GCSE Specification and Revision Notes

Last Modified: 10/04/2015

The left column is the complete Edexcel Mathematics A (1MA0) specification. A few items I have merged together (where there was duplication). A few items I have created, either because they weren't explicitly referenced in the specification (e.g. proof), or where I felt a few sub-subtopics deserved an item of their own (e.g. simplifying algebraic fractions).

The second column contains notes I have written and 'Test Your Understanding' questions (a mixture of past paper questions and my own). Use the last column to tick items that you feel you have fully grappled.

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General Tips:

1. You MUST show full workings for each answer. 'Method marks' can usually be obtained when your answer is wrong, but not if there are no workings.
2. Do not give answers to anything less than 3 significant figures. Note that 0.0043 is only to 2 significant figures.
3. Be wary about copying errors when going from one line of working to the next. Has a 'minus' accidentally become a 'plus'?
4. Spot when different units have been used in the same problem, and ensure they are converted to the same unit.
5. Don't ever use 'trial and error' for questions where an algebraic approach is expected – you won't get any credit.
6. Take special care when punching numbers into a calculator and copying results off the display.
7. Check your answer looks 'plausible' given the context. If it costs £11500 to seed a garden you've probably gone wrong.
8. Check that you've actually answered the question. Often, once you've calculated the correct value, some 'conclusion' is needed, e.g. "Therefore Bob will not have enough money. He is 50p short."

Common General Algebraic Errors:

- $(a+b)^2 \rightarrow a^2+b^2$. Writing out the bracket twice we actually find $(a+b)(a+b)=a^2+2ab+b^2$
- Similarly $\sqrt{a^2+b^2} \rightarrow a+b$. You can see this is not true when $a=3, b=4$ for example.
- $\frac{x^2+3x+2}{x^2-4} \rightarrow \frac{3x+2}{-4}$. When 'cancelling' fractions, we can only divide, whereas in this example we've incorrectly *subtracted* x^2. If we factorised the example, it would be OK to cancel $\frac{(x+1)(x+2)}{(x+2)(x-2)}$ to $\frac{x+1}{x-2}$ because we have indeed *divided* by $x+2$.
- $x(x-1) \rightarrow x^2-1$. Oops!
- $\frac{x}{3}+a=y \quad \rightarrow \quad x+a=3y$. The a hasn't been multiplied by 3.
- $c-b(a-b) \quad \rightarrow c-ab-b^2$. Sign error at the end.
- $x(x+1)-(x+2)^2 \rightarrow \ x^2+x-x^2+4x+4$. A lack of brackets when subtracting expanded expression leads to sign errors. See (53i).
- $a+3x=b \ \rightarrow \ 3x=b+a$ Sign not changed when a moved to other side of equation.
- $\frac{x+2b}{3}=y \rightarrow \frac{x}{3}=y-2b$ or $\sqrt{x+2b}=y \rightarrow \sqrt{x}=y-2b$ ($2b$ is trapped inside fraction/root so we have to deal with the $\div 3$ and $\sqrt{}$ first)
- $\sqrt{x}=2x \ \rightarrow \ x=2x^2$ When $2x$ is squared, you get $4x^2$ not $2x^2$ as $2x \times 2x = 4x^2$

Number

Integers and Decimals		
1. Understand and order integers and decimals	**Test Your Understanding:** Order the following: $\frac{1}{3}, 0.33, 0.303, 30\%$	
2. Use brackets and the hierarchy of operations (BIDMAS)	• BIDMAS is actually (B)(I)(DM)(AS), i.e. Division and Multiplication have the same 'priority', and Addition and Subtraction have the same priority. When you have say a mix of addition and subtraction, evaluate left-to-right. E.g. $9 - 3x + 5x$ simplifies to $9 + 2x$ NOT $9 - 8x$: in the latter you had done the addition first, when there was no reason to do so. • Note that due to BIDMAS, negative numbers to a power require bracketing: -4^2 would produce -16 on a calculator because it does the square ('indices') first. You want $(-4)^2$. This is highly important for the b^2 in the Quadratic Formula when substituting numbers in.	
3. Add, subtract, multiply and divide integers, negative numbers and decimals	• When multiplying two decimals, first multiply them as if they were whole numbers, then put the decimal point back in the result by counting the number of jumps in decimal point in the original numbers. • When subtracting negative numbers, ensure numbers are lined up in the units column, and fill in any 'gaps' with 0s. • We like dividing by whole numbers. Hence if you're dividing by a decimal, multiply both numbers by 10 until you're dividing by a whole number, e.g. $3.678 \div 0.09 \rightarrow 367.8 \div 9$ **Test Your Understanding:** a. $3 - (-4)$? b. 396×48? c. $5.7 - 2.89$? d. $(-7.1)^2$? e. $508.97 \div 1.1$	
4. Understand and use positive numbers and negative integers, both as positions and translations on a number line	This just means that you understand $-5 + 7$ for example as starting at -5 on a number line and 'moving'/translating 7 up. This won't specifically be tested.	
5. Round whole numbers to the nearest, 10, 100, 1000, …		
6. Round decimals to appropriate numbers of decimal places or significant figures	Be wary that any 0s after the first non-zero digit count as significant. e.g. 3.40204 to 3sf is 3.40, NOT 3.4. Similarly it is 3.40 to 2dp, not 3.4. But 0.0020413 is 0.00204 to 3sf. Note that 3.9853 to 1dp is 4.0. **Test Your Understanding:** Write the following to the indicated number of significant figures or decimal places: a. 24703 to 2sf b. 15.0849 to 1dp c. 25.969403 to 3sf d. 495.18473 to 3dp	
7. Multiply and divide by any number between 0 and 1	You should appreciate that multiplying by a number between 0 and 1 makes it smaller, and dividing by it makes it bigger. You should recognise that $\div\, 0.5$ is the same as $\times\, 2$ and $\div\, 0.25$ the same as $\times\, 4$ and so on (see Fractions): this is particularly useful in estimation (see below)	
8. Check their calculations by rounding, eg 29 × 31 ≈ 30 × 30	For estimation questions, the general rule of thumb is to round each number to 1 significant figure, unless it is close to some other nice number (such as $0.26 \rightarrow 0.25$ because it's a quarter), e.g: $$\frac{4.98 \times 31}{0.49} \approx \frac{5 \times 30}{0.5} = 300$$ **Test Your Understanding:** Estimate the following. a. $\frac{79 \times 6.89}{0.52}$ b. $\frac{4.03 \times 4.94}{0.24}$	
9. Check answers to a division sum using multiplication eg use inverse operations	Not tested as such.	
10. Multiply and divide whole numbers by a given multiple of 10		
11. Put digits in the correct place in a decimal number		

Fractions		
12. Find equivalent fractions and write a fraction in its simplest form.	**Test Your Understanding:** Put $\frac{45}{54}$ in its simplest form.	
13. Compare the sizes of fractions	The strategy is to find a common denominator for all fractions, so that we can just easily compare the numerators, e.g. $\frac{5}{7}$ and $\frac{3}{4}$ can be converted to $\frac{20}{28}$ and $\frac{21}{28}$, thus $\frac{3}{4}$ is larger. Sometimes the ordering will be obvious by thinking of the fractions on a number line. **Test Your Understanding:** Which of $\frac{4}{5}$ and $\frac{5}{7}$ is bigger?	
14. Find fractions of an amount	**Test Your Understanding:** Find $\frac{3}{7}$ of 35	
15. Convert between mixed numbers and improper fractions	For mixed number to improper fractions, to get the new numerator times the whole part with the denominator and add the numerator. The denominator stays the same. e.g $3\frac{2}{5} \rightarrow \frac{17}{5}$ For improper fractions to mixed numbers, see how many times the denominator goes into the numerator and find the remainder also. $\frac{39}{7} \rightarrow 5\frac{4}{7}$	
16. Add and subtract fractions	The 'foolproof' way is to cross-multiply: multiply the two denominators, then times each numerator by the other fraction's denominator and add. e.g. $\frac{3}{7}+\frac{1}{4}=\frac{12+7}{28}=\frac{19}{28}$. However, sometimes you only need to change one of the fractions, e.g. $\frac{1}{3}+\frac{4}{9} \rightarrow \frac{3}{9}+\frac{4}{9}=\frac{7}{9}$ **Test Your Understanding:** a. $\frac{4}{9}+\frac{3}{5}$ b. $1\frac{4}{7}-\frac{1}{2}$ c. $5\frac{1}{4}+4\frac{4}{5}$	
17. Multiply and divide fractions including mixed numbers.	If any whole numbers, put over 1: $4 \rightarrow \frac{4}{1}$. If any mixed numbers, convert to improper fractions first. When multiplying, just multiply numerators and denominators separately. When dividing, 'flip' (reciprocate) the <u>second</u> fraction and instead multiply. **Test Your Understanding:** a. $\frac{3}{5}\times\frac{4}{9}$ b. $2\frac{4}{5}\times 3$ c. $1\frac{1}{4}\div 2\frac{1}{5}$ d. $4 \div 1\frac{1}{5}$	

Factors, Multiples, Primes, Roots, Powers		
18. Identify factors, multiples and prime numbers	Ensure you don't confuse factors and multiples. Factors of 6: 1, 2, 3, 6. Multiples of 6: 6, 12, 18, 24, …	
19. Find the prime factor decomposition of positive integers	Use a prime factor tree. Don't forget to write $\times$ between each prime factor. Try to collect the same prime factors together using power notation. e.g. $120 = 2\times 2\times 2\times 3\times 5$, but it would be better to write $2^3\times 3\times 5$ **Test Your Understanding:** Express 240 as the product of its prime factors.	
20. Find the common factors and common multiples of two numbers	See below.	
21. Find the Highest Common Factor (HCF) and the Least Common Multiple (LCM) of two numbers	There are two ways you could find the Lowest Common Multiple: • Write out multiples of the larger number until you see a multiple of the smaller number. e.g. for 60 and 54, write out multiples of 60: 60, 120, 180, 240, 300, 360, 420, 480, 540. And 540 is the first multiple of 54 in this list so is the LCM. • Find the prime factorisation of each number. Then see which things 'wins' in each factor. E.g. $60 = 2^2\times 3\times 5$ and $54 = 2\times 3^3$. Of the 2s the 2^2 'wins' over the 2 so we use 2^2. The 3^3 'wins' over the 3. And the '5' beats nothing. So the LCM is $2^2\times 3^3\times 5 = 540$. To find the HCF: • Write out the factors of each number, and look for the highest number in both lists. • Alternatively you can again use the prime factorisation of each number, but this time see which factor 'loses' (where 'nothing' loses against anything). So in $60 = 2^2\times 3\times 5$ and $54 = 2\times 3^3$, the 2 loses, the 3 loses and 'nothing' loses against 5, so the HCF is $2\times 3 = 6$. The previous method is faster however.	

	More Notes: LCM questions often come in an applied setting. e.g. Two buses come every 5 minutes and 7 minutes respectively: if they both come now, when will they next both come at the same time? (i.e. in 35 minutes time) **Test Your Understanding:** a. Find the Lowest Common Multiple of 120 and 96. b. Find the Highest Common Factor of 48 and 60? c. Two train services both arrive at 9am. The first comes every 20 minutes. The second comes every 25 minutes. What is the time when the two train services both come at the same time? d. Cookies come in packs of 12 and chocolate bars in packs of 9. I want to have the same number of cookies and chocolate bars. What's the smallest number of packs of cookies can I buy?	
22. Recall integer squares from 2 × 2 to 15 × 15 and the corresponding square roots. Recall the cubes of 2, 3, 4, 5 and 10.	This is just saying you should know these of-by-heart. Of course, in exam, you could work out 14^2 say by long multiplication, but it's highly useful to memorise these if you haven't already.	
23. Use standard form, expressed in conventional notation. Be able to write very large and very small numbers presented in a context in standard form. Convert between ordinary and standard form representations. Convert between ordinary and standard form representations. Interpret a calculator display using standard form	When the power of 10 is negative (i.e. we have a small number), the number of 0s on the front (including the one before the decimal place) is the number in the index. So $0.00147 = 1.47 \times 10^{-3}$ as there's 3 leading 0s. But if ever in doubt count the number of times the decimal place has to move until you get to a number between 1 and $9.\dot{9}$. When converting a number not in standard form, if you divide the first part by 10, you need to multiply the second part by 10 (i.e. by increasing the power by 1) to cancel out the effect. When adding or subtracting two numbers in standard form, it's often easiest to just convert the numbers into normal numbers, then add/subtract, then convert back to standard form. Example: $(3 \times 10^7) \times (4 \times 10^2) = 12 \times 10^9 = 1.2 \times 10^{10}$ $(2 \times 10^5) \div (8 \times 10^9) = 0.25 \times 10^{-4} = 2.5 \times 10^{-5}$ $(1.2 \times 10^5) + (3 \times 10^4) = 120000 + 30000$ $= 150000 = 1.5 \times 10^5$ Test Your Understanding: a. Express 367 000 in standard form. b. Express 0.00048 in standard form. c. Express 2.67×10^{-2} as an ordinary number. d. Express 5.2×10^3 as an ordinary number. e. Calculate $(5.2 \times 10^3) \times (2 \times 10^7)$ giving your answer in standard form. f. Calculate $(2.4 \times 10^8) \div (4 \times 10^5)$ giving your answer in standard form. g. Calculate $(4 \times 10^5) + (3 \times 10^4)$	
24. Use index laws to simplify and calculate the value of numerical expressions involving multiplication and division of integer powers, and powers of a power *(Covered later)*		

Fractions, Decimals and Percentages		
25. Understand that a percentage is a fraction in hundredths	e.g. $34\% = \frac{34}{100}$	
26. Convert between fractions, decimals and percentages	To convert (non-recurring) decimals to fractions, just put over 10/100/1000 depending on how many digits are after the decimal point, then simplify. e.g. $0.56 = \frac{56}{100} = \frac{14}{25}$ and $0.035 = \frac{35}{1000} = \frac{7}{200}$ **Test Your Understanding:** a. Convert 0.84 to a fraction in its simplest form. b. Express $\frac{3}{8}$ as a percentage.	
27. Convert between recurring decimals and exact fractions as well as understanding the proof	To convert from a fraction to recurring decimal, just use long division. You will need to add ".00000" to the number you're dividing to get extra digits for which to put your remainders on. Make sure you put the decimal point in the same place on the result, and stop once you see the repetition in digits.	

	Example: Convert $\frac{3}{7}$ to a recurring decimal. $$\begin{array}{r} 0\,.\,4\;2\;8\;5\;7\;1\;4 \\ 7\,\overline{\big)\,3\,.\,{}^{3}0\;{}^{2}0\;{}^{6}0\;{}^{4}0\;{}^{5}0\;{}^{1}0\;{}^{3}0} \end{array}$$ Thus $\frac{3}{7} = 0.\dot{4}2857\dot{1}$ To convert from recurring decimal, do the following. e.g. For $0.2\dot{3}\dot{4}$ 1. Write $x =$ your number with the repeating digits written out explicitly. $$x = 0.234343434 \ldots.$$ 2. See how often your digits a repeating. If it's just 1 repeating digit, times by 10, if 2, times by 100, if 3, times by 1000 and so on. $$100x = 23.43434343 \ldots$$ 3. Subtract the first equation from the second. If you lined up the decimal points on your first two lines of working, this will make the subtraction easier. $$99x = 23.2$$ (Noting that everything from the second digit after the decimal place onwards is the same) 4. Divide to find x. If you have a decimal in the fraction, times by 10 until it's a whole number. Simplify if the question asked you to. $$x = \frac{23.2}{99} = \frac{232}{990} = \frac{116}{495}$$ **Test Your Understanding:** a. Convert $\frac{3}{16}$ to a decimal. b. Convert $\frac{3}{11}$ to a recurring decimal. c. Convert $\frac{6}{7}$ to a recurring decimal. d. Convert $0.\dot{4}$ to a fraction. e. Convert $0.\dot{4}0\dot{1}$ to a fraction. f. Convert $0.6\dot{3}1\dot{5}$ to a fraction.	
28. Write one number as a percentage of another number	Just find the proportion one number is of the other, and we times by 100 to convert a fraction to a percentage. **Example:** Express 38 as a percentage of 70. $\frac{38}{70} \times 100 = 54.3\%$	
29. Calculate the percentage of a given amount	The method depends on whether or not you have a calculator. • Calculator Method: Convert the percentage to a decimal then multiply. e.g. 38% of 40 = $0.38 \times 40 = 15.2$ • Non-Calculator Method: Find more manageable chunks such as 10%, 5%, 1% and combine as necessary. e.g. 35% of 60? 10% = 6 thus 30% = 18 and 5% = 3. Then 35% = 21 **Test Your Understanding**: Without using a calculator, determine 46% of 360.	
30. Find a percentage increase/decrease of an amount	If not using a calculator, just find the percentage of this number, and add or subtract as necessary. If using a calculator, you should identify an appropriate 'decimal multiplier' and times by this. e.g. Find 40%: $\times 0.4$ Increase by 5%: $\times 1.05$ Decrease by 25%: $\times 0.75$ **Example:** Find the cost of a £14 tshirt after a 2.5% rise: $£14 \times 1.025 = £14.35$ **Test Your Understanding:** a. One bank account offers 5% interest in your first year followed by 1% in the second. Another bank account offers 2% interest followed by 3% interest the next year. If I am investing my money for the two years, which bank account should I choose?	
31. Find a reverse percentage, eg find the original cost of an item given the cost after a 10% deduction	It's important for *any* percentage question to first identify whether you're trying to find the *new* amount or the *original* amount. **Example**: A cake is reduced in a sale by 20% to £16. Find the original amount. *Method 1*: Work your way back to 100%. 80% -> £16. Therefore 10% -> £2. Therefore 100% -> £20	

	Method 2: Let the original price be x $$x \times 0.8 = £16$$ $$x = \frac{£16}{0.8} = £20$$ **Test Your Understanding:** a. A hamster is bought, and sold at a 30% profit for £11.70. Find the original cost of the hamster. b. A pink hat is sold in a sale for 25% less at a sale price of £13.50. But how much has the price been reduced from the original?	
32. Use a multiplier to increase by a given percent over a given time , eg 64×1.1^8 increases 64 by 10% over 8 years	Just use appropriate decimal multipliers raised to the correct power. Examples: • The price of a goldfish starts at £300 and rises by 10% for 5 years. What's the final price of the goldfish? $£300 \times 1.1^5 = £483.15$ • A polar bear population starts at 6500 and diminishes at a rate of 5% each year. What's the population after 10 years? $$6500 \times 0.95^{10} = 3892\ bears$$ **Test Your Understanding:** a. The price of a Ferrari falls from £180 000 by 25% each year for 6 years. What is the new value of the car?	
33. Calculate simple and compound interest	Note: Simple interest is where the interest each year is based on the **original** amount. E.g. If £1000 accumulate 3% interest each year for 4 years, you end up with £1000 + (4 x £30) = £1120, whereas with compound interest, you'd have $£1000 \times 1.03^4 = £1125.51$. Notice you get slightly more with compound interest. **Test Your Understanding:** a. A bank account offers 3.5% interest per annum. I invest £3500. How much do I have after 10 years. b. [Harder] The annual healthcare costs of a Labrador dog has fallen by 10% for 5 years so the cost is now £1000. What was the original cost 5 years ago?	

Ratio and Scale		
34. Write ratios in their simplest form	Ratios simplify just like fractions. $6:9 \rightarrow \quad 2:3$	
35. Divide a quantity in a given ratio	For most ratio questions, you need to decide whether the quantity given is (i) the total of the parts (ii) one of the parts or (iii) the difference of the parts. Then find 1 part. **Example:** Alice and Bob shares £40 in the ratio 5:3. How much does Bob get? The £40 represents the 'total' parts. So: $$8\ parts = £40$$ $$1\ part = £5$$ $$3\ parts = £15$$	
36. Solve a ratio problem in a context. Solve word problems about ratio and proportion	**Test Your Understanding:** a. The ratio of cats to dogs in a home is 5:7. There are 56 dogs. How many cats? b. Some money is shared between Alice and Bob in the ratio 4:7. Bob gets £12 more than Alice. How much did Alice get? c. To make magic, you require mixing unicorn horn, fairydust and orphan meat in the ratio 3:5:2. I want to make 5kg of magic. If I have 1600g of unicorn horn, 2400g of fairydust and 1200g of orphan meat, can I make enough magic?	
37. Use and interpret maps and scale drawings. Read and construct scale drawings drawing lines and shapes to scale. Estimate lengths using a scale diagram	Before forming a ratio put the distances in the same unit. **Test Your Understanding:** a. 10cm on a map represents 5km in real life. Represent this scale in the form $1:n$ b. A map scale is 1:20 000. What does 5.4cm on the map represent in real life?	
38. Calculate an unknown quantity from quantities that vary in direct or inverse proportion	First represent the sentence as an equation, using a constant of proportionality k. Write "$= k \times$" for "is directly proportional to" and "$= k \div$" for "is inversely/indirectly proportional to". e.g. "y is directly proportional to the square of x" -> $y = kx^2$ e.g. "y is inversely proportional to the square root of x" -> $y = \frac{k}{\sqrt{x}}$	

Then substitute the given values to work out k, and then use your now full formula (with known k) to work out the answer.

Example: "y is inversely proportional to the cube of x. When $x = 4$, $y = 20$. To 3sf, find y when $x = 7$."

$$y = \frac{k}{x^3} \qquad 20 = \frac{k}{4^3} \rightarrow k = 20 \times 4^3 = 1280$$

When $x = 7$, $y = \frac{1280}{7^3} = 3.73$

Test Your Understanding:

a. q is directly proportional to r. When $q = 5, r = 6$. What is q when $r = 18$?
b. m is indirectly proportional to n. When $m = 5, n = 11$. What is m when $n = 4$?
c. y is directly proportional to the square of x. When $x = 8, y = 10$. What is y when $x = 20$?
d. y is inversely proportional to the square root of x. When $x = 10, y = 10$. What is x when $y = 20$?

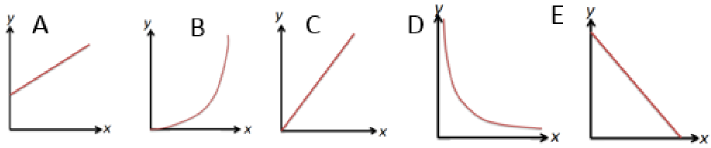
39. Set up and use equations to solve word and other problems involving direct proportion or inverse proportion and relate algebraic solutions to graphical representation of the equations

By "graphical representation", it means a line graph. When y is directly proportional to x, from above, we get the equation $y = kx$. This is the equation of a straight line *that goes through the origin*.

If y is inversely proportion to x then from above, $y = \frac{k}{x}$. This give a reciprocal graph (see equations of graphs in Algebra).

Test Your Understanding: Of these graphs, identify the one where:

a. y is directly proportional to x.
b. y is inversely proportional to x.



Index Notation and Surds

40. Manipulate and simplify expressions using laws of indices.

Laws of indices:

• $a^b \times a^c = a^{b+c}$ • $\frac{a^b}{a^c} = a^{b-c}$ • $a^{-b} = \frac{1}{a^b}$	• $a^0 = 1$ • $a^{\frac{1}{n}} = \sqrt[n]{a}$

Examples:

- Simplify $\frac{x^7}{x^{-4}}$. x^{11} (as 7 - -4 = 11)
- Simplify $(m^{-2})^5$ $= m^{-10}$ or $\frac{1}{m^{10}}$
- If $2^m = 2\sqrt{2}$, find m. $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$ so $m = \frac{3}{2}$
- If $a = 2^m$ and $b = 2^n$, find (i) 2^{m+n} in terms of a and b and (ii) 2^{2m} in terms of a. Just use laws of indices backwards: (i) $2^{m+n} = 2^m \times 2^n = ab$ (ii) $2^{2m} = (2^m)^2 = a^2$

Test Your Understanding:

a. Simplify $\frac{(y^3 \times y^4)^3}{y^2}$
b. Simplify $\frac{x^{-3}}{x^{-5}}$
c. If $3^x = 27\sqrt{3}$, find x
d. Simplify $\frac{(a+1)^2}{a+1}$
e. If $x = 3^a$ and $y = 3^b$, find (i) 3^{a-b} in terms of x and y and (ii) 3^{a+2b} in terms of x and y.

41. Simplify a whole term raised to a power.

When asked to simplify something like $(2x^2y)^3$, just do each item in the brackets to the outer power. i.e. $8x^6y^3$. Another example: $(25x^6y^5)^{\frac{1}{2}} = 5x^3y^{\frac{5}{2}}$

Common error: $(27x^6y)^{\frac{1}{3}} = 9x^2y^{\frac{1}{3}}$ (where the 27 has been multiplied by 1/3 rather than raised to the power of 1/3). $(3x^3)^4 = 12x^{12}$ would also be wrong (answer is $81x^{12}$).

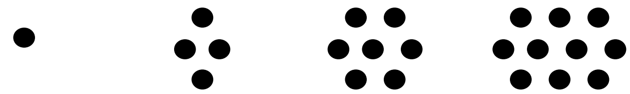
	Test Your Understanding: Simplify the following. a. $(3xy^4)^3$ b. $(5x^2y^4)^2$ c. $(36x^4y^6)^{\frac{1}{2}}$ d. $(64x^8y^3)^{\frac{1}{3}}$	
42. Use index laws to simplify and calculate numerical expressions involving powers, eg $(2^3 \times 2^5) \div 2^4$, 4^0, $8^{-2/3}$	• Firstly, raising a positive number to a power NEVER gives you a negative number. • $a^{-n} = \frac{1}{a^n}$, e.g. $4^{-1} = \frac{1}{4}$, $5^{-2} = \frac{1}{25}$, $\left(\frac{5}{7}\right)^{-1} = \frac{7}{5}$ • $a^{\frac{1}{n}} = \sqrt[n]{a}$, for example: $25^{\frac{1}{2}} = 5$, $81^{0.25} = 81^{\frac{1}{4}} = 3$ • $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$, for example: $8^{\frac{2}{3}} = 2^2 = 4$. i.e. Deal with denominator first (by taking that root) but leave the numerator in the power. • Further examples: $\left(\frac{25}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{25}\right)^{\frac{3}{2}} = \left(\frac{3}{5}\right)^3 = \frac{27}{125}$ $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{2^3} = \frac{1}{8}$ **Test Your Understanding:** Evaluate the following. a. 3^0 b. 7^{-2} c. $64^{\frac{1}{3}}$ d. $9^{-\frac{3}{2}}$ e. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$	
43. Calculate with surds (including simplifying surds, multiplying, dividing, and expanding brackets)	Laws of surds: $\sqrt{a}\sqrt{b} = \sqrt{ab}$ $\qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ • To simplify a surd, find the largest square number that goes into it. It's best to put the square number first: e.g. $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$ (this one is particularly common), $\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$ • To add/subtract surds, simplify first: $\sqrt{50} + \sqrt{18} = \sqrt{25}\sqrt{2} + \sqrt{9}\sqrt{2} = 5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$ • To expand brackets, it's best to expand first THEN simplify. Remember that $\sqrt{x} \times \sqrt{x} = x$. For example: $\left(3 - \sqrt{2}\right)\left(\sqrt{8} + 1\right) = 3\sqrt{8} + 3 - \sqrt{16} - \sqrt{2}$ $= 3\sqrt{4}\sqrt{2} + 3 - 4 - \sqrt{2}$ $= 6\sqrt{2} - 1 - \sqrt{2} \qquad = 5\sqrt{2} - 1$ **Test Your Understanding:** a. Simplify $\sqrt{80}$ b. Simplify $\sqrt{75} - \sqrt{48}$ c. Expand and simplify $\left(\sqrt{27} - 2\right)\left(4 + \sqrt{3}\right)$ d. Expand and simplify $\left(\sqrt{5} - 3\right)^2$ e. A rectangle has width $\sqrt{2} + 3$ and height $3\sqrt{18} - 1$. Determine (i) its area and (ii) its perimeter.	
44. Rationalise the denominator, e.g. write $(\sqrt{18} + 10) \div \sqrt{2}$ in the form $p + q\sqrt{2}$	Multiply top and bottom by the surd at the bottom to rationalise the denominator. Then simplify if possible. Examples: $\frac{\sqrt{18}+10}{\sqrt{2}} = \frac{\sqrt{2}(\sqrt{18}+10)}{2} = \frac{\sqrt{36}+10\sqrt{2}}{2} = 3 + 5\sqrt{2}$ $\frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$ **Test Your Understanding:** a. Rationalise the denominator of $\frac{8}{\sqrt{2}}$ b. Write $\frac{\sqrt{80}+10}{\sqrt{5}}$ in the form $a + b\sqrt{5}$	

Using a Calculator		
45. Be able to use a calculator efficiently.	Ensure that you use brackets when appropriate, particularly when raising a negative number to a power. For example, to find "-3 squared on a calculator", you must input "$(-3)^2$", otherwise your calculator will give you the incorrect answer of -9 (as the calculator does the squared first according to BIDMAS). A nice trick is to exploit the ANS key. Suppose we wanted to evaluate $3x^2 - x^3 + 2$ when $x = -3.4$. To avoid having to use brackets, first type "-3.4 = ". Now you can use the ANS key: "3 ANS^2 – ANS^3 + 2". **Test Your Understanding:** • Find the value of $\sqrt{\frac{\sin(55)+3}{51.4^2-2^3}}$ giving all the figures on your calculator display. • Using your calculator, evaluate $1-(9-2x^2)^2$ when $x=-3$.	

Algebra

Algebraic Expressions		
46. Form an algebraic expression from a description	If c is the cost of a cat and d the cost of a dog, then "the cost of four cats and three dogs" could be represented as $4c + 3d$	
47. Collect like terms	Terms are only considered 'like' if they have both the same variables and the same powers. e.g. $x^2 + x + x + 2 \rightarrow \ x^2 + 2x + 2$ **Test Your Understanding:** Simplify the following: a. $x^2y + xy^2 - 2x^2y$ b. $x + 3y + 2x - 2y$	
48. Multiply a single term over a bracket	Just multiply each term in the bracket by the one on the front. Be careful of double negatives. $xy(x+y) = x^2y + xy^2$ $1 - 2(3 - x) = 1 - 6 + 2x = 2x - 5$ **Test Your Understanding:** Expand and simplify the following. a. $2(x+4) - 3(2-2x)$ b. $x(x-y) - y(y-x)$	
49. Factorise algebraic expressions by taking out common factors	Think both about the numbers and the variables you can factorise out. If the expression in the bracket still has a factor you can take out, then you haven't fully factorised and you'd lose marks. Check your solution by expanding it out and seeing if it matches the original expression. **Examples**: $2x^2 + 4x = 2x(x+2)$ $\quad xy^2 - y^2 = y^2(x-1)$ **Test Your Understanding:** Factorise the following. a. $3x - 6$ b. $6x^2y + 9xy^2$ c. $8ab^3 - 12a^2bc$	
50. Expand the product of two linear expressions	Just multiply each thing in the first bracket by each in the second – this will result in 4 terms. Again, <u>be very careful with double negatives</u>. And remember that say $(3x)^2 = 9x^2$ not $3x^2$. Examples: $(x+1)(x-4) = x^2 + x - 4x - 4 = x^2 - 3x - 4$ $(1-2x)^2 = (1-2x)(1-2x) = 1 - 2x - 2x + 4x^2$ **Test Your Understanding:** Expand the following: a. $(y+6)(y-7)$ b. $(2x-5)(3x+4)$ c. $(4x+1)^2$ d. $(xy^2+1)(xy^2-1)$	
51. Factorise expressions of the form $ax^2 + bx + c$	Notes: • When the coefficient of x^2 (i.e. the number in front of x^2) is 1, then simply find two numbers which add to give the middle number and times to give the last. To do this consider factor pairs of the last number and see which pairs have a sum or difference of the middle. e.g. $x^2 + 2x - 15$ Factor pairs of 15 are 5×3 and 15×1. Notice that 5 and 3 have a difference of 2, which tells us our two numbers are 5 and -3. Thus: $x^2 + 2x - 15 = (x+5)(x-3)$ • When the coefficient of x^2 then we can either 'intelligently guess' the factorisation (which I particularly encourage if the coefficient of x^2 is prime, limiting the possibilities to try) or 'split the middle term'. **Example:** $2x^2 - x - 6$ $\quad \oplus\ -1 \quad \otimes\ -12$ We again find two numbers which add to give the middle number (-1) but now, which times to give the <u>first times the last number</u>, -12 (rather than just the last number). These numbers are 3 and -4, so we 'split the middle term' using these numbers: $2x^2 + 3x - 4x - 6$ Now factorise the first half and the second half separately. The bracket will be the same for the two, so as soon as you've got your first bracket, you can duplicate this. $= x(2x+3) - 2(2x+3)$ $= (2x+3)(x-2)$ The last step was because $2x + 3$ was common to both terms.	

	Test Your Understanding: Factorise the following. a. x^2+2x+1 b. x^2+6x+8 c. $x^2-7x+10$ d. $x^2-3x-10$ e. $x^2+ax+bx+ab$ f. $2x^2+3x+1$ g. $3y^2+11y-4$ h. $12x^2-x-1$	
52. Factorise quadratic expressions using the difference of two squares	You know you have the difference of two squares where, unsurprisingly, you have: (i) two terms (ii) the difference between them (iii) each term looks like something squared. Then write out two bracket, one with +, one with −, and write the square root of each term before and after each. Note that the <u>order matters</u>, so for $1-x^2$, $(1+x)(1-x)$ is a correct factorisation, but $(x+1)(x-1)$ is not. **Examples**: $x^2-9=(x+3)(x-3)$ $\quad 4x^2-b^2=(2x+b)(2x-b)$ **Test Your Understanding:** Factorise the following. a. $4-x^2$ b. x^2y^2-1 c. $25y^4-36z^2$	
53i. Simplify rational expressions by cancelling, adding, subtracting, and multiplying	Be careful of sign errors in expanding. Whenever subtracting the product of two brackets (or is squared), <u>leave it in brackets first to avoid a sign error</u>, e.g.: $3(x+2)-4(3-x) \quad \rightarrow 3x+6-12+\mathbf{4x}$ $1-(1-x)^2 \quad \rightarrow \quad 1-(1-2x+x^2)$ $=1-1+2x-x^2$ $=2x-x^2$ **Test Your Understanding:** Simplify the following (expanding where necessary). a. $2x \times 3y$ b. $3xy^2 \times 4y$ c. $\frac{xy^2}{2xy}$ d. $(x+y)^2+(x-y)^2$ e. $(x+1)^2-3(x+2)$ f. $(2x+1)^2-(x+1)(x-3)$ g. $(3x+1)(2x-3)-(1-2x)^2$	
53ii. Simplify algebraic fractions.	The key is to *factorise* top and bottom of the fraction before *cancelling* common factors. **Examples:** $\frac{x^2+x}{x+1}=\frac{x(x+1)}{x+1}=x$ $\frac{x^2+3x+2}{x^2-1}=\frac{(x+1)(x+2)}{(x+1)(x-1)}=\frac{x+2}{x-1}$ **Test Your Understanding:** Simplify the following algebraic fractions. a. $\frac{2x^2+3x+1}{x^2+2x+1}$ b. $\frac{2x^2+5x-3}{x^2-9}$ c. $\frac{x^2+2xy+y^2}{2x+2y}$	
53iii. Add and subtract algebraic fractions	The principle of adding/subtracting fractions is equally applicable to algebraic fractions. e.g. for $\frac{1}{3}+\frac{2}{7}$, we'd multiply the denominators, then 'cross multiply' the numerators, i.e. $(1\times 7)+(2\times 3)$. Examples: $\frac{1}{x}+\frac{2}{y}=\frac{y+2x}{xy}$ $\quad \frac{x}{x-1}-\frac{2}{x+2}=\frac{x(x+2)-2(x-1)}{(x-1)(x+2)}=\frac{x^2+2x-2x+2}{(x-1)(x+2)}=\frac{x^2+2}{(x-1)(x+2)}$ Notice in the latter example that we need not expand out the denominator – it makes it no 'simpler'. Also note that as always we need to be careful with double negatives. Note that it's not always necessary to multiply the denominators, e.g: $\frac{1}{x}+\frac{1}{x^2}=\frac{x}{x^2}+\frac{1}{x^2}=\frac{x+1}{x^2}$ This is similar to how with $\frac{1}{2}+\frac{3}{4}$ we can use 4 as the common denominator rather than 8. **Test Your Understanding:** Express the following as a single fraction. a. $\frac{1}{2}+\frac{3}{x}$ b. $\frac{2}{x}+\frac{1}{x^2}$ c. $\frac{1}{x}+\frac{1}{x+1}$ d. $\frac{2}{x-1}-\frac{3}{x}$ e. $\frac{x+1}{x-1}-\frac{x-1}{x+1}$	

Patterns and Sequences		
54. Generate simple sequences of numbers, squared integers and sequences derived from diagrams	Example: Find the number of dots in the 100th diagram in this sequence:  The number of dots is 1, 4, 7, 10, …. The formula for the nth term is therefore $3n-2$. Therefore the 100th term is $3(100)-2=297$ **Test Your Understanding:** Find the number of matchsticks required for the 50th image in this sequence. 	
55. Describe the term-to-term definition of a sequence in words	e.g. “The terms double each time” or “The terms decrease by 3 each time”. ‘Term-to-term’ just means that the rule to generate new terms is based on the previous term (or multiple terms, in the case of the Fibonacci sequence) rather than on the position n. So “add 3 each time” would be a term-to-term rule, whereas $3n+1$ would be a position-to-term rule.	
56. Identify which terms cannot be in a sequence	This may be because: • Using the formula for the nth term would lead to a non-whole number of n; it’s not possible to say have the 13.4th term! Suppose the rule for the nth term is $3n+2$ and we’re establishing if 76 is in the sequence. $3n+2=76$ $3n=74$ $n=24\frac{2}{3}$ which is not possible as n is not whole, so 76 is not in the sequence. • The terms all end in certain digits (e.g. 6, 11, 16, 21, … always ends in 1 or 6) and the term given does not end in this digit. **Test Your Understanding:** Establish with explanation whether the following terms are in the sequence with the given formula. a. 24 where nth term is $2n+1$ b. 255 where sequence is 7, 12, 17, 27, … c. 104 where nth term is $7n-3$ d. 2792 where nth term is $6n+2$	
57. Generate specific terms in a sequence using the position-to-term and term-to-term rules	**Example:** Write the first four terms of the sequence whose nth term is n^2-2n+3 For the first term, $n=1$, so $1^2-2(1)+3=2$ The second term is $2^2-2(2)+3=3$ Using this method, first four terms are $2, 3, 6, 11$ **Test Your Understanding**: Find the first three terms of the following sequences. a. The sequence whose nth term is $7n+3$ b. The sequence whose nth term is $10-2n$ c. The sequence whose nth term is 3^n-n^2 d. The sequence whose first term is 3 and each term is twice the previous term.	
58. Find and use the nth term of an arithmetic sequence	You will only need to find the n^{th} term of a linear sequence (where the difference between terms is constant), NOT quadratic sequences (where the second difference is constant). **Example:** “Find the formula for the nth term of the sequence 5, 8, 11, 14, …. Hence find the 100th term of the sequence.” The terms increase by 3 each time, so our formula starts $3n$. Then imagine the 3 times table (as this is the sequence $3n$ would give). We need to add 2 to ‘correct’ it, so the formula is $3n+2$. Therefore the 100th term is $(3\times 100)+2=302$ **Test Your Understanding**: For the following sequences, find the formula for the nth term, and hence determine the 100th term of the sequence. a. 2, 5, 8, 11, … b. 5, 6, 7, 8, … c. 11, 9, 7, 5, … d. $5\frac{1}{2}, 6\frac{1}{6}, 6\frac{5}{6}, 7\frac{1}{2}$	

Formulae and Solutions to Linear Equations		
59. Substitute numbers into a formula. Substitute positive and negative numbers into expressions such as $3x^2 + 4$ and $2x^3$	The key to avoiding errors with substitution are: • Observe the laws of BIDMAS. If $x = 4$, then $2x^2$ would be $2 \times 4^2 = 32$, NOT $(2 \times 4)^2 = 64$ (a very common student error). • Be careful with negatives (particularly when there are squared/cubed terms). If $a = 1, b = -2$, then $a - b^2 = 1 - 4 = -3$, and $a - 2b = 1 + 4 = 5$. Note that when evaluating "minus four squared" on a calculator, you need to use $(-4)^2$ and not -4^2 as previously discussed. **Test Your Understanding:** Given that $a = -1$ and $b = 2$ and $c = -3$ determine the value of: a. $a - bc$ b. $a^2 + bc$ c. $b^2 - c^2$ d. $bc - ac$ e. $c^2 - 4bc$	
60. Solve linear equations, including where the unknown appears on both sides, and where the equation may including negative signs and brackets.	• If the unknown is on one side then collect on the side where the number on front of the unknown is greater. Collect anything that doesn't involve the unknown on the other side. e.g. $2 - 3x = 5x + 5 \quad \rightarrow \quad -3 = 8x$ • Ensure you do the 'opposite' when moving something to the other side. • The result may be negative and/or fractional. Simplify the fraction if necessary, but there is no need to convert to a decimal. • Expand out any brackets present first. • Example: $3(4x - 2) = 2(1 - 3x) + x$ $12x - 6 = 2 - 6x + x$ $12x - 6 = 2 - 5x$ $17x = 8$ $x = \frac{8}{17}$ **Test Your Understanding:** Solve the following. a. $3x + 4 = 6$ b. $3(2x + 6) = 2(5x - 7)$ c. $1 - 2(3 - 2y) = y$	
61. Set up linear equations from word problems and geometric contexts.	**Examples:** • "The angles in a triangle are $2x + 6, 3x + 10$ and $4x - 7$. Determine x." The angles in a triangle add to 180, so: $2x + 6 + 3x + 10 + 4x - 7 = 180$ $9x + 9 = 180$ $9x = 171$ $x = 19°$ • "In 4 years time I will be 4 times as old as I was 11 years ago. How old am I?" If your age now is a, then your age in 4 years time is $a + 4$, and 4 times your age 11 years ago is $4(a - 11)$ $a + 4 = 4(a - 11)$ $a + 4 = 4a - 44$ $48 = 3a$ $x = 16$	
62. Solve linear equations in one unknown, with fractional coefficients	If there is a fraction in your equation, your instinct should be to times both sides of the equation by the denominator of this fraction. Example: $\frac{x}{3} + 4 = x$ $x + 12 = 3x$ $12 = 2x$ $x = 6$ A common error is to forget to multiply one of the terms, e.g. writing $x + 4 = 3x$ More examples: (both past paper questions) • Solve $\frac{2-y}{5} = 1 \quad \rightarrow \quad 2 - y = 5 \quad \rightarrow \quad 2 - 5 = y \quad \rightarrow \quad y = -3$	

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- Solve $\frac{5(2x+1)^2}{4x+5} = 5x - 1$

$$5(2x+1)^2 = (5x-1)(4x+5)$$
$$5(2x+1)(2x+1) = 20x^2 - 4x + 25x - 5$$
$$5(4x^2 + 4x + 1) = 20x^2 + 21x - 5$$
$$20x^2 + 20x + 5 = 20x^2 + 21x - 5$$
$$20x + 5 = 21x - 5$$
$$10 = x$$

Notice in expanding $5(2x+1)^2$, we maintained an outer bracket while expanding the $(2x+1)^2$ part, to avoid error (see (53i)).

Test Your Understanding: Solve the following.

a. $2 - \frac{x}{3} = 6$
b. $\frac{3-4p}{6} = p$
c. $\frac{2(3x-1)^2}{9x-1} = 2x + 1$

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<td>63. Solve simple linear inequalities in one variable, and represent the solution set on a number line</td>
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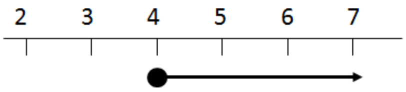
The principle is exactly the same as with solving equalities. The only difference is that when you divide by a negative number, the direction of the inequality is reversed. This is avoidable by making sure the x term is positive (and moved to the other side if not), so that we only ever divide/times by a positive number.

Example: $1 - 3x > 5$

$$1 > 5 + 3x$$
$$-4 > 3x$$
$$-\frac{4}{3} > x$$

(which is the same as $x < -\frac{4}{3}$)

Example: "Represent $x \geq 4$" on a number line.



Remember that a filled circle means "including 4", and an empty circle would mean "excluding 4". Note that arrow, indicating the line continues to infinity.

Test Your Understanding:

a. Solve $2x - 1 < 5$, representing your solution set on a number line.
b. Solve $3(4 - 2x) \geq 7$

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<td>64. Use the correct notation to show inclusive and exclusive inequalities</td>
<td>This is checking you understand the difference between say $x < 3$ (all values strictly less than 3) and $x \leq 3$ (all values at most 3, which can include 3).</td>
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<td>65. Change the subject of a formula including cases where the subject is on both sides of the original formula, or where a power of the subject appears</td>
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When the subject appears multiple times, the approach is "*collect and factorise*", i.e. collect all the terms involving the subject on one side of the equation (expanding brackets first if necessary), factorising this subject out, and then dividing through.

Example: Make x the subject of $y = \frac{a+ax}{2-3x}$

$y(2-3x) = a + ax$
$2y - 3yx = a + ax$
$2y - a = ax + 3yx$ (Collected x terms on RHS, since this keeps terms positive)
$2y - a = x(a + 3y)$ (Factorise out x)

$$x = \frac{2y - a}{a + 3y}$$

For other expressions where the subject only appears once, you just need to "undo the last thing done to x" (or whatever the subject is) each time.

Test Your Understanding: In each case make x the subject.

a. $b = \sqrt{\frac{4x-1}{3}}$
b. $a - 3x = b$
c. $a = \pi x + \frac{x}{3}$ (hint: times both sides by 3 first)
d. $a = \frac{2x+1}{2x-1}$
e. $\frac{1}{x} + \frac{1}{a} = \frac{1}{b}$ (hint: what should you times both sides by to have no fractions?)

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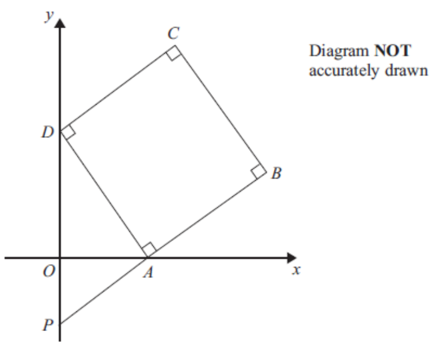
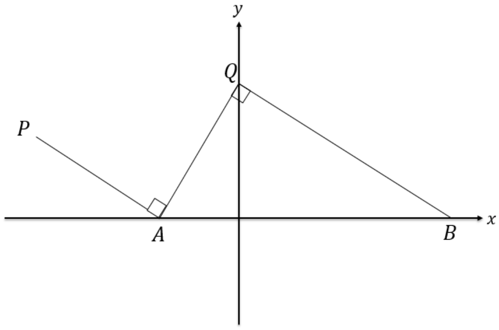
Linear Graphs		
66. Recognise that equations of the form *y* = *mx* + *c* correspond to straight-line graphs in the coordinate plane	You should understand that m is the gradient and c is the y-intercept.	
67. Draw straight line graphs for real-life situations – ready reckoner graphs – conversion graphs – fuel bills, eg gas and electric – fixed charge (standing charge) and cost per unit	For a 'fixed charge' graph, e.g. a builder charges a fixed charge of £50 and an hourly rate of £25/hr, then if total cost is plotted with time, then the £50 will be the y-intercept (i.e the cost when the time is 0) and the £25 will be the gradient. To plot a straight line representing the charges, just calculate the total charge for two different amounts of time, plot the points and join with a straight line. You may have to compare the charges of two different builders, e.g. draw a straight line for each, and find their intersection to work out for what amount of time the total cost will be the same.	
68. Plot and draw graphs of straight lines with equations of the form *y* = *mx* + *c*	The trick here is to pick just two suitable values of x (preferably at each of end of your provided axis) and work out the corresponding y value by substituting into the equation. Then just join up these two points (ensuring your line goes to the end of the available space, to indicate clearly that your line is infinitely long). For example, if the equation was $x+2y=4$, the choosing $x=0$ gives you $y=2$, hence plot the point $(0,2)$. Plot one more point and hey presto. We could also choose $y=0$, so that $x=4$, giving us the point $(4,0)$. This method avoids any problems when the scales of the x and y axis are different. **Test Your Understanding:** a. With x-axis varying from -3 to 3 and y axis from -8 to 8, plot the line with equation $y=2x-1$. b. With x-axis and y-axis varying from 0 to 6, plot the line with equation $x+2y=6$	
69. Find the gradient of a straight line from a graph or given two points on the line.	If you're given a graph, find the coordinates of any two points on the graph. Gradient is then: $$m=\frac{change\ in\ y}{change\ in\ x}$$ Ensure you are consistent in what point you're considering the change from. e.g. If $A(3,6)$ and $B(5,-8)$ are two points on the line, then the y change is -14 and the x change is 2 (not 14 and 2), thus the gradient is 7. **Test Your Understanding:** a. A line goes through the points $(2,5)$ and $(6,3)$. Determine the gradient of the line.	
70. Be able to identify the y and x intercept of a straight line with the axis.	Notes: • When a line crosses the y-axis then $x=0$. Just substitute into your equation. e.g. "Find where $2x+3y=4$ intercepts the y-axis." When $x=0$, $0+3y=4$, thus $y=\frac{4}{3}$. The point is therefore $\left(0,\frac{4}{3}\right)$ • When a line crosses the x-axis then $y=0$. Again substitute and solve. **Test Your Understanding:** a. Find the coordinates of the points at which the line with equation $y=3x+4$ crosses (i) the y-axis and (ii) the x-axis.	
71. Analyse problems and use gradients to interpret how one variable changes in relation to another	If the gradient is negative, as one variable increases, the other decreases. If the gradient is positive, as one variable increases, the other increases. As the gradient increases in magnitude, one variable rises (or falls) more sharply as the other increases.	
72. Find the gradient of a straight line from its equation	Make y the subject so that your equation is in the form $y=mx+c$. Then m is the gradient. **Example**: "Find the gradient of the line with equation $x+2y=1$ $$x+2y=1$$ $$2y=-x+1$$ $$y=-\frac{1}{2}x+\frac{1}{2}$$ Thus the gradient is $-\frac{1}{2}$ **Test Your Understanding:** Find the gradient of the lines with equations, a. $y=1-3x$ b. $x+y=1$ c. $2x-y=4$ d. $3x+4y=5$	

73. Explore the gradients of parallel lines and lines perpendicular to each other	• Lines which are parallel have the same gradient. • If two lines with gradients m_1 and m_2 are perpendicular then $m_1 \times m_2 = -1$ (this is useful to state if you have to prove two lines are perpendicular). If you have one gradient and want to work out the other, find the *negative reciprocal*. So $2 \to -\frac{1}{2}$, $\frac{1}{3} \to -3$, $-\frac{4}{5} \to \frac{5}{4}$ • If two lines a parallel they will never intersect. Otherwise they will. **Examples:** • "Line A has the equation $x + 2y = 1$. Line B passes through the points $(0,3)$ and $(4,11)$. Are the lines parallel, perpendicular or neither?" For line A rearranging gives $y = -\frac{1}{2}x + \frac{1}{2}$ so $m_1 = -\frac{1}{2}$. The gradient of line B is $m_2 = \frac{8}{4} = 2$. Since $-\frac{1}{2} \times 2 = -1$, the lines are perpendicular.	
74. Write down the equation of a line parallel or perpendicular to a given line, or the line which also passes through a given point.	If asked to find "a" line parallel or perpendicular to another, this suggests there are multiple possibilities. The y-intercept can be anything you like. **Examples:** • "Find the equation of a line parallel to $y = 3x + 4$". $y = 3x + 1$ or even just $y = 3x$ will do, as we only require the gradients are the same! • "Find the equation of a line perpendicular to $y = -2x + 1$". Find the 'negative reciprocal' of the gradient, in this case $\frac{1}{2}$. Thus $y = \frac{1}{2}x + 5$ is a suitable example. If the line also has to pass through a given point, there's only ONE possible line. Example: "Find the equation of the line perpendicular to $y = 2x + 1$ and passes through the point $(4,1)$." We know our equation will starts $y = -\frac{1}{2}x + \cdots$ There's two methods to determine the correct y-intercept: 1. The quick ('mental') way: If we evaluate $-\frac{1}{2}x$ for our point $(4,1)$, then $-\frac{1}{2}(4) = -2$. Then we have to 'correct' this by adding 3 to get to the y value of 1. Thus $y = -\frac{1}{2}x + 3$ 2. $y = -\frac{1}{2}x + c$. Substituting using the given point: $$1 = -\frac{1}{2}(4) + c = -2 + c$$ $$c = 3$$ $$y = -\frac{1}{2}x + 3$$ These methods are effectively the same: it's just (1) is a mental way of approaching (2). **Test Your Understanding:** a. Find the equation of a line which is perpendicular to $y = -5x + 2$ b. Find the equation of the line which is parallel to another line $y = 3x + 1$, and goes through the point $(4,2)$ c. Find the equation of the line which is perpendicular to the line with equation $y = 3x - 1$ and goes through the point $(9,1)$	
75. Find the equation of a line which passes through two given points.	First use the two points to find the gradient, then repeat as above. **Example**: "Find the equation of the line which passes through the points $(2,5)$ and $(4,4)$" Gradient $m = -\frac{1}{2}$ Then using either of the points (let's say $(4,4)$) then $y = -\frac{1}{2}x + 6$ **Test Your Understanding**: Find the equation of the line which passes through $(4,5)$ and $(6,4)$	
76. Solve more difficult problems involving straight line equations	**Example:** *"ABCD is a square.* *P and D are points on the y-axis.* *A is a point on the x-axis. PAB is a straight line.* *The equation of the line that passes through the points A and D is y = –2x + 6* *Find the length of PD."* We need to find points D and P. D is just the y-intercept of the equation so is (0,6) To find P, we first need to coordinate of A, i.e. the x-intercept of $y = -2x + 6$. This occurs at (3,0) (by setting y to be 0). The line PB is perpendicular to $y = -2x + 6$ and goes through the	

point $(3,0)$. Using the theory above, we work out the equation as $y = \frac{1}{2}x - \frac{3}{2}$
Thus P has coordinate $\left(0, -\frac{3}{2}\right)$.
PD = 6 + 1.5 = 7.5.



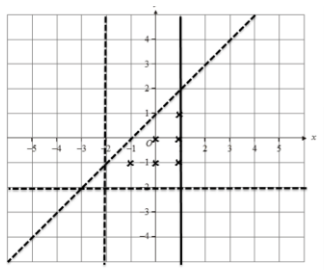
Test Your Understanding:

Line PA has the equation
$y = -\frac{1}{2}x - 4$.
$\angle PAQ = 90°$, Q is a point on the y-axis and $\angle AQB = 90°$.
Determine the length of AB.

77. Show the solution set of several inequalities in two variables on a graph

Start by plotting each line *as if it was an equality*, using a dotted line if < or > (to indicate the values on the line are not included), and a solid line if ≤≥ (to indicate the values on the line are included). The way we can tell the region we want is above or below the line is by looking at whether y is on the smaller or greater side of the inequality. So if $x + 2y > 4$, then we are above the line. If y wasn't positive, move it first: $1 - y > 3 \quad \rightarrow \quad 1 > 3 + y$ so we're below the line as y is on the smaller side.

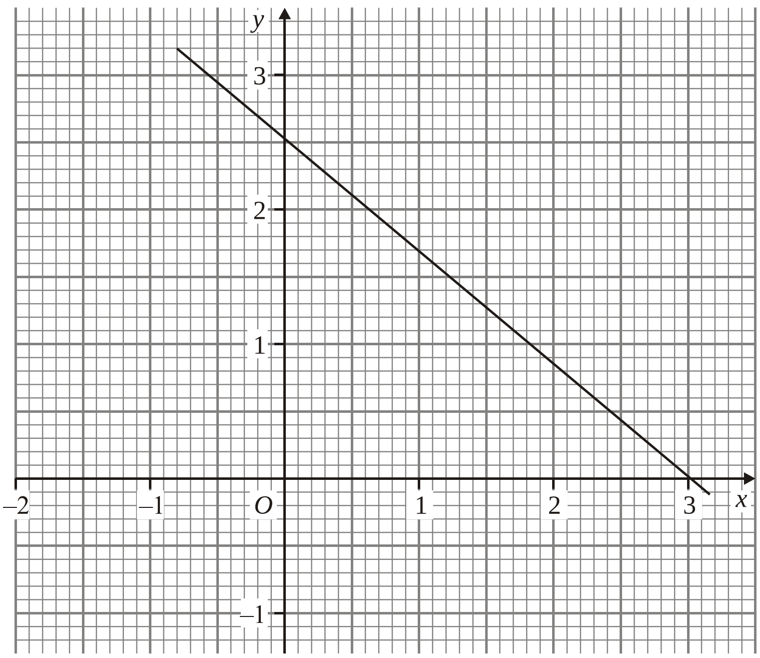
Example:
$-2 < x \leq 1$
$y > -2$
$y < x + 1$
x and *y* are integers. On the grid, mark with a cross (✖), each of the six points which satisfies **all** these 3 inequalities.



Notice on the solution to the right that we have put crosses on the line but not on the dotted lines. Also note that:

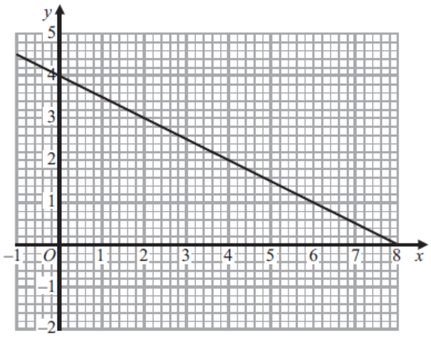
- Some lines are sometimes already drawn for you, but will always be solid. You might want to have some visual note that this actually represents a dotted line if the inequality is < >.
- The region is almost always the one enclosed by all your lines. So if ever in doubt, choose this one.

Test Your Understanding:
On the grid, shade the region of points whose coordinates satisfy the four inequalities
$y > 0, \quad x > 0, \quad 2x < 3, \quad 6y + 5x < 15$



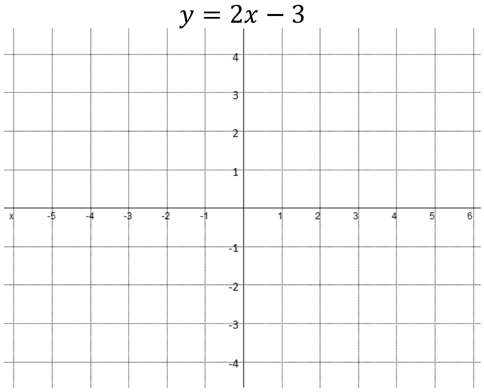
Label this region ***R***. *P* is a point in the region ***R***. The coordinates of *P* are both integers. Write down the coordinates of *P*.

<table>
<tr><th colspan="2">Simultaneous Equations</th><th></th></tr>
<tr><td>78. Use elimination or substitution to solve simultaneous equations</td><td>To solve by elimination, scale each equation so that either the x terms or the y terms are the same. If they have different signs, adding the equations will make them cancel. If they're the same sign, subtracting the equations will make them cancel.
e.g. "Solve the simultaneous equations
$4x + 7y = 1$
$3x + 10y = 15$
Scaling so that say y is the same:
$40x + 70y = 10$
$21x + 70y = 105$
Subtracting to eliminate y, being very careful that you subtract the same way round:
$19x = -95$
$x = -5$
Substituting this into one of original equations:
$4(-5) + 7y = 1$
$7y = 1 + 20 = 21$
$y = 3$
You can check your answer by substituting both your values into the other equation:
$3(-5) + 10(3) = 15$
$-15 + 30 = 15$
$15 = 15$

Test Your Understanding:
a. Solve the following simultaneous equations:
$5x - y = 27$
$3x - y = 17$
b. Solve the simultaneous equations:
$5x + 6y = 12$
$3x - 4y = 11$</td><td></td></tr>
<tr><td>79. Interpret a pair of simultaneous equations as a pair of straight lines and their solution as the point of intersection. Consider the real life applications, eg mobile phone bills</td><td>Example: The graph of the straight line x + 2y = 8 is shown on the grid.
(a) On the grid, draw the graph of $y = \frac{x}{2} - 1$


(b) Use the graphs to find estimates for the solution of
$x + 2y = 8$
$y = \frac{x}{2} - 1$

By correctly drawing the line in (a), then the solution to the simultaneous equations is just the coordinate of their intersection, which is (5,1.5) thus $x = 5$ and $y = 1.5$.

Test Your Understanding:
On the axis provided, sketch the lines $x + y = 3$ and $y = 2x - 3$. Hence determine the solutions to the simultaneous equations:
$x + y = 3$
$y = 2x - 3$

</td><td></td></tr>
</table>

80. Set up a pair of simultaneous equations in two variables

Are you able to take a problem in words and turn it into simultaneous equations?
Example: “Four Aardvarks and five Buffalo cost £2.50. Two Aardvarks and one Buffalo costs 80p. How much is one Aardvark?”
We can form two equations (letting a be the cost of an aardvark and b the cost for a buffalo):

$$4a + 5b = 250$$
$$2a + b = 80$$

Then solve.

Test Your Understanding: “In 1999 the concert had 3 readings and 9 songs. It lasted 120 minutes. In 2000 the concert had 5 readings and 5 songs. It lasted 90 minutes. In 2001 the school plans to have 5 readings and 7 songs. Use simultaneous equations to estimate how long the concert will last.”

Trial and Improvement

81. Solve cubic functions by successive substitution of values of x. Use systematic trial and improvement to find approximate solutions of equations where there is no simple analytical method of solving them.

Example: Suppose you wish to solve $x^3 + 2x = 30$, *correct to 1dp*. The key is to:

- Have tried the solution to 1dp (or the specific accuracy) either side of the precise solution.
- Have then tried the midpoint of the two. This is crucial to get full marks.

It may help to lay your workings like follows:

$$x = 3 \ \rightarrow \ 3^3 + 2(3) = 33 \qquad \mathit{Too\ large}$$
$$x = 2.7 \rightarrow \ 2.7^3 + 2(2.7) = 25.083 \quad \mathit{Too\ small}$$
$$x = 2.9 \rightarrow 2.9^3 + 2(2.9) = 30.189 \quad \mathit{Too\ large}$$
$$x = 2.8 \rightarrow 2.8^3 + 2(2.8) = 27.552 \quad \mathit{Too\ small}$$
$$x = 2.85 \rightarrow 28.849 \ldots \qquad \mathit{Too\ small}$$

Therefore the solution to 1dp is 2.9.
Note that it was <u>not</u> sufficient to try 2.8 and 2.9 and choose 2.9 because 30.189 is closer to 30. It’s theoretically possible that 2.8 was closer. But by trying 2.85 and observing it gives a value too small, we know the solution lies between 2.85 and 2.9. Any number in this range is guaranteed to be 2.9 to 1dp.

Test Your Understanding:
Find the solution to $x^3 - 3x = 10$ correct to 1dp.

82. Understand the connections between changes of sign and location of roots

Suppose we were solving $x^3 + x - 11 = 0$ and found that $x = 2.0$ gives -1 on the LHS and $x = 2.1$ gives 0.361. Since this passes 0, we know that if we sketched $y = x^3 + x - 11$, the line would cross the x-axis (i.e. we’d have a ‘root’) somewhere between $x = 2.0$ and $x = 2.1$
You are unlikely to be tested on this.

Quadratic Equations, Functions and Graphs

83. Generate points and plot graphs of simple quadratic functions, then more general quadratic functions

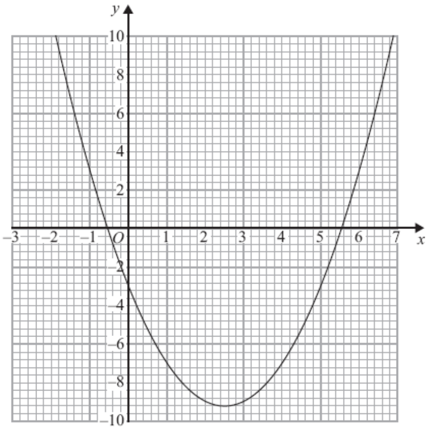
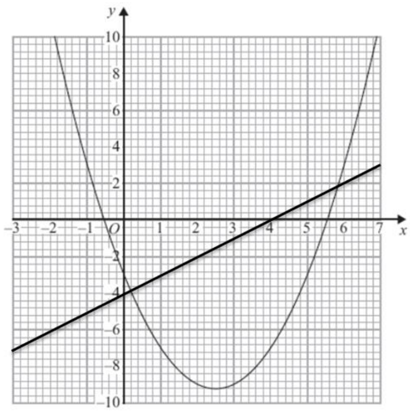
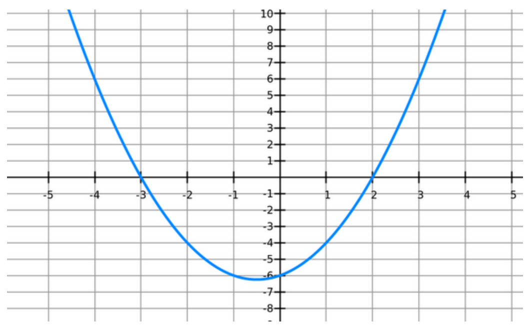
You’re just being tested on your ability to substitute values into an equation, plot them on a graph, and join with a curved line (not straight lines between points!). The mark scheme allows your line to stray at most 1mm from your plotted points, so beware.

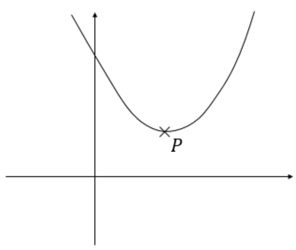
The ‘TABLE’ mode on your calculator is very helpful. e.g. To generate a table of points for $y = x^2 + 2x$, press MODE -> TABLE, which should bring up “$f(X) =$”. Then type in $X^2 + 2X$: you can get X using the ‘alpha’ button at the top and find the red X. Press =, then say type -3 for ‘start’, 3 for ‘end’ (so that your table goes from $x = -3\ to\ 3$) and step size 1 (so that your x value goes up by 1 each time). Press = and you should have a table, which you can navigate through using your cursor keys. Use MODE -> COMP to go back to computation mode.

Test Your Understanding:
Given that $y = x^2 - 2x - 3$, complete the following table.

x	-2	-1	0	1	2
y		0	-3		

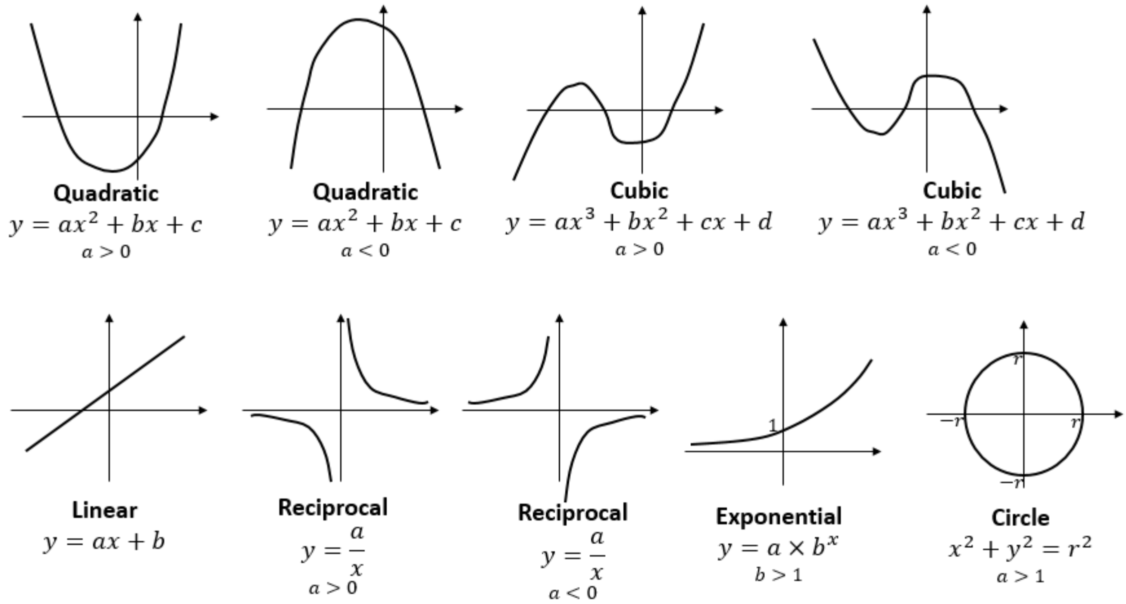
Hence on suitable axis sketch the graph with equation $y = x^2 - 2x - 3$.

84. Find approximate solutions of a quadratic equation from the graph of the corresponding quadratic function	**Example:** The diagram shows the graph of $y = x^2 - 5x - 3$  (a) Use the graph to find estimates for the solutions of (i) $x^2 - 5x - 3 = 0$ (ii) $x^2 - 5x - 3 = 6$ For (i), compare this equation to the original. We're interested where $y = 0$. By looking at the graph, we see this is roughly where $x = -0.5$ and $x = 5.5$ **Test Your Understanding**: Do part (ii)	
85. Find the intersection points of the graphs of a linear and quadratic function, knowing that these are the approximate solutions of the corresponding simultaneous equations representing the linear and quadratic functions	(Continued from above) (b) Use the graph to find estimates for the solutions of the simultaneous equations $y = x^2 - 5x - 3$ $y = x - 4$  We saw earlier that the solution to two simultaneous equations can be found by plotting the two lines and finding their intersection(s). If we draw $y = x - 4$ (by using the 'two points' method described earlier), then we can see the solutions are: x = 0.2, y = -3.8 x = 5.8, y = 1.8 **Test Your Understanding:** This is a sketch of $y = x^2 + x - 6$ (a) Estimate the values of x at which $x^2 + x - 6 = 5$ (b) By using a suitable graph, estimate the solutions to the simultaneous equations: $y = x^2 + x - 6$ $y = x + 1$ 	
86. Understand when an 'exact' solution is required to an equation.	When the question asks for an 'exact' answer, it means it wants you to leave the answer in surd form. $\sqrt{2}$ is 'exact' (because it precisely represents the result) whereas $1.41\ldots$ is not, because you would have to write all the decimal places to infinity.	
87. Solve simple quadratic equations by factorisation.	You must first get 0 on one side. Ideally, you want your x^2 term to be positive, otherwise your quadratic will be difficult to factorise. Example: Solve $x^2 - 5x - 6 = 0$ $(x + 1)(x - 6) = 0$ $x = -1 \text{ or } x = 6$ Remember that if one side is 0, then either $x + 1 = 0$ (i.e. $x = -1$) or $x - 6 = 0$. **Test Your Understanding**: Solve the following. a. $x^2 + 5x - 14 = 0$ b. $x^2 + 4x = 0$ c. $y^2 + 8y = 20$ d. $2y^2 + 3y + 1 = 0$	

88i. Be able to complete the square.	'Completing the square' means to get your equation in the form $a(x+b)^2+b$ (at GCSE, a is usually, but not necessarily, 1) **Examples:** Put x^2+6x-1 in the form $(x+a)^2+b$. • Half the coefficient of x -> $(x+3)^2$ But because the expansion of this would give a +9 term we don't want, we 'throw it away'. Hence: $x^2+6x-1=(x+3)^2-9-1=(x+3)^2-10$ i.e. $a=3$ and $b=-10$ • Similarly: $x^2-8x+2 \quad = \quad (x-4)^2-16+2 \quad = \quad (x-4)^2-14$ Notice we always *subtract* the square of this halved term, regardless of whether the number we squared was positive or negative. • "Put $2x^2+16x+4$ in the form $a(x+b)^2+c$." First factorise out the number on front of the x^2. Then complete the square for what's inside, ensuring the outer brackets remain untouched for now. $=2(x^2+8x+2)$ $=2((x+4)^2-16+2)$ $=2((x+4)^2-14)$ Lastly expand out the outer brackets. $=2(x+4)^2-28$ **Test Your Understanding:** a. Put $y^2+10y-3$ in the form $(y+a)^2+b$ b. Given $x^2-14x+5=(x+a)^2-b$, find the values of a and b. c. Put $3x^2+18x-6$ in the form $a(x+b)^2+c$ d. Put $2x^2-4x+1$ in the form $a(x-b)^2+c$	
88ii. Be able to identify the minimum/maximum value of an expression and the value of x for which this minimum/maximum occurs. Be able to identify the minimum/maximum point of a quadratic graph.	**Example**: *"By completing the square, find the minimum value of $x^2+6x+15$ ".* $x^2+6x+15=(x+3)^2+6$ Squared things are always at least 0 (since negative times negative is positive). Thus the smallest the $(x+3)^6$ can be is 0. Thus the minimum value of the expression is 6.  *"Find the value of x for which this minimum occurs."* To make $(x+3)^2$ zero, just make $x=-3$. *"The sketch shows the line with equation $y=x^2-8x+21$. Find the coordinate of the minimum point P."* $y=(x-4)^2+5$ Letting $x=4$ makes the squared term zero, so the y value is 5. Thus $P=(4,5)$. **Test Your Understanding:** a. Determine (i) the minimum value of x^2+8x-1 and (ii) the value of x for which this minimum occurs. b. Determine the coordinate of the minimum point of $y=x^2+2x+10$.	
89. Solve quadratic equations by completing the square.	It's recommended when given the choice to solve either by factorisation or by the quadratic formula, to use the quadratic formula (since the quadratic formula is derived from completing the square anyway!). However, if you've already been asked to complete the square in the first part of a question and are subsequently asked to solve an equation involving the expression, you can now quickly solve. **Example:** Given that $x^2+4x-1=(x+2)^2-5$, solve the equation $x^2+4x-1=0$ This means: $(x+2)^2-5=0$ $(x+2)^2=5$ $x+2=\pm\sqrt{5}$ $x=-2\pm\sqrt{5}$ **Test Your Understanding:** Given that $x^2-6x-5=(x-3)^2-14$, determine the exact solutions to $x^2-6x-5=0$.	

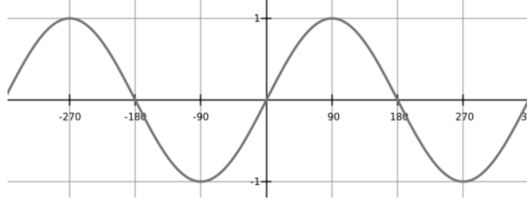
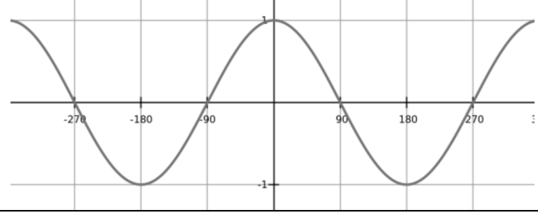
90. Solve simple quadratic equations by using the quadratic formula	The quadratic formula is in your formula booklet, but it's useful to memorise it: $$If\ ax^2 + bx + c = 0$$ $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ *You know you will need to use the formula when the question specifies "to 3 significant figures" or "to 2 decimal places"* as it implies the answer will not be a nice whole number, and thus factorisation is not an option. Common student errors: • When b is negative, then forgetting that $-b$ will be positive. • When b is negative (say -5), writing -5^2 on your calculator to represent b^2. As mentioned, whenever squaring negative numbers, you require brackets, i.e. $(-5)^2$ **Example**: Solve $2x^2 - 4x - 1 = 0$, giving your answer in exact form. $a = 2, b = -4, c = -1$ (it's helpful to explicitly right out a, b, c. Ensure that you observe the sign of each number) $$x = \frac{4 \pm \sqrt{16 - (4 \times 2 \times -1)}}{4}$$ $$= \frac{4 \pm \sqrt{16+8}}{4} = \frac{4 \pm \sqrt{24}}{4}$$ $$= \frac{4 \pm 2\sqrt{6}}{4} = 1 + \frac{1}{2}\sqrt{6}\ or\ 1 - \frac{1}{2}\sqrt{6}$$ **Test Your Understanding:** a. Find solutions to $3x^2 - x - 5 = 0$, giving your solutions to 3 significant figures. b. Find solutions to $x^2 + 4x - 44 = 0$, giving your solutions in the form $a \pm b\sqrt{3}$. c. Find solutions to $2x^2 + 5x - 1 = 0$, giving your solutions in exact form.	
91. Select and apply algebraic and graphical techniques to solve simultaneous equations where one is linear and one quadratic	The 'graphical' approach to approximating the solutions is to sketch the lines representing the two equations, and find the point(s) of intersection. See (85) The 'algebraic' approach is as such: • Step 1: Rearrange your linear equation to make either x or y the subject. If for example you had $x + 2y = 1$, it would be easier to rearrange to make $x = 1 - 2y$ than it would be to have $y = \frac{1-x}{2}$ • Step 2: Substitute this expression into the quadratic equation. • Step 3: Simplify and solve (ensuring you get 0 on one side first) • Step 4: For each solution (for either x or y), work out the value of the other variable (e.g. using your rearranged linear equation). • Step 5: Check your solutions by substituting them into the original equations. **Example**: Solve the simultaneous equations: $$x^2 + y^2 = 5$$ $$x - y = 1$$ Step 1: $x = 1 + y$ Step 2: $(1 + y)^2 + y^2 = 5$ Step 3: $(1 + y)(1 + y) + y^2 = 5$ (notice we repeated the brackets) $1 + 2y + y^2 + y^2 = 5$ (a <u>VERY common error</u> here is to forget the original y^2) $$2y^2 + 2y - 4 = 0$$ $$y^2 + y - 2 = 0$$ $$(y + 2)(y - 1) = 0$$ $$y = -2\ or\ y = 1$$ Step 4: Using $x = 1 + y$, $x = -1\ or\ x = 2$ Step 5: Checking. When $x = 2$ and $y = 1$: $2^2 + 1^2 = 5$ (correct) and $2 - 1 = 1$ (correct) When $x = -1$ and $y = -2$: $(-1)^2 + (-2)^2 = 5$ (correct) and $-1 - (-2) = 1$ (correct) **Test Your Understanding:** Solve the following simultaneous equations, giving each answer to 3 significant figures where relevant. a. $x^2 + y^2 = 8$ $y = x + 4$ b. $x^2 + y^2 = 5$ $x - 3y = 5$ c. $y = x^2 - x - 2$ $x + 2y = 10$ (giving your solutions to 3 significant figures)	

92. Solve equations involving algebraic fractions which lead to quadratic equations	Just times both sides of your equation by any denominators there might be. **Example**: "Solve $\frac{2}{x^2}-\frac{3}{x}=4$, giving your answer to 3sf." If we times both sides by x^2, we get: $2-3x=4x^2$ $4x^2+3x-2=0$ Then we proceed to use the quadratic formula. **Test Your Understanding:** a. Find the solutions to $\frac{1}{x^2}+\frac{1}{x}=1$, giving your solutions to 3sf. b. Find the solutions to $\frac{2}{x}+3=\frac{5}{x^2}$, giving your solutions to 3sf. c. Find the solutions to $\frac{(2x+2)^2}{x+1}=5x+1$	
93. Derive the quadratic equation by completing the square	You do not need to know this for the exam.	

Further Graphs and Functions		
94. Plot and recognise cubic, reciprocal, exponential and circular functions.	You need to be able to recognise the shapes of different graphs. Recall that a line is all the points which satisfy some given equation. For example if the equation of the line is $x^2+y^2=9$, then $(0,3)$ would be a point on your line because $0^2+3^2=9$.  In the above equations, x and y are variables and a,b,c,d are constants. Properties: • **Quadratic:** Either U shapes of ∩ shaped depending on whether the coefficient of x^2 is positive or negative respectively. Quadratics either have a single minimum point or a single maximum point. They may or may not cross the x axis. • **Cubic:** Depending on whether the coefficient of x^3 is positive or negative, you either get an 'uphill rollercoaster' shape or a 'downhill rollercoaster' respectively, each with two 'turns'. Cubics always have at least one root because the line must pass the x-axis at some point. • **Reciprocal:** There is no point on the line when $x=0$ because in $y=\frac{2}{x}$ for example, you can't divide by 0. Similarly y can't be 0 because you would have to divide 2 by infinity. Make sure you can distinguish reciprocal graphs when the numerator is positive and when it is negative (if you forget, just consider what you'd get say if say $x=1$ and consider where this point would be on the graph). • **Exponential:** Note firstly that the y value is always positive (assuming we restrict the base to a positive value, which is usually the case), hence the graph is always above the x-axis. If $y=a\times b^x$, then when $x=0$, $y=a\times b^0=a$. • **Circle:** If $x^2=y^2=r^2$, the circle is centred at the original and has radius r. So for example if $x^2+y^2=5$, then the radius is $\sqrt{5}$.	

<table>
<tr>
<td>95. Plot and recognise trigonometric functions $y = \sin x$ and $y = \cos x$, within the range $-360°$ to $+360°$</td>
<td>
You need to be able to draw the graphs of $y = \sin x$ and $y = \cos x$. You should start by labelling your y-axis with just -1 and 1 (as both graphs' y-values can only vary between -1 and 1) and x-axis with multiples of 90° (these gives points of interest as the y values at these points will either be -1, 0 or 1).

sin starts at 0 and initially goes up (oscillating between 0, 1, 0, -1, 0) whereas cos starts at 1 and therefore must initially go down (oscillating between 1, 0, -1, 0, 1). Both graphs then repeat (and thus can be duplicated if you have to sketch the graph when x is negative)

<table>
<tr><td>$y = \sin x$

</td><td>$y = \cos x$

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<tr><td></td><td></td></tr>
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<tr>
<td>96. Use the graphs of these functions to find approximate solutions to equations, eg given x find y (and vice versa)</td>
<td>You did this in (85) with quadratics – the principle is exactly the same for other graphs (e.g. a reciprocal graph intersecting with a linear one).</td>
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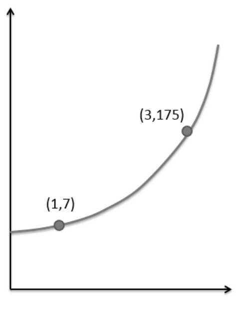
<tr>
<td>97. Find the values of p and q in the function $y = pq^x$ given the graph of $y = pq^x$</td>
<td>
$y = pq^x$ is an exponential function. The key to finding p and q is using strategic points on the graph.

Example: Suppose $y = pq^x$ goes through the point $(0,3)$ and $(2,48)$, where p, q are positive constants.
Notes: For $(0,3)$, $x = 0$, $y = 3$, so substituting into the equation:
$$3 = pq^0 = p \times 1 = p$$
Thus $p = 3$. Then substituting $x = 2, y = 48$ (and $p = 3$) using the second point into $y = pq^x$:
$$48 = 3 \times q^2$$
$$q^2 = 16$$
$$q = 4$$
Thus our function is $y = 3 \times 4^x$.



The questions are slightly harder if x is not 0 for one of the points:
Example: "The graph shows two points (1,7) and (3,175) on a line with equation:
$$\boldsymbol{y = ka^x}$$
Determine k and a (where k and a are positive constants)."

Substituting using our points, we get two simultaneous equations:
$$7 = ka \qquad (1)$$
$$175 = ka^3 \quad (2)$$
With simultaneous equations in the past you've eliminated by adding or subtracting. But it's also possible to divide! Dividing (2) by (1):
$$27 = a^3$$
$$a = 3$$
Then substituting back into equation (1):
$$7 = k \times 3$$
$$k = \frac{7}{3}$$
Test Your Understanding:
a. Given that (2,6) and (5,162) are points on the curve $y = ka^x$, find the value of k and a.
b. Given that $(3, 45)$ and $\left(1, \frac{9}{5}\right)$ are points on the curve $y = a^2 b^x$ where a and b are <u>positive</u> constants, find the value of a and b.
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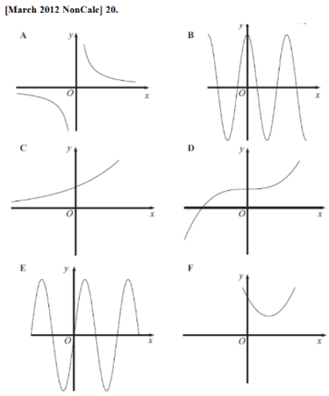
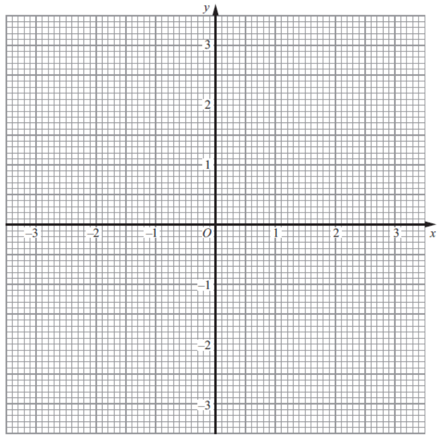
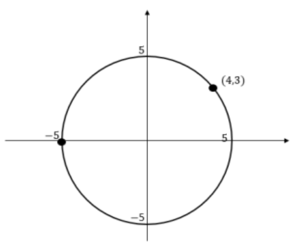
</table>

98. Match equations with their graphs	Use the shapes given above. In the case where you have multiple graphs of the same type, e.g. two cubics where both the x^3 terms are positive, then you need to use other features to distinguish them: the most obvious feature is their y-intercept, which recall you can find by just setting $x = 0$ to see what y value it gives, and comparing this to the graph. In (iii) below for example, you can see the y-intercept is 5.  **Test Your Understanding:** Match the graphs with their equations: i. $y = 4\sin x$ ii. $y = 4\cos x$ iii. $y = x^2 - 4x + 5$ iv. $y = 4 \times 2^x$ v. $y = x^3 + 4$ vi. $y = \frac{4}{x}$	
99. Find the intersection points of a given straight line with this circle graphically	**Test Your Understanding:** (a) Construct a graph of $x^2 + y^2 = 9$  (b) By drawing the line $x + y = 1$ on the grid, solve the equations $x^2 + y^2 = 9$, $x + y = 1$ x = ... , y = ... or x = ... , y = ...	
100. Select and apply construction techniques and understanding of loci to draw graphs based on circles and perpendiculars of lines	You may for example be asked to construct the perpendicular bisector of two points on the circumference of the circle. Due to the circle theorem ("perpendicular bisector of a chord passes through the centre of the circle"), you should find your constructed line passes through the centre of the circle.  **Test Your Understanding:** The graph shows the line with equation $x^2 + y^2 = 25$. The points $(4,3)$ and $(-5,0)$ lie on the line. Construct the perpendicular bisector of these points.	

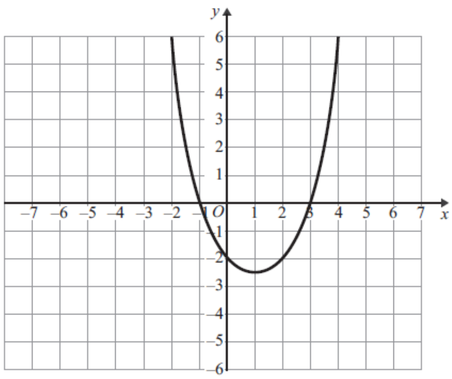
Transformations of Functions		
101i. Apply to the graph of $y = f(x)$ the transformations $y = f(x) + a$, $y = f(ax)$, $y = f(x + a)$, $y = a\,f(x)$ for linear, quadratic, sine and cosine functions	If for example we had the function $y = f(x)$ then $y = f(x + a)$ is a new function because we're modifying the input to the function. This will affect how we draw the graph. If you were to think of this as a 'number machine' where f is a box in your number machine that transforms some input into some output, then $f(x + a)$ represents adding a to the input before it goes through the f number machine. All function transformations can be summarised using the following table:	

	Affects which axis?	What we expect or opposite?
Change <u>inside</u> $f(\ )$	x	Opposite
Change <u>outside</u> $f(\ )$	y	What we expect

Examples:

- $f(x+2)$: Change is inside brackets, so we do the opposite to x, i.e. shift the graph left 2 units.
- $f(2x)$: Halves the x values (so 'squashed' horizontally)
- $3f(x)$: Triples the y values.
- $f(x-2)+1$: Shift right 2 units and shifts up 1 unit.
- $2f\left(\frac{x}{3}\right)-4$: Stretch horizontally be a factor of 3, stretch vertically by factor of 2 (i.e. double y values) and move down 4 units.
- $f(-x)$: By the table above, the x values switch from negative to positive and vice versa. This is a reflection in the y-axis.
- $-f(x)$: Similarly, the y values switch between positive and negative, so this is a reflection in the x-axis.
- $-f(-x)$: By reflecting in both axis, this is equivalent to a 180° rotation.

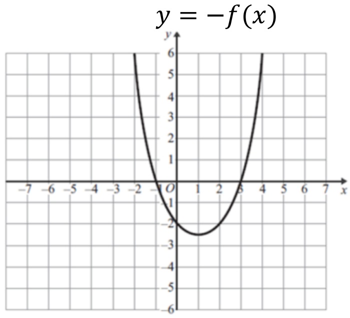
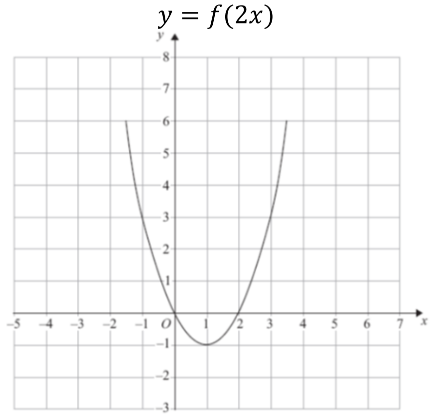
Very important point: When transforming graphs that have already been drawn for you, **<u>you MUST start by transforming the points which are exactly on the gridlines</u>** (i.e. have integer coordinates), as these points will be checked in the mark scheme.



Example: "The graph of $y=f(x)$ is shown on the grid. On the same axis, draw $y=f(x-3)$"

By our rules above, this is a translation 3 units right. The points that are exactly on the grid points on $y=f(x)$ are
$(-2,5),(-1,0),(0,-2),(2,-2),(3,0),(4,5)$
Translating each of these points 3 right, we get:
$(1,5),(2,0),(3,-2),(5,-2),(6,0),(7,5)$
Plot each points then join them up with a curved line to match the original graph. Using this method, you're guaranteed full marks for your drawing.

Test Your Understanding:

a. On this grid, sketch the graph of $y=-f(x)$ 	b. On this grid, sketch the graph of $y=f(2x)$ 

101ii. Determine the effect of transformations on specific points.	Some questions just give you points in isolation and ask you to calculate the coordinate after a transformation. Just use the rules above.

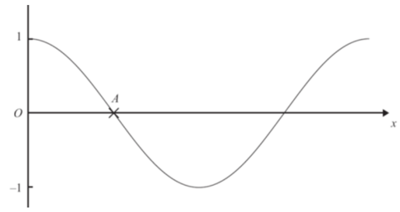
Some questions just give you points in isolation and ask you to calculate the coordinate after a transformation. Just use the rules above.

Examples:

- "If $(3,-2)$ is a point on $y=f(x)$, what is the transformed point on $y=f(x+2)$?" We subtract 2 from x and y is unaffected, so $(1,-2)$
- "If $(3,-2)$ is a point on $y=f(x)$, what is the transformed point on $y=f(2x)+1$? x value is halved and we add 1 to y value, so $\left(\frac{3}{2},-1\right)$
- "If $(3,-2)$ is a point on $y=f(x)$, what is the transformed point on $y=-f(x)$?" y value is negated and x value is unchanged, giving $(3,2)$.

Test Your Understanding: The graph $y=f(x)$ has a maximum value of $(-4,3)$. Calculate the maximum point for the graph with equation $y=f(-x)$.

<table>
<tr>
<td>101iii. Consider the effect of transformations on trigonometric graphs.</td>
<td>
Exactly the same rules as above apply. Recall earlier where we chose points exactly on the grid to transform before joining in between. This time, you should consider the coordinates of the peaks, troughs and the intercepts with the x and y axis, and transform each.

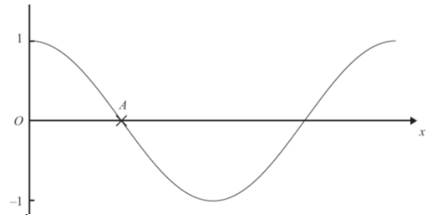


Example: The diagram shows a sketch of $y = \cos x$.
a) Determine the coordinates of the point A.
This is $(90,0)$
b) On the same diagram, draw a sketch of $y = 2\, cos\, x$.

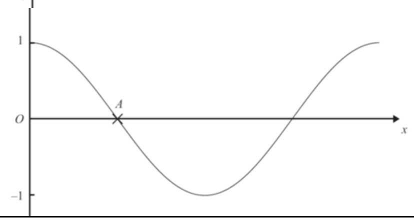
This modification of "$\times$ 2" is outside the cos function, so it affects the y axis and does what we expect. As per the advice in the notes, consider the points $(0,1), (90,0), (180,-1), (270,0), (360,1)$. After transformation these become $(0,2), (90,0), (180,-2), (270,0), (360,2)$. Plot these and join up with the same curved shape.

More examples: Consider the graph of $y = \sin(x)$. What transformation would you need for:

$y = \sin(x) + 1$: The graph shifts up 1 unit, so that the y values vary between 0 and 2.
$y = \sin(2x)$: The x values are halved.
$y = -\sin(x)$: Reflected in the x-axis.

Test Your Understanding:


a) The diagram shows a sketch of $y = \cos x$. On the same diagram, draw a sketch of $y = \cos(2x)$



b) The diagram shows a sketch of $y = \cos x$. On the same diagram, draw a sketch of $y = \frac{1}{2}\cos(\frac{1}{2}x)$
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<tr>
<td>102. Select and apply the transformations of reflection, rotation, enlargement and translation of functions expressed algebraically</td>
<td>
We can represent some of our geometric transformations on a function $y = f(x)$:
<table>
<tr><td>Rotation 180° about the origin</td><td>$y = -f(-x)$ (as this reflects in both the x and y axis).</td></tr>
<tr><td>Enlargement by scale factor a about the origin.</td><td>$y = af\left(\frac{x}{a}\right)$</td></tr>
<tr><td>Translation by $\binom{a}{b}$</td><td>$y = f(x - a) + b$</td></tr>
<tr><td>Reflection in the y-axis</td><td>$y = -f(x)$</td></tr>
<tr><td>Reflection in the x-axis</td><td>$y = f(-x)$</td></tr>
</table>
Note that you are unlikely to be asked say the transformation for a 180° rotation about the origin (i.e. you do not need to learn the above table). However, you may be given such a transformation and asked to work out the resulting graph. (Or see 103i below)
</td>
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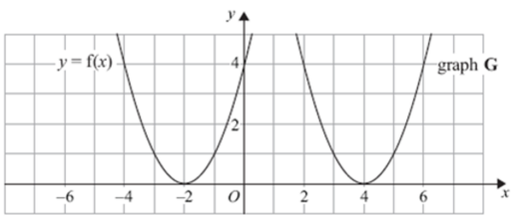
</tr>
<tr>
<td>103i. Interpret and analyse transformations of functions and write the functions algebraically</td>
<td>
This is the opposite of (102): given a drawn graph and its drawn transformation, can you specify what the transformation was using $y = f(\ )$ notation?



Example: Graph G is a translation of $y = f(x)$, as pictured below. Write the equation of G.
The translation is 6 units right. Thus the equation is $y = f(x - 6)$ (remembering that the change inside the function brackets is reversed)
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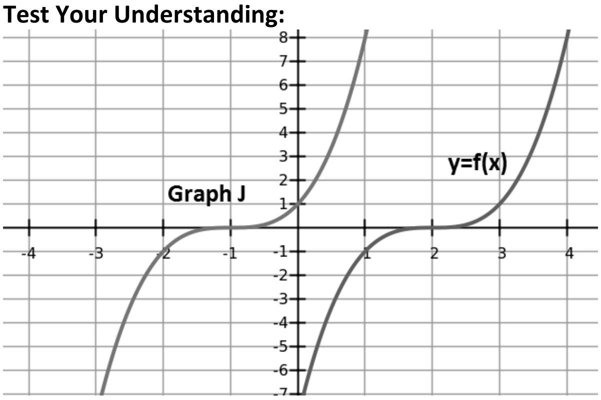
</table>

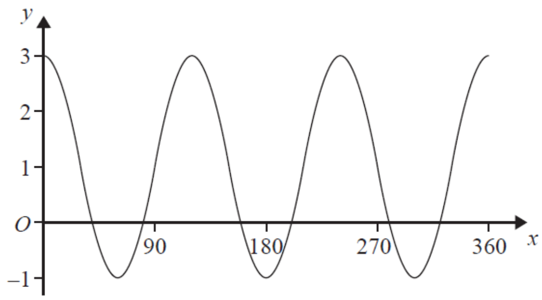
Test Your Understanding:



Graph J is a translation of $y = f(x)$. Write the equation of J.

103ii. Determine unknown parameters within a transformation.

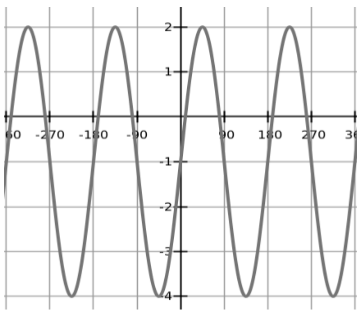
Example: "*Here is a sketch of the curve y = a cos bx° + c,   0 ≤ x ≤ 360*



Find the values of a, b and c." (June 2014)

The key here is to think how the graph of $y = \cos x$ has been transformed. The graph of $y = \cos x$ repeats every 360°, but this repeats 3 times every 360, thus it has been squashed by a factor of 3. Thus $b = 3$.
Similarly, the y value usually varies between -1 and 1, but the y value here varies between -1 and 3, i.e. with double the range. This doubling of range indicates that $a = 2$. This would mean y would vary between -2 and 2, so the graph must have been shifted up 1. Thus $c = 1$.

Test Your Understanding:



The graph shows a sketch of $y = a\sin(bx) + c$
Determine the values of a, b and c.

Proof (not explicitly in specification but implied by it)		
104. Understand parity arguments (i.e. how odd and even numbers combine) and be able to prove that an expression is always odd or even.	You should know that: • $odd \times odd = odd$ $odd \times even = even$ • $even \times even = even$ $odd + odd = even$ • $odd + even = odd$ $even + even = even$ **Example:** $m = n(n+1)$ where m and n are integers. Explain why m is always even. Do a *case analysis*, i.e. consider when m is odd, and when m is even. "If n is odd, then $n+1$ is even, and $odd \times even = even$. If n is even, then $n+1$ is odd, and so $even \times odd = even$. Therefore m is always even." **Test Your Understanding:** Prove that $n^2 + n + 1$ is odd for all integers n.	
105. Be able to generically represent consecutive numbers, even numbers, consecutive even numbers, and so on.	If you have to form a proof which starts something like "for any two consecutive integers", you need to represent ANY possible consecutive integers – it is not sufficient to prove for specific examples. We represent the numbers ALGEBRAICALLY. • Two consecutive integers: $x,\quad x+1$ • Three consecutive integers: $x,\quad x+1,\quad x+2$ However, it often makes the maths easier if the numbers are $x-1, x,\ x+1$ as you find terms often cancel. • Even integers: $2x$ • Odd integers: $2x+1$ (this works because all odd integers are 1 more than a multiple of 2) • Two consecutive odd integers: $2x-1$, $2x+1$ (or $2x+1, 2x+3$)	
106. Know how to prove that an expression is a multiple of some number.	**Example**: To show that $24n$ is a multiple of 8, we'd just need to rewrite it as $8(3n)$, i.e. we explicitly need to show the factor of 8 by factorising it out of our expression. **Test Your Understanding:** Show that $3n+9$ is a multiple of 3.	
107. Form proofs about related numbers and expressions.	**Example:** "*Prove that the difference of the squares of two consecutive odd integers is a multiple of 8*". As above, we need to show it's possible for ANY two consecutive odd integers, so we need to represent them algebraically: • First odd integer: $2x-1$ • Second odd integer: $2x+1$ • Difference of squares: $(2x+1)^2 - (2x-1)^2$ Then expanding and simplifying: $(2x+1)(2x+1) - (2x-1)(2x-1)$ $= 4x^2 + 4x + 1 - (4x^2 - 4x + 1)$ $= 4x^2 + 4x + 1 - 4x^2 + 4x - 1$ $= 8x$ which is a multiple of 8. **Test Your Understanding:** a. Prove that the sum of three consecutive numbers is a multiple of 3. b. Prove that the difference of the squares of two consecutive integers is the sum of the two integers. c. Prove that $(2n+3)^2 - (2n-3)^2$ is a multiple of 8 for all values of n	

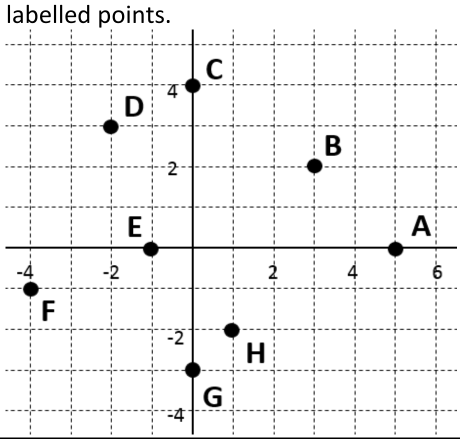
Shape, Space and Measures

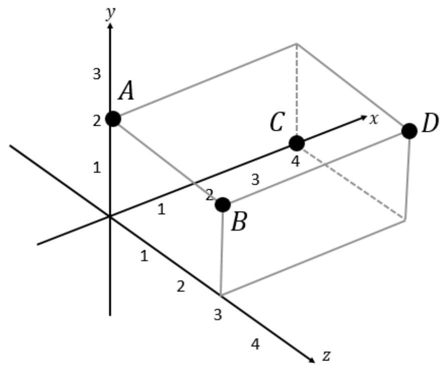
Coordinates

108. Use axes and coordinates to specify/identify points in all four quadrants in 2-D and 3-D

Test Your Understanding:

Identify the coordinates of each of the labelled points.	Identify the coordinates of each of the labelled points (remember that points in 3D go (x, y, z))





109. Find the coordinates of the midpoint of a line segment, *AB*, given the coordinates of *A* and *B*

Simply find the average of each of x and y (and z if a 3D coordinate), by adding the two values and dividing by 2.

Example: The midpoint of $(0,7)$ and $(4,10)$ is $(2, 8.5)$.

Test Your Understanding:

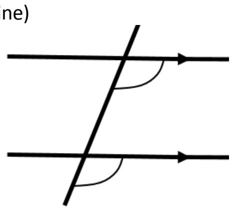
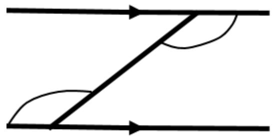
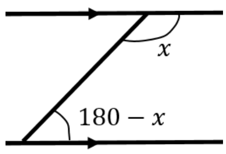
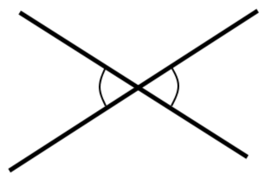
a) Find the midpoint of $(2,6)$ and $(3,-2)$
b) Find the midpoint of $(1,0,6)$ and $(9,-2,-10)$

Shape and Angle

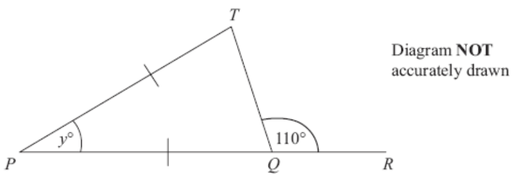
110. Recall and use properties of angles

- Angles at a point
- Angles at a point on a straight line
- Perpendicular lines
- Vertically opposite angles
- Corresponding angles
- Alternate angles
- Cointerior angles

Ensure you can quote the following when asked to justify your steps. These form 'communication marks' in the mark scheme. You WILL NOT be credited for using "Z angles" (instead of alternate), "F angles" (instead of corresponding), or "C angles" (instead of cointerior)

"Angles at a point sum to 360°"	"Angles in a triangle sum to 180°"
"Angles on a straight line sum to 180°"	
"Corresponding angles are equal" (the way to spot these is to notice when an angle can be 'shifted' from parallel line to another across a connecting line) 	"Alternate angles are equal" (note that the two angles must be on the inside of the parallel lines, and that the angles on alternate (i.e. opposite) sides of the line connecting the parallel ones) 
"Cointerior angles add to 180°" (you can avoid using this one by combining say corresponding angles with angles on a straight line as you reasons, but it's nicer to use one step instead of two) 	"Vertically opposite angles are equal" 

Example:



PQR is a straight line. *PT* = *PQ*.
(i) Work out the value of *y*.
$y = 180 - (2 \times 70) = 40°$
(ii) Give reasons for your answer.
"Angles on straight line add to 180. Base angles of isosceles triangle are equal. Angles in triangle sum to 180."

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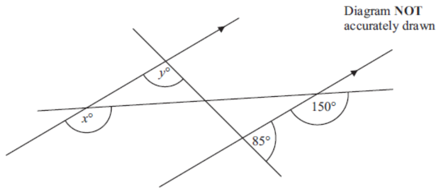
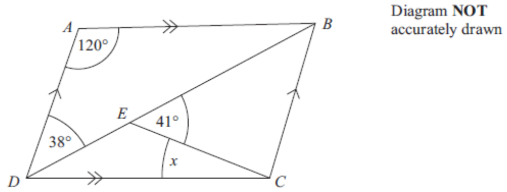
Test Your Understanding:

a) (i) Determine the angle x. (ii) Determine the angle y, giving a reason for your answer.



b) Calculate x. You must give reasons for your answer.



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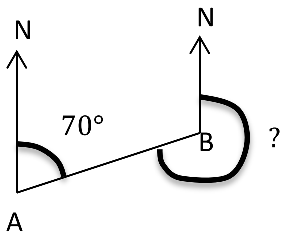
<tr>
<td>111. Understand draw and measure bearings. Calculate bearings and solve bearings problems.</td>
<td>

Bearings are measured <u>clockwise</u> from <u>North</u>.

- When measuring a bearing, ensure the $0°$ on your protractor is pointing North and that you use the right ring of angles on your protractor (outer vs inner). To measure an angle of more than $180°$, measure the angle anticlockwise from North and subtract from $360°$.
- Be very careful about the wording "the bearing of B from A". The bearing is being measured at A". Similarly with "the bearing of A to B", the bearing is again being measured at A".
- For questions such as "If the bearing of B from A is $70°$, what is the bearing of A from B?", then the following diagram solves this: (which need not be drawn using a protractor, as you'll be using laws of angles to solve)



Because the smaller angle at B is 110° (cointerior angles sum to 180), the bearing of A from B must be $360° - 110° = 250°$.
The simple way to going from "A from B" to the opposite, "B from A", is to add $180°$ if the original bearing is $180°$, and add subtract $180°$ otherwise.

Test Your Understanding:

a) The bearing of B from A is $300°$. What is the bearing of A from B?".

b) A ship S is 50km away and a bearing of $65°$ from a lighthouse L. A whale W is 60km from the lighthouse, at a bearing of $200°$ from the lighthouse. (i) Using a scale of 1cm : 10km, draw a the positions of the ship, whale and lighthouse. (ii) Hence estimate the distance from the whale to the ship.

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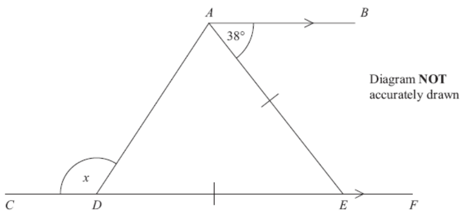
<tr>
<td>112. Distinguish between scalene, isosceles, equilateral, and right-angled triangles.
Use the size/angle properties of isosceles and equilateral triangles.</td>
<td>

Recall:

- Equilateral triangle: All sides the same length (and angles all $60°$)
- Isosceles triangle: Two of the sides (and two of the angles) are the same.
- Scalene triangle: All sides and angles are different.
- Right-angled triangle: One angle is $90°$, opposite the hypotenuse. The remaining two angles sum to $90°$.

We get two extra sentences we can use for justify angle calculations:

- "Angles in an equilateral triangle are $60°$"
- "<u>Base</u> angles of an isosceles triangle are equal."



Test Your Understanding:
Work out the size of angle x. You must give reasons for your answer.

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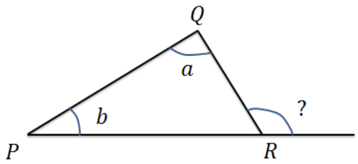
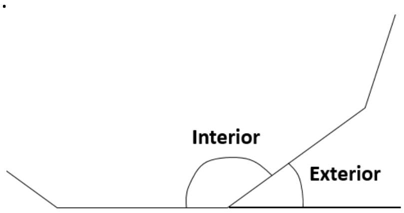
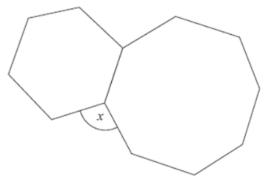
113. Understand and use the angle properties of quadrilaterals. Explain why angle sum is 360.	The angles in a quadrilateral add to $360°$. (This stems from the formula, explored later, for the sum of the interior angles of a polygon in general) **Test Your Understanding:** a. The angles of a quadrilateral are $3x$, $2x-5$, $x+10$ and $2x+15$. Determine x.	
114. Understand a proof that the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices.	Suppose that, as per the diagram below, $\angle PQR = a$ and $\angle QPR = b$. Then $\angle QRP = 180 - a - b$ (angles in a triangle sum to 180). Then the other angle at R is $180-(180-a-b) = a+b$ (angles on a straight line add to 180). Thus the exterior angle of the triangle is the sum of the interior angles at the other two vertices. 	
115. Give reasons for angle calculations.	Learn these verbatim to quote word for word in an exam: - "Angles in a triangle sum to 180" - "Base angles of an isosceles triangle are equal" - "Angles on a straight line sum to 180" - "Alternate angles are equal" (don't just write "alternate angles") - "Corresponding angles are equal". - "Vertically opposite angles are equal". - "Cointerior angles sum to 180"	
116. Understand what is meant by an interior angle and exterior angle, and that the two at any given point sum to 180°	Recall that an exterior angle is the angle between an extended side of the polygon and the adjacent side of the polygon. As can be seen from the diagram, the exterior and interior angles clearly add to $180°$. 	
117. Calculate the sum of the interior angles of an n-sided polygon. Use that sum of exterior angles of polygon is 360.	• The sum of the exterior angles of ANY polygon is $360°$. You can remember this by imagining yourself walking around the polygon. Each 'turning angle' is the exterior angle, and you would have made a full spin by the time you get back to the start. • The total interior angle is: $$180(n-2)$$ (because the polygon can be divided up into $n-2$ triangles) **Test Your Understanding:** a. What is the sum of the interior angles of a regular 15 sided shape? b. Four of the interior angles in a pentagon are $100°$. What is the fifth interior angle? c. Five of the exterior angles of a hexagon are $50°$. Find the remaining exterior angle.	
118. Find the size of each interior angle or the size of each exterior angle or the number of sides of a regular polygon	**Examples:** • *"What is each exterior angle of a regular hexagon?"* Total exterior angle is 360° therefore: $$\frac{360}{6} = 60$$ • *"What is each interior angle of a regular hexagon?"* Interior and exterior angles sum to 180, therefore: $$180 - 60 = 120°$$ • *"The interior angle of a regular polygon is $150°$. How many sides does it have?"* If 360 divided by the number of sides gives each exterior angle, then 360 divided by the exterior angle must give the number of sides. Exterior angle $= 180 - 150 = 30°$ Number of sides $= \frac{360}{30} = 12°$ **Test Your Understanding:** a. A regular polygon has an exterior angle of $3°$. How many sides does it have? b. A regular polygon has an interior angle of $175°$. How many sides does it have? c. A regular polygon has 15 sides. What is each interior angle?	



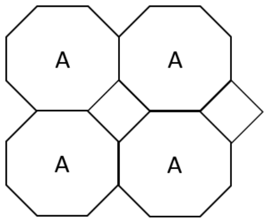
d. The diagram on the left shows a regular hexagon and a regular octagon. Calculate the angle x.

119. Use geometric language appropriately and recognise and name pentagons, hexagons, heptagons, octagons and decagons

Simply a case of memorisation.

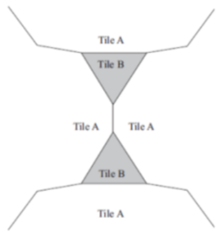
Num sides	**Name**	**Num sides**	**Name**
4	Quadrilateral	8	Octagon
5	Pentagon	9	Nonagon
6	Hexagon	10	Decagon
7	Heptagon (not 'septagon'!)		

120. Understand tessellations of regular and irregular polygons and combinations of polygons. Explain why some shapes tessellate when other shapes do not.



Shapes tessellate (whether multiple copies of the same shape or different shapes) when the interior angles joining at a point sum to $360°$. For example, a hexagon tessellates with itself as we can join 3 hexagons each with interior angle $120°$, whereas a pentagon does not tessellate because its interior angle of $108°$ is not a factor of 360.

More complicated problems might require to you to find the interior angle first before calculating the number of sides
e.g. "*Two copies of a regular polygon A and a square tessellate as pictured. Prove that A is an octagon.*"
At any point in the diagram, there is one square and two copies of A. Thus interior angle of A is $\frac{360-90}{2} = 135°$.
Exterior angle of $A = 180 - 135 = 45°$
Number of sides of $A = \frac{360}{45} = 8$, thus A is an octagon.



Test Your Understanding: A pattern is made up of two tiles, A and B, as pictured on the right. Both tiles are regular polygons. Work out how many sides Tile A has.

2D and 3D Shapes

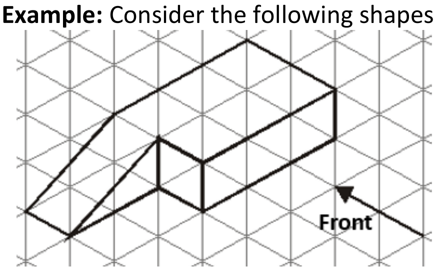
121. Use 2-D representations of 3-D shapes. Use isometric grids. Draw nets and show how they fold to make a 3-D solid

An isometric grid is one which consists of equilateral triangles, inside of the usual squares.

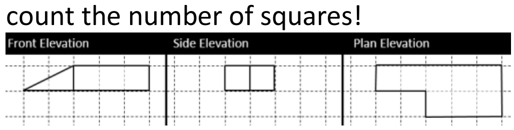
122. Understand and draw front and side elevations and plans of shapes made from simple solids. Given the front and side elevations and the plan of a solid, draw a sketch of the 3-D solid

Recall that the plan is the view from the top, the front elevation is the horizontal view from the designated 'front' (which will be indicated), and the side elevation the horizontal view from the side.

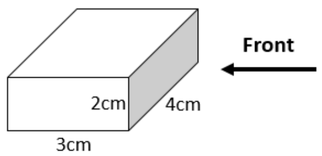
Example: Consider the following shapes sketched isometrically:

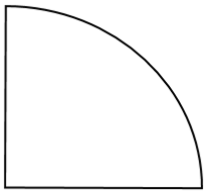
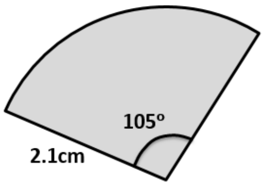
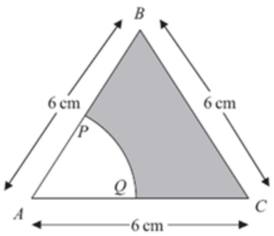
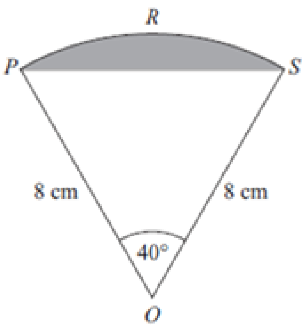


Then the plan, front elevation and side elevation are as follows. Ensure you correctly count the number of squares!



Test Your Understanding: A cuboid is $4cm \times 2cm \times 3cm$ as pictured. On a square grid, draw the plan, front elevation and side elevation.



Perimeter, Area		
123. Calculate perimeters of shapes made from triangles and rectangles		
124. Recall and use the formulae for the area of a triangle, rectangle and a parallelogram	Area of triangle: $\frac{1}{2} \times base \times perpendicular\ height$ Area of parallelogram: $base \times perpendicular\ height$ Area of kite: $\frac{1}{2} \times width \times height$	
125. Calculate perimeter and area of compound shapes made from triangles, rectangles and other shapes		
126. Find circumferences of circles and areas enclosed by circles	• Area of circle: πr^2 • Circumference of circle: $2\pi r$ (ensure that if provided the diameter, you halve it)	
127. Appreciate how to leave answers 'in terms of π'.	This simply means that you appreciate that answers can be left "in terms of π", e.g. an answer may be given as "5π". This allows certain answers to be expressed exactly. We can collect like terms where appropriate: $\pi + 6\pi \rightarrow 7\pi$	
128. Find the perimeters and areas of semicircles and quarter circles	 6cm **Example:** "Find the area and perimeter of this shape." It's a quarter circle so $Area = \frac{\pi \times 6^2}{4} = 9\pi\ cm^2$ Perimeter consists of quartile circle and two straight lines: $Perimeter = 6 + 6 + \frac{2 \times \pi \times 6}{4} = 12 + 3\pi\ cm$ **Test Your Understanding:** a. Find the area and perimeter of a semicircle with diameter 10cm. b. Find the area and perimeter of a quarter circle of radius 10cm.	
129. Calculate the lengths of arcs and the areas of sectors of circles	• Area of sector: $fraction\ of\ circle \times area\ of\ whole\ circle$ $= \frac{\theta}{360} \times \pi r^2$ • Length of arc: $fraction\ of\ circle \times circumference\ of\ whole\ circle$ $= \frac{\theta}{360} \times 2\pi r$  **Example:** $Area = \frac{105}{360} \times \pi \times 2.1^2 = 4.04cm^2$ $Perimeter = 2.1 + 2.1 + \left(\frac{105}{360} \times 2 \times \pi \times 2.1\right) = 8.05cm$ (Notice that perimeter includes the straight lengths)  **Test Your Understanding:** Determine the area of the shaded region (note: you will need the formula for the area of a non-right angle triangle: $A = \frac{1}{2}ab \sin C$)	
130. Find the area of a segment of a circle given the radius and length of the chord	 (Note: Revise this only after you've covered non-right angle triangles) To find the area of a segment: (i) Start with the area of the sector then (ii) 'cut out' the triangle, using the formula $A = \frac{1}{2}ab \sin C$ (i.e. the area of a non-right angle triangle). **Example**: "Calculate the area of the shaded segment, and the length of the chord PS." Area of sector $= \frac{40}{360} \times \pi \times 8^2 = \frac{64}{9}\pi$ Area of triangle $= \frac{1}{2} \times 8 \times 8 \times \sin 40 = 20.5692$ $\therefore$ Area of segment $= \frac{64}{9}\pi - 20.5692 = 1.77cm^2$ Use cosine rule to find length of chord PS: $PS^2 = 8^2 + 8^2 - (2 \times 8 \times 8 \times \cos 40)$ $PS = 5.47cm\ to\ 3sf$ **Test Your Understanding:** a. Find the area of segment with radius 5.2cm and angle at the circle centre of 70°. b. Find the length of this chord.	

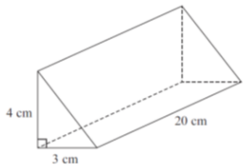
131. Convert between units of area	Since a $1m \times 1m = 1m^2$ square is the same as a $100cm \times 100cm = 10\ 000cm^2$ square, therefore $1m^2 = 10\ 000cm^2$. That is, whenever we convert between area units, we have to multiply/divide by the scale factor squared. **Test Your Understanding:** a. What is $4.5m^2$ in cm^2? b. What is $3cm^2$ in mm^2?	

Surface Area and Volume

132. Know and use formulae to calculate the surface areas and volumes of cuboids and right-prisms

For surface area, simply calculate the area of each face individually and add up.
For volume of a prism, use $Volume = Area\ of\ cross\ section \times length$

Example:



a) Find the surface area of the following triangular prism.
b) Find its volume.

Using Pythagoras, the longer length is $\sqrt{3^2+4^2} = 5cm$

Surface area $= \left(\frac{1}{2} \times 4 \times 3\right) + \left(\frac{1}{2} \times 4 \times 3\right) + (3 \times 20) + (4 \times 20) + (5 \times 20) = 252cm^2$

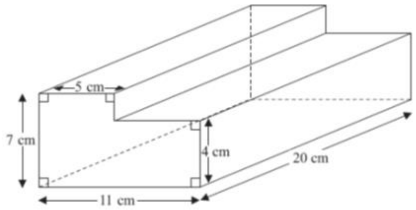
Ensure you count the number of sides to check you haven't missed any, and that you use the correct unit.

For volume: Cross-sectional area $= \frac{1}{2} \times 4 \times 3 = 6$

Therefore $Volume = 6 \times 20 = 120cm^3$

Common errors: Forgetting to halve when finding the area of the triangle, or getting wrong unit.

Test Your Understanding: Find the volume of this prism.



133. Find the volume of a cylinder and surface area of a cylinder

Since a cylinder is a prism with circular ends, its volume is:

$$V = area\ of\ cross\ section \times length$$
$$= \pi r^2 h$$

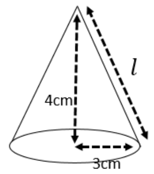
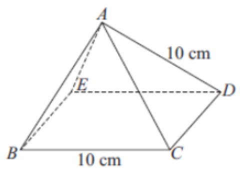
For surface area (which is not given in your formula booklet), we need area of the two ends (two lots of πr^2) and the curved surface. For the latter, imagine curving a piece of paper so it forms a hollow cylinder. The area of the paper gives the curved area we want. This is a rectangle with length $2\pi r$ (the circumference of a circle) and height h.

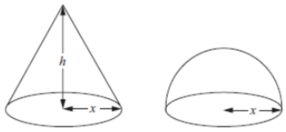
Thus:

$$surface\ area\ of\ cylinder = 2\pi r^2 + 2\pi rh$$

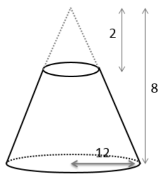
Test Your Understanding:

a. Find the surface area and volume of a cylinder with radius 6cm and length 5cm.
b. A cylindrical vase with radius 8cm and height 20cm is filled with cups of water, where each cup is a cylinder with radius 3cm and height 10cm. How many cups will fill the vase?

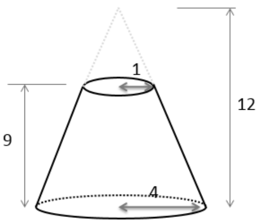
134. Find the surface area and volume of cones, spheres and hemispheres.	These formulae are given in the formula booklet: • Volume of sphere: $\frac{4}{3}\pi r^3$ • Surface area of sphere: $4\pi r^2$ • Curved surface area of cone (i.e. excluding the bottom): $\pi r l$, where l is the slant height. • Volume of cone: $\frac{1}{3}\pi r^2 h$ (where h is the perpendicular height of the cone) **Examples:** • *"A solid cone has radius 3cm and height 4cm. Find its surface area."* The area consists of the curved face and the flat bottom. We need l the slant height. We can see from the diagram we can use Pythagoras.  $l = \sqrt{3^2 + 4^2} = 5$ $Surface\ area = (\pi \times 3 \times 5) + (\pi \times 3^2)$ $= 24\pi$ • *"A solid hemisphere has radius 5cm. Determine its volume and surface area."* The volume is simply half the volume of a sphere: $\frac{1}{2} \times \left(\frac{4}{3}\pi \times 5^3\right) = \frac{250}{3}\pi = 261.799cm^3$ The surface area is half the surface area of a sphere, but we have to <u>include the flat face as well</u> (a circle) given that the hemisphere is solid: $Area\ of\ surface\ surface = \frac{1}{2} \times (4\pi \times 5^2) = 50\pi$ $Area\ of\ flat\ top = \pi \times 5^2 = 25\pi$ $Total\ surface\ area = 75\pi$ **Test Your Understanding:** a. Determine the total surface area of a solid hemisphere with radius 10m. b. Determine the volume and surface area of a cone with radius 5cm and slant height 13cm (hint: you will need to work out the perpendicular height).	
135. Find the volume of a pyramid.	**Volume of a pyramid = $\frac{1}{3} \times \boldsymbol{base} \times \boldsymbol{perpendicular\ height}$** **Example:** Find the volume of a square-based pyramid with base of side 5cm and height of 12cm. $Volume = \frac{1}{3} \times 5^2 \times 12 = 100cm^3$ More difficult problems might be where the slant height of the pyramid is given rather than the height perpendicular to the base. We have to use 3D Pythagoras. **Example:** "*The square based pyramid has sides all of length 10cm. Determine its volume.*" Suppose the centre of the base was O. We could use triangle AOD to find the height of the pyramid OA. However, we don't know the length OD. But OD is half of BD, which by  Pythagoras, is $\sqrt{10^2 + 10^2} = 10\sqrt{2}$. Then using Pythagoras on triangle AOD: $AO^2 + \left(5\sqrt{2}\right)^2 = 10^2$ $AO = \sqrt{100 - 50} = \sqrt{50}$ Thus: $Volume = \frac{1}{3} \times 10^2 \times \sqrt{50} = 235.7cm^3$ **Test Your Understanding:** a. Determine the volume of a pyramid with a rectangular base of width 6cm and length 8cm, and a slant height of 13cm (your answer should turn out to be a whole number).	
136i. Solve a range of problems involving surface area and volume, eg given the volume and length of a cylinder find the radius	**Example:** "The volume of a cylinder is $100cm^3$ and its length $5cm$. Determine its radius." Just using the relevant formula: $100 = \pi \times r^2 \times 5 \qquad r^2 = \frac{100}{5\pi}$ $r = 2.52cm$ **Test Your Understanding**: A cone has a volume of $100cm^3$ and a height of 10cm. Determine its radius.	

<table>
<tr><td>136ii. Solve problems in which the surface area or volume of two shapes is equated.</td><td>

Example: “Pictured are a solid cone and a solid hemisphere. The surface area of the cone is equal to the surface area of the hemisphere. Express h in terms of x.”
We’ll need the slant height l first of the cone as it is required in the surface area formula.
$l = \sqrt{x^2 + h^2}$
$SA_{cone} = \pi x\sqrt{x^2 + h^2} + \pi x^2$
$SA_{hemisphere} = \left(\frac{1}{2} \times 4\pi x^2\right) + \pi x^2$
Equating:
$\pi x\sqrt{x^2 + h^2} + \pi x^2 = 2\pi x^2 + \pi x^2$
$\pi x\sqrt{x^2 + h^2} = 2\pi x^2$
$\sqrt{x^2 + h^2} = 2x$
$x^2 + h^2 = 4x^2$
$h^2 = 3x^2$
$h = \sqrt{3}\, x$
Test Your Understanding:
a. A solid hemisphere with radius x has the same surface area as a cylinder with radius x and height h. Determine the height of the cylinder in terms of x.
b. A solid sphere of radius x is melted down to form a cone of radius x and height h. Determine the height of the cone in terms of x.</td><td></td></tr>
<tr><td>137. Convert between volume measures, including cubic centimetres and cubic metres</td><td>Since a $1m \times 1m \times 1m = 1m^3$ cube is the same as a $100cm \times 100cm \times 100cm = 1000\ 000cm^3$ cube, therefore $1m^3 = 1\ 000\ 000cm^3$. That is, whenever we <u>convert between volume units</u>, <u>we have to multiply/divide by the scale factor cubed</u>!

Test Your Understanding:
a. What is 300 000cm^3 in m^3?
b. What is $4.2m^3$ in cm^3?
c. What is $20cm^3$ in mm^3?</td><td></td></tr>
<tr><td>138. Solve problems involving more complex shapes and solids, including segments of circles and frustums of cones</td><td>A frustum is a cone with the top chopped off.


Example: “Work out the volume of this frustum.”
Note that if the ‘chopped off’ cone is a quarter of the size of the overall cone, it must have a quarter the radius, i.e. 3. Thus:
$V = \left(\frac{1}{3}\pi \times 12^2 \times 8\right) - \left(\frac{1}{3}\pi \times 3^2 \times 2\right) = 384\pi - 6\pi = 378\pi$

Test Your Understanding: Determine the volume of this frustum.

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<tr><td>139. Construct triangles including an equilateral triangle</td><td>

- To construct a triangles with sides of different lengths, say 4cm, 5cm and 6cm: Start by drawing a straight line with one of the lengths, say 6cm. Set your compass to one of the other two lengths, say 4cm, and draw an arc with the compass at one of the ends of your straight line. Now putting the point of the compass at the other end of the straight line, draw an arc of the last length (5cm), ensuring your arc crosses the one you drew earlier. Then connect your original straight line to this point of intersection.
- To construct an equilateral triangle, start with a straight line of any length. Simply set your compass to be the length of your line, draw an arc at each end as above, and join your line to the point of intersection.
- Note that you <u>MUST show your construction lines</u>, or you will likely receive <u>NO MARKS</u>.

Test Your Understanding:
a. Construct a triangle with lengths 10cm, 7cm and 5cm.</td><td></td></tr>
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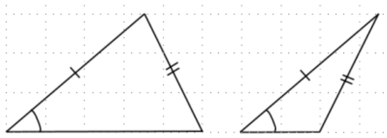
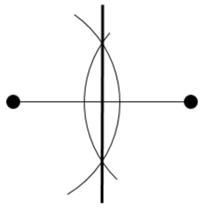
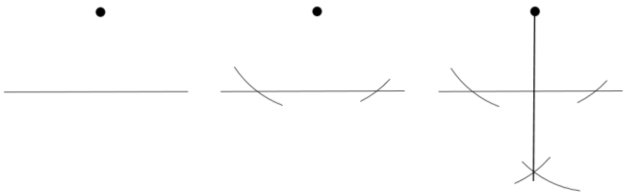
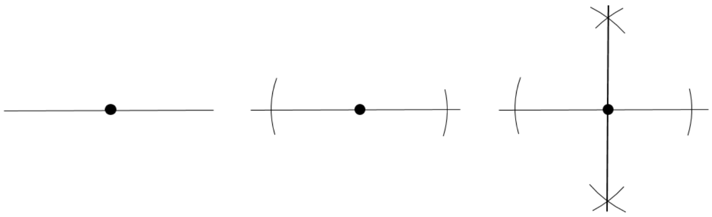
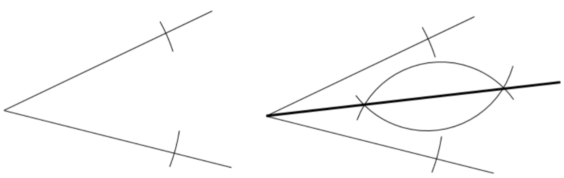
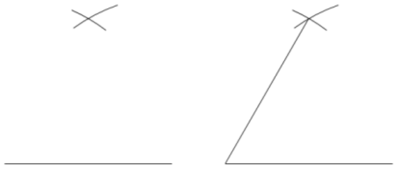
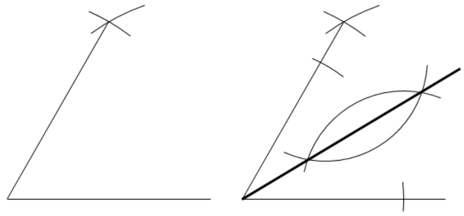
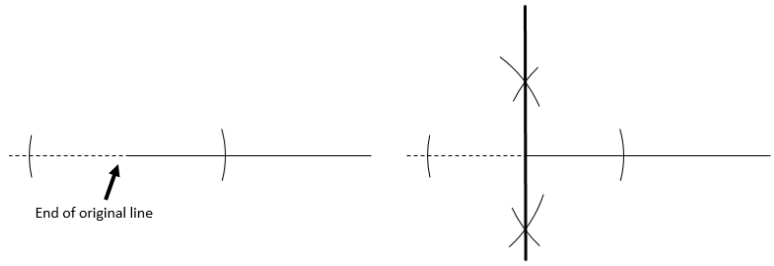
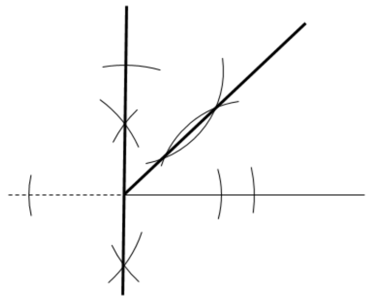
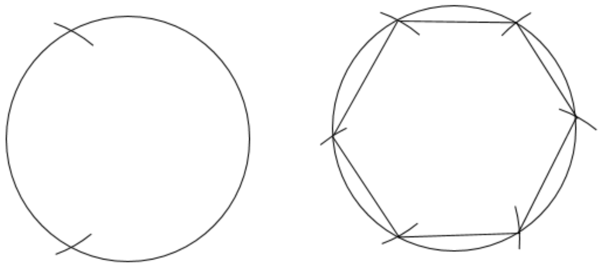
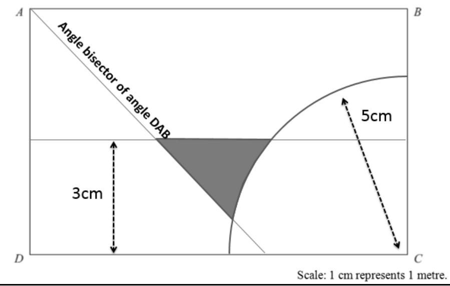
	b. Construct an equilateral triangle of length 8cm.	
140. Understand, from the experience of constructing them, that triangles satisfying SSS, SAS, ASA and RHS are unique, but SSA triangles are not	This will be covered later in ‘congruent triangle proofs’, but observe in the diagram below that we have two triangles which are the same in terms of ‘SSA’ (side-side-angle), but are clearly not congruent: 	
141. Construct the perpendicular bisector of a given line	Recall that the perpendicular bisector of two points is the line consisting of all points which are equidistant (i.e. the same distance) from these two points.  Steps are: (i) Put your compass tip on one end of the line and set the distance slightly over halfway, but not too close to halfway. (ii) Draw an arc, and then putting your compass on the other end of the line, draw the parts of a new arc necessary to overlap the first arc. (iii) Draw a line connecting the two points of intersection. Important note: You will lose marks if either (a) your perpendicular bisector isn’t long enough or (b) your two arcs don’t overlap sufficiently – for example if they only merely ‘touched’ rather than crossed at two points, the straight line will then be difficult to draw.	
142. Construct the perpendicular from a point to a line	You have a point not on the line, and want to draw a line that goes through this point that is perpendicular to a line. (i) Put your compass at the point, and using a suitable distance, mark two little arcs that intersect with the line. (ii) The perpendicular bisector of these two points will be the desired line. Thus, set your compass slightly over halfway the distance between these two points of intersection, and two intersecting arcs. Join this point of intersection with the original point.  **Test Your Understanding:** Draw a line and a point not on the line. Construct the perpendicular from this point to the line.	
143. Construct the perpendicular from a point on a line	This time, the point is on the line. Simplify use any distance with your compass to draw two little arcs either side, then find the perpendicular bisector of these two points. If the resulting line doesn’t go through the original point, you’ve gone wrong!  **Test Your Understanding:** Draw a line and a point on the line. Construct the perpendicular.	
144. Construct the bisector of a given angle	The bisector of an angle is a line which divides the angle exactly into two. (i) Put the tip of the compass at the point where the lines join, and setting your distance on the compass suitably far, draw two little arcs on the lines (the same distance), then (ii) find the perpendicular bisector of these two new points in the usual way. If this line doesn’t go through the point of intersection of the original two lines, you’ve gone wrong. As always, you MUST leave your construction lines. 	
145. Construct angles of 60º, 90º , 30º, 45º	• To construct 60°, use the same method as constructing an equilateral triangle.	

	 • To construct $30°$, draw $60°$ then construct the perpendicular bisector.  • To construct $90°$, extend your line out with a light line, and use your compass to mark two points the same distance from the end of your original line. Then construct the perpendicular bisector of these two points.  • To construct $45°$, first construct $90°$, then construct the angle bisector.  **Test Your Understanding:** Practice constructing each of these angles.	
146. Construct a regular hexagon inside a circle	Draw a circle with the compass. Do not change the distance on the compass. Starting with the tip of the compass at a point on the circumference, mark two arcs on the circumference either side. Now put the compass at one of these points of intersection, and repeat, until you have 6 points around the circle. Join up to form a hexagon. 	
147. Construct: -a region bounded by a circle and an intersecting line - a given distance from a point and a given distance from a line - equal distances from 2 points or 2 line segments - regions which may be defined by ‘nearer to’ or ‘greater than’	A locus of points is a set of points that satisfies some property, e.g. the set of all points that are equidistant from two lines A and B. Different common loci correspond to different common constructions: 1. The perpendicular bisector of two points A and B is the locus of points equidistant from the two points. 2. The angle bisector of two lines A and B is the locus of points equidistant from the two lines. 3. A circle with centre A and radius r is the locus of points r away from A. **Example:** “*Find the region closer to AB than to AD, at most 3cm away from CD, and at least 5cm from point C.*”	

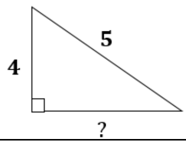
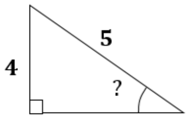
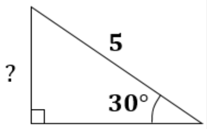
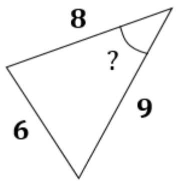
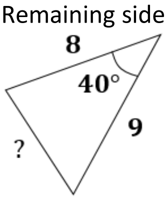
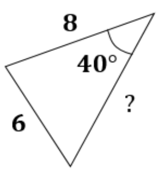
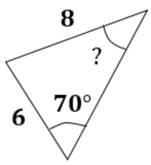


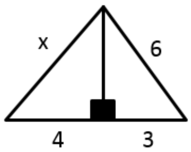
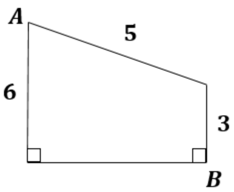
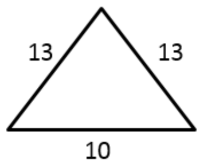
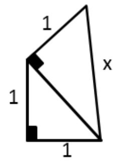
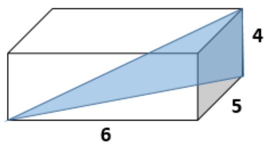
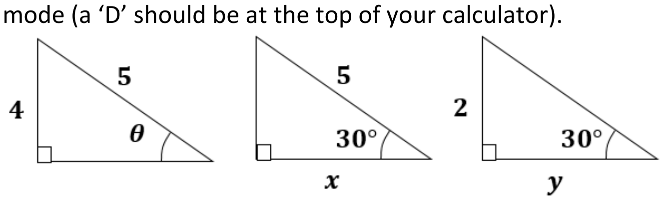
The common student mistake is to draw the angle bisector of AB and AD as the line connecting the two corners of the rectangle, AC. But this is not the angle bisector, which needs to be at $45°$! Ensure constructions lines are used for the angle bisector, and don't forget to indicate the final region.

148. Find and describe regions satisfying a combination of loci (see above)

Pythagoras and Trigonometry

149. Be able to decide which of Pythagoras, trigonometry or sine/cosine rules to use based on available information.

You have	You want	Use
Right-angled triangle with two sides known	Other side 	Pythagoras
Right-angled triangle with two sides known	An angle 	SOH/CAH/TOA
Right-angled triangle with side and angle known	Another side 	SOH/CAH/TOA
Non-right-angled triangle with all three sides known	An angle 	Cosine rule
Non-right-angled triangle with two sides known and a missing side opposite a known angle	Remaining side 	Cosine rule
Non-right-angled triangle with two sides known and a missing side <u>not</u> opposite the known angle	Remaining side 	Sine rule twice (we'll get to this)
Non-right-angled triangle with an angle and opposite side known, and another side-angle pair…	…although either the angle or the side in that second pair is missing. 	Sine rule

150. Understand, recall and use Pythagoras' theorem in 2-D. Give an answer in the use of Pythagoras' Theorem as √13	• Ensure you first identify the hypotenuse – this is the term on its own on one side of the equation in Pythagoras. • You can ONLY use Pythagoras if your triangle is right-angled. • The 'quick' way to use Pythagoras is to see if the missing side is the hypotenuse or one of the shorter sides. We want hypotenuse h: Do square root of **sum** of squares. $h=\sqrt{a^2+b^2}$ We want shorter side a: Do square root of **difference** of squares: $a=\sqrt{h^2-b^2}$ • If you're asked to give your answer in 'exact' form, leave it as a square root, as surds cannot be represented 'exactly' in decimal form. • Check your answer looks sensible. • It can help to leave a side in surd form if you will need to use it in a subsequent calculation…  **Example:** Determine x. Central length: $y=\sqrt{6^2-3^2}=\sqrt{27}$ Then: $x=\sqrt{27+4^2}=\sqrt{43}$ Notice that by leaving the central length as $\sqrt{27}$, when we used it again in the left triangle, squaring it conveniently gave us 27.  **Test Your Understanding**: a. Determine the length AB (hint: it may help to draw a line horizontally in the middle of the shape, which has the same length as the bottom length). b. Find the height of this isosceles triangle.   c. Determine x.	
151. Use Pythagoras to solve 3D problems, including the diagonal of a cuboid and the height of a pyramid.	We encountered the use of Pythagoras to find the height of a squared-based pyramid earlier in Volumes of Solids.  **Example:** "Find the internal diagonal of a cuboid with sides 4cm, 5cm and 6cm." The key with most 3D problems is to form a 2D triangle inside the shape. The bottom length of this triangle, using Pythagoras on the base of the cuboid, is $\sqrt{6^2+5^2}=\sqrt{61}$. Then using this length length on the blue triangle: $$diagonal=\sqrt{61+4^2}=\sqrt{77}=8.77cm$$ **Test Your Understanding:** a. What is the length of the internal diagonal of a cube of unit length? b. What is the length of the internal diagonal of a cuboid with side lengths 3cm, 4cm, 12cm.	
152. Recall and use the trigonometric ratios to solve 2-D and 3-D problems	Remember that trigonometry only applies to right-angled triangles. Ensure your calculator is set to 'degrees' mode (a 'D' should be at the top of your calculator).  **Examples:** Determine θ, x, y. First example: Opposite and hypotenuse are involved, so use sin (remember "soh cah toa"). $$\sin\theta=\frac{4}{5}$$ $$\theta=\sin^{-1}\left(\frac{4}{5}\right)=53.1°$$ Remember that we're trying to make θ the subject, and $\sin^{-1}$ 'undoes' the sin.	

Second example: Adjacent and hypotenuse involved, so cos.

$$\cos 30 = \frac{x}{5}$$
$$x = 5\cos 30 = 4.33$$

Third example: Opposite and adjacent involved, so tan.

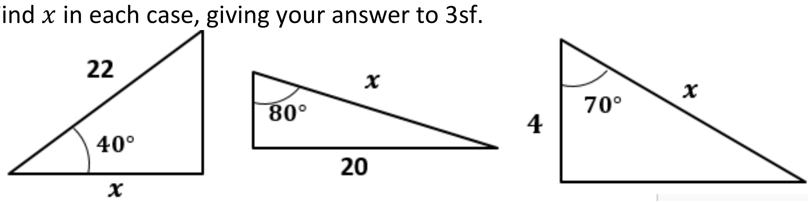
$$\tan 30 = \frac{2}{y}$$
$$y = \frac{2}{\tan 30} = 3.46$$

I call the rearrangement from the first to the second line the 'swapsie' trick. Since $\frac{10}{5} = 2$ and $\frac{10}{2} = 5$, it means we can swap the thing we're dividing by and the result.

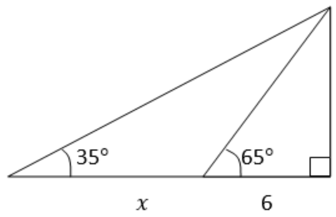
Test Your Understanding:

a. I put a ladder 1.5m away from a tree. The ladder is inclined at 70° above the horizontal. What is the height of the tree?

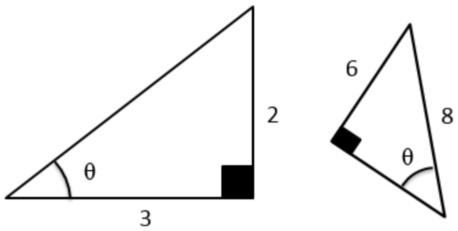
b. Find x in each case, giving your answer to 3sf.



c. Determine x (Hint: Use the smaller triangle first to determine the length of the rightmost side.



d. Determine the angle θ in each case.



153. Find angles of elevation and angles of depression

The angle of elevation is the angle something is inclined above the horizontal. The angle of depression is the angle below the horizontal.

Example: "A plane flies for some time ending up 100 miles due North. In that time it has ascended 3 miles. Find its angle of elevation."

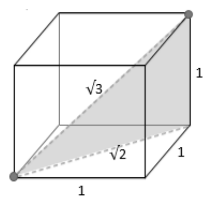
Forming an appropriate triangle, we find the angle of elevation $\theta = \tan^{-1}\left(\frac{3}{100}\right) = 1.72°$ (i.e. a very shallow angle of elevation)

154. Understand the language of planes, particularly in the context of diagonals of a cuboid. More generally, find the angle between a line and a plane (but not the angle between two planes or between two skew lines)

A plane is the 2D equivalent of a line: it is a flat 2D surface which exists in 3D.

Example: "A unit cube lies on a level surface. Determine the angle of elevation above the *horizontal plane* of the diagonal of the cube which goes from one of the bottom corners to the opposite top corner."

Using 3D Pythagoras, we obtain the following triangle. Then the angle of elevation $\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3°$



Test Your Understanding: A cuboid with base lengths 4cm and 5cm, and height 10cm, lies on a level surface. Determine the angle of elevation above the horizontal plane of the diagonal of the cuboid which goes from one of the bottom corners to

	the opposite top corner."	

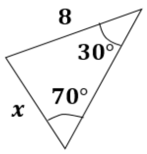
Non-Right Angled Triangles

155. Find the unknown lengths, or angles, in non right-angle triangles using the sine and cosine rules

As previously explained, there are a number of cases to consider. Ensure your answer looks sensible based on the diagram. With one exception, the way to tell between cosine and sine rule is see how many angles are involved (whether known or needs to be determined). <u>If one angle only, use cosine rule, otherwise use sine rule</u>.

1. <u>Two pairs of sides and opposite angles, where in one pair the **side** is missing.</u>
   Use the version of the sine rule where the unknown side appears at the top:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



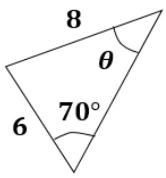
Example: Find x.

$$\frac{x}{\sin 30} = \frac{8}{\sin 70}$$
$$x = \frac{8 \sin 30}{\sin 70} = 4.26 \ to\ 3sf$$

2. <u>Two pairs of sides and opposite angles, where in one pair the **angle** is missing.</u>
   Use the version of the sine rule where the unknown angle appears at the top.

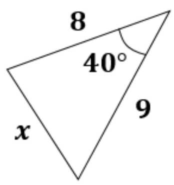
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Example: *Find θ in the diagram on the right.*



$$\frac{\sin \theta}{6} = \frac{\sin 70}{8}$$
$$\sin \theta = \frac{6 \sin 70}{8}$$
$$\theta = \sin^{-1}\left(\frac{6 \sin 70}{8}\right) = 44.8°$$

3. <u>Two sides given and angle given, and unknown side opposite given angle.</u>
   Use cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$. Since the only angle in the formula is A, label the angle A and its opposite side a, and so forth.

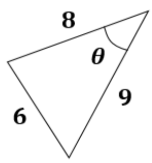


Example: *Determine x in the diagram on the right.*

$$x^2 = 8^2 + 9^2 - (2 \times 8 \times 9 \times \cos 40)$$
$$= 34.6896 \ldots$$
$$x = 5.89$$

Common student error: Because of BIDMAS, the cosine rule should be interpreted as $a^2 = b^2 + c^2 - (2bc \cos A)$
However, some incorrectly evaluate $(b^2 + c^2 - 2bc) \cos A$ instead, calculating $b^2 + c^2 - 2bc$ before multiplying by $\cos A$. The brackets in the example above are to avoid this confusion.

4. <u>Three sides given, and an unknown angle.</u>
   Again, cosine rule is used. This time, more rearrangement is required.



mine θ.

Labelling the angle as A and the opposite side as a and so on:
$6^2 = 8^2 + 9^2 - (2 \times 8 \times 9 \times \cos \theta)$
$36 = 145 - 144 \cos \theta$ (simplified a little)
$144 \cos \theta = 145 - 36$ (used 'swapsie trick': we can swap thing we're

subtracting and result)

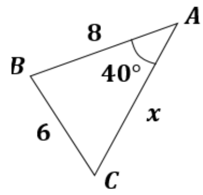
$$\cos\theta = \frac{109}{144}$$
$$\theta = \cos^{-1}\left(\frac{109}{144}\right) = 40.8°$$

5. <u>Two sides given and angle given, and unknown side **NOT** opposite given angle.</u>

Suppose we had the following triangle. An attempt to use cosine rule yields the following:

$$6^2 = x^2 + 8^2 - (2 \times x \times 8 \times \cos 40)$$

The difficulty here is that we have a quadratic equation (i.e. with both an x term and an x^2 term). We could technically rearrange to $ax^2 + bx + c = 0$ and use the quadratic formula. However, it's easier to use sine rule twice, first to find the angle at C, and then to find x.



Using sine rule to find C:

$$\frac{\sin C}{8} = \frac{\sin 40}{6}$$
$$\dots$$
$$C = \sin^{-1}\left(\frac{8 \sin 40}{6}\right) = 58.986969 \dots$$

Use fact that angles in triangle add to 180:

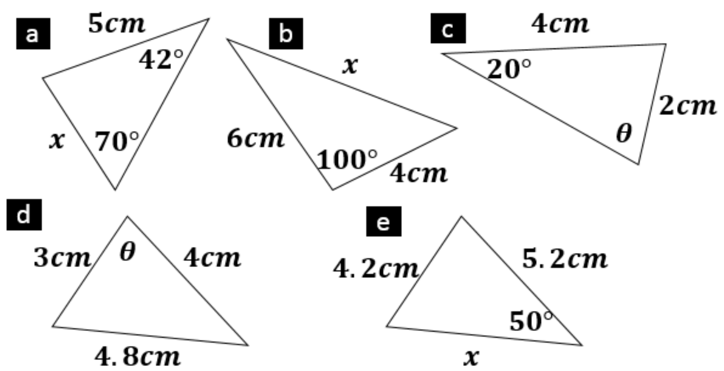
$$B = 180 - 40 - 58.9869\dots = 81.013\dots$$

Use sine rule again (we can either use the $A - a$ pair of the $C - c$ pair, but $A - a$ is easier as 6 and 40° are whole numbers):

$$\frac{x}{\sin 81.013\dots} = \frac{6}{\sin 40}$$
$$x = \frac{6 \sin 81.013\dots}{\sin 40} = 9.22$$

Tips: Use the 'ANS' key where possible to use your previous result directly. Give any intermediate values to lots of decimal places to avoid rounding errors.

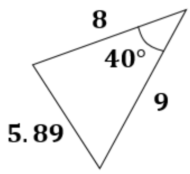
Test Your Understanding: Find the value of the variable in each triangle.



156. Find the area of triangles given two lengths and an included angle

$Area = \frac{1}{2}ab\sin C$. However, there is no need to label the sides/angles to use the correct ones: remember that the angle is between the two sides used (i.e. the 'included angle').

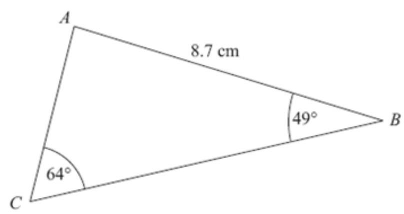
Example 1: *Determine the area of this triangle:*



The angle is between the sides with lengths 8 and 9. Therefore:

$$A = \frac{1}{2} \times 8 \times 9 \times \sin 40 = 23.1$$

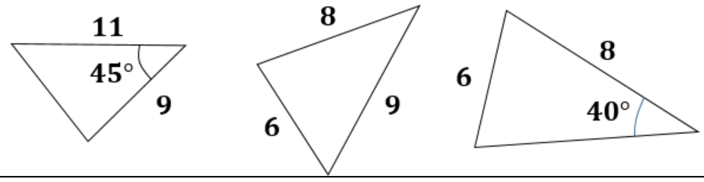
Example 2: *Determine the area of this triangle.*



We need the length either side of one of the angles (49° seems like a more sensible choice) to get the area.

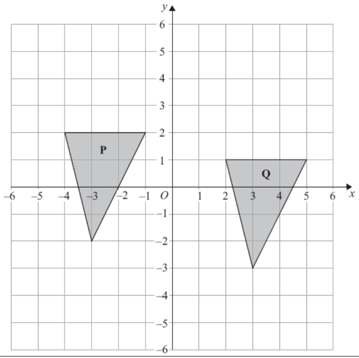
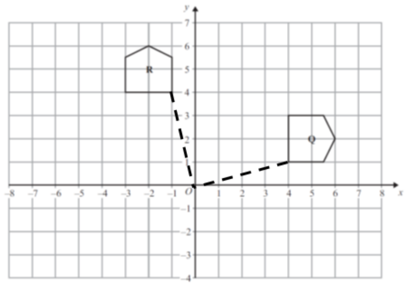
Remaining angle $= 180 - 49 - 64 = 67°$

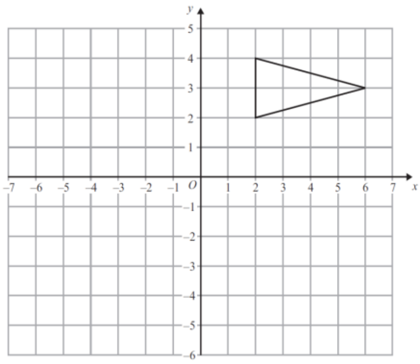
Using sine rule to get BC:

	$\frac{BC}{\sin 67} = \frac{8.7}{\sin 64}$ $BC = 8.91015 \ldots$ Then: $Area = \frac{1}{2} \times 8.7 \times 8.91015 \ldots \times \sin 49 = 29.25cm^2$ **Test Your Understanding:** Determine the area of each of these triangles. 	

Compound Measures		
157. Convert between units of measure in the same system. (NB: Conversion between imperial units will be given. metric equivalents should be known	You should know all of the obvious: 1m = 100cm, 1cm = 10mm, 1km = 1000m, 1l = 1000ml, 1kg = 1000g, 1 ton = 1000kg. A less obvious conversion is 1l = $1000cm^3$ (i.e. 1 litre is equivalent to a cube with sides 10cm).	
158. Know rough metric equivalents of pounds, feet, miles, pints and gallons: Metric /Imperial Convert between imperial and metric measures	These are (which you unfortunately MUST MEMORISE): • 1 kg ≈ 2.2 pounds (≈ means 'approximately equal to') • 1 litre ≈ 1.75 pints • 4.5 litres ≈ 1 gallon • 8 km ≈ 5 miles • 30cm ≈ 1 foot **Example:** Convert 4.59km to miles. Just identify the scale factor to get from km to miles, ensuring you apply a 'common sense' check in terms of dividing vs multiplying, to see whether you should be making the value larger or smaller. $\frac{8}{5} = 1.6$ $4.59\ km \div 1.6 = 2.87\ miles\ to\ 3sf$ **Check Your Understanding:** a. Convert 6.7 litres to gallons. b. Convert 6.7 litres to pints.	
159. Use the relationship between distance, speed and time to solve problems. Calculate speed when, eg fractions of an hour must be entered as fractions or as decimals	Use your speed-distance-time triangle. To convert time in hours from decimal form to hours and minutes, just multiply the decimal part by 60. e.g. 6.2 hours: $0.2 \times 60 = 12\ minutes$, therefore 6.2 hours = 6 hours 12 minutes. Similarly, when time is given in minutes, you can divide by 60. **Example:** "A car journey of 180km takes 4 hours and 20 minutes. Determine the average speed." $t = 4\frac{1}{3} \qquad d = 180$ $s = \frac{180}{4\frac{1}{3}} = 41.54\ km/h$ **Test Your Understanding:** a. A dolphin swims 50km at a speed of 6.3km/h. Determine the time it takes the dolphin in hours and minutes.	
160. Convert between metric units of speed eg km/h to m/s	The key here is to (a) convert one unit at a time and (b) always check whether you should be making the value smaller or larger (using common sense!) **Example**: Convert 8km/h to m/s. 8km/h = 8000m/h = $\frac{8000}{3600} = 2.22\ m/s$ We could tell we needed to divide by 3600 (rather than multiply) because you travel *less* metres in a second than you do in an hour. **Test Your Understanding:** Convert 6.7 km/h to m/s.	
161. Construct and interpret	• When the line is horizontal, the object is not moving.	

distance time graphs	• The steeper the line, the greater the speed. The speed is given by the gradient.	
162. Know that density is found by mass ÷ volume. Use the relationship between density, mass and volume to solve problems, eg find the mass of an object with a given volume and density. Convert between metric units of density eg kg/m^3 to g/cm^3	You can use the 'density-mass-volume' triangle in the same way you can use a 'speed-distance-time' triangle. **Example**: *"A gold bar is in a cuboid shape of 15cm by 5cm by 6cm. Its mass is 10kg. Calculate the density of gold."* $$density = \frac{10}{15 \times 5 \times 6} = 0.0222kg/cm^3 = 22.2g/cm^3$$ Note that we can change the units (see 'Convert between metric of units of speed' above) **Test Your Understanding:** a. A block of zinc has density 6g/cm^3. If its volume is $2m^3$ what is its mass? b. A slab of Unobtanium is in the shape of a triangular prism, with length 30cm and whose cross section has base 5cm and height 4cm. If its mass is 2500g, what is its density?	
163. Calculate the upper and lower bounds of calculations, particularly when working with measurements. Find the upper and lower bounds of calculations involving perimeter, areas and volumes of 2-D and 3-D shapes. Give the final answer to an appropriate degree of accuracy following an analysis of the upper and lower bounds of a calculation	To find the lower or upper bound of a value, <u>subtract and add half the accuracy</u>. e.g. "3.6cm correct to 1dp" gives lower bound 3.55cm and upper bound 3.65cm (recall that while technically the highest possible value which rounds down is $3.64\dot{9}$, we don't write this). To find the lower bound and upper bound of a quantity which is the result of a multiplication, just use common sense. If $a = b \times c$, then to get the upper bound of a, we use the upper bound of b and the upper bound of c to make the value of a as large as possible. Similarly if $a = \frac{b}{c}$, to get the upper bound we divide the upper bound of b by the lower bound of c (since we want to divide by a small a value as possible to end up with a value as high as possible). To give your answer to "an appropriate degree of accuracy", give as many decimal places as possible where the upper bound and lower bound would be the same to this degree of accuracy. **Example:** "$m = \frac{\sqrt{s}}{t}$ $s = 3.47$ correct to 2 decimal places. $t = 8.132$ correct to 3 decimal places. By considering bounds, work out the value of m to a suitable degree of accuracy. You must show all your working and give a reason for your final answer." $$s_{lower} = 3.465 \quad s_{upper} = 3.475$$ $$t_{lower} = 8.1315 \quad t_{upper} = 8.1325$$ $$m_{upper} = \frac{\sqrt{s_{upper}}}{t_{lower}} = \frac{\sqrt{3.475}}{8.1315} = 0.2292486 \ldots$$ $$m_{lower} = \frac{\sqrt{s_{lower}}}{t_{upper}} = \frac{\sqrt{3.465}}{8.1325} = 0.2288903 \ldots$$ The answer is 0.229, 'as both the lower and upper bound of m are this value to 3 decimal places'. Use that last sentence word-for-word to guarantee the final mark! The point is that both bounds are still the same to 3dp, but not once you specify to 4dp, and thus it would be inappropriate to give this level of accuracy. A common student error is to forget other operations in the equation, in this case the square root. Another common error is to take the value of m just to be the midpoint of the lower and upper bound. **Test Your Understanding:** a. $x = y \times z$. $y = 4.5$ correct to 1 decimal place. $z = 3.68$ correct to 2 decimal places. Work out the lower and upper bound of x. b. $q = \frac{r^2}{s}$. $r = 2.87$ correct to 2 decimal places. $s = 3.584$ correct to 3 decimal places. Work out the value of q to a suitable degree of accuracy, giving a reason for your answer.	

Transformations		
Overview: Be able to both describe transformations and carry out transformations involving translation, rotation, reflection and enlargement.	How marks are allocated for describing transformations: a. "A translation (1 mark) by the vector ____ (1 mark)" b. "A rotation (1 mark) by ___° clockwise/anticlockwise (1 mark) about the point ___ (1 mark)" c. "A reflection (1 mark) in the line ____ (1 mark)" d. "An enlargement (1 mark) by the scale factor ____ (1 mark) about the point ___ (1 mark)" For practice questions, see my "GCSE Revision: Transformations" questions.	
164. Understand translation as a combination of a horizontal and vertical shift including signs for directions. Translate a given shape by the vector	 *"Describe the transformation from P to Q"* By choosing the top-left corner and counting squares to the top-left corner of Q, we can see x increases by 6 and y decreases by 1. Thus: "A translation by the vector $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$"	
165. Understand rotation as a (anti clockwise) turn about a given origin. Describe and transform 2-D shapes using single rotations. Find the centre of rotation	 *"Describe the transformation from P to Q"* Fortunately, almost always in exams, the centre of rotation is the origin, allowing you to say "about the origin" or "about the point (0,0)". Always check this as the centre first. If it isn't, guess a sensible centre point and check it as follows: (i) choose any point on the original shape and the equivalent point on the image (ii) draw lines from these points to your centre [e.g. see diagram] (iii) see if this angle looks like 90°. You can count squares from each point to the centre. Notice from P to the centre we want 1 square across and 4 squares down. From the centre to Q, we went 4 across and 1 up. The across movements and up/down movements swap for 90° rotations. If the rotation was $180°$, choose a point on the original shape and equivalent point on the resulting shape. Draw a line between them: the centre is then the midpoint. The same approach can be used to rotate a given shape. For each vertex on the original shape, draw a line to the centre. Using 'common sense' to have a rough sense of where the point will end up, count squares from the point to the centre. If a $90°$ rotation, the counts swap. If a $180°$ rotation, just go this same number of squares past the centre. Repeat.	
166. Reflect shapes in a given mirror line; parallel to the coordinate axes and then y = x or y = –x.	To reflect a shape in a line, for each point, draw a line directly towards the line of reflection, counting squares (whether horizontal, diagonal, or diagonally). Then continue this number of squares beyond the line of reflection and leave a point. Repeat.	
167. Enlarge shapes by a given scale factor from a given point; using positive and negative scale factors greater and less than one. Find the centre of enlargement	• To identify scale factor, just compare two lengths on your original and enlarged shape. E.g. If the width was 2 and is now 6, the scale factor is 3. If the shape has 'flipped' the scale factor will be negative. • To identify the centre of enlargement, pick a point on the original shape and the equivalent point on the large shape. Join up with a line, extending the line out. Repeat with another pair of points. The centre of enlargement is the intersection of the two lines. • To carry out an enlargement, for each point on the original shape, count the squares across and up/down from the centre of enlargement. Times each of these values by the scale factor. Then go this number of squares across and up/down from the centre again. If the value if negative, go in the opposite direction.	



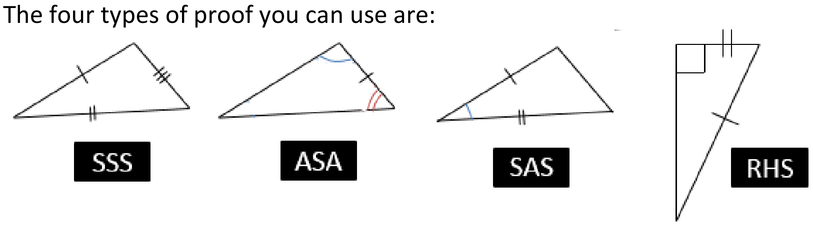
Example: “*Enlarge by scale factor* $-\frac{1}{2}$*, centre* $(0,-2)$”
Using top-left corner $(2,4)$: Squares across from centre: 2, Squares up: 6. Multiplying each by $-\frac{1}{2}$, we get 1 square left and 3 squares down (from the centre). Therefore plot the point $(-1,-5)$. Repeating with the other points, we get $(-1,-4)$ and $(-3,-4.5)$. Join up your new points.

168. Understand that shapes produced by translation, rotation and reflection are congruent to the image. Use congruence to show that translations, rotations and reflections preserve length and angle, so that any figure is congruent to its image under any of these transformations. Recognise that enlargements preserve angle but not length, linking to similarity

Similarity and Congruence

169. Understand and use SSS, SAS, ASA and RHS conditions to prove the congruence of triangles using formal arguments, and to verify standard ruler and a pair of compasses constructions. Formal geometric proof of similarity of two given triangles

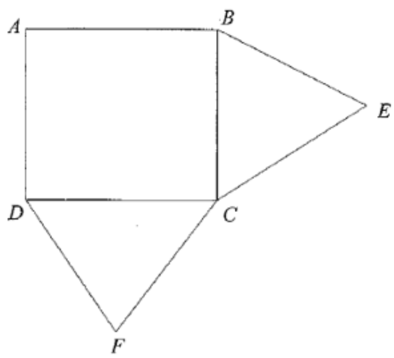
The four types of proof you can use are:



The first means for example that two triangles are congruent if all their sides are the same. In the last one “R” stands for Right Angle, “H” for hypotenuse and “S” for ‘another side’, i.e. both triangles must have a right angle, hypotenuse of same length, and another side of the same length.
Note that the order of the letters matters. ‘SAS’ means that the angle is in between the two sides (i.e. the included angle). As seen earlier ‘ASS’ meanwhile would not necessarily lead to congruent triangles.
‘AAS’ however is equivalent to ‘ASA’ because if two angles are the same, the third angle would be the same, so in ASA, the side need not be in between the two angles.

To do congruence proofs, it may help to structure your proof in the following way:

- Put four bullet points. For the first three, label to the left of each bullet point each letter of the proof you want to use (e.g. ‘R’, ‘H’ and ‘S’) and leave the final bullet point for your conclusion.
- For each of the first three bullet points, justify (whether equating angles, sides, mentioning midpoints, laws of angles or using circle theorems) why your sides or angles are the same.
- In the conclusion, write: “Therefore, triangles [ABC] and [DEF] are congruent by [SAS]” (obviously replacing […] as appropriate)



Example: “*ABCD is a square and CDF and BCE are equilateral triangles. Prove that triangle BCF is congruent to triangle DCE.*”

SAS seems like the best proof to use as we can work out the angles, and know from the fact that we have regular polygons that all sides are the same.

- *(S)* FC = CD as CDF is equilateral.
- *(A)* $\angle DCE = \angle FCB = 150°$
- *(S)* CB = CE as BCE is equilateral.
- Therefore, BCF is congruent to DCE by SAS.

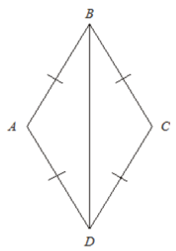
Once we’ve proven that two triangles are congruent, then any sides or angles that we hadn’t previously shown were equal as part of the proof, we have now proven are equal as well. This is useful for follow up questions, where we can use “… = … as … and … are congruent”.

Example: (Continuing) “G is the point such that BEFG is a parallelogram. Prove that ED = EG.”

Because we now know BCF is congruent to DCE, we know that the remaining sides of the triangles not involved in our proof, DE and BF, are also equal. Thus:

- EG = BF, as opposite sides of a parallelogram are equal.
- BF = DE, as triangles DCE and BCF are congruent.
- Therefore, ED = EG.

Example 2: "In the diagram, $AB = AD = BC = CD$. Prove that triangles ABD and BCD are congruent."
It's possible to use SAS here (as $\angle BAD = \angle BCD$ as opposite angles of a parallelogram), however SSS is easier.



- *(S)* AD = DC.
- *(S)* AB = BC
- *(S)* BD is common.
- Therefore, ABD is congruent to BCD by SSS.

Note that '… is common' is the justification you should use when sides are shared.

Test Your Understanding:

a. AC and BD are diameters of the circle, and O is the centre (left diagram below). Prove that triangles ABD and ACD are congruent (Hint: you could use RHS, SAS or even ASA here)
b. (Right diagram below) ABC is an equilateral triangle. D lies on BC, and AD is perpendicular to BC. (i) Prove that triangles ABD and ADC are congruent [note: we DON'T yet know that D is the midpoint of B and C, so can't use this in our proof] (ii) Prove that $BD = \frac{1}{2}AB$

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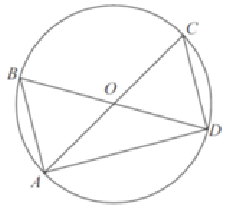
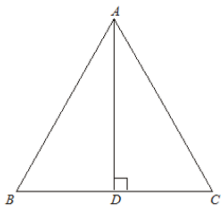


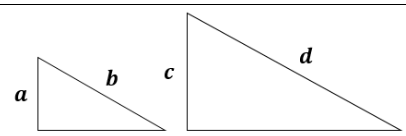


170. Recognise that all corresponding angles in similar figures are equal in size when the lengths of sides are not

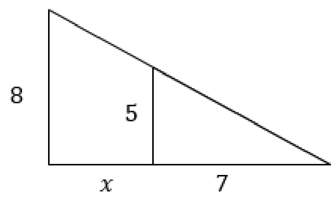
Remember that:

- <u>Similar</u> means same shape (all angles will be the same)
- <u>Congruent</u> means same size and shape.

171. Solve problems involving finding lengths in similar shapes.



If two shapes are similar then the scale factor of lengths is the same:
i.e. $\frac{d}{b} = \frac{c}{a}$.



Example: Determine x.
$\frac{8}{5} = 1.6$ which is the scale factor. $7 \times 1.6 = 11.2$ gives whole bottom length.

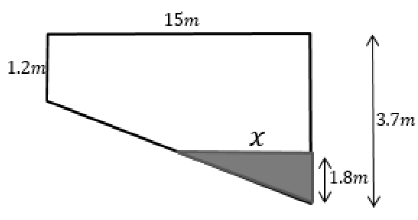
$$x = 11.2 - 7 = 4.2$$

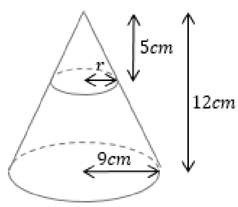
Alternatively using a more algebraic approach:

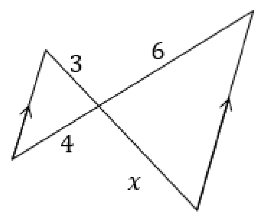
$$\frac{x+7}{7} = \frac{8}{5}$$

Then solve.

Test Your Understanding: Find the value of the variable in each case.

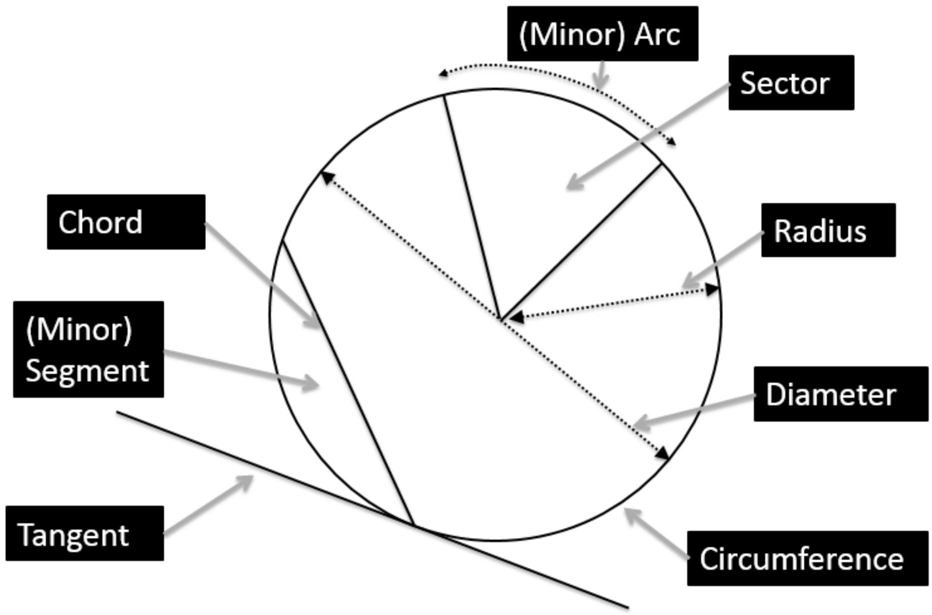




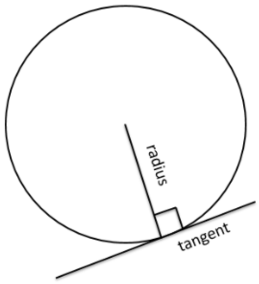
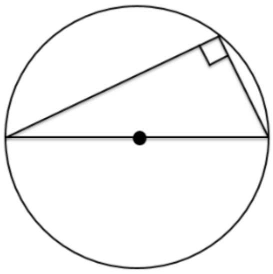
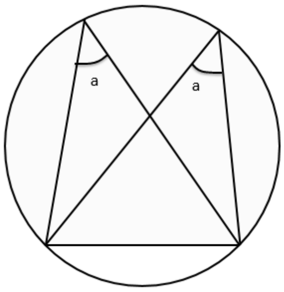
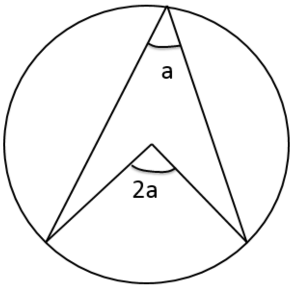
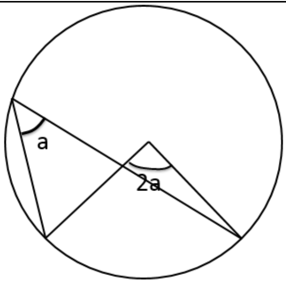
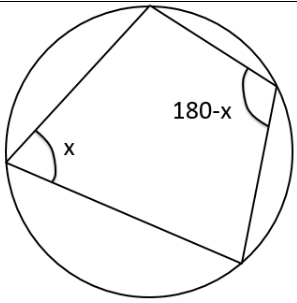
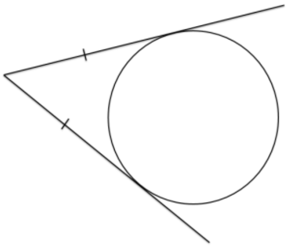
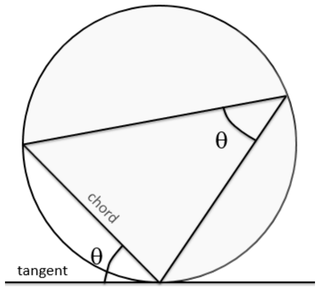


172. Understand the effect of enlargement for perimeter,

Whatever the scale factor for length is, the scale factor is squared for area and cubed for volume. I find it helpful to lay out all information in a 3 by 2 table:

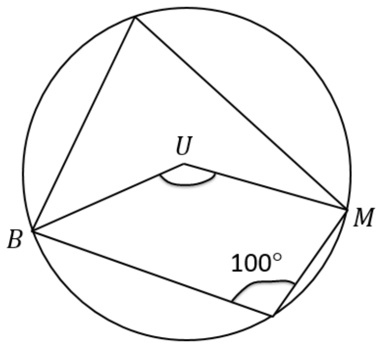
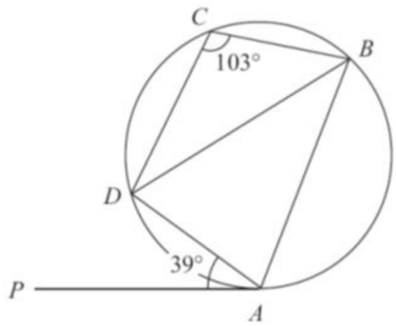
area and volume of shapes and solids. Know the relationships between linear, area and volume scale factors of mathematically similar shapes and solids	$Length: \quad \rightarrow$ $Area: \quad \rightarrow$ $Volume: \quad \rightarrow$ **Example**: *"Shape A is enlarged to form shape B. The surface area of shape A is 30cm^2 and the surface area of B is 270cm^2. If the volume of shape A is 80cm^3, what is the volume of shape B?"* Fill in initial information, working out the scale factor for area: $Length: \quad \rightarrow$ $Area: \quad 30 \xrightarrow{\times 9} 270$ $Volume: \quad 80 \rightarrow$ Since the scale factor of length is squared to get area, the scale factor of length is $\sqrt{9} = 3$. This allows us to work out the scale factor of volume, $3^3 = 27$. This in turn allows us to work out the new volume. $Length: \quad \xrightarrow{\times 3}$ $Area: \quad 30 \xrightarrow{\times 9} 270$ $Volume: \quad 80 \xrightarrow{\times 27} 2160cm^3$ Final note: If the question talks about mass instead of volume, note that mass can be treated as volume since it's proportional to it (for a fixed density). **Test Your Understanding:** a. Two cylinders A and B haves volumes $10cm^3$ and $640cm^3$. If the surface area of solid A is 20cm^2, what is the surface area of solid B? b. Cylinder A and cylinder B are mathematically similar. The length of cylinder A is 4 cm and the length of cylinder B is 6 cm. The volume of cylinder A is 80cm^3. Calculate the volume of cylinder B.	
Circle Theorems		
173. Recall the definition of a circle and identify (name) and draw the parts of a circle	 An arc would be a 'major arc' if it goes around at least half the circle. The 'major segment' would be the larger segment on the other side of the chord.	
174. Know, use and quote the following circle theorems: - The angle between the radius and tangent is 90. - Tangents from an external point are equal in length. - The angle at the centre	You should be able to quote the theorems on the left word for word when asked to justify your reasoning. Use the information given to decide on what circle theorems might be relevant. Have a tangent? Then if there's a radius connected you have a 90° angle. If there's a chord connected you can use the Alternate Segment Theorem. Similarly reflect like this if the centre of the circle is indicated.	

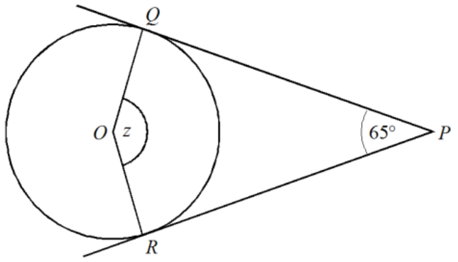
is twice the angle at the circumference.
- The angle in a semicircle is 90.
- Opposite angles of a cyclic quadrilateral are 180.
- Angles in same segment are equal.
- Alternate Segment Theorem.

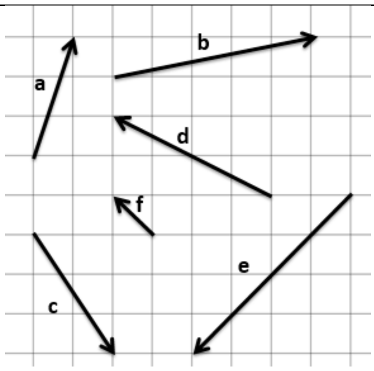
 Angle between radius and tangent is 90°	 Angle in semicircle is 90°	 Angles in same segment are equal.
 Angle at centre is twice angle at circumference.	 (The same as the left, except here the lines cross)	 Opposite angles of a cyclic quadrilateral add to 180
 Lengths of the tangents from a point to the circle are equal.		Alternate Segment Theorem (i.e. angle between tangent and chord is equal to angle in alternate segment, but you only need to write 'Alternate Segment Theorem' as your reason)

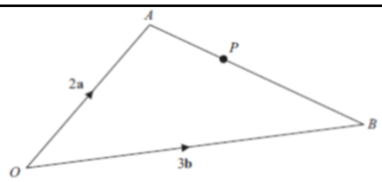
The following laws are often relevant also:
- Angles in a quadrilateral (not necessarily a cyclic quadrilateral) sum to 360.
- A triangle where two of its vertices are at the circumference of a circle and one at the centre is isosceles (reason to quote: "Base angles of isosceles triangle are equal").

Check Your Understanding:

a. U is the centre of the circle. Determine the angle BUM  b. Determine angle QOR, giving reasons for your answer.	c. Determine angle ADB, giving reasons for your answer. 

Vectors		
175. Understand that $2\boldsymbol{a}$ is parallel to $\boldsymbol{a}$ and twice its length, and understand that $\boldsymbol{a}$ is parallel to $-\boldsymbol{a}$ and in the opposite direction	Two vectors are <u>parallel</u> if <u>one is a multiple of the other</u> (e.g. if you have two vectors $\overrightarrow{AB} = \frac{3}{2}(a+b)$ and $\overrightarrow{CD} = \frac{1}{2}(a+b)$, you could say "$\overrightarrow{AB}$ is a **<u>multiple</u>** of $\overrightarrow{CD}$").	
176. Use and interpret vectors as displacements in the plane (with an associated direction)	This just means that just as a coordinate represents a 'position' in space, a vector represents a 'movement' (or displacement).	
177. Use standard vector notation to combine vectors by addition, e.g. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ and $\underline{a} + \underline{b} = \underline{c}$	You should use this notation in vector proofs (see examples below): it's important so that you can show which vectors you combined together.	
178. Represents vectors, and combination of vectors, on a plane.	 Vectors are written like $\begin{pmatrix}2\\-3\end{pmatrix}$ whereas coordinates are written like $(2,-3)$. In this diagram $a = \begin{pmatrix}1\\3\end{pmatrix}$, $b = \begin{pmatrix}5\\1\end{pmatrix}$, $c = \begin{pmatrix}2\\-3\end{pmatrix}$ **Test Your Understanding:** Find the remaining vectors.	
179. Solve geometrical problems involving vectors.	This includes: - Being able to use ratios to find a fraction of some vector. E.g. If $\overrightarrow{OA} = a$ and P is a point on the line such that $OP:PA = 3:4$, then $\overrightarrow{OP} = \frac{3}{7}a$ - Being able to prove three points A, B, C form a straight line (by showing that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel (and share a common point B). Note that vectors questions at GCSE tend to have a very set structure: • Part (a) gets you to find some vector in the diagram. This is usually easy to determine. • A harder follow up question. You can almost always use your answer to part (a), so your 'route' through the diagram should include this vector. **Examples:**	



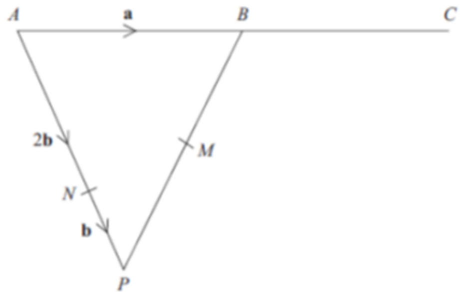
P is the point on AB such that $AP{:}PB = 2{:}3$.

a) Find $\overrightarrow{AB}$
$= -2a + 3b$ (as we have to go backward along the $2a$ vector)

b) Prove that $\overrightarrow{OP}$ is parallel to $a + b$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{5}\overrightarrow{AB}$$
$$= 2a + \frac{2}{5}(-2a + 3b)$$
$$= 2a - \frac{4}{5}a + \frac{6}{5}b$$
$$= \frac{6}{5}a + \frac{6}{5}b$$
$$= \frac{6}{5}(a + b)$$

Note: The factorisation at the end is important because it shows you expression is a multiple of $a + b$.



a) *Find the vector $\overrightarrow{PB}$.*

$$\overrightarrow{PB} = -3b + a$$

b) *B is the midpoint of AC and M is the midpoint of PB. Prove that NMC is a straight line.*

NMC is a straight line if $\overrightarrow{NM}$ and $\overrightarrow{MC}$ are parallel. Recall that to show two vectors are parallel, we need to show one is 'a multiple' of the other.

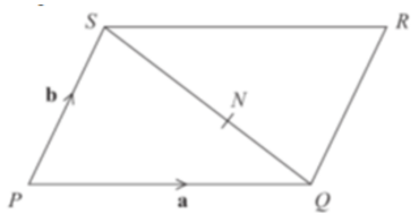
$$\overrightarrow{NM} = \overrightarrow{NP} + \frac{1}{2}\overrightarrow{PB}$$

(notice that we're using our answer to part (a))

$$\overrightarrow{NM} = b + \frac{1}{2}(-3b + a)$$
$$= b - \frac{3}{2}b + \frac{1}{2}a$$
$$= \frac{1}{2}a - \frac{1}{2}b = \frac{1}{2}(a - b)$$
$$\overrightarrow{MC} = \frac{1}{2}\overrightarrow{PB} + \overrightarrow{BC}$$
$$= \frac{1}{2}(-3b + a) + a$$
$$= -\frac{3}{2}b + \frac{1}{2}a + a$$
$$= \frac{3}{2}a - \frac{3}{2}b = \frac{3}{2}(a - b)$$

"$\overrightarrow{NM}$ is parallel to $\overrightarrow{MC}$ (and share a common point M, therefore NMC is a parallel line".
Notice that by factorising out the fraction when determining each vector, it was easier to show that one vector is a multiple of the other

Test Your Understanding:



a. $PQRS$ is a parallelogram. N is a point on SQ such that $SN{:}NQ = 3{:}2$.
(i) Write down, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, an expression for $\overrightarrow{SQ}$.
(ii) Express $\overrightarrow{NR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(More questions can be found in my Vectors past paper questions compilation)

Data Handling & Probability

To practice these questions, refer to my compiled past paper Data Handling questions.

<table>
<tr><th colspan="2">Collecting Data</th><th></th></tr>
<tr><td>180. Discuss how data relates to a problem, identify possible sources of bias and plan to minimise it. Consider fairness Understand how different sample sizes may affect the reliability of conclusions drawn Understand sample and population.
Design and criticise questions for a questionnaire</td><td>Particularly common criticisms of questionnaire questions (it’s worth memorising the wording of these):
• “Overlapping regions” (e.g. one response box is 1-3 times and another 3-5 times)
• “Non-exhaustive response boxes” (e.g. there’s no option for ‘0 times’)
• “No timescale” (is it per week or per month?)
• “Response labels too vague” (what does ‘occasionally’ mean?)
Common problems with bias and sampling:
• “Question is biased” (i.e. is pressuring respondents to give a particular answer)
• “Sample size too small”
• “Only people/things in a particular area were asked” (i.e. sample was not random)

When asked to rewrite a questionnaire question:
• You must have a timeframe.
• You must have “at least 3 non-overlapping response boxes”. The response boxes should also cover all possibilities (e.g. ‘0 times’)</td><td></td></tr>
<tr><td>181. Select and justify a sampling scheme and a method to investigate a population, including random and stratified sampling. Use stratified sampling.</td><td>Random sampling:
• Definition: “Random sampling is when each thing in the population has an <u>equal chance</u> of being chosen” – the ‘equal chance’ is the important bit here.
• “How would you achieve a random sample?” – Any method which would ensure each thing is equally likely to be chosen. Best way: “Assign each person a number, then use a random number generator to select a person.”

Stratified sampling:
• Definition: “The population is divided into groups (e.g. by ethnicity or year group) and the same proportion are randomly sampled from each group.”
• Example Question: In a school of 100 people, 40 people are in Class X, 50 people in class Y and 10 people in Class Z. I wish to obtain a sample of 20 people. How many do I sample from each class?”
Proportion of people sampled $= \frac{20}{100} = \frac{1}{5}$
Class X: $\frac{1}{5} \times 40 = 8$. Class Y: $\frac{1}{5} \times 50 = 10$. Class Z: $\frac{1}{5} \times 10 = 2$

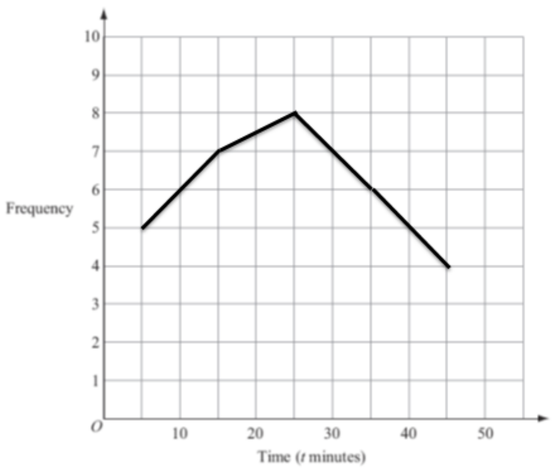
Test Your Understanding: In a school there are 200 students, and students can study one of Geography, History and Maths. Of the boys, 45 study Geography, 10 study History and 25 study Maths. Of the girls, 70 study Geography, 15 study History and 35 study Maths. I want a stratified sample of 50 people stratified by gender and subject. How many boys studying Geography do I sample?</td><td></td></tr>
<tr><td>182. Design and use data-collection sheets for grouped, discrete and continuous data. Sort, classify and tabulate data.</td><td>For a tally sheet, the three marks will be for three column headings in your table:
1. The values you’re taking a tally of, e.g. football teams, favourite colour.
2. “Tally”
3. “Frequency” (i.e. the data collector would add the tallies up after collection)</td><td></td></tr>
<tr><td>183. Group discrete and continuous data into class intervals of equal width.</td><td>This means you turn for example this data:
Height: $47cm, 55cm, 59cm, 63cm, 68cm, 69cm, 69cm, 73cm, 74cm$
Into a grouped frequency table like this:
<table><tr><td>Height (h) in cm</td><td>Frequency</td></tr><tr><td>$40 \le h < 50$</td><td>1</td></tr><tr><td>$50 \le h < 60$</td><td>2</td></tr><tr><td>$60 \le h < 70$</td><td>4</td></tr><tr><td>$70 \le h < 80$</td><td>1</td></tr></table></td><td></td></tr>
<tr><td>184. Design and use two-way tables, and use information provided to complete a two-way table.</td><td>Filling out a two-way table is simply common sense. These are often used to derive probabilities.
<table><tr><td></td><td>Boys</td><td>Girls</td><td>Total</td></tr><tr><td>Likes pie</td><td>20</td><td>25</td><td>45</td></tr><tr><td>Does not like pie</td><td>40</td><td>15</td><td>55</td></tr><tr><td>Total</td><td>60</td><td>40</td><td>100</td></tr></table>Calculate:</td><td></td></tr>
</table>

	1. The probability that a randomly selected student is a boy: $\frac{60}{100} = \frac{2}{5}$ 2. The probability that a randomly selected student likes pie: $\frac{45}{100} = \frac{9}{20}$ 3. The probability that a randomly selected girls likes pie: $\frac{25}{40} = \frac{5}{8}$ 4. The probability that a randomly selected person who likes pie, is a boy: $\frac{20}{45} = \frac{4}{9}$

Displaying Data

185. Produce: composite bar charts, comparative and dual bar charts, pie charts, frequency polygons, and frequency diagrams for grouped discrete data, scatter graphs, line graphs, frequency polygons for grouped data, grouped frequency tables for continuous data

- To produce a pie chart, simply work out the angle for each thing. If for example there were 50 students and 15 studied history, then find $\frac{15}{50}$ of 360°: $\frac{15}{50} \times 360 = 108°$
- To plot a frequency polygon, use the **MIDPOINT** of each interval and plot against frequency. Join up each pair of dots with a straight line (and don't join to the origin).



Time (t minutes)	Frequency
$0 < t \leq 10$	5
$10 < t \leq 20$	7
$20 < t \leq 30$	8
$30 < t \leq 40$	6
$40 < t \leq 50$	4

186. Interpret: composite bar charts, comparative and dual bar charts, pie charts, scatter graphs, frequency polygons and histograms

187. Recognise simple patterns, characteristics and relationships in line graphs and frequency polygons

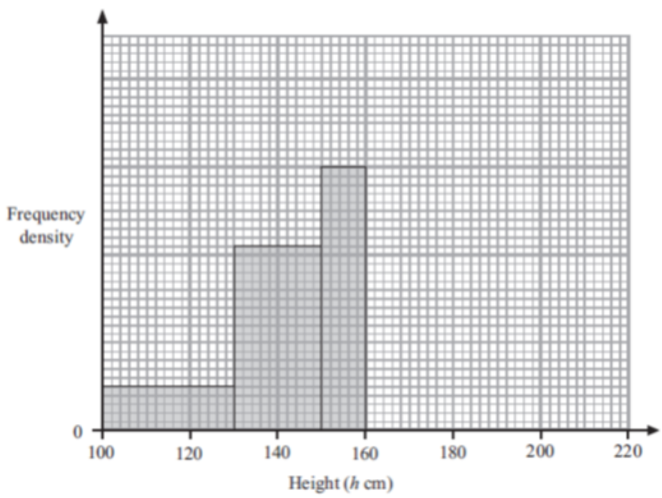
188. For histograms:
- Produce a histogram from a grouped frequency table.
- Find the median from a histogram or any other information from a histogram, such as the number of people in a given interval
- Complete a grouped frequency table and understand and define frequency density.

You just need to know that:

- $frequency\ density = \frac{frequency}{class\ width}$
  Class width is the width of the interval (e.g. the width of $5 < w \leq 20$ is 15.
  Frequency density can be interpreted as 'the number of things/people per each value'. So if in a run there are 20 runners who ran between 10s-14s, then the frequency density of 5 represents '5 runners (on average) for each second interval'.
- For the purposes of GCSE, the area of each bar = frequency (the exception is problems in (189))
- Sometimes the frequency density scale on the histogram is not given. If not, you can work it out using an existing bar on the histogram by using the information in the table (see example). Once you have your scale, you can then add missing bars.
- When a grouped frequency table is given, always add a frequency density column.

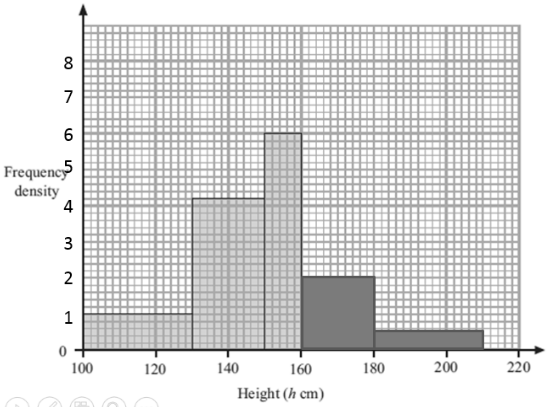
Example: "*Use the histogram to complete the table, and use the table to complete the histogram*."

Height (h cm)	Frequency
$100 < h \leq 130$	30
$130 < h \leq 150$	
$150 < h \leq 160$	
$160 < h \leq 180$	40
$180 < h \leq 210$	18



First add a FD column to your table. Notice that we have complete information for the first bar: the bar is in the histogram, and the frequency is known. Using the formula, $FD = \frac{30}{30} = 1$. This means that we can put 1 on the Frequency Density axis in line with the top of the first bar, and use this to fill in the rest of the previously unknown Frequency Density scale.
The area of the next two bars is $20 \times 4.2 = 84$ (be careful with reading of your scale) and $10 \times 6 = 60$, which are the frequencies (recall area = frequency). The frequency density of the last two rows is $\frac{40}{20} = 2$ and $\frac{18}{30} = 0.6$, so draw your last two bars with these heights of the bars (and with the correct width using the ranges $160 < h \leq 180$ and $180 < h \leq 210$.

Height (h cm)	Frequency	Frequency Density
$100 < h \leq 130$	30	30 ÷ 30 = 1
$130 < h \leq 150$	84	4.2 (using graph)
$150 < h \leq 160$	60	6 (using graph)
$160 < h \leq 180$	40	40 ÷ 20 = 2
$180 < h \leq 210$	18	18 ÷ 30 = 0.6

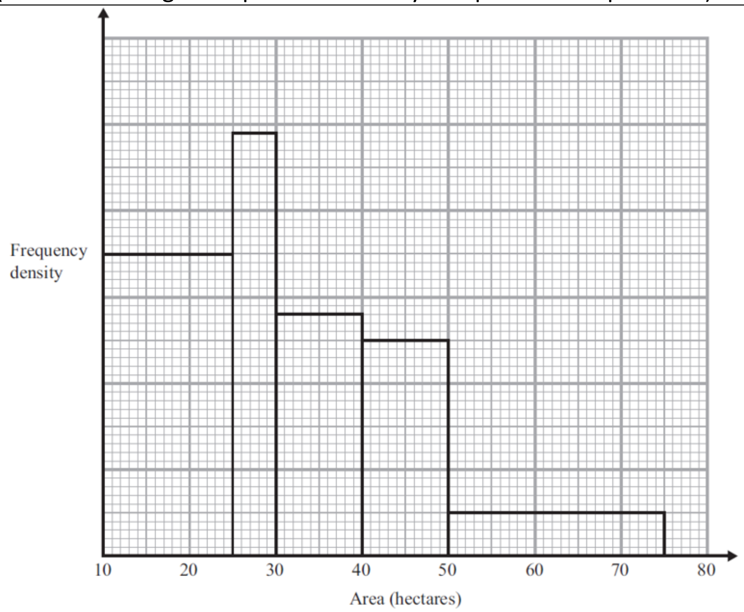


Example: "*Using the same histogram, estimate the number of people with a height between 170cm and 200cm*."
There's two ways of thinking about this.

- Use the histogram. As before, area = frequency. The slight difficulty here is that we need to use the areas of parts of bars.
  Area $= (10 \times 2) + (20 \times 0.6) = 32$. So 32 people.
- Alternatively, just looking directly at the frequency table, 170cm is halfway across the $160 < h \leq 180$ range so use half of 40 which is 20. And 200cm is two-thirds across the next interval, and two thirds of 18 is 12. Again, we have a total of 32 people.

(For more histograms questions see my compiled exam questions)

189. Solve histogram problems involving finding proportion of the total area.

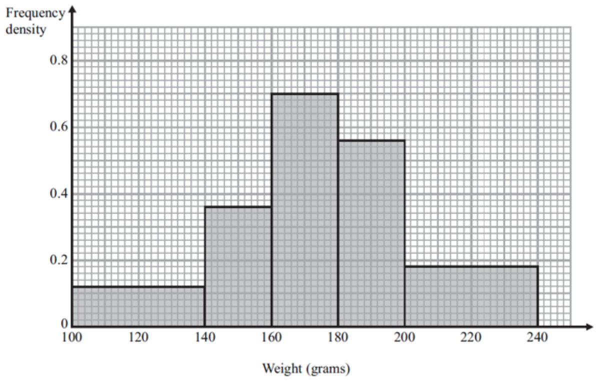
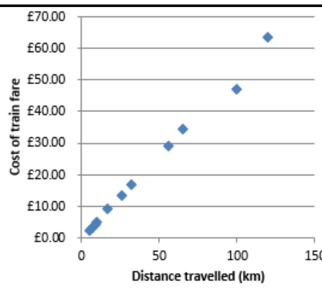
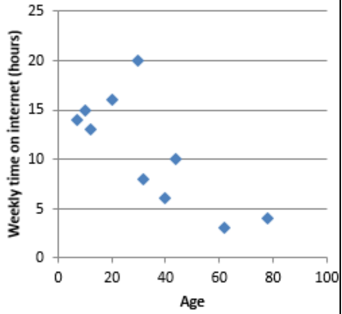
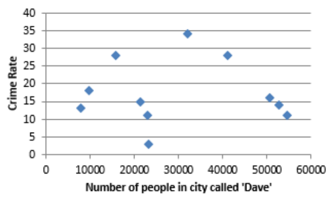


Example: This histogram shows information about the sizes of 285 farms.
Work out an estimate for the number of farms with an area above 38 hectares.

This is a different kind of histogram problem because we have no grouped frequency table. Since area indicates frequency, the strategy is to find the proportion of the total area in our range.
We have a 3 or 4 step process:

1. Label frequency density axis with any sensible scale (in this case 10 per big square is easiest to avoid decimals). (Note: Although your arbitrary choice of scale will likely mean it is not the case that area = frequency [only proportional to], it will not affect your answer to the third step below.)
2. Find the total area of the histogram, and the area in your range (i.e. above 38 hectares).
3. Express the area in your range as a fraction of the total area. If the question asked for "the proportion of things in a range", you're done.
4. If you're given the total frequency, take the fraction you found of this frequency.

Applying this to the question: Labelling big squares 10, 20, 30, … on frequency density axis, areas of each bar are: $15 \times 35 = 525$, 245, 280, 250, 125. Thus total area = 1425. Area above 38 hectares:

	$(2 \times 28) + 250 + 125 = 431$ Estimate of number of farms above 38 hectares: $\frac{431}{1425} \times 285 = 86.2$ farms.  **Test Your Understanding:** The histogram shows information about weights of apples in a sample. Use the histogram to find the proportion of apples in the sample with a weight between 140g and 200g.	
190. From line graphs, frequency polygons and frequency diagrams: read off frequency values, calculate total population, find greatest and least values		
191. From pie charts: find the total frequency and find the frequency represented by each sector	**Example:** *"A pie chart represents the favourite colour of 120 people. It has three slices, 'Red', 'Blue' and 'Green' with angles of* $147°, 168°, 45°$ *respectively. Determine how many people the 'Red' slice represents."* Find how many degrees one person represents: $\frac{120}{360} = \frac{1}{3}$ $147 \times \frac{1}{3} = 49$ people.	
192. Look at data to find patterns and exceptions, explain an isolated point on a scatter graph	An extreme value on a scatter graph is one which is far away from the line of best fit.	
193. Draw lines of best fit by eye, understanding what these represent. Use a line of best fit, or otherwise, to predict values of one variable given values of the other variable	A scatter graph is when each data point has two values, for example, each point could represent a person, each with two values for 'English mark' and 'Maths mark'. By drawing a line of best fit, you can use it as a line graph in the usual way to estimate one value from the other.	
194. Distinguish between positive, negative and zero correlation using lines of best fit	 (Strong) positive correlation  (Weak) negative correlation  No correlation	
195. Understand that correlation does not imply causality. Appreciate that correlation is a measure of the strength of the association between two variables and that zero correlation does not necessarily imply 'no relationship'	The number of deaths due to cancer has decreased over the years. Similarly, the number of mobile devices has increased. This means the two are negative correlated, but it is clear that more mobile devices doesn't cause less cancer deaths.	

<table>
<tr><th colspan="2">Averages and Range</th><th></th></tr>
<tr><td>196. Calculate mean, mode, median and range for small data sets</td><td>Example: Here are some heights of people: $162cm \quad 168cm \quad 171cm \quad 171cm$
Range = 171 – 162 = 9cm. Mode = 171cm (most common value).
Median = $\frac{168+171}{2} = 169.5cm$ (the middle value: if two middle, take average of two)
Mean = $\frac{(162+168+171+171)}{4} = 168cm$</td><td></td></tr>
<tr><td>197. Determine the quartiles and interquartile range for listed data.</td><td>Example: Find the interquartile range of the following data:
$1cm\ 2cm\ 2cm\ 3cm\ 4cm\ 7cm\ 8cm\ 10cm\ 12cm\ 17cm\ 20cm$
When you need to find quartiles of listed data in a GCSE exam, you will always find they give you a number of items one less than a multiple of 4 (in this case 11).
A trick is to add one to the number of items, then find quarters of this. Thus we want the 3rd, 6th and 9th items for the Lower Quartile, Median and Upper Quartile respectively, i.e. 2cm 7cm, 12cm.
$$Interquartile\ Range = Upper\ Quartile - Lower\ Quartile$$
$$= 12cm - 2cm = 10cm$$</td><td></td></tr>
<tr><td>198. Recognise the advantages and disadvantages between measures of average</td><td>Mode is the only average available if the values are categorical (i.e. not numerical), e.g. favourite colour.
Mean is best from the perspective of taking all values into account.
“Explain why the median may be a more appropriate average than the mean”: “The median is <u>not affected by extreme values</u>”.</td><td></td></tr>
<tr><td>199. Produce ordered stem and leaf diagrams and use them to find the range and averages</td><td>Example: “Here are the times, in minutes, taken to solve a puzzle.
5 10 15 12 8 7 20 35 24 15
20 33 15 24 10 8 10 20 16 10
In the space below, draw a stem and leaf diagram to show these times.”
<table><tr><td>0</td><td>5 7 8</td></tr><tr><td>1</td><td>0 0 0 0 2 5 5 5</td></tr><tr><td>2</td><td>0 0 0 4 4</td></tr><tr><td>3</td><td>3 5</td></tr></table>Key: 3|7 means 37 minutes

The ‘stem’ is the first digit and the ‘leaf’ is the second. You need a key to indicate what each value means, e.g. “4|1 means 4.1cm”. Leaves must be in ascending order.

Example: “Use your stem and leaf diagram to determine the median.”
Now that your values are in ascending order, it’s easier to find the middle one. It may help to put a dot next to the first and last leaf (05 and 35), and move inwards one leaf at a time until you get to the middle. Alternatively, because there are 20 items, you known the median lies between the 10th and 11th, so count this many in. This gives a median of 15.</td><td></td></tr>
<tr><td>200i. Calculate averages and range from frequency tables (Use Σf and Σfx)</td><td><table><tr><th>Num children</th><th>Frequency</th></tr><tr><td>0</td><td>15</td></tr><tr><td>1</td><td>20</td></tr><tr><td>2</td><td>5</td></tr></table>The above table shows how many children different families have.
As always the mean is the sum of the values divided by the number of values. If there’s 15 families with 0 children, that’s 0 children. If there’s 20 families with one child, that’s 20 children, and if there’s 5 families with 2 children, that’s 10 children.
There’s 15 + 20 + 5 = 40 families, so mean is $\frac{30}{40} = 0.75$
Thus, the approach is to sum the products of the values each with its frequency (i.e. Σfx, where x means the number of children, and Σ means ‘sum of’), and divide by the total frequency (Σf).
i.e. $mean = \frac{\Sigma fx}{\Sigma f}$ (But I recommend you remember the method rather than the formula)

Common student errors: Dividing by 3 instead of 40, because you think 3 rows means there’s 3 values. No: the values are each duplicated (e.g. there are 15 zeros!)</td><td></td></tr>
<tr><td>200ii. Solve problems involving mean.</td><td>Example: “The mean of mark of 20 students in a class is 70. A new person joins the class and the mean mark rises to 71. What was his mark?”
The key with these questions is just to <u>consider totals and differences of totals</u>.
Total mark of 20 students: $20 \times 70 = 1400$ Total mark of 21 students: $21 \times 71 = 1491$
So mark of 21st student $= 1491 - 1400 = 91$</td><td></td></tr>
</table>

Example: "A class consists of 20 boys and 10 girls. The mean mark of the boys is 62 and the mean mark of all the students was 65. What is the mean mark of the girls?"
Total mark of all students: $65 \times 30 = 1950$
Total mark of boys: $62 \times 20 = 1240$
Total mark of girls: $1950 - 1240 = 710$
Mean mark of girls: $\frac{710}{10} = 71$

Test Your Understanding:

a. The mean weight of 10 pancakes is 140g. When an additional pancake is made the mean weight decreases to 135g. How heavy was the additional pancake?
b. Galapagos tortoises can only be found on two islands: A and B, where there are 25 and 15 tortoises respectively. The mean age of the tortoises on the two islands is 130 and 120 years. What is the average age across the two islands?
c. On Farmer Frost's farm there are 20 cats. On Farmer Ingall's farm there are 50 cats. This year, Farmer Frost's cats have caught 120 mice on average. Across the two farms each cat caught 115 mice on average. What is the average number of mice caught by each of Ingall's cats?

201. Estimate the mean for large data sets with grouped data (and understand that it is an estimate)

Time taken (t minutes)	Frequency
$0 < t \leq 10$	6
$10 < t \leq 20$	11
$20 < t \leq 30$	8
$30 < t \leq 40$	5

If the data is grouped, the only difference from above is that you use the midpoints of each interval as a representative value.

Example: Estimate the mean time taken using the table on the left.

Total time: $(5 \times 6) + (15 \times 11) + (25 \times 8) + (35 \times 5) = 570$
Total frequency = $6 + 11 + 8 + 5 = 30$
Mean = $\frac{570}{30} = 19$ minutes.

Example: "*Explain why this mean is an estimate*"
Because we don't know the exact times within each range.

Test Your Understanding: The table shows the IQs of some people from Norfolk. Estimate the mean IQ.

IQ	Frequency
$70 < i \leq 90$	6
$90 < i \leq 100$	20
$100 < i \leq 130$	7
$130 < i \leq 160$	1

202. Find the median class interval and modal class interval from a grouped frequency table.

Time (t seconds)	Frequency
$60 < t < 70$	12
$70 < t < 80$	22
$80 < t < 90$	23
$90 < t < 100$	24
$100 < t < 110$	19

To find the <u>modal class interval</u>, just specify the interval with the highest frequency: $90 < t < 100$

To find the <u>median class interval</u>, find in what interval the middle item would occur, using cumulative frequencies. There are 100 items (i.e. total of frequencies) so we want the 50^{th}. This doesn't occur within first 12 items, nor the first 34 (12+22) but does occur within the first 57 (12+22+23), thus median class interval is $80 < t < 90$

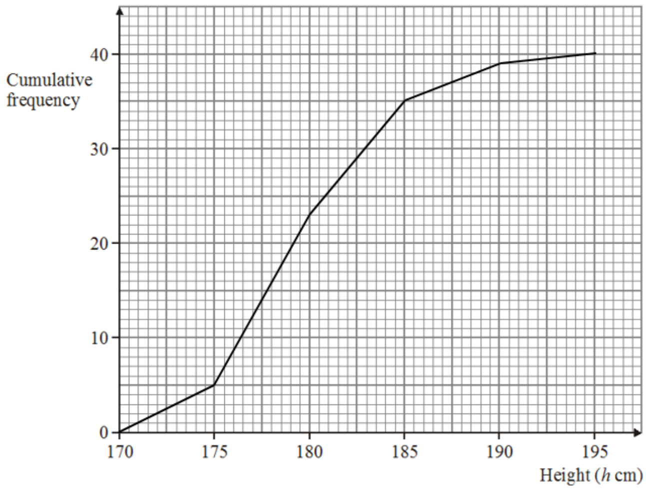
Test Your Understanding: Using the table on the right, (a) what is the modal class interval? (b) What is the median class interval?

Pocket money (£x)	Frequency
$0 < x \leq 2$	1
$2 < x \leq 4$	10
$4 < x \leq 6$	23
$6 < x \leq 8$	14
$8 < x \leq 10$	2

203. Draw cumulative frequency tables and graphs. Use cumulative frequency graphs to find median, quartiles and interquartile range

'Cumulative' means 'running total'. A cumulative frequency graph allows you to see how many people have <u>some value OR LESS</u>.
If you're given a frequency table, you can get the cumulative frequency for each row for the total value up to that value so far.
For plotting points, you use the <u>**end**</u> of each interval (unlike a frequency polygon, where you use the midpoint) because for example (using the table below), you don't know where the 5 people are in the 170-175cm interval, but you do know 5 people have a height of 175cm or less.
This means the plots you point are (175,5), (180,23) and so on. You ALSO plot a 'zero point': You know 0 people have a height of 170 or less, so plot (170,0). Join up with straight lines between each pair (a curve is also accepted).

Height (h cm)	Cumulative Frequency
$170 \le h < 175$	5
$170 \le h < 180$	23
$170 \le h < 185$	35
$170 \le h < 190$	39
$170 \le h < 195$	40

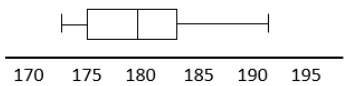


Example: "*Use your cumulative frequency graph to estimate (i) the median and (ii) the interquartile range*."
There's 40 people. So the 20th person will give the median. Drawing a horizontal line across from 20 to our graph, then down, the median is 179cm.
For the Lower Quartile, use the 10th person: this gives 176.5cm.
For the Upper Quartile, use the 30th person: this gives 183cm.
Therefore Interquartile Range = 183 – 176.5 = 6.5cm.

204. Draw box plots from a cumulative frequency graph

We can use the values from above to construct a box plot. The minimum and maximum values will always be given to you (because it's impossible to tell where the minimum for example occurs within the 170-175cm range, and likewise for the maximum).

Example: (From above) "The minimum height was 173cm and the maximum height was 192cm. Construct a box plot."

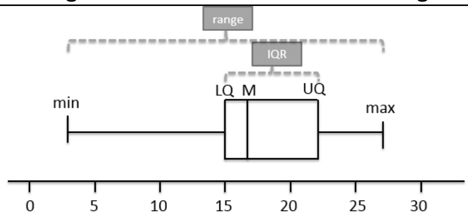


205. Compare the measures of spread between a pair of box plots/cumulative frequency graphs

There will always be two marks for this:

1. Compare the **<u>medians</u>** of the two box plots/box plot and CF graph. <u>It is not sufficient to simply state the two values</u>: you need to say which is bigger, or state they are the same. e.g. "The boys' median time was greater than the girls." If you wrote "the boy's median time was 15.6s and the girls 15.8s", then you will NOT get a mark.
2. Compare a measure of spread: either the **<u>range</u>** or the **<u>interquartile range</u>**. E.g. "The boys' interquartile range of times was the same as the girls."

206. Interpret box plots to find median, quartiles, range and interquartile range



207. Compare distributions and make inferences, using the shapes of distributions and measures of average and spread, including median and quartiles

From the box plot above, you could state there are more extreme values below the median (because of the long whisker to the left). You could also say the values are overall more spread out above the mean, because the right box is wider.

Probability

208. Write probabilities using fractions, percentages or decimals

209. Compare experimental data and theoretical probabilities. Compare relative frequencies from samples of different sizes

- Theoretical probabilities are the exact <u>true probability of something happening</u>, e.g. the theoretical probability of rolling a five on a fair die is $\frac{1}{6}$.
- Experimental probabilities (also known as relative frequencies) are probabilities <u>based on observed counts</u>. e.g. If I roll a die 120 times and see 25 sixes, the relative frequency of heads if $\frac{25}{120} = 0.208$. Relative frequencies tend to be given as decimals.

<table>
<tr><td></td><td>

- The more times an experiment is repeated, the close the relative frequency will be to the theoretical probability. e.g. If I throw a die repeatedly and count fives, the proportion of throws which will land five will get increasingly closer to $\frac{1}{6}$ as I throw more and more times. i.e. Relative frequencies become more reliable when the sample size increases.

</td><td></td></tr>
<tr><td>210. Find the probability of successive events, such as several throws of a single dice. Identify different mutually exclusive outcomes and know that the sum of the probabilities of all these outcomes is 1</td><td>

- If events are <u>independent</u>, it means they don't affect each other (e.g. "winning the lottery" and "owning a garden gnome"). If we want the probability that "A happened **<u>AND</u>** B happened", then we **<u>multiply</u>** the probabilities.
- If events are <u>mutually exclusive</u>, it means they can't happen at the same time. If we want the probability that "A happened **<u>OR</u>** B happened", we **<u>add</u>** the probabilities.

Example: "What's the probability of throwing 3 Heads in a row with a fair coin?"

$$P(H_1 \text{ } and \text{ } H_2 \text{ } and \text{ } H_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Example: "A spinner has three colours: red, green and blue. The probability of getting red is 0.2 and getting blue 0.3. What is the probability of getting green or blue?"
Colours are mutually exclusive as we can't get two colours at once. $P(blue) = 1 - 0.2 - 0.3 = 0.5$. So $P(green \text{ } or \text{ } blue) = 0.3 + 0.5 = 0.8$

</td><td></td></tr>
<tr><td>211. Estimate the number of times an event will occur, given the probability and the number of trials</td><td>

Simply multiply probability by number of trials.
Example: "The probability I get a six on an unfair die is 0.15. I throw the coin 120 times. How many sixes do I expect to see?"

$$120 \times 0.15 = 18 \text{ } times$$

</td><td></td></tr>
<tr><td>212. List all outcomes for single events, and for two successive events, systematically.
Use and draw sample space diagrams</td><td>

The **<u>sample space</u>** is the list of all possible outcomes. You can either present in list form or (in the case of two things happening) table form, known as a sample space diagram.

Example: "*You throw three coins. (a) List the possible outcomes and (b) hence determine the probability of throwing exactly two heads*."
Outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT (note that listing them in a systematic order prevents you from forgetting any possibilities)
In 3 of the 8 of these, you have two heads, thus $P(2 \text{ } Heads) = \frac{3}{8}$

Example: "*You roll two dice and add the values. Draw a sample space, diagram, and hence determine the probability that the total of the two dice is 8.*"

<table>
<tr><td></td><td></td><td colspan="6">1st Die</td></tr>
<tr><td></td><td>+</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td rowspan="6">2nd Die</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td></tr>
<tr><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td></tr>
</table>

Out of the 36 outcomes, five are 8, so probability is $\frac{5}{36}$

Test Your Understanding:

a. You roll two dice and multiply their outcomes. Determine the probability that the product is at least 7.
b. At a restaurant I have a choice of Avacado (A) or Beetroot (B) for starter, and Cod (C), Dog (D) or Eggs (E) for main course. List all the possible combinations of dishes I can have.

</td><td></td></tr>
<tr><td>213. Understand conditional probabilities. Use a tree diagram to calculate conditional probability</td><td>

Suppose the probability it rains today is 0.3 (irrespective or previous weather). If it rained yesterday, the probability it rains today will be higher. So probabilities can change depending on what events previous occurred.

Example: "*The probability I win a tennis game today (W) is 0.7. If I win today, the probability I win tomorrow (T) is 0.9, and if I didn't win today, the probability I win tomorrow is 0.4.*
(a) Draw a tree diagram to represent this information. (b) Hence determine the probability that I win a game tomorrow."
Diagram shown above. To find a probability from a tree: (i) Multiply the probabilities across each matching path across the tree and (ii) add these probabilities.
Therefore probability is $(0.7 \times 0.9) + (0.3 \times 0.4) = 0.75$

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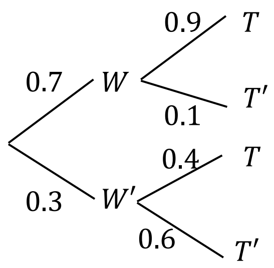
</table>

		
214. Solve more complex problems involving combinations of outcomes.	**Example**: "*The probability I buy avocado today is 0.7. The probability I buy asparagus is 0.6. Find the probability I buy either (but not both)*" For such problems, I'd advise: (i) listing the matching combinations of outcomes, (ii) finding the probability of each (by multiplying) and (iii) adding these together. A suitable tree diagram would also work. Avocado and not asparagus: $0.7 \times 0.4 = 0.28$ Not avocado and asparagus: $0.3 \times 0.7 = 0.18$ Probability of either (by not both): $0.28 + 0.18 = 0.46$ **Test Your Understanding**: The probability I pass my maths test is 0.68. The probability I pass my English test is 0.87. What's the probability I pass: (i) neither (ii) either (but not both).	
215. Understand selection with or without replacement. Draw a probability tree diagram based on given information	I generally advise using the above method rather than a probability tree, as you don't wastefully have to worry about unused parts of your tree. Be careful to note whether the item is not replaced (which changes BOTH the overall count of objects and the count of that type of object) or replaced. If you're just told "you take 3 sandwiches" then non-replacement is clearly implied. **Example**: "*A shop has 3 cheese sandwiches, 5 ham sandwiches and 2 dog sandwiches. I buy 2 sandwiches at random. Determine the probability I have two sandwiches of the same type*." As above, list out outcomes (the ordering of the sandwiches in the selection matters!), multiply to find probability of each and then add. DON'T simplify your probabilities until the end, otherwise you'll make it more difficult to add your fractions. CC: $\frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$ HH: $\frac{5}{10} \times \frac{4}{9} = \frac{20}{90}$ DD: $\frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$ $$\frac{6}{90} + \frac{20}{90} + \frac{2}{90} = \frac{28}{90} = \frac{14}{45}$$ **Test Your Understanding:** a. I have seven 10p coins and three 5p coins. I select three coins at random. Determine the probability that I have 25p in total. b. I have a bag of 6 green balls and 3 blue balls. I take a ball at random, note the colour then put it back. I take another ball. Determine the probability I have balls of different colours. c. I go to Battersea Dogs and Cats Home, which has 5 black cats, 3 white cats and 3 green cats. As a crazy cat person I take home three cats home with me at random. Determine the probability the cats are all of different colours.	

Answers

Number

1	30%, 0.303, 0.33, $\frac{1}{3}$ (since $\frac{1}{3} = 0.3333 \ldots$)
3	a. 7 b. 19008 c. 2.81 d. 50.41 e. 462.7
6	a. 25000 b. 15.1 c. 26.0 d. 495.185
7	a. $\frac{80\times 7}{0.5} = 1120$ b. $\frac{4\times 5}{0.25} = 80$ or $\frac{4\times 5}{0.2} = 100$
12	$\frac{5}{6}$
13	Two fractions are $\frac{28}{35}$ and $\frac{25}{35}$ thus $\frac{4}{5}$ is bigger.
14	15
16	a. $\frac{47}{45}$ b. $\frac{15}{14}$ c. $\frac{201}{20} = 10\frac{1}{20}$
17	a. $\frac{4}{15}$ b. $\frac{42}{5}$ c. $\frac{25}{44}$ d. $\frac{10}{3}$
19	$2^4 \times 3 \times 5$
21	a. 480 b. 12 c. 10.40am d. 3 packs of cookies (and 4 of chocolate bars)
23	a. 3.67×10^5 b. 4.8×10^{-4} c. 0.0267 d. $5\ 200$ e. 1.04×10^{11} f. 6×10^2 g. 4.3×10^5
26	a. $\frac{21}{25}$ b. 0.375
27	a. 0.1875 b. $0.\dot{2}\dot{7}$ c. $0.\dot{8}5714\dot{2}$ d. $\frac{4}{9}$ e. $\frac{401}{999}$ f. $\frac{701}{1110}$
29	165.6
30	Suppose initial amount was £1. Then in first bank account, we'd have $1 \times 1.05 \times 1.01 = £1.0605$. In second $1 \times 1.02 \times 1.03 = £1.0506$. So first is better.
31	a. $£11.70 \div 1.3 = £9$ b. $£13.50 \div 0.75 = £18$ $£18 - £13.50 = £4.50$

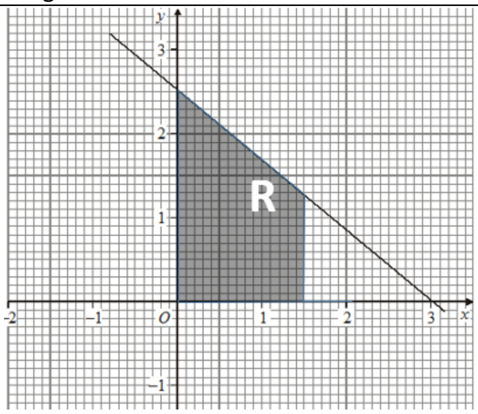
32	$£180\ 000 \times 0.75^6 = £32\ 036.13$
33	a. $£3500 \times 1.035^{10} = £4937.10$ b. $£1000 \div 0.9^5 = £1693.51$
36	a. 40 b. 3 parts = £12 1 part = £4 Thus 4 parts = £16 c. 10 parts = 5000g 1 part = 500g Unicorn: 1500g (so enough) Fairydust: 2500g (so NOT enough)
37	a. 10cm : 5 000m = 10cm : 500 000cm = 1 : 50 000 b. $5.4cm \times 20\ 000 = 108\ 000cm$ $= 1080m$ $= 1.08km$
38	a. $q = kr$ $5 = 6k$ $k = \frac{5}{6}$ $q = \frac{5}{6} \times 18 = 15$ b. $m = \frac{k}{n}$ $5 = \frac{k}{11} \to k = 55$ $m = \frac{55}{4} = 13.75$ c. $y = kx^2$ $10 = k \times 8^2 \to k = \frac{10}{64}$ $y = \frac{10}{64} \times 20^2 = 62.5$ d. $y = \frac{k}{\sqrt{x}}$ $10 = \frac{k}{\sqrt{10}} \to k = 10\sqrt{10}$ $20 = \frac{10\sqrt{10}}{\sqrt{x}}$ $\sqrt{x} = \frac{10\sqrt{10}}{20}$ $x = 2.5$
39	a. C, because $y = kx$ is the equation of a straight line which goes through the origin. b. D, because $y = \frac{k}{x}$ is a reciprocal graph.
40	a. y^{19} b. x^2 c. $27\sqrt{3} = 3^3 \times 3^{\frac{1}{2}} = 3^{\frac{7}{2}}$ So $x = \frac{7}{2}$ d. $a + 1$ e. (i) $3^{a-b} = \frac{3^a}{3^b} = \frac{x}{y}$ (ii) $3^{a+2b} = 3^a \times 3^{2b}$ $= 3^a \times (3^b)^2$ $= xy^2$
41	a. $27x^3y^{12}$ b. $25x^4y^8$ c. $6x^2y^3$ d. $4x^{\frac{8}{3}}y$

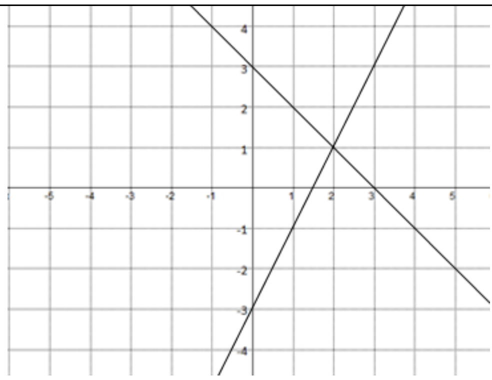
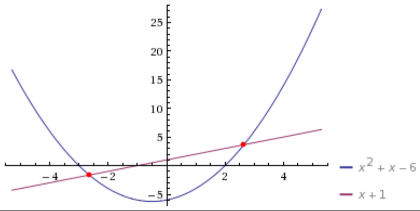
42	a. 1 b. $\frac{1}{49}$ c. 4 d. $\frac{1}{27}$ e. $\frac{4}{9}$
43	a. $4\sqrt{5}$ b. $\sqrt{3}$ c. $1+10\sqrt{3}$ d. $14-6\sqrt{5}$ e. (i) $15+26\sqrt{2}$ (ii) $4+20\sqrt{2}$
44	a. $=\frac{8\sqrt{2}}{2}=4\sqrt{2}$ b. $=\frac{\sqrt{400}+10\sqrt{5}}{5}=\frac{20+10\sqrt{5}}{5}$ $=4+2\sqrt{5}$
45	a. $0.0380784\ldots$ b. -80

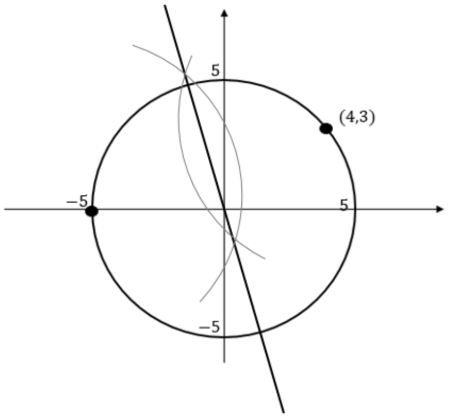
Algebra

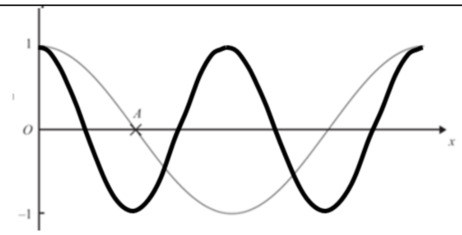
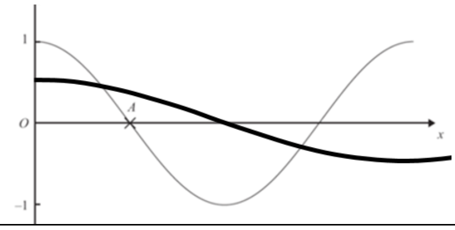
47	a. $-x^2y+xy^2$ b. $3x+y$
48	a. $2x+8-6+6x$ $=8x+2$ b. x^2-xy-y^2+xy $=x^2-y^2$
49	a. $3(x-2)$ b. $3xy(2x+3y)$ c. $4ab(2b^2-3ac)$
50	a. y^2-y-42 b. $6x^2-7x-20$ c. $16x^2+8x+1$ d. x^2y^4-1
51	a. $(x+1)^2$ b. $(x+4)(x+2)$ c. $(x-5)(x-2)$ d. $(x+2)(x-5)$ e. $(x+a)(x+b)$ f. $(2x+1)(x+1)$ g. $(3y-1)(y+4)$ h. $(4x+1)(3x-1)$
52	a. $(2+x)(2-x)$ b. $(xy+1)(xy-1)$ c. $(5y^2+6z)(5y^2-6z)$
53i	a. $6xy$ b. $12xy^3$ c. $\frac{y}{2}$ d. $x^2+2xy+y^2+x^2-2xy+y^2$ $=2x^2+2y^2$ e. x^2-x-5 f. $(2x+1)(2x+1)-(x+1)(x-3)$ $=4x^2+4x+1-(x^2-2x-3)$ $=4x^2+4x+1-x^2+2x+3$ $=3x^2+6x+4$ g. $(3x+1)(2x-3)-(1-2x)(1-2x)$ $=6x^2-7x-3-(1-4x+4x^2)$ $=6x^2-7x-3-1+4x-4x^2$ $=2x^2-3x-4$

53ii	a. $=\frac{(2x+1)(x+1)}{(x+1)(x+1)}=\frac{2x+1}{x+1}$ b. $=\frac{(2x-1)(x+3)}{(x+3)(x-3)}=\frac{2x-1}{x-3}$ c. $\frac{(x+y)(x+y)}{2(x+y)}=\frac{x+y}{2}$
53iii	a. $\frac{x+6}{2x}$ b. $\frac{2x+1}{x^2}$ c. $\frac{2x+1}{x(x+1)}$ d. $\frac{-x+3}{x(x-1)}$ e. $\frac{(x+1)^2-(x-1)^2}{(x+1)(x-1)}=\frac{4x}{(x+1)(x-1)}$
54	nth term is $3n-2$ So $(3\times 50)-2=148$
56	a. No, as n would be 11.5 b. No, as number does not end in 2 or 7. c. No as $n=15.2857\ldots$ d. Yes, $n=465$
57	a. 10, 17, 24 b. 8, 6, 4 c. 2, 5, 18
58	a. $3n-1$ b. $n+4$ c. $13-2n$ d. $\frac{2}{3}n+\frac{29}{6}$
59	a. $-1--6=5$ b. $1-6=-5$ c. $4-9=-5$ d. $-6-3=-9$ e. $9--24=33$
60	a. $x=\frac{2}{3}$ b. $6x+18=10x-14$ $32=4x$ $x=8$ c. $1-6+4y=y$ $3y=5$ $y=\frac{5}{3}$
62	a. $6-x=18$ $x=-12$ b. $3-4p=6p$ $3=10p$ $p=\frac{3}{10}$ c. $2(3x-1)^2=(2x+1)(9x-1)$ $2(3x-1)(3x-1)=18x^2+7x-1$ $2(9x^2-6x+1)=18x^2+7x-1$ $18x^2-12x+2=18x^2+7x-1$ $-12x+2=7x-1$ $3=19x$ $x=\frac{3}{19}$
63	a. $x<3$ b. $12-6x\geq 7$ $5\geq 6x$ $x\leq\frac{5}{6}$
65	a. $x=\frac{3b^2+1}{4}$ b. $x=\frac{a-b}{3}$ c. $3a=3\pi x+x$

	$3a = x(3\pi + 1)$ $x = \dfrac{3a}{3\pi + 1}$ d. $a(2x - 1) = 2x + 1$ $2ax - 2a = 2x + 1$ $2ax - 2x = 1 + 2a$ $x(2a - 2) = 1 + 2a$ $x = \dfrac{1 + 2a}{2a - 2}$ e. $ab + bx = ax$ $ab = ax - bx$ $ab = x(a - b)$ $x = \dfrac{ab}{a - b}$
68	a. Your line should go through $(0, -1)$ and $(3,5)$ b. Your line should go through $(0,3)$ and $(6,0)$
69	$m = \dfrac{-2}{4} = -\dfrac{1}{2}$
70	(i) $(0,4)$ (ii) $(-\frac{4}{3}, 0)$
72	a. -3 b. -1 c. 2 d. $-\frac{3}{4}$
74	a. $y = \frac{1}{5}x + c$ (where c is anything) b. $y = 3x - 10$ c. $y = -\frac{1}{3}x + 4$
75	$y = -\dfrac{1}{2}x + 7$
76	Point A is $(-8,0)$ Equation of AQ: $y = 2x + 16$ Point Q is therefore $(0,16)$ QB is parallel to PA so has gradient $-\frac{1}{2}$ Equation of QB: $y = -\frac{1}{2}x + 16$ Thus point B: $(32,0)$ Length $AB = 8 + 32 = 40$
77	 $P(1,1)$
78	a. $x = 5, y = -2$ b. $x = 3, y = -\frac{1}{2}$

79	 $x = 2, y = 1$
80	$3r + 9s = 120$ $5r + 5s = 90$ Solving: $r = 7, s = 11$ Therefore $5r + 7s = 112$ minutes
81	$x = 2.6$ to 1dp You MUST have tried $x = 2.6, 2.7, 2.65$ to gain full marks.
83	Missing values: 5, -4, -3
84	Using graph: $x = -1.4$ *or* $x = 6.4$ (to 1dp)
85	$x = 2.6, x = -2.6$ 
87	a. $(x + 7)(x - 2) = 0$ $x = -7, x = 2$ b. $x(x + 4) = 0$ $x = 0, x = -4$ c. $y^2 + 8y - 20 = 0$ $(y + 10)(y - 2) = 0$ $y = -10, y = 2$ d. $(2y + 1)(y + 1) = 0$ $y = -\dfrac{1}{2}$ *or* $y = -1$
88i	a. $(y + 5)^2 - 28$ b. $(x - 7)^2 - 44$ c. $3(x^2 + 6x - 2)$ $= 3((x + 3)^2 - 11)$ $= 3(x + 3)^2 - 33$ d. $2\left(x^2 - 2x + \frac{1}{2}\right)$ $= 2\left((x - 1)^2 - \dfrac{1}{2}\right)$ $= 2(x - 1)^2 - 1$
88ii	a. $(x + 4)^2 - 17$ Thus minimum value is -17. Occurs when $x = -4$ b. $y = (x + 1)^2 + 9$ Thus minimum point is $(-1,9)$
89	$(x - 3)^2 - 14 = 0$ $(x - 3)^2 = 14$ $x - 3 = \pm\sqrt{14}$ $x = 3 \pm \sqrt{14}$
90	a. $x = -1.14, x = 1.47$ b. $x = -8.93, x = 4.93$ c. $x = -2.69, x = 0.186$ (note that 0.19 would not have been to 3sf)

91	a. $x = -2, y = 2$ b. $x = -1, y = -2$ $x = 2, y = -1$ c. $x = -2.41, y = 6.20$ $x = 2.91, 3.55$
92	a. $1 + x = x^2$ $x^2 - x - 1 = 0$ $x = -0.62, 1.62$ b. $2x + 3x^2 = 5$ $3x^2 + 2x - 5 = 0$ $x = -\frac{5}{3}, x - 1$ c. $(2x + 2)^2 = (5x + 1)(x + 1)$ $4x^2 + 8x + 4 = 5x^2 + 6x + 1$ $x^2 - 2x - 3 = 0$ $x = -1, x = 3$
97	a. $6 = k \times a^2$ $162 = k \times a^5$ Dividing: $27 = a^3$ $a = 3$ $k = \frac{6}{3^2} = \frac{2}{3}$ b. $45 = a^2 b^3$ $\frac{9}{5} = a^2 b$ Dividing: $25 = b^2$ $b = 5$ $45 = a^2 \times 125 \; \rightarrow \; a = \frac{5}{3}$
98	i = E, ii = B, iii = F, iv = C, v = D, vi = A
99	Circle centred at the origin and going through all the 3s on the axes. Using points of intersection of circle and line: $x = -1.6, y = 2.6$ $x = 2.6, y = -1.6$
100	 Note that the perpendicular bisector goes through the centre of the circle.
101i	Line MUST go through points a. $(-2,-6), (-1,0), (0,2), (2,2),$ $(3,0), (4,-6)$ b. Must go through points: $(-0.5, 3), (0,0), (0.5,-1),$ $(1,0), (1.5,3)$
101ii	$(4,3)$
101iii	a.

	 b. 
103i	$y = f(x + 3)$
103ii	Instead of 1 oscillation per 360°, there's 2, so $b = 2$ Range of y values is 6 instead of usual 2 (i.e. -1 to 1), so $a = 3$ This would make y vary from -3 to 3, and it's been shifted down 1, so $c = -1$
104	If n is even we have: $(even \times even) + even + odd$ $= even + even + odd$ $= odd$ If n is odd we have: $(odd \times odd) + odd + odd$ $= odd + odd + odd$ $= odd$ Thus $n^2 + n + 1$ is odd for all n.
106	$= 3(n + 3)$
107	a. $x + (x + 1) + (x + 2)$ $= 3x + 3$ $= 3(x + 1)$ b. Difference of squares: $(x + 1)^2 - x^2$ $= x^2 + 2x + 1 - x^2$ $= 2x + 1$ Sum of two numbers: $x + (x + 1) = 2x + 1$ These are equal. c. $= (2n + 3)(2n + 3) - (2n - 3)(2n - 3)$ $= 4n^2 + 12n + 9 - (4n^2 - 12n + 9)$ $= 4n^2 + 12n + 9 - 4n^2 + 12n - 9$ $= 24n$ $= 8(3n)$

Shape, Space and Measures

108	$A(5,0)$ $B(3,2)$ $C(0,4)$ $D(-2,3)$ $E(-1,0)$ $F(-4,-1)$ $G(0,-3)$ $H(1,-2)$ $A(0,2,0)$ $B(0,2,3)$ $C(4,0,0)$ $D(4,2,3)$
109	a. $(2.5, 2)$ b. $(5,-1,-2)$
110	a. i) $x = 150°$ (corresponding angles), ii) $y = 95°$ (could have "corresponding angles are equal and angles on a straight line sum to 180" or "angles on a straight line sum to 180 and alternate angles are equal")

	b. $\angle ADC = 60°$ (cointerior angles sum to 180) $\angle BDC = 60 - 38 = 22°$ $\angle DEC = 139°$ (angles on straight line sum to 180) $x = 180 - 139 - 22 = 19°$ (angles in triangle sum to 180)
111	a. $120°$
112	$\angle AED = 38°$ (alternate angles are equal) $\angle ADE = \frac{180-38}{2} = 71°$ (base angles of isosceles triangle are equal) $x = 180 - 71 = 109°$ (angles on straight line sum to 180)
113	$3x + 2x - 5 + x + 10 + 2x + 15 = 360$ $8x + 20 = 360$ $8x = 340$ $x = 42.5°$
117	a. $180(15 - 2) = 2340°$ b. $140°$ c. $360 - (5 \times 50) = 110°$
118	a. $\frac{360}{3} = 120$ sides b. Exterior = $180 - 175 = 5°$ $\frac{360}{5} = 72$ sides c. $\frac{360}{15} = 24$ $180 - 24 = 156°$ d. $360 - 120 - 135 = 105°$
120	Angles at the bottom of the top triangle: Interior angle of A = $\frac{360-60}{2} = 150$ Exterior angle = 180 – 150 = 30 Num sides = $\frac{360}{30} = 12$
122	Front elevation: A rectangle of width 4cm and height 2cm Plan: A rectangle of sides 3cm and 4cm Side elevation: A rectangle of width 3cm and height 2cm.
128	a. $Area = \frac{\pi \times 5^2}{2} = 39.27cm^2$ $Perimeter = 10 + \frac{2 \times \pi \times 5}{2} = 10 + 5\pi$ b. $Area = \frac{\pi \times 10^2}{4} = 25\pi$ $Perimeter = 10 + 10 + \frac{2 \times \pi \times 10}{4}$ $= 20 + 5\pi$
129	Area of whole triangle: $\frac{1}{2} \times 6 \times 6 \times \sin 60 = 9\sqrt{3}$ Area of sector: $\frac{60}{360} \times \pi \times 3^2 = \frac{3}{2}\pi$ Shaded area: $9\sqrt{3} - \frac{3}{2}\pi$
130	$\left(\frac{70}{360} \times \pi \times 5.2^2\right) - \left(\frac{1}{2} \times 5.2 \times 5.2 \times \sin 70\right)$ $= 3.81cm^2$ b) $5.97cm$
131	a. $4.5m^2 = 45\,000cm^2$ b. $3cm^2 = 300mm^2$
132	Cross-sectional area: $44 + 15 = 59cm^2$
	Volume: $V = 59 \times 20 = 1180cm^2$
133	a. $Surface\ Area = 2(\pi \times 6^2) + (2 \times \pi \times 6 \times 5) = 72\pi + 60\pi = 132\pi$ $Volume = \pi \times 6^2 \times 5 = 180°$ b. Volume of vase = $\pi \times 8^2 \times 20 = 1280\pi$ Volume of cup = $\pi \times 3^2 \times 10 = 90\pi$ Cups = $\frac{1280\pi}{90\pi} = 14\frac{2}{9}$
134	a. $= \frac{4\pi \times 10^2}{2} + (\pi \times 10^2) = 300\pi$ b. Height = $\sqrt{13^2 - 5^2} = 12$ Volume = $\frac{1}{3}\pi \times 5^2 \times 12 = 100\pi$ Surface area = $(\pi \times 5^2) + (\pi \times 5 \times 13)$ $= 90\pi$
135	Diagonal across base: $\sqrt{6^2 + 8^2} = 10$ Thus halfway across base is 5. Height = $\sqrt{13^2 - 5^2} = 12$ Volume = $\frac{1}{3} \times (6 \times 8) \times 12 = 192cm^3$
136i	$\frac{1}{3} \times \pi \times r^2 \times 10 = 100$ $10\pi r^2 = 300$ $r^2 = \frac{300}{10\pi}$ $r = \sqrt{\frac{300}{10\pi}}$
136ii	a. Surface area of solid hemisphere: $\frac{4\pi x^2}{2} + \pi x^2 = 3\pi x^2$ Surface area of cylinder: $2\pi x^2 + 2\pi xh$ Equating: $3\pi x^2 = 2\pi x^2 + 2\pi xh$ $\pi x^2 = 2\pi xh$ $h = \frac{\pi x^2}{2\pi x} = \frac{x}{2}$ b. 'Melted down' means volume is preserved. Volume of sphere: $\frac{4}{3}\pi x^3$ Volume of cone: $\frac{1}{3}\pi x^2 h$ Equating volumes: $\frac{4}{3}\pi x^3 = \frac{1}{3}\pi x^2 h$ $4\pi x^3 = \pi x^2 h$ $4x = h$
137	a. $300\,000 \div 100^3 = 0.3m^3$ b. $4.2 \times 100^3 = 4\,200\,000cm^3$ c. $20 \times 10^3 = 20\,000mm^3$
138	Volume of full cone: $\frac{1}{3}\pi \times 4^2 \times 12 = 64\pi$ Volume of cut off cone: $\frac{1}{3}\pi \times 1^2 \times 3 = \pi$ Volume of frustum: 63π
150	a. $\sqrt{5^2 - (6-3)^2} = 4$ $AB = \sqrt{4^2 + 6^2} = \sqrt{80}$ b. Splitting the triangle into two: $h = \sqrt{13^2 - 5^2} = 12$

	c. $\sqrt{1^2+1^2}=\sqrt{2}$ $x=\sqrt{1^2+2}=\sqrt{3}$
151	a. $\sqrt{3}$ b. $13cm$
152	a. $x=1.5\tan 70=4.12$ b. i) $x=22\cos 40=16.9$ ii) $x=\frac{20}{\sin 80}=20.3$ ii) $x=\frac{4}{\cos 70}=11.7$ c. Right-most side: $6\tan 65=12.86704\ldots$ Bottom side of big triangle: $=\frac{12.86704}{\tan 35}=18.376$ $x=18.376-6=12.376$ d. i) $\theta=\tan^{-1}\left(\frac{2}{3}\right)=33.69°$ ii) $\theta=\sin^{-1}\left(\frac{6}{8}\right)=48.6°$
154	$\theta=\tan^{-1}\left(\frac{10}{\sqrt{41}}\right)=57.4°$
155	a. $\frac{x}{\sin 42}=\frac{5}{\sin 70}\qquad x=\frac{5\sin 42}{\sin 70}=3.56cm$ b. $x=\sqrt{6^2+4^2-(2\times 6\times 4\times\cos 100)}$ $=7.77cm$ c. $\frac{\sin\theta}{4}=\frac{\sin 20}{2}$ $\theta=\sin^{-1}\left(\frac{4\sin 20}{2}\right)=43.2°$ d. $4.8^2=3^2+4^2-(2\times 3\times 4\times\cos\theta)$ $23.04=25-24\cos\theta$ $24\cos\theta=1.96$ $\theta=\cos^{-1}\left(\frac{1.96}{24}\right)=85.3°$ e. First finding angle at left of triangle: $\frac{\sin\theta}{5.2}=\frac{\sin 50}{3.6}$ $\theta=\sin^{-1}\left(\frac{5.2\sin 50}{4.2}\right)$ $=71.5203°$ Ag Angle at top: $180-50-71.5203=58.4797°$ Using sine rule again: $\frac{x}{\sin 58.4797}=\frac{4.2}{\sin 50}$ $x=\frac{4.2\sin 58.4797}{\sin 50}=4.67cm$
156	a. $A=\frac{1}{2}\times 11\times 9\times\sin 45=35.00$ b. Find angle say between 8 and 9: $6^2=8^2+9^2-(2\times 8\times 9\times\cos\theta)$ $\theta=40.8044°$ Then $A=\frac{1}{2}\times 8\times 9\times\sin 40.8044$ $=23.5$ c. Working out the bottom side in the same method as 155(e): 9.21975 $A=\frac{1}{2}\times 9.21975\times 8\times\sin 40=23.7$
158	a. $6.7\div 4.5=1.49\ gallons$ b. $6.7\times 1.75=11.7\ pints$
159	$t=\frac{d}{s}=\frac{50}{6.3}=7.936=7\ hours\ 56\ mins$
160	$6.7km/h=6700\ m/h=1.86\ m/s$

162	a. $2m^3=2\,000\,000\ cm^3$ $m=2\,000\,000\times 6=12\,000\,000\ g$ $=12\,000\ kg=12\ tons$ b. $V=\left(\frac{1}{2}\times 5\times 4\right)\times 30=300\ cm^3$ $d=\frac{2500}{300}=8.33g/cm^3$
163	a. $x_{lower}=4.45\times\ 3.675=16.35375$ $x_{upper}=4.55\times 3.685=16.76675$ b. $q_{lower}=\frac{2.865^2}{3.5845}=2.289921886\ldots$ $q_{upper}=\frac{2.875^2}{3.5835}=2.306578\ldots$ So $q=2.3$ to $1dp$ as both bounds are the same to this level of accuracy.
169	a. Option 1: SAS $BD=CA$ as both are diameters. $\angle OAD=\angle ODA$ as ΔOAD is isosceles. AD is common. Therefore ABD and ACD are congruent by SAS. Option 2: RHS $\angle BAD=\angle CDA=90°$ as angle in semicircle is 90°. $BD=CA$ as both are diameters. AD is common. Therefore ABD and ACD are congruent by RHS. Option 3: ASA $\angle ABD=\angle ACD$ as angles in same segment are equal. $BD=CA$ as both are diameters. $\angle OAD=\angle ODA$ as ΔOAD is isosceles. Therefore ABD and ACD are congruent by ASA. b. By RHS: $\angle ABD=\angle ADC=90°$ as AD is perpendicular to BC. $AB=AC$ as ΔABC is equilateral. AD is common. Therefore ABD and ADC are congruent by RHS. We could have also use ASA or SAS given that $\angle ABD=60°$ and $\angle BAD=180-90-60=30°$. Only SSS is not a valid proof here as we can't yet show $BD=DC$. (ii) Given that ABD and ADC are congruent, $BD=DC$ thus $BD=\frac{1}{2}BC=\frac{1}{2}AB$.
171	a. $\frac{x}{15}=\frac{1.8}{2.5}\qquad x=10.8m$ b. $\frac{r}{9}=\frac{5}{12}\qquad r=3.75cm$ c. $\frac{x}{3}=\frac{6}{4}\qquad x=4.5$
172	a. $320cm^2$ b. $80\times 1.5^3=270cm^3$
174	a. $\angle BUM=160°$ (opposite angles of cyclic quadrilateral add to 180, angle at centre is twice angle at circumference) b. $z=115°$ (angle between radius and tangent is 90, angles in quadrilateral sum to 360) c. $\angle DBA=39°$ (Alternate Segment Theorem)

	$\angle DAB = 77°$ (opposite angles of cyclic quadrilateral add to 180) $\angle ADB = 180 - 77 - 39 = 64°$ (angles in triangle sum to 180)
178	$d = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad e = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad f = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
179	(i) $\overrightarrow{SQ} = -b + a \text{ or } a - b$ (ii) $\overrightarrow{NR} = \overrightarrow{NQ} + \overrightarrow{QR}$ $= \frac{2}{5}(-b + a) + b$ $= -\frac{2}{5}b + \frac{2}{5}a + b$ $= \frac{2}{5}a + \frac{3}{5}b$

Data Handling and Probability

181	$\frac{50}{200} \times 45 = 11.25$ so 11 boys
200ii	a. $(135 \times 11) - (140 \times 10) = 85g$ b. $\frac{(25\times130)+(15\times120)}{40} = 126.25$ c. $\frac{(60\times115)-(20\times120)}{50} = 113$
201	$Mean = \frac{(80 \times 6) + (95 \times 20) + (115 \times 7) + (145 \times 1)}{34}$ $= 97.9$
202	(a) Modal class interval = $4 < x \leq 6$

	(b) Median class interval: $4 < x \leq 6$ (as 25th item occurs within this interval)
212	a. $\frac{22}{36} = \frac{11}{18}$ b. AC, AD, AE, BC, BD, BE
214	(i) $0.32 \times 0.13 = 0.0416$ (ii) $(0.68 \times 0.13) + (0.32 \times 0.87) = 0.3668$
215	a. 10p 10p 5p: $\frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{7}{40}$ 10p 5p 10p: … $= \frac{7}{40}$ 5p 10p 10p: … $= \frac{7}{40}$ Therefore probability is $3 \times \frac{7}{40} = \frac{21}{40}$ b. GB: $\frac{6}{10} \times \frac{3}{10} = \frac{18}{100}$ BG: $\frac{3}{10} \times \frac{6}{10} = \frac{18}{100}$ Probability is $\frac{18}{100} + \frac{18}{100} = \frac{9}{25}$ c. BWG: $\frac{5}{11} \times \frac{3}{10} \times \frac{3}{9} = \frac{1}{22}$ BGW: … $= \frac{1}{22}$ WBG, WGB, GWB, GBW also give $\frac{1}{22}$ So probability is $\frac{1}{22} \times 6 = \frac{3}{11}$