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Differential Contra Algebraic Invariants: Applications to Classical Algebraic Problems

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Abstract—In this paper we discuss an approach to the study of orbits of actions of semisimple Lie groups in their irreducible complex representations,which is based on differential invariants on the one hand, and on geometry of reductive homogeneous spaces on the other hand. According to the Borel—Weil—Bott theorem, every irreducible representation of semisimple Lie group is isomorphic to the action of this group on the module of holomorphic sections of some one—dimensional bundle over homogeneous space. Using this, we give a complete description of the structure of the field of differential invariants for this action and obtain a criterion which separates regular orbits.

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1. INTRODUCTION

Problem of studying the orbit space Ω/G for an action $G : \Omega$ of a group G on a space Ω is one the most important problems, which has a lot of different applications in many areas (representation theory, geometry, differential equations, etc.). Most of examples of this problem can be separated into the following cases:

- Ω is a smooth manifold and G is a Lie group (geometric situation);
- Ω is an algebraic manifold and G is an algebraic Lie group, acting algebraically on Ω (algebraic situation).

In the first case it was proved by J.L. Koszul and R. Palais, that if the action $G : \Omega$ is proper and free, then the orbit space Ω/G is smooth manifold and $\Omega \to \Omega/G$ is a principal G-bundle. Moreover, G-orbits are separated by *smooth invariants*.

In this paper we consider the second case of algebraic action. This case has very long and interesting history. Namely, if the group G is semi-simple, Ω is a vector space and the action $G : \Omega$ is algebraic and linear, it was proved by D. Hilbert that the orbit space Ω/G is an algebraic manifold and regular G-orbits are separated by *polynomial invariants*. Also he proved that if G is reductive, then regular G-orbits are also separated by polynomial invariants and the space of closed G-orbits is the spectrum of polynomial invariant ring.

In 1960-th D. Mumford generalized Hilbert's results and created his *geometrical invariant theory* (see [13]). Mumford applied his theory to solution of so-called *module space problem*: he described module spaces of algebraic curves, Abel manifolds, vector bundles on curves, etc. These results made it possible to reinterpret basic methods and problems in invariant theory and stimulate a lot of new works in this area.

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In his famous talk on the Second mathematical Congress in Paris Hilbert formulated 23 great problems for the XX century. His 14th problem states, that polynomial invariant ring is finitely generated for *every algebraic* group G (and hence regular G-orbits are separated by polynomial invariants).

In 1954 Nagata constructed the counterexample to Hilbert 14th problem. Nagata group G has dimension 18 and it linearly acts on vector space V. Further this counterexample was simplified by Steinberg.

On the other hand, the classical result of M. Rosenlicht [16] claims, that the regular G-orbits of algebraic action of algebraic group G on an algebraic manifold are separated by *rational invariants* in opposite to polynomial ones.

There is no effective way to calculate rational invariants, and the algorithm for calculation of polynomial invariants works very slow. This observation makes it very difficult to use Hilbert and Rosenlicht theorems even in very simple problems.

This paper is a survey of our results [3]–[6] on applications and on using rational differential invariants in order to solve the problem of orbit separation.

Firstly, we reformulate algebraic problems under consideration as a problem on solutions of a suitable differential equation.

Secondly, we use rational differential invariants instead of algebraic ones. It allows us to use the Lie–Tresse theorem (see [11]) which gives constructive and effective way to find the field of rational differential invariants. The combination of the Lie–Tresse [11] theorem and the Rosenlicht [16] theorem shows that the field separates regular orbits.

Now let us discuss three examples.

1.1. Binary Forms

Let $V_n = \mathrm{S}^n(\mathbb{C}^2)^*$ be the space of binary forms of degree n over the field $\mathbb{C}$. Consider an action of the group $\mathrm{GL}_2(\mathbb{C})$ on the space V_n such that the subgroup $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ acts by linear coordinate transformations, and center $\mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{C})$ acts by the homotheties $f \mapsto \lambda f$, where $f \in V_n$ and $\lambda \in \mathbb{C}^*$.

This problem is closely connected with other classification problems in invariant theory such that the classification problem for the action of projective group $\mathrm{PGL}_2(\mathbb{C})$ on the projective line and with the classification of hyperelliptic curves of genus g (see [18]). The $\mathrm{SL}_2(\mathbb{C})$-invariant algebras $\mathbf{B}_n := \mathbb{C}[V_n]^{\mathrm{SL}_2(\mathbb{C})}$ of binary forms of degree $n \leqslant 8$ are known so far (see [8, 9, 2]).

The case $n = 3$ was solved by Bool in 1841. Namely, if

$$V_3 = \{ax^2 + 2bxy + cy^2 : a, b, c \in \mathbb{C}\},$$

then

$$\mathbf{B}_3 = \mathbb{C}\left[(ad - bc)^2 - 4\left(ac - b^2\right)\left(bd - c^2\right)\right].$$

The first nontrivial case $n = 4$ was solved by Bool, Cayley and Eisenstein in 1841–1850 and initiated the classical invariant theory (note that the problem of classification of binary forms of degree 4 is closely related to the cross ratio of the four projective points on projective line, and to the j-invariant of elliptic curve; see [18]).

If $V_4 = \{ax^3 + 3bx^2y + 3cxy^2 + dy^3 : a, b, c, d \in \mathbb{C}\}$, then $\mathbf{B}_4 = \mathbb{C}[I, J]$, where

$$I = ae - 4bd + 3c^2,\ J = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The case $n = 5$ was studied by Cayley. In this case the invariant algebra is generated by four homogeneous polynomials of degrees 4, 8, 12 and 18, which satisfy a homogeneous relation of degree 36 (note that the invariant of degree 18 consists of more than 800 monomials). There is also one syzygy between these invariants: $I_4I_8^4 + 8I_8^3I_{12} - 2I_4^2I_8^2I_{12} - 72I_4I_8I_{12}^2 - 432I_{12}^3 + I_4^3I_{12}^2 - 16I_{18}^2 = 0$. Note that the equation $f(x, 1) = 0$, where $f \in V_5$, is solvable in radicals if and only if $I_{18}(f) = 0$. The cases $n = 6, 7, 8$ were studied by Gordan, Shioda, Dixmier and Lazard. The explicit form of generators of the invariant algebra for $n = 7$ were found by Bedratyuk in 2007 only (see [2]).

In 1982 V. Kac proved, that the number of basic polynomial invariants grows exponentially with degree n, so it is impossible to calculate these invariants with the help of computer for sufficiently big n.

1.2. Ternary Forms

In the case, when $n = 2$, the invariant algebra is freely generated by the Hessian of ternary form.

The case, when $n = 3$, was studied by Weierstrass. In this case the invariant algebra is equal to $\mathbb{C}[S;T]$, where the invariants S and T respectively have degrees 4 and 6 (see [18, 8]).

Rational $\mathrm{GL}_3(\mathbb{C})$-invariant $j = S^3/T^2$ is called *j-invariant of ternary cubic form*, and two non-singular forms of degree 3 are $\mathrm{GL}_3(\mathbb{C})$-equivalent if and only if their j-invariants coincide.

It is known that any non-singular ternary form f of degree 3 is equivalent to the Weierstrass normal form $y^2z + x^3 + pxz^2 + qz^3$, where $p = S(f)$ and $q = T(f)$. It is also known (see [18]) that any elliptic curve is isomorphic to the curve $X(f) = \{f = 0\} \subset \mathbb{C}P^2$ for some ternary form f of degree 3 and two elliptic curves $X(f)$ and $X(\widetilde{f})$ are projectively isomorphic if and only if the ternary forms f and $\widetilde{f}$ are $\mathrm{GL}_3(\mathbb{C})$-equivalent.

In the case, when $n = 4$, it is known that the invariant algebra contains invariants I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}, where the indices of invariants show their degrees. It is worth to note that Emmy Noether found more than 300 concomitants (see [8]). But it is unknown if they generate the invariant algebra or not.

1.3. p-Forms

In the case of any p and $n = 2$, the algebra of polynomial invariants is freely generated by one polynomial, namely, by the Hessian of the form. In the case of $p = 4$ and $n = 3$, the algebra of invariants is $\mathbb{C}[A, B, C, D, E, F]$, where the basis invariants have degrees 8, 16, 24, 32, 40, 100, respectively, $F^2 \in \mathbb{C}[A, B, C, D, E]$, and the invariants A, B, C, D and E are algebraically independent (see [8]).

2. BASIC IDEAS OF A NEW APPROACH

In this paper we suggest a new approach to these classical problems based on ideas of differential geometry, jet spaces and differential invariants.

First of all, every algebraic action $G : \Omega$ can be linearized in the following sense. According to Sumihiro's linearization theorem, each algebraic G-manifold Ω can be embedded into a G-invariant submanifold in an irreducible finite-dimensional G-module V. So, it is enough to study invariants of the action of group G in its irreducible representation.

Let G be a connected semisimple complex Lie group, and let $\rho_\lambda \colon G \to \mathrm{GL}(V)$ be its irreducible representation with highest weight λ (see [10]). First, let us fix a Borel subgroup B in group G and consider homogeneous complex flag manifold $M := G/B$. Then, consider the action $B : G$ of Borel group B on G by the right shifts: $g \mapsto gb^{-1}$, where $g \in G$ and $b \in B$. Finally, let us define the bundle product $E := G \times_B \mathbb{C} = G \times \mathbb{C}/\sim$, where the equivalence relation $\sim$ is defined by the following: $(g, c) \sim (gb^{-1}, \chi_\lambda(b)c)$, and where $\chi_\lambda \in \mathfrak{X}(T)$ is the character corresponding to the highest weight λ of the maximal torus $T \subset B$.

We introduce one-dimensional bundle $\pi^\lambda \colon E \to M, \qquad \pi^\lambda(g, c) = gB$. Holomorphic sections of this bundle are just holomorphic functions $f \colon G \to \mathbb{C}$, which satisfy the relation $f(gb) = \chi_\lambda(b)f(g)$, for all $g \in G$ and $b \in B$.

Group G acts in bundle π^λ by left shifts. This action prolongs to the action on the space of holomorphic sections of bundle π^λ: $g(f)(g') = f(g^{-1}g')$.

According to the Borel–Weil–Bott theorem (see, for example, [7]), if λ is a dominant weight of the group G, then this action is isomorphic to the representation ρ_λ. Therefore, the study of orbits of irreducible representations of semisimple complex Lie groups with the highest weight λ is equivalent to the study of the orbits of these actions on the space of holomorphic sections of bundle π^λ.

Let us illustrate this idea in case $G = \mathrm{SL}_2(\mathbb{C})$ (see also [3]). It is known (see, for example, [10, 15]), that the dominant weights of the group $\mathrm{SL}_2(\mathbb{C})$ are $\lambda = \frac{n}{2}\alpha$, where α is the positive root of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and $n \geqslant 0$ is a non-negative integer. The Borel group $B = \mathrm{B}_2(\mathbb{C})$ consists of upper triangular matrices, and the character χ_λ acts on B in the following way:

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^n.$$

Then $M = \mathrm{SL}_2/\mathrm{B}_2 \simeq \mathbb{C}P^1$.

If we denote the homogeneous coordinates on M by $(x : y)$, then the holomorphic sections of bundle π^λ are just the homogeneous polynomials of degree n in variables x and y. Thus, the study of invariants of representations of group $\mathrm{SL}_2(\mathbb{C})$ is reduced to the classification $\mathrm{SL}_2(\mathbb{C})$-orbits of binary forms. This case was considered in [3], where binary forms were considered as solutions of the Euler differential equation. Let us recall the main results.

3. CLASSIFICATION OF BINARY FORMS

In this section we consider the $\mathrm{GL}_2(\mathbb{C})$-action on the space V_n of binary forms.

Let $\mathbb{C}^2$ be the plane with the coordinates (x, y). Denote by $J^k\mathbb{C}^2$ spaces of k-jets of analytical functions with canonical coordinates x, y, u, u_{10}, u_{01}, …. The group $\mathrm{GL}_2(\mathbb{C})$ acts on the space of analytic functions $C^\infty(\mathbb{C}^2)$ in following way. Namely, the subgroup $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ acts by linear coordinate transformations, and the center $\mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{C})$ acts by the homotheties $f \mapsto \lambda f$, where $f \in V_n$ and $\lambda \in \mathbb{C}^*$. This action prolongs to actions in the k-jet spaces $J^k\mathbb{C}^2$.

Recall that the space V_n of binary forms of degree n is identified with the space of smooth solutions of the Euler equation $xu_x + yu_y = nu$. The corresponding algebraic manifold $\mathcal{E} \subset J^1\mathbb{C}^2$ is given by the equation $xu_{10} + yu_{01} = nu$.

By a differential invariant of order k of binary form we mean a $\mathrm{GL}_2(\mathbb{C})$-invariant function on the manifold $\mathcal{E}^{(k)}$, which is polynomial in u_σ and u^{-1}.

In a similar way one defines an *invariant derivation* as a linear combination of total derivatives $\nabla = A\frac{d}{dx} + B\frac{d}{dy}$ (where A, $B \in C^\infty(J^\infty\mathbb{C}^2)$ and $\frac{d}{dx}, \frac{d}{dy}$ are the total derivatives), which is invariant with respect to the prolonged action of the group $\mathrm{GL}_2(\mathbb{C})$.

Note that for such derivations functions $\nabla(I)$ are differential invariants (generally, of order higher than the order of I) for any differential invariant I. This observation allows us to construct new differential invariants from known ones by differentiations only.

Theorem 1. *The algebra of differential invariants of the* $\mathrm{GL}_2(\mathbb{C})$*-action on the manifold* $\mathcal{E}^{(\infty)}$ *is freely generated by the differential invariant* $H := \frac{u_{20}u_{02} - u_{11}^2}{u^2}$ *and the invariant derivation* $\nabla = \frac{u_{01}}{u}\frac{d}{dx} - \frac{u_{10}}{u}\frac{d}{dy}$.

Remark. The numerator $f_{xx}f_{yy} - f_{xy}^2$ of the restriction $H(f)$ is the Hessian of the form f. It is well-known (see [18]) that the Hessian is a covariant of form f. It is easy to see that the numerators of all differential invariants $\nabla^k H$ are covariants. Hence, we can construct new covariants using the invariant derivation ∇.

Now consider the invariants $I_1 = H$, $I_2 = \nabla H$ and $I_3 = \nabla^2 H$. The restrictions of these invariants to the graph $L_f^4 \subset J^4\mathbb{C}^2$ of a form $f \in V_n$ are homogeneous polynomials in x and y. Then they are algebraically dependent and $F(I_1(f), I_2(f), I_3(f)) = 0$ for some irreducible polynomial F. Let us order the variables I_k by the requirement that $I_1 \prec I_2 \prec I_3$ and assume that the polynomial F has the minimal degree with respect to this order and is defined up to non zero scalar.

Definition. We say that a binary form $f \in V_n$ is *regular*, if $(2I_1I_3 - 3I_2^2)(f) \neq 0$.

Theorem 2. *Let* f_1, $f_2 \in V_n$ *be binary forms and* F_1, F_2 *be the corresponding dependencies between the invariants* I_k. *Then the forms* f_1 *and* f_2 *are* $\mathrm{GL}_2(\mathbb{C})$*-equivalent if and only if* $F_1 = F_2$.

Remarks. 1. It can be proved that the singularity condition $(2I_1I_3 - 3I_2^2)(f) = 0$ is equivalent to the condition that the form $f \in V_n$ has no more than two roots without taking multiplicity, or that the form f is equivalent to the form x^ky^{n-k} for some $k \leqslant n$, or that the orbit of form f has the dimension less than 4.

2. It is obvious that the ratios of coefficients of the polynomial F are rational algebraic invariants. Moreover, they separate regular orbits of binary forms. Hence (see [18]) they generate the field of rational invariants. So we obtain a method to calculate generators of rational invariant field for binary forms of arbitrary degree.

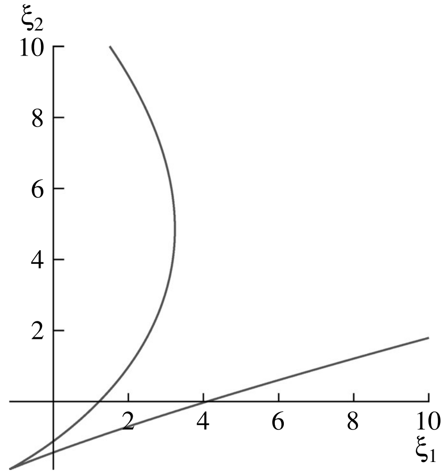

Fig. 1. Graph of the function $F(1,\xi_1,\xi_2)=0$ for the form $f(x,y)=xy(x+y)(-2x+y)$, where $\xi_1=I_2^2/I_1^3$ and $\xi_2=I_3/I_1^2$.

Examples. 1. Consider the forms

$$f_1(x,y)=xy(x+y)(-2x+y) \quad \text{and} \quad f_2(x,y)=xy(x+y)(-3x+y).$$

The corresponding dependencies between invariants I_k are

$$\begin{aligned} F_1(I_1,I_2,I_3) &= 3087I_2^4-(12348I_3+16464I_1^2)I_1I_2^2 \\ &-(800I_3^3-7548I_2^2I_3^2-23328I_1^4I_3-15552I_1^6), \\ F_2(I_1,I_2,I_3) &= 19773I_2^4-(79092I_3+105456I_1^2)I_1I_2^2 \\ &-(9800I_3^3-20292I_2^2I_3^2-93312I_1^4I_3-62208I_1^6). \end{aligned}$$

Then $F_1\neq F_2$ and the forms f_1 and f_2 do not belong to the same orbit. From the point of view of classical invariant theory, the orbits of binary forms of degree 4 are determined by the j-invariant (see [18]). In our case the j-invariant of the first form equals 343/36 and of the second one $-13^3/144$.

2. Consider the forms

$$f_1(x,y)=xy(x+y)(-x+y) \quad \text{and} \quad f_2(x,y)=xy(x+y)(2x+y).$$

The corresponding dependencies between the invariants I_k are

$$F_1(I_1,I_2,I_3)=F_2(I_1,I_2,I_3)=3I_2^2-6I_1I_3-8I_1^3.$$

Therefore, these forms belong to the same orbit. In this case the j-invariants of forms are equal to 27/4.

3. Consider the forms

$$f_1(x,y)=x^{10}-x^5y^5+y^{10} \quad \text{and} \quad f_2(x,y)=x^2y^2(x^6+y^6).$$

The corresponding dependencies between invariants I_k are

$$\begin{aligned} F_1(1,\xi_1,\xi_2) &= 1565515579392\xi_2-1464571772928\xi_1-544563247104\xi_1\xi_2+329450323968\xi_2^2 \\ &+140150452800\xi_1\xi_2^2-782467500\xi_1\xi_2^3+207664515328\xi_1^2-130753566000\xi_1^2\xi_2 \\ &-44018323200\xi_2^3+1173701250\xi_2^4+1391569403904+36422690625\xi_1^3, \\ F_2(1,\xi_1,\xi_2) &= -729482240\xi_2-201958400\xi_1-69766200\xi_1\xi_2-244800000\xi_2^2 \\ &+26471025\xi_1^2-27378000\xi_2^3-724451328, \end{aligned}$$

where $\xi_1:=I_2^2/I_1^3$ and $\xi_2:=I_3/I_1^2$. Then $F_1\neq F_2$ and the forms f_1 and f_2 do not belong to the same orbit.

4. The results obtained above may be generalized on space of *rational binary forms*. Let us consider the form

$$f(x,y) = xy\frac{(x+y)(x+2y)}{(x-y)(x-2y)}.$$

Then one has

$$\begin{aligned}
F(1,\xi_1,\xi_2) &= 154229410578567349395511105024{\xi_2}^2{\xi_1}^3 - 694032347603553072279799726088\xi_2{\xi_1}^4 \\
&+ 28918014483481378011658321922{\xi_2}^4{\xi_1}^2 - 556321829102390137076475494440{\xi_2}^3{\xi_1}^3 \\
&+ 1768551552623627792077490749440{\xi_2}^2{\xi_1}^4 - 983836776601619115865662711398 4\xi_2{\xi_1}^5 \\
&+ 308458821157134698791022100048{\xi_1}^4 + 655876613096096885560686359347 2{\xi_1}^5 \\
&+ 8148627609559524732432072572928{\xi_1}^6 - 2141112434058460368562244026368{\xi_1}^7 \\
&+ 753073293840660885720268 8{\xi_2}^8 - 384842775202908363970841935872{\xi_1}^8 \\
&- 4707117626354335668083097 6{\xi_2}^9 + 333176614158536204776243 2{\xi_2}^{10} \\
&+ 2409834540290114834304860 16{\xi_2}^6\xi_1 - 136673243790348732281735086 08{\xi_2}^5{\xi_1}^2 \\
&+ 179471560000206825234030919680{\xi_2}^4{\xi_1}^3 - 415512808557102055287769006080{\xi_2}^3{\xi_1}^4 \\
&+ 253859862561286373869983694848{\xi_2}^2{\xi_1}^5 + 1600854930279013658005444755456\xi_2{\xi_1}^6 \\
&+ 1010209174972661988360153464832\xi_2{\xi_1}^7 - 1351285292406540377215991808{\xi_2}^7\xi_1 \\
&+ 6464391879609645678578368512{\xi_2}^6{\xi_1}^2 - 944676378576871101008393011 2{\xi_2}^5{\xi_1}^3 \\
&- 384095333678579343591533568 00{\xi_2}^4{\xi_1}^4 + 291902384978808793261785612288{\xi_2}^3{\xi_1}^5 \\
&- 811280868499898987186679840768{\xi_2}^2{\xi_1}^6 + 936640250679895007106170 88{\xi_2}^8\xi_1 \\
&+ 355316241870148823167795 20{\xi_2}^7{\xi_1}^2 - 41543293854822374346999091 20{\xi_2}^6{\xi_1}^3 \\
&+ 153766637296759559895182622 72{\xi_2}^5{\xi_1}^4 - 240702404340643391537170268 16{\xi_2}^4{\xi_1}^5 \\
&+ 917805465324176383561764044 8{\xi_2}^3{\xi_1}^6 + 260311768127198330698137 60{\xi_2}^9\xi_1 \\
&- 153050307987937513189613760{\xi_2}^8{\xi_1}^2 + 368761639995304206203713536{\xi_2}^7{\xi_1}^3 \\
&- 1328951619623517097285509 12{\xi_2}^6{\xi_1}^4 - 267582051540050491871755 2{\xi_2}^{10}\xi_1 \\
&+ 153271233631275820267968{\xi_2}^9{\xi_1}^2 + 569130016731947834259456{\xi_2}^{11} \\
&+ 14553484508474084880243{\xi_2}^{12}.
\end{aligned}$$

4. CLASSIFICATION OF p-FORMS

Classification of homogeneous p-forms with respect to the linear action of group $\mathrm{GL}_p(\mathbb{C})$ can be provided in a way similar to the case of binary forms. Namely, let us consider the space $\mathbb{C}^p$ with coordinates x_1, …, x_p and the k-jet space $J^k\mathbb{C}^p$ with canonical coordinates $(x_1,\dots,x_p,u,u_\sigma)$. The homogeneous p-forms of degree n are solutions of the Euler equation $\mathcal{E} = \{\sum_{i=1}^p x_i u_i = nu\} \subset J^1\mathbb{C}^p$ (here, $u_i = u_{0\dots1\dots1} - 1$ occupies the i-th position).

4.1. Invariants Horizontal Forms

Before constructing differential invariants and invariant differentiations, we specify a set of invariant tensors for the action of the group $\mathrm{GL}_p(\mathbb{C})$ on the Euler equation.

Theorem 3. *The horizontal symmetric forms* $Q_k = \sum_{|\sigma|\leqslant k} \frac{u_\sigma}{u}\frac{(dx)^\sigma}{\sigma!}$ *given on the space* $J^k\mathbb{C}^p$ *are* $\mathrm{GL}_p(\mathbb{C})$*-invariant for all* $k \geqslant 1$.

Remark. We refer to the forms Q_k as *invariant k-forms*. Note that these forms are invariant for the linear action of arbitrary group G.

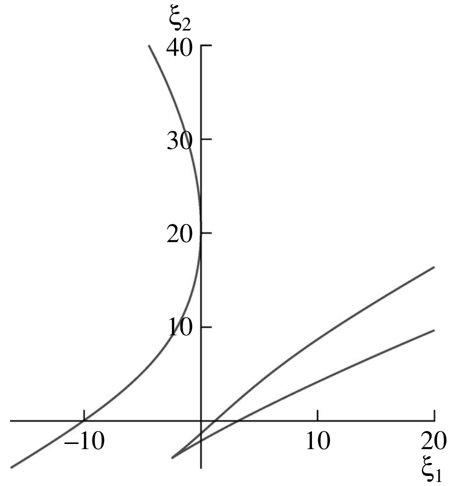

Fig. 2. Graph of the function $F(1, \xi_1, \xi_2) = 0$ for the form $f(x, y) = x^{10} - x^5y^5 + y^{10}$, where $\xi_1 = I_2^2/I_1^3$ and $\xi_2 = I_3/I_1^2$.

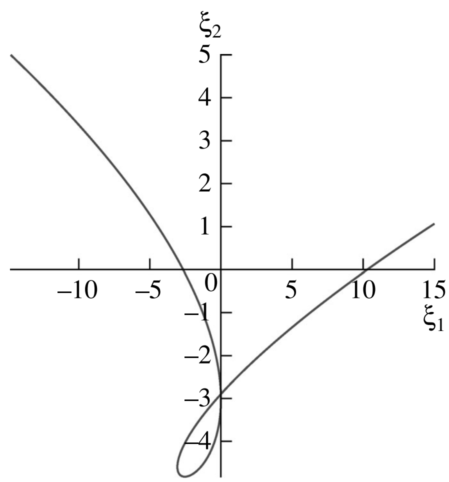

Fig. 3. Graph of the function $F(1, \xi_1, \xi_2) = 0$ for the form $f(x, y) = x^2y^2(x^6 + y^6)$, where $\xi_1 = I_2^2/I_1^3$ and $\xi_2 = I_3/I_1^2$.

4.2. Invariant Derivatives

Now we will describe a basis of invariant differentiations. Recall that, by an infinite jet, we mean a sequence $\{\theta_k\}$ of k-jets projected onto each other, i.e., such that $\pi_{k+1,k}(\theta_{k+1}) = \theta_k$, where $\pi_{k+1,k}\colon J^{k+1} \to J^k$ is a natural projection. Geometrically, each $(k+1)$-jet θ_{k+1} can be represented in the form of a k-jet θ_k and an R-plane $L_{(}\theta_{k+1}) \subset T_{\theta_k}J^k$ (see [1]).

A tangent vector in the space J^∞ of infinite jets is a sequence of pairs $\{(\theta_k, v_k)\}$, where $\theta_k \in J^k$ and $v_k \in L(\theta_{k+1})$, projected onto each other. If this sequence begins with a k_0-jet, then we say that k_0 is *the order* of the tangent vector.

The tangent space T to an infinite jet $\{\theta_k\}$ of order k_0 is the R-plane $L(\theta_{k_0+1})$ at the point θ_{k_0}. The tangent vectors are elements of the induced bundle $\tau_{k_0} := \pi_{k_0}^*(\tau)$, where $\tau\colon T\mathbb{C}^p \to \mathbb{C}^p$ is the tangent bundle of $\mathbb{C}^p$. By a vector field on the space of infinite jets of order k_0 we understand a section of the bundle τ_{k_0}. Now, take an infinite jet $\{\theta_k\}$ of order k_0 and let T denote the tangent space of this jet. All further considerations are in the space T.

Note that the invariant k-forms Q_k can be regarded as symmetric k-forms on the space T. We assume that the quadric Q_2 is non-degenerate on T.

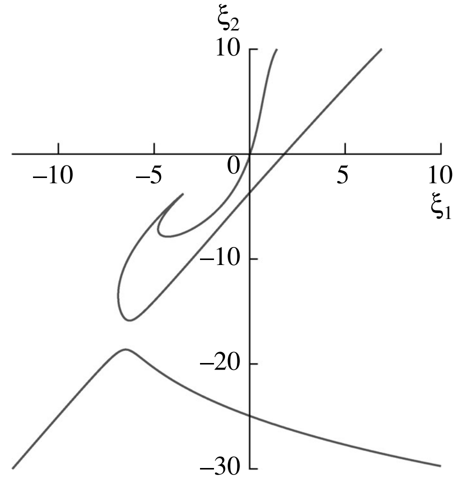

Fig. 4. Graph of the function $F(1,\xi_1,\xi_2)=0$ for the form $f(x,y)=xy\frac{(x+y)(x+2y)}{(x-y)(x-2y)}$, where $\xi_1=I_2^2/I_1^3$ and $\xi_2=I_3/I_1^2$.

For each tensor v, let v^* denote the tensor dual with respect to Q_2. By $\langle v, w*\rangle$ we denote the convolution of tensors. Consider the radial differentiation

$$r=\sum_{i=1}^{p} x_i\frac{d}{dx_i}\in T.$$

It is invariant and determines the decomposition of the space T into the direct sum $\langle r\rangle\oplus U$ of subspaces orthogonal with respect to Q_2. In what follows, all considerations are in the subspace U.

Take the tensor $Q_2^*\in \mathrm{S}^2(U)$ dual to Q_2 and consider a vector $\nabla_1\in U$ dual to the tensor obtained by pairing Q_3 and Q_2^*, that is, $\nabla_1=\langle Q_3,Q_2^*\rangle^*$. Consider also the linear operator $D\colon U\to U$ which sends each vector v to the vector dual to the convolution of cubic Q_3 with the symmetric product of ∇_1 and v, i.e., defined by $D\colon v\mapsto\langle\langle Q_3,\nabla_1\rangle,v\rangle^*$. We also set $\nabla_i:=D^{i-1}\nabla_1$.

Note that the vectors ∇_i depend on a point in space of 3-jets and are linearly independent in a Zariski open subset of the fiber of the projection $\mathcal{E}_3\to\mathcal{E}_2$.

Varying the point in the space of 3-jets, we obtain the set of differentiations ∇_1, ∇_2, …, ∇_{p-1}.

Theorem 4. *Differentiations r, ∇_1, …, ∇_{p-1} are invariant and form a basis in the space of invariant differentiations.*

Finally, we are ready to describe the entire field of rational invariants.

Note that the values of k-forms Q_k on the set of k invariant differentiations are differential invariants. Let $I_\alpha:=Q_3(\nabla_{\alpha_1},\nabla_{\alpha_2},\nabla_{\alpha_3})$, where $\alpha:=(\alpha_1,\alpha_2,\alpha_3)$ is an unordered set of indices, i.e., consider the coefficients[1)] of form Q_3 in the invariant basis $\{r,\nabla_1,\dots,\nabla_{p-1}\}$.

Theorem 5. *The field of differential invariants of an action of group $\mathrm{GL}_p(\mathbb{C})$ on manifold $\mathcal{E}_\infty$ is generated by the differential invariant H of order 2, by the differential invariants I_α of order 3 and by the invariant differentiations ∇_1, …, ∇_{p-1}. Moreover, this field is algebraically generated by the invariant H and by the derivatives of the form $\nabla^\sigma I_\alpha$. It separates the $\mathrm{GL}_p(\mathbb{C})$-orbits of jets of maximal dimension.*

Finally, we apply Theorem 3 in order to describe explicitly the $\mathrm{GL}_p(\mathbb{C})$-orbits of homogeneous p-forms with nonzero Hessian. For this purpose, consider the fourth-order differential invariants H, I_α, $\nabla_j I_\alpha$. Their restrictions to the graph L_f^4 of a form f with nonzero Hessian (this requirement is necessary,

[1)] It can be proved that all coefficients before r^* vanish, and therefore, the differentiation r does not participate in the definition of the invariants I_α.

because otherwise, the denominators of some invariants vanish) are homogeneous rational functions in the variables $x_1, \ldots, x_p$ and determine the rational mapping

$$\pi_f \colon \mathbb{C}^p \to \mathbb{C}^N, \quad \pi_f(a) = (H([f]_a^4), I_\alpha([f]_a^4), \nabla_j I_\alpha([f]_a^4))$$

(here N is the number of the chosen invariants). Thus, there are algebraic dependencies between these restrictions. We denote the set of such dependencies by $\mathcal{D}_f$ and the closure of image of mapping π_f by X_f. We refer to $\mathcal{D}_f$ as the ideal of dependencies of the p-form f.

Theorem 6. 1. *Forms f and $\widetilde{f}$ with nonzero Hessian are* $\mathrm{GL}_p(\mathbb{C})$*-equivalent if and only if* $X_f = X_{\widetilde{f}}$.

2. *Forms f and $\widetilde{f}$ with nonzero* Hessian *are* $\mathrm{GL}_p(\mathbb{C})$*-equivalent if and only if their ideals of dependencies coincide:* $\mathcal{D}_f = \mathcal{D}_{\widetilde{f}}$.

4.3. Examples: Ideal Dependencies for Ternary Forms

Example. Consider $f(x, y, z) = (x + y + z)(x^2 + y^2 + z^2)$. Then the ideal of dependencies $\mathcal{D}_f$ looks as follows (here $I := I_{(1,1,1)}$, $J := I_{(1,1,2)}$, $K := I_{(1,2,2)}$, $L := I_{(2,2,2)}$, $\nabla := \nabla_1$ and $\delta := \nabla_2$):

$$\begin{gathered}
\mathcal{D}_f = \langle \nabla I, \nabla J, \nabla K, \nabla L, -50JL + 189K^2 + 51K\delta L - 4\delta L^2,\\
HL + IL - 15JK + J\delta L,\\
123HK - 4H\delta L - 3IK + 4I\delta L - 30J^2,\\
2205IL^2 - 30303JKL + 1814JL\delta L + 12K\delta L^2 + 4\delta L^3,\\
1323IKL - 84IL\delta L - 4810J^2L + 7149JK\delta L - 452J\delta L^2,\\
40HJ\delta L + 305I^2L - 4923IJK + 334IJ\delta L + 115440K - 6400\delta L,\\
55I^2L - 873IJK + 54IJ\delta L + 40J^3 + 16752K - 896\delta L,\\
4845I^2L^2 - 73167IJKL + 4726IJL\delta L + 40J^2\delta L^2 + 1038960KL - 51200L\delta L,\\
12200960IJL^2 + 3319599IKL\delta L + 1153783IL\delta L^2 - 22658370J^2L\delta L\\
+ 255150J\delta L^3 - 1925654640L^2,\\
-185IJL + 6IK\delta L + 2I\delta L^2 + 2835J^2K - 210J^2\delta L - 3360L,\\
85050H\delta L^2 - 209920IJL + 178977IK\delta L + 50209I\delta L^2 - 468510J^2\delta L + 33131280L,\\
18450H^2\delta L + 18695HI\delta L - 138990HJ^2 + 6426I^2K - 163I^2\delta L - 2980IJ^2 - 1210320J,\\
215775I^2KL - 1785I^2L\delta L - 203490IJ^2L + 2310IJ\delta L^2 - 91697840JL\\
+ 136288536K\delta L - 8616928\delta L^2\\
-28160I^2JL^2 + 47346I^2KL\delta L - 3118I^2L\delta L^2 + 420IJ^2L\delta L - 6379791285IL^2\\
+ 87737639535JKL - 5275532430JL\delta L - 13778100\delta L^3,\\
3280H^2J^2 + 450HI^2\delta L - 3280HIJ^2 + 153I^3K + I^3\delta L - 110I^2J^2\\
+ 118080HJ - 26240IJ - 1889280,\\
9260I^3L^3 - 127332I^2JKL^2 + 7656I^2JL^2\delta L + 20I^2L\delta L^3 + 17193345IKL^2\\
+ 5659625IL^2\delta L - 62467470J^2L^2 - 287550JL\delta L^2 + 14580\delta L^4\rangle.
\end{gathered}$$

The dependencies become much more simple in the variables $\xi = -576I/H$, $\eta = -32JH/3$, $\zeta_1 = -16H^3K/27$, $\zeta_2 = 8H^5L/81$, $\zeta_3 = -(\nabla J)^2/9216H$, $\zeta_4 = -3962711310336(\nabla K)^2H^3$ (which are homogeneous of degree 0).

Example. Consider the form $f(x, y, z) = x^2yz$. Then the dependencies look as follows: $F_1(\xi, \eta) = \xi - 9\eta + 12$, $F_2(\eta, \zeta_1) = 3\eta + 9\zeta_1 - 4$, $F_3(\eta, \zeta_2) = 3\zeta_2 - \eta$. All other dependencies are also linear. Hence, in this case the manifold X_f is a 2-dimensional plane (in coordinates $\xi, \ldots, \zeta_4$).

Example. Consider the form $\widetilde{f}(x,y,z)=z^4+(xy)^2$. The dependencies look as follows:

$$\widetilde{F}_1(\xi,\eta,\zeta_1)=(-2\xi^2+32\xi-128){\zeta_1}^2+(-256\eta+2\eta^2\xi+8\xi^2 \\ +16\eta^2+8\eta\xi-\eta\xi^2)\zeta_1+8\eta^3-128\eta^2-8\eta^2\xi+\eta^3\xi,$$

$$\widetilde{F}_2(\xi,\eta,\zeta_2)=(4\xi^3+768\xi-2048-96\xi^2){\zeta_2}^2+(4\eta^3\xi+1536\eta^2-96\eta^2\xi+\eta\xi^3-96\eta^3-8\xi^3+128\eta\xi \\ +128\xi^2-8\eta\xi^2-4096\eta)\zeta_2+384\eta^3-24\eta^3\xi+128\eta^2\xi+\xi\eta^4-2048\eta^2-24\eta^4,$$

$$\widetilde{F}_3(\xi,\eta,\zeta_3)=(64\xi+192\eta+1024){\zeta_3}^2+(256\xi-12\eta^5+224\eta^4-768\eta^3-64\xi^3+512\eta-1088\eta^2\xi \\ -384\eta\xi^2+224\eta^3\xi-16\xi\eta^4-\eta^2\xi^3+16\eta\xi^3+96\eta^2\xi^2-7\eta^3\xi^2+8192)\zeta_3+16384+\eta^3\xi^4+32\eta^5 \\ +64\eta\xi^3+640\eta^4+360\eta^3\xi^2-3328\eta^2\xi+208\xi\eta^4+64\eta\xi^4-16\eta^2\xi^4+704\eta^3\xi-256\xi^3 \\ -1024\eta\xi^2-48\eta^5\xi+9\eta^5\xi^2+6\eta^4\xi^3-2560\eta^3-80\eta^3\xi^3+252\eta^2\xi^3-112\eta^4\xi^2+320\eta^2\xi^2,$$

$$\widetilde{F}_4(\xi,\eta,\zeta_4)=(-256\xi^3-49152\eta+49152\xi-262144+12288\eta\xi-768\eta\xi^2){\zeta_4}^2+(131072\eta+7\eta^3\xi^4 \\ +768\eta^5-3072\eta\xi^3-8192\eta^4+3264\eta^3\xi^2-4096\eta\xi-1024\eta^2\xi-4096\xi^2+3072\xi\eta^4+\xi^5\eta^2-16\xi^5\eta \\ +320\eta\xi^4+8192\eta^2-96\eta^2\xi^4-11264\eta^3\xi-256\eta^3\xi^3+64\xi^5-2048\eta^3-192\eta^5\xi+12\eta^5\xi^2 \\ +16\eta^4\xi^3+1472\eta^2\xi^3-6144\eta^2\xi^2-384\eta^4\xi^2)\zeta_4-64\eta^3\xi^4+2560\eta^5+1024\eta^3\xi^2+80\eta^5\xi^3+16\eta^4\xi^4 \\ -16384\eta^2-704\eta^5\xi-360\eta^5\xi^2-252\eta^4\xi^3+256\eta^2\xi^3-320\eta^4\xi^2-208\eta^6\xi+48\eta^7\xi-\eta^5\xi^4 \\ -9\eta^7\xi^2+3328\xi\eta^4-64\eta^3\xi^3-32\eta^7-640\eta^6+112\eta^6\xi^2-6\eta^6\xi^3.$$

The other dependencies are much more complicated, and we do not write them.

5. DEFINITIONS AND NOTATIONS

Now we suggest another approach to the study of invariants of irreducible representations for semisimple Lie groups based on the Borel–Weil–Bott theorem. Namely, we consider jet space of the section of bundle π^λ, then we describe the differential invariant field of the G-action on the jets of sections and, finally, obtain the criterion, which separates G-orbits of the regular sections of bundle π^λ.

In this section we introduce basic notations and recall necessary definitions.

5.1. Compact Real Form

To study invariants of the group G on the module of holomorphic sections of bundle π^λ, we use the following trick.

Let K be the compact real form of the group G (see [18]), $\mathfrak{k}$ be its Lie algebra and $T:=K\cap B$ be its maximal torus. Then $M\simeq K/T$, $E\simeq K\times_T\mathbb{C}$, and holomorphic sections of $\pi^\lambda\colon E\to M$ can be considered as functions $f\colon K\to\mathbb{C}$ such that $f(kt)=\chi_\lambda(t)f(k)$ for all $k\in K$ and $t\in T$.

It follows from the unitary trick (see, for example, [18]) that the rational differential G-invariants coincide with the rational differential K-invariants. Hence, we shall study the invariants of K-action on the module of holomorphic sections.

5.2. Decomposition of Lie Algebras $\mathfrak{k}$ and $\mathfrak{m}$

Note that the torus $\mathfrak{t}$ defines the decomposition of algebra $\mathfrak{k}$:

$$\mathfrak{k}=\mathfrak{t}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{k}_\alpha=\mathfrak{k}_-\oplus\mathfrak{t}\oplus\mathfrak{k}_+,\quad \mathfrak{m}=\bigoplus_{\alpha\in\Phi_+}\Pi_\alpha,$$

where

$$\mathfrak{k}_\pm:=\bigoplus_{\alpha\in\Phi_\pm}\mathfrak{k}_\alpha,\quad \Pi_\alpha:=\mathfrak{k}_\alpha\oplus\mathfrak{k}_{-\alpha},$$

and $\mathfrak{m}$ is the tangent space to manifold M at T, Φ, $\Phi_\pm$ are the root system and the sets of positive/negative roots.

5.3. Jet Bundles

Here we introduce some notations and definitions connected with jet bundles and differential invariants. For more details see [1, 11].

Let us consider the bundle π^λ which corresponds to highest weight λ. By $J^k(\pi^\lambda)$ we denote the k-jet space of holomorphic sections of the bundle π^λ. Canonical projections will be denoted as $\pi^\lambda_{k,k-1}\colon J^k(\pi^\lambda) \to J^{k-1}(\pi^\lambda)$ and $\pi^\lambda_k\colon J^k(\pi^\lambda) \to M$. Let also $J^\infty(\pi^\lambda) = \lim_{\longleftarrow} J^k(\pi^\lambda)$ be the infinite jet space.

The action of the group K on the sections of the bundle π^λ canonically prolongs to the action on all jet spaces $J^k(\pi^\lambda)$ and on $J^\infty(\pi^\lambda)$.

Definition (see also [11]). A rational function J on the k-jet space $J^k(\pi^\lambda)$ is called a *differential invariant of the K-action of order $\leqslant k$* if J is invariant under the prolonged action of the group K on the space $J^k(\pi^\lambda)$.

Similarly, a total derivation $\nabla\colon C^\infty(J^\infty(\pi^\lambda)) \to C^\infty(J^\infty(\pi^\lambda))$ is called *invariant* if it commutes with the prolonged action of the group K. We will consider only invariant derivations with rational components.

Remark. Recall that, according to the results of paper [11], the field of differential invariants of algebraic K-actions is finitely generated over invariant derivations, i.e., each differential invariant can be represented as a rational function of invariant derivations of a finite number of basic differential invariants.

Our aim is to describe the field of differential invariants of the action of the group K on the space $J^\infty(\pi^\lambda)$, i.e., to find the set of basic invariants and invariant derivations, which generate the entire field.

6. DECOMPOSITION OF JET BUNDLES

6.1. Invariant Connections

In this section we describe the so-called *Nomizu and Wang invariant connections*. These connections will be used in the next section for construction of invariant tensors.

Let $\varkappa$ be the Killing form on the group K. The restriction of the Killing form to the Lie algebra $\mathfrak{t}$ of the torus T is non-degenerate. Hence, there exists an orthogonal decomposition of the Lie algebra $\mathfrak{k}$ of the group K: $\mathfrak{k} = \mathfrak{t} \oplus_\perp \mathfrak{m}$. Moreover, the subspace $\mathfrak{m}$ is ad($\mathfrak{t}$)-invariant: $[\mathfrak{m}, \mathfrak{t}] \subseteq \mathfrak{m}$. Therefore, M is a reductive homogeneous space and π^λ is a homogeneous vector bundle over reductive homogeneous space (see [14, 17]). The decomposition $\mathfrak{k} = \mathfrak{t} \oplus_\perp \mathfrak{m}$ defines an invariant torsion-free connection Γ on M in the following way:

$$\Gamma_X Y = \frac{1}{2} \cdot \mathrm{pr}_{\mathfrak{m}}[X, Y],$$

where X, Y are vector fields on M and $\mathrm{pr}_{\mathfrak{m}}$ denotes the projection onto the subspace $\mathfrak{m}$. This connection is called the *Nomizu connection* (see [14]).

There is also an invariant connection Δ in the bundle π^λ, which is defined as follows: $d_\Delta f = df - \omega_\lambda f$ for all sections f, where ω_λ is an invariant differential 1-form on K such that $\omega_\lambda\,|_{\mathfrak{t}} = \lambda$ and $\omega_\lambda\,|_{\mathfrak{k}_\pm} = 0$. This connection is called the *Wang connection* (see. [17]). Remark, that these two invariant connections can be effectively calculated.

6.2. Invariant Tensors

In this section we construct K-invariant tensors on the jet space $J^\infty(\pi^\lambda)$ (some more simple analogs of these tensors were obtained in [5]).

The following short sequence of bundles

$$0 \to \mathrm{S}^k\tau^* \otimes \pi^\lambda \to \pi_k^\lambda \to \pi_{k-1}^\lambda \to 0,$$

where $\tau^*\colon T^*M \to M$ is the cotangent bundle of the base M and $\mathrm{S}^k\tau^*$ is its k-th symmetric power, is exact. Using the invariant connections Γ and Δ, we now a construct K-invariant splitting of this sequence. To this end we construct a differential operator $\pi^\lambda \to \mathrm{S}^k\tau^* \otimes \pi^\lambda$ of order k and with identity symbol. First of all let us consider the case $k = 1$. Then the covariant differential $d_\Delta\colon \pi^\lambda \to \tau^* \otimes \pi^\lambda$ is our differential operator.

Now let $k = 2$. Then the tensor product of connections defines the operator

$$d_{\Gamma\otimes\Delta}\colon \tau^* \otimes \pi^\lambda \to \tau^* \otimes \tau^* \otimes \pi^\lambda.$$

Let

$$d^s_{\Gamma\otimes\Delta} := \mathrm{Sym} \circ d_{\Gamma\otimes\Delta}\colon \tau^* \otimes \pi^\lambda \to \mathrm{S}^2\tau^* \otimes \pi^\lambda,$$

where Sym is the operator of symmetrization. In the same way, taking symmetric product we get the operators

$$d^s_{\Gamma\otimes\Delta}\colon \mathrm{S}^{k-1}\tau^* \otimes \pi^\lambda \to \mathrm{S}^k\tau^* \otimes \pi^\lambda.$$

Finally, we get the operator

$$\delta_k := \underbrace{d^s_{\Gamma\otimes\Delta} \circ \ldots \circ d^s_{\Gamma\otimes\Delta}}_{k-1} \circ d_\Delta\colon \pi^\lambda \to \mathrm{S}^k\tau^* \otimes \pi^\lambda$$

with identity symbol. This operator generates the morphism $\phi_{\delta_k}\colon \pi_k^\lambda \to \mathrm{S}^k\tau^* \otimes \pi^\lambda$ of jet bundles (see [1]) and defines the decomposition of k-jet bundle π_k^λ into direct sum: $\pi_k^\lambda = \pi_{k-1}^\lambda \oplus (\mathrm{S}^k\tau^* \otimes \pi^\lambda)$. Note that every transformation which preserves the connections Γ and Δ (in particular, the action of our group K), preserves the morphisms ϕ_{δ_k} and the corresponding decomposition of the bundle π_k^λ.

Now let us consider the induced bundles $\pi_l^{\lambda\,*}(\mathrm{S}^k\tau^* \otimes \pi^\lambda)$ over $J^l(\pi^\lambda)$. Lifting the operators d_Δ and $d^s_{\Gamma\otimes\Delta}$ to the induced bundle over $J^l(\pi^\lambda)$, we obtain the *total differentials* $\widehat{d}^s_\Delta\colon \pi_l^{\lambda\,*}(\pi^\lambda) \to \pi_{l+1}^{\lambda\,*}(\tau^* \otimes \pi^\lambda)$ and

$$\widehat{d}^s_{\Gamma\otimes\Delta}\colon \pi_l^{\lambda\,*}(\mathrm{S}^k\tau^* \otimes \pi^\lambda) \to \pi_{l+1}^{\lambda\,*}(\mathrm{S}^{k+1}\tau^* \otimes \pi^\lambda).$$

Finally, we identify the morphisms of bundles $\phi_{\delta_k}\colon \pi_k^\lambda \to \mathrm{S}^k\tau^* \otimes \pi^\lambda$ with sections of the bundle $\pi_k^{\lambda\,*}(\mathrm{S}^k\tau^* \otimes \pi^\lambda)$ over $J^k(\pi^\lambda)$ and put $Q_k := \phi_{\delta_k}$. These sections can be also viewed as horizontal symmetric k-forms on $J^k(\pi_k^\lambda)$ with values in π^λ.

Theorem 7. *Symmetric tensors Q_k are K-invariant for all $k \geqslant 0$. The following equalities hold: $Q_1 = \widehat{d}^s_\Delta Q_0$ and $Q_{k+1} = \widehat{d}^s_{\Gamma\otimes\Delta} Q_k$, for $k \geqslant 1$. Each k-*jet *θ_k can be represented in the following form: $\theta_k = (Q_0(\theta_k), Q_1(\theta_k), \ldots, Q_k(\theta_k))$.*

6.3. Invariant Tensors on Differential Equations

Below we will use the invariant tensors Q_k for classification of holomorphic sections of the bundle π^λ. A section is holomorphic if and only if it satisfies the Cauchy–Riemann equations, therefore we will classify solutions of this equation system. So we need to restrict invariant tensors Q_k on these equations. Here we describe this restriction in a general situation, and after that we will apply it to our problem of classification of holomorphic sections.

Let $\mathcal{E} = \{\mathcal{E}_k \subset J^k(\pi^\lambda) : k \geqslant 1\}$ be a formally integrable system of differential equations (see [1]), where $\mathcal{E}_{k+1} := \mathcal{E}_k^{(1)}$ are prolongations. We assume that if the system $\mathcal{E}$ has order l, then $\mathcal{E}_k = \pi_k^\lambda$ for all $k < l$. The symbol of this system is denoted by $g := \{g_k \subset \mathrm{S}^{k+l}\tau^* \otimes \pi^\lambda\}$.

Definition. The system $\mathcal{E}$ is said to be *concordant with connections* $\{\Gamma, \Delta\}$, if $\phi_{\delta_{k+l}}(\mathcal{E}_k) \subseteq g_k$ for all k.

If the system $\mathcal{E}$ is concordant with connections $\{\Gamma, \Delta\}$, then the equations $\mathcal{E}_k$ can be decomposed into the direct sum $\mathcal{E}_k = \bigoplus_{i \leqslant k} g_i$, and the above construction defines tensors Q_k on $\mathcal{E}_k$.

Now we return to our problem of classification of holomorphic sections of the bundle π^λ. Let $\mathcal{E} := \mathcal{E}_{CR}$ be the Cauchy–Riemann system. Then its symbols are $g_k = \mathrm{S}^k\tau^*_{1,0} \otimes \pi^\lambda$, where $\tau^* = \tau^*_{1,0} \oplus \tau^*_{0,1}$ is the decomposition of cotangent bundle into holomorphic and anti-holomorphic parts defined by the complex structure on the homogeneous space M. Note that the Cauchy–Riemann system $\mathcal{E}$ is concordant with connections $\{\Gamma, \Delta\}$. Hence one can restrict invariant morphisms ϕ_{δ_k} and tensors Q_k to the equations $\mathcal{E}_k$. The corresponding restrictions will be denoted by $Q_k^{\mathcal{E}}$.

6.4. Invariant Derivations

In this section we construct a basis of invariant derivations by using the invariant tensors $Q_k^{\mathcal{E}}$.

Let us fix an infinite jet $\theta \in J^\infty(\pi^\lambda)$ and consider the Cartan space $\mathcal{T}_\theta = \mathcal{T}_\theta^{1,0} \oplus \mathcal{T}_\theta^{0,1}$ (recall that the Cartan plane is a subspace of $T_\theta J^\infty(\pi^\lambda)$ generated by the plane tangent to the graphs L_f^∞ of sections at θ such that $\theta \in L_f^\infty$). Then the value $Q_0^{\mathcal{E}}$ at θ is just a vector from the one-dimensional space π^λ and the value of each tensor $Q_k^{\mathcal{E}}$ (here $k \geqslant 1$) can be presented as $Q_k^{\mathcal{E}} = q_k^{\mathcal{E}} \otimes Q_0^{\mathcal{E}}$, where $q_k^{\mathcal{E}} \in \mathrm{S}^k\mathcal{T}_\theta^{*1,0}$ is a horizontal symmetric complex-valued k-form. We assume that $Q_0^{\mathcal{E}} \neq 0$.

It follows from Theorem 7 that the forms $q_k^{\mathcal{E}}$ satisfy the recurrent relation

$$\widehat{d}^s_\Gamma q_k^{\mathcal{E}} = q_{k+1}^{\mathcal{E}} + q_k^{\mathcal{E}} \cdot q_1^{\mathcal{E}},$$

where $\widehat{d}^s_\Gamma$ is the total symmetric differential due to the Nomizu connection Γ.

Let us consider the decomposition of space $\mathfrak{m}$ into the direct sum: $\mathfrak{m} = \bigoplus_{\alpha>0} \Pi_\alpha$ of 1-dimensional complex spaces. Let us project the invariant 1-form $q_{1,\theta}^{\mathcal{E}}$ to all planes Π_α. We get the set of invariant 1-forms $q_{1,\alpha}^{\mathcal{E}}$, where $\alpha \in \Phi_+$. Assume that all of them are non-trivial. Then, taking the dual vectors to these forms in each invariant plane Π_α, we obtain invariant vectors $\nabla_{i,\theta}$ on $\mathfrak{m}$, where $i = 1, \ldots, m := \dim_{\mathbb{C}} M$. Finally, after variation of infinite jet θ, we get the set of horizontal vector fields $\nabla_1, \ldots, \nabla_m$.

Definition. We say that a 1-jet θ is *regular*, if the vectors $Q_0^{\mathcal{E}}(\theta)$ and $\nabla_{1,\theta}, \ldots, \nabla_{m,\theta}$ are non-zero.

An arbitrary k-jet is said to be *regular*, if it projects on a regular 1-jet.

7. FIELD OF DIFFERENTIAL INVARIANTS

In this section we give a complete description for the field of differential invariants of the action of the group K on the infinite jet space $J^\infty(\pi^\lambda)$. First of all, we consider invariants of order 1 and 2.

7.1. Invariants of Order 1

Let $\{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ be the set of simple roots (here $l = \mathrm{rk}\,\Phi$). Define $n_\alpha := (n_1, \ldots, n_l)$, for each root $\alpha \in \Phi$, where $\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l$.

Let now θ_1 be a regular 1-jet and let $\theta_0 := \pi^\lambda_{1,0}(\theta_1)$. The stabilizer of 0-jet θ_0 is the stabilizer of the weight λ in the torus T. Hence, differential K-invariants of pure order 1 are just invariants of the stabilizer-action on the fiber of $\pi^\lambda_{1,0}$ over θ_0. Each $t \in T$ and $\alpha \in \Phi$ induces an action of $t_\alpha \colon \Pi_\alpha \to \Pi_\alpha$ on the plane Π_α. Let $t_i := t_{\alpha_i}$, for simple roots α_i (where $i = 1, \ldots, l$). Then $t_\alpha = t_1^{n_1} \ldots t_l^{n_l} =: t^{n_\alpha}$. Recall that, for each $\alpha \in \Phi$, we denoted the projection of the invariant 1-form $q_1^{\mathcal{E}}$ onto the invariant plane Π_α^* by $q_{1,\alpha}^{\mathcal{E}}$. Then $t(q_{1,\alpha}^{\mathcal{E}}) = t^{n_\alpha - n_\lambda} q_{1,\alpha}^{\mathcal{E}}$.

Let $I_\alpha := \varkappa(q_{1,\alpha}, q_{1,\alpha})$. Then $t(I_\alpha) = t^{2(n_\alpha - n_\lambda)} I_\alpha$. Therefore the functions $J_\lambda := I^{n_\lambda}$ and $J_\alpha := I_\alpha / I^{n_\alpha}$, where $I^{n_\alpha} := I_{\alpha_1}^{n_1} \ldots I_{\alpha_l}^{n_l}$, are differential invariants of order 1, for each $\alpha \in \Phi_+$, $\alpha \neq \alpha_i$. So, we have found $m - l + 1$ independent differential invariants of order 1. These invariants separate K-orbits of regular 1-jets and generate the field of K-invariants of order 1.

7.2. Invariants of Order 2

To obtain differential invariants of order 2 we write the quadric $q_2^{\mathcal{E}}$ in the invariant basis. Then the coefficients $\mathcal{Q}_{ij} := q_2^{\mathcal{E}}(\nabla_i, \nabla_j)$ are differential invariants of order 2. These invariants together with invariants J_λ, J_α generate the field of K-invariants of order $\leqslant 2$ and separate regular orbits.

7.3. The Field of All Invariants

Finally, we are able to find the field of all differential invariants.

Theorem 8. *The field of differential invariants of the action of group G is generated over field $\mathbb{C}$ by the differential invariants J_λ and J_α of order 1, the differential invariants $\mathcal{Q}_{ij}$ of order 2 and the invariant derivations $\nabla_1, \ldots, \nabla_m$. This field separates K-orbits of regular jets.*

Remark. Our construction of basic differential invariants (and of the whole field of invariants) is universal and does not depend either on group G, or on its representation ρ_λ. Moreover, this construction is effective and makes it possible to calculate differential invariants and invariant derivations. On the other hand, there are no methods for calculating fields of *algebraic rational invariants* for the actions of semisimple groups in irreducible representations.

8. SEPARATION OF K-ORBITS OF THE REGULAR SECTIONS

In this section we provide a criterion for the separation of K-orbits of sections of the bundle π^λ, and hence of G-orbits of the irreducible representation ρ_λ.

Let us consider an arbitrary holomorphic section s of the bundle π^λ. The section s is said to be *regular*, if the set $M_s^{\text{reg}} := \{x \in M : [s]_x^2 \text{ is regular}\}$ is dense in M. The restrictions of the basic invariants J_λ, J_α and $\mathcal{Q}_{ij}$ to the regular section s are holomorphic functions on M_s^{reg} and define the map

$$h_s\colon M_s^{\text{reg}} \to \mathbb{C}^N, \quad h_s(x) = (J_\lambda([s]_x^2), J_\alpha([s]_x^2), \mathcal{Q}_{ij}([s]_x^2)),$$

where $N := (m - l + 1) + \binom{m}{2}$ is the number of the basic invariants J_λ, J_α and $\mathcal{Q}_{ij}$. Let $\mathcal{H}_s := \operatorname{Im}(h_s)$ be the image of the map h_s.

Theorem 9. *Regular sections s and $\widetilde{s}$ of the bundle π^λ are K-equivalent if and only if $\mathcal{H}_s = \mathcal{H}_{\widetilde{s}}$.*

Remark. The results of this paper can be easily generalized to the case when the representation ρ_λ is multiplicity free, i.e., ρ_λ is a sum of irreducible representations with multiples 1.

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