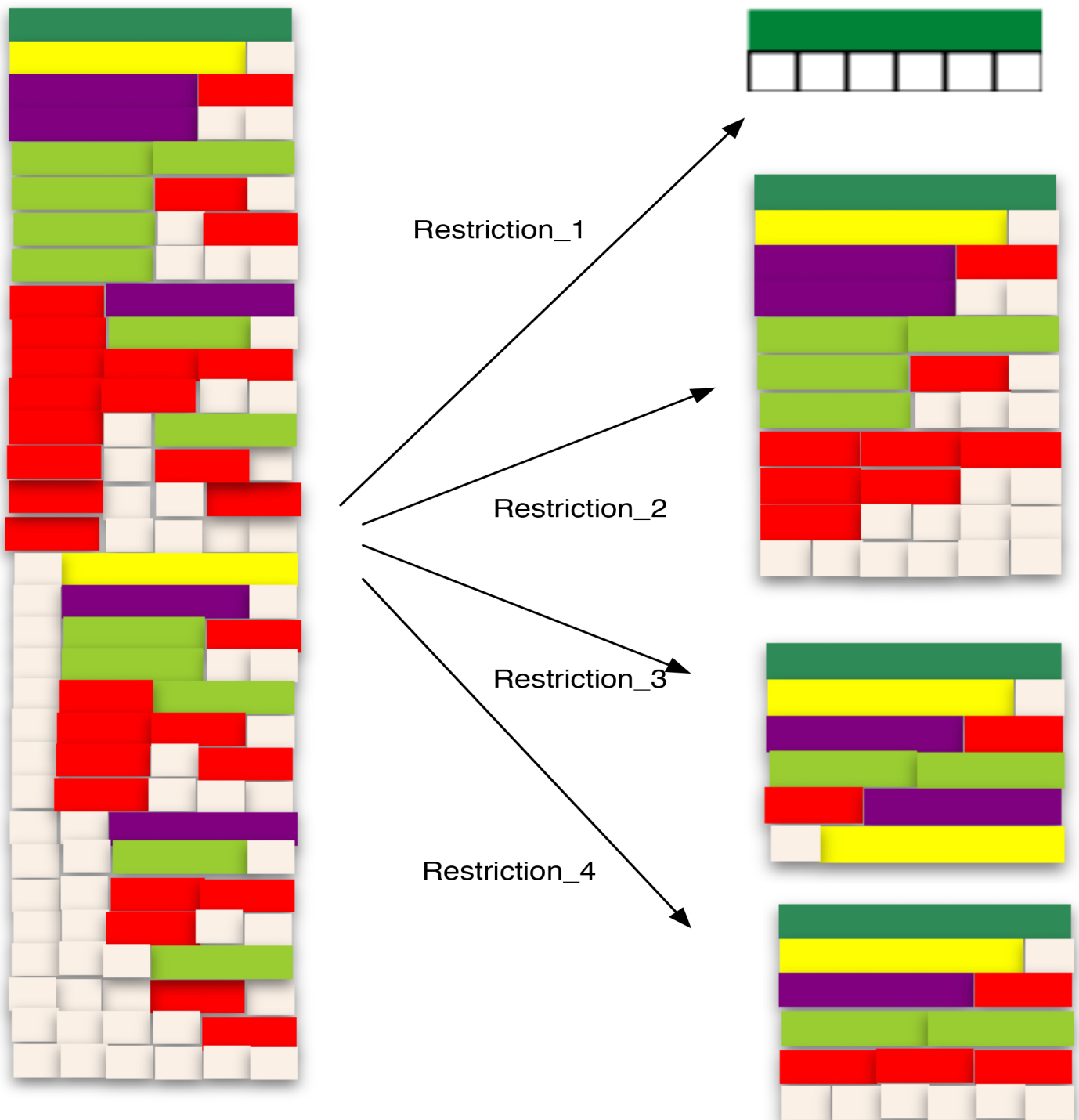


Working with rods and why

an ATM open resource



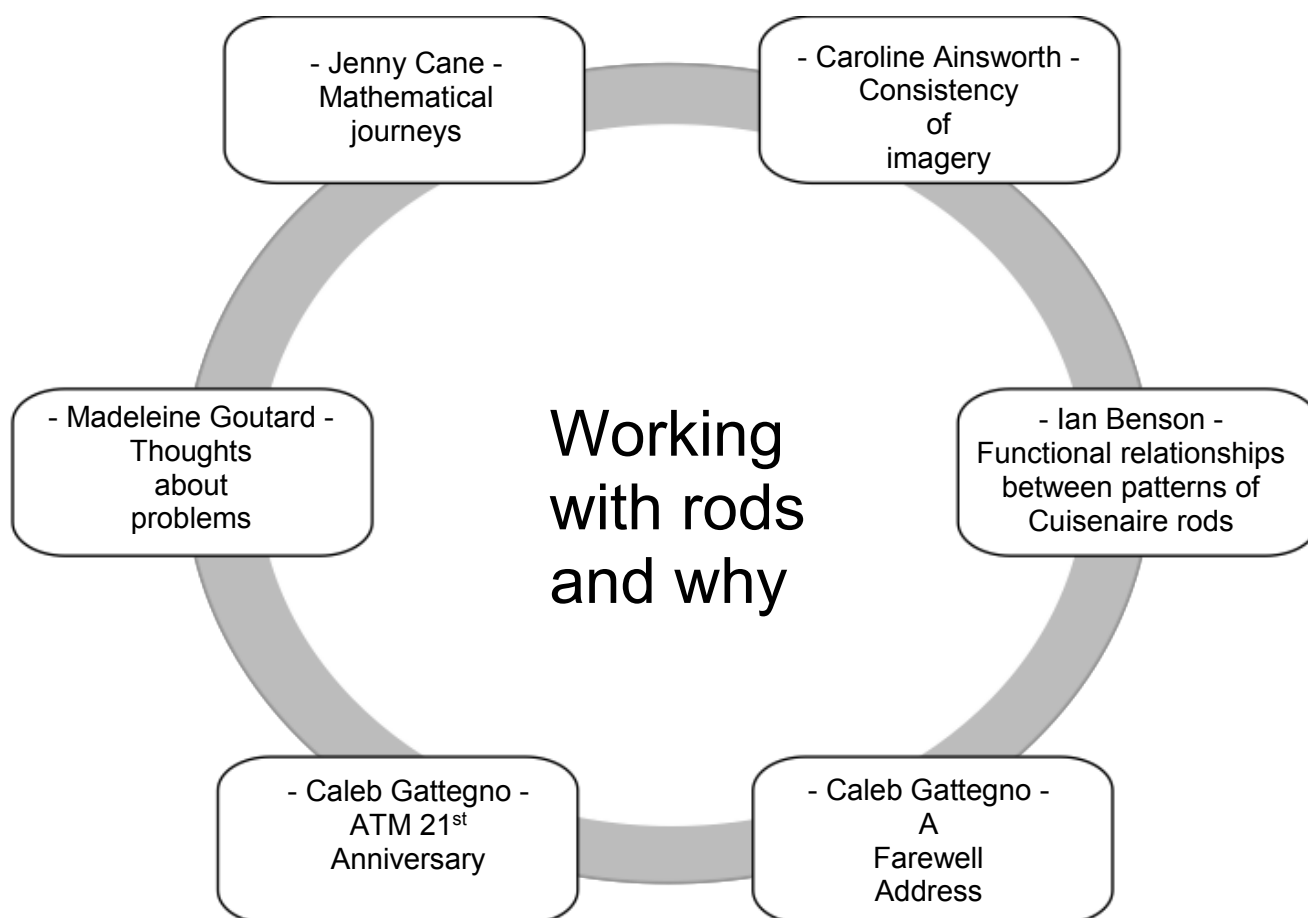
Gattegno Cuisenaire[®] Reader

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Gattegno Cuisenaire Reader

A resource for all interested in better teaching of mathematics



"I wrote in the late fifties, early sixties, a series of textbooks for children. And, I suppose some teachers see them, but their children never saw them. They are for children. They contain what I know about children's powers. And, I try to help them. By using their powers, but I finish the first 6 years of the course in two. Where is the teacher who would accept that doing the work of 6 years in 2. But if you are at all touched by truth, find out. Find out if it is possible that it only takes two years to be master of the arithmetic."

- Caleb Gattegno, A Farewell Address-

The articles in this booklet bring together inspirational writings on the theory of reforming mathematics education together with articles by primary teachers who exemplify the Cuisenaire-Gattegno approach in practice.

Sixty years after Cuisenaire, Gattegno and Goutard embarked on this journey, new demands on mathematics teachers and new developments in conceptual mathematics and computer languages make reform both more urgent and more tractable.

The 2014 national curriculum is one of the first in the world to mandate that all four arithmetic operations and fractions as operators be studied from Year 1. Gattegno's textbooks propose an algebra of colour coded Cuisenaire rods to do just this.

In his Science of Education Gattegno proposed " subordinating teaching to learning" to harness mental powers present in every child. He set out a new role for the teacher - to create game-like situations that the learner can experience mathematically, while supplying the labels and notations they cannot invent themselves. He proposed a way of talking about their learning in terms of awarenesses - which could be subconscious, conscious, named or categorized.

Madelaine Goutard illustrates this approach to lesson design in her article.

The articles from Jenny Cane and Caroline Ainsworth show how inspiring teaching can emerge from a close reading of Gattegno and Goutard's books. They contain open questions to guide learners exploring whole number and fractional relationships through permutations and combinations of Cuisenaire rods. Ian Benson discusses why these books differ from traditional textbooks and how this approach can be extended to further enrich school mathematics.

The recent revitalisation of the Cuisenaire-Gattegno tradition was initiated through a network of schools by Charles Clarke, Secretary of State for Education in 2004. We are grateful to Stockland and Bursted Wood primary schools, founding members of the network, for permission to reproduce children's writing. Cuisenaire® Rods is a trademark registered to Educational Solutions (UK) Ltd. It is used with permission. The cover illustration is copyright Sociality Mathematics, 2017

We invite the reader to join the ATM in writing a new chapter in this exciting journey.

Ian Benson

Jim Thorpe

October 2017

Mathematical journeys: Our journey in colour with Cuisenaire rods

This article draws on Jenny Cane's plenary presentation to the ATM conference.

The title of the 2017 ATM conference, *Mathematical journeys* immediately made me think about my own mathematical journey as a primary school teacher. I had been navigating my way through the Gattegno textbooks *Numbers in colour* whilst using Cuisenaire rods with key stage 1 children. For the teachers at Bursted Wood Primary, the journey so far had been one of excitement, wonder and discovery. Putting together the plenary allowed me to reflect, as well as giving me the chance to look forward. The aim of the plenary was to both chart the progress of our work and to show how we had started to use Gattegno's curriculum graph (Figure 1) as a way of navigating through a primary mathematics curriculum alongside the aims of the current National Curriculum. I realised that the curriculum graph was going to be the map of my journey, which I could share.

The graph shows mathematical routes that cross each other. Some of the routes we embarked on felt more complete than others, but each had the Gattegno textbooks at the heart of it. I wanted to show how we were trying to shift the study of mathematics from a focus on instrumental understanding towards supporting children to make their own creative connections. Gattegno calls this "subordinating teaching to learning".

Why did we start using Cuisenaire rods and the Gattegno textbooks at Bursted Wood?

Before we began using the rods, our mathematics lessons drew on a wide range of resources. Often the resources did not meet the needs of the children nor the intention of the lesson. Sometimes the children needed to learn how to use the resource to make efficient use of it and perhaps, in some cases, the teachers themselves were not aware of the impact the resources could have on the lesson. Problem solving and reasoning were not high priorities in lessons and too often the children were simply watching their teacher think. I was reminded of Madeleine Goutard's comment, (Goutard 1958), "The kind of problems that we are in the habit of inventing for children to solve are concerned with determined situations ... there is only one possibility and the children are asked to fill in the blanks". (p. 9)

In 2013, the new National Curriculum was published. At that time, the teachers in the school were questioning how we were going to be able to help our key stage one children understand fractions in a meaningful way. We did not want children to work through tasks in a procedural way. A meeting with Ian Benson guided us to material written by Gattegno and we decided to trial this material with our Year 1 cohort. At first I did not know what to expect but I was willing to experiment as I had never had experience of teaching key stage one mathematics before. The textbook was laid out clearly and the exercises were easy to understand. There were no suggested timings and I felt a sense of freedom. This might have been overwhelming for some teachers, but for me, it offered the freedom and the creativity I had been looking for.

The children became immediately engaged with the Cuisenaire rods and the lessons had a buzz about them. There was a fantastic mix of conversation, concentration and excitement. At first the children were using the rods alongside Book 1 for two sessions a week, but I realised quickly that the work undertaken in the sessions was providing a far richer and more meaningful environment for mathematics than I had ever seen before. Children were using the language of equivalence and fractions expertly and naturally and never tired of discovering new things about the patterns they had made. Towards the end of the trial we made the decision to use Book 1 as a medium-term plan and the material contained in the book provided the ideas for our daily mathematics lesson. This was a leap of faith as this meant that the children would not be writing digits for well over half a term, but we felt this was going to play to our advantage because the children's understanding of the structure of the number system and their developing forms of representation of their thinking through engaging with early algebra work would be strong. I am grateful for a forward-thinking leadership team that allowed me to follow my professional judgement and move away from routines that we had always followed. Now, four years on, every child in years 1-3 has their own set of Cuisenaire rods (sharing the resource would not have worked) and the teachers in these year groups are using the textbooks to plan.

The Gattegno curriculum graph helped us see how the textbooks fit together to provide us with a starting point for our planning. It has multiple entry points and logical dependencies. Gattegno showed how the study of permutations and combinations of Cuisenaire rods in key stage 1 could yield awarenesses about sets, equivalence, equation, function and correspondence. Children speak and write with a mathematical vocabulary that begins when we use playful activities, then make connections to actions in the rod world and virtual actions in the mind. Zulie Catir and Caleb Gattegno constructed a chart (see Figure 2), which helped to explain this further and has helped with our thought processes and planning.

Algebra before arithmetic	
Action	using the rods
Virtual Action	using the image generated by the action of using the rods
Speaking	using language to describe the imagery
Writing	using symbols to express

Figure 2: The Catir and Gattegno chart.

In the rest of this article, I want to illustrate how the journeys within the graph provide us with the language and experiences we feel our children need and thrive on in their mathematics lessons.

Journey 1: First encounters

The first chapter of *Numbers in colour* is entitled free play. This title is followed by a blank page. I loved this because of its simple message, play. As the children enter year 1 they each receive their own box of rods. It is a moment of pure excitement and the children love unwrapping the box and playing with the rods. In terms of a transition from reception to year 1 it is perfect. It is during this time that the teacher can begin noticing what the children are making and can immediately establish the game-like structure of lessons. The chance to observe the children's creativity also allows for conversations about what they see, aiming to build the choices they can make with the rods. The children take pride in owning their box of rods. Even packing away the resource can provide much in the way of mathematics. This beginning is accessible for all and not overwhelming for young children. Gattegno (1974) writes:

Indeed, the wise teacher will just use the rods for long stretches of free play and will be in no hurry to introduce any trace of direction. Free play is, precisely, free. Organisation or direction

destroys its essential character. It may need faith to stand by watching children building an Eiffel Tower or making doorways ... especially when the construction seems to have no particularly mathematical significance; but six to eight weeks spent in this free play will pay handsome dividends ... While thus engaged children meet, without words or other intermediary between their own spirits and reality, a rich variety of relationships inherent in the rods.

This journey of free play, initially with rods and later with free writing, is one journey that continues across all paths on the curriculum graph. It is essential and an activity that the children never tire of, no matter how old they are.

Journey 2: Equivalence

Children begin this next journey by sorting, naming, ordering and using equivalence to form patterns using the rods. The work is taken from Book I, Chapter 2, titled *Qualitative work*. Once again it is up to the teacher to maintain the game-like structure set from free play to motivate the children to explore more. Here Gattegno introduces 'trains', which are two or more rods placed end to end. He asks the learner to make as many trains as they can and describe them. He follows this by asking the learner to take one rod and find trains that are equivalent in length to that rod. It is at this point we have children as young as five using the word equivalent with ease and with understanding.

Initial equivalence work is predominantly random and is described as 'empirical' by Madeleine Goutard in *Mathematics and children*. She notes that children are often kept at the empirical stage using concrete resources and goes on to observe that perhaps when a teacher feels a child is struggling to understand a mathematical concept they are simply given objects to count with. Gattegno's treatment using Cuisenaire rods allows the child to play and converse whilst the teacher can help to steer a conversation. If a child does not understand something, there are easy constructions to revert back to and familiarity can be re-established. The empirical stage moves swiftly, through repetition, to systematic thinking by asking children to sort rods into patterns of equivalent length, thus promoting higher-order thinking skills. By the time the children move to Book II, the foundation of pattern building is taken a step further and the children are able to explore patterns within patterns. Some are even able to plan the internal parts of a pattern before completing them with the rods.

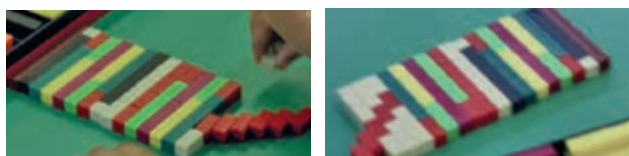


Figure 3: Year 2 children building patterns within patterns.

It is clear to me that the children have started to internalise the patterns they have made and are able to construct them virtually as mental images. Whereas previously in my mathematics teaching, I may have focused on number bonds to 10 or 20 and given the children just two numbers to put together to construct these, I notice that when the children use the Cuisenaire rods their only restriction is the rule of equivalent length and they find it easy to understand that a number can be partitioned in many ways. They are also quick to rule out rods that will not be equivalent to each other or can 'see' what the missing gap in a pattern would be without having to test it. On reflection, I believe it is at this point that I started to see how so much could be gained from a seemingly simple task. A single resource and the idea of pattern can be returned to again and again, first through language and then through writing. Gattegno writes about "yielding a lot from a little". These tasks show exactly what he means.

Journey 3: Discovering notation

This journey builds on the idea of free play and equivalence and is focused on children finding ways to describe their awarenesses through notation. Children move freely between the rod world and the algebraic world, translating the rod work into writing and interpreting their notation through rod constructions. The rods are given a letter name based on the colour code (see Figure 4). This offers opportunity for noticings such as there is a 'b' and 'B' or discussion of why the brown rod is called tan.

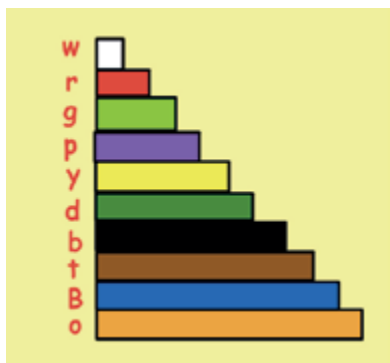


Figure 4: Letter names of the rods.

Children begin working on notation by going back to the initial patterns they have made, this time describing them in writing using the letter code. Goutard refers to this writing as a 'paper train'.

Crucially, the trains remain in the rod domain and the set of names are in the algebra domain. The children move with ease between the domains. Quickly, the signs for plus and equivalence help the children to form equations. Children were confidently able to use the sign for equivalence at the beginning or end of an equation. Crucially they do not use '=' simply to signal an answer, they use it to signify equivalence.

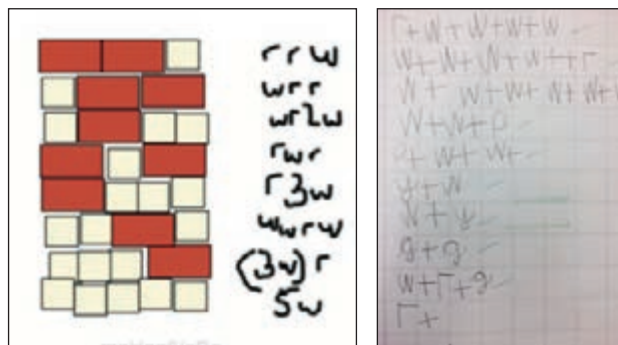


Figure 5: Children's recording using letter codes.

Number study follows this work and once again the familiar patterns are returned to for the children to write about. The relationships established in the qualitative chapter are so strongly internalised that using number becomes straightforward and natural. The rods themselves are not manufactured with a value attached to them, so it is important that the teacher emphasises the lack of a fixed value early on. Gattegno creates an early exercise where the rods are taken in turn and measured by the white rods to name them algebraically (*Gattegno Mathematics Text Book 1. Part 3. Activity 19*). Later, by calling white the unit, he names the other rods by the number of whites needed to make an equivalent length. (*Gattegno Mathematics Text Book 1. Part 3. Activity 1*).

What becomes clear, once the number study is in full swing, is that the children are comfortable establishing different names for the same rod. The red rod, for example, could be red, two, half of four, a fifth of ten depending on its context. One activity that I have developed from the books that works well across all stages is called the 'name game'. Here the children generate a pattern, maybe suggested by the books or a creation of their own, and write as many different names for each row as possible. Here they are moving smoothly across the domains in speech, action and virtual action. When working with Book 1, I notice that my lessons are about the children generating their own mathematics through what they see. The combinations, possibilities and answers vary but the constructions remain simple. I feel as if I am planting mathematical seeds that will blossom over the course of the books and there is no rush to see them bloom straight away.

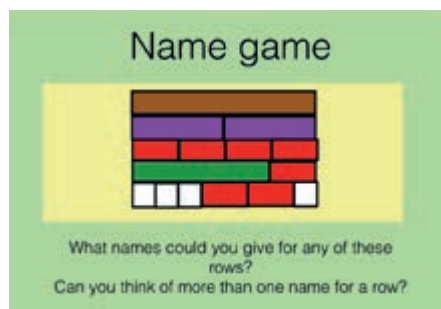


Figure 6: The name game.

Journey 4: Structure and relationships

At this point, a new manipulation is introduced where the firmly established trains become crosses. These crosses represent products and the thirty seven most important products are laid out on a visual chart. Instead of children simply learning products in sequence and by rote, here they construct or deconstruct them with ease. Once the new manipulation is established, I find that this is mainly carried out virtually and that the children understand the process surprisingly quickly. Gattegno continues to develop the learner's appreciation of how numbers can be related and, more importantly, provides exercises to allow the learner to think about how to use these relationships to calculate with other known facts. Gattegno describes this phase of Book II as "milestone numbers" and they are a springboard to deeper learning about geometric progressions.

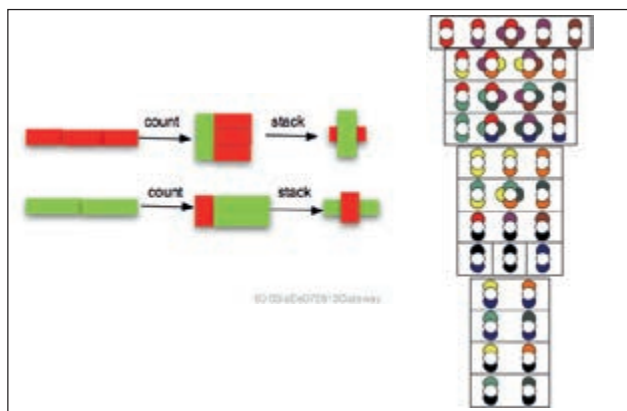


Figure 7: Products.

Final thoughts

It is exciting to think how far we have come with the materials and to know that we have a guide to help us navigate through future material. I feel that my approach to teaching has changed a great deal and I would go as far as to say I feel rejuvenated. I enjoy noticing and observing more and do not feel the need to talk as much in my lessons. The mathematics comes from the children and their ability to generate conversations about what they see or the way they write about their patterns allows me to find out what they understand. I feel like I am becoming the teacher I have always wanted to be as Gattegno's concept of subordinating teaching to learning becomes a reality.

The tasks are accessible to all but are deep enough to challenge. Within the textbooks, I find the pedagogy I need but also a sense of creative freedom. It does not feel like a scheme. I love the notion of game-like structures, and by thinking like this, I have been able to cut out unnecessary planning and gimmicks that perhaps I might once have used. There is always a buzz in my classroom from the children using the rods.

Another important aspect of this project has been the professional conversations that I have been able to engage in. I am regularly in contact with other teachers and academics who support me and allow me to ask questions or, in turn, support them. There is a sense that we are creating a portfolio of work that is unique to each school but shows the power of the textbooks. Above all, I realise what Goutard meant when she said, "you don't teach Cuisenaire rods". As she writes in *Mathematics and children*:

Between two classes both using the rods, enormous difference in level may exist which can only be accounted for by the extent teachers benefit from what the material has to offer. Teachers must drop the idea that pieces of wood have magic powers. The transformation of education can only result from a renewal of the teachers as persons and this, in turn, must stem from creating awareness on their part (p. 3).

The journey so far has been exciting and inspirational. There is much more to do and further progress to be made, but we have a strong sense of direction. We truly believe that we are inspiring our children and not simply informing them.

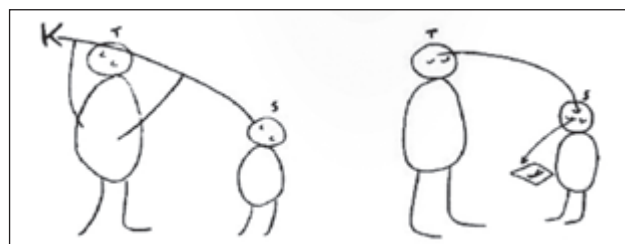


Figure 8: Illustration from *What we owe children: The subordination of teaching to learning*, 1987.

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Consistency of imagery

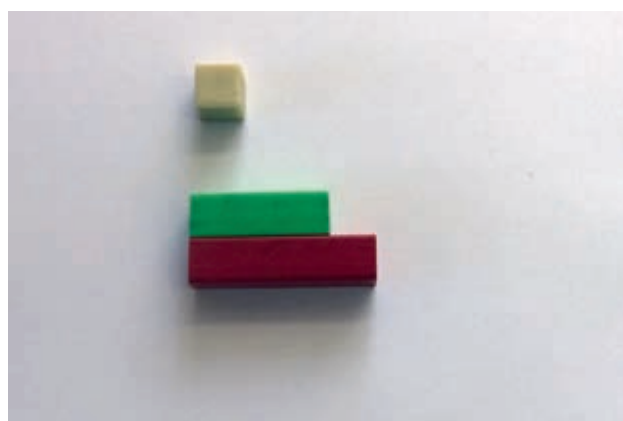
Caroline Ainsworth describes how she uses Cuisenaire rods to develop children's understandings of the structure of mathematics.

I work as a specialist mathematics teacher in a small primary school in Devon. As a school we use Cuisenaire rods from Reception to Year 6 (10 to 11 years old) to develop children's understanding of the structure of mathematics. For the past ten years, we have been developing an approach based on Caleb Gattegno and Madeleine Goutard's methods to support our mathematics teaching. Although we use other manipulatives and resources, schemes of work and textbooks, we rely on our use of the rods to give consistency of imagery. We believe this helps us to be more efficient with our teaching, to build on children's learning from one area of mathematics to another, to see connections and, most importantly, to make generalisations. In this article I reflect on how this imagery is created, sustained, and used to support calculating, reasoning and generalising. I begin and end with a narrative to give insight into how we work on mathematics in the classroom and, as an interlude, reflect on how we use the rods in developing imagery.

A narrative of calculating with fractions – part 1

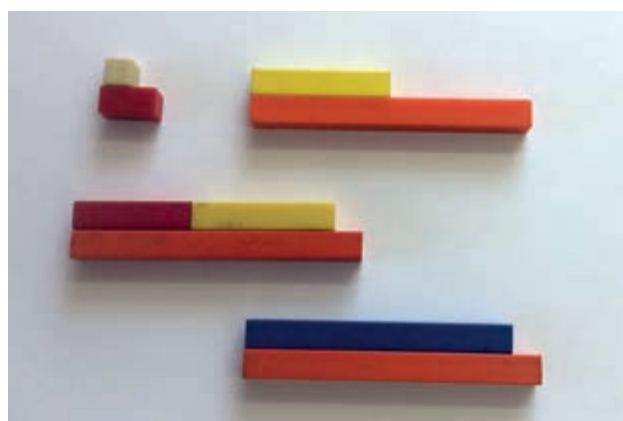
With three weeks to go before the Standardised Assessment Tasks (SATs), I am contemplating revising calculating with fractions with Year 6. The new test paper will require the children to add, subtract, multiply and divide using fractions and mixed numbers. The grid next to each calculation on the sample paper seems to imply the expectation of an efficient, written method. But I do not want to reduce my children to following a set procedure, however tempting it is as a way of gaining marks, as children can be easily confused if they do not understand the procedures. I know many children in the class are efficient with these procedures, but I want to retain their spirit of reasoning and articulating their ideas. So, conscious of time constraints, but determined to value deeper learning, I set a range of fraction calculations and tell the class, "I am not as interested in the answers as I am in your thinking. Can you use the rods to explain your thinking to me?"

Mark is calculating $\frac{1}{3} \times \frac{3}{4}$. He has chosen to set out these rods:



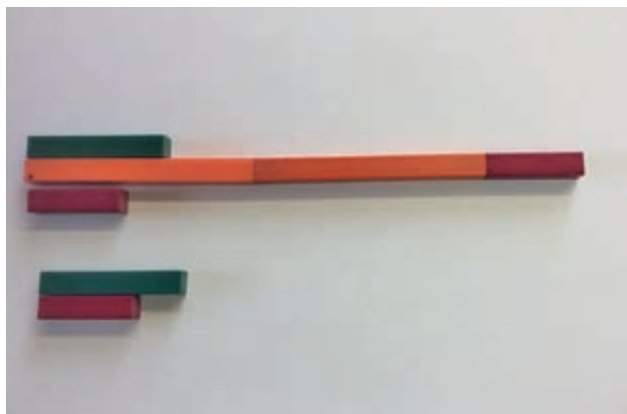
I ask Mark to explain his thinking. "Look, that is three-quarters of that [pointing to green and pink] and this [white] is one third green. So, one third of three-quarters must be one quarter." He places the white on the pink for emphasis. He continues working, satisfied that he has convinced me he is right.

Emily is working on $\frac{2}{5} + \frac{1}{2}$, she has set out this:



"This is half [picking up white and red] and so is this [yellow and orange]. This is one-fifth, [she taps red along the orange with no attempt at accuracy as we both know its true], so this must be two-fifths [pink]." She places the pink and yellow end to end. "So it makes nine-tenths altogether."

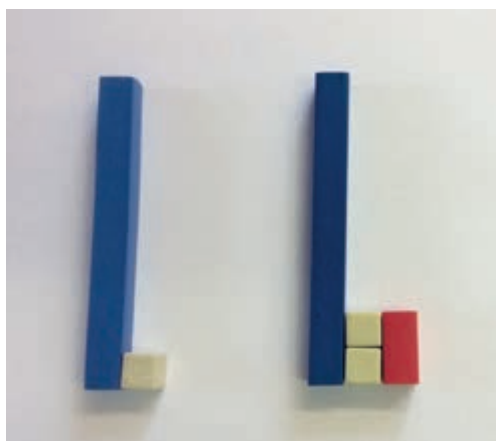
Richard is working on $\frac{1}{6}$ divided by $\frac{1}{4}$, he has chosen to set this out:



He picks up dark green and explains, “this is one quarter of that [2 oranges and pink] because it fits in 4 times, and this is one sixth of it [showing me pink] because it fits in 6 times. So really it’s how many of these [pink] are there in that [dark green],” he pauses, “no,” he corrects himself, “it’s the other way around.” He places dark green against pink and states, “it doesn’t.” I ask, “does some of it fit in?” He compares the pink and dark green, “two-thirds of it does,” he pauses again, “yes, that’s right because quarters are bigger than sixths so its going to be less than one.”

Neither Mark, Emily or Richard need me to confirm their answers are correct, but the rods give us a shared context and image of the calculation. They support the child when articulating the situation and allow me a window on their thinking.

Meanwhile, Alfie, who joined the class in September, is telling me that one-ninth plus one-ninth is two-eighths because, “you add the ones and you add the nines.” I can see that even Alfie himself is not convinced by his own reasoning. He says, “it is, because you add them, don’t you, you add the top and the bottom?” I set out the blue and ask him if we can agree to call it 1.



I ask him what he would then call white. He replies “one-ninth.” I pass him a second white and ask him what he has now. Instantly he replies, “two-ninths ... oh yes, that’s right, I get it now.” But with only a few weeks to SATs I worry that, under pressure, he will jump to the same incorrect calculating method. My quick intervention will not be enough to establish understanding. This is not to imply that none of the children who have consistently used rods at our school develop or persist with misconceptions, but familiarity with the rods gives us a way of constantly challenging any misconceptions, not by a teacher-led explanation, but by carefully guiding the child through reasoning, supported by the rods, towards a new awareness of the situation.

As a class, we examine Richard’s idea and I ask everyone why they think Richard chose 2 oranges and a pink. Benn says, “We know they could be 24 and we know 4×6 is 24 so it makes sense to use 24.” I ask him how we know this. He replies, “we’ve used the rods a lot so you just know it.” I think Benn is referring to the fluency children gain when using a manipulative as simple, yet powerful, as the rods for building not only number bonds and times tables facts, but a deeper understanding. Somehow, the rods come to contain both numerical possibilities and a generalisation of the situation which can be shared by the teacher and children.

Interlude: Developing imagery

The children in the narrative above are Year 6 (10 to 11 years old) and have invested time and effort into constructing a sense of both number and operations with the rods since Reception. How has this happened? It may be useful here to describe how we go about using rods to build and develop this consistent imagery.

To guide our teaching we use the approach Goutard describes in *Mathematics and Children* (1964). She shows four ways of presenting number relationships which may be studied with children in order to gain depth of understanding about calculating. These ways form the basis of many of our lessons and are used to introduce the signs. I offer simplified versions of the four ways, in turn. The following image, the table of partitions of lengths allows the child to see commutativity (children say “you can swap the rods around”) and the compensating dynamic of partitions (children say “one side gets smaller the other gets bigger” or “there’s a staircase going up and one going down,”) without referring to specific numbers.



Repeated and varied activities using this arrangement of rods can lead to flexibility with calculating, including understanding how calculations might be adjusted, for example $299 + 327$ can become $300 + 326$, which for many is easier to calculate mentally. The children have agreed that the '+' sign can be used to represent rods being placed end-to-end (the children often say "stuck together"). The compensating dynamic can be used to help children memorise number bonds. This can be number specific, so if we see that $5 + 5 = 10$ (using 2 yellows and an orange), then $6 + 4$ must also be 10 because one rod has increased in length, so the other must decrease, by the length of a white. In this way the actual rods, or a retained image of them, can help children move from known to unknown facts. But the same image can be used to work from $50 + 50 = 100$ to $60 + 40 = 100$ or $600 + 400 = 1000$. We notice that once the children realise the rods can represent any number, they generalise for themselves. The teacher's job is to devise activities where children can use this awareness. In order to assess their understanding of the situation, I ask, "If $a + b = c$, which of these is also true?"

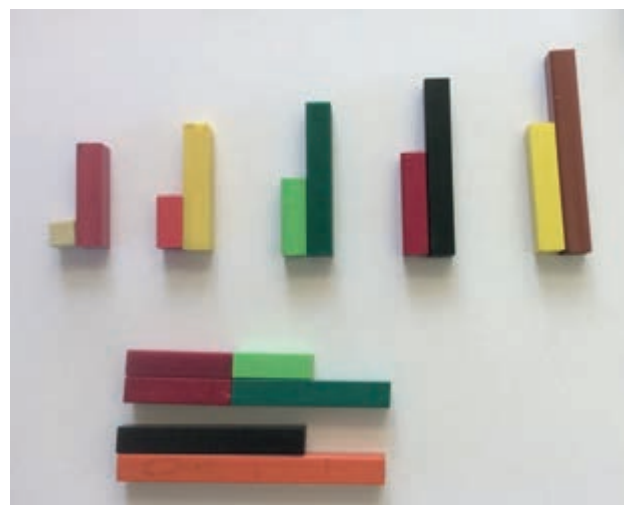
$(a+1) + (b-1) = c + 1$? Or $(a+1) + (b-1) = c$?

Somehow the rods, as they do not relate to specific numbers, can be used by the teacher to create a bridge from children's awareness of the compensating dynamic to an understanding of this in written notation.

Families of equivalent differences

Similarly, activities based around study of a family of equivalent differences can generate understanding that the family is infinite and that as long as the same

quantity is added to or subtracted from both numbers, a calculation can be adjusted to make it easier, for example, $657 - 298$ can become $659 - 300$.



The image shows that we are referring to the difference between two lengths, very different from the image of an overall length made when rods are stuck together. In order to bring this to children's awareness, Goutard uses an activity in which children are shown a pair of rods by the teacher and asked to make pairs of rods with an equivalent difference, children often call it "the same gap". They can see what is important, that the gap remains the same, and what can be changed, that is the lengths of the rods. We use the question, "what can you do to my pair of rods to turn it into your pair of rods?" to make children aware of the addition or subtraction of an equivalent length.

We can ask, "If $a - b = c$, which of these is also true? $2a - 2b = 2c$ or $2a - 2b = c$?" Some children quickly grasp the idea and use algebra to express what we have seen, others try this out with specific rods. The key seems to be to try to work towards this type of question whenever subtraction is being studied, giving all children as much chance as possible to reach this mastery level. Here the rods are used to create an image of a relationship.

Tables of factors or divisors (equivalent products)

This is not to say that using the rods as numbers cannot be a good starting place for seeing those relationships, but we find it useful to move quickly to generalisations, before children slip into thinking what they are noticing applies only to this particular number or group of numbers.

For example, when studying families of equivalent products, we often ask the children to set out a number with many factors such as 24 then to write

number sentences.



$$12 \times 2 = 24, 6 \times 4 = 24, 8 \times 3 = 24$$

We gather facts together on the board and ask the children what they see. Children quickly say that one number is doubling, the other halving, yet the product is staying the same. For me this is not as important as the children articulating the answer to the next question, “why is this?”. They often say something to the effect of, “because those are half the size, you need twice as many of them”. When everyone has had the chance to explain this relationship, we will move on to asking them to set out another number such as 18 and repeat the exercise. We might then gather together $2 \times 9 = 18$ and $6 \times 3 = 18$ to draw out that one number is being tripled and the other divided by 3. In answer to the question, “what do you notice?”, we are looking for more general explanations about the inverse operations being applied.

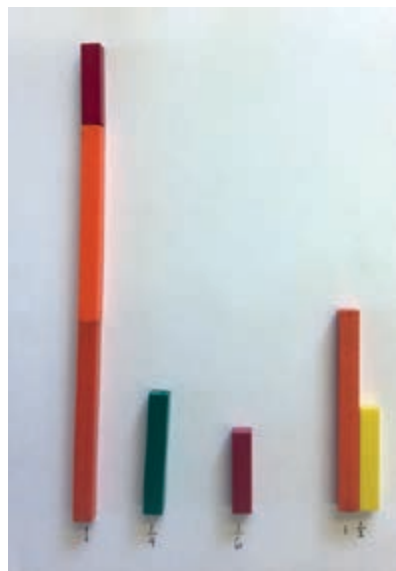
Again, we can use this awareness to adjust calculations (5×126 can become 10×63) and check understanding using algebra: “if $a \times b = c$ is this true? $2a \times 2b = 2c$?; $2a \times \frac{1}{2}b = c$?; $\frac{1}{2}a \times 2b = c$?; $\frac{1}{2}a \times \frac{1}{2}b = c$?”.

These activities can be repeated many times with different arrangements of rods. Children’s explanations of what they see and how they generalise shifts subtly each time they are asked, their reasoning becoming clearer. They benefit from listening to each other’s explanations, and through explaining aloud the image becomes useful and memorable to the child.

Working in this way, always looking for how we can generalise the situation, leads children to anticipate these generalisations. I find the children are often a step ahead of what I am saying. If I pause, leave a gap, remember to be quiet, children will fill the silence with, “So does that mean?”; “What if we used big numbers?”; “Would it work with fractions?”; “Does it always work?”; “How would you do it with decimals?” On a really good day, I resist the temptation to answer immediately, allowing another child to suggest an answer or at least a means by which we can check.

Family of equivalent fractions (or quotients)

Goutard uses the rods to create an image of the family of equivalent fractions (or quotients). This awareness may well have helped Mark and Richard to select rods for their calculations. Richard would be familiar with 2 oranges and pink as 24, dark green as 6 and pink as 4, but could simultaneously see dark green as $\frac{1}{4}$ of the longer train of rods and pink as $\frac{1}{6}$. Mark saw yellow and orange simultaneously as $\frac{5}{10}$ and as $\frac{1}{2}$.



In order to use the rods effectively to create flexible imagery, we have learnt that these images need to be used repeatedly, approached in different ways and the children given plenty of opportunities to discover and articulate for themselves the generalisations they contain. It is not enough to ‘see’ commutativity in an arrangement of rods, it needs to then be tested with fractions, decimals, mixed numbers, more than 2 numbers, and do on. The consistency of imagery is only useful if we go beyond agreeing the way we label certain rod arrangements and are constantly aware of the need to draw out generalisations.

A narrative of calculating with fractions – part 2

The lesson continues and the Year 6 children show me many varied and creative ways to visualise calculating with fractions that I store away for use with future classes. The lesson ends and I move on to teach fractions to Year 2. Also heading towards SATs, I would like these young learners to have a depth of understanding about quarters and halves in a range of contexts. I plan to use Goutard’s approach to teaching fraction notation, but again I do not want to limit their thinking and risk missing opportunities for generalising. I am looking for what Goutard calls, “creativity at the level of symbols”. I do not want to

limit their ideas to labelling an image with a fraction just as they begin to grasp generalisations about fraction notation.

We have set out a pink rod and I suggest it could be called 1 today. I ask for a name for red, someone suggests a half. I ask what we might call white, someone offers a quarter but I remember to be careful to allow other ideas to follow. Someone says they see it as half of a half, another one quarter plus one quarter, another one minus three-quarters and many others. The children begin to record their ideas and explain their thinking to their group, comparing rods and reasoning about possible names. We vary the rod we call 1.



Functional relationships between patterns of Cuisenaire rods

Ian Benson suggests how appropriate new conventions can support generalization

The study of permutations and combinations of Cuisenaire rods has proved to be a rich source of mathematical tasks motivating abstraction through algebraic symbol systems as well as mathematical generalisation. (Gattegno, 1963a, Ainsworth 2011, Benson 2011, 2014) Cuisenaire-Gattegno is a unique approach that can meet the aim set by the new national curriculum, that learners *“need to be able to move fluently between representations of mathematical ideas.”* (DfE, p53)

In this exercise I take a Key Stage 3 rod permutation problem posed on the nrich site as <http://nrich.maths.org/4338> (go to www.atm.org.uk/mt245 for link.)

I show how teachers can use this task to support generalisation by employing a new formalism and diagramming convention that records the functional relationships between patterns of Cuisenaire rods. This convention was suggested by William Lawvere and Stephen Schanuel as an *“external and internal diagram”* for mappings between typed sets. (Lawvere, Schanuel 1997, 2004).

The exercise requires the learner to construct specific patterns for trains equivalent in length to each rod in turn. They are restricted to using only white or red rods.

Figure 1 shows a solution to this exercise for the pink rod.

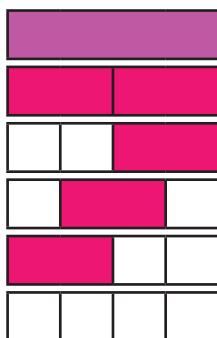


Figure 1. The pattern of white and red trains for the pink rod

Learners are asked to construct such a pattern interactively for the smaller rods, light green, pink, and yellow. Then they are asked to hypothesise how many trains would be in the pattern for the next longest - dark green - and check their work online. Finally learners are asked to say how many trains are in the pattern for orange, the longest rod, and to explain their work.

The solution is a Fibonacci series. The new formalism enables the learner to demonstrate succinctly why this is the case, in particular why there are 89 trains in the pattern for orange.

Gattegno held strong views on how to communicate mathematically (1973, p. 2). He wrote that mathematical activity unfolds as:

1. ACTION: using the number array, the set of fingers, rods ...
2. VIRTUAL ACTION: using imagery generated by the action
3. SPEAKING: language to describe imagery
4. WRITING: symbols and notation

At first he argued that this meant that pupils didn't need text-books at all.

“Many people know that from 1953 to 1956 I refused to follow any suggestion from teachers that I should write text-books. My reasons were:

1. *that I wanted the teachers to use the rods according to their lights and not to mine*
2. *that, just as I did not want to interfere with the learning process of children, so I did not wish to interfere with the teacher's freedom of work.”* (Gattegno, 1963b, p. 107)

However, he continued to be pressed by teachers, and eventually found the solution in the form of a log, recording a set of lessons with actual children, *“A set of questions – what better basis could be put into a book?”*. This preserved the experimental basis of the approach, as there would be no need to give answers, leaving teachers and pupils free to explore. He hoped that everyone coming to the new text-books would know that they were to be used with Cuisenaire rods, which he colour-coded in order of size w for white; r for red; g for light green; p for pink; y for yellow etc. Without the rods his books, like this note, are meaningless, *“but they become child's play when used in conjunction with the rods.”* (op. cit., p. 108)

The next decision that Gattegno had to make was whether to include diagrams in his books. He chose not to do this in the books written for children between 1956 and 1960. He wrote,

... *“anyone who inserts diagrams has in my opinion*

1. *completely misunderstood the components of learning with the rods, which is a mathematisation of actions;*
2. *mixed an old-fashioned and no longer justified approach, based on images suggested by illustrations with a dynamic approach based on actions with objects, thus reducing the efficiency of both;*
3. *slowed down the learning process by spending time on irrelevant work;*
4. *fostered habits of thought which are hybrid, thus under-cutting the integrity of the learners mind.”* (op. cit., p. 109)

By 1973 in 'Common Sense of Teaching Mathematics' Gattegno had relented and he includes as an illustration a diagram of a set of complete patterns that he uses to define what he means by an integer. He writes, "the set of all ways of making a particular length with rods produces an equivalence class: all trains in the set are equivalent since they have the same length" (1974, p. 51). Figure 2 reproduces his drawing, with annotations to show the correspondences they suggest.

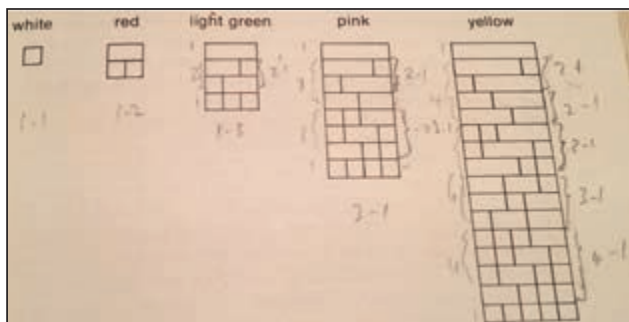


Figure 2. A set of complete patterns of rods from white to yellow

The diagram arranges the complete patterns into product groups, that is, groups of trains with the same number of cars. Looking at the number of trains in each group we see a pattern emerge corresponding to the coefficients in Newton's Binomial expansion or the sequence known conventionally as Pascal's triangle: 1, 1-1, 1-2-1, 1-3-3-1 etc.

The *nrich* exercise studies red and white trains only. From Figure 2 we can see that these patterns can be represented from left to right as sets of trains using Gattegno's coding scheme for the rods, choosing a capital letter to name each pattern.

$$\begin{aligned}
 W &= \{ w \} & G &= \{ rw, wr, www \} \\
 R &= \{ r, ww \} & P &= \{ rr, wwr, wrw, rww, wwww \}
 \end{aligned}$$

G and each subsequent pattern can be related to the earlier patterns by drawing a line between the members of each set as shown in Figure 3. This is called an internal diagram. An arrowhead is used to indicate the direction of the functional mapping between the sets. In this case the functions depend on the first car of the source train. In one direction trains starting with a white car map onto the previous set, and trains starting with a red car map onto the set two sets earlier. These maps are called surjections. They are shown with a closed arrowhead.

In the other direction the functions prepend a white (red) car to the source train to form the target train. This is shown with the open arrowhead. These functions are injections, or 1-to-1 mappings.

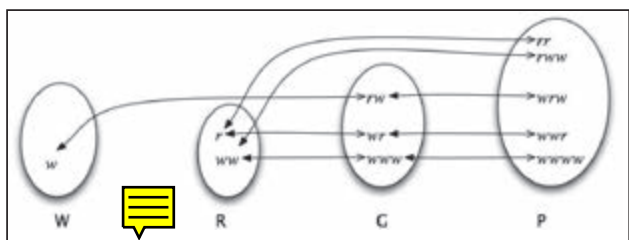


Figure 3. Functional relationships between the patterns (internal diagram)

A study of the sequence in Figure 3 shows that after G each subsequent pattern can be mapped onto the disjoint sum of the previous two sets. The number of members in each set is therefore the sum of the number of members of these prior sets.

Figure 4 is an external diagram. It extends our series to the remaining patterns (where K, N and E code for the patterns for the black, brown and blue rods). Arrows again indicate the functional relationships between the patterns.

By inspection the number of members in the pattern for orange is 89.

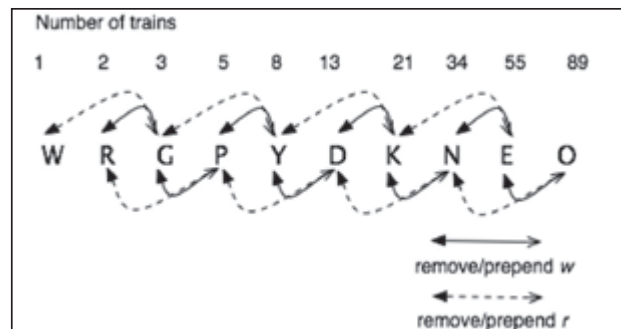


Figure 4. Functional relationships between the patterns (external diagram)

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Note Key Stage 3 rod permutation problem can be accessed at <http://nrich.maths.org/4338>



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21st Anniversary

Caleb Gattegno – reflections on 21 Years of ATM

To be of age is a symbol rather than a full reality. ATM was, in a way, of age when it was born as ATAM, because it filled a need: it re-juvenated a number of mathematics teachers in Great Britain and the Commonwealth. Has ATM at twenty-one the inspirational value it had in 1952? Does anybody today think of inspiration, or of touching teachers affectively and effectively?

These questions may be answered by each reader personally.

I, as the convener of the Committee which funded ATAM, still feel that what made our movement acceptable in 1952 can still make us recognisable as contributing vital ingredients to the schools of Britain because we are moved by our vision and our thoughts, because we feel that teaching is a giving of ourselves rather than the handing out of old or new mathematics. What is the job which I recognise as mine, but which, perhaps, not everybody sees as theirs? The words have changed since my first manifestoes but the spirit is the same. Our students or pupils or school children (according to age) can be made to discover that mathematics is a mental activity which is everybody's birth right, and that they can engage in it if their elders know it or of it. The activity of mathematicians is to mathematise, not to rediscover the mathematics of the past. So the activity of students of existing mathematics should be to look at what exists as if it were being made for the first time. This will give dignity to errors, trials, tentative conclusions, guesses or assumptions, working out an example thoroughly before finding that there are classes of examples which display the same phenomena and before being able to enunciate the theorem that stands for the phenomenon.

This will generate new methods of teaching which are more serious than those offered today under various names. In particular it will give teachers an enhanced dignity through an enhanced responsibility. For they will have to know a great deal more about learning and learning mathematics than the literature provides.

Man only finds freedom at the end of a quest.

He cannot be clumsy, insensitive, and yet be free.

Teachers will feel the blessings of freedom as they manage to reach that true competence as teachers which is indissolubly related to their pupils' learning.

Anyone who finds that in order to understand what goes on in his class he needs to consult other people or books, knows that he cannot be in contact with his pupils' learning, for each of us is a "learning system", is a retaining system" by construction; and every one of us delights in the feeling of mental expansion and spiritual growth.

If only teachers really knew this the climate of classes

would be that of a joyful, busy, keen set of minds finding that they can mathematise the universe. As we learn and enjoy learning so do our classes – when we accept the responsibility of presenting them with meaningful challenges not too far beyond their reach, not so easy as to appear trivial nor so mechanical as to be soul killing but assuredly capable of exciting them.

The games children play have these attributes, and all of us could learn from observing how children play.

The aim of a game is always a particular mastery.

Rules are explicit and errors are related to these rules governing the acquisition of some skill. Once a skill is mastered it becomes instrumental and higher level challenges then emerge in the shape of new games.

Over the last twenty-one years I have studied, on my behalf and that of all teachers, how games can be invented which take care of the level reached by a searching mind, and have proposed challenges linked to the various levels which expand the self in the dialogues it entertains with the objective situations.

My games are related to the mathematical structures I want to bring to the students' notice. Simple structures require naked materials, like coloured rods or prisms or geoboards. More complex structures require the insertion of various mental dynamics into the manipulations of the materials.

Games are austere and ritualistic, just the opposite of the qualities that attract teachers today in the so-called "open education", where what happens is not on the whole under anybody's control, and therefore not anyone's responsibility. In my games the openness comes from the fact that there exist entities in the universe which are independent of each other and not hierarchically linked, but once engaged on a road the inner dynamics get hold of the players and the mathematisation is acted out ritually.

If, for instance, we play a game of transformation in the field of the addition of integers in a particular base, the vision of the transform is the substance of the game, but the game ends which the transform displays the name of the sum in that base. The game aims at the mastery of changing by one's own means which was given into an addition which immediately yields the name of the sum by simple inspection of the addend.

The job that remains to be done by teachers of mathematics includes the re-examination of all that we ask children to do in order to acquire a certain mastery of mathematisation of some areas – a job comparable to that done by the Bourbakis for the edifice of 20th century mathematics. To this I dedicated myself when we started ATM.

How much have you, reader, done in that direction?

Thoughts about Problems

M.Goutard (translated from French)

The kind of problems that we are in the habit of inventing for children to solve are, it seems to me, concerned with very precisely determined situations: in them a number of terms are brought together in a certain relationship which is very set and fixed. From this static scheme there is the one possibility only – that of finding a single missing term. Actually, any one of the terms may be omitted and in each case the child is asked only to fill in the blank. We could compare such problems, and the words in which they are stated, to a jigsaw puzzle which the child must reconstruct in order to specify which piece is missing. Or, to change the metaphor, s/he is given the ends of a chain and must assume that the other links exist.

Unless an analysis of the mathematics involved, and of the dynamism of the relationships, has been learnt previously, the child finds that s/he cannot solve these problems: this is made worse because the facts are buried in a concrete form, which adds to the confusion by making it difficult for the child to find the relationships, since they are not visible as such. This explains the successful results which can be obtained by a teaching method that Cuisenaire rods make possible, for a mastery of dynamic schemes is gained which can be applied to any concrete situation. The child is at ease in using their own knowledge, and *immediately* finds the answers to the problems posed.

From this one could say that the question of problems is solved. I ask myself, however, whether in a such a case one is justified in speaking of 'problems'. Rather, it seems to me that we have arrived at a point at which the problem has disappeared: at this level there is no challenge to the mind which deserves to be called a problem.

What strikes me is that whenever we set a problem which is exactly determined

(which it is, if only one solution is possible) we imply a mind capable of its *instant* solution. In proposing a problem to a child, we are assuming that s/he is capable of interpreting it independently. But as soon as a problem is correctly posed in the mind, it is both understood and solved by the same act of thought. All that is required is a little time in which to formulate the reply. But this time is not being used creatively. It is occupied only in shifting from one point to another by means of acquire mechanisms. Just as one cannot instantaneously transfer oneself to the other side of the road, so does it take the mind a little time to perform certain operations. But the problem itself was solved as soon as it was put and seen. And it is a well-known fact that, in the history of science, an unsolved problem is one that is wrongly posed and that the person to solve it will be the one who asks the right questions.

In putting any particular question to a child, we assume that, if s/he is able to put it to themselves, their mind will work at just this level of reality. If this is so, the solution will be found immediately, in which case there is no more taking place than the exploration of a certain horizontal level. We are giving training (certainly useful) in some dexterity, but are not concerned with true creativeness. If, on the other hand, the child does not move on this level, the problem cannot be expressed in their mind: s/he must first make the preliminary analysis referred to above, and not until s/he has done this will s/he be able to rise to the appropriate level.

We seem to be faced with these alternatives:

1. We may pose to a child problems which s/he can solve only by rising to a higher level through 'inventing' the mathematics s/he needs. But, besides the dangers of affective blockage that such a method carries, it is also suspect in that it does not give the opportunity for a sufficiently free analysis of the mathematics involved, since it provides only one static example of a

dynamic scheme; a necessarily constrained aspect of what is merely a contingent state.

2. We can make the algebraic analysis separately and in the abstract. From fruitful awarenesses, the child acquires such ease and freedom that thereafter s/he will have no difficulty in applying knowledge to problems. But now that s/he is applying knowledge, the child's mind is no longer being used creatively. S/he has reached the point at which the 'problem' has disappeared.

Consequently, it seems to me that in order to find a way out of this alternative we must form a different conception of what a problem should be.

Let us bear in mind that what appeared to be the most important thing was not the solution of a problem but the way of posing it. Instead of requiring that the child shall solve problems, mathematical education ought to stimulate the child to ask their own questions (clearly more is envisaged than the writing out in their own words of little stories on a familiar theme: instead we are contemplating a reversal of what is usually done). The need to give children a sense of problem arises because we need creative mathematicians and not only people who know mathematics and can express clearly what they know. Suppose that, instead of having problems set for them, children meet, invent and construct their own problems! For what matters is the questioning: reflection is only a question in action. Questioning should not come from the object but from the subject. It is man who asks questions of nature; there is a response if the questions are well framed. But in our classrooms what happens is just the contrary: objective reality is used to put questions to the child, and the questions are put in such a way as to leave the possibility of only one answer.

I want to show that a problem taken at the 'object' level, about a reality already fixed, cannot be other than artificial since the

facts given already include the solution. The most banal example will prove this. "Some goods cost 100 francs. They are sold for 120 francs, what is the profit?" When reality questions in such a way, it does it artificially; it is pretending to ignore what is already known, since it is obvious that the profit is fixed *at the same time* as the selling price. If there was a problem at all, it occurred before this. What may have been the possibility of a problem lies in the 'fact' that the goods costing 100 francs had to be sold at a certain price: one might here have paused to consider if it would have been better to sacrifice a little profit in order to assure a larger sale. But it is made clear that as soon as one term is fixed so is the other also, and that thus there is no problem. In this we also see that the 'time' of the resolution of such a classic problem is only arbitrary and artificial time, which has no bearing on the time which relates to the solving of the real problem as it is experienced by the solver.

We guess now how it will be possible to find an issue to the alternative above. Instead of starting with problems situated at an objective level in a reality completely determined, we must start with problems situated at the level of the subject in a reality not completely determined. The problem will consist in determining it and in turning it into a 'solved reality'. The time factor entering into the solution will be a creative time used mentally and actually: having a proper function and not (as in the trivial example) just as an artifice.

It is obvious that such a way of looking at a problem accords with true mental activity. In life we are faced with problems which are at first more or less undetermined. People live at different levels of awareness of what reality is, which gives an illusion that another person could enter a situation that is from the first perfectly determined. To their awareness it is at first incompletely determined; it is through the effort to increase this awareness that the other person determines it. Initially, there is nothing but a 'fact': e.g. the necessity of

constructing a house; or the presence of a certain number of objects; or the apparent movements of the sun, etc... These facts offer a base for the progressive development of all that is possible.

In all fields of education and especially with young children, we must start with indefinite situations (for such is the reality in which they live). If in such a situation it is the size or number of objects that must be considered, the problem becomes mathematics. With the little ones who are just beginning to count, I suggest that problems arise from facts as simple as this: someone has given us a certain number of things (pictures shall we say). What shall we do with them? Each child starts off in any way they like, to find out what can be done from the fact they have a number of pictures. All kinds of problems arise spontaneously. The harvest cannot fail to be rich. From this lively beginning with its possibilities of flying off in all directions, however, certain especially interesting avenues can be found and explored: for example, collecting sets of cards given out in succession. This provides an endless series of possibilities obtained in a precise manner. In another direction, on the contrary, it can happen that the possibilities become rarer and rarer, that we tread ways more and more restricted and impose on ourselves conditions more and more difficult, until we reach the point at which the large or small number of terms left are in a state of completely static equilibrium. To take a simple example, let us suppose that we decide to take two sets of cards: if we fix only the difference between the number of cards in the two sets (say by giving three more cards to one child than the other), there is an infinite number of possibilities; if only the sum is fixed (the cards of one set being shared between two children) the possibilities form a definite series; if both sum and difference are fixed, then only one solution is possible.

So, placed in an indefinite situation which he wishes to determine, the child uses their imagination towards an aim that he will

reach through action. In the process he finds the data of his problems. According as he wishes to go in one direction or another, he sees which terms he needs to support actions and adjust operations accordingly. But the terms relative to the ones he uses are implied by the child's choices and he becomes aware both of the terms and of the relations. A child's deliberate constructive action gives them the power to analyse a reality which at first was only potential. Having introduced more and more 'necessities' until he has reached a certain point in the field of determination, he can go back, unlinking the related terms and freeing the possibilities, until there remains only one thing given, the number of possibilities becoming again infinite – the return gives a rational development of all possibilities.

The traditional problem is found at the final stage of the first movement in this scheme: the one that goes from relative indetermination into complete determination. To bombard the child with disguised versions of some fixed types of problems is quite unnecessary, whereas from any real fact an infinite number of problems can be developed.

It is worth saying that if we start from what is indefinite the child ceases to be obsessed by *the* answer, a magic figure which s/he must find at all costs. Since s/he is no longer confined in a maze from which s/he must find one exit, any issue is valuable when s/he can justify it. The child's mind is concentrated on their own aims and actions. Each child can respond at their own level, giving their own solutions and explaining them freely. Also, the data which s/he finds and brings in are those which suit the growing demands for precision corresponding to the child's own awareness, and hence s/he has full understanding of their function. S/he is no longer embarrassed with more terms than s/he can deal with, having no idea why they are included.

Perhaps all this will throw some light on problems about unequal parts, as in mixtures, alloys, etc. These are supposed to be difficult and many methods are used to solve them, e.g. graphs, reasoning, mnemonics, although it is considered preferable to use algebra in a mechanical way once the formation of equations has been learnt. I quote a typical example from a book intended for teaching in the fourth year of secondary school: "Two casks contain equal quantities of wine. The wine in the first cask costs 70 francs per litre, and that in the second costs 75 francs per litre. Half the contents of the second cask are poured into the first one, and then half of this mixture is poured back into the second cask. The contents of the latter then have a total value of 730 francs. Find what quantity of wine each cask held initially." The way the statement is made almost leads one to believe that the mixture has been made at random and as it happens one becomes aware of its price, then wondering what the composition might be. Certainly, some children do understand it like that. As if the composition had not been known first in order for the price to be evaluated! Since this is the case, the real problem requires that we shall find the proportions in which the elements must be combined so that a predetermined price can be charged. So, I want to show that it is only when children are given the opportunity to 'live' the whole problem just as it appears to anyone who actually wants to make mixtures of any kind, that they will be able to understand and master the compensating dynamics involved in such problems.

What we want, when we make mixtures, is to obtain intermediate quantities. As it is easier to realise what happens when the elements preserve their identity within the mixture, a good beginning could come from imagining oneself to be a carpenter who has to make a floor with blocks of wood of different lengths. First, we decide on the total length (and discover how the elements can be combined in various ways to make up this length), next we consider the

number or size of the elements (trying to find out how to combine and arrange what we have, in order to make another new length each time). In each case, we examine the field of possibilities, consider new demands and make a choice that will fit the concrete situation. The analysis takes into account the factors involved and the part they play: the greater the variety of elements, or the smaller the elements, or the smaller the differences between them, the greater is the number of possible lengths that can be formed. Then if we become interested in the work of chemists or salesmen who want to mix liquids of varying price or consistency, we can start off by finding the composition or the price, of mixtures which can be made, and discover the limits of variation that we can choose from. Finally, we may investigate how to obtain a mixture having such and such a consistency or price that we presume to be possible. It is soon seen that if the number chosen is the arithmetic mean of the given elements, the mixture can easily be made. Other cases may be more difficult. Suppose we start with liquids of value $3u$ and $10u$ and decide to make a mixture from these that has any other of the intermediate values we happen to choose. Cuisenaire rods will provide a particularly useful aid in the work. A child decides to make a mixture of value $6u$. She begins by mixing one litre of each so that he gets $10u + 3u = 13u$ (an orange and a light green rod), when the liquid she wants should be only $2 \times 6u = 12u$ (2 dark green rods), so she finds that she has too much. If she adds a litre of the lower rate, then for three litres of the mixture, $13u + 3u = 16u$ (an orange and two light greens); although she should obtain $3 \times 6u = 18u$ (3 dark greens). Now she has not enough, so decides to add a litre of the higher value. She continues in this way until she reaches the exact mixture she wants (such that the line of orange and light green rods is the same length as the line of dark green rods). This happens after mixing seven litres: four of value $3u$ and three of value $10u$ (4 light

green rods and 3 orange rods = 7 dark green rods).

When we examine all the solutions obtained by the various members of the class, we see that each of the different mixtures chosen was found by combining seven litres but in different proportions. This comparison confirms the awareness that what concerns us is the difference between two quantities, and as a result the pupil can appreciate the way in which intermediate quantities could be formed. With seven litres of value $3u$ (a train of 7 light green rods) and seven litres of value $10u$ (a train of 7 orange rods) we can make all possible mixtures. By interchanging litres of $3u$ and $10u$ (which is to replace a light green rod by an orange rod and vice versa) our first mixtures have values of $4u$ and $9u$ per litre. Explaining in terms of the rods: to use an orange instead of a light green adds "7" to the total length which initially was of 7 light greens. This is equivalent to supplying one white rod to each or to exchanging light green rods for crimson rods. Similarly, to remove one of the orange rods and put a light green one in its place can be considered, alternatively, as changing to a train of blue rods. At the next exchange of a light green rod and an orange one, the new mixtures, 5 light greens + 2 oranges and 2 light greens + 5 oranges are equivalent to trains of 7 yellow and 7 brown (and have values $5u$ and $8u$ per litre). We continue in this way until we have made the total of all possibilities (between 3 and 10). The development gives insight into the whole scheme so that a child will see at once that if she wants a mixture of value $8u$, say, she combines the liquids $3u$ and $10u$ in the proportions of $2/7$ and $5/7$.

On returning to the former problem that was supposed to be difficult, we find that the children have become experts on mixtures of all kinds so that they find it already resolved (as, truly, it is) without their even having had to pick up a pencil. From the values combined, of 70 fr. and 75 fr., the first thing we see is that the result

lies between these, and the second obvious fact is that it will be a multiple of 5 if people dealt with integers. Seeing the figure 730 fr. suggests at once that we are concerned with 10 litres at 73 fr. per litre, composed of $2/5$ wine at 70 fr. (4 litres) and $3/5$ wine at 75 fr. (6 litres). All that remains to be done is to take into account that the second cask contains half of the 70 fr. wine, which tells us immediately that both casks originally held 8 litres.

Problems of this kind are often stated with one extra fact that is irrelevant (which may be the total number of litres of the mixture), while a direct insight makes for greater economy of labour. There is now no laborious forming of equations followed by blind mechanical resolution; no tedious arguments based on false assumptions; no imaginary substitutions which are so long winded and involve repeating the same process over and over again. 'Reasoning' is brought in because one does not 'see'. But the child who can see is only the one who is allowed to experience the true problem in all its vicissitudes and not its a *priori* resolution: that is, the child who mixes and who provides himself with the changing reality on which he works so that he can observe the effects of what he does to it, creating various and interesting fields of determination. The highly structured 'traditional' problem should be used only as a final test by which a child proves that mastery of mixture problems is such that he can also 'see' when the problem is reversed. This reversibility is now possible and time does not enter any more, since the whole problem has been exhausted through true problems clearly seen.

My thoughts now turn in another direction. I mentioned earlier that all teaching must begin with situations that are not precisely determined, but which need to be formed, and that by taking a certain point of view on it, the problem becomes mathematics. I now want to discuss this *instant* at which the problem changes into mathematics.

If a problem is concerned with a concrete reality, this might be of such a quality that it could not at once be dealt with as mathematics without imposing arbitrarily on the child. Telling the time is a good example. It is not a foregone conclusion that a child is ready to learn to tell the time simply by virtue of the fact she knows how to multiply by 5. There is the risk of teaching nothing but a meaningless ritual. I will describe an experience I had with five-year-olds that shows at what moment such knowledge can be introduced.

We were in a new classroom which had an electric clock; in the children's stories this clock often entered as a mythical person of evil repute. I then became aware that the children did not realise that the clock hands moved uniformly, and that for them it lived between stopped moments. They thought the hands made a small movement. Stopped for a longer period, moved on a little, stopped again, and so on. "But," I asked, "what about when it is completely stopped?" one sensitive and intelligent child gave an answer to which the other children assented: "When we turn around and round we get giddy and then have to stop. It is the same with the clock, for although the hands don't go around so quickly, the force of turning makes it giddy so that it has to stop to recover." Interest being aroused, we watched carefully to see when the hands moved. To pass the time the children began to count. Each time the hand moved when they had reached a different number, since their speed of counting was not regular and they also lost count in bursts of laughter when they saw the hands move. They noticed the variability, but attributed it to the moodiness of the clock or its degree of giddiness. This continued for some time, until one child said firmly: "No, the clock's hands move evenly, it is we who count too quickly." What a turning of the tables! From measuring its movement, we had discovered instead that it measured our own! We sat at the table with our eyes glued to the clock and beat out a measure,

determined to make it as regular as we possibly could: now we found the same result every time to within one count, this sudden deep awareness of the relation of the subject to the children's world led to endless questions about clock-making, etc and to further widening of the field of consciousness of time (what would happen if clocks did not mark the passing of time, first thinking of the disturbance that it would cause in our little schoolchildren's world and then going on to imagine the chaos that would enter the whole of social life). It showed me that it is not until children are conscious of problems of time, and of the correspondence of the different movements in the world that they are ready to consider a mathematical point of view.

Further reflections on this revealing example make one wonder if the difficulties which some adolescents have in applying mathematics to practical problems are not caused by their having been plunged too soon into quantitative aspects of it. A qualitative study should first clear the ground, sweep away obstacles and provide the opportunity to develop awarenesses that are essential to a proper understanding of the practical situation involved.

To summarise, what we are recommending is an existential conception of problem: that problems should form and mould themselves from the spontaneous contours of the child's thought, favouring its movement towards greater maturity. It is no longer a question of puzzles perfectly complete, and independent of the subject who has to deal with them. The individual should be called upon to interfere effectively and efficaciously in order to raise reality to a higher level of structure. This is why a true problem of consciousness must be a problem having true duration, requiring time for its solution: which means that the mind is alert and, from its growing awareness, is constructing the world.

Mathematics Teaching No. 7 July 1958

A Farewell Address

(a large meeting on the last evening of ATM Conference, Winchester, 1988. This is a slightly curtailed transcript of Dr Gattegno's talk, prepared by Dick Tahta and agreed with him)

... I am very glad to be with you. I see three or four faces that I have known for many years, a few that I have known for fewer years, and many I've never met. I won't know you because I am talking to you, but I hope that I will be able to say something worthwhile for you.

In 1950, before ATM, I held a weekend seminar at Braziers Park in Oxfordshire. People came from the schools where I had students from the London Institute of Education. It happened that I suggested that maybe we could consider that one day there would be a request for the metric system to be adopted by the British. I didn't finish my sentence before I had a riot. (*laughter*) That was excluded forever. You see, I was so surprised that something as mild as 'let's talk of the metric system' created so much turmoil.

Well you have the metric system now, and it was almost 40 years ago that I asked teachers to consider it. I've been asking teachers to consider other matters for the same length of time and I haven't had a riot. But I've had silence, except from the one or two people tonight who let me know that it happens that I have had some input into their lives. Well, since nobody tells me how would I know? I am not sure that what I am going to say tonight will not create a riot but, if it does, wait 40 years and you will see what...(*laughter*)

I have no pleasure in saying that the fields in which I work hardly interest anybody, anywhere. I call myself the greatest failure in the field of education because I tried so much and achieved so little...I see that nobody's prepared to find one of the most important things that I have put in front of you, several times through your journal. That is, that when we were little children,

we had tremendous mental powers; and we still treat them all as if they had none. This is what I know to be true. Why is no one trying to know whether indeed it is true or not, to put me in my place if it is not true, or else go on with research that I started and try to add something to it? This is the most important thing. I have repeatedly written articles, that you may not have read, in which I say: don't waste your time, please, doing all sorts of things that people have been doing for so long, with slightly better results, in terms of having two children in a class who understand it instead of one....

I did not prepare this talk. I had thought I was going to meet with perhaps one or a dozen people who were interested in discussing some question or other. Now I see a variety of people sitting together and I would like to put to you a number of issues. The first is that there is only one instrument in research in order to find answers. One instrument and that is to raise questions, to ask questions. To question is the instrument. And if you don't question, then don't be astonished that you don't find anything.

There are questions that are trivial and don't serve any purpose: What's the time? Five to nine. Its finished, it's a useless question. What's your name? it's a useless question. Now, there are important questions and we have to learn to ask them. One important question is: why is it that every one of us did not speak until the ninth month or tenth or twenty-fourth month after birth? This is an important question. Why is it that we don't speak for many months and then a year, or a few months, later, we do speak? It is disturbing to have a question of this kind: why is it that I did not speak? The answer is so rich, in terms of your own education, that I hope you will put it to yourselves.

Another question is: why do I sleep every night? This is one of the most stupid questions, yet one which has the greatest profundity and a tremendous yield. Why is

it that I sleep every night? Another question is: what is the cost of learning anything? Is there an objective way in which we can know what it costs to be as good at addition as Weierstrass? Weierstrass was a great mathematician and he must have been good at addition. *(laughter)* Why is it that it takes so long to teach long division and that it's not understood by so many? The cost of learning anything is an important question. It is included in a wider challenge, which you may perhaps want to work on: what is the economics of education? ...

I say something that is going to be true in 2021 or 28 let's say, to be more optimistic. And that is that every child, in every school, will learn two new languages every year. I know it to be true. And for you it's just nonsense, one more sign of the craziness of that man. I know it is true. Therefore, if you wait to 2028, you will see it. And if you work for it, it will be known in 2027, perhaps.

Among the people that I have met in my life, and I have met perhaps a hundred thousand teachers, in forty-eight countries, I haven't met one who believes that there is a need for him or her to find out whether I'm crazy or speaking the truth. I don't mind if you say I am crazy. What does it mean? If I was that crazy, you wouldn't be here. There must be some attenuation of that craziness of mine that makes you come and spend a few minutes in my presence.

We haven't yet begun, ladies and gentlemen, to see that before we can improve education, we must study it. And we must study it seriously, not go to courses and wait for someone to tell us what we think. I hear that there was one session today about the teacher as researcher. If you read what I wrote in the late 40s, early 50s, this is spelt out there. And it is spelt out with a certain force, in which I said: 'Only you, who are in the classroom, can do this study. If you don't do it, it will not be done'. You don't have to be researchers with degrees, wanting

publications. You have to be researchers because you are interested in the truth of the situation in which you find yourselves.

You are there with n children, a small number if they are meant to be handicapped in some way or another, in larger numbers if they are supposed to be normal. You are paid by the LEA to do a certain job. You have to separate this requirement that you have to perform to justify your cheque from the actual opportunity that you have of knowing something that is not yet known. It is not yet known because university researchers only spend a few hours a week watching a certain group of people. You spend thirty hours a week, if you are in a primary school, or five hours a week with each class if you are in a secondary school, if I am correct in my recollections of what I did in the late 40s. I can assure you that what I know, I learnt in these situations. I learnt it because I allowed my students to enlighten me.

People often say: 'I teach them but they don't learn'. Well, if you know that, stop teaching. *(laughter)* Not resign from your job: stop teaching in the way that doesn't reach people, and try to understand what there is to do for you to become daily more skilled in helping these youngsters furnish their minds with things that are so elementary that, where they take five years today, I can do them in 18 months, sometimes less.

It's so elementary, what we teach in the BSc or in the high school. We take years to do something which could be done, when it's well understood, in weeks, days or hours. You see, I said to you a few moments ago, they can learn two languages every year, and yet they take five years not to learn a language. *(laughter)* They do several terms in geometry and have no idea of what it is. Not everybody; those that pass the exam get a social sanction that, perhaps they know. The others – those that acknowledge the fact – they tell you: 'I don't understand

anything in algebra, anything in geometry. I don't understand'. How is it they don't understand? This is a good question: how is it that they don't understand?

Behind all these few words that I am telling you, there are years of reflection, experiment and trials, with sometimes a correct answer found in a reasonable time, sometimes not found, ever...The opportunity we have of educating ourselves has been wasted year after year after year. I did not waste it; but I say that to myself because nobody believes it, nobody knows it. I didn't waste it. I know that every time I was in a classroom, working in a certain way, when I finished I could tell myself: 'you have found this that you did not find earlier in similar circumstances, and it belongs to the situation, it is part of it'.

Why do I have to be respectful of tradition when I don't even know how much struggle those who imposed, or made, this tradition available, had to go through so many years ago? There are those who believe that God wrote the Bible. Are you among those who believe that the curriculum has been written by God? Or written by some people who, may be extremely learned, or know the little that is sufficient to get on committees which allow them to make proposals which are passed by the hierarchy? For me, every one of these is a question that I have to put to myself. I'm not answering. I have to put questions...

Why is it that we are so respectful of tradition? Do you remember 1543? This is the year of the death of Copernicus. A few months earlier, he had published his treatise and he had the good taste to die immediately afterwards. *(laughter)* Copernicus told us one thing. Against all the beliefs of everybody up to that time, he was alone, just a monk in a Polish monastery. Today, we don't know any of the others and we still think that Copernicus made a gift to us of a vision of the world that is true. Not totally true, since Kepler had to make some changes, and Newton had to make some explanation, Laplace had to

make something, Einstein had to do something, there was room for changes. But Copernicus changed our mind. He was the one who said: ... sit on the sun and look at the world. He could do it at his desk in his cell. He sat on the sun and looked, and he saw.

Don't you think there is a lesson to learn, that the whole of humanity can be wrong and one person not? If you are carried by tradition, well, nobody will say anything, nobody will notice if you put the brakes on, and stop and say why, why am I going that way? You see, all the foundations of mathematics, as we know them in the literature, are based on the axioms that Peano, Hilbert, Landau, and so on, have made known to us. And, these foundations of mathematics, I say, are wrong. Why are they wrong? You see, these names – I didn't add Russell, I didn't add the great names of this country, Whitehead and others – well, if you add them, they are all wrong together. *(laughter)*

They are all wrong together because they only worked on the content of their minds. They didn't work on the reality, the reality of the minds of learners. They weren't concerned with that, they were concerned with writing a set of axioms and then to use the machinery and obtain the consequences which we have integrated into our curriculum. They were so little concerned with reality that they did not notice that counting is a complex activity. We have, for centuries, taught people, by offering counting as the basis of elementary arithmetic. It's wrong! Shall I say it louder? It's wrong. Not because I say so, but because counting is a complex activity. It's a complex activity asking of children more than is required in order to give them a better foundation.

You all know that it takes a little time for little children to say: one, two, three, four, five...They say: one, seven, three, two, four. Don't they sometimes say this? And after a while they say: one, two, three... And you are very very happy that they say

one, two, three, like you. But you didn't ask yourselves: why do they say: one, seven, three, two? Why? It happens so often; so many people know that their own children do that. But they don't stop to ask the question. Once the children get it right – which means they do what everybody does – then we are satisfied. It doesn't strike anyone that they do it, that they have spent some time sorting things out by themselves. Does it make you wonder, how do they work, these children? How do they use their minds in order to get what we are going to build our arithmetic on?

When I travelled in Europe, I discovered that in Czechoslovakia, you teach the first year up to 5, the second up to 10, the third up to 20. In Yugoslavia, you never mention numbers until children are seven. So that they don't know that they live at 25 King's Road, or 28 King's Road. They are not allowed, you see. And when you go to other countries, you find the same thing. In the preface of Piaget's *Child's Conception of Number*, he says children keep learning integers every year up to the age of six. Nonsense, nonsense, my child, at the age of three, would bring the parchisi set and say: can we play it in base 3? If you are not interested in understanding how the minds of your students work, maybe it's time to start. Maybe it's time for you to ask questions that will help you.

Have you ever noticed that children learn to speak their mother tongue by themselves? And that you are evading questions in saying: they do it by imitation. 'By imitation', indeed. The greatest nonsense, I have ever heard, and everybody repeats it. It's absolutely wrong. No-one can learn to speak the mother tongue by imitation. So, you have to ask the question: how did we, because we were babies, how did we learn our mother tongue? What sort of powers of the mind did we have to sort these things out by themselves?

These questions are possible. They were possible for me, they are possible for you. Jerome Bruner wanted to study how young

children learn. So, he interviewed mothers. Can you imagine? Can you imagine wanting to learn how children learn, and saying it must be the mother who teaches? And then he wanted, with his assistants, to spend a lot of time in knowing how mothers connect with their children. And the problem – the challenge – he ignored. It did not occur to him to ask: how did I learn to speak my language?

The answer that you all give is: I can't remember; therefore, I can't do this study. True, you can't remember. But it's not true that you can't do the study. If only you could, in your work, get the support, of the reality of how we, as very young children, as babies, have used our powers, our human powers. If you relate to them, in the proper manner, then the future will be so wonderful. You'll see that all the things that are oppressive, creating depression in so many teachers, burning them out and so on, have no reason to be there.

There's a Copernican revolution to be made. Let us begin with the learner, not with the teacher and the teachers' courses, or the regulations and the curriculum. Let's begin with the learners. And you don't all have to start at birth...You can be with children at any stage and you will discover things about learning which are going to help you more than you can believe today, more than you can hope for tomorrow, or ever know, until you do it again and again with different groups on different subjects in different ways. It's a whole area that belongs to education: to reach the learning process in human beings.

You know what they did in Harvard, Yale, Edinburgh, and perhaps London? They went to animals to learn about human learning. They found somethings that animals do and then they had the problem, the very difficult problem: how does it relate to what humans do? But if you start with humans and you say: I am sure of one thing, and that is that everybody, who is not deaf, or has some other trouble, learns to speak the language. Whatever it is...any

language can be learned by little ones. They must do some things with their mind. They must have the mind to do those things. And since the parents don't know what they are, the parents can't teach their little children. If they did know, they wouldn't know how to teach it. Then children would not speak, just as they don't learn to read or write or do arithmetic. *(laughter)*

I'm trying to say very few things and I hope that one out of all of you will take it seriously and you will look into what is the reality of human learning, that, in which I, you, everyone, has been involved. Why is it that I didn't speak after I was born? My environment was speaking all the time, so it can't be the environment that does it. Why is it that I spent months sorting things out? With what did I sort things out? Mothers are satisfied to say, when children make a mistake: I tell them again and again and again what they should say. Well, they can say that, you say it, every day. You repeat so many times what you do in your teaching. The result is, the students get bored, they don't listen to you anymore. Not that they learn better because you repeat...

You are prepared to say: if people are serious and equip themselves in the proper manner and give themselves special instruments, we can know a great deal about this and that – in physics, chemistry, biology and so on. You are prepared to say that. Why aren't you prepared to say, if we equipped ourselves in a certain way, we would be able to enter the challenges knowing what learning really is?

One of the instruments, that you all have, is to ask questions. You don't need microscopes or telescopes... You only need to open up your mind, to say: I can't remember therefore I can't recall it, I'm going to find out what it is – and I'm going to find out in truth what it is. And slowly, you educate yourselves. Slowly you find out that no baby walks before he or she can learn to stand. No baby runs until he or she can walk. There are hierarchies in time.

Why is it that the child who learnt to stand does not stand like a statue the rest of his life? He has achieved something extraordinary. And instead he says, well I know how – what – I have done. I am going to use this to conquer the next stage, and the next stage is to walk. And once I have learnt to walk, I am not going to walk the rest of my life, I'm going to recognise that there is a component of my walk which is called speed, and I'm going to speed it up. And the result will be that something new happens and I can...run.

This is the state of mind of the learners, that they don't rest after each mastery. They aim at mastery: when they have it, they use it to conquer the next thing. This is what human learning is. It's already found in some animals in some areas, but human beings do it all the time. They go on and on doing this same work on something till they master it. One they master it, they apply it, they transfer it. This is the important thing about human learning that has allowed us to go on and on finding more of what is ours, what belongs to our capabilities, our mind...

Once at my desk in Addis Ababa in 1957, I blushed, I was so ashamed of myself. 1957, twenty years after I got my doctorate in mathematics, I understood what we do when we add two fractions... I did not know that to add two fractions involves addition. I said it but I didn't know it. I could write it, I could get the answer, but I didn't know what it meant to add two fractions. And suddenly, I realised that, whenever I have pears and apples, two pears and three apples, I don't have five apples or five pears. I have something altered, I have five pieces of fruit. So why did I do that? Because I wanted to find how to get them together, I had to raise myself to another level where the pears, pearness and appleness are replaced by fruitness. And at that moment, I can say five. And I never realised that 'common denominator' meant 'give the same name' to both. And in the middle of the word 'denominator' I see a French word 'nom' which I knew very well.

It didn't strike me, ever, that it is addition that forces me to get denominators, common denominators, not fractions. That was my shame...

The logicians believe that there is only one 1, and how could they make 2, with one 1?...they say: we add 1 to itself...and so I go and repeat to my students there is only one 1, and you add 1 to itself.. when in fact you need bags of 1's (*laughter*), bags of 2's and bags of 3's. You know you'll be able to move ahead, to take any step. And if you have bags in which there are as many as you would ever need, you don't want. But if you only had one 1: what a life.

When I was young, when I was reading the works of so-called great men, great mathematicians and so on, I was impressed by what they said. So, I know that one can be impressed by what one reads. But this is no reason to punish the youngsters who come to us by trying to get them impressed. Well, when you work on these questions with an open mind, you stop being intimidated, you stop being carried away. You recover your independence, your autonomy. And you know that what you have to do, what you're doing belongs to the challenge, it's a discipline you have to give yourself – a spiritual discipline. This is part of the education of teachers. To make them know that, without spiritual discipline, they can't be free – they can't free themselves. Once they discover how to free themselves, of this or that, they see that there is a reality which can be presented to others in the way they have received it...so I would say, if I have criteria, my job is to make them have criteria, not knowledge. Knowledge will come out of that....

I wrote in the late fifties, early sixties, a series of textbooks for children. And, I suppose some teachers saw them, but their children never saw them. They are for children: they contain what I know about children's powers. And, I try to help them, by using their powers: but I finish the first 6 years of the course in two? But if you are at all touched by truth, find out. Find out if it is

possible that it only takes two years to be master of the arithmetic.

(at this stage various remarks were made inviting comment on what had been said, and taking up some further issues)

The challenge is still there. Is it true? Is it true that if the mind is clear, if the children know what they are doing, it goes fast, they remember everything forever? Is it true? Find out.

There are a number of articles in MT over the last fifteen years which tell a number of these things. If you still have the collection, try to find out by reading. Find whether these articles, that were written for you, have any power or inspiration. This is the way of teaching, to inspire, not to inform. Don't forget that, when you stand in front of your children. Unless they have already got certain experiences, your height is the justification for their worship. They think you are strong. And it is reinforced when they think that you have so much knowledge, and they don't. So, there is a discrepancy – a spiritual difference – between where you stand and where they stand. And this difference, by itself, creates the inspiration. This is why it is what you are that matters in teaching, not what you know...

Thirty-seven, or thirty-six, years ago, we were a handful when we started. A few months later, we were a platoon. A year later, perhaps we were a company. Now you are a brigade. You can do so much more. By your numbers alone, you can do so much more to transform the teaching of mathematics.

Goodnight. (*applause*)

From Gattegno Anthology (ATM, 1988) and audio clip https://www.atm.org.uk/write/MediaUploads/About/History_Gattegno/It_only_takes_two_years_to_be_master_of_arithmetic_1m01s_1.2Mb.mp3

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- Electronic access to thousands of online MT journal articles
- Access to all online member-only resources
- Professional support and enrichment – being part of a community where ideas are generated and shared
- Regular ATM e-newsletters, containing current news and activities
- A network of local branches offering regular meetings
- Accreditation - ATM is proud to offer members the opportunity to apply for the CMathTeach Designation, making ATM membership the route to Chartered Mathematics Teaching status
- Influence and having a voice - eligibility to vote on resolutions that shape the direction of ATM

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