Gene Golub SIAM Summer School 2012

Numerical Methods for Wave Propagation
Finite Volume Methods
Lecture 1

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Outline for 3 lectures on FVM

Main goals:

- Some theory of hyperbolic problems in one dimension
- Focus on linear theory (+some nonlinear)
- Godunov-type finite volume methods, Riemann solvers
- High-resolution shock capturing via limiters
- Application to Shallow water equations

Note: Slides will be posted and green links can be clicked.

The Clawpack software (Version 4.6.2) is installed on the Virtual Machine (VM) and will be used for some examples. For documentation, see **www.clawpack.org**.

Outline

This lecture

- First order hyperbolic equations $q_t + f(q)_x = 0$.
- Derivation of conservation law, integral form
- Advective flux, pressure terms
- Linear systems: $f(q) = Aq \implies q_t + Aq_x = 0$.
- Diagonalization, characteristics, Riemann problems
- Motivating examples
  - Advection, flow in a pipe.
  - Linear acoustics, sound waves
  - Linearized shallow water equations

Linear and nonlinear waves

A wave is a disturbance or displacement that propagates.

Examples:

- Water waves (disturbance of depth)
- Sound waves (disturbance of pressure)
- Seismic waves (displacement of elastic material)

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linear partial differential equations

Solutions are often continuous, smooth functions

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linear partial differential equations

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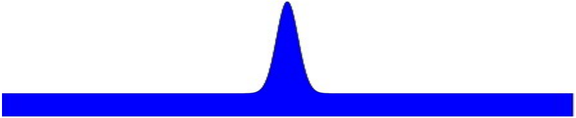
Larger displacements require nonlinear equations
Solutions may be discontinous: shock waves

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Computational challenges!

Need to capture sharp discontinuities.

PDE breaks down, standard finite difference approximation to $q_t + f(q)_x = 0$ can fail badly: nonphysical oscillations, convergence to wrong weak solution.

First order hyperbolic PDE in 1 space dimension

Linear: $q_t + Aq_x = 0, \qquad q(x,t) \in \mathbb{R}^m,\ A \in \mathbb{R}^{m\times m}$

Conservation law: $q_t + f(q)_x = 0, \qquad f : \mathbb{R}^m \to \mathbb{R}^m$ (flux)

Quasilinear form: $q_t + f'(q)q_x = 0$

Hyperbolic if A or $f'(q)$ is diagonalizable with real eigenvalues.

Models wave motion or advective transport.

Eigenvalues are wave speeds.

Note: Second order wave equation $p_{tt} = c^2 p_{xx}$ can be written as a first-order system (acoustics).

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity, so

$$\int_{x_1}^{x_2} q(x,t)\,dx = \text{total mass in interval}$$

changes only because of fluxes at left or right of interval.

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = F_1(t) - F_2(t)$$

where

$$F_j = f(q(x_j,t)), \qquad f(q) = \text{flux function.}$$

Derivation of Conservation Laws

If q is smooth enough, we can rewrite

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

as

$$\int_{x_1}^{x_2} q_t\,dx = -\int_{x_1}^{x_2} f(q)_x\,dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x)\,dx = 0$$

True for all $x_1,\ x_2 \implies$ differential form:

$$q_t + f(q)_x = 0.$$

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Advection equation

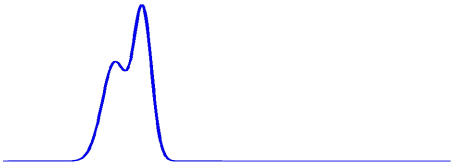
Flow in a pipe at constant velocity

$u =$ constant flow velocity

$q(x,t) =$ tracer concentration, $\quad f(q) = uq$

$\implies \quad q_t + uq_x = 0.$

True solution: $q(x,t) = q(x - ut, 0)$



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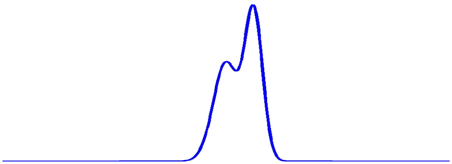

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Advection example

Some examples solving the advection equation
with periodic boundary conditions

Using Clawpack and various numerical methods...

www.clawpack.org/g2s3/claw-apps/advection-1d-3/README.html

Advective flux

If $\rho(x,t)$ is the density (mass per unit length),

$$\int_{x_1}^{x_2} \rho(x,t)\,dx = \text{total mass in } [x_1, x_2]$$

and $u(x,t)$ is the velocity, then the advective flux is

$$\rho(x,t)u(x,t)$$

Units: mass/length $\times$ length/time $=$ mass/time.

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Continuity equation (conservation of mass):

$$\rho_t + (\rho u)_x = 0$$

Momentum flux

$\rho(x,t)u(x,t)$ is the momentum density
(momentum per unit length),

$$\int_{x_1}^{x_2} \rho(x,t)u(x,t)\,dx = \text{total momentum in } [x_1, x_2]$$

The advective flux of momentum is

$$(\rho(x,t)u(x,t))u(x,t) = \rho u^2.$$

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Conservation of momentum:

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

This includes another term:
Pressure variation $\Longrightarrow$ acceleration.

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Sound speed: $c = \sqrt{P'(\rho)}$ varies with ρ.

System is hyperbolic if $P'(\rho) > 0$.

Compressible gas dynamics

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Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Isentropic: $P(\rho) = \hat{\kappa} \rho^\gamma$ ($\gamma \approx 1.4$ for air)

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Linearized shallow water equations

$h(x,t) = h_0 + \tilde{h}(x,t)$ (with $|\tilde{h}| \ll h_0$)
$u(x,t) = 0 + \tilde{u}(x,t)$ (linearized about ocean at rest)

Insert into the nonlinear equations

$$h_t + (hu)_x = 0$$
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Then ignore quadratic terms like $\tilde{u}\tilde{h}_x$ to obtain:

$$\tilde{h}_t + h_0\,\tilde{u}_x = 0$$
$$h_0\,\tilde{u}_t + gh_0\,\tilde{h}_x = 0$$

$$\Longrightarrow \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_t + \begin{bmatrix} 0 & h_0 \\ g & 0 \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix}_x = 0.$$ Eigenvalues: $\pm\sqrt{gh_0}$

Same structure as linear acoustics.

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + \kappa u_x &= 0 \qquad \text{Change in pressure due to compression} \\ \rho u_t + p_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where $K =$ bulk modulus, and $\rho =$ unperturbed density of gas.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Suppose hyperbolic:

- Real eigenvalues $\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$,
- Linearly independent eigenvectors $r^1,\ r^2,\ \ldots,\ r^m$.

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Let $R = [r^1|r^2|\cdots|r^m] \quad m \times m$ matrix of eigenvectors.

Then $Ar^p = \lambda^p r^p$ means that $AR = R\Lambda$ where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).$$

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$AR = R\Lambda \implies A = R\Lambda R^{-1}$ and $R^{-1}AR = \Lambda$.

Similarity transformation with R diagonalizes A.

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Multiply system by R^{-1}:

$$R^{-1}q_t(x,t) + R^{-1}Aq_x(x,t) = 0.$$

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Use $R^{-1}AR = \Lambda$ and define $w(x,t) = R^{-1}q(x,t)$:

$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$

Since R is constant!

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Since R is constant!

This decouples to m independent scalar advection equations:

$$w^p_t(x,t) + \lambda^p w^p_x(x,t) = 0. \qquad p = 1,\ 2,\ \ldots,\ m.$$

Solution to Cauchy problem

Suppose $q(x,0) = \overset{\circ}{q}(x)$ for $-\infty < x < \infty$.

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The solution to the decoupled equation $w_t^p + \lambda^p w_x^p = 0$ is

$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}{}^p(x - \lambda^p t).$$

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We can rewrite this as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p = \sum_{p=1}^{m} \overset{\circ}{w}{}^p(x - \lambda^p t)\, r^p$$

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$ (acoustics relative to flow with speed u_0)

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0 + c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

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Note: Eigenvectors are independent of u_0.

Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}^1_x(x - \lambda^1 t) r^1$$

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In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

Physical meaning of eigenvectors

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Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

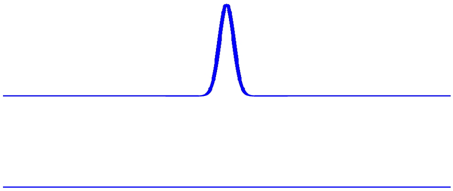
The pressure variation is Z_0 times the velocity variation.

Acoustic waves

$$q(x,0) = \begin{bmatrix} \overset{\circ}{p}(x) \\ 0 \end{bmatrix} = -\frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$= w^1(x,0)r^1 + w^2(x,0)r^2$$

$$= \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ -\overset{\circ}{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \overset{\circ}{p}(x)/2 \\ \overset{\circ}{p}(x)/(2Z_0) \end{bmatrix}.$$



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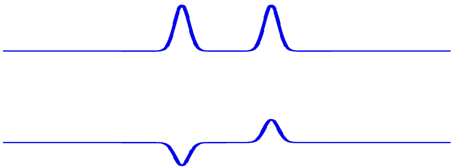

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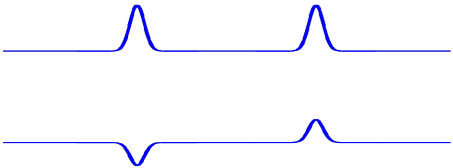

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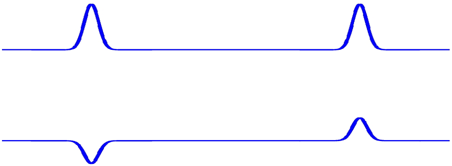
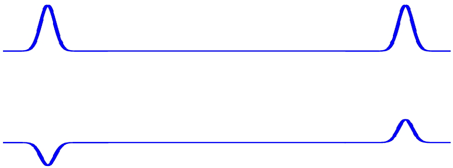
$$= w^1(x,0)r^1 + w^2(x,0)r^2$$

$$= \begin{bmatrix} \mathring{p}(x)/2 \\ -\mathring{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \mathring{p}(x)/2 \\ \mathring{p}(x)/(2Z_0) \end{bmatrix}.$$



Acoustic waves

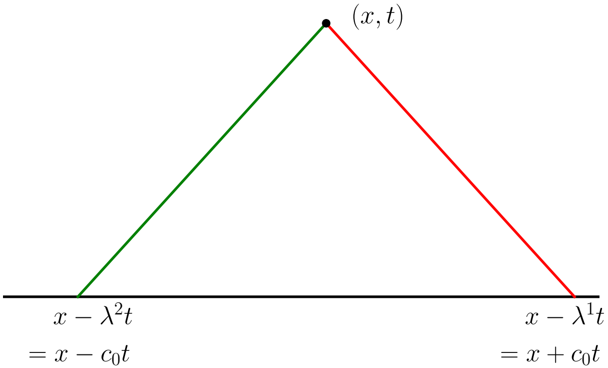
$$
\begin{aligned}
q(x,0) = \begin{bmatrix} \mathring{p}(x) \\ 0 \end{bmatrix} \quad &= \quad -\frac{\mathring{p}(x)}{2Z_0} \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} \quad + \quad \frac{\mathring{p}(x)}{2Z_0} \begin{bmatrix} Z_0 \\ 1 \end{bmatrix} \\
&= \quad w^1(x,0) r^1 \quad + \quad w^2(x,0) r^2 \\
&= \begin{bmatrix} \mathring{p}(x)/2 \\ -\mathring{p}(x)/(2Z_0) \end{bmatrix} + \begin{bmatrix} \mathring{p}(x)/2 \\ \mathring{p}(x)/(2Z_0) \end{bmatrix}.
\end{aligned}
$$



Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t, 0)r^1 + w^2(x-\lambda^2 t, 0)r^2 \\ &= w^1(x+c_0 t, 0)r^1 + w^2(x-c_0 t, 0)r^2 \end{aligned}$$



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Recall that $w(x,0) = R^{-1}q(x,0)$, i.e.

$$w^1(x,0) = \ell^1 q(x,0), \qquad w^2(x,0) = \ell^2 q(x,0)$$

where ℓ^1 and ℓ^2 are rows of R^{-1}.

$$R^{-1} = \begin{bmatrix} \ell^1 \\ \ell^2 \end{bmatrix}$$

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Note: ℓ^1 and ℓ^2 are left-eigenvectors of A:

$$\ell^p A = \lambda^p \ell^p \qquad \text{since} \;\; R^{-1}A = \Lambda R^{-1}.$$

Solution by tracing back on characteristics

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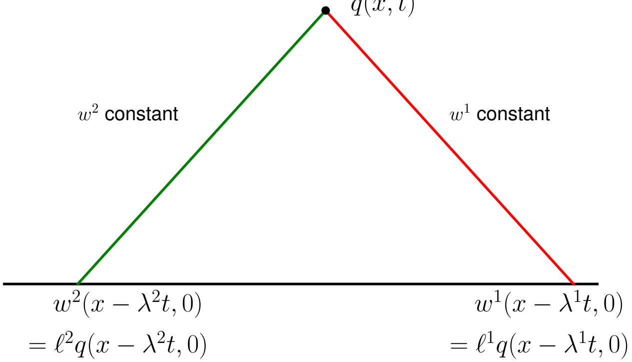


The Riemann problem

The Riemann problem consists of the hyperbolic equation under study together with initial data of the form

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \ge 0 \end{cases}$$

Piecewise constant with a single jump discontinuity from q_l to q_r.

The Riemann problem is fundamental to understanding

- The mathematical theory of hyperbolic problems,
- Godunov-type finite volume methods

Why? Even for nonlinear systems of conservation laws, the Riemann problem can often be solved for general q_l and q_r, and consists of a set of waves propagating at constant speeds.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \geq ut \end{cases}$$

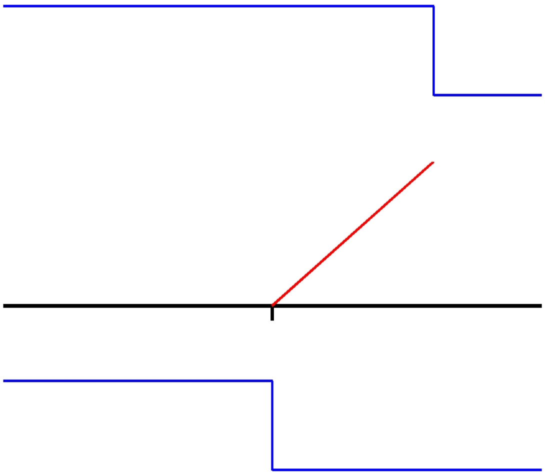
consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection

$q(x,T)$

x–t plane

$q(x,0)$



Discontinuous solutions

Note: The Riemann solution is not a classical solution of the PDE $q_t + uq_x = 0$, since q_t and q_x blow up at the discontinuity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Integrate in time from t_1 to t_2 to obtain

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} uq(x_1,t)\,dt - \int_{t_1}^{t_2} uq(x_2,t)\,dt.$$

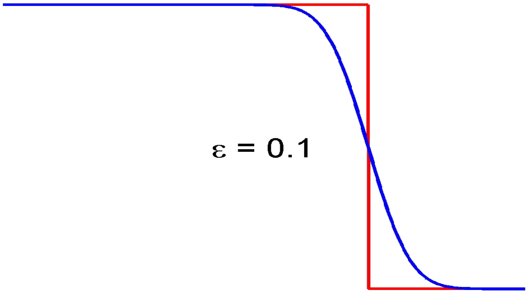
The Riemann solution satisfies the given initial conditions and this integral form for all $x_2 > x_1$ and $t_2 > t_1 \geq 0$.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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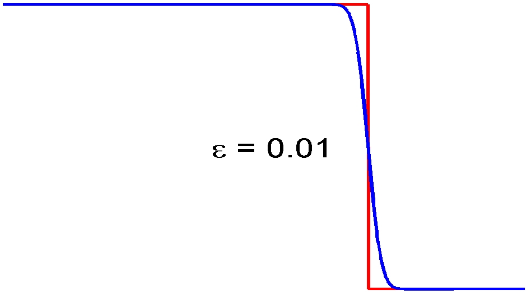

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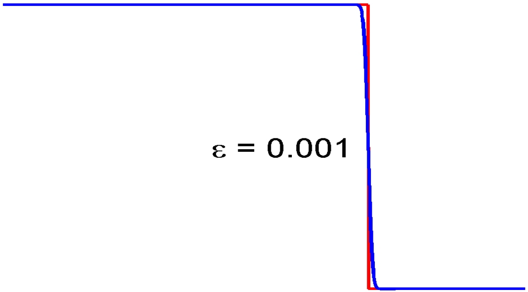


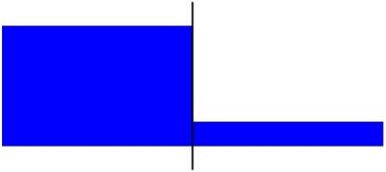
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram ($u_l = u_r = 0$)

Pressure:



Acoustic waves propagate with speeds $\pm c$.

Riemann Problem

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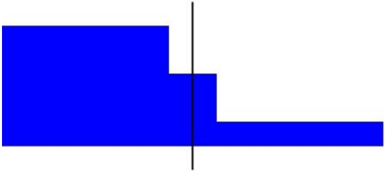

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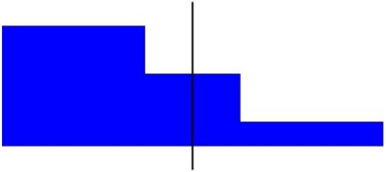

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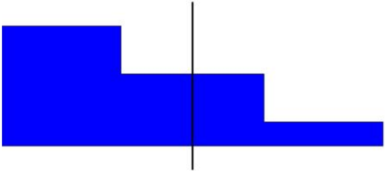
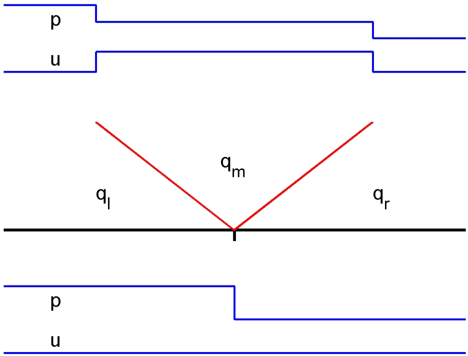


Acoustic waves propagate with speeds $\pm c$.

Riemann Problem for acoustics

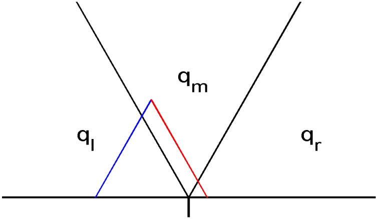
Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Riemann Problem for acoustics

In x–t plane:



$$q(x,t) = w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2$$

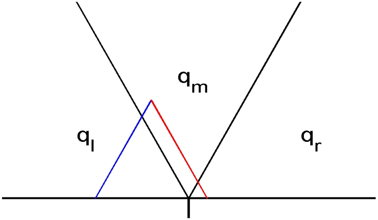
Decompose q_l and q_r into eigenvectors:

$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

Riemann Problem for acoustics



$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

So the waves $\mathcal{W}^1$ and $\mathcal{W}^2$ are eigenvectors of A:

$$\mathcal{W}^1 = q_m - q_l = (w_r^1 - w_l^1) r^1$$
$$\mathcal{W}^2 = q_r - q_m = (w_r^2 - w_l^2) r^2.$$

Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
General Riemann problem data $q_l,\ q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

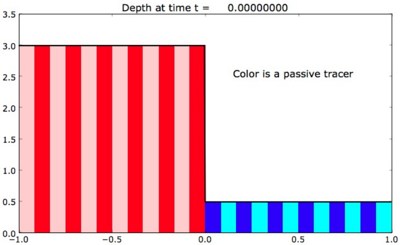
$$\mathcal{W}^p = \alpha^p r^p, \qquad \text{propagating with speed } s^p = \lambda^p.$$

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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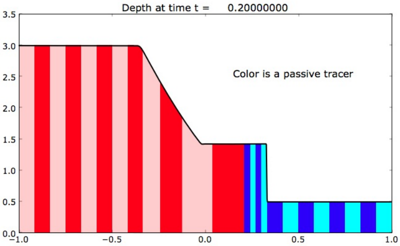

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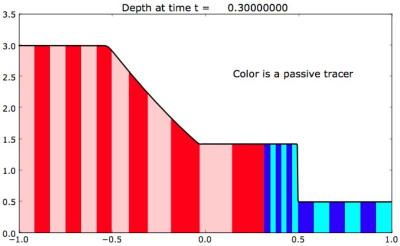

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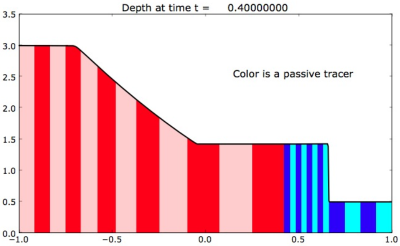

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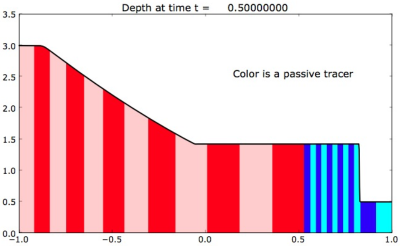
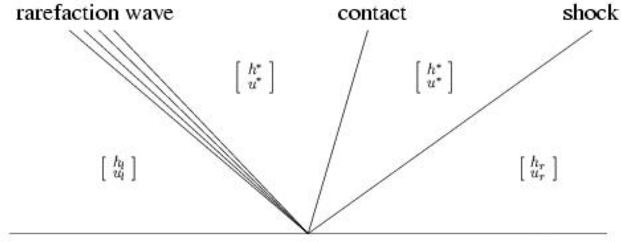


Riemann solution for the SW equations in x-t plane



Similarity solution:
Solution is constant on any ray: $q(x,t) = Q(x/t)$

Riemann solution can be calculated for many problems.
Linear: Eigenvector decomposition. Nonlinear: more difficult.

In practice “approximate Riemann solvers” used numerically.

Extra Material

You might want to work through

the following slides on your own!

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

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Heat equation: Same form, where

$q(x,t) =$ density of thermal energy $= \kappa T(x,t)$,
$T(x,t) =$ temperature, $\kappa =$ heat capacity,
flux $= -\beta T(x,t) = -(\beta/\kappa) q(x,t) \implies$

$$q_t(x,t) = (\beta/\kappa) q_{xx}(x,t).$$

Advection-diffusion

$q(x,t) =$ concentration that advects with velocity u
and diffuses with coefficient β:

$$\text{flux} = uq - \beta q_x.$$

Advection-diffusion equation:

$$q_t + uq_x = \beta q_{xx}.$$

If $\beta > 0$ then this is a parabolic equation.

Advection dominated if u/β (the Péclet number) is large.

Fluid dynamics: “parabolic terms” arise from

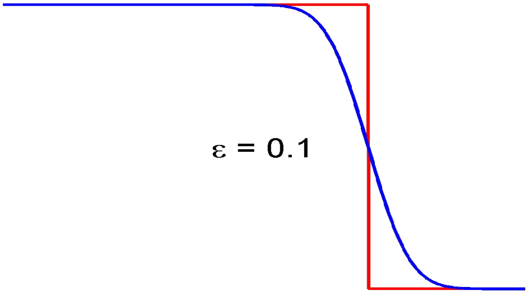
- thermal diffusion and
- diffusion of momentum, where the diffusion parameter is the viscosity.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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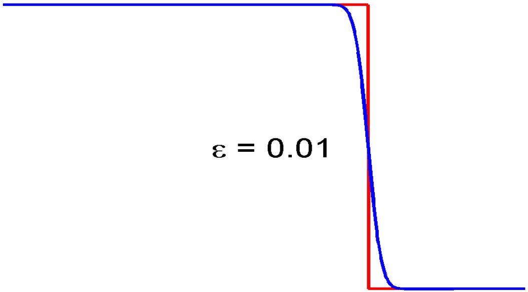

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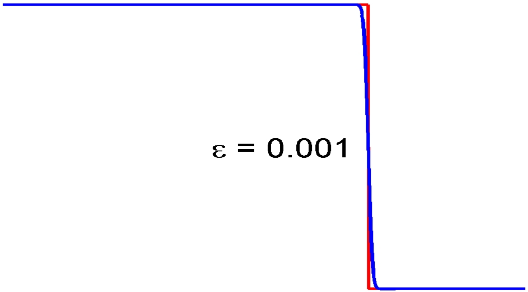


Coupled advection–acoustics

Flow in pipe with constant background velocity $\bar{u}$.
$\phi(x,t) =$ concentration of advected tracer
$u(x,t),\ p(x,t) =$ acoustic velocity / pressure perturbation

Equations include advection at velocity $\bar{u}$:

$$\begin{array}{llllllll} p_t & + & \bar{u}p_x & + & Ku_x & & & = 0 \\ u_t & + & (1/\rho)p_x & + & \bar{u}u_x & & & = 0 \\ \phi_t & & & & & + & \bar{u}\phi_x & = 0 \end{array}$$

This is a linear system $q_t + Aq_x = 0$ with

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

Coupled advection–acoustics

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

eigenvalues: $\lambda^1 = u - c, \qquad \lambda^2 = u \qquad \lambda^3 = u + c,$

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} Z \\ 1 \\ 0 \end{bmatrix},$

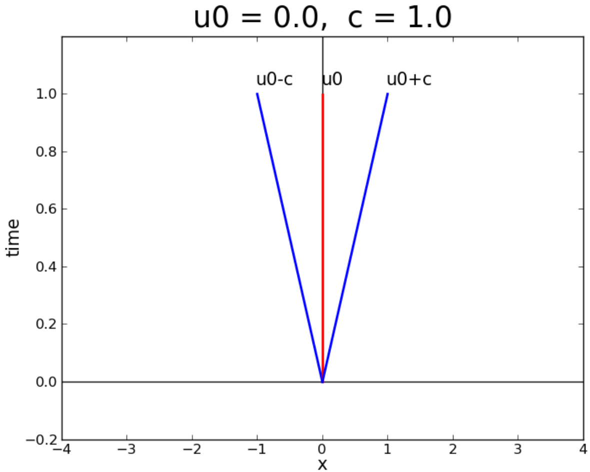
where $c = \sqrt{\kappa/\rho}, \;\; Z = \rho c = \sqrt{\rho\kappa}$.

$$R = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} -1 & Z & 0 \\ 0 & 0 & 1 \\ 1 & Z & 0 \end{bmatrix}.$$

Coupled advection–acoustics

Wave structure of solution in the x–t plane

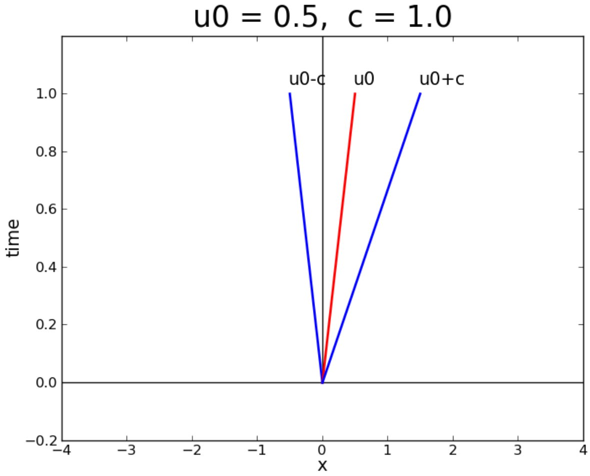
With no advection:



Coupled advection–acoustics

Wave structure of solution in the x–t plane

Subsonic case ($|u_0| < c$):



Coupled advection–acoustics

Wave structure of solution in the x–t plane

Supersonic case ($|u_0| > c$):