 Equilibrium concepts

Exercise 1 (Training and payment system, By Kim Swales)

Two players: The employee (Raquel) and the employer (Vera). Raquel has to choose whether to pursue training that costs £$1,000$ to herself or not. Vera has to decide whether to pay a fixed wage of £$10,000$ to Raquel or share the revenues of the enterprise 50:50 with Raquel. The output is positively affected by both training and revenue sharing. Indeed, with no training and a fixed wage total output is £$20,000$, while if either training or profit sharing is implemented the output rises to £$22,000$. If both training and revenue sharing are implemented the output is £$25,000$.

1. *Construct the pay-off matrix*
2. *Is there any equilibrium in dominant strategies?*
3. *Can you find the solution of the game with Iterated Elimination of Dominated Strategies?*
4. *Is there any Nash equilibrium?*

Solution.

1. This game has the following characteristics:
   - Players: Raquel and Vera
   - Strategies:
     - Raquel's: pursue training (costly to herself: £$1,000$), or not
     - Vera's: give revenue sharing (50:50), or fixed wage (£10,000)
   - payoffs: depend on total output and the way it is split. Output depends positively upon two factors: whether Raquel has training and if Vera adopts profit sharing.
     - Fixed wages + no training: output = 20,000

– Add either training or revenue share: output = 22,000
– Both training and revenue share: output = 25,000

We can then build the payoff matrix: with unit of account: £′000

		Vera	
		Revenue sharing	Fixed wage
Raquel	Training	$11.5, 12.5$	$9, 12$
	No training	$11, 11$	$10, 10$

2. No, there is no equilibrium in dominant strategies because Raquel has no dominant strategy. She prefers to train only if Vera gives revenue sharing, while prefers not to train with a fixed wage.

3. Yes. Fixed wage is a dominated strategy for Vera. Assuming that players are rational and that this information is common knowledge, Raquel knows that Vera will never choose a fixed wage. Then she will choose to train because No training is a dominated strategy after the elimination of Vera's dominated strategy.

4. Yes. Every equilibrium identified by Iterated Elimination of Dominated Strategies is a Nash equilibrium.

■

Exercise 2 (Simultaneous-move games)

Construct the reaction functions and find the Nash equilibrium in the following normal form games.

Will and John 1

		Will	
		Left	Right
	Up	$9, 20$	$90, 0$
John	Middle	$12, 14$	$40, 13$
	Down	$14, 0$	$17, -2$

Will and John 2

			Will	
		Left	Centre	Right
John	Up	$2, 8$	$0, 9$	$4, 3$
	Down	$3, 7$	$-2, 10$	$2, 15$

Will and John 3

		Will	
		Left	Right
	Up	$9, 86$	$7, 5$
John	Middle	$6, 5$	$10, 6$
	Down	$15, 75$	$4, 90$

Solution.

1. Will and John 1

   The reaction functions are the following

	John	**Will**
John's R.F.	Down	Left
	Up	Right

	Will	**John**
Will's R.F.	Left	Up
	Left	Middle
	Left	Down

   The Nash equilibrium is defined by mutually consistent best responses: therefore {down, left} is the unique Nash equilibrium of the game.

2. Will and John 2

   The reaction functions are the following

	John	**Will**
John's R.F.	Down	Left
	Up	Centre
	Up	Right

	Will	**John**
Will's R.F.	Centre	Up
	Right	Down

   The Nash equilibrium is defined by mutually consistent best responses: therefore {up, centre} is the unique Nash equilibrium of the game.

3. Will and John 3

   The reaction functions are the following

	John	**Will**
John's R.F.	Down	Left
	Middle	Right

	Will	**John**
	Left	Up
Will's R.F.	Right	Middle
	Right	Down

The Nash equilibrium is defined by mutually consistent best responses: therefore {middle, right} is the unique Nash equilibrium of the game.

■

Exercise 3 (by Kim Swales)

The table below represents the pay-offs in a one-shot, simultaneous move game with complete information. (Player As pay-offs are given first)

		Player B		
		Left	Middle	Right
	Top	$7, 17$	$21, 21$	$14, 11$
Player A	Middle	$10, 5$	$14, 4$	$4, 3$
	Bottom	$4, 4$	$7, 3$	$10, 25$

- *Find the Nash equilibria in pure strategies for the game whose pay-offs are represented in the table above.*
- *What is the likely focal equilibrium and why?*

Exercise 4 (by Kim Swales)

Companies A and B can compete on advertising or R+D. The table below represents the pay-offs measured in profits (£, million) in a one-shot simultaneous move game, with complete information. Company A's profits are shown first.

		Company B	
		Advertising	R&D
Company A	Advertising	$50, 25$	$10, 70$
	R&D	$20, 40$	$60, 35$

1. *Find the mixed strategy equilibrium.*
2. *What are the expected pay-offs for both firms?*

2 Prisoners' Dilemma games

Exercise 5 (A prisoner's dilemma game, by Kim Swales)

Firms Alpha and Beta serve the same market. They have constant average costs of £2 per unit. The firms can choose either a high price (£10) or a low price (£5) for their output. When both firms set a high price, total demand = 10,000 units which is split evenly between the two firms. When both set a low price, total demand is 18,000, which is again split evenly. If one firm sets a low price and the second a high price, the low priced firm sells 15,000 units, the high priced firm only 2,000 units.

Analyse the pricing decisions of the two firms as a non-co-operative game.

1. *In the normal from representation, construct the pay-off matrix, where the elements of each cell of the matrix are the two firms' profits.*
2. *Derive the equilibrium set of strategies.*
3. *Explain why this is an example of the prisoners' dilemma game.*

Solution.

1. The pay-off for firm $i = \alpha, \beta$ is total profits, Π_i, which equals total revenue, TR_i minus total cost, TC_i. Therefore, for the following sets of strategies:

   (a) {High price, High price}. Total demand is equal to $10,000$ and so each firm sells $5,000$ units.

$$
\begin{aligned}
TR_i &= 5,000 \times 10 = 50,000 \\
TC_i &= 5,000 \times 2 = 10,000
\end{aligned}
$$

$$\Pi_i = 50,000 - 10,000 = 40,000 \quad \forall i = \alpha, \beta$$

   (b) {Low price, Low price}. Total demand is equal to $18,000$ and so each firm sells $9,000$ units.

$$
\begin{aligned}
TR_i &= 9,000 \times 5 = 45,000 \\
TC_i &= 9,000 \times 2 = 18,000
\end{aligned}
$$

$$\Pi_i = 45,000 - 18,000 = 27,000 \quad \forall i = \alpha, \beta$$

   (c) {High price, Low price}. Firm α sells $2,000$ and while firm β sells $15,000$ units.

$$
\begin{aligned}
TR_\alpha &= 2,000 \times 10 = 20,000 \\
TC_\alpha &= 2,000 \times 2 = 4,000
\end{aligned}
$$

$$\Pi_\alpha = 20,000 - 4,000 = 16,000$$

$$
\begin{aligned}
TR_\beta &= 15,000 \times 5 = 75,000 \\
TC_\beta &= 15,000 \times 2 = 30,000
\end{aligned}
$$

$$\Pi_\beta = 75,000 - 30,000 = 45,000$$

(d) {Low price, High price}. Firm α sells $15,000$ and while firm β sells $2,000$ units.

$$\begin{aligned} TR_\alpha &= 15,000 \times 5 = 75,000 \\ TC_\alpha &= 15,000 \times 2 = 30,000 \\ \Pi_\alpha = 75,000 - 30,000 = 45,000 & \\ TR_\beta &= 2,000 \times 10 = 20,000 \\ TC_\beta &= 2,000 \times 2 = 4,000 \\ \Pi_\beta = 20,000 - 4,000 = 16,000 & \end{aligned}$$

The pay-off matrix therefore is:

		Beta	
		High price	Low price
Alpha	High price	$40,40$	$16,45$
	Low price	$45,16$	$27,27$

where Alpha's pay-off is first, and Beta's pay-off is second and both are given in £ thousands.

2. Each player has a dominant strategy, low price. The equilibrium is therefore (low price, low price) with pay-offs $\{27,27\}$.

3. It has the two crucial characteristics of the Prisoner's Dilemma game: each player has a dominant strategy, low price. But where both players play their dominated strategy (high price), the outcome $\{40,40\}$ is a Pareto improvement on the outcome where they both play their dominant strategies $\{27,27\}$. It also has the characteristic often found in Prisoners' Dilemma games that the equilibrium outcome is the one that gives the lowest joint pay-off.

■

Exercise 6 (An example of the *Tragedy of Commons*, by Kim Swales)

Show how the phenomena of overfishing can be represented as a Prisoners' Dilemma. (hint: set up the game with two players, each of which can undertake low or high fishing activity).

Solution. The case of overfishing should be set up in a manner similar to this:

		Scotland	
		High fishing activity	Low fishing activity
Spain	High fishing activity	$1,1$	$3,0$
	Low fishing activity	$0,3$	$2,2$

The sustainable fishing catch is higher when both nations undertake low fishing activity. However, there is then an incentive for both to increase fishing. In fact, high fishing is a dominant strategy for both players. We therefore, end up with the worst outcome. ■

Game Theory
Solutions & Answers to Exercise Set 2

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Exercise 1 (Cournot duopoly)

Market demand is given by

$$P(Q) = \begin{cases} 140 - Q & if\ Q < 140 \\ 0 & otherwise \end{cases}$$

There are two firms, each with unit costs = £20. Firms can choose any quantity.

1. *Define the reaction functions of the firms;*

2. *Find the Cournot equilibrium;*

3. *Compare the Cournot equilibrium to the perfectly competitive outcome and to the monopoly outcome.*

4. *One possible strategy for each firm is to produce half of the monopolist quantity. Would the resulting outcome be better for both firms (Pareto dominant)? Explain why this is not the equilibrium outcome of the Cournot game.*

Solution.

1. In a Cournot duopoly the reaction function of Firm A identifies its optimal response to any quantity produced by Firm B. In the presence of private firms, the optimal quantity is the one that maximizes Π_A, Firm A's profit, where

$$\begin{aligned} \Pi_A &= P(Q)q_A - cq_A \\ &= (140 - q_A - q_B)\, q_A - 20q_A \end{aligned}$$

The first order condition for profit maximization is:

$$\begin{aligned} \frac{\partial \Pi_A}{\partial q_A} &= 140 - 2q_A - q_B - 20 = 0 \\ \hat{q}_A(q_B) &= \frac{120 - q_B}{2} \end{aligned}$$

Since the game is symmetric (firms have identical cost functions), the reaction function of firm B is:

$$\hat{q}_B\left(q_A\right)=\frac{120-q_A}{2}$$

2. Cournot equilibrium is identified by the quantities that are mutually best responses for both firms; so, they are obtained by the solution of the following two-equation system:

$$\begin{cases} q_A = \frac{120-q_B}{2} \\ q_B = \frac{120-q_A}{2} \end{cases}$$

The equilibrium quantities are

$$q_A^\star = q_B^\star = 40$$

$Q^\star = 80$ and the equilibrium price is

$$P(Q^\star) = 140 - 80 = 60$$

and firms' profits are:

$$\Pi_B^\star = \Pi_A^\star = 60q_A^\star - 20q_A^\star = 1600$$

3. The competitive equilibrium outcome is characterized by $P(Q) = c = 20$. So total quantity should be:

$$P(Q) = 140 - Q = 20 \quad Q = 120$$

In such a case firms' profits are zero.

The quantity produced by a monopoly is obtained by the usual first order condition:

$$\begin{aligned} \Pi_M &= \left(140 - q_M\right)q_M - 20q_M \\ \frac{\partial \Pi_M}{\partial q_M} &= 140 - 2q_M - 20 = 0 \\ q_M &= 60 \end{aligned}$$

The price under monopoly is $P(q_M) = 140 - 60 = 80$ and profits are

$$\Pi_M = 80 \times 60 - 20 \times 60 = 3600$$

So, profits under monopoly are higher than the sum of firms' profits under cournot competition; i.e., $\Pi_M > \Pi_A^\star + \Pi_B^\star$.

4. If the each firm agreed to produce half of the monopolist quantity ($\frac{q_M}{2} = 40$), their profits would be $\frac{\Pi_M}{2} = 1800$, larger than the Cournot profits. So, a Pareto improvement with respect to Cournot equilibrium would be possible. However, this cannot be an equilibrium since firms' strategies are not mutually consistent best response; that is

$$\hat{q}_i\left(\frac{q_M}{2}\right) \neq \frac{q_M}{2} \quad \forall i = A, B$$

■

Exercise 2 (Cournot duopoly with asymmetric firms)

In a market characterized by the following (inverse) demand function

$$P = 40 - Q$$

two firms compete à la Cournot. Firm A has production cost described by the cost function $c_A(q_A) = 20q_A$*, while firm B's cost function is* $c_B(q_B) = q_B^2$*.*

1. *Which firms has increasing marginal cost? Which one has constant marginal cost?*
2. *Define the reaction functions of the firms.*
3. *Compute the Cournot equilibrium quantities and price.*

Solution.

1. The marginal cost of firm A is $\frac{\partial c_A(q_A)}{\partial q_A} = 20$ that is constant and independent of q_A. The marginal cost for firm B is $\frac{\partial c_B(q_B)}{\partial q_B} = 2q_B$ that is incresing in q_B.
2. In a Cournot duopoly the reaction function of Firm A identifies its optimal response to any quantity produced by Firm B. The optimal quantity is the one that maximizes Π_A, Firm A's profit, where

$$\begin{aligned}\Pi_A &= P(Q)q_A - c(q_A) \\ &= (40 - q_A - q_B)\, q_A - 20q_A\end{aligned}$$

The first order condition for profit maximization is:

$$\begin{aligned}\frac{\partial \Pi_A}{\partial q_A} &= 40 - 2q_A - q_B - 20 = 0 \\ \hat{q}_A(q_B) &= \frac{20 - q_B}{2}\end{aligned}$$

The reaction function of firm B identifies the quantity q_B that maximizes firm's B profits helding constant q_A. Firm's B profit is given by:

$$\begin{aligned}\Pi_B &= P(Q)q_B - c(q_B) \\ &= (40 - q_A - q_B)\, q_B - q_A^2\end{aligned}$$

The first order condition is:

$$\begin{aligned}\frac{\partial \Pi_B}{\partial q_B} &= 40 - 2q_B - q_A - 2q_B = 0 \\ \hat{q}_A(q_B) &= \frac{40 - q_A}{4}\end{aligned}$$

3. Cournot equilibrium is identified by the quantities that are mutually best responses for both firms; so, they are obtained by the solution of the following two-equation system:

$$\begin{cases} q_A = \frac{20-q_B}{2} \\ q_B = \frac{40-q_A}{4} \end{cases}$$

The equilibrium quantities are

$$q_A^{\star} = \frac{40}{7}; q_B^{\star} = \frac{60}{7}$$

$Q^{\star} = \frac{100}{7}$ and the equilibrium price is

$$P(Q^{\star}) = 40 - \frac{100}{7} = \frac{180}{7}$$

and firms' profits are:

$$\begin{aligned} \Pi_A^{\star} &= \frac{180}{7} q_A^{\star} - 20 q_A^{\star} = \frac{1600}{49} \\ \Pi_B^{\star} &= \frac{180}{7} q_B^{\star} - q_B^{\star 2} = \frac{7200}{49} \end{aligned}$$

■

1 Bertrand oligopoly

Exercise 3 (Competition à la Bertrand)
Market demand is given by

$$P(Q) = \begin{cases} 100 - Q & if\ Q < 100 \\ 0 & otherwise \end{cases}$$

Suppose that two firms both have average variable cost $c = \$50$. Assuming that firms compete in prices, then:

1. *Define the reaction functions of the firms;*
2. *Find the Bertrand equilibrium;*
3. *Would your answer change if there were three firms? Why?*

Solution.

1. The construction of the reaction function in a competition à la Bertrand proceeds in the following manner.

   Since firms compete in prices, we need to use the direct demand function where Q is a function of P. From the inverse demand:

$$Q(P) = \begin{cases} 100 - P & if\ P < 100 \\ 0 & otherwise \end{cases}$$

There are two firms, firm 1 and firm 2. Consider now the effect of the price choice of any firm i on its own profits for any given price chosen by firm j with $i, j = 1, 2$ and $i \neq j$.

(a) $P_i > P_j : \Pi_i = 0$

In this case, firm 1 sells no output and therefore gets zero profits.

(b) $P_i = P_j : \Pi_i = \frac{1}{2}\left[(P_j - 50)(100 - P_j)\right]$

This is where firm i shares the total profits in the industry. The total profits of the industry are here determined in the following way. Price minus average cost gives the profit per unit of output $(P_j - 50)$ and this is multiplied by the total output $(100 - P_j)$.

(c) $P_i < P_j : \Pi_i = \left[(P_i - 50)(100 - P_i)\right]$

here firm i gets the whole profits of the industry.

In order to firm the reaction function of firm i we can distinguish 3 different cases.

(a)

$$\hat{P}_i(P_j) = 50 \qquad if \quad P_j \leq 50$$

In such a case, any price $P_i < P_j$ will make negative profits, so firm i will prefer to lose the race rather than beating j on price. More precisely we can say that firm i will never set a price below $c = 50$. This is usually portrayed as the strategy that if $P_j \leq 50$, $P_i = 50$.[1]

(b)

$$\hat{P}_i(P_j) = P_j - \epsilon \quad if \quad 50 < P_j \leq 75 = P^M$$

where P^M is the monopolistic price. If $P_j > 50$, there is potential for positive profits, as long as $P_j < 100$, at which point quantity demanded falls to zero, so that profits would be zero too. The first issue is, should firm i match the price of firm j, or attempt to undercut its rival? Intuition suggests that the firm should undercut its rival rather than match P_j. This can be shown formally by comparing the profits for firm i if it matches P_j, Π_i^A, with the profits, Π_i^B, it gets if undercuts P_j by a small amount ϵ so that $P_i = P_j - \epsilon$, .

$$\begin{aligned}
\Pi_i^A &= \frac{1}{2}\left[(P_j - 50)(100 - P_j)\right] \\
\Pi_i^B &= \left[(P_j - \epsilon - 50)(100 - P_j + \epsilon)\right] \\
\Pi_i^B &= 2\Pi_i^A - \epsilon(150 - 2P_j - 2\epsilon)
\end{aligned}$$

[1] From a more formal game-theoretic point of view $P_i = 50$ is only one of the possible infinite best responses to $P_j \leq 50$. However, for the sake of finding the Bertrand equilibrium, there is no loss in considering only the strategy $P_j = 50$ as best response to $P_j \leq 50$.

This means:

As $\epsilon \to 0$, that is as the Pi_i gets closer and closer to Pi_j, $\Pi_i^B \to 2\Pi_i^A$: if firm i sets its price just below Pi_j, its profit will be roughly double the profit from matching P_j. However, the fact that firm i will do better by just undercutting firm j than it does by matching firm j does not mean that just undercutting is always the best strategy. When the firm has the whole market, its optimal price is the monopoly price. Therefore, when Pi_j is greater than the monopoly price, firm j should set the monopoly price. In this example, the monopoly price $P^M = 75$. Following this discussion:

(c)

$$\hat{P}_i(P_j) = 75 \quad if \quad P_j > P^M$$

The reaction function of any firm i is then the following:

$$\hat{P}_i(P_j) = \begin{cases} 50 & if \quad P_j \leq 50 \\ P_j - \epsilon & if \quad 50 < P_j \leq 75 \\ 75 & if \quad P_j > 75 \end{cases}$$

2. The Nash equilibrium is where the two reaction functions intersect, which is where $P_1 = P_2 = 50$. In such a case the best response of the two firms are mutually consistent. This is actually the only point in which this is so. (Please check this result by your own. Can you find another Nash equilibrium?) This is the same as the competitive solution. Price is set equal to average (=marginal) cost and zero profits are made.

3. No, because an increase in competition does not change the price set by the firms that cannot be lower than marginal cost. This is the essence of the so-called Bertrand paradox: two firms are enough to achieve the competitive outcome.

■

Exercise 4 (Bertrand game with differentiated products)

If two firms have the same constant marginal cost, they earn zero profits in the Bertrand equilibrium. This depends crucially on the feature that the goods involved are perfect substitutes. If products are differentiated instead, then the Bertrand equilibrium can lead to positive profits. The products are differentiated when consumers consider them only imperfect substitutes Whilst a consumer may be unwilling to buy the product of one producer, she will have the incentive to do this if the price of their favourite product becomes too high. To model this we allow the demand for each good to depend not only on its own price but also on the price of the other good.

Assume for example that the demand for the good produced by $Firm1$, q_1, *and the demand for the good produced by* $Firm2$, q_2, *are described by the following functions:*

$$\begin{aligned} q_1 &= 180 - p_1 - (p_1 - \bar{p}) \\ q_2 &= 180 - p_2 - (p_2 - \bar{p}) \end{aligned}$$

where $\bar{p}$ is the average price that is taken over the prices of the two firms. Each firm has average (and marginal) cost $c = 20$. Suppose the firms can only choose between the three prices $\{94, 84, 74\}$.

1. *Compute the profits of the firms under the 9 different price combinations that are possible in the model.*
2. *Using you answer to the previous point, construct the* 3x3 *payoff matrix for the normal form game where the payoffs are given by the profits of the firms*
3. *Find the (Bertrand-)Nash equilibrium of this game. What are the profits at this equilibrium?*

Solution.

The profits of both firms will depend on both prices and can be written in the following way:

$$\begin{aligned}\Pi_1(p_1, p_2) &= \left(180 - p_1 - \left(p_1 - \frac{p_1 + p_2}{2}\right)\right)(p_1 - 20) \\ \Pi_2(p_1, p_2) &= \left(180 - p_2 - \left(p_2 - \frac{p_1 + p_2}{2}\right)\right)(p_2 - 20)\end{aligned}$$

Substituting for the different firms' price choices profits are obtained and reported in the payoff matrix of the Bertrand game with discrete choices. It is easy to show that both firms

		Firm 2		
		$p_2 = 74$	$p_2 = 84$	$p_2 = 94$
	$p_1 = 74$	$5724, 5724$	$5994, 5824$	$6264, 5624$
Firm 1	$p_1 = 84$	$5824, 5994$	$6144, 6144$	$6464, 5994$
	$p_1 = 94$	$5624, 6264$	$5994, 6464$	$6364, 6364$

have a $p_i = 84$ as a dominant strategy, and so the (Bertrand-)Nash equilibrium is given by the pair of strategies $\{84, 84\}$ and by the payoffs $\{6144, 6144\}$. ■

Game Theory
Solutions & Answers to Exercise Set 3

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Exercise 1 (Sustainable cooperation in the long run)

Two farmers, Joe and Giles, graze their animals on a common land. They can choose to use the common resource lightly or heavily and the resulting strategic interaction may be described as a simultaneous-move game. The payoff matrix is the following:

		Giles	
		light	heavy
Joe	light	$40, 40$	$20, 55$
	heavy	$55, 20$	$30, 30$

1. *Find the Nash equilibrium of the game and show that it is an example of "Prisoners' Dilemma" games.*
2. *Suppose that the same game is repeated infinitely.*
   Is the {light, light} outcome a Nash equilibrium if both players play a Grim strategy and have a discount factor of 0.7?

Solution.

1. The Nash equilibrium is {heavy, heavy} with a payoff of 30 for both players. Indeed, both Joe and Giles have a dominant strategy to use the common land heavily. In addition, the Nash equilibrium is Pareto dominated by the outcome {40,40} arising when both players choose the dominated strategy *light*. The two features are characteristic of Prisoners' Dilemma games. The game describes the so-called "tragedy of commons" in which the users of a common resource have an incentive to over-use it.
2. When the game is repeated infinitely the trigger strategy is the following:
   - Start playing cooperatively (*light*)
   - Play cooperatively as long as the other player chooses *light*

- Whenever the other player chooses *heavy*, switch to *heavy* and play it forever.

The couple of strategies {light, light} can be sustained as Nash equilibrium when both players play a Grim trigger strategy only if they care enough about future payoffs; i.e., if the discount factor δ is high enough.

In fact the cooperative strategy will be preferred against a grim trigger strategy if the net present value of cooperation is larger than the net present value of deviation.

If Joe always plays *light* against Giles who follows a grim trigger strategy, Joe's payoff will be 40 forever. The present value of cooperation is then the present value of this infinite sequence; i.e.,

$$\Pi_{coop} = 40 + \delta 40 + \delta^2 40 + \delta^3 40 + ... = \frac{40}{1-\delta}$$

By deviating from cooperation and playing *heavy*, Joe will get 55 in the first period but will face the reaction of his opponent. From the following period Gilles will always play *heavy*. So, after his own deviation, the best option for Joe is to stick with *heavy* forever. The net present value of this deviating strategy is then:

$$\begin{aligned} \Pi_{dev} &= 55 + \delta 30 + \delta^2 30 + ... = 55 + \delta \left(30 + \delta 30 + \delta^2 30 + ...\right) \\ &= 55 + \delta \frac{30}{1-\delta} \end{aligned}$$

Playing cooperatively against a player adopting a grim trigger strategy is sustainable as a Nash equilibrium from Joe's viewpoint only if $\Pi_{coop} > \Pi_{dev}$;

$$\frac{40}{1-\delta} > 55 + \delta \frac{30}{1-\delta}$$

Multiplying both sides by $(1-\delta)$, the inequality becomes

$$\begin{aligned} 40 &> 55\,(1-\delta) + \delta 30 \\ 40 &> 55 - 25\delta \end{aligned}$$

By adding $(25\delta - 40)$ to both sides, the inequality becomes

$$\begin{aligned} 25\delta &> 15 \\ \delta &> \delta^\star = \frac{15}{25} = 0.6 \end{aligned} \tag{1}$$

Since the game is symmetric, condition (1) holds for both Joe and Giles. Given that their discount factor is $0.7 > \delta^\star$, the cooperative outcome {light, light} is sustainable as a Nash equilibrium of the infinitely repeated game when both players play the grim trigger strategy.

This threat of punishment, therefore, may represent a solution to the tragedy of commons.

■

Exercise 2

Distinguish simultaneous-move games and dynamic games in terms of information. Explain why in dynamic games Nash equilibria may not be subgame perfect. Using examples, show how non-credible threats are ruled out using backward induction.

Solution.

Students should discuss the difference between complete and perfect information in order to answer to the first question properly. Nash equilibria are not always subgame perfect because the concept on Nash equilibrium does not take into account the temporal dimension of the decision process in a dynamic game. So, menaces that at the beginning of the game might be used to threaten the opponent, would never be used when the time comes. Subgame perfection can therefore be considered a refinement applied to the Nash equilibrium concept that selects equilibrium strategies that are credible in every subgame of the whole game. The entry game is a good example to show the way in which backward induction works. ■

1 Finitely repeated games

Exercise 3 (Entry Deterrence)

A market is characterized by a demand function $Q = 1 - P$ and by a single firm with constant marginal cost c. The monopolist is facing potential entry from a new firm having the same marginal cost but an additional fixed cost of entry $F = 0.1$. If the incumbent accepts the entry passively, then Cournot competition is played. However, the monopolist can threaten to produce the competitive output (i.e., the quantity such that $P = c$) so that the new entrant will make losses if it enters the market. If the new firm does not enter, the incumbent behaves as a monopolist.

1. *Assume that $c = 0$ and compute the payoffs of both firms (i.e., profits) in the cases of Monopoly, Cournot duopoly, and aggressive behaviour.*

2. *Using the extensive form (game tree) representation, describe this entry game as a two-stage game where in the first stage the new entrant decides whether to enter or not, and in the second stage the incumbent firm decides whether to be passive or aggressive in case of entry, while it does nothing in case the new firm does not enter the market.*

3. *Is the threat of aggressive behaviour by the monopolist credible? Answer the question by finding the subgame perfect equilibrium of the game.*

4. *Describe the dynamic game using the normal form (pay-off matrix) representation and find the Nash equilibria of the game. Does the game have any Nash equilibria that are not subgame perfect?*

Assume that incumbent is monopolist in 10 different national markets and it faces the threat of entry in all the markets sequentially (i.e., in stage 1 of the game the monopolist plays the game with a potential entrant in the Belgian market; in stage two the same game is played with another potential entrant for the Dutch market, and so on up to 10 rounds).

5. *Find the SPNE of the entire dynamic game by backward induction.*

6. *Is there room for building a reputation? Explain your answer.*

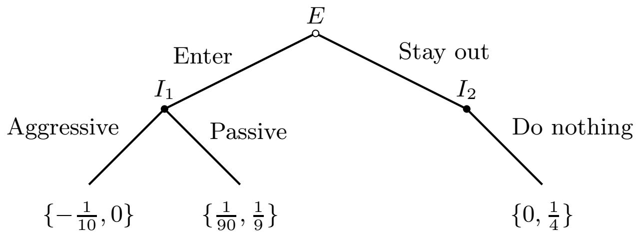
Solution.

1. There are three possible outcomes of the game depending on the choices of the new entrant (entry vs no entry) and of the incumbent firn (aggressive vs passive behaviour). In case of no entry, the incumbent will set the monopolist level of output. Let i be the incumbent label and e the new firm label. The monopolist quantity is $q_i^m = \frac{1}{2}$ and the payoffs are $\Pi_i^m = \frac{1}{4}$ and $\Pi_e^m = 0$ (you should always show how to get there).

   If the new firm enters the market and the incumbent plays passively, Cournot competition arises with $q_i^c = q_e^c = \frac{1}{3}$ and $\Pi_i^c = \frac{1}{9}$ while $\Pi_e^c = \frac{1}{9} - \frac{1}{10} = \frac{1}{90}$.

   The two firms produce the same quantity but earn different profits given the fixed cost the new firm has to pay in order to enter the market.

   Finally, if the new firm enters the market and the incumbent behaves aggressively (i.e., it produces the quantity suc that $p = c$), then the new entrant pays the fixed cost but it is not able to produce a positive quantity since firm i serves all the market at $p = 0$. So, $q_i^a = 1$, $q_e^a = 0$, $\Pi_i^a = 0$ and $\Pi_e^a = -\frac{1}{10}$.



2. The figure above depicts the extensive form of the two-stage dynamic game of entry deterrence we are analyzing.

3. To assess the credibility of of the aggressive behaviour we rely on the notion of Subgame Perfect Nash Equilibrium. This is the Nash equilibrium of the dynamic game in which the equilibrium strategies of the players are optimal in each subgame since in each subgame the players are maximizing their payoff. In order to find the SPNE we use the backward induction technique by analyzing the last subgame, solving for the best choices, and going backward.

   Starting from the node I_1 (that is, we are assuming that in the first stage the new firm enter the market) the incumbent firms strictly refers to be passive rather than aggressive;

in fact the payoff for the incumbent is $\frac{1}{9}$ in the first case and 0 in the latter. At node I_2 there is no choice and firm i simply behaves as monopolist given the no-entry decision of the new firm.

Given these solutions to the last stage of the game, in the first stage the new entrant has to choose whether to enter the market. If it does not enter, its payoff is zero while be entering it gets its Cournot profits. The new entrant will choose to enter because it anticipates that incumbent will be passive because the threat of aggressive behaviour is not credible (i.e., it is not in the interest of the incumbent when it has to make the choice).

4. In order to use the normal form to find the Nash equilibria of the dynamic game it is necessary to specify the strategies of the two firms. Recall that in a dynamic game a strategy for a player is a set of instructions identifying what decision to take at each relevant node. So, player e has two strategies: enter and not enter. Firm i has two strategies: {aggressive, do nothing} and {passive, do nothing}.

   The normal for representation of the dynamic game is:

		Incumbent	
		aggressive, do nothing	passive, do nothing
Entrant	enter	$-\frac{1}{10}, 0$	$\underline{\frac{1}{90}}, \underline{\frac{1}{9}}$
	stay out	$\underline{0}, \underline{\frac{1}{4}}$	$0, \underline{\frac{1}{4}}$

   It is easy to see that there are two Nash equilibria (stay out, {aggressive, do nothing}) and (enter, {passive, do nothing}). The first one is not credible for the reason stated in the previous point.

5. The unique SPNE of this repeated game consists of the repetition in each of the 10 stages of the game of the SPNE of the one-shot sequential game described in the solution to the previous point. This is because the game is finite and in the last stage the incumbent has no reason not to accomodate entry because there will be no further stage and the payoff of behaving passively is larger than the payoff of behaving aggressively. At stage 9 the only reason to behave aggressively is to threaten the new firm in the stage 10. But knowing that entry will be accommodated anyway this is not a useful (and so credible) strategy for the incumbent. Then, the incumbent will be passive in stage 9, too. The same reasoning may be applied to all the previous stages in order to get the SPNE of the full repeated game.

6. No, it is not possible for the incumbent to build an aggressive reputation because fighting is not a credible option after entry at any strage of the repeated game. This is because the game is finite and there is no reason to build a reputation of tough player in a finite game. Only if there is no certain endpoint (i.e., the game has a possibly infinite number

of stages) it might be possible (and useful) for the incumbent to build a reputation of tough player.

■

Exercise 4 (Centipede Game) *Trinny and Susannah are playing the "Centipede" game. At each node in the game the player can either move down, which means the game stops, or move right, which means that the game is passed to the other player. The two cycle game is depicted in Figure 1. Trinny's pay-off is given first.*

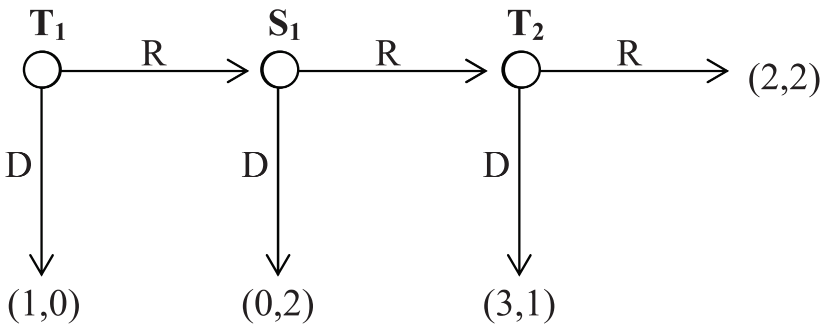


Figure 1:

1. *Solve the game by backward induction*

2. *Show that the outcome is inefficient.*

3. *If the game was extended to 100 cycles, with the pay-offs increasing in a similar manner, as represented in Figure 2, how would this affect the outcome of the game?*

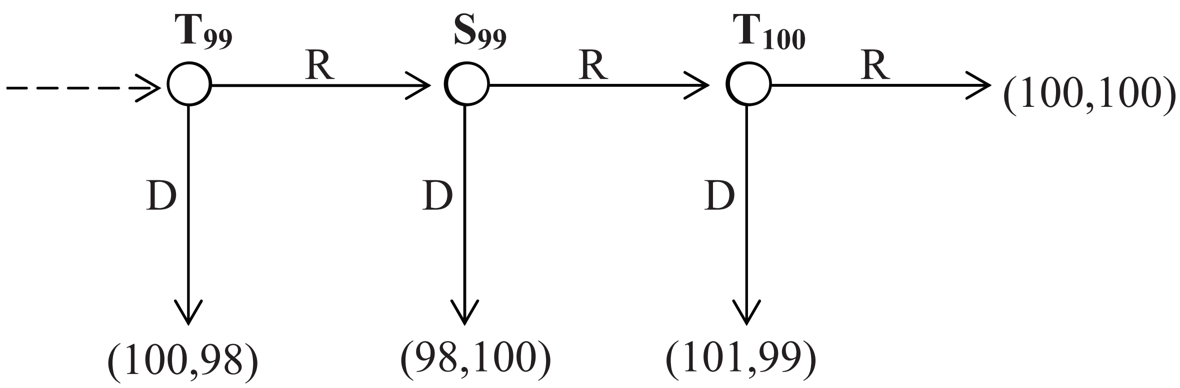


Figure 2:

Solution.

1. We solve by going to the end and working backwards.

- At T2 Trinny plays down (D) as her pay-off of 3 will be greater than the corresponding pay-off of 2 when she plays right (R). We can therefore replace the sub-game at T2 with the pay-off (3,1).
- Next consider the Susannah's decision at node S1. She will choose D, as her resulting pay-off of 2 is greater then the pay-off of 1 if she passes the game to Trinny. Therefore we can replace the sub-game starting at S1 by the pay-off (0,2).
- Finally consider Trinny's decision at T1. She gets a pay-off of 1 if she plays down and a pay-off of 0 if she passes the game onto Susannah. She therefore plays down.

The solution to the game is therefore that at T1 Trinny plays down (D) and the game ends with the pay-off (1,0)

2. The outcome is clearly inefficient in that there are alternative outcomes that Pareto dominate the equilibrium outcome (that is, there are outcomes where both players receive a higher pay-off). An example would be if both players always played right, R, which gives a pay-off of (2,2).

3. If the game was extended to 100 cycles, the outcome of the game should be exactly as previously: it finishes with the very first move with Trinny playing down, and a pay-off of (1,0).

   When it is played experimentally there is usually some co-operation though not right to the final round.

   How should Susannah interpret a move by Trinny at T1 to the right? This is an interesting question: how should one player interpret moves which are off the equilibrium path in a dynamic game? There are a number of possibilities.

   - One is that Trinny doesn't understand the game.
   - A second is that Trinny understands the game but hasn't fully worked out the implications (maybe there is some bounded rationality).
   - Third, it could be that Susannah is mistaken about Trinny's pay-offs, and a move which appears irrational is in fact rational. For example, the pay-offs given at present might simply be the money pay-offs. But Trinny might get positive utility from Susannah getting a high money pay-off and therefore is prepared to pass the game to Susannah, even though that might mean that her own money pay-off is lower. This would be a form of altruistic behaviour.
   - Finally it might be that Trinny is offering co-operation even though this co-operative strategy appears to be unsustainable.

   How Susannah should react will depend on which of these she thinks is correct.

■

Game Theory
Questions & Answers to Exercise Set 4

Giuseppe De Feo

May 10, 2011

Exercise 1 *Define the concepts of expected value, expected utility and explain the relationship between this two concepts.*

Solution. Please refer to the lecture notes for the definition of expected value and expected utility. The expected value and the expected utility are different concepts since the latter is a measure of utility and the former is just a sum of money. But if we compare the utility of the expected value (UEV) of a lottery with the expected utility (EU) of the same lottery we can infer the attitude toward risk of the individual whose utility we are considering. More precisely, if the $EU > UEV$ then the individual likes risk and prefers uncertain to certain outcomes; if $EU < UEV$ the individual is risk averse; and if $EU = UEV$ (s)he is risk neutral. In fact EU denotes the utility gained by playing a certain lottery while UEV denotes the utility of a "degenerate" lottery that pays with certainty the expected value of the same lottery. So, the only difference between the two concept is just risk; if the individual is risk adverse then the utility that (s)he gains from playing the lottery must be lower than the utility of getting the expected value with certainty. If (s)he is risk neutral there should be no difference between the two measures, while the lottery gives a higher utility to a risk loving individual. ■

Exercise 2 *Mark wants to maximise his expected utility. His preferences are represented by the utility function* $U(y) = y^{\frac{1}{2}}$ *where* y *is a monetary payoff. Mark is offered the following bet on the toss of a coin by Amanda;*

- *If the coin comes up tails Amanda pays Mark* $£1,000$
- *If the coin comes up heads Mark pays Amanda* $£1,000$

Mark's initial capital is $£10,000$ *which he retains in its entirety if he does not take the bet.*

1. *What is Mark's expected utility if he accepts the bet?*
2. *Will he accept the bet? Explain your answer.*
3. *Is Mark risk averse, risk neutral or risk loving? Explain your answer.*

Amanda offers Mark an alternative bet whereby if the coin comes up tails Amanda gives him £$10,000$ *but if the coin comes up heads Mark gives Amanda his entire* £$10,000$.

4. *Show that Mark does not accept this bet.*

Amanda offers Mark yet another alternative bet whereby Mark still loses his entire £$10,000$ *if the coin comes up heads, but if the coin comes up tails Amanda pays him*£$50,000$.

5. *Does Mark accept this new alternative? Explain your answer.*

6. *Given that Mark loses his entire*£$10,000$ *if the coin comes up heads, what is the smallest amount that Amanda has to pay Mark in the event of tails in order to persuade him to take the bet?*

Solution.

1. The expected utility of the bet for Mark is:

$$\begin{aligned}
EU(Bet) &= 0.5U(9000) + 0.5U(11000) \\
&= 0.5\sqrt{9000} + 0.5sqrt11000 \\
&= 0.5 \times 94.87 + 0.5 \times 104.88 \\
&= 99.87
\end{aligned}$$

2. Assuming that mark is an expected utility maximizer, he will not accept the bet since the expected utility from the bet is lower that the expected utility of retaining his wealth. In fact: $EU(NoBet) = \sqrt{10000} = 100 > 99.87 = EU(Bet)$.

3. it would have been not necessary to compute the expected utilities to answer the previous point. Since the utility function describing Mark's preferences is concave (i.e., the marginal utility is decreasing), Mark is risk averse and therefore he would never accept a fair gamble such as the one proposed by Amanda. It is a fair gamble since the expected value of the gamble is zero.

4. By the previous point, the second bet put forward by Amanda is again a fair gamble. So, Mark will never accept it since he is risk averse. The difference between the expected utility of the bet and the utility of not accepting the bet is even larger:

$$\begin{aligned}
EU(Bet_2) &= 0.5U(0) + 0.5U(20000) \\
&= 0.5\sqrt{20000} = \frac{\sqrt{2}}{2}100 \approx 70.71
\end{aligned}$$

5. The third bet is no longer a fair gamble but is a gamble that has a positive expected value for Mark

$$EV(Bet_3 = -10000 \times 0.5 + 50000 \times 0.5 = +20000 > 0.$$

So, in order to answer the question we have to compute the Mark's expected utility if he accepts the bet. That is:

$$\begin{aligned} EU(Bet_3) &= 0.5U(0) + 0.5U(50000) \\ &= 0.5\sqrt{50000} = \frac{\sqrt{5}}{2}100 \approx 111.80 > 100 = EU(NoBet) \end{aligned}$$

The latter bet is therefore accepted by Mark since his expected utility is larger than the utility of not betting.

6. We have to find the amount of money that Mark should get in case he wins the bet that makes him indifferent between rejecting and accepting the bet, knowing that in case of loss he would lose £10000. We have to find the x such that:

$$\begin{aligned} EU(Bet_4) \equiv 0.5U(0) + 0.5U(10000 + x) &= U(10000) \\ 0.5\sqrt{10000 + x} &= 100 \\ \sqrt{10000 + x} &= 200 \\ 10000 + x &= 40000 \\ x &= 30000. \end{aligned}$$

Therefore x can be interpreted as the amount of money such that the certainty equivalent of the bet is £10000.

■

Exercise 3 *The local government wants to hire a manager to undertake a public project. If the project fails, it will lose £20,000. If it succeeds, the project will earn £100,000. The manager can choose to "work" or to "shirk". If she shirks, the project will fail for sure. If she works, the project will succeed half of the times but will still fail half of the times. Measured in monetary terms, the manager's utility is £10,000 lower if she works than if she shirks. In addition, the manager could earn £10,000 in another job (where she would shirk). Assume further that both parties are risk neutral.*

1. *Describe the dynamic game using the extensive form representation.*
2. *Which is the compensation scheme that maximizes government's utility? Which is the expected wage of the manager in any equilibrium of the game?*
3. *Do you think the solution will change with a risk-averse manager?*
4. *Assume that the government prefers to sell the project to the manager. What should be the fee paid to the government?*

Exercise 4 *Discuss the trade-off between insurance and incentives in the presence of moral hazard when the agent is risk-averse. Do you think it is true that a risk-neutral principal should pay more to hire a risk-averse agent than a risk neutral agent? Explain your answer.*

Advanced Microeconomics
Question & Answers to Exercise Set 5

Giuseppe De Feo

May 10, 2011

Exercise 1 (MIT Sloan School of Management) *Consider the market for health insurance. Suppose that the market is comprised of 4 groups of people of differing risk categories. There are a large and equal number of people in each group, but insurers cannot tell which group a person belongs to (i.e. this is a situation of asymmetric information). Each group faces a risk of requiring medical treatment of value $10,000.*

Suppose that the willingness to pay of people in each group is as follows:

Risk	0.2	0.4	0.6	0.8
willingness to pay	2,500	5,200	6,800	8,500
Actuarially fair premium				
Risk Premium				

a. *Complete the table of actuarially fair insurance premiums that could be charged to each group separately by an insurance company large enough to diversify the risks. How do these compare to the willingness to pay?*

b. *Suppose now that the risk category is private information. What is the average riskiness of a person seeking insurance? What premium would an insurance company have to charge to break even?*

c. *Will all the agents participate at this price? If not what would be the composition of risks facing the insurer? Would the premium found above be sufficient to cover the risks taken by the insurer?*

d. *Continue with this logic. What will be the price of insurance in the equilibrium and which groups will participate?*

e. *Is this an efficient outcome?*

Solution.

Risk	0.2	0.4	0.6	0.8
willingness to pay	2,500	5,200	6,800	8,500
Actuarially fair premium	**2,000**	**4,000**	**6,000**	**8,000**
Risk Premium	**500**	**1,200**	**800**	**500**

a. The actuarially fair premium is equal to the expected loss $= 10,000\times$ risk. It is always lower than the willingness to pay since people exhibit risk aversion.

b. If all agents participate the chance of a loss in the population as a whole is 50%. The insurance company will have to charge at least 5,000 to avoid a loss, and in a competitive market this will be the price of insurance.

c. At this price the lowest risk 20% category would not participate as the premium is too high. If only the 40%, 60%, 80% risks are in the market then the chance of a loss is 60%. The insurer would need to charge 6,000.

d. Continuing as above, when the price is 6,000 only the 60% and 80% risks remain with overall risk 70%. The insurer would raise the premium to 7,000 driving out the 60% group and thus the equilibrium price will be 8,000 and only the highest risk agents will insure.

e. This is not an efficient outcome if insurance could be provided on actuarially fair terms to each group individually this would generate a surplus since they are risk averse. Instead, the adverse selection problem leads to the collapse of the insurance market leaving many agents uninsured.

   Can we do any better? If there is no way to obtain the private information then the policy maker faces the same constraints as the market and the insurance market creates only 500 of surplus per high risk person.

   Making participation mandatory can create more total surplus. In this case insurers would know that the aggregate risk is 50% and the competitive price of insurance would be 5,000. At this price the surplus of the agents is:

Risk	0.2	0.4	0.6	0.8
willingness to pay	2,500	5,200	6,800	8,500
Actuarially fair premium	**2,000**	**4,000**	**6,000**	**8,000**
Pooled premium	**5,000**	**5,000**	**5,000**	**5,000**
Surplus	**-2,500**	**200**	**1,800**	**3,500**

Note, however, that even though the total surplus has increased this is NOT A PARETO IMPROVEMENT the low risk agents are worse off with mandatory participation and there is no way to compensate them since that would require that we can somehow identify them, which by the assumption of private information is impossible.

■

Exercise 2 (Hindriks and Myles, 2006, 9.3) *Are the following statements true or false?*

a. *An insurance company must be concerned about the possibility that someone will buy fire insurance on a building and then set fire to it. This is an example of moral hazard.*

b. *A life insurance company must be concerned about the possibility that the people who buy life insurance may tend to be less healthy than those who do not. This is an example of adverse selection.*

c. *In a market where there is separating equilibrium, different types of agents make different choices of actions.*

d. *Moral hazard refers to the effect of an insurance policy on the incentives of individuals to exercise care.*

e. *Adverse selection refers to how the magnitude of the insurance premium affects the types of individuals that buy insurance.*

Solution.

a. True, moral hazard arises when the insured party can take actions not observable by the insurer (hidden action) that affect the probability of an accident or loss. In this case the hidden action is the fact of setting fire to a building once you know that you are fully insured. Thus, moral hazard refers to the effect of an insurance policy on the incentives for individuals to exercise care. The insurance company should worry that in providing better coverage it reduces the incentive to exercise care and increases the overall likelihood of an accident or loss.

b. True, adverse selection arises when the insured party has hidden information about its risk of an accident or loss. In this case, those who buy life insurance are better informed about their health than the insurance company. They have greater preference for life insurance if they know they are in bad health. So the insurance company should worry about how the insurance premium affects the types of individuals that buy insurance. In particular, increasing the insurance premium drives out the low-risk individuals and thus increases the overall riskiness of the set of individuals who buy insurance.

c. True, different types of agents separate from each other by acting differently. In contrast, under a pooling equilibrium it is impossible to distinguish different types of agents because they act the same. For instance, in the insurance market, low-risk individuals can separate from high-risk individuals by buying less insurance.

d. True, see part a.

e. True, see part b.

■

Exercise 3 (Watson, 2008, p.319) *Consider the following static game of incomplete information. Nature selects the type* (c) *of player 1, where* $c = 2$ *with probability* $2/3$ *and* $c = 0$ *with probability* $1/3$. *Player 1 observes c (he knows his own type), but player 2 does not observe c. Then players make symultaneous and independent choices and receive payoffs as described by the following matrix.*

		Player 2	
		X	Y
Player 1	A	$0, 1$	$1, 0$
	B	$1, 0$	$c, 1$

a. *Draw the Bayesian normal form matrix of the game*

b. *Compute the Bayesian Nash equilibrium*

Exercise 4 (Watson, 2008, p.346) *Consider an extensive-form game in which player 1 is one of two types: A and B. Suppose that types A and B have* exactly *the same preferences; the difference between these types has something to do with the payoff of another player. Is it possible for such a game to have a separating PBE, where A and B behave differently?*