Gauss Jordan Method
&
Inverse of Matrix

3 Variables SLEs in Matrix Form

$a11x + a12y + a13z = b1$
$a21x + a22y + a23z = b2$
$a31x + a32y + a33z = b3$

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

A X = B

Gauss Jordan Method,

Augmented Matrix,

$$[A/B] = \begin{bmatrix} a11 & a12 & a13 & b1 \\ a21 & a22 & a23 & b2 \\ a31 & a32 & a33 & b3 \end{bmatrix} \begin{matrix} \rightarrow R1 \\ \rightarrow R2 \\ \rightarrow R3 \end{matrix}$$

R1 → R1/a11, $[A/B] \cong \begin{bmatrix} 1 & a12' & a13' & b1' \\ a21 & a22 & a23 & b2 \\ a31 & a32 & a33 & b3 \end{bmatrix}$

R2 → R2 – a21R1, $$[A/B] \cong \begin{bmatrix} 1 & a12' & a13' & b1' \\ 0 & a22' & a23' & b2' \\ a31 & a32 & a33 & b3 \end{bmatrix}$$

R3 → R3 – a31R1, $$[A/B] \cong \begin{bmatrix} 1 & a12' & a13' & b1' \\ 0 & a22' & a23' & b2' \\ 0 & a32' & a33' & b3' \end{bmatrix}$$

R2 → R2/a22’, $$[A/B] \cong \begin{bmatrix} 1 & a12' & a13' & b1' \\ 0 & 1 & a23'' & b2'' \\ 0 & a32' & a33' & b3' \end{bmatrix}$$

R3 → R3 – a32’R2, $$[A/B] \cong \begin{bmatrix} 1 & a12' & a13' & b1' \\ 0 & 1 & a23'' & b2'' \\ 0 & 0 & a33'' & b3'' \end{bmatrix}$$

Case 1 $a33'' \neq 0$ **Unique Solution**

$R1 \rightarrow R1 - a12'R2,$

$R3 \rightarrow R3/a33'',$

$$[A/B] \cong \begin{bmatrix} 1 & 0 & a13'' & b1'' \\ 0 & 1 & a23'' & b2'' \\ 0 & 0 & 1 & b3''' \end{bmatrix}$$

$R1 \rightarrow R1 - a13''R3,$

$R2 \rightarrow R2 - a13''R3,$

$$[A/B] \cong \begin{bmatrix} 1 & 0 & 0 & b1''' \\ 0 & 1 & 0 & b2''' \\ 0 & 0 & 1 & b3''' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b1''' \\ b2''' \\ b3''' \end{bmatrix}$$

$x = b1'''$ $y = b2'''$ $z = b3'''$

Case 2 $a33'' = 0$ $b3'' \neq 0$ $r(A) = 2$ & $r(A/B) = 3$ $r(A) \neq r(A/B)$ **No Solution**

Case 3 $a33'' = 0$ $b3'' = 0$ $r(A) = r(A/B) = 2$ **Infinitely Many Solutions**

Example 1 Test the Consistency and Solve the following SLEs using Gauss Jordan Method if possible:

$$x + y + z = 6$$
$$x - y + 2z = 5$$
$$3x + y + z = 8$$

In Matrix form SLEs is, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$ ⇨ **A X = B**

Augmented Matrix, $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix}$

R2 → R2 – R1, $[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 3 & 1 & 1 & 8 \end{bmatrix}$

R3 → R3 – 3R1, $[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix}$

R3 → R3 – R2, $[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{bmatrix}$

R2 → R2/(-2), $[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & -3 & -9 \end{bmatrix}$

r(A) = r(A/B) = 3 = n, Therefore, System is Consistent & It has Unique Solution.

R1 → R1 – R2,

R3 → R3/-3,

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/2 \\ 1/2 \\ 3 \end{bmatrix}$$

R1 → R1 – 3/2 R3,

R2 → R2 + ½ R3,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

x = 1 y = 2 z = 3

Therefore, Solution of the system is unique and it is (x, y , z) = (1, 2, 3).

Example 2 Test the Consistency and Solve the following SLEs using Gauss Jordan Method if possible:

$$3x + 2y - 5z = 4$$
$$x + y - 2z = 1$$
$$5x + 3y - 8z = 6$$

In Matrix form SLEs is, $\begin{bmatrix} 3 & 2 & -5 \\ 1 & 1 & -2 \\ 5 & 3 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$ ⇨ **A X = B**

Augmented Matrix, $[A/B] = \begin{bmatrix} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{bmatrix}$

R1 $\leftrightarrow$ R2, $$[A/B] \cong \begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & -5 & 4 \\ 5 & 3 & -8 & 6 \end{bmatrix}$$

R2 $\rightarrow$ R2 – 3R1, $$[A/B] \cong \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 5 & 3 & -8 & 6 \end{bmatrix}$$

R3 $\rightarrow$ R3 – 5R1, $$[A/B] \cong \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 2 & 1 \end{bmatrix}$$

R2 $\rightarrow$ R2/(-1), $$[A/B] \cong \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix}$$

R3 → R3 + 2R2, $$[A/B] \cong \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

r(A) = 2 & r(A/B) = 3 ∴r(A) ≠ r(A/B)

Therefore, System is inconsistent & It has no solution.

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

by row (3), 0 z = -1

0 = -1 which Is not possible

Therefor, Solution of given SLE is not possible.

Example 3 Test the Consistency and Solve the following SLEs using Gauss Jordan Method if possible:

$$2x + 2y + 2z = 0$$
$$-2x + 5y + 2z = 1$$
$$8x + y + 4z = -1$$

In Matrix form SLEs is, $\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ ⇒ **A X = B**

Augmented Matrix, $[A/B] = \begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$

R1 → R1/2, $$[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

R2 → R2 + 2R1, $$[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

R3 → R3 – 8R1, $$[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$

R3 → R3 + R2, $$[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R2 → R2/7, $[A/B] \cong \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

r(A) = r(A/B) = 2 < n,

Therefore, System is Consistent & It has Infinitely Many Solutions.

R1 → R1 – R2, $[A/B] \cong \begin{bmatrix} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

z = k, k ϵ R

y + 4/7 z = 1/7 ⇨ y = (1/7) - (4/7) (z) = (1/7) - (4/7) (k) = $\frac{1}{7} - \frac{4}{7}k$, k ϵ R

x + 3/7 z = -1/7 ⇨ x = $-\frac{1}{7} - \frac{3}{7}k$

∴(x, y , z) = {($-\frac{1}{7} - \frac{3}{7}k$, $\frac{1}{7} - \frac{4}{7}k$, k) / k ϵ R}.

Do you think there is a need of new method for solving Inverse of a Matrix?

1. Yes
2. No

VOTE INDIVIDUALLY IN CHAT BOX [30 sec]

Would you like to use Adjoint Method to find inverse of 4^{th} Order and higher order Matrix?

1. Yes

2. No

VOTE INDIVIDUALLY IN CHAT BOX [30 sec]

Inverse of a Matrix by Gauss Jordan Method

The inverse of an $n \times n$ matrix **A** is an $n \times n$ matrix **B** having the property that

AB = BA = I

$[A / I] \xrightarrow{\text{RREF}} [I / A^{-1}]$

B is called the *inverse* of **A** and is usually denoted by $\mathbf{A}^{-1}$.

If a square matrix has no zero rows in its Row Echelon form or Reduced Row Echelon form then inverse of Matrix exists and it is said to be *invertible* or *nonsingular Matrix*.

If Row Echelon form and Reduced Row Echelon form of Matrix possess zero row then inverse of Matrix does not exist and the Matrix is said to be *singular*.

One can solve System of Linear Equations **AX = B** if Inverse of A exists, which yields $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Example 1 Find the Inverse of Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ **using Gauss Jordan Method if possible.**

Let Augmented Matrix be, $[A/I] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$

R2 → R2 – R1,
R3 → R3 – 3R1,

$$[A/I] \cong \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -2 & -2 & -3 & 0 & 1 \end{bmatrix}$$

R3 → R3 – R2,
R2 → R2/(-2),

$$[A/I] \cong \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{bmatrix}$$

As given square matrix A has no zero rows in its Row Echelon form or Reduced Row Echelon form the inverse of Matrix exists.

R1 → R1 – R2,
R3 → R3/-3,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & -1/3 \end{bmatrix}$$

R1 → R1 – 3/2 R3,
R2 → R2 + ½ R3,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 5/6 & -1/3 & -1/6 \\ 0 & 0 & 1 & 2/3 & 1/3 & -1/3 \end{bmatrix} = [I / A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Example 2 Solve the following SLEs using Matrix Inversion Method if possible:

$x + y + z = 6$

$x - y + 2z = 5$

$3x + y + z = 8$

OR

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

$\Rightarrow$ **A X = B**

$\Rightarrow$ **X = A^{-1} B**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ **x = 1** **y = 2** **z = 3**

Example 3 Find the Inverse of Matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$ **using Gauss Jordan Method if possible.**

$[A/I] = \begin{bmatrix} 2 & 2 & 2 & 1 & 0 & 0 \\ -2 & 5 & 2 & 0 & 1 & 0 \\ 8 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$ **R1 → R1/2,** $[A/I] \cong \begin{bmatrix} 1 & 1 & 1 & 1/2 & 0 & 0 \\ -2 & 5 & 2 & 0 & 1 & 0 \\ 8 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$

R2 → R2 + 2R1,
R3 → R3 – 8R1, $[A/I] \cong \begin{bmatrix} 1 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 7 & 4 & 1 & 1 & 0 \\ 0 & -7 & -4 & -4 & 0 & 1 \end{bmatrix}$

R3 → R3 + R2,
R2 → R2/7, $[A/I] \cong \begin{bmatrix} 1 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 4/7 & 1/7 & 1/7 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{bmatrix}$

As given square matrix A has zero row in its Row Echelon form or Reduced Row Echelon form the inverse of Matrix does not exist.

Example 4 Find the Inverse of Matrix $A=\begin{bmatrix}1&3&3&1\\2&3&4&4\\2&2&1&4\\1&5&5&1\end{bmatrix}$ **using Gauss Jordan Method if possible.**

Let Augmented Matrix be, $[A/I]=\begin{bmatrix}1&3&3&1&1&0&0&0\\2&3&4&4&0&1&0&0\\2&2&1&4&0&0&1&0\\1&5&5&1&0&0&0&1\end{bmatrix}$

R2 → R2 – 2R1,
R3 → R3 – 2R1,
R4 → R4 – R1,

$$[A/I]\cong\begin{bmatrix}1&3&3&1&1&0&0&0\\0&-3&-2&2&-2&1&0&0\\0&-4&-5&2&-2&0&1&0\\0&2&2&0&-1&0&0&1\end{bmatrix}$$

R2 → R2/ –3,

$$[A/I]\cong\begin{bmatrix}1&3&3&1&1&0&0&0\\0&1&2/3&-2/3&2/3&-1/3&0&0\\0&-4&-5&2&-2&0&1&0\\0&2&2&0&-1&0&0&1\end{bmatrix}$$

R1 → R1 – 3R2,
R3 → R3 + 4R2,
R4 → R4 – 2R2,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & -2/3 & 2/3 & -1/3 & 0 & 0 \\ 0 & 0 & -7/3 & -2/3 & 2/3 & -4/3 & 1 & 0 \\ 0 & 0 & 2/3 & 4/3 & -7/3 & 2/3 & 0 & 1 \end{bmatrix}$$

R3 → 3R3/–7,
R4 → 3R4,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & -2/3 & 2/3 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 2/7 & -2/7 & 4/7 & -3/7 & 0 \\ 0 & 0 & 2 & 4 & -7 & 2 & 0 & 3 \end{bmatrix}$$

R1 → R1 – R3,
R2 → R2 - 2R3/3,
R4 → R4 – 2R3,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 0 & 19/7 & 5/7 & 3/7 & 3/7 & 0 \\ 0 & 1 & 0 & -6/7 & 6/7 & -5/7 & 2/7 & 0 \\ 0 & 0 & 1 & 2/7 & -2/7 & 4/7 & -3/7 & 0 \\ 0 & 0 & 0 & 24/7 & -45/7 & 6/7 & 6/7 & 3 \end{bmatrix}$$

R4 → 7R4/24,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 0 & 19/7 & 5/7 & 3/7 & 3/7 & 0 \\ 0 & 1 & 0 & -6/7 & 6/7 & -5/7 & 2/7 & 0 \\ 0 & 0 & 1 & 2/7 & -2/7 & 4/7 & -3/7 & 0 \\ 0 & 0 & 0 & 1 & -15/8 & 2/8 & 2/8 & 7/8 \end{bmatrix}$$

As given square matrix A has no zero rows in its Row Echelon form or Reduced Row Echelon form the inverse of Matrix exists.

R1 → R1 – 19R4/7,
R2 → R2 + 6R4/7,
R3 → R3 – 2R4/7,

$$[A/I] \cong \begin{bmatrix} 1 & 0 & 0 & 0 & 35/8 & -2/8 & -2/8 & -19/8 \\ 0 & 1 & 0 & 0 & -6/8 & -4/8 & 4/8 & 6/8 \\ 0 & 0 & 1 & 0 & 2/8 & 4/8 & -4/8 & -2/8 \\ 0 & 0 & 0 & 1 & -15/8 & 2/8 & 2/8 & 7/8 \end{bmatrix} = [I / A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{35}{8} & -\frac{2}{8} & -\frac{2}{8} & -\frac{19}{8} \\ -\frac{6}{8} & -\frac{4}{8} & \frac{4}{8} & \frac{6}{8} \\ \frac{2}{8} & \frac{4}{8} & -\frac{4}{8} & -\frac{2}{8} \\ -\frac{15}{8} & \frac{2}{8} & \frac{2}{8} & \frac{7}{8} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 35 & -2 & -2 & -19 \\ -6 & -4 & 4 & 6 \\ 2 & 4 & -4 & -2 \\ -15 & 2 & 2 & 7 \end{bmatrix}$$

Next Lecture : All Different Problems of Unit 1