General Plasma Physics I Notes
AST 551

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0 Introduction

These notes are intended to summarize and explain the topics discussed during class in the Fall 2017 section of AST551, General Plasma Physics I. I have written these notes primarily as a learning experience. I find that in order to learn something, I need to explain it to someone else. I also need to focus on the details of the subject, going through each step in detail. You will see that I try not to skip steps as much as possible. I've found that with many of the plasma physics books I've looked at, I understand the math and the derivations, but walk away without an understanding of the physics. My goal with these notes is for that not to be the case, both for me and for the reader. I have done my best to not only include the math, but to do my best to explain the physics behind the math, opting for wordiness over brevity. If I can't explain something simply, it's because I don't understand it well enough.

Although my original intention was to cover only the topics covered in class, I ultimately decided that there are a few topics not covered in class which I would have liked to learn during GPP1. To a first-order approximation, however, these notes cover the topics from class. I've divided the notes into 6 chapters, not necessarily correlated with the order the topics were covered in class. The first chapter covers the most basic topics in plasma physics, including plasma oscillations, Debye shielding, space-time scales, and a bit on collisions. The second chapter covers single particle motion, including particle drifts, adiabatic invariants, mirror machines, and the iso-rotation theorem. The third chapter introduces kinetic theory, the Vlasov equation and discuss collision operators. The fourth chapter derives fluid equations, MHD, and ideal MHD. Chapter 5 covers some fundamental waves in plasmas, from kinetic, fluid, and MHD perspectives. Chapter 6 covers Landau damping, to the extent it was covered in class.

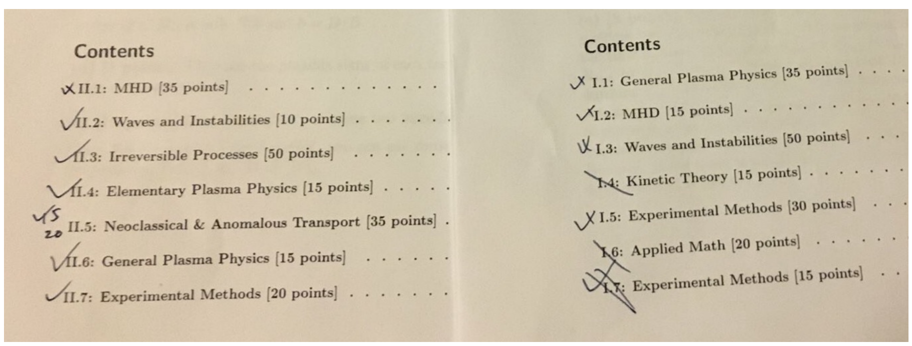
So far, the best resource I have found for learning introductory plasma physics is Paul Bellan's book, *Fundamentals of Plasma Physics*. Every derivation is done step-by-step in great detail, so that the reader is not lost, and each concept is explained thoroughly and usually with good physical insight. The downside of the book is that it is quite long.[1] Everything Professor Bhattacharjee does is exceptional,[2] and his textbook *Introduction to Plasma Physics with Space, Laboratory, and Astrophysical Applications* is no exception. His book covers many of the same topics covered in these notes, plus many more. It would be a good reference book for this course, and less time and algebra intensive than Bellan. It's a *great* reference book for GPP2. *Physics of Fully Ionized Plasmas* by Lyman Spitzer is a really old, fairly short book, with an old-fashioned take to the fundamentals of plasmas. Sam Cohen once told me it's the only book I need to read to understand plasma physics. I don't believe him.

[1] After writing these notes summarizing the topics covered in class, I've realized that the first 5 chapters of Bellan's book are remarkably similar to the 6 chapters of these notes. What that suggests to me is that the topics covered in this course haven't changed all that much since fall of 1970 when Bellan, now a professor at Caltech, took the course.

[2] Except perhaps ping pong.

Introduction to Plasma Physics and Controlled Fusion by Francis Chen is often referenced as a good book for beginning students - however, I think the level is appropriate for an undergraduate starting a summer of research into plasma physics, not for a graduate student concentrating in plasma physics. These books have all been helpful for me in reviewing this material. However, I should give a big thanks to the professors for this course, Nat Fisch and Hong Qin, for not only teaching me this material but generously answering my questions as I've tried to figure this stuff out.

Among the many things I know very little about, one is what one should do to prepare oneself to be a plasma physicist. However, I do know that the process of writing these notes has been enormously helpful to me in understanding this material. My hope is that these notes might also be useful for other students as they take AST551 or prepare for generals. These notes will be *more* useful if they do not contains errors or typos, so if you are reading these notes and find a typo or error, no matter how small, please let me know so I can fix it. You can reach me at mcgreivy@princeton.edu.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I.6: Applied Math [20 points]
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Speaking of generals, I've attached a picture with the cover of the a previous written section of the generals exam. Of the 360 points in the written section of this generals exam, 50 of the points come directly from this course. Everything on the test, with the exception of the applied math section, builds upon or asks directly about the introductory material covered in this course. I think it's important we learn it, and learn it well. To put it another way - you can't learn how to dance unless you know how to move your hips. Let's boogie.

1 Basics

It's unbelievable how much you don't know about the game you've been playing all your life.

Mickey Mantle

Greg Hammett imbued us first-year students with three pieces of wisdom during the first lecture for GPP1 way back in September. I figure I should pass that advice on. The first piece of advice is to remember how fortunate we are to be at this wonderful university, and to make the most of this experience. The second piece of advice is to find meaning and purpose in our lives outside of work. The third piece of advice is to get some sleep.

Plasma physics, as you may or may not have been told,[3] is a rich, varied subject. This richness comes mathematically, experimentally, as well as through the numerous applications of plasma physics research.

Research in plasma physics draws knowledge from of a huge number of areas of physics, including electromagnetism, thermodynamics, statistical mechanics, nuclear physics, and atomic physics. Experiments in plasma physics often involve vacuum systems, superconducting coils, cryogenic systems, complex optical instruments, advanced materials for plasma-facing components, waveguides, and much much more. Computational plasma physics involves developing and implementing numerical algorithms, linking computational work to physical models, theory, and experiment, and often uses some of the most powerful supercomputers in existence.

There are lots of applications of plasma physics. A few of the numerous applications of plasma physics include astrophysics (where over 99% of the visible universe is in the plasma state), plasma thrusters, water processing, industrial processes and fusion energy. Fusion energy, which is easily one of the most challenging scientific endeavors today, also holds one of the greatest rewards. The long-term promise and allure of fusion energy comes from the immense energy bound up in the atomic nucleus. There are readily available fuel sources[4] which release that energy and which could power humanity for many millions of years. Fusion power is carbon-dioxide free, has no risk of nuclear meltdown, doesn't require large land usage, and is a steady power source. It's a big goal, with big challenges.

Throughout these notes, we will start to see some of the mathematical and physical richness come to play. GPP1, however, focuses on the theoretical foundations of the subject rather than concentrate on any particular application of plasma physics.

[3] Once you are in the field for long enough, you will inevitably be told this at some point.

[4] Deuterium is readily available in seawater. It should be emphasized that tritium, while theoretically capable of being generated from lithium, does not exist in significant quantities naturally and the process of creating tritium has not been demonstrated on a large scale. This is one of the most challenging tasks facing developers of future D-T reactor.

1.1 Finals words before the onslaught of equations

One important question has not been answered so far - what is a plasma? Most briefly, a plasma is an ionized gas. But of course this response leaves much to the imagination. How ionized does it need to be to be a plasma? A gas of what?

As Nat points out, states of matter are really approximations of reality. Take, for example, a closed box stuffed chock-full of gravel. Each individual rock in that gravel certainly behaves like a solid when we observe it. If we were to take that box and throw it in the air, it would rotate approximately like a solid body. But when we open that box and pour that gravel into a funnel, the behavior of the gravel is probably better described with a fluid approximation. Similarly, the tectonic plates which makes up the earth's continents are certainly solid when we look at them over the course of a day or a month or a year. But when we look at them over a timescale of millions of years, the plates travel, flow, and merge, certainly unlike a solid.

What we've learned from these examples is that whether some real physical system can be treated as one of the idealized states of matter depends on how we are observing that system. Alternatively, in the language of plasma physics, the state of matter some system is in depends on the the timescales and length scales which we are observing the system over. For example, in gas clouds in the interstellar medium, the degree of ionization is very low and the magnetic fields are very small, but over large enough scales and over long enough times, their evolution is apparently well-described by the equations of plasma physics.

In some sense, plasmas fit somewhere along an energy spectrum, where the spectrum ranges over the energy per particle (i.e. temperature). At one end of the spectrum is condensed matter physics, i.e. solids. These are at the lowest temperature. As we increase the temperature, eventually the solids become fluids, fluids become gases, and at some point they become plasma-like. In the temperature range where the gas becomes fully ionized, we have an ideal classical plasma ($\sim$10eV to $\sim$100KeV). If we were to turn up the temperature even further, then at $\sim$1MeV positrons start to become produced, and we have a relativistic QED plasma, so that we have to develop other equations to understand this system. In this energy range, we are already out of the realm of classical plasma physics. If we really crank up the energy dial, up to $\sim$100MeV, then we'll have a quark-gluon plasma, which is confined by the strong force rather than the Electromagnetic force. What we see from this discussion is that plasma physics is the physics of matter within a certain restricted temperature range.

This still doesn't answer our question of "what is a plasma"! It turns out that this definition is a bit technical, but I'll state it here. Some system is a plasma if the number of plasma particles in a Debye sphere is much greater than 1, or $n_0 \frac{4}{3}\pi\lambda_D^3 \gg 1$. Often, this is just written as $n\lambda_D^3 \gg 1$ In effect, this means that the plasma is electrically neutral on scales larger than the Debye length. We will explore these ideas more in sections 1.5 and 3.3.2.

1.1.1 Logical framework of Plasma Physics

Here is a half-truth: Plasma physics has been fully solved. Suppose we have a large number of particles, each of which has charge q_i and mass m_i. These particles interact via the Lorentz force,

$$m_i \frac{d^2\vec{x}_i}{dt^2} = q_i(\vec{E} + \frac{d\vec{x}_i}{dt} \times \vec{B}) \tag{1.1}$$

The initial conditions for the electric and magnetic fields are given by two of Maxwell's equations,

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.2}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1.3}$$

while the time-evolution of the electric and magnetic fields are determined by the other two Maxwell equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.4}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{1.5}$$

Given a set of initial conditions for the particles, we can solve for the time evolution of these particles. If we really wanted to be precise, we could even use the Lorentz-invariant force laws, and calculate the time-evolution of the plasma particles to arbitrary precision. Unfortunately, this simplistic approach doesn't work for a myriad of reasons, both *practical* and *physical*.

Practically, such an approach is not solvable analytically with $O(10^{23})$ particles, and to solve it with a computer requires so much computing power that even on the most powerful supercomputers, it would take something on the order of the age of the universe to rigorously simulate even the most basic plasma configurations. And this is only the simplest model of a plasma possible! To make any progress in plasma physics, we obviously need a description of a plasma which can be practically solved. Thus, we will need to *approximate* somehow in order to get a tractable solution.

Physically, this simple model described in equations 1.1 through 1.5 is wrong. *Firstly*, there is no consideration of boundary conditions. In any terrestrial plasma, the plasma will be confined to some region by a solid[5] boundary, and the plasma particles will interact with the boundary in some complex way. Much of plasma physics research involves understanding the effects of plasmas as they interact with their boundary. In astrophysical plasmas, the boundaries are either ignored, not well-defined, or do not exist. In practice, periodic or open boundary conditions are often used to understand astrophysical plasmas. *Secondly*, not every particle is ionized, and neutral particles would (rigorously) need to be treated with quantum mechanics. Indeed, a proper treatment of collisions

[5] Or, in some applications, a fluid boundary

between even ionized particles in plasmas would (rigorously) involve quantum mechanics. Atomic physics, including collision cross sections and reaction rates, needs to be included to understand collisions as well as the ionization and recombination of particles. We also need to account for the radiation emitted through these atomic processes and through particle acceleration.

As you can imagine, a rigorous, complete description of a plasma would get extremely complicated pretty quickly. For this reason, *approximation will be our friend* as we study plasma physics. With the approximation schemes we make in these notes and throughout our study of plasmas, we will need to keep track of when the approximations we make are valid, so as not to apply some equation to a physical situation where it is not applicable.

A ‘model’ is an approximation scheme. There are a number of models in plasma physics. These include considering only single-particle dynamics (particle drifts, the subject of chapter 2), many particle dynamics (particle-in-cell computing, not covered in these notes), statistical models, and fluid models.

The main statistical model we use in plasma physics involves smoothing our distribution function to get a 6-dimensional - 3 spatial dimensions, 3 velocity dimensions - time-evolving distribution function f, which tells us the number of particles at a given position with a given velocity. This distribution function f is called the Vlasov distribution and is described by the Vlasov equation. This model is sometimes known as ‘kinetic theory’. We’ll discuss kinetic theory in chapter 3. Really, the Vlasov equation is an equation for non-equilibrium statistical mechanics.

Another model involves treating each species[6] in the plasma as a fluid. This requires taking velocity-integrals or moments of f, and replacing an arbitrary velocity distribution with, at each position in space, a mean velocity, temperature, and a pressure tensor. Alternatively, instead of treating each species as a fluid, we can treat the plasma as a single fluid, and calculate an overall mean velocity, a total current, and a single temperature and pressure tensor. This approximation is called Magnetohydrodynamics, or MHD. These fluid models of a plasma will be derived from the Vlasov equation in chapter 4.

1.2 Plasma Oscillations

We will start our investigation of plasma physics by looking at plasma oscillations. We start here for a couple reasons. Firstly, plasma oscillations illustrate many of the equations and techniques used throughout our study of this field. Secondly, plasma oscillations are the most simple example of what is called collective dynamics. Dynamics is the study of how a system evolves over time. Collective dynamics simply means that when interacting, many plasma particles can conspire to create macroscopic behavior which is different than what would be observed if the particles were not interacting. Our study of waves and Landau damping are other examples of collective behavior. We’ll have a lot more

[6] Species means a category of particles - so ions, electrons, and neutral atoms are species. I use the symbol σ to define species, following Bellan. Others write this with s or α or even j, so don’t get confused.

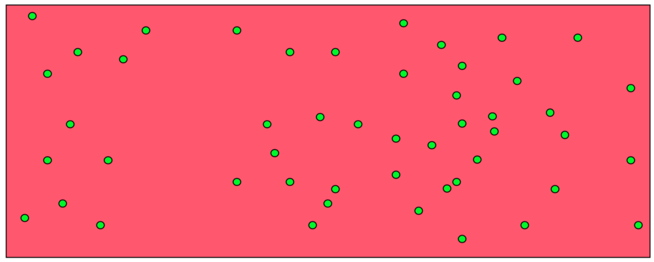

Figure 1: An initial electron density configuration. Perturbation is exaggerated for illustration.

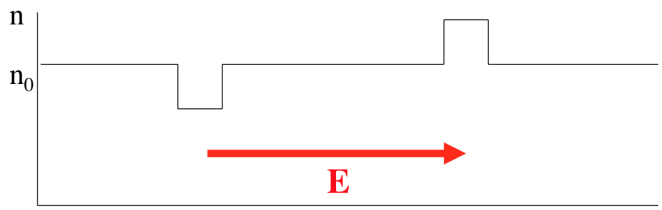

Figure 2: Electric field due to density perturbation. Over time, these bumps in density will rise and fall but stay at constant position in space in a cold plasma.

to say about plasma oscillations in chapter 5. For now, let's look at a simplified model of plasma oscillations, where the ions are assumed to be motionless and only the electrons are allowed to move. This is, as a matter of fact, the way you start on any complicated or unfamiliar problem: You first get a rough idea, and then you go back when you understand it better and do it more carefully.

Intuitively, plasma oscillations arise due to the electrostatic force which arises when electrons are displaced from an equilibrium. Suppose some number of electrons are displaced to the right, as in figure 1. Since there is now a positive charge density to the left and a negative charge density to the right, an electric field is setup which points towards the right, as in figure 2. The electrons on the right side will feel a force to the left, and will be accelerated leftwards. As they are accelerated leftwards, eventually $\vec{E} = 0$, but the electrons have developed a velocity leftwards. This leftward velocity eventually creates a higher electron density on the left, and a lower electron density on the right. This process will repeat itself, and the net effect is that the density perturbations will oscillate in time but not in space. These oscillations are called plasma oscillations.

To derive these plasma oscillations, we have to start somewhere. We'll start with the multi-fluid equations. We derive them in chapter 4. Instead of looking at individual particles, we will treat the density of particles as a continuous smooth field. Thus, for each species we have a continuity equation and a momentum equation. We will also use Poisson's equation and assume that any electric fields are electrostatic ($\vec{E} = -\vec{\nabla}\phi$), and that the magnetic field is zero.

We're looking for oscillations of the electrons, which we expect to be much faster than any oscillations of the ions because the electrons are much lighter[7]. We will therefore assume that the ions are stationary ($\vec{u}_i = 0$) and have a constant and static density n_0. Our equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} e(n_0 - n_e) \tag{1.6}$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0 \tag{1.7}$$

$$m_e n_e \frac{\partial \vec{u}_e}{\partial t} + m_e n_e (\vec{u}_e \cdot \vec{\nabla})(\vec{u}_e) = -e n_e \vec{E} \tag{1.8}$$

This is our first experience with fluid equations, which we have obviously not derived. Equation 1.7 is a continuity equation for electron density, similar to the charge conservation equation in electrodynamics. It just means that if the electron density inside a fixed infinitesimal volume changes in time, it is because there are electrons flowing across the boundary of that infinitesimal volume. Equation 1.8 is a momentum equation for the electrons. Essentially, it means that the mass times acceleration of electrons is equal to the force they feel due to the electric field.

For those who have seen fluid equations before, note that there is no pressure term in the momentum equation. Pressure, as you will remember from elementary kinetic theory of gases, is an effect which comes about due to the motion of molecules. Thus, whatever results we will derive are technically applicable only in the approximation of a zero-temperature (cold) plasma, where the molecules do not have thermal velocities. For a plasma with a non-zero temperature (warm plasma), we will see in chapter 5 that the wave dispersion relation changes.

With these equations, we will introduce a method, called linearization, which will prove useful throughout our understanding of plasma waves. With this method, we take some equilibrium ($\frac{\partial}{\partial t} \to 0$) solution to the equations, and call the equilibrium values the 0th order solution. From there, we will assume there is some perturbation to the equilibrium solution, and call the perturbations the first-order quantities. We plug the first-order quantities into the equations we have and ignore any terms which are second-order or higher. This gives us linearized equations. We'll then look for oscillatory solutions.

Let's see linearization in action. For plasma oscillations, we start with the most basic equilibrium possible: a zero-velocity plasma ($\vec{u}_0 = 0$), with a uniform density of electrons and ions ($n_0(\vec{x}) = n_0$) and zero electric field ($\phi_0 = \text{constant}$). Then, we apply a small perturbation to all relevant quantities, except ion density which is assumed to be constant over the timescales we are interested in. This gives $\vec{u}_e = \vec{u}_1$, $n_e = n_0 + n_{e1}$, and $\phi = \phi_1$. By ignoring all terms second-order or higher, we have

$$\vec{\nabla}^2 \phi_1 = -\frac{e}{\epsilon_0}\Big(n_0 - n_0 - n_{e1}\Big) = e n_{e1}/\epsilon_0 \tag{1.9}$$

[7]This differentiation of timescales between electrons and ions due to their different masses will be a recurring theme throughout our study of plasma physics.

$$\frac{\partial n_{e1}}{\partial t} = -\vec{\nabla}\cdot(n_0\vec{u}_1) = -n_0\vec{\nabla}\cdot\vec{u_1} \tag{1.10}$$

$$m_e n_0 \frac{\partial \vec{u}_1}{\partial t} = e n_0 \vec{\nabla}\phi_1 \tag{1.11}$$

Now taking the divergence of the linearized momentum equation, we have

$$m_e n_0 \frac{\partial \vec{\nabla}\cdot\vec{u}_1}{\partial t} = -m_e\frac{\partial^2 n_{e1}}{\partial t^2} = e n_0 \vec{\nabla}^2\phi_1 = \frac{e^2 n_0}{\epsilon_0} n_{e1} \tag{1.12}$$

$$\frac{\partial^2 n_{e1}}{\partial t^2} = -\omega_p^2 n_{e1}(\vec{x},t) \tag{1.13}$$

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e} \tag{1.14}$$

Because the derivative is a partial derivative with respect to time, this equation gives a solution for the density perturbation which oscillates in time, but not in space. The way we visualize this is as follows: imagine we take some electrons from one point in space and displace them slightly to another position. This electron density is shown graphically in figure 2. At each point in space, the perturbation will oscillate sinusoidally, so at some later time $t = 2\pi/4\omega_p$, the density will be instantaneously constant, and another quarter-period later the leftmost electron density perturbation will have a higher electron density. This is the same physical picture we described earlier.

Remember: physically, we can think of plasma oscillations as arising due to electrostatic forces which cause the electrons to accelerate back and forth. In between the leftward and rightward bumps in Figure 2, there is an electrostatic electric field which pushes the electrons between the two bumps back and forth, creating the forces which drive the oscillation. Most simply, we can say that plasma oscillations are the result of an electrostatic restoring force combined with electron inertia. Although we are focused on the density here, note that the electron fluid velocity oscillates in time, as does ϕ.

1.3 Debye Shielding

As we remember from electromagnetism, the electric field inside a conductor is 0. Otherwise, charges would move around, causing the electric field to change, until the electric field eventually became 0.

Plasmas, in general, are highly conducting. Thus, we should expect that the electric field inside a plasma is 0, right? Well, not exactly. Indeed plasmas, like conductors, screen external electric fields quite well. However, the electric field inside a plasma is not necessarily zero. If we place a charge Ze in a warm plasma and make it stay there, then the electric potential a distance r away from the charge is

$$\phi = \frac{Ze}{4\pi\epsilon_0 r} e^{-r/\lambda_D} \tag{1.15}$$

where λ_D is a constant called the Debye length which depends on, among other things, temperature. Note that the potential falls off in a plasma faster than

$1/r$, due to the exponential dependence. This faster-than-exponential falloff of the plasma potential is what is called Debye shielding or Debye screening. Over distances significantly longer than a couple Debye lengths, the plasma potential due to a charge in the plasma is very small. Loosely speaking, plasmas are net neutral over distances longer than a Debye length.

Physically, Debye shielding is an effect which arises due to the balance between the electric force on particles and particle's random thermal velocities. If we put a test charge in a plasma, all of the particles will feel a force due to that charge, causing them to move towards the charge. If particles had zero thermal velocity, they would move towards the charge until they had no electric force on them, and eventually settle into an equilibrium where all of the forces on the particles are zero. However, particles *do* have thermal velocity, so they don't just sit nicely at rest around a test charge, but instead fly about randomly, preventing the potential from a charge in a plasma from being totally shielded. The Debye length is the length scale over which a plasma shields electric fields. Based on this picture, we expect a higher-temperature plasma to have a larger Debye length because the random motion will be faster. We also expect a plasma full of particles with larger charge q_σ to have a smaller Debye length, because the particles will feel a stronger electrostatic force due to any test charge.

Let's derive equation 1.15, the potential for a test charge $+Ze$ in a plasma. Imagine inserting a test particle of infinitesimal charge Q into a plasma. Assume that each species (represented by σ) in the plasma is in thermal equilibrium with temperature T_σ, and that each species can be treated as a fluid with density n_σ. Now, it is true that in thermal equilibrium

$$n_\sigma = n_0 e^{\frac{-q_\sigma \phi}{k_B T_\sigma}} \tag{1.16}$$

This is simply the Boltzmann distribution of statistical mechanics, where the energy level $E = q_\sigma \phi$. Alternatively, we can derive this from the fluid equations, assuming thermal equilibrium. We use the equation of motion,

$$m_\sigma \frac{d\vec{u}_\sigma}{dt} = q_\sigma \vec{E} - \frac{1}{n_\sigma}\vec{\nabla} P_\sigma \tag{1.17}$$

Assuming the inertial term on the left-hand side (LHS) is negligible (which physically means the changes in the plasma are slow), the electric field is electrostatic ($\vec{E} = -\vec{\nabla}\phi$), the temperature is spatially uniform, and the ideal gas law $P_\sigma = n_\sigma k_B T_\sigma$ holds, then this reduces to

$$0 = -n_\sigma q_\sigma \vec{\nabla}\phi - k_B T_\sigma \vec{\nabla} n_\sigma \tag{1.18}$$

which has the solution

$$n_\sigma = n_0 e^{-q_\sigma \phi / k_B T_\sigma} \tag{1.19}$$

The assumptions we just made are all consistent with the plasma being in thermal equilibrium, which is the assumption used to derive the Boltzmann distribution from statistical mechanics. So our work checks out. Now, we assume that $k_B T_\sigma \gg q_\sigma \phi$, which is true when the test particle's charge is small so that the ϕ created by this charge is small as well, so $q_\sigma \phi$ is doubly small.[8] Taylor expanding equation 1.19 in this limit, we get

$$n_\sigma \approx n_0 \Big(1 - \frac{q_\sigma \phi}{k_B T_\sigma}\Big) \tag{1.20}$$

Remember what we're trying to do: solve for the potential of a test charge in a plasma. To solve for electric potential, we need to use Poisson's equation. Assuming the test charge is at the origin, Poisson's equation gives us

$$-\vec{\nabla}^2 \phi = \frac{1}{\epsilon_0} \Big(Q\delta^{(3)}(\vec{r}) + \sum_\sigma n_\sigma(\vec{r}) q_\sigma\Big) \tag{1.21}$$

Using equation 1.20 and the fact that the plasma is net neutral to zeroth order, this simplifies to

$$-\vec{\nabla}^2 \phi + \frac{\phi}{\lambda_D^2} = \frac{1}{\epsilon_0} Q\delta^{(3)}(\vec{r}) \tag{1.22}$$

where

$$\frac{1}{\lambda_D^2} = \sum_\sigma \frac{1}{\lambda_{D\sigma}^2} \tag{1.23}$$

$$\lambda_{D\sigma} = \sqrt{\frac{\epsilon_0 k_B T_\sigma}{q_\sigma^2 n_0}} \tag{1.24}$$

Now, $\vec{\nabla}^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial \phi}{\partial r})$. We look for a solution of the form

$$\phi = \frac{f(r) Q}{4\pi\epsilon_0 r} \tag{1.25}$$

where $f(0) = 1$. This is an inspired guess, which gives us the potential for a test charge in the $r \to 0$ limit but diverges from the typical $\frac{1}{r}$ dependence as $r > 0$. Plugging this in to equation 1.22, we have

$$-\frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \Big(r^2 \frac{\partial}{\partial r} \Big(\frac{f(r)}{r}\Big)\Big) + \frac{f(r) Q}{4\pi\epsilon_0 r \lambda_D^2} = \frac{1}{\epsilon_0} Q\delta^{(3)}(\vec{r}) \tag{1.26}$$

This first term is a bit tricky to simplify. From the chain rule, we can see that we'll have four terms when we expand this. One of the terms will be $-\frac{Q}{4\pi\epsilon_0} f(r) \vec{\nabla}^2 \frac{1}{r}$, a second will be $-\frac{Q}{4\pi\epsilon_0 r} f''(r)$, and the third and fourth will be cross-terms, proportional to $f'(r)$. It turns out that the cross-terms will cancel each other, as we see below.

$$\frac{\partial}{\partial r}\Big(r^2 \frac{\partial}{\partial r}\Big(\frac{f(r)}{r}\Big)\Big) = \frac{\partial}{\partial r}\Big(r^2 \Big(\frac{f'(r)}{r} - \frac{f(r)}{r^2}\Big)\Big) = f'(r) - f'(r) + \text{Other terms}$$

[8] This doesn't works in the $r \to 0$ limit, because the field of the test charge ϕ goes as $\frac{1}{r}$. So technically equation 1.15, which we are deriving now, breaks down very close to our charge, and becomes a better approximation as we get further away from the charge.

With the cancellation of the cross-terms, equation 1.26 becomes

$$-\frac{Qf(r)}{4\pi\epsilon_0}\vec{\nabla}^2\frac{1}{r}-\frac{Q}{4\pi\epsilon_0 r}f''(r)+\frac{f(r)Q}{4\pi\epsilon_0 r\lambda_D^2}=\frac{1}{\epsilon_0}Q\delta^{(3)}(\vec{r}) \tag{1.27}$$

Remembering from Griffiths E+M that $\vec{\nabla}^2(\frac{1}{r})=-4\pi\delta^{(3)}(\vec{r})$, the first term on the LHS and the RHS cancel. This leaves us with

$$f''=\frac{f}{\lambda_D^2} \tag{1.28}$$

which has the solution

$$f(r)=e^{-r/\lambda_D} \tag{1.29}$$

The positive exponential solution is ruled out due to boundary conditions at infinity - the potential at infinity can't be infinity. Plugging $f(r)$ into our inspired guess from equation 1.25, we have our Debye shielding equation for a test charge Q in a plasma.

$$\phi(r)=\frac{Q}{4\pi\epsilon_0 r}e^{-r/\lambda_D} \tag{1.30}$$

where, as before,

$$\lambda_{D\sigma}=\sqrt{\frac{\epsilon_0 k_B T_\sigma}{q_\sigma^2 n_0}}$$

Note that the Debye length is larger for larger values of T_σ, and smaller for larger values of q_σ. This makes sense, since Debye screening is an effect we see due to the thermal motion of charged particles in tandem with the electrostatic forces they feel. Loosely speaking, a species' charge causes it to want to stay close to any test charge Q in the plasma, thus large q_σ should decrease the Debye length, by increasing the electrostatic force on these particles. On the other hand, a species' thermal motion causes it to zoom around randomly, so large T_σ should increase the Debye length by increasing these random speeds. A zero temperature plasma has zero Debye length, because (in equilibrium) the particles will have no thermal velocity and thus exactly cancel the potential due to any test charges. Note also that the Debye length *doesn't* depend on the particle's mass. While low-mass particles (i.e. electrons) will have larger thermal energy, they will also be accelerated more easily the electric forces. It turns out that the factors of m_σ cancel in giving us our Debye length. Note also the following nifty little relation, neglecting a factor of $\sqrt{3}$ in the thermal velocity $V_{T\sigma}$.

$$\lambda_{D\sigma}=\frac{V_{T\sigma}}{\omega_{p\sigma}}=\sqrt{\frac{k_B T_\sigma}{m_\sigma}}\sqrt{\frac{m_\sigma\epsilon_0}{q_\sigma^2 n_0}} \tag{1.31}$$

We can actually use the physical intuition we've developed to remember this in a slightly different way. The plasma frequency $\omega_{D\sigma}$ comes about as the interaction between electric forces and particle inertia. The thermal velocity $V_{T\sigma}$ is, roughly, the balance between particle thermal energy and particle inertia. The Debye length $\lambda_{D\sigma}$ is the balance between particle thermal energy and electric forces. We can write this, roughly, as

$$\lambda_{D\sigma} = \frac{\text{Thermal energy}}{\text{Electric forces}} = \frac{\frac{\text{Thermal energy}}{\text{Inertia}}}{\frac{\text{Electric forces}}{\text{Inertia}}} = \frac{V_{T\sigma}}{\omega_{P\sigma}}$$

I find remembering which factors go where in the Debye length and the plasma frequency can be tricky, unless I'm using some sort of physical intuition like this to help me figure it out.

Here's something seemingly contradictory that confused me: the electric field inside a plasma is not always 0! But we learned in freshman physics that the electric field inside a conductor is 0. We also know that plasmas are highly conducting. So what is it about a plasma which is different than a typical conductor, such as a metal? Actually, in terms of shielding of electric fields, nothing! In an idealized metal, the electrons are at a temperature T and are free to move around as they please. Their behavior obeys Poisson's equation and the Boltzmann relation. Therefore, we must see Debye shielding in a metal! In fact, if we were to put a test charge in a metal and hold it there, we would see a potential quite like the Debye potential. At the edge of a charged conductor, where there is a surface charge, the electric field will not go to 0 immediately inside the conductor! Instead, it will fall to from σ/ϵ_0 to 0 over a couple debye lengths.

1.4 Collisions in Plasmas

Collisions are one of the most subtle, challenging, and important topics in plasma physics. In this section, we'll just do the very basics. I'll first cover the (very) brief tutorial we had on collisions in class. I'll then go on to discuss collisions in a bit more depth, in an attempt to give us some physical intuition for how collisions work in a plasma.

The distance of closest approach between two particles in a collision is approximately the distance at which the average kinetic energy equals the electrostatic potential energy. This would occur if we had a particle with energy $\frac{3}{2}k_BT$ moving directly towards a stationary particle, until the electrostatic potential energy is $\frac{1}{4\pi\epsilon_0}\frac{q^2}{b}$, where b is the distance of closest approach. Solving for b, we get

$$b = \frac{q^2}{6\pi\epsilon_0 k_B T} \tag{1.32}$$

The collision cross-section is roughly $\sigma = \pi b^2$, so

$$\sigma = \frac{q^4}{36\pi\epsilon_0^2(k_BT)^2} \tag{1.33}$$

= The mean free path is defined as

$$l = \frac{1}{\sigma n} \tag{1.34}$$

This discussion, from the beginning of the class, captures the basic physics of collisions in a plasma and the general scaling of b, σ, and l. However, we're going to want to leave GPP1 with a much better understanding of how collisions in a plasma work. In the rest of this subsection, I'll say a few basic things which weren't covered in class but which I think are important.[9]

Firstly, in a plasma where the number of particles in a Debye sphere is much greater than 1, grazing collisions dominate large-angle collisions. Why is this? Essentially, it comes down to the fact that the coulomb force is a long-range force. Unlike in neutral gases, where we don't have the Coulomb interaction between particles, the long-range interactions between charged plasma particles are important. In a plasma, the electric force between two charged particles goes like $\sim -\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda_D}}{r}\right)$. This faster-than exponential falloff of the electric field means that in a plasma, the force between charges particles is important at small r, and until about $r \sim \lambda_D$. If the number of charged particles in a Debye sphere is much greater than 1, then there will be *many* grazing (small-angle) collisions between particles. The cumulative effect of these many grazing collisions, as Bellan calculates, is greater than the effect of the much rarer large-angle collisions between particles that are close together. So in a plasma, grazing collisions dominate large-angle collisions.

Secondly, the enormous difference in masses between electrons and ions ($\frac{m_p}{m_i} \approx 2000$) means that collisions between electrons and electrons are very different from collisions between electrons and ions are very different from collisions between ions and ions. Additionally, the timescale of collisions for energy-equilibration and momentum-equilibration are not necessarily the same as each other. We'll try to gain some physical intuition for the different collision timescales in a plasma. We'll need to be very careful when we do this - it isn't as simple as it might seem.

Because electrons are so much less massive, they move more quickly *and* their momentum and energy is changed more easily. We'd therefore expect that the collision frequency of electrons with other electrons and with ions is much larger than the collision frequency of ions with electrons and even of ions with other ions. The relative size of these collision frequencies, assuming the ion and electron temperatures are not too different, is summarized in the table below. I've stolen this table from Bellan. He defined $\nu_{\sigma\alpha}$ to be the frequency which species σ transfers all of it's momentum to species α. Similarly, he defines $\nu_{E\sigma\alpha}$ as the frequency which species σ transfers all of it's energy to species α.

~ 1	$\sim (m_e/m_i)^{1/2}$	$\sim m_e/m_i$
ν_{ee}	ν_{ii}	ν_{ie}
ν_{ei}	ν_{Eii}	ν_{Eei}
ν_{Eee}		ν_{Eie}

For collisions between the same species (like-particle collisions), we see that the scattering frequency $\nu_{\sigma\sigma}$ is the same as the energy transfer frequency $\nu_{E\sigma\sigma}$.

[9] These ideas are covered in more depth in Bellan chapter 1.

This is essentially because when a particle collides elastically with a particle of the same species, in the frame where one of the particles is at rest, we have (in 1-D) that $\Delta v_1 = v_1 = \Delta v_2$. So all of the momentum and energy of the first particle is transferred to the second particle in that frame. This suggests that for each species, there is a single frequency at which that species transfers all of its momentum *and* energy to itself. For electrons, this frequency is a factor of $(m_e/m_i)^{1/2} \sim 40$ higher than for ions. The reason ions have a lower collision frequency with other ions is because ions have a thermal velocity which is slower than the electrons. The thermal velocity of ions is slower by a factor $(m_e/m_i)^{1/2}$, so ions collide with other ions less than electrons collide with other electrons by this same factor.

For energy transfer between different species, we see that energy is transferred from electrons to ions at the same rate energy is transferred from ions to electrons. This makes sense - if energy was transferred at different rates between the two species, then we wouldn't have conservation of energy between populations of electrons and ions which are coming into thermal equilibrium with each other. The reason the energy transfer rate is so slow is because when electrons and ions collide with each other, the velocities of the two species don't change much. In the frame of reference where the ion starts at rest, the ion velocity after a collision with an electron is $\sim 2\frac{m_e}{m_i}v_e$, so the ion energy after the collision, in this frame, is $E_i \sim \frac{m_i v_i^2}{2} \sim \frac{2m_e^2 v_e^2}{m_i}$. It takes of order $m_i/m_e \sim 2000$ of these collisions before the total amount of kinetic energy in the electron, $\frac{m_e v_e^2}{2}$ is translated to an ion, and vice versa - making the energy transfer time slower than ν_{ei} by about factor.

For momentum transfer between different species, we don't necessarily have the same momentum transfer frequency for electrons colliding with ions as for ions colliding with electrons. Imagine we had electrons initially moving with some velocity $\vec{u}_0$ relative to stationary ions. When the electrons collide with the much heavier ions, the electrons change their velocity significantly due to each collision with an ion. It takes only one collision for all of the electron momentum to be imparted to the ions - so ν_{ei} is just the frequency with which an electron collides with an ion, which is about the same as the frequency which an electron collides with an electron, ν_{ee}. Thus, the electron momentum is scattered quickly due to collisions with ions. Now imagine we have ions moving with some velocity $\vec{u}_0$ relative to stationary electrons. The momentum of the ions doesn't change much due to collisions with the light electrons, so the ion momentum won't be scattered quickly due to collisions with electrons. In fact, it takes about $\frac{m_i}{m_e}$ collisions with an electron before an ion has imparted all of it's momentum to electrons. This is why $\nu_{ie} \sim \frac{m_e}{m_i}\nu_{ei}$.[10] This is why ν_{ie} is so much smaller than ν_{ei}.

I find keeping track of all these little ν's and what they mean to be extremely confusing. Let's talk through a couple examples to make sure we understand

[10] If this statement confuses you, remember the definition of ν_{ie}: the frequency which ions transfer all of their momentum to electrons. Since it takes so many collisions for an ion to transfer all of it's momentum to electrons, this frequency is slow.

these ideas - because like I said, they can be quite subtle. Suppose we have a plasma where the ions are initially cold and at rest, while the electrons are initially hot and have some net velocity $\vec{u}_0$. What happens? The first things which happen are on the ν_{ee} timescale. The electrons collide with the stationary ions, and on the timescale $\nu_{ei} \sim \nu_{ee}$ impart all of their momentum to the ions. After a time $\sim \nu_{ei}^{-1}$, the plasma has picked up a net velocity $\vec{u} \approx \frac{m_e}{m_i}\vec{u}_0$, so that the electrons and ions have the same net velocity and the total momentum of the plasma is conserved. On the larger ν_{Eei}^{-1} timescale, the electrons and ions will reach thermal equilibrium, so the thermal energy of the electrons will be transferred to the ions, until eventually the temperature reaches an equilibrium $(T_{e0} + T_{i0})/2$.

Suppose we have the same situation but viewed in a different frame of reference. Now, the electrons are initially hot and at rest, while the ions are cold and have a net velocity $-\vec{u}_0$. What happens? Since this is just the same situation but viewed in a different frame of reference, then of course the same physics must take place. The energy equilibrium again goes on the ν_{Eei}^{-1} timescale. This time, however, we have ions transferring momentum to electrons. It seems like that should happen on the ν_{ie}^{-1} timescale, instead of the ν_{ei}^{-1} timescale, right? Wrong. ν_{ie} is the frequency at which ions transfer *all* of their momentum to electrons. Here, ions only need to transfer a tiny fraction $\frac{m_e}{m_i}$ of their momentum to the electrons for the electrons to have the same momentum as the ions. After a time ν_{ei}^{-1}, the ions have given the electrons a net velocity $-\vec{u}_0(1 - \frac{m_e}{m_i})$, while their velocity has decreased slightly to this value as well. If we transfer back to the frame of reference of the previous example, this gives us the same results, as it must.

Let's do another two examples, which Bellan introduces in section 1.9 of his book. They are great examples, because they really get to the essence of the physics of collisions in plasmas.[11] Here is the first example: Suppose we have a plasma where the ions begin completely stationary, half of the electrons begin at rest, and half of the electrons begin with velocity $\vec{u}_0$. On the ν_{ee}^{-1} timescale, the electrons with initial velocity $\vec{u}_0$ will scatter off the other electrons, and be deflected in velocity space. On the ν_{ei}^{-1} timescale, which is of the same order as ν_{ee}^{-1}, the electrons will transfer all of their momentum to the ions. It only takes a single collision for electrons to transfer their momentum to ions. If the electrons *weren't* colliding with ions, then they would have velocity $\vec{u}_0/2$ at this point. However, they are giving up all of their momentum to ions on this timescale, so the total momentum of the plasma needs to be conserved. This means that after a time of order ν_{ei}^{-1}, the electrons and ions will have the *same* net velocity, which from conservation of momentum has to be $\vec{u}_0(\frac{m_e}{m_e+m_i})$. The ions are already in thermal equilibrium and momentum equilibrium with each other, so nothing changes on the ν_{ii}^{-1} timescale. On the very long ν_{Eei}^{-1} timescale, the electrons and ions thermalize, and approach the same temperature as usual.

Here is the second example: Suppose we have a plasma where the electrons

[11] Bellan gets both of these examples wrong, because he got mixed up about what ν_{ei} and ν_{ie} really mean. I told you this stuff was subtle!

begin completely stationary, half of the ions begin at rest, and half of the ions begin with velocity $\vec{u}_0$. This example is a bit trickier. The ions transfer all of their momentum to the electrons on the very long ν_{ie}^{-1} timescale. However, the ions only need to transfer a small fraction of their momentum to the electrons in order for the two species to have the same net velocity. This means that the electrons will have the same net velocity as the ions on the ν_{ei}^{-1} timescale. Because there are two populations of ions, one at rest and one with velocity $\vec{u}_0$, the electrons will reach a velocity between the two on this fast timescale, or about $\vec{u}_0/2$. On the slower ν_{ii}^{-1} timescale, the ions will transfer momentum and energy between the two populations and reach a velocity set by conservation of momentum, a bit less than $\vec{u}_0/2$. Once again, on the very long ν_{Eei}^{-1} timescale the electrons and ions will reach thermal equilibrium with each other.

1.5 Plasma Length and Time Scales

There are numerous length scales in plasmas:

- Distance of closest approach, $b = \frac{e^2}{6\pi\epsilon_0 k_B T}$
- Interparticle spacing, $n^{-1/3}$
- Mean free path, $\lambda_{mfp} = \frac{1}{n\pi b^2}$
- Electron gyroradius, $\rho_e = \frac{m_e V_{Te}}{eB} = \frac{\sqrt{k_B T_e m_e}}{eB}$
- Ion gyroradius, $\rho_i = \frac{\sqrt{k_B T_i m_i}}{ZeB}$
- Debye Length, $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n_0}}$

The electron and ion gyroradius size depends on the local magnetic field, which can vary dramatically between different plasmas. We can, however, say that the ion gyroradius is nearly always significantly higher than the electron gyroradius, as long as the electron temperature is not dramatically larger than the ion temperature. In a plasma where the number of particles in a Debye sphere is much greater than 1, we have the following ordering of scale lengths:

- $b \ll n^{-1/3} \ll \lambda_D \ll \lambda_{mfp}$

This is one of those things which all plasmas share in common, as long as they have a large number of particles in a Debye sphere. We can prove this as follows: Suppose we define Λ as the number of particles in a debye sphere, $\Lambda = \frac{4\pi}{3} n\lambda_D^3 \gg 1$. We have that

$$\lambda_D^2 = \frac{\epsilon_0 k_B T}{e^2 n_0} = \frac{1}{6\pi n_0 b} \tag{1.35}$$

so

$$\frac{b}{\lambda_D} = \frac{1}{6\pi n_0 \lambda_D^3} = O(1/\Lambda) \tag{1.36}$$

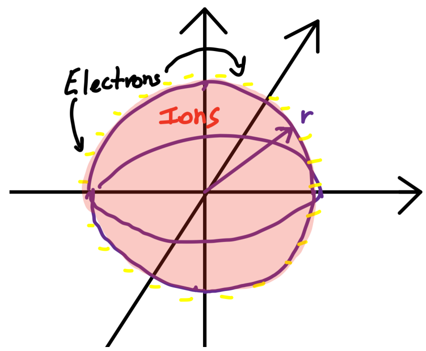

Figure 3: Illustration of the electrons (in yellow, on the surface of the sphere) evacuating a region of space with radius r, leaving a density of ions (in red) behind. This calculation is used to show why the number of particles in a Debye sphere being much greater than 1 implies overall quasineutrality over distances larger than a Debye length.

λ_D is bigger than the distance of closest approach b by a factor of order Λ. We also have that

$$\frac{n^{-1/3}}{b} = \frac{n^{-1/3}}{\lambda_D}\frac{\lambda_D}{b} = O(\Lambda^{-1/3})O(\Lambda) = O(\Lambda^{2/3}) \tag{1.37}$$

The interparticle spacing $n^{-1/3}$ is bigger than the distance of closest approach by a factor of $\Lambda^{2/3}$. Finally, we have that

$$\frac{\lambda_{mfp}}{\lambda_D} = \frac{1}{n\pi b^2 \lambda_D} = \frac{\lambda_D^2}{\pi b^2}\frac{1}{n\lambda_D^3} = O(\Lambda^2)O(1/\Lambda) = O(\Lambda) \tag{1.38}$$

Putting it all together, we have (in units where $b = 1$), $b : 1$, $n^{-1/3} : \Lambda^{2/3}$, $\lambda_D : \Lambda$, $\lambda_{mfp} : \Lambda^2$, which gives our ordering of scale lengths in a plasma. We know the mean free path in a plasma is much longer than the Debye length, which is much larger than the interparticle spacing which is much larger than the distance of closest approach. This ordering of scales lengths suggests that a plasma is a sparse, low-collisionality gas of interacting particles. I should emphasize again that all of this relies on the fact that the number of particles in a Debye sphere, Λ, is much greater than 1.

We can see now why we've chosen our definition of a plasma to be where the number of particles in a Debye sphere, Λ, is much greater than 1. If we choose this definition, then we have a definite ordering of scale lengths, which gives qualitatively similar behavior for a variety of different plasma environments.

This condition, the number of particles in a Debye sphere, also implies that the plasma is quasineutral over length scales larger than a Debye length. How do we know this? Well, there's an ingenious calculation we can do which shows that this is the case. Imagine that all of the electrons in some region of space were to all move radially outwards from a point until their velocity becomes zero, as in figure 3. How large of a spherical region could the electrons evacuate, such that we are quasineutral over that region? Well, the trick is to set the thermal energy of the electrons inside the volume (which all end up at the surface of the sphere) equal to the energy stored in the electromagnetic field created by ions left behind in the absence of the electrons. The electric field is created by the ions left behind, and from Gauss's law

$$E_r 4\pi r^2 = \frac{ne4\pi r^3}{3\epsilon_0}$$

so

$$E_r = \frac{ner}{3\epsilon_0}$$

The electromagnetic field energy is

$$\int \frac{\epsilon_0}{2} E^2 dV = \frac{2\pi n^2 e^2}{9\epsilon_0} \int r'^4 dr' = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0}$$

The thermal energy per particle is $\frac{3}{2} k_B T$, so the total thermal energy of the electrons in that volume is $2\pi n k_B T r_{max}^3$. Setting the two energies equal, we have

$$2\pi n k_B T r_{max}^3 = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0}$$

so the maximum radius r_{max} that the thermal energy of the electrons could evacuate is

$$r_{max} = \sqrt{\frac{45\epsilon_0 k_B T}{ne^2}} \approx 7\lambda_D \tag{1.39}$$

The largest region of space that can be evacuated of electrons is a few Debye length. Now, the logic goes like this. If the number of particles in a Debye sphere is much greater than 1, then the situation required to make this happen (where all of the electrons are moving radially outwards) would be extremely unlikely, as the number of particles is large. Thus, if there are many particles in a Debye sphere, our plasma is extremely likely to be quasineutral. If there is less than 1 particle in a Debye sphere, then it is possible that the plasma will be non-neutral over length scales larger than a Debye sphere.

We've talked about spatial scales in a plasma. What about timescales? Here are some of the most important frequencies, although there are many other important frequencies. The associated timescales are just the inverse of the frequencies.

- Electron gyrofrequency, $\Omega_e = \frac{eB}{m_e}$
- Ion gyrofrequency, $\Omega_i = \frac{q_i B}{m_i}$
- (Electron) plasma frequency, $\omega_{pe} = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_e}}$
- Electron-ion momentum collision frequency, ν_{ei}

2 Single Particle Motion

I therefore don't have much to say. But I will talk a long time anyway.

Richard Feynman

As we have seen, the mean free path of particles in a plasma is significantly longer than any of the other scale lengths, assuming the number of particles in a debye sphere is much greater than 1 (which is how we are defining a plasma). Particles often travel a long distance before colliding with other particles. For many plasmas, the collision timescale ν^{-1} is much longer than other relevant timescales. In these plasmas, an ion or electron might $\vec{v} \times \vec{B}$ rotate (sometimes called gyromotion or Larmor motion) many times before it collides with another plasma particle, changing it's trajectory. Thus, analyzing the motion of individual charged particles gives valuable insight into the behavior of the plasma as a whole.

We will first investigate the motion of particles in prescribed electric and magnetic fields. We'll see that four main particle drifts show up: the $\vec{E} \times \vec{B}$ drift (called the E cross B drift), the grad-B drift, the curvature drift, and the polarization drift. We will also see that periodic motion in slowly changing fields leads to the existence of conserved quantities for individual particles, which can be helpful for analyzing the motion of particles in complicated electromagnetic fields. These conserved quantities are called adiabatic invariants. We'll look at two adiabatic invariants, μ and $\mathcal{J}$. We will then analyze the magnetic mirror machine, the classic example of single-particle motion. Lastly, we'll discuss the isorotation theorem, an example of single-particle motion not typically found in textbooks which Nat covered in class.

2.1 Guiding Center Drifts

Imagine we have a constant, static magnetic field in the z-direction, $\vec{B} = B_0\hat{z}$. If we put a charged particle of charge q and mass m in that magnetic field, then the particle will spiral around the magnetic field, while it's velocity in the z-direction will remain constant. Let's see this. The force on the particle will be $q\vec{v} \times \vec{B}$, which always points perpendicular to the motion. The force being perpendicular to the motion is the condition for uniform circular motion. Thus, we have a centripetal acceleration $v_\perp^2/R = qv_\perp B_0/m$. This is easily solved, as in freshman physics, to give a frequency $\Omega = qB_0/m$ and a gyroradius $\rho = \frac{mv_\perp}{qB_0}$.

In many plasmas, there exists some sort of uniform background magnetic field[12]. Thus, the most basic, ubiquitous behavior of single particles in a plasma is gyromotion around this background magnetic field. However, the behavior of particles in spatially, time-varying fields is much more complicated. We will

[12] Usually due to some external magnetic coils, internal current, or a background field in outer space.

see that the guiding center motion (center of the orbit) involves various drifts in addition to the gyro-orbiting around the magnetic field.

Suppose there exists a charged particle of mass m and charge q in arbitrary electric and magnetic fields, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$[13]. The equation of motion for the charged particle is

$$\ddot{\vec{r}} = \frac{q}{m}(\vec{E}(\vec{r},t) + \dot{\vec{r}} \times \vec{B}(\vec{r},t)) \tag{2.1}$$

Let us assume that $\vec{E}$ and $\vec{B}$ are known. In general, this expression cannot be integrated exactly to solve for the motion. However, we will make a few approximations for this problem to become solvable. Firstly, we will assume that the particle gyro-orbits around the magnetic field, and that the gyroradius of the particle is small relative to the length scales ($\frac{B}{\vec{\nabla} B}$) over which the electric and magnetic fields change. Thus, $\vec{r}(t) = \vec{r}_{gc}(t) + \vec{\rho}(t)$[14] where $\vec{r}_{gc}$ is the position of the guiding center of the particle, and $\vec{\rho}$ is the vector from the guiding center to the particles position. We will see that there are a number of drifts of $\vec{r}_{gc}$ which add to each other in the limit that the gyroradius is much smaller than the relevant lengthscales of the magnetic and electric fields.

We define $\vec{\rho}$ as $\frac{m\hat{b}\times\dot{\vec{r}}}{qB}$, where the magnetic field is evaluated at the position of the guiding center. Note that this definition makes sense intuitively[15]. Why? Well, remember when we had a constant magnetic field, such that the gyroradius ρ was $\frac{mv_\perp}{qB}$? Well, notice that our definition here is essentially the same - $\vec{\rho}$ is perpendicular to both $\hat{b}$ and $\vec{v}$, points in the right direction, and reduces to our previous expression in the limit that the magnetic field is constant in space.

2.1.1 E Cross B Drift

The first guiding center drift we will examine is called the E cross B drift.

$$\vec{v}_{E\times B} = \frac{\vec{E} \times \vec{B}}{B^2} \tag{2.2}$$

First off, we know that if there is an electric field parallel to the local magnetic field, a particle will accelerate in that direction without feeling any magnetic force. What about for electric fields perpendicular to the local magnetic field direction? Well, this gives rise to the $\vec{E} \times \vec{B}$ drift, which we are going to study in this section. How should we understand this drift intuitively? Well, imagine we have a static constant B field in the z-direction, and a static constant E field in the y-direction. Now imagine at $t = 0$ putting a charge $+q$ at the origin

[13]Bellan's book does a wonderfully rigorous, although not particularly physically enlightening derivation of the drift equations. Spitzer and Chen, on the other hand, give wonderfully intuitive but less rigorous explanations of these drifts. These notes aim for somewhere in the middle.

[14]The 'gc' stands for 'guiding center' while the ρ is the gyroradius.

[15]How we decide to split $\vec{r}$ between the guiding center drift and the cyclotron orbit is somewhat arbitrary, but this definition makes things easier mathematically.

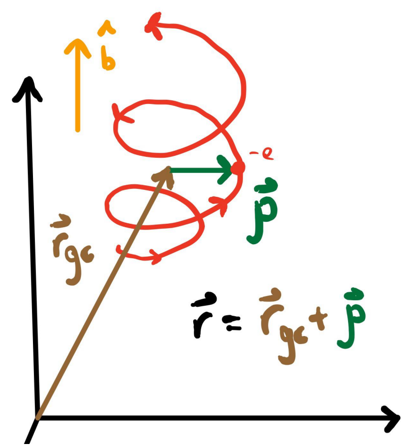

Figure 4: A negatively charged particle moving in a magnetic field. The gyroradius, $\vec{\rho}$, and guiding center position, $\vec{r}_{gc}$, are shown.

with zero initial velocity. What will happen? The electric field will cause the charge to initially accelerate in the y-direction. As the charge picks up speed in the y-direction, the magnetic field puts a force in the x-direction on the charge, causing it to turn in the positive x-direction. As the particle turns, eventually it's velocity is entirely in the x-direction. Now, the magnetic force will be in the negative y-direction, and it turns out that this force will be stronger than the electric force in the positive y-direction. Thus, the particle starts to curve downwards, in the negative y-direction. At some point, the particle will come to rest at $y = 0$,[16] and then the process will repeat itself. However, the particle will have been displaced in the x-direction, which is also the $\vec{E} \times \vec{B}$ direction. This process is illustrated in figure 5.

For negatively charged particles (i.e. electrons), they will initially accelerate in the opposite direction, but the magnetic force will cause them to curve towards the right in figure 5, again creating a drift in the $\vec{E} \times \vec{B}$ direction.

Mathematically, we derive this as follows. For simplicity, we will assume that the magnetic field is constant over the gyroorbit of the particle. We have

$$\vec{r}_{gc}(t) = \vec{r} - \vec{\rho} = \vec{r} - \frac{m\hat{b} \times \dot{\vec{r}}}{qB}$$

Taking the time-derivative, we have

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}} - \frac{m\hat{b}}{qB} \times \ddot{\vec{r}}$$

[16] It must come to rest at $y = 0$, by conservation of energy.

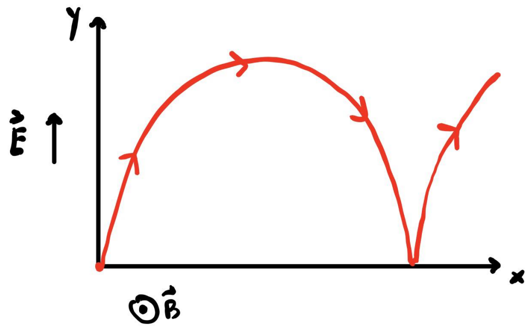

Figure 5: Illustration of the $\vec{E} \times \vec{B}$ drift for a particle which starts at rest at the origin. For particles with other initial velocities, the motion will have a different trajectory but the same net guiding center velocity.

Inserting the equation of motion equation 2.1 into $\ddot{\vec{r}}$, we have

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}} - \frac{\hat{b} \times \vec{E}}{B} - \hat{b} \times (\dot{\vec{r}} \times \hat{b}) \tag{2.3}$$

Now, we can recognize that the rightmost term is $\dot{\vec{r}}_\perp$, the velocity perpendicular to the local magnetic field. We also know that $\dot{\vec{r}} = \dot{\vec{r}}_\parallel + \dot{\vec{r}}_\perp$. The two $\dot{\vec{r}}_\perp$s cancel and we are left with

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}}_\parallel + \frac{\vec{E} \times \vec{B}}{B^2} \tag{2.4}$$

This perpendicular drift of the guiding center is the E cross B drift we described earlier. Notice that if we replaced $q\vec{E}$ with an arbitrary force $\vec{F}$, we would get a drift velocity

$$\vec{v_F} = \frac{\vec{F} \times \vec{B}}{qB^2} \tag{2.5}$$

For example, this force could be the force of gravity, $F = m\vec{g}$. In laboratory plasmas, this equation tells us that gravity causes positive and negative particles to drift in opposite directions, creating a current.[17] This effect is very small, so in general we can neglect it.

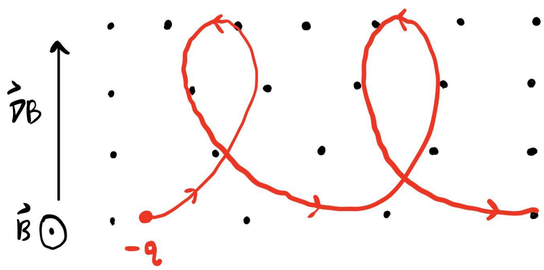

Figure 6: Illustration of the motion of a negatively charged particle in a magnetic field gradient.

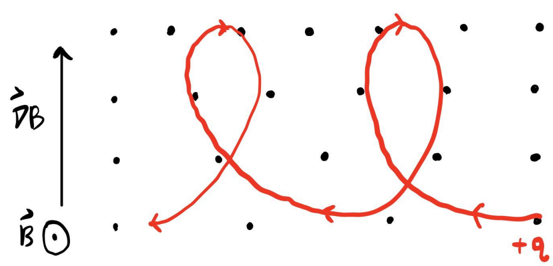

Figure 7: Illustration of the motion of a positively charged particle in a magnetic field gradient.

2.1.2 Grad-B drift

The grad-B drift is an effect that arises due to changes in the magnetic field strength perpendicular to the magnetic field direction. The grad-B drift is equal to

$$\vec{v}_{\vec{\nabla}B} = \frac{v_\perp}{2}\frac{\rho\hat{b}\times\vec{\nabla}B}{B^2} = \frac{mv_\perp^2}{2}\frac{\vec{B}\times\vec{\nabla}B}{qB^3} \tag{2.6}$$

The grad-B drift arises due to the decreased radius of curvature in regions of stronger magnetic field. This effect is shown in Figures 6 and 7. The grad-B drift was derived in class by getting an equation for the particle drift to 1st order, then plugging in the 0th-order motion to that equation. This is a classic technique from classical mechanics - solving the 1st-order equation of motion using the 0th-order solution. Suppose we integrate the vector gyroradius $\vec{\rho}$ over one orbit. To 0th order in $\epsilon = \rho/L$, where L is the length scale over which the magnetic field changes, we have the relation

$$\int_0^{2\pi} \vec{\rho}d\theta = \hat{b}\times\Delta\vec{r} \tag{2.7}$$

where $\Delta\vec{r}$ is the distance the guiding center travels in one rotation along the magnetic field. You could argue this on geometric grounds, as the sum over the gyroradius vectors tells us how much the gyrocenter has shifted. However, this geometric interpretation wasn't immediately obvious to me, so it might be easier to justify it algebraically. Equation 2.7 is true because

$$\vec{\rho} = \frac{m\hat{b}\times\dot{\vec{r}}}{qB} = \frac{m\hat{b}}{qB}\times\frac{d\vec{r}}{dt}$$

so taking the integral with respect to θ

$$\int_0^{2\pi} \vec{\rho}d\theta = \frac{m\hat{b}}{qB}\times\int\frac{d\vec{r}}{dt}d\theta = \frac{m\hat{b}}{qB}\times\int\frac{d\vec{r}}{dt}(\frac{d\theta}{dt})^{-1}d\theta \tag{2.8}$$

since $\hat{b}$ and B are constant over a gyroorbit to 0th order in ϵ. But $\frac{d\theta}{dt} = \Omega = \frac{qB}{m}$ to 0th order in ϵ. This gives equation 2.7.

$$\int_0^{2\pi} \vec{\rho}d\theta = \hat{b}\times\Delta\vec{r} \tag{2.9}$$

This is an equation to first-order of the change in ρ due to an inhomogenous magnetic field. Let's get another 1st order equation, where we explicitly allow

[17]Goldston has a nice discussion of the gravitational drift. Basically, in a finite plasma (let's say a tokamak), the gravitational current points in the horizontal direction and will lead to a buildup of charge on the side walls of a machine. This buildup of charge leads to an electric field in the horizontal direction, creating a $\vec{E}\times\vec{B}$ drift of both species downwards. This is the same thing that happens in a tokamak without a poloidal magnetic field - the grad-B and curvature drifts cause particles to drift in opposite directions vertically, creating a vertical electric field which causes particles to drift out of the tokamak.

the possibility that the magnetic field is not constant over a gyroperiod. We have $B(\vec{r}_{gc}+\vec{r}_c) \approx B(\vec{r}_{gc}) + (\vec{r}_c \cdot \vec{\nabla})B$. Assuming the magnetic field direction is constant over a gyroperiod, to 1st order in ϵ, we get

$$\vec{\rho}(\vec{r}) = \frac{m}{q}\frac{\hat{b}\times\dot{\vec{r}}}{B(\vec{r}_{gc}+\vec{\rho})} \approx \frac{m}{qB(\vec{r}_{gc})}(\hat{b}\times\dot{\vec{\rho}})\Big(1-\frac{(\vec{\rho}\cdot\vec{\nabla})B}{B(\vec{r}_{gc})}\Big) \tag{2.10}$$

Equations 2.7 and 2.10 together give equations for our drift motion to 1st order in ϵ. Now, we plug in equation 2.10 to equation 2.7 and use our 0th order solution for ρ to solve the 1st order equations for the perturbed motion. This gives

$$\hat{b}\times\Delta\vec{r} = \int \frac{m}{qB(\vec{r}_{gc})}(\hat{b}\times\dot{\vec{\rho}})\Big(1-\frac{(\vec{\rho}\cdot\vec{\nabla})B}{B(\vec{r}_{gc})}\Big)d\theta \tag{2.11}$$

We need to figure out $\vec{\rho}$ and $\dot{\vec{\rho}}$. Assume we have a positive particle and we set our coordinate system to point along the local magnetic field, such that $\vec{B} = B_z(\vec{r})\hat{z}$. To 0th order, $\vec{\rho}(\theta) = \rho(\cos\theta\hat{x} - \sin\theta\hat{y})$. Similarly, $\dot{\vec{\rho}} = v_\perp(-\sin\theta\hat{x} - \cos\theta\hat{y})$. We plug these into equation 2.11 and perform the integration over θ. The first term in equation 2.11 integrates to 0 because it is linear in $\sin\theta$ and $\cos\theta$. The second term, which is non-linear in $\sin\theta$ and $\cos\theta$, doesn't integrate to zero. Let's solve for it. We have

$$(\vec{\rho}\cdot\vec{\nabla})B = \rho\cos\theta\frac{\partial B_z}{\partial x} - \rho\sin\theta\frac{\partial B_z}{\partial y}$$

$$-(\hat{b}\times\dot{\vec{\rho}})(\vec{\rho}\cdot\vec{\nabla})B = v_\perp\rho\hat{b}\times\Bigg[\Big(\sin\theta\cos\theta\frac{\partial B_z}{\partial x} - \sin^2\theta\frac{\partial B_z}{\partial y})\hat{x} + \Big(\cos^2\theta\frac{\partial B_z}{\partial x} - \sin\theta\cos\theta\frac{\partial B_z}{\partial y}\Big)\hat{y}\Bigg]$$

We're going to integrate this expression over 2π. The first and fourth terms of this expression integrate to 0, because they go like $\sin\theta\cos\theta$. $\sin^2\theta$ and $\cos^2\theta$ integrated over 2π give π. This leaves us with

$$\hat{b}\times\Delta\vec{r} = \int \rho d\theta = \frac{\pi m\rho v_\perp}{qB^2}\hat{b}\times(-\frac{\partial B_z}{\partial y}\hat{x} + \frac{\partial B_z}{\partial x}\hat{y}) \tag{2.12}$$

We're getting close. We can see pretty easily that

$$-\frac{\partial B_z}{\partial y}\hat{x} + \frac{\partial B_z}{\partial x}\hat{y} = \hat{b}\times\vec{\nabla}B$$

so from equation 2.12 this becomes

$$\Delta\vec{r} = \frac{\pi m\rho v_\perp}{qB^2}\hat{b}\times\vec{\nabla}B \tag{2.13}$$

Now,

$$\vec{v}_{\vec{\nabla}B} = \Delta\vec{r}/T = \frac{\Delta\vec{r}qB}{2\pi m}$$

where T is the period of a gyroorbit, $\frac{2\pi}{\Omega}$. This gives the result we've been trying to derive, equation 2.6.

$$\vec{v}_{\vec{\nabla} B} = \frac{\rho v_\perp}{2B}\hat{b} \times \vec{\nabla} B = \frac{m v_\perp^2}{2}\frac{\hat{b} \times \vec{\nabla} B}{qB^2} \tag{2.14}$$

Let's recap what the grad-B drift is physically and what we've done. The grad-B drift is a particle drift which arises in magnetic fields which have gradients perpendicular to the direction of the field. Physically, the grad-B drift arises because particles have a smaller gyroradius where the field is larger, and a larger gyroradius where the field is smaller. The net effect of a gradient in B is shown in figures 6 and 7.

We've derived the grad-B drift expression using a technique from classical mechanics where we found an equation describing the change in gyrocenter position due to a magnetic field gradient to first order, and then plugging in the zeroth-order solution to this equation to estimate the first-order drift. To get the first-order equation for the change in gyrocenter position[18], I expanded the equation $\vec{\rho} = \frac{\hat{b}\times\vec{v}}{qB}$ around the gyrocenter, using $B(\vec{r}_{gc} + \vec{\rho}) \approx B(\vec{r}_{gc}) + \vec{\rho}\cdot\vec{\nabla} B|_{\vec{r}_{gc}}$. I then integrated the gyroorbit around 2π to get the change in guiding center position.

2.1.3 Curvature Drift

In a magnetic field which changes direction slowly, charged particles (approximately) follow the field lines[19]. Although this statement is fundamental to plasma physics, it really is a remarkable fact when you think about it! Why should particles trajectories curve and twist with a magnetic field?

Frankly, I don't have a compelling answer for this, and if someone does please let me know! Here is the best answer I could come up with: Imagine that in some region in space, the local magnetic field curves. If a charged particle travels along the field line with components parallel and perpendicular to the magnetic field, any motion perpendicular to the field will get washed out by the gyromotion, while any motion parallel to the field will not be affected. Try a particle might to move perpendicular to the magnetic field line, it can't get very far away from the field line it was originally on because any motion in one of the directions perpendicular to the field will be transferred into motion in the *other* perpendicular direction by the Lorentz force. Thus, the vast majority of the displacement by a particle in a curved magnetic field will be in the parallel direction.

Now that we understand why particles more or less follow field lines, let's try to understand the curvature drift. The curvature drift is a drift which arises whenever the magnetic field lines in some region are not straight. Unlike for

[18] More precisely, I would say "to get the equation for the change in gyrocenter position due to a gradient in the magnetic field to first order in $\epsilon = \frac{\rho}{L}$ where L is the scale length of the change in the magnetic field strength".

[19] Technically, their guiding centers follow the field lines to 0th order in $\epsilon = r_c/L$.

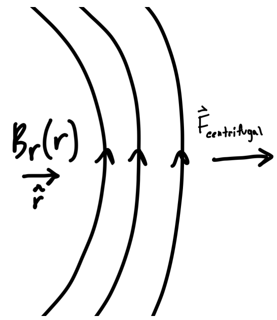

Figure 8: Curved magnetic field lines. Particles drifting along the field lines will drift upwards or downwards depending on their charge due to the curvature drift.

the $\vec{E} \times \vec{B}$ drift or the grad-B drift, I haven't seen a clear physical picture for what the particle does as it drifts due to the curvature drift. The statement everyone gives you about the curvature drift is something along the lines of "". However, inspired by the derivation for the curvature drift in Spitzer's book, we can see that there *indeed is* a simple, physically intuitive way of thinking about the curvature drift. Let's do Spitzer's derivation first, then we will discuss the physics of the curvature drift.

Imagine a particle spiraling around field lines in a curved, constant-strength magnetic field, as in figure 8. In the rotating frame of the particle, there is some centrifugal pseudo-force in the radial direction, equal to $\vec{F} = m\frac{v_\parallel^2}{R}\hat{r}$. Plugging this force into the $\frac{\vec{F}\times\vec{B}}{qB^2}$ drift, we get the drift

$$\vec{v}_D = \frac{mv_\parallel^2}{qB}\frac{\hat{r}}{R} \times \hat{b} \tag{2.15}$$

This is the curvature drift for a magnetic field which has a radius of curvature R. For a more general magnetic field, the curvature drift is

$$\vec{v}_c = \frac{mv_\parallel^2}{qB}\hat{b} \times (\hat{b} \cdot \vec{\nabla})\hat{b} \tag{2.16}$$

So what is the physical picture we should have in mind? Imagine we have a particle moving with parallel velocity $v_\parallel$ along curved field lines, as in figure 8. In the rotating reference frame of the particle, they feel a centrifugal pseudo-force outwards. Because of that centrifugal force outwards, they initially move 'outwards' slightly (in the same way that the electric field in the $\vec{E} \times \vec{B}$ drift causes particles to initially move slightly in the direction along the electric field). As the particle accelerates radially outwards, the $qv_{Dr} \times \vec{B}$ force causes the particle's velocity to change either upwards or downwards, depending on the sign. The dynamics of the motion are now the same as in the $\vec{E}\times\vec{B}$ drift, except the displacement is now vertical. In summary, particles begin to accelerate radially due to the centrifugal force, and the magnetic field turns this radial velocity into vertical displacement.

Let's derive the curvature drift more formally, as was done in class. Suppose we split the particles position between the guiding center drift $\vec{r}_{gc}$ and the gyroradius, $\vec{\rho}$, and we expand $\vec{B}$ around the gyrocenter, such that $\vec{B}(\vec{r}) = \vec{B}(\vec{r_{gc}}) + (\vec{\rho}\cdot\vec{\nabla})\vec{B}$. Also suppose that the $\vec{E} = 0$. Thus, our equation of motion, equation 2.1, becomes

$$\ddot{\vec{r}}_{gc} + \ddot{\vec{\rho}} = \frac{q}{m}\Big[\Big(\dot{\vec{r}}_{gc} + \dot{\vec{\rho}}\Big) \times \Big(\vec{B}(\vec{r}_{gc}) + (\vec{\rho}\cdot\vec{\nabla})\vec{B}(\vec{r}_{gc})\Big)\Big] \tag{2.17}$$

We can, in the limit $\frac{\rho}{L} = \epsilon \ll 1$, define the gyro-motion to be the solution to the equation

$$\ddot{\rho} = \dot{\vec{\rho}} \times \vec{B}(\vec{r}_{gc}) \tag{2.18}$$

Subtracting the gyromotion equation from equation 2.17, we have

$$\ddot{\vec{r}}_{gc} = \frac{q}{m}\Big[\dot{\vec{r}}_{gc} \times \vec{B}(\vec{r}_{gc}) + \dot{\vec{\rho}} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B}(\vec{r}_{gc}) + \dot{\vec{r}}_{gc} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B}(\vec{r}_{gc})\Big] \tag{2.19}$$

Now, let's average this equation over one gyroperiod. The third term, because it is linear in $\vec{\rho}$, will integrate to 0 to first order in ϵ. Thus we have

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{q}{m}\Big[\langle\dot{\vec{r}}_{gc}\rangle \times \vec{B}(\vec{r}_{gc}) + \langle\dot{\vec{\rho}} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B}(\vec{r}_{gc})\rangle\Big] \tag{2.20}$$

The first term on the RHS will end up contributing to the curvature drift, while the second term on the RHS will end up contributing to the $\vec{\nabla}B$ drift. Since we've already calculated the grad-B drift, we won't calculate this second term. If we were to do this calculation,[20] we could show that this term equals $\frac{-\mu\vec{\nabla}B}{m}$ where $\mu = \frac{mv_\perp^2}{2B}$.

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{q}{m}(\langle\dot{\vec{r}}_{gc}\rangle \times \vec{B}(\vec{r}_{gc})) + \frac{-\mu\vec{\nabla}B}{m} \tag{2.21}$$

The first term on the right will end up contributing to the curvature drift. Let's calculate that now. Crossing equation 2.21 with $\hat{b}$, we have

$$\langle\ddot{\vec{r}}_{gc}\rangle \times \hat{b} = \frac{\mu\hat{b}\times\vec{\nabla}B}{m} + \frac{qB}{m}(\langle\dot{\vec{r}}_{gc}\rangle \times \hat{b}) \times \hat{b} \tag{2.22}$$

Now, this last term equals $-\frac{qB}{m}\langle\dot{\vec{r}}_{gc,\perp}\rangle$. This result is actually easy to see - use the geometric interpretation of the cross product to convince yourself of this. Solving for $\langle\dot{\vec{r}}_{gc,\perp}\rangle$, we have

$$\langle\dot{\vec{r}}_{gc,\perp}\rangle = \frac{\mu\hat{b}\times\vec{\nabla}B}{qB} - \frac{m\langle\ddot{\vec{r}}_{gc}\rangle \times \hat{b}}{qB} \tag{2.23}$$

[20] If we wanted to do this calculation, we would use the zeroth order solution to the gyromotion, $\vec{\rho} = \cos\theta\hat{x} - \sin\theta\hat{y}$, and plug it in to this term.

Hey look, our first term is the $\vec{\nabla}B$ drift, as promised! The second term simply requires solving for $\ddot{\vec{r}}_{gc}$. Well, to first-order in $\epsilon = \vec{r}_c/L$, we have

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{d}{dt}\dot{\vec{r}}_{gc} = \frac{d}{dt}(v_\parallel \hat{b} + \vec{v}_{drift}) + O(\epsilon^2) = \frac{dv_\parallel}{dt}\hat{b} + v_\parallel \frac{d\hat{b}}{dt} + O(\epsilon^2) \tag{2.24}$$

Note that since there is no 0th order electric field, the drift velocity is 1st order in ϵ. Since we're assuming the fields change slowly in time and space, it's time derivative will be second order in ϵ, and can be ignored. We can also conclude that $\frac{dv_\parallel}{dt} = 0$, averaged over a cyclotron period, is 0, because there is no 0th order electric field. So

$$\langle\ddot{\vec{r}}_{gc}\rangle = v_\parallel \frac{d\hat{b}}{dt} + O(\epsilon^2) \tag{2.25}$$

Now,

$$\frac{d\hat{b}}{dt} = v_\parallel(\hat{b}\cdot\vec{\nabla})\hat{b}$$

Why is this true? Well,

$$\frac{d\hat{b}}{dt} = \frac{\partial\hat{b}}{\partial s}\frac{\partial s}{\partial t} = v_\parallel \frac{\partial\hat{b}}{\partial s} \tag{2.26}$$

where s is the distance along a magnetic field line. Now,

$$\frac{\partial\hat{b}}{\partial s} = (\hat{b}\cdot\vec{\nabla})\hat{b} \tag{2.27}$$

This is not proved directly in Bellan or in the class notes, but it makes sense geometrically. We could also convince ourselves of this by looking at a point in space where the magnetic field is instantaneously in the z-direction. Then, $\frac{\partial\hat{b}}{\partial s} = \frac{\partial\hat{b}}{\partial z}$, and $(\hat{b}\cdot\vec{\nabla})\hat{b} = (\frac{\partial}{\partial z})\hat{b}$. So equations 2.11 and 2.12 are equivalent, and we have our result $\frac{d\hat{b}}{dt} = v_\parallel(\hat{b}\cdot\vec{\nabla})\hat{b}$. So to first order in ϵ

$$\langle\ddot{\vec{r}}_{gc}\rangle = v_\parallel^2(\hat{b}\cdot\vec{\nabla})\hat{b} \tag{2.28}$$

Plugging this into equation 2.23,[21] we get

$$\dot{\vec{r}}_{gc,\perp} = \frac{\mu\hat{b}\times\vec{\nabla}B}{qB} + \frac{mv_\parallel^2}{qB}\hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b} \tag{2.29}$$

The second term is the curvature drift, as promised! Why is it called the curvature drift? Well, imagine we had a magnetic field which at some point in space, was in the $\hat{\phi}$ direction with a radius of curvature R. Then $(\hat{b}\cdot\vec{\nabla})\hat{b} = (\frac{1}{R}\frac{d}{d\phi})\hat{\phi} = -\hat{r}/R$. The curvature drift in that situation is $\frac{mv_\parallel^2}{qBR}\hat{r}\times\hat{b}$. As we discussed at the beginning of this section, we see that the curvature drift causes a drift in curved magnetic fields, perpendicular to both $\hat{b}$ and the radius of curvature vector.

[21] Here we remove the brackets because it is understood that this velocity is a drift velocity, which is by definition a time-averaged quantity.

2.1.4 Polarization Drift

The polarization drift is a drift that arises due to a time-dependent $\vec{v}_{drift}$. However, if our time-variation in $\vec{v}_{drift}$ is mostly due to a time-dependent $v_{\vec{E}\times\vec{B}}$, then we get a polarization drift due to a time-dependent electric field. This is

$$\vec{v}_p = \frac{d\vec{E}_\perp}{dt}\frac{m}{qB^2} = \frac{1}{\Omega B}\frac{d\vec{E}_\perp}{dt} \tag{2.30}$$

Chen explains the polarization drift. He writes "The physical reason for the polarization current is simple. Consider an ion at rest in a magnetic field. If a field E is suddenly applied, the first thing the ion does is to move in the direction of E. Only after picking up a velocity v does the ion feel a Lorentz force $ev \times B$ and begin to move downward. If E is now kept constant, there is no further v_p drift but only a v_E drift. However, if E is reversed, there is again a momentary drift, this time to the left. Thus v_p is a startup drift due to inertia and occurs only in the first half-cycle of each gyration during which E changes. Consequently, v_p goes to zero with ω/Ω."

The polarization drift is intimitely related to the polarization current in a dielectric medium. If we remember the polarization current, we know that $\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$. If the charges respond infinitely quickly to an applied electric field, such that $\vec{P} = \chi\vec{E}$, then $J_P = \chi\frac{\partial \vec{E}}{\partial t}$. This will probably make more sense to us when we learn about the plasma dielectric tensor in our waves course.

In class, the polarization drift for a time-changing electric field was derived as follows: Imagine we have a time-dependent electric field, and a static, constant magnetic field. Our equation of motion for a single particle, equation 2.1, becomes

$$\ddot{\vec{r}}_{gc} + \ddot{\vec{\rho}} = \frac{q}{m}(\vec{E}(\vec{r},t) + (\dot{\vec{r}}_{gc} + \dot{\vec{\rho}}) \times \vec{B}(\vec{r})) \tag{2.31}$$

Keeping only the terms involving the guiding center motion, and ignoring any 1st order spatial variation in $\vec{E}$ relative to the 0th order electric field, we have

$$\ddot{\vec{r}}_{gc} = \frac{q}{m}(\vec{E}(\vec{r}_{gc},t) + \dot{\vec{r}}_{gc} \times \vec{B}(\vec{r}_{gc})) \tag{2.32}$$

Crossing this with $\hat{b}$ gives, following the same steps as in the derivation of the curvature drift,

$$\dot{\vec{r}}_{gc,\perp} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{m}{qB}\ddot{\vec{r}}_{gc} \times \hat{b} \tag{2.33}$$

This equation is only true for an electric field which doesn't vary much over the course of a gyroorbit. Mathematically, this is equivalent to $\frac{1}{E\Omega}\frac{\partial \vec{E}}{\partial t} \ll 1$, $\frac{r_c}{E}\vec{\nabla}E \ll 1$. Now, let's solve this equation iteratively. To 0th order,

$$\dot{\vec{r}}_{gc} = v_{gc\parallel}\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2} \tag{2.34}$$

Taking the time derivative of this 0th order solution, because our magnetic field is constant and static, we get $\ddot{\vec{r}}_{gc} = \frac{dv_{gc\parallel}}{dt}\hat{b} + \frac{\frac{d\vec{E}}{dt}\times\vec{B}}{B^2}$. Plugging this into equation 2.33 gives

$$\dot{\vec{r}}_{gc\perp} = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt} \tag{2.35}$$

The second term is the polarization drift. Note that if we had not assumed the magnetic field was constant, the curvature and $\vec{\nabla}B$ drift terms would show up in this equation as well.

$$v_P = \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt} \tag{2.36}$$

While this derivation gives us the correct form of the polarization drift, I don't think it really gives us any intuition for what the polarization drift *is*. I find Chen's explanation of the physical origin of the polarization drift to be pretty confusing. I'll try instead to explain the polarization drift a different way. Here goes. Suppose we have a positively charged particle which starts at rest at the origin, in a region of crossed electric and magnetic fields, as in figure 5. Even though the particle starts at rest at the origin, it's guiding center position is at positive-y, due to the acceleration it will feel from the electric field. This means that if we were to suddenly turn on an electric field, the guiding center of the particle at rest would immediately go from $y = 0$ to some position $y > 0$. Let's calculate this and show that the change in guiding center due to the $\vec{E}\times\vec{B}$ drift equals $\int \vec{v}_p dt$. To start the calculation, suppose we have a magnetic field in the $\hat{z}$-direction and a particle at rest at the origin. Suppose we turn on an electric field in the $\hat{y}$-direction suddenly, such that $\frac{d\vec{E}}{dt} = \vec{E}_0\delta(t)$ and $\vec{E} = \vec{E}_0 H(t)$ where $H(t)$ is the Heaviside step function. Now, for the $\vec{E}\times\vec{B}$ drift we can transform ourselves into a frame moving at velocity $\vec{v}_{\vec{E}\times\vec{B}}$ where in that frame, the particle is simply exhibiting cyclotron motion around a fixed point. Since our particle starts at rest in the laboratory frame, then in the drift frame it must start with velocity $\vec{v}_\perp = -\vec{v}_{\vec{E}\times\vec{B}}$. This means that as soon as we turn on the electric field, our gyroradius (in the drift frame) becomes

$$\rho = \frac{v_\perp}{\Omega} = \frac{mE_0}{qB^2}$$

However, the gyroradius of the particle is also the change in guiding center position, since the particle starts at rest at the origin. So the change in guiding center position is $\Delta\vec{r}_{gc\perp} = \frac{m\vec{E}_0}{qB^2}$. It turns out that if we calculate the change in guiding center position using the polarization drift, we get the same result. Using the polarization drift, we have

$$\Delta\vec{r}_{gc\perp} = \int \vec{v}_P dt = \frac{m}{qB^2}\int \frac{d\vec{E}}{dt}dt = \frac{m\vec{E}_0}{qB^2}$$

We see that if we immediately turn on an electric field, the change in the guiding center from the polarization drift and the $\vec{E}\times\vec{B}$ drift are *the same*. Do you see

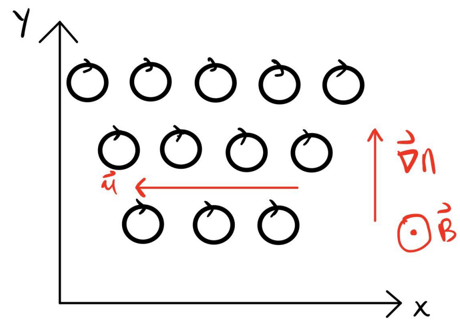

Figure 9: Magnetized particles in a density gradient have a net velocity $\vec{u}$ due to their cyclotron motion in a density gradient. Here, the velocity points in the $q\hat{b} \times \vec{\nabla} n$ direction.

now what the polarization drift represents? When a particle undergoes $\vec{E} \times \vec{B}$ motion, it has a gyroradius which changes depending on the strength of $\vec{E}_\perp$. When $\vec{E}_\perp$ changes in time, so does the position of the gyro-center, and hence there is a velocity of the gyrocenter proportional to $\frac{d\vec{E}_\perp}{dt}$.

2.1.5 Magnetization Drift and Magnetization Current

There is actually a fourth type of particle drift which arises due to single-particle effects, called the magnetization drift.[22] However, the magnetization current is unlike any of the other particle drifts we've seen. The reasons for that is that magnetization current isn't a single-particle drift, but rather an averaged motion of particles due to the effects of magnetized particles in a density gradient. Figure 9 illustrates the physical origin of this drift. In this figure, we have a magnetic field pointing out of the page, and a density gradient in the vertical direction. We're looking at positively charged particles, so they orbit *clockwise* in this region.[23] As we can see, each individual particle simply orbits around the magnetic field, so the perpendicular velocity of each particle's guiding center is 0. However, when we look at the *average* velocity of the plasma, the average velocity $\vec{u}$ is to the left because there are more particles at higher y, and those particles orbit to the left at the bottom of their orbits, while there are less particles at smaller y and those particles orbit to the right at the top of their orbits.

Suppose we have a plasma of species σ in a magnetic field $\vec{B}$ and with density n_0. Suppose our particles have charge q_σ and mass m_σ, and the plasma has a

[22] The magnetization drift wasn't covered in class. However, the magnetization *current* was.

[23] The direction of orbit of charged particles *doesn't* follow a right-hand rule, it follows the opposite. So for positively charged particles, to determine the direction they orbit, point your thumb in the direction opposite the magnetic field and curl your fingers to get the direction they orbit. For electrons, point your thumb in the direction of the magnetic field and curl your fingers to get the direction they orbit.

temperature T_σ. The magnetization drift $\vec{u}_{M\sigma}$ is

$$\vec{u}_{M\sigma} = \frac{-1}{n_\sigma}\vec{\nabla} \times \left(\frac{n_\sigma k_B T_{\sigma\perp}\hat{b}}{q_\sigma B}\right) \tag{2.37}$$

Note that for a plasma with constant temperature in a straight, constant magnetic field, this reduces to

$$\vec{u}_{M\sigma} = \frac{k_B T_{\sigma\perp}}{n_\sigma q_\sigma B}\hat{b} \times \vec{\nabla} n_\sigma \tag{2.38}$$

which is what we would calculate for the magnetization drift in figure 9. Because they orbit in opposite directions, electrons and ions are going to have a magnetization drift velocity $\vec{u}_M$ in opposite directions, so this drift velocity will create a current. Let's calculate what this magnetization current is.

$$\vec{J}_M = \sum_\sigma q_\sigma n_\sigma \vec{u}_{M\sigma} = \vec{\nabla} \times \left(\frac{-\sum_\sigma n_\sigma k_B T_\sigma \hat{b}}{B} = \vec{\nabla} \times \left(\frac{-P_\perp \hat{b}}{B}\right)\right.$$

$$\vec{J}_M = \vec{\nabla} \times \vec{M} \tag{2.39}$$

where $\vec{M} = \frac{-P_\perp \hat{b}}{B}$. The magnetization current is, like the magnetization drift, due to the collective effects of a large number of charged particles orbiting around a magnetic field. Note that the physical reason for the magnetization current in a plasma is the same as for the magnetization current in a magnetized material. In a plasma, we have particles acting as magnetic dipoles due to a magnetic field, and the curl of the dipole moment of the particles gives us a current. This is illustrated schematically in figure 10, where the purple line represents the magnetization current.

If you are going to remember only one thing from this section of the magnetization drift and magnetization current, remember this: whenever we apply the results of single-particle guiding-center drifts to understand the results of the plasma as a whole, we need to include the magnetization drift (or magnetization current) in that analysis. If we don't, then we'll get the wrong results.

2.1.6 Drift Currents

So far, we've derived four types of single-particle guiding-center drifts in magnetic fields, and one type of averaged fluid drift. These are:

- $\vec{v}_{E\times B} = \frac{\vec{E}\times\vec{B}}{B^2}$
- $\vec{v}_{\vec{\nabla} B} = \frac{mv_\perp^2}{2}\frac{\hat{b}\times\vec{\nabla}B}{qB^2}$
- $\vec{v}_c = \frac{mv_\parallel^2}{qB}\hat{b} \times (\hat{b}\cdot\vec{\nabla})\hat{b}$
- $\vec{v}_p = \frac{m}{qB^2}\frac{d\vec{E}}{dt}$

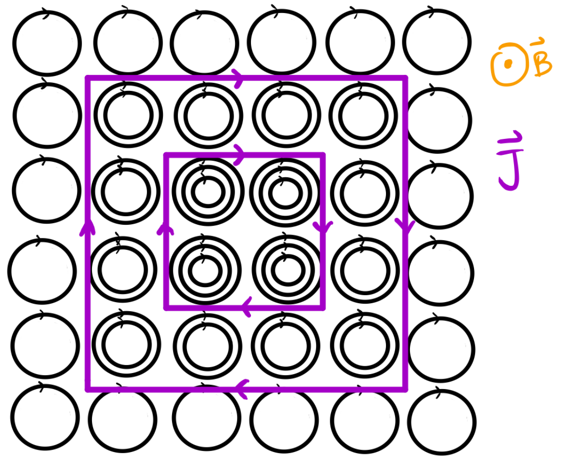

Figure 10: The orbits of charged particles around a magnetic field act as magnetic dipoles and can create a magnetization current (purple line) in a plasma.

- $\vec{u}_{M\sigma} = \frac{-1}{n_\sigma}\vec{\nabla} \times \left(\frac{n_\sigma k_B T_{\sigma\perp}\hat{b}}{q_\sigma B}\right)$

With the exception of the $\vec{E} \times \vec{B}$ drift, each of these drifts is linear in q_σ. This means that particles of opposite charge will go in different directions due to those drifts. When particles of opposite charge don't travel at the same velocity, we have a current! So four of the five drifts contribute to a current in the plasma.

The single-particle guiding-center drift model plus the magnetization current is equivalent to the perpendicular part of the fluid equations. We can show this by summing over all of the perpendicular currents due to particle drifts to get $J_{\perp Total}$, and cross that with $\vec{B}$. After some manipulations, we can show that the single-particle picture which gives us $\vec{J}_{\perp Total}$ is equivalent to the perpendicular component of the doubly adiabatic MHD momentum equation.[24] This takes a lot of algebra, and I won't do the full calculation in these notes. I'll just do the first couple steps so you get the idea.

It turns out, as we will see in chapter 4, that pressure for the fluid description of a plasma is a tensor defined as

$$\overleftrightarrow{P} = \sum_\sigma m_\sigma \int \vec{v}'\vec{v}' f_\sigma d^3\vec{v}$$

where $\vec{v}'$ is defined as $\vec{v} - \vec{u}$ where $\vec{u}$ is the mean fluid velocity and f is the distribution function in the Vlasov treatment, to be introduced in chapter 3. Now, if we have a Maxwellian distribution, such that we have a well-defined

[24]Doubly adiabatic MHD is the same as MHD where the equation of state is adiabatic, except the pressures in the perpendicular and parallel directions are different. Hopefully this makes more sense after chapter 4.

temperature in the direction parallel and perpendicular to the local magnetic field, our pressure tensor becomes diagonal

$$\overleftrightarrow{P} = \begin{bmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{bmatrix} \tag{2.40}$$

$$P_\parallel = \sum_\sigma n_\sigma \kappa T_{\sigma\parallel} = \sum_\sigma n_\sigma m_\sigma \langle v_{\sigma\parallel}^{'2} \rangle$$

$$P_\perp = \sum_\sigma n_\sigma \kappa T_{\sigma\perp} = \sum_\sigma \frac{1}{2} n_\sigma m_\sigma \langle \vec{v}_{\sigma\perp}^{'2} \rangle$$

The factor of 1/2 in $P_\perp$ comes from the fact that there are two perpendicular directions. Using the definition $\vec{J} = \sum_\sigma q_\sigma n_\sigma \vec{u}_\sigma$, we find

- $\vec{J}_{E\times B} = 0$
- $\vec{J}_{\vec{\nabla}B} = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma \langle v_{\perp,\sigma}^2 \rangle}{2q_\sigma} \frac{\hat{b}\times\vec{\nabla}B}{B^2} = \frac{\vec{B}\times\vec{\nabla}B}{B^3} \sum_\sigma \frac{1}{2} n_\sigma m_\sigma \langle v_{\perp,\sigma}^2 \rangle = \frac{\vec{B}\times\vec{\nabla}B}{B^3} P_\perp$
- $\vec{J}_c = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma \langle v_\parallel^2 \rangle}{q_\sigma B} \hat{b} \times (\hat{b}\cdot\vec{\nabla})\hat{b} = \frac{\hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b}}{B} P_\parallel$
- $\vec{J}_p = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma}{q_\sigma B^2} \frac{d\vec{E}}{dt} = (\frac{\rho}{B^2}) \frac{d\vec{E}}{dt}$
- $\vec{J}_M = \vec{\nabla} \times (\frac{-P_\perp \hat{b}}{B})$

Summing the currents, we get $\vec{J}_{\perp total} = \vec{J}_{\vec{\nabla}B} + \vec{J}_c + \vec{J}_p + \vec{J}_M$. If we take $\vec{J}_{\perp total} \times \vec{B}$, after prolific amounts of algebra we find that it gives us the perpendicular component of the doubly adiabatic MHD equations,

$$\rho \frac{d\vec{u}_\perp}{dt} = \left[\vec{J}_{\perp total} \times \vec{B} - \vec{\nabla} \cdot \left[P_\perp \overleftrightarrow{\boldsymbol{I}} + (P_\perp - P_\parallel)\hat{b}\hat{b} \right] \right]_\perp \tag{2.41}$$

Remember the conclusion we draw from this: the single-particle guiding-center drift model contains the same information as (the perpendicular component of) the fluid model, as long as we are sure to include the magnetization current.[25]

2.2 Adiabatic Invariants

There are lots of invariants we know of. Energy and momentum are the simplest examples - in any closed system, the total energy and total momentum are constant. It turns out that for collisionless plasma particles, there are a couple *adiabatic* invariants which are enormously useful in understanding the

[25] I had a lot of trouble with this derivation, because $n_\sigma m_\sigma \langle v_\parallel^2 \rangle \neq P_\parallel$ and $n_\sigma m_\sigma \langle v_\perp^2 \rangle \neq P_\perp$, even if the pressure is Maxwellian. This comes down to the fact that $\vec{v} = \vec{u} + \vec{v}'$, so $\langle v_\perp^2 \rangle = u_\perp^2 + \langle v'^2 \rangle$ where the second term contributes to a Maxwellian. Hong seems to think that the conclusion is right, but that the derivation is wrong. Question: ask Nat

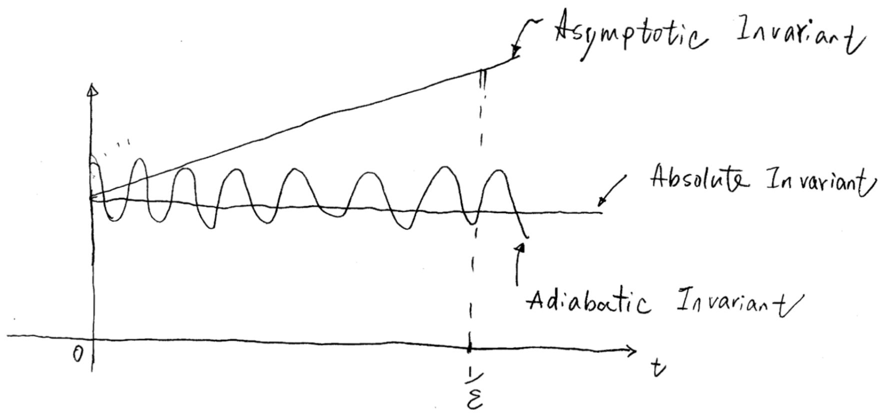

Figure 11: The change in time of an adiabatic invariant, compared with an absolute invariant (such as energy) and an asymptotic invariant.

motion of plasma particles in complicated electromagnetic fields. Before we discuss these adiabatic invariants, we must ask ourselves the obvious question: *What even is an adiabatic invariant*? Suppose we have a system with some canonical[26] coordinate Q and it's canonical momentum P, and a Hamiltonian H for that system such that $\frac{\partial H}{\partial P} = \dot{Q}$, and $-\frac{\partial H}{\partial Q} = \dot{P}$. This is what we call a Hamiltonian system - a dynamical system governed by Hamilton's equations.[27] The most obvious example of a Hamiltonian system relevant to plasma physics is that of a charged particle in some electric and magnetic fields.

Now, suppose we have some slowly changing parameter, $\lambda(t)$, in the Hamiltonian, so that $H(Q, P, \lambda(t))$. Also suppose that the canonical coordinates of the system undergo some nearly periodic motion. Then, the integral

$$I = \oint P dQ \tag{2.42}$$

is constant over any one period of motion.[28] This integral is the general form of any adiabatic invariant. This explanation, while brief and to the point, overlooks many of the details (which are certainly important!). Let's try to understand these details, before we derive this result.

What does it mean for a system to be nearly periodic? We don't have a good mathematical definition of this. Intuitively, however, we have some idea of

[26]Don't worry too much about this word 'Canonical'. It basically just means a set of coordinates which we can use the Lagrangian or Hamiltonian formalism with.

[27]Really, I should consider the case of multiple coordinates, such that Q and P are vectors. However, I'm not sure how to prove adiabatic invariance in this case, since defining the beginning and end of the nearly periodic motion is trickier. Thus, I've kept them as non-vectors.

[28]This was not proven in class, unless I missed it in my notes, which is possible. It is proven in Hong's supplemental notes. I find Bellan's derivation easier to understand, thus these notes will prove this result using his method.

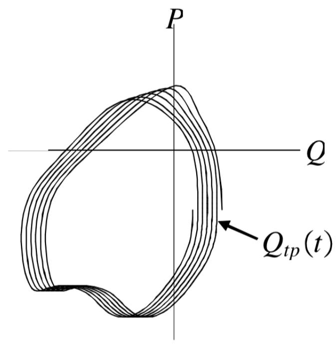

Figure 12: Nearly periodic motion in the $P - Q$ plane.

what this might mean. As an example, consider the simple pendulum with no energy losses. It's frequency is $\sqrt{g/l}$, and it certainly undergoes periodic motion. Now imagine we slowly change the length of the pendulum, $l(t)$. Although the canonical $P-Q$ coordinates of the pendulum will not be exactly the same after each oscillation, the pendulum will nearly return to it's starting point after each oscillation. If the length of the pendulum changes slowly, it's period is $\sqrt{g/l(t)}$ Thus, we say that the motion is nearly periodic.

When we say that $\lambda(t)$ changes slowly, how slow is slowly? Well, the derivation we will do depends on $\lambda(t)$ being differentiable from one period to the next. So the result that the integral in equation 2.42 is constant will be exact in the limit that the change in $\lambda(t)$ over any one period is infinitesimal. In the more plausible limit that $\frac{T}{\lambda}\frac{d\lambda}{dt} = \epsilon \ll 1$, where T is the period of the system's periodic motion, then the change in I at any time never becomes greater than some small value, $O(\epsilon)$. Mathematically, this is $\frac{I(t)-I(0)}{I(0)} < O(\epsilon)$ for $0 < t < \frac{1}{\epsilon}$ where $\epsilon \ll 1$. Hong has a nice visualization of this in figure 11. What this means is that if $\lambda(t)$ changes at a small rate (ϵ), then the total change in I stays small for a very long time ($O(\epsilon)$), no matter how much λ changes by. All that is required is that the rate of change of λ is small.

If we are integrating over a period, how do we define the beginning and end of a period if the endpoint is not the same the starting point? Well, in the $P-Q$ plane the motion will have some periodic behavior like that in figure 12. We integrate from one turning point Q_{tp} to the next, where Q_{tp} is defined as the location during the cycle where $\frac{dQ}{dt} = 0$ and Q has it's maximum value during the period. In addition to being physically reasonable, this definition is mathematically convenient, as we will see now. Let's derive the invariance of I in equation 2.42. We want to show that $\frac{dI}{dt} = 0$ - this will require us to be very careful with our partial derivatives. So pay attention.

$$\frac{dI}{dt} = \frac{d}{dt}\oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} PdQ \tag{2.43}$$

Now, since $E(t) = H(P, Q, \lambda(t))$, we can in principle invert this to write $P(E(t), Q, \lambda(t))$. So

$$\frac{dI}{dt} = \frac{d}{dt} \oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} P(E(t), Q, \lambda(t)) dQ \tag{2.44}$$

To simplify this, we have to use something called the 'Leibniz integral rule'. This rule says that

$$\frac{d}{dt}\Big(\int_{x=a(t)}^{x=b(t)} f(t,x)dx \Big) = f(t, b(t))\frac{db(t)}{dt} - f(t, a(t))\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,x)dx \tag{2.45}$$

This is the same form as our integral, with $a(t) = Q_{tp}(t)$, $b(t) = Q_{tp}(t + \tau)$, $f(t,x) = P(E(t), Q, \lambda(t))$, and $dx = dQ$. Using the Leibniz integral rule, equation 2.44 becomes

$$\frac{dI}{dt} = P(E(t), Q_{tp}(t+\tau), \lambda(t))\frac{dQ_{tp}(t+\tau)}{dt} \tag{2.46}$$

$$-P(E(t), Q_{tp}(t), \lambda(t))\frac{dQ_{tp}(t)}{dt} + \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \frac{\partial P}{\partial t} dQ$$

By definition, $\frac{dQ_{tp}(t+\tau)}{dt} = \frac{dQ_{tp}(t)}{dt} = 0$ because those are defined to be the turning points. From the definition of the partial derivative, we can also write $\frac{\partial P}{\partial t} = (\frac{\partial P}{\partial t})_Q$. Our integral is now

$$\frac{dI}{dt} = \int \left(\frac{\partial P}{\partial t}\right)_Q dQ \tag{2.47}$$

Now, let's attack this remaining term.

$$\left(\frac{\partial P}{\partial t}\right)_Q = \left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} \frac{d\lambda}{dt} + \left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} \frac{dE}{dt} \tag{2.48}$$

Now, we can use some tricks to simplify these two terms. Since $E(t) = H(P, Q, \lambda(t))$, then we can say that

$$\left(\frac{dH}{dE}\right)_{Q,\lambda} = 1 = \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda} \left(\frac{\partial P}{\partial E}\right)_{Q,\lambda}$$

so

$$\left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} = \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda}^{-1}$$

Since E is a function only of time,

$$\left(\frac{dE}{d\lambda}\right)_Q = 0 = \left(\frac{\partial H}{\partial \lambda}\right)_{Q,P} + \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda} \left(\frac{\partial P}{\partial \lambda}\right)_{Q,E}$$

Thus,

$$\left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} = -\left(\frac{\partial H}{\partial \lambda}\right)_{Q,P} \Big/ \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda}$$

Plugging these results into equation 2.48 and then into equation 2.47, we get

$$\frac{dI}{dt} = \int \frac{1}{(\frac{\partial H}{\partial P})_{Q,\lambda}} \left[\frac{dE}{dt} - \frac{d\lambda}{dt}\left(\frac{\partial H}{\partial \lambda}\right)_{Q,P}\right] dQ \tag{2.49}$$

Hey, this looks promising! Let's solve for $\frac{dE}{dt}$, using $E(t) = H(P, Q, \lambda(t))$ and Hamilton's equations $\frac{\partial H}{\partial P} = \frac{dQ}{dt}$ and $\frac{\partial H}{\partial Q} = -\frac{dP}{dt}$.

$$\frac{dE}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial H}{\partial Q}\frac{dQ}{dt} + \frac{\partial H}{\partial P}\frac{dP}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} \tag{2.50}$$

The second and third terms have canceled from Hamilton's equations. Plugging this into equation 2.49 gives $\frac{dI}{dt} = 0$. This completes our proof of the adiabatic invariance of I.

A lot just happened - it might be helpful to recap. Our Hamiltonian depends on some slowly varying parameter, $\lambda(t)$. Some particle executes nearly periodic motion in the $P - Q$ plane. We've shown, by carefully keeping track of partial derivatives, that the integral $I = \int P dQ$ is invariant, where the integral goes from $Q_{tp}(t)$ to $Q_{tp}(t + \tau)$ and we've defined the turning points to be where $\frac{dQ}{dt} = 0$. We first used the Leibniz integral rule to turn the total derivative of the integral into a partial derivative of P. We then used some clever manipulations of the partial derivatives, as well as Hamilton's equations, to eliminate the remaining term.

If you're paying attention closely, this proof might give you some concern. After all, we've shown that $\frac{dI}{dt} = 0$ *exactly*, haven't we? But shouldn't $\frac{dI}{dt}$ only be small if λ changes sufficiently slowly? Go back to the proof and see if you can figure out where in the proof it is required that λ change sufficiently slowly - it isn't obvious. The issue lies with the factor of τ in $Q_{tp}(t + \tau)$: τ depends on $\lambda(t)$! This means that when we apply the Leibniz integral rule, $\frac{db}{dt}$ becomes $\frac{\partial Q_{tp}(t+\tau)}{\partial t} + \frac{\partial Q_{tp}(t+\tau)}{\partial \tau}\frac{\partial \tau}{\partial \lambda}\frac{d\lambda}{dt}$. So as long as $\frac{d\lambda}{dt}$ is sufficiently small, then we've proven that $\frac{dI}{dt}$ is small.[29]

2.2.1 First Adiabatic Invariant μ

The magnetic moment of a charged particle in a plasma μ,

$$\mu = \frac{mv_\perp^2}{2B} = \frac{KE_\perp}{B} \tag{2.51}$$

[29]Hong points out that this proof isn't rigorous, in the sense that it doesn't prove the behavior of an adiabatic invariant shown in figure 11. To prove that an adiabatic invariant doesn't change over a long period of time requires some more advanced math. Hong can point you to the proper resources if you're interested in learning a more rigorous proof.

is an adiabatic invariant for single plasma particles. This is the quantity that is adiabatically conserved due to the periodic motion of particles gyrating around magnetic field lines. Before we prove the conservation of μ for slowly varying fields, let's ask ourselves *why* is μ called the magnetic moment? From classical mechanics, the magnetic moment of a loop carrying current I with area A is IA. We can think of charged particles in a magnetic field as being little loops of current carrying material. The area of that loop is $\pi\rho^2 = \pi\frac{v_\perp^2}{\Omega^2}$ and the current is $\frac{q}{T} = \frac{q\Omega}{2\pi}$. Putting this together, and using $\Omega = \frac{qB}{m}$ as usual, we have

$$IA = \frac{\pi v_\perp^2}{\Omega^2}\frac{q\Omega}{2\pi} = \frac{mv_\perp^2}{2B} = \mu$$

Note that from classical electromagnetism, the force on a magnetic moment in a changing magnetic field is

$$F_\parallel = -\mu\vec{\nabla}_\parallel B$$

This is essentially the reason why the magnetic moment of charged particles is conserved: as particles move to regions of stronger field, they feel a force slowing them down in the parallel direction, and due to conservation of energy their velocity in the perpendicular direction needs to increase.

Let's prove μ-conservation the same way it was proven in class. Imagine we have a particle in a magnetic field, $\vec{B}(\vec{r},t)$ which changes slowly in space and time. If this particle does not collide with other particles, then we have conservation of energy.

$$0 = \frac{d}{dt}(mv_\perp^2/2 + mv_\parallel^2/2) = \frac{d}{dt}(\mu B + \frac{1}{2}mv_\parallel^2) \tag{2.52}$$

Expanding this, we have

$$\frac{d\mu}{dt}B + \mu\frac{dB}{dt} + mv_\parallel\frac{dv_\parallel}{dt} + O(\epsilon^2) = 0 \tag{2.53}$$

As we argued for in the curvature drift derivation, $\frac{dB}{dt} = \frac{\partial s}{\partial t}\frac{\partial B}{\partial s} = v_\parallel(\hat{b}\cdot\vec{\nabla})B$. Now, we can take the dot product of $\hat{b}$ with equation 2.21 to get

$$\hat{b}\cdot(\langle\ddot{\vec{r}}_{gc}\rangle) = \langle\frac{dv_\parallel}{dt}\rangle = \frac{-\mu\hat{b}\cdot\vec{\nabla}B}{m} \tag{2.54}$$

Plugging these results into equation 2.53 averaged over a gyro-period, we get

$$0 = \frac{d\mu}{dt}B + \mu v_\parallel\hat{b}\cdot\vec{\nabla}B - mv_\parallel\frac{\mu\hat{b}\cdot\vec{\nabla}B}{m} = \frac{d\mu}{dt}B \tag{2.55}$$

Thus, μ does not change in time for a single particle moving in a slowly varying magnetic field. It turns out that μ is also conserved for particles in slowly varying electromagnetic fields as well, not just slowly varying magnetic fields.

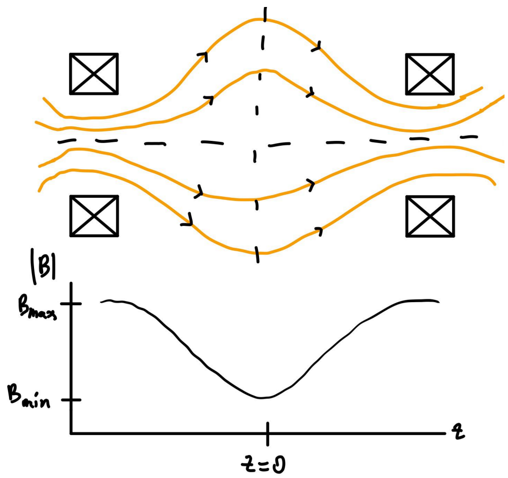

Figure 13: Illustration of the magnetic mirror (top) and the magnetic field magnitude as a function of z (bottom). The magnetic field is cylindrically symmetric.

2.2.2 Second Adiabatic Invariant $\mathcal{J}$

Imagine we have a particle in arbitrary electromagnetic fields which vary slowly in time whose guiding center undergoes some approximately periodic motion. This periodic motion is most often a particle bouncing back and forth between two regions. Here, we define "slowly" to mean that the timescale over which the electromagnetic fields change is much longer than the particle's bounce period τ_B, so that $\frac{\tau_{bounce}}{B}\frac{d\vec{B}}{dt} \ll 1$. In such a circumstance, then the quantity $\mathcal{J}$ is constant in time for each particle.

$$\mathcal{J} = \oint v_{\parallel} dt \tag{2.56}$$

This was not derived in class. However, the conservation of this quantity should not be surprising, as it fits the bill in terms of our general adiabatic invariant discussed earlier - we have an approximately periodic motion in some slowly changing Hamiltonian system.

2.3 Mirror Machine

The classic illustration of single-particle motion is the magnetic mirror. The most basic magnetic mirror consists of two cylindrically symmetric current-carrying coils which set up a cylindrically symmetric magnetic field. This is shown in Figure 13.

The crucial thing to realize - the trick for solving magnetic mirror problems - is that when it comes to magnetic mirrors, we use (adiabatic) invariants for collisionless single particles. For the classic mirror machine, these (adiabatic)

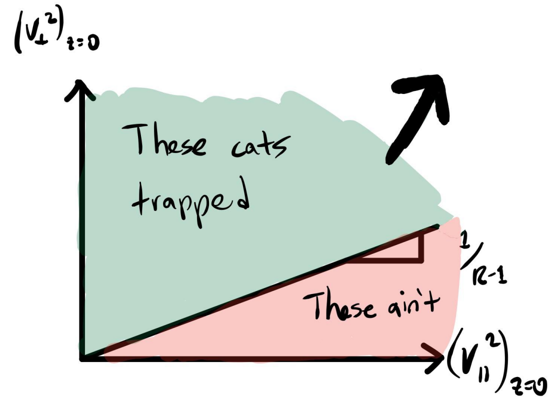

Figure 14: Plot of equation 2.59 showing the particles which are trapped and which aren't. As we can see, particles with larger perpendicular velocities are trapped.

invariants are energy $E = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2 + q\phi$ and $\mu = \frac{mv_\perp^2}{2B}$. The second adiabatic invariant $\mathcal{J}$ is sometimes used as well when the fields in the mirror machine are changing slowly relative to the bounce time of particles between the ends of the mirror.

The strategy is to equate the invariants E and μ at the midplane ($z = 0$) of the magnetic mirror to the motion at the maximum z, where $v_\parallel^2 = 0$. If the electric potential $\phi = 0$, and B is minimum at the midplane, then from

$$E = \big(\frac{1}{2}mv_\perp^2\big)_{z=0} + \big(\frac{1}{2}mv_\parallel^2\big)_{z=0} = \big(\frac{1}{2}mv_\perp^2\big)_{B=B_{max}} \tag{2.57}$$

and

$$\mu = \frac{1}{B_{min}}\big(\frac{1}{2}mv_\perp^2\big)_{z=0} = \frac{1}{B_{max}}\big(\frac{1}{2}mv_\perp^2\big)_{B=B_{max}} \tag{2.58}$$

we can easily (I promise) solve (try it!) for the condition on the trapped particles,

$$\left(\frac{v_\perp^2}{v_\parallel^2}\right)_{z=0} \geq \frac{1}{R-1} \tag{2.59}$$

where $R = \frac{B_{max}}{B_{min}}$. Make sure you know how to do this calculation, as you will be asked to do it multiple times over this course, and you will have to do slightly more complex versions on the homework. They might, for example, introduce an electric field, or prescribe a magnetic field shape that changes in time, etc etc.

Our intuition for the trapping equation, equation 2.59, is relatively simple: particles with high perpendicular velocities are trapped in the mirror, while particles with high parallel velocities are lost from the mirror. The physical picture to have in your head is the following: as a particle goes into a region with higher magnetic field, it's parallel velocity gets converted into larger perpendicular velocity, increasing the gyroradius. If the particle has enough parallel velocity to begin with, then the perpendicular velocity will increase, but not enough to bring the parallel velocity to zero at B_{max}. Note also: if the mirror ratio $R = 1$, then the ratio of perpendicular velocity to parallel velocity required for trapping goes to infinity, and we don't have any trapped particles. Which makes sense, because we don't have any magnetic field variation to trap the particles with.

What is the actual mechanism for the force which causes charged particles to be accelerated away from regions of higher $|B|$? It's the Lorentz force, of course. Let's see how that works in practice. As an example, let's use the magnetic mirror machine of figure 13, with $\vec{B} = B_r\hat{r} + B_z\hat{z}$. Let's imagine our particle of charge $+q$ starts at $z = 0$ and orbits the $r = 0$ axis towards positive-z, where the magnetic field increases in strength. Since it's a positive particle, $v_\perp$ will be in the negative-θ direction, as shown in the figure. As we can see, $\vec{v}_\perp \times B_r\hat{r}$ gives us a force in the negative-$\hat{z}$ direction, pushing our particle away from the region of high-B. Similarly, $v_z\hat{z} \times \vec{B}_r$ gives us a force in the negative-$\hat{\theta}$ direction, increasing the perpendicular velocity of the particle. The conclusion we draw from these forces is that as a charged particle orbits towards positive-z, it's z-velocity decreases, while it's $v_\perp$ increases. This example demonstrates how the Lorentz force, acting in a region of converging field lines, is consistent with μ-conservation.

2.4 Isorotation Theorem

The iso-rotation theorem is not usually seen in introductory textbooks, but Nat covers it because it is a relatively simple application of single-particle motion which has a simple result.

The statement of the isorotation theorem is as follows: in a cylindrically symmetric region of magnetic fields where $B_\phi = 0$, where $\vec{E} \times \vec{B}$ motion dominates the perpendicular particle motion and magnetic surfaces are equipotential surfaces, then for all the particles on a given magnetic surface, the rotation rate is constant.

There is a corrolary of the isorotation theorem which is proved in class as well. The corrolary says that under the same set of assumptions, then as particles drift from one surface to another, they gain in potential energy equal to exactly twice the energy lost in azimuthal drift energy, so as to climb up the potential. I found this to be a rather confusing statement, so we'll unpack this corrolary more as we go on.

Why would we expect magnetic surfaces to be equipotential surfaces? Here is Nat's answer: It comes down to the ability of particles to stream along field lines, while their motion is confined perpendicular to the field lines. If $E_\| \neq 0$, then the ions and electrons will quickly move in opposite directions to get rid of that $E_\|$. So we would more or less expect $E_\|$ to be 0, which means that $\vec{E}$ would be perpendicular to magnetic field lines. If $\vec{E}$ is perpendicular to magnetic field lines, then magnetic surfaces are equipotential surfaces.

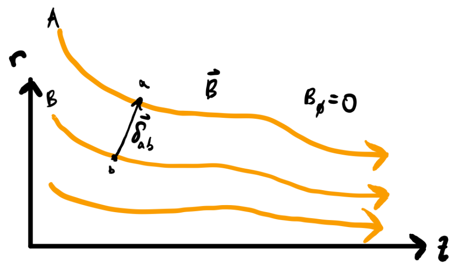

Figure 15: An illustration of the geometry considered for the isorotation theorem. Here, $B_\phi = 0$, while B_r and B_z are cylindrically symmetric. A and B are two magnetic surfaces, and a and b are two points on these surfaces separated by a vector $\vec{\delta}_{ab}$ with magnitude δ_{ab}. a and b are chosen such that $\vec{\delta}_{ab}$ lies in the r-z plane and is perpendicular to the local magnetic field.

The applications of the isorotation theorem are unclear. I suspect Nat had been thinking about it for his 'electric tokamak' idea, which a couple grad students have worked on recently, and he thought it would be fun to discuss in the course. If we have a 'straight' electric stellerator (where $\frac{\partial}{\partial z}$ is not necessarily 0) in a cylindrical geometry, and the poloidal magnetic field is 0, then the poloidal rotation rate is constant if the particles rotate primarily due to $\vec{E} \times \vec{B}$ rotation.

2.4.1 Magnetic Surfaces

Before we prove the isorotation theorem, we should discuss magnetic surfaces, since they are introduced in class along with the isorotation theorem and are referred to in the theorem. The most general definition of a magnetic surface is **a surface in space where all the magnetic field lines on the surface stay on the surface**.

It is also important to note that magnetic field lines are not in any way guarenteed to form magnetic surfaces. The most general behavior of magnetic field lines is stochastic (i.e. random) behavior, meaning a given magnetic field line, if followed forever, will fill a volume in space.

We should also note that magnetic field lines do not necessarily close in on themselves, even though $\vec{\nabla} \cdot \vec{B} = 0$ and even in the special case where we have magnetic surfaces. A magnetic field line on a magnetic surface might go around the surface forever, never closing on itself but filling the surface. In principle, magnetic field lines can close on themselves after some finite distance, but this is certainly not required or even expected in non-symmetric fields.

When would we expect to see magnetic surfaces in the first place? Good question! Magnetic surfaces are, in ideal MHD, predicted to arise in equilibrium in magnetized plasmas. Starting with the MHD equilibrium equation $\vec{J} \times \vec{B} = \vec{\nabla} P$, we can dot this equation with $\vec{B}$ to get that $\vec{B} \cdot \vec{\nabla} P = 0$. Let's think about this equation for a second. The change in P is always perpendicular to $\vec{B}$, which implies that as we follow $\vec{B}$, our field line will have constant P. If the gradient of P does not vanish anywhere, this implies that $\vec{B}$ field lines lie on surfaces of constant P. We can perform the same procedure with $\vec{J}$ instead of $\vec{B}$, to conclude that the vector field $\vec{J}$ lies on surfaces of constant P as well. The result is that in ideal MHD equilibrium, a surface of constant pressure is also a magnetic surface.

Now, here is a fun result, which is covered in GPP2 but I thought I'd include in these notes since it's relatively straightforward mathematically but conceptually fun. A theorem from topology says that the simplest topological form for a non-vanishing vector field which lies on a smooth surface is a torus. I'm not exactly sure what the word 'simplest' means in this context, but for our purposes, that isn't what is important. What is important is that if we have an MHD equilibrium where $\vec{J} \times \vec{B} = \vec{\nabla} P$, then we have our $\vec{B}$ field on a surface of constant P, meaning we have a magnetic surface. And if we have magnetic surfaces, this theorem says the simplest surface we can have is a toroidal one! In other words, if we want to create a plasma in an MHD equilibrium, we're pretty much stuck using a torus.

In cylindrically symmetric systems (where ϕ is ignorable), a magnetic surface is defined as the surface defined by constant rA_ϕ. This comes from the result that $\vec{B}\cdot\vec{\nabla}(rA_\phi) = 0$, implying (using the same logic we used earlier with P) that $\vec{B}$ lies on a surface of constant rA_ϕ. This result is easily proved, as shown now. We can arbitrarily write $\vec{A}$ for cylindrically symmetric $\vec{B}$ as $\vec{A} = A_r(r,z)\hat{r} + A_\phi(r,z)\hat{\phi} + A_z(r,z)\hat{z}$. We have for $\vec{B}$,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \big(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\big)\hat{\phi} - \frac{\partial A_\phi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\hat{z} \tag{2.60}$$

With $\vec{B}$ in hand, we simply perform a dot product to show that $\vec{B}$ lies on a surface of constant rA_ϕ.

$$\vec{B} \cdot \vec{\nabla}(rA_\phi(r,z)) = -\frac{\partial A_\phi}{\partial z}\frac{\partial}{\partial r}(rA_\phi) + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\frac{\partial (rA_\phi)}{\partial z} = 0 \tag{2.61}$$

Note that the ϕ-component of $\vec{B}$ doesn't show up in the dot product because of cylindrical symmetry. This proves that for cylindrically symmetric systems, surfaces of constant rA_ϕ are magnetic surfaces.

2.4.2 Proof of Iso-rotation Theorem

Remember our assumptions here: we've assumed that we have a cylindrically symmetric region of electromagnetic fields where $B_\phi = 0$, that $\vec{E} \times \vec{B}$ motion dominates the drift motion and that we have magnetic surfaces which are also equipotential surfaces. We'll first prove the isorotation theorem, before proving it's corrolary.

The isorotation theorem says that all particles on a given magnetic surface isorotate, i.e. they all rotate at the same frequency. The rotation rate $\Omega = \frac{v_\phi}{r}$, so we'll want to find v_ϕ. We have $\vec{B} = \vec{\nabla} \times \vec{A}$, and we can write $\vec{A}$ as $\vec{A} = A_\phi(r,z)\hat{\phi}$, so that $B_\phi = 0$ and we have cylindrical symmetry. Because we assume that the drift velocity is dominated by the $\vec{E} \times \vec{B}$ velocity, we have $v_\phi = \frac{E}{B}$, where these are just the magnitudes since we assume our flux surface is also an equipotential surface and hence $\vec{E}$ is perpendicular to $\vec{B}$. This is the point in the derivation where we require that $B_\phi = 0$, for if B_ϕ were not 0, then a particle's parallel velocity would give it some component in the ϕ-direction, so v_ϕ would not be $\frac{E}{B}$.

Now imagine, as in figure 15, that there are two nearby magnetic surfaces A and B and two points a and b on these surfaces separated by a vector $\vec{\delta}_{ab}$. Suppose that the distance δ_{ab} is small. Here, a and b are chosen such that the vector between the two points lies in the r-z plane and is perpendicular to the local B-field. This choice will make our integration easier in a moment. In this case,

$$E = \frac{-\Delta V_{ab}}{\delta_{ab}}$$

where $\Delta V_{ab} = V_a - V_b$ is the difference in electric potential between points a and a. With this sign convention, an inwardly pointing $\vec{E}$ will give us a positive E, compatible with positive Ω based on the right hand rule. This sign convention will be important in the corrolary of the isorotation theorem. Now imagine taking the vector $\vec{\delta}_{ab}$ and rotating it around the z-axis to form a surface which resembles a bent annulus (i.e. a hollow disk). If we integrate $\vec{B} \cdot d\vec{S}$ over this surface, we get

$$\int_S \vec{B} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l} = \Delta \Big[2\pi A_\phi r \Big] \approx B 2\pi r \delta_{ab} \tag{2.62}$$

Note that this is the magnitude of $\vec{B}$, because we chose points a and b such that the vector $\vec{\delta}_{ab}$ points perpendicular to B, meaning the area integral picks out the magnitude of $\vec{B}$. Solving for B, we have

$$B = \frac{\Delta[2\pi A_\phi r]}{2\pi r \delta_{ab}}$$

Solving for Ω, we get

$$\Omega = \frac{v_\phi}{r} = \frac{E}{Br} = \frac{\frac{-\Delta V_{ab}}{\delta_{ab}}}{\frac{r\Delta[2\pi A_\phi r]}{2\pi r \delta_{ab}}} = \frac{-\Delta V_{ab}}{\Delta[A_\phi r]} \tag{2.63}$$

Since magnetic surfaces are, for cylindrically symmetric systems, constant-$(A_\phi r)$ surfaces, then the denominator is going to be the same regardless of which points on A and B we choose. Since we are assuming that A and B are equipotential surfaces, then the numerator is going to be the same regardless of which points on A and B we choose. Thus, Ω will be approximately the same for all particles near surfaces A and B, assuming of course that the distance between A and B is small. So all particles between those two surfaces isorotate!

Note that the isorotation theorem does *not* say that the rotation rate is the same for all of the particles *everywhere* in the system. The electric field might be very strong on one magnetic surface, but very weak on another magnetic surface. This means that for two different surfaces separated by a large distance, the rotation rate might not be the same on each surface.

Now it's time to prove the corollary of the isorotation theorem. Remember, the statement of the corrolary is the following: as particles drift from one surface to another, they gain in potential energy equal to exactly twice the energy lost in azimuthal drift energy, so as to climb up the potential.

We start our proof with conservation of rotational angular momentum, $p_\phi = mrv_\phi + qrA_\phi$. This is a result of Lagrange's equations in cylindrically symmetric systems.[30] Since $\Delta p_\phi = 0$ as a particle drifts between a and b as in figure 15, we have that $m\Delta[rv_\phi] = -q\Delta[rA_\phi]$. We have that $r = \frac{v_\phi}{\Omega}$. Using the constancy of Ω between surfaces A and B, we find that

$$m\Delta[rv_\phi] = \frac{1}{\Omega}(mv_{\phi,b}^2 - mv_{\phi,a}^2) = -q\Delta[rA_\phi]$$

Using equation 2.63, we have that

$$\Delta[A_\phi r] = -\frac{\Delta V_{ab}}{\Omega}$$

so

$$\frac{1}{\Omega}(mv_{\phi,b}^2 - mv_{\phi,a}^2) = \frac{q(V_a - V_b)}{\Omega}$$

We can cancel Ω from both sides.

$$(mv_{\phi,b}^2 - mv_{\phi,a}^2) = q(V_a - V_b)$$

The LHS is $2(W_{\phi,b} - W_{\phi,a}) = -2\Delta W_{\phi,ab}$ where $W_{\phi,a}$ is the azimuthal drift energy at a due to azimuthal $\vec{E} \times \vec{B}$ rotation and ΔW_{ab} is the change in W_ϕ in going from b to a. The RHS is the change in electrostatic potential energy due to our particle moving from b to a, ΔU_{ab}. This gives us

$$\Delta U_{ab} = -2\Delta W_{\phi,ab} \tag{2.64}$$

This proves the corrolary of the isorotation theorem, which says that as particles drift from one surface to another, they gain in potential energy equal to exactly twice the energy lost in azimuthal drift energy, so as to climb up the potential.

I didn't make a fuss over the minus signs while going through this derivation, but it turns out they are important in deriving the corrolary and you need to keep track of them! Here, I picked a convention for positive angular velocity Ω, and made sure all my signs were consistent with that.

[30] Lagrange and Hamilton's equations for a particle in a magnetic field are *not* covered in class or in these notes, and probably should be. Note to future Nick: If I ever go back and revise these notes make sure to add a section on Lagrange's and Hamilton's equations for a plasma. It's pretty important and basic stuff that I wish I understood.

Whether or not a particle gains or loses electrostatic potential energy as it climbs up or down the potential depends crucially on the curvature and grad-B drifts on the particle. A particle could either gain azimuthal drift energy, or lose azimuthal drift energy and hence slow it's rotation down depending on it's sign, the direction of the electron field, and the direction of the particle drifts.

Question: None of this seems consistent - how can particles slow down, if their rotation rate is just set by $\vec{E} \times \vec{B}$? Something weird is going on that I don't understand.

3 Kinetic Theory

It is only the plasma itself which does not understand how beautiful the theories are and absolutely refuses to obey them.

Hannes Alfvén

Let's forget about plasma physics for a second, and think about the field of classical fluid mechanics. Fluids, like all states of matter, are made up of individual molecules or atoms. The most fundamental assumption made in fluid mechanics[31] is called the continuum assumption. Under this assumption, we treat all quantities as continuous and well-defined at each point in space. Strictly speaking, this requires for each quantity that we set the value of that quantity at each point in space equal to the average value of that quantity over a volume large enough to contain many molecules but much smaller than the relevant macroscopic lengths of the fluid.

Wikipedia phrases this similarly: "The continuum assumption is an idealization of continuum mechanics under which fluids can be treated as continuous, even though, on a microscopic scale, they are composed of molecules. Under the continuum assumption, macroscopic (observed/measurable) properties such as density, pressure, temperature, and bulk velocity are taken to be well-defined at "infinitesimal" volume elements – small in comparison to the characteristic length scale of the system, but large in comparison to molecular length scale." Also, the Navier-Stokes equations for fluids "are based on the assumption that the fluid, at the scale of interest, is a continuum, in other words is not made up of discrete particles but rather a continuous substance." Thus, in an ideal fluid, there is a well-defined, smooth mass distribution at each point in space, $\rho(\vec{x}, t)$, as well as a well-defined, smooth field which represents the mean velocity, $\vec{u}(\vec{x}, t)$.

However, there are cases in classical fluid mechanics where the continuum assumption is not valid. Wikipedia has this to say: "Those problems for which the continuum hypothesis fails, can be solved using statistical mechanics. To determine whether or not the continuum hypothesis applies, the Knudsen number, defined as the ratio of the molecular mean free path to the characteristic length scale, is evaluated. Problems with Knudsen numbers below 0.1 can be evaluated using the continuum hypothesis, but (sic) molecular approach (statistical mechanics) can be applied for all ranges of Knudsen numbers." Well, as we showed in chapter 1, in a plasma the mean free path is significantly longer than the Debye length, which is the scale length over which a plasma is electrically neutral. In fact, for a fusion-relevant plasma with number density $n \approx 10^{20}/m^3$ and temperature 1KeV, we have a mean free path of roughly 3km, much longer than the relevant scale lengths. Thus, our Knudsen number is very large in most plasmas, which motivates us to abondon the simple approach used in fluid

[31] Really, this is the unifying assumption for all of continuum mechanics.

mechanics.[32] Instead, we will use an approach called kinetic theory.

Here is a preview of where we are going with kinetic theory. We will examine the time-evolution of particles in 6-D phase space[33], describing the evolution of particles with a function called N. We will write down an equation for the time-evolution of N, called the Klimontovich equation. Combined with the Lorentz force law and Maxwell's equations, this set of equations is exactly equivalent to a bunch of charged particles interacting through electromagnetic forces.

At this point, we go from an approach which tracks each individual particle to a smooth distribution function which tracks the density of particles in phase space. We replace N, which is a non-continuous function of delta functions in phase space (see figure 16), with a smooth, continuous function called f. To get from N to f, we average N over the ensemble corresponding to N. An ensemble is defined as all of the possible microstates corresponding to a given macrostate. *Kinetic theory involves the study of f.* The equation describing the evolution of f is called the Vlasov equation, or in other fields the Boltzmann equation. What separates the Vlasov approach from those used for simple classical fluids is that we are accounting for the distribution of velocities. We're allowing for the possibility that our plasma is not in a Maxwellian state, i.e. we are doing non-equilibrium statistical mechanics. By accounting for the distribution of velocities, we are keeping a continuum assumption but allowing for low collisionality.

Wikipedia has a lot of interesting things to say about the Boltzmann (Vlasov) equation. "The Boltzmann equation or Boltzmann transport equation (BTE) describes the statistical behaviour of a thermodynamic system not in a state of equilibrium, devised by Ludwig Boltzmann in 1872. The classic example of such a system is a fluid with temperature gradients in space causing heat to flow from hotter regions to colder ones, by the random but biased transport of the particles making up that fluid. In the modern literature the term Boltzmann equation is often used in a more general sense, referring to any kinetic equation that describes the change of a macroscopic quantity in a thermodynamic system, such as energy, charge or particle number. The equation arises not by analyzing the individual positions and momenta of each particle in the fluid but rather by considering a probability distribution for the position and momentum of a typical particle—that is, the probability that the particle occupies a given very small region of space (mathematically written $\mathrm{d}^3\vec{x}$ where d means "differential") centered at the position $\mathbf{x}$, and has momentum nearly equal to a given momentum vector $\mathbf{p}$ (thus occupying a very small region of momentum space $\mathrm{d}^3\mathbf{p}$), at an instant of time. The Boltzmann equation can be used to determine how physical quantities change, such as heat energy and momentum, when a fluid is in transport. One may also derive other properties characteristic to fluids such as viscosity, thermal conductivity, and electrical conductivity (by treating the charge carriers in a material as a gas). See also convection-diffusion equation.

[32] Actually, the picture is more complex than this. In some plasmas, a fluid approximation is justified. When a fluid approximation is justified will be explained in chapter 4.

[33] Phase space simply means that each particle is labeled by it's 3 spatial components and 3 velocity components, if you remember your undergraduate classical mechanics course.

The equation is a nonlinear integro-differential equation, and the unknown function in the equation is a probability density function in six-dimensional space of a particle position and momentum. The problem of existence and uniqueness of solutions is still not fully resolved, but some recent results are quite promising." Wikipedia: "Since much of the challenge in solving the Boltzmann equation originates with the complex collision term, attempts have been made to "model" and simplify the collision term. The best known model equation is due to Bhatnagar, Gross and Krook. The assumption in the BGK approximation is that the effect of molecular collisions is to force a non-equilibrium distribution function at a point in physical space back to a Maxwellian equilibrium distribution function and that the rate at which this occurs is proportional to the molecular collision frequency. The Boltzmann collision operator is therefore modified to the BGK form: $(\frac{\partial f}{\partial t})_{coll} = \nu(f_m - f)$." Wikipedia: "Exact solutions to the Boltzmann equations have been proven to exist in some cases; this analytical approach provides insight, but is not generally usable in practical problems. Instead, numerical methods (including finite elements) are generally used to find approximate solutions to the various forms of the Boltzmann equation. Example applications range from hypersonic aerodynamics in rarefied gas flows to plasma flows. Close to local equilibrium, solution of the Boltzmann equation can be represented by an asymptotic expansion in powers of Knudsen number (the Chapman-Enskog expansion). The first two terms of this expansion give the Euler equations and the Navier-Stokes equations. The higher terms have singularities. The problem of developing mathematically the limiting processes, which lead from the atomistic view (represented by Boltzmann's equation) to the laws of motion of continua, is an important part of Hilbert's sixth problem." Enough Wikipedia, let's move forward.

In a classical fluid the velocity distribution function is replaced by 3 components representing the mean velocity (the mean velocity vector $\vec{u}(\vec{x}, t)$) of the velocity distribution at each point in space and time and nine components of the pressure tensor (actually six, since it's an antisymmetric tensor), which are found by averaging over the microscopic velocity distribution. By averaging over the velocity distribution, we remove potentially important information about the plasma (by going from infinity degrees of freedom regarding the velocity distribution to 1 (n) + 3 $(\vec{u})$ + 6 $(\overleftrightarrow{P})$ degrees of freedom), and limit ourselves to the range of behaviors we can study.

The Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f + \vec{a} \cdot \vec{\nabla}_v f = C(f) \tag{3.1}$$

where $C(f)$ is the collision operator, representing the effects of collisions between particles. First, we will ensemble-average the Klimontovich equation to get the Vlasov equation. Next, we will examine some of the properties of the Vlasov equation and examples of the collision operator.

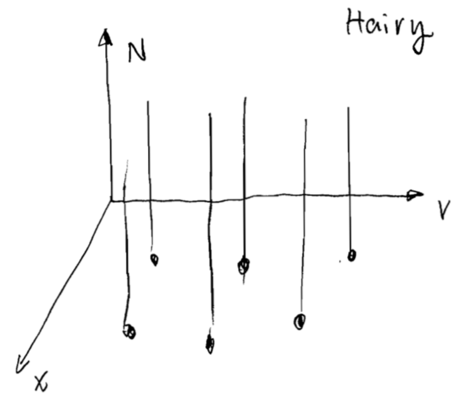

Figure 16: Visualization of N. Each delta function represents the trajectory of a particle in phase space.

3.1 Klimantovich Equation

Suppose we have N_0 particles in some region of space. Suppose $N(\vec{x}, \vec{v}, t)$ describes the evolution of those N_0 particles in phase space. Thus,

$$N(\vec{x}, \vec{v}, t) = \sum_{i=1}^{N_0} \delta^{(3)}(\vec{x} - \vec{x}_i(t))\delta^{(3)}(\vec{v} - \vec{v}_i(t)) \tag{3.2}$$

where $\vec{x}_i(t)$ and $\vec{v}_i(t)$ represent the position and velocity of the ith particle. Note that the units of N are $\frac{1}{L^3V^3}$.

Taking the partial derivative with respect to time of N and using the chain rule we get

$$\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial t} = \sum_{i=1}^{N_0} \frac{\partial \vec{x}_i}{\partial t} \cdot \frac{\partial N}{\partial \vec{x}_i} + \frac{\partial \vec{v}_i}{\partial t} \cdot \frac{\partial N}{\partial \vec{v}_i} \tag{3.3}$$

Using our definition for N (equation 3.2), we get

$$\frac{\partial N}{\partial \vec{x}_i} = \frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}_i}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) = -\frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) \tag{3.4}$$

Similarly,

$$\frac{\partial N}{\partial \vec{v}_i} = -\delta^{(3)}(\vec{x} - \vec{x}_i(t))\frac{\partial \delta^{(3)}(\vec{v} - \vec{v}_i(t))}{\partial \vec{v}} \tag{3.5}$$

In equation 3.3, we can replace $\frac{\partial \vec{x}_i}{\partial t}$ with $\vec{v}_i$, and $\frac{\partial \vec{v}_i}{\partial t}$ with $\vec{a}_i$. Plugging in 3.4 and 3.5 to 3.3, we get

$$\frac{\partial N}{\partial t} = -\sum_{i=1}^{N_0} \vec{v}_i \cdot \frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) + \vec{a}_i \cdot \delta^{(3)}(\vec{x} - \vec{x}_i(t))\frac{\partial \delta^{(3)}(\vec{v} - \vec{v}_i(t))}{\partial \vec{v}} \tag{3.6}$$

We pull the gradients out of the equation first. This is a legal move, because $\vec{v}_i$ and $\vec{a}_i$ are coordinates representing the position of a single particle in phase space, and thus commute just fine with the derivatives in equation 3.6.

After we do this, we can simplify the $\vec{v}_i$ and $\vec{a}_i$. Because of the delta functions, $\vec{v}_i(t)$ will become $\vec{v}$ and $\vec{a}_i(t)$ will become $\vec{a}$. Making these replacements we can next pull the dot products out of the equation and replace the delta functions with N.[34] Moving everything to the left hand side, we have the Klimontovich equation

$$\frac{\partial N}{\partial t} + \vec{\nabla}_x \cdot (\vec{v}N) + \vec{\nabla}_v \cdot (\vec{a}N) = 0 \tag{3.7}$$

Physically, this equation actually has a fairly simple meaning. It comes from conservation of particles, in the same way that the continuity equation comes from the conservation of charge. Geometrically, the Klimontovich equation is equivalent to the idea that the number of particles leaving a region in phase space is the number of particles flowing across the border of that region in phase space. Figure 17 shows one such region in phase space. Mathematically, this is

$$\frac{\partial}{\partial t}\left[\int_V N(\vec{x},\vec{v},t)d^3\vec{x}d^3\vec{v}\right] = -\int_S N\vec{v}_6 \cdot d\vec{A} = -\int_V \vec{\nabla}_6 \cdot (N\vec{v}_6)d^3\vec{x}d^3\vec{v} \tag{3.8}$$

where the 6 represents the 6 dimension of phase space, V is a volume of interest and S is the surface of that volume. Moving the right hand side over to the left gives the Klimontovich equation, as promised.

Since $\vec{x}$ and $\vec{v}$ are independent variables, $\vec{\nabla}_x \cdot (\vec{v}N) = \vec{v} \cdot \vec{\nabla}_x N$. If $\vec{\nabla}_v \cdot \vec{a} = 0$, which is sometimes true, we can therefore write the Klimontovich equation as

$$\frac{\partial N}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N + \vec{a} \cdot \vec{\nabla}_v N = 0 \tag{3.9}$$

Perhaps the most important acceleration example to consider is the Lorentz force, $\vec{a} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$. Here, $\vec{\nabla}_v \vec{a} = 0$. We can prove this as follows: $\vec{\nabla}_v = \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} + \frac{\partial}{\partial v_z}\hat{z}$, so $\vec{\nabla}_v \cdot \vec{a} = \frac{\partial a_x}{\partial v_x} + \frac{\partial a_y}{\partial v_y} + \frac{\partial a_z}{\partial v_z}$. Now, $\frac{\partial a_i}{\partial v_i} = \frac{q}{m}(\frac{\partial E_i}{\partial v_i} + \frac{\partial}{\partial v_i}(\vec{v} \times \vec{B})_i) = 0$ where the last step is because the ith component of $\vec{v} \times \vec{B}$ does not include v_i, but rather the other two components of $\vec{v}$. Thus, equation 3.9 can be used for the Lorentz force.

One last comment before we average over the ensemble to get the Vlasov equation. Using the chain rule, the total derivative of N with respect to time, $\frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial N}{\partial \vec{x}} + \frac{\partial \vec{v}}{\partial t} \cdot \frac{\partial N}{\partial \vec{v}}$. By inspection, we see that setting this equal

[34] Why can't we apply the delta function first to equation 3.6, pull the dot products out of the equation, and then pull the gradients out of the sum next to get $\frac{\partial N}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N + \vec{a} \cdot (\vec{\nabla}_v N) = 0$? This is a bit subtle. The derivative with respect to $\vec{x}$ or $\vec{v}$ on the delta function means that there is no longer a delta function which sets all the $\vec{x}_i$ to $\vec{x}$ or the $\vec{v}_i$ to $\vec{v}$. Basically, the derivative of a delta function is not a delta function. It is only when we remove the derivative that we again have a delta function which makes $\vec{a}_i(\vec{x}_i, \vec{v}_i, t) = \vec{a}(\vec{x}, \vec{v}, t)$. So we cannot change the order we perform these operations.

Figure 17: 3-D Visualization of the 6-D volume V (labeled C) in phase space which we consider.

to 0 gives us equation 3.9. What does this imply? Well, you might notice that $\frac{dN}{dt}$ is the same as the convective derivative in 6 dimensions. Thus, the total derivative follows the motion of a plasma particle in phase space. Thus, the phase space density is constant following a particles trajectory. Therefore, if $\vec{\nabla}_v \vec{a} = 0$, then $\frac{dN}{dt} = 0$ and we can say that the phase space density is constant as individual elements of N move around in phase space. While this doesn't have much consequence in the delta-function representation of N, it will be more meaningful and less obvious when we replace N with f, a smooth function.

We said earlier that the Klimontovich equation combined with the Lorentz Force Law and Maxwell's equations is exactly equivalent to a number of charged particles interacting through electromagnetic forces. Let us see now how that works for a fully-ionized plasma.

Because our plasma is fully-ionized, we know that for each species of plasma particles σ with charge q and mass m, the acceleration is $\vec{a} = \frac{q_\sigma}{m_\sigma}(\vec{E}_m + \vec{v} \times \vec{B}_m)$. Ignoring gravity, this is the only other force which can act outside of the nucleus. In this form, we label $\vec{E}_m$ and $\vec{B}_m$ with the subscript m (which stands for microscopic) to represent the fact that on a microscopic level, $\vec{E}$ and $\vec{B}$ fluctuate significantly from place to place. Thus, our Klimontovich equation becomes

$$\frac{\partial N_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E}_m + \vec{v} \times \vec{B}_m) \cdot \vec{\nabla}_v N_\sigma = 0 \tag{3.10}$$

Now, in order to solve this partial differential equation, we obviously need an initial condition and boundary conditions on N_σ. However, we also need to know $\vec{E}_m$ and $\vec{B}_m$ and how they evolve in time. Thus, we need microscopic formulations of Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E}_m = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int d^3\vec{v} N_\sigma(\vec{x}, \vec{v}, t) \tag{3.11}$$

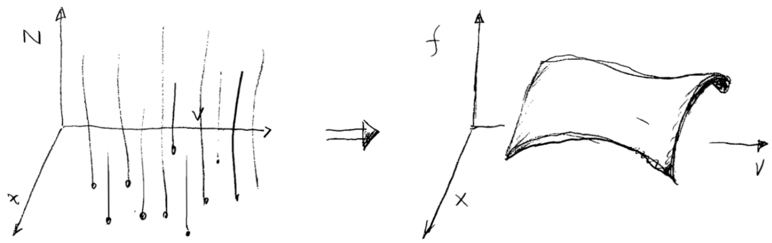

Figure 18: Visualization of ensemble-averaging N to get f.

$$\vec{\nabla} \cdot \vec{B}_m = 0 \tag{3.12}$$

$$\vec{\nabla} \times \vec{E}_m = -\frac{d\vec{B}_m}{dt} \tag{3.13}$$

$$\vec{\nabla} \times \vec{B}_m = \mu_0 \sum_\sigma q_\sigma \int d^3\vec{v}(\vec{v}N_\sigma(\vec{x},\vec{v},t)) + \mu_0\epsilon_0 \frac{\partial \vec{E}_m}{\partial t} \tag{3.14}$$

This system of equations is equivalent to a number of charged particles interacting through electromagnetic forces.

3.2 Vlasov Equation

Now, instead of tracking each individual particle, we want to replace N with a smooth function f accounting for the number of particles at a given position with a given velocity. An example of this is shown in Figure 18. We also want to replace our microscopically varying electric and magnetic fields $\vec{E}_m$ and $\vec{B}_m$ with smooth vector fields $\vec{E}$ and $\vec{B}$. Let us define, where brackets represent an average over the ensembles, $f(\vec{x},\vec{v},t) \equiv \langle N(\vec{x},\vec{v},t)\rangle$, $\vec{B} \equiv \langle \vec{B}_m \rangle$, $\vec{E} \equiv \langle \vec{E}_m \rangle$. Let us also define $\delta N \equiv f - N$, $\delta\vec{E} \equiv \vec{E}_m - \vec{E}$, and $\delta\vec{B} \equiv \vec{B}_m - \vec{B}$.[35]

When we perform these averages, Maxwell's equations are the same as our normal Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int d^3\vec{v} f_\sigma(\vec{x},\vec{v},t) \tag{3.15}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.16}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.17}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \sum_\sigma q_\sigma \int d^3\vec{v} f_\sigma(\vec{x},\vec{v},t) + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{3.18}$$

[35]Note that these definitions are slightly different from the definitions used by Hong in his notes.

These are the same as Maxwell's equations before ensemble averaging, just with m removed. Why is this the case? Well, it comes from the fact that Maxwell's equations are linear. We can set $\vec{B}_m = \vec{B} + \delta\vec{B}$, do the same with $\vec{E}_m$, and when we take the ensemble average, subtract off the microscopic portions of Maxwell's equations from both sides without making any approximations.

Ensemble-averaging the Klimontovich equation, we get the Vlasov-Maxwell equation. Starting with the Klimontovich equation, equation 3.10, we ensemble average. Using $\langle \delta N \rangle = 0$, we have

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_\sigma + \Big\langle \frac{q_\sigma}{m_\sigma}((\vec{E} - \delta\vec{E}) + \vec{v} \times (\vec{B} - \delta\vec{E})) \cdot \vec{\nabla}_v (f_\sigma - \delta N) \Big\rangle = 0$$

This would be the same as our Klimontovich equation with f replacing N, except we have an additional non-linear correlation term which survives the ensemble-averaging which comes to the right side.

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma = -\Big\langle \frac{q_\sigma}{m_\sigma}(\delta\vec{E} + \vec{v} \times \delta\vec{B}) \cdot \vec{\nabla}_v \delta N_\sigma \Big\rangle \quad (3.19)$$

Note that terms first-order in the fluctuations average to 0 by definition, but terms second-order in fluctuations, i.e. correlation terms, do not automatically average to 0. The correlation term on the right hand side accounts for the effect of particle-particle interactions, i.e. collisions. How can we see this intuitively? Without being rigorous at all, we can see that the ensemble-average of $q_\sigma \delta N \delta\vec{E}$ will not go to 0, as where there is some δN there will also be a correlated $\delta\vec{E}$ because the charged particle will create an electric field. Because this term is non-linear, these correlations won't average to zero.

As we know from Wikipedia[36], the hardest part about solving the Vlasov-Maxwell equation is the complex collision term. In most applications, we simply set this term to be equal to some simplified collision operator, $C(f)$, or 0. Solving the Vlasov-Maxwell equation in practice requires choosing a good collision operator which approximates well the right-hand side of equation 3.19, while being practically solvable (by a computer, at least).

Ben (Israeli), while studying for the final for this course, asked a great question in our first-year plasma physics group chat: "What is the distinction between the Klimontovich and collisionless Boltzmann equations?" There is a lot we could say in response. See if you can answer it for yourself before reading my response. Here is what I would say: Firstly, the Klimontovich equation is *not* collisionless. Particles can collide, and those collisions are well-accounted for in the microscopic fluctuations of the electric and magnetic fields. Ignoring that fact, the main distinction is with the ensemble-averaging. The Klimontovich equation describes the evolution of a *single* microstate. The Vlasov equation averages over all possible microstates. Thus f is a smooth function and the collisions, instead of being between discrete particles, represent the averaged effect of collisions over all the microstates consistent with that particular macrostate.

[36] Yale cites Wikipedia. I guess Princeton does too.

Do the Klimontovich and Vlasov equations contain the same information? Almost, but not exactly. An analogy would be whether or not quantum mechanics and statistical mechanics contain the same information about the macroscopic state of a classical ideal gas. We could use quantum mechanics to solve for the wave function of the molecules. Or we could forget about the microscopic state, and take the statistical mechanics approach. Either way, we get more or less the same results when we look at our gas on a macroscopic level, but we have more information about the microscopic state with quantum mechanics (or with the Klimontovich equation). Where the two descriptions of a plasma *don't* contain the same information is whenever we have a chaotic system, like turbulence. Chaos is extreme sensitivity to initial conditions. If we change the microstate of some configuration, while maintaining the same macrostate, this extremely small change in the initial conditions (even on a microscopic level!) can change the macrostate in future times. This is the whole business of a butterfly flapping its wings in Texas and changing the weather patterns in China. If we have a microstate in a chaotic system, and use the Klimontovich equation to simulate the evolution of that system in time, we will get one answer. If we take the macrostate compatible with that microstate, and use the Vlasov equation to evolution of that system in time, we will get another answer. This is because of the extreme sensitivity to initial conditions in chaotic systems.

At this point, you might be asking yourself if we can even use the Vlasov equation to study chaotic systems like turbulence, if they give the wrong answers. The answers are both yes and no. No, we can't, because to get the correct result in a chaotic system requires complete knowledge of the initial conditions. *However*, the answer is also *yes*, we can use the Vlasov equation to study turbulence (and we do!) because when we study turbulence we are interested in the *average* fluctuation wavenumber/frequency/intensity/etc, not in the specific behavior of any particular turbulent eddy at some point in space. When we look at a turbulence simulation, we are seeing one of the many possibilities of what might happen given the initial (macro)state of the system.

3.2.1 Some facts about f

- The total number of particles of species σ in our plasma, N_σ, is $\int d^3\vec{x}d^3\vec{v}f_\sigma(\vec{x},\vec{v},t)$
- The particle density of species σ is $n_\sigma(\vec{x},t) = \int f_\sigma(\vec{x},\vec{v},t)d^3\vec{v}$. In words, the particle density, also called number density, at $\vec{x}$ is equal to the integral of f over all possible velocities.
- The mean velocity of the species σ, $\vec{u}_\sigma = \frac{1}{n_\sigma}\int \vec{v}f_\sigma(\vec{x},\vec{v},t)d^3\vec{v}$. In words, the mean velocity $\vec{u}$ is the first moment of f with respect to velocity, divided by the density.[37]

[37] Throughout these notes, I've chosen u to represent the mean fluid velocity, while I've used the symbol v to represent a whole bunch of things: the thermal velocity $V_{T\sigma}$, the parallel and perpendicular velocities of a single particle $v_\parallel$ and $v_\perp$, the phase space velocity $\vec{v}$, etc. The distinction is important, because u refers to the averaged velocity over an entire plasma, while v refers to the velocity of a single particle or a group of particles, but not the average velocity

- The plasma energy per volume in the particle kinetic energy for species σ is $\int d^3\vec{v}\frac{1}{2}m_\sigma v^2 f_\sigma$. This accounts for both the thermal energy of the plasma as well as the kinetic energy of the mean plasma flow.

3.2.2 Properties of Collisionless Vlasov-Maxwell Equations

Suppose that $C(f) = 0$, such that our plasma is collisionless. We expect our Vlasov-Maxwell equation, $\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{\nabla} f = 0$, to have a few basic properties, such as particle, energy, and momentum conservation.

Particle conservation comes about automatically because f is conserved as we follow a section of f around in phase space. $\frac{df}{dt}$ is the convective derivative in phase space, and since it equals zero then the value of f is constant as a particle travels around in phase space. This means that the total integral of f over velocity space and real space, which represents the total number of particles, doesn't change with time.

The total energy (plasma plus electromagnetic) is conserved through the evolution of f under the Vlasov-Maxwell equation. The total energy

$$\mathcal{E} = \frac{1}{2}\int d^3\vec{x}\bigg[\epsilon_0 E^2 + \frac{B^2}{\mu_0} + \sum_\sigma \int d^3\vec{v} m_\sigma v^2 f_\sigma\bigg] \tag{3.20}$$

is constant in time - we prove this on a homework assignment. The same is true with total momentum,

$$\vec{\mathcal{P}} = \int d^3\vec{x}\bigg[\epsilon_0 \vec{E}\times\vec{B} + \sum_\sigma \int d^3\vec{v} m_\sigma \vec{v} f_\sigma\bigg] \tag{3.21}$$

This is also proved in the homework. Now, it turns out that if $c_i(\vec{x}, \vec{v}, t, \vec{B}(\vec{x}, t), \vec{E}(\vec{x}, t))$ is a constant of motion for a single particle, such that

$$\frac{dc_i}{dt} = \frac{\partial c_i}{\partial t} + \vec{v}\cdot\vec{\nabla} c_i + \frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{\nabla}_v c_i = 0 \tag{3.22}$$

then any function $f(c_1, c_2, ...,)$ which is a function of c_i's is a solution of the collisionless Vlasov equation. This is easily shown, as

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial c_i}\frac{dc_i}{dt} = 0 \tag{3.23}$$

Mathematically, this statement is pretty obviously true. But physically, it isn't exactly obvious what this means. Let me try to explain. The collisionless Vlasov equation tells us how the distribution of particles evolves in phase space. The convective derivative of f, $\frac{df}{dt}$, equals zero in the collisionless regime. This means that the elements of f follow single-particle trajectories in phase space. Now, if $\vec{B}(x)$ and $\vec{E}(x)$ are not functions of t, then there are a few constants of motion

of the plasma as a whole. So whenever you see u, I'm talking about a fluid velocity, and whenever you see v I'm talking about something else.

that single-particle trajectories are required to be consistent with (for example, energy). That means that if our distribution function is *only* a function of these conserved quantities (and not, for example, $\vec{x}$ and $\vec{v}$ unless they are also part of a conserved quantity), then our distribution function is consistent with particles following collisionless single-particle trajectories. Let's do a few examples to get a sense of this. For all of these examples, we will assume $\vec{B} = 0$ so we can understand the physics without complicating the analysis too much.

- A spatially uniform f where $\vec{E} = 0$, with some arbitrary velocity distribution. Here f *isn't* a function of constants of motion, but does satisfy the Vlasov equation. Of course, this example doesn't illustrate the whole nonsense with the c_i's, I just wanted to make sure we recognized this.

- $f(\vec{x}, \vec{v}, t) = n(\vec{x} - \vec{v}t)g(\vec{v})$, where $g(\vec{v})$ is some arbitrary function of $\vec{v}$, and $\vec{E} = 0$. This *does* satisfy the Vlasov equation, because $\vec{x} - \vec{v}t$ is a constant of motion for a single-particle in a zero-field plasma.[38]

- Suppose $\vec{E} = E_0\hat{x}$ and $f(\vec{x}, \vec{v}, t) = f(\frac{1}{2}mv^2 + qE_0x)$. Yes, as weird as this looks, it solves the Vlasov-Maxwell equation.

- Suppose $\vec{E} = 0$ and $f = Ce^{-(x^2+y^2+z^2)/a^2}e^{-v^4/b^4}$ where C is some normalization constant and a and b are arbitrary. This doesn't satisfy the Vlasov-Maxwell equation, the reason being that the particles with non-zero velocity will move. This just assumes the density stays constant, which isn't true. Instead, the particles with non-zero velocity will move in straight lines. If $f = Ce^{-((x-v_xt)^2+(y-v_yt)^2+(z-v_zt)^2)/a^2}e^{-v^4/b^4}$, then the Vlasov-Maxwell equation is satisfied.[39]

- If $\vec{E} = E_0\hat{x}$ then $f = n(x)\delta(v_x - \frac{qE_0}{m}t, v_y, v_z)$ is a solution to the Vlasov equation. What this represents physically is the situation where all of the particles start at arbitrary positions with zero velocity, and are accelerated due to an electric field in the x-direction.

3.2.3 Entropy of a distribution function

This isn't covered in class, but I wanted to introduce the concept of entropy in plasmas briefly. The definition of the entropy of a distribution function f is

$$S = -\int d^3\vec{v}\int d^3\vec{x} f(\vec{x}, \vec{v})\ln f(\vec{x}, \vec{v}) \tag{3.24}$$

Why is this true? Forget about distribution functions for a second, and imagine we have N distinguishable pegs we can put into N holes, so that exactly 1 peg goes into each hole. Since the pegs are distinguishable, we have $N!$ ways of

[38] Technically, our f isn't only a function of constants of motion here, because of the $g(\vec{v})$ factor. The reason it still solves the Vlasov equation here is that there aren't any fields, so the gradient with respect to $\vec{v}$ doesn't matter.

[39] Again, because the fields are arbitrary then the e^{-v^4} factor is allowed.

ordering the N pegs into the N holes (N options for the first peg, $N-1$ options for the second peg, etc until there is 1 option for the last peg).

Now suppose that we group together the N holes into M groups, such that the ith group has $f(i)$ holes in that group, and $\sum_i^M f(i) = N$. Now suppose we want to place the pegs in the holes again, such that we care about the ordering of the pegs within each group of holes and any peg can go into any hole. In that case, then the number of ways of arranging the pegs into these M groups must also equal $N!$ Why? If any peg can be in any group and the order matters within the group, then there are N ways to place the first peg, $N-1$ ways to place the second, etc etc. Therefore the number of ways of arranging these N pegs is the same as if we had no groups in the first place.

Now suppose we don't care about the internal arrangement of the pegs within each of the M groups, but we do care about which pegs go into which group. How many ways can we arrange the pegs into these M groups such that we don't care about the internal arrangement of the pegs within each group? Well, we don't know it yet, but let us call this result C. We know that since there are $f(i)!$ ways of arranging the pegs within group i, then the number of ways of arranging the pegs into these M groups such that we do care about the ordering of the pegs within each group is

$$C \times f(1)! \times f(2)! \times ... \times f(M)! \tag{3.25}$$

But from our previous paragraph, we know this equals $N!$ Solving for C, we get

$$C = \frac{N!}{f(1)! \times f(2)! \times ... \times f(M)!} \tag{3.26}$$

How does this relate to entropy? This next sentence is important, so buckle up and pay attention. C is the number of microscopic states corresponding to the macrostate given by the $f(i)$'s. So using our definition of entropy from statistical mechanics, $S = k_B \ln\Omega$, we have

$$\begin{aligned} S &= k_B \ln C \\ &= \ln\left(\frac{N!}{f(1)! \times f(2)! \times ... \times f(M)!}\right) \\ &= \ln N! - \ln f(1)! - \ln f(2)! - ... - \ln f(M)! \end{aligned} \tag{3.27}$$

Now suppose $f(i) \gg 1$ for all i, such that we can use Stirling's formula $\ln N! \approx N\ln N - N$ to simplify the entropy. Using $\sum_i^M f(i) = N$, we cancel the N term to get

$$S = N\ln N - \sum_i^M f(i)\ln f(i) \tag{3.28}$$

Since N is a constant, we can drop it from the entropy to get

$$S = -\sum_i^M f(i)\ln f(i) \tag{3.29}$$

What does this have to do with distribution functions? Well, suppose we have a known distribution function $f(\vec{x}, \vec{v}, t)$. This is our macrostate. Each point $(\vec{x}, \vec{v})$ in phase space can be thought of as a group of holes, and each particle can be thought of as a peg. We have a known number of particles in each point in phase space, analogous to having a known number of holes in each group. The microstate is the particular arrangement of particles (pegs) which gives us our macrostate $f(\vec{x}, \vec{v}, t)$ (the number of holes in each group of holes).

If we have a known number of pegs $f(i)$ in each group of holes, then the entropy is given by equation 3.29. Therefore, if we have a known number of particles in each point in phase space, $f(\vec{x}, \vec{v})d^3\vec{x}d^3\vec{v}$, then the entropy (turning the sum over i into an integral over $\vec{x}$ and $\vec{v}$) is

$$S = -\int d^3\vec{v} \int d^3\vec{x} f(\vec{x}, \vec{v}) \ln f(\vec{x}, \vec{v}) \tag{3.30}$$

the same as equation 3.24. This is the entropy of a distribution function for a plasma. If I wanted to take this a little further,[40] I could show that the maximum entropy distribution function is a Maxwellian.

3.3 Collisions in the Vlasov Description

3.3.1 Heuristic Estimate of Collision Operator

When we go from the Klimontovich equation to the Vlasov equation by ensemble-averaging, we get a collision operator term

$$C(f_\sigma) = -\Big\langle \frac{q_\sigma}{m_\sigma}(\delta\vec{E} + \vec{v} \times \delta\vec{B}) \cdot \vec{\nabla}_v \delta N_\sigma \Big\rangle \tag{3.31}$$

In general, we can't solve exactly for this term. For now, let's try to get a heuristic estimate of what this might be. Let's look only at the $\delta\vec{E} \cdot \vec{\nabla}_v \delta N_\sigma$ term and ignore the term with the magnetic field fluctuations, for simplicity. We can write N heuristically as $\frac{\overline{N}}{L^3 V_T^3}$, where $\overline{N}$ is the total number of particles in the system and L is the length scale of the system. From the law of large numbers, $\delta\overline{N} \sim \sqrt{\overline{N}}$ on average for a given microstate. From Gauss's equation $\vec{\nabla} \cdot \delta\vec{E} = \sum_\sigma \frac{q_\sigma}{\epsilon_0} \int \delta N_\sigma d^3\vec{v}$. $\vec{\nabla} \cdot \delta\vec{E} \sim \delta E/\lambda_D$, because the distance scale over which $\vec{E}$ changes is the Debye length. Continuing with our hand-wavy algebra by solving for δE, we have $\delta E \sim \frac{\lambda_D q V_T^3 \delta N_\sigma}{\epsilon_0} \sim \frac{q \lambda_D \delta\overline{N}}{\epsilon_0 L^3}$. We can plug in these estimates to equation 3.31 to get a heuristic estimate of $C(f)$.

$$C(f) \sim \delta E \frac{q\delta\overline{N}}{m V_T^4 L^3} \sim \frac{q^2 \lambda_D \delta\overline{N}^2}{\epsilon_0 m V_T^4 L^6} \sim \frac{q^2 \overline{N}}{\epsilon_0 m \omega_P L^3} \frac{1}{L^3 V_T^3} \sim \frac{\omega_P}{L^3 V_T^3} \tag{3.32}$$

[40] Bellan does this in his textbook. Note to future Nick: this would be a good thing to add if I update these in more depth. I should also add a section on the method of characteristics, on the basic plasma sheath, on langmuir probes, and on hamiltonian formulism of single-particle motion.

where $\omega_P^2 = \frac{q^2 n}{\epsilon_0 m}$, $\frac{V_T}{\lambda_D} = \omega_P$ and $\frac{\overline{N}}{L^3} = n$ have been used. We also have that

$$\frac{\partial N}{\partial t} \sim \omega_P N \sim \frac{\omega_P \overline{N}}{L^3 V_T^3} \tag{3.33}$$

We've made this estimate using Gauss's law. Remember, though, that we only feel the effects of an electric field up to roughly a Debye length. This means that the system we consider should in fact be a Debye sphere. This means that $\overline{N} \sim \Lambda$, the number of particles in a Debye sphere, and $L \sim \lambda_D$. This means that

$$\frac{C(f)}{\frac{\partial f}{\partial t}} \sim \Lambda^{-1} \tag{3.34}$$

This is a cool conclusion! If the number of particles in a Debye sphere is much greater than 1, then the collision operator is negligible relative to the terms on the LHS of the Vlasov equation, and we can use the collisionless Vlasov equation. Similarly, if the number of particles in a Debye sphere is less than 1, then the collision operator dominates over the LHS terms of the Vlasov equation.

3.3.2 Strongly and Weakly Coupled Plasmas

We're finally ready to understand why a plasma is defined to be where the number of particles in a Debye sphere, $\Lambda = n\lambda_D^3$, is much greater than 1. Actually, this is really quite a bad definition for a plasma: a plasma is an ionizezd gas. A plasma where $\Lambda \gg 1$ is a *weakly coupled plasma*. The opposite limit is a *strongly coupled plasma*. What I've defined to be a plasma is really a weakly coupled plasma, and I haven't mentioned strongly coupled plasmas.

If $\Lambda \gg 1$, then our weakly-coupled plasma has the following properties:

- The collision operator in the Vlasov equation, $C(f)$, is small relative to the terms on the LHS of the Vlasov equation. We can therefore use the Vlasov equation to understand our plasma, even if we can't calculate $C(f)$ exactly.
- Large-angle collisions are less important than small-angle collisions.
- The plasma is net neutral over scales larger than a Debye length.
- The kinetic energy per particle, $\frac{1}{2}k_B T_\sigma$, is much greater than the potential energy between any two particles, $\frac{q_\sigma^2}{4\pi\epsilon_0\lambda_D}$.[41]
- There is a definite ordering of scale lengths (see section 1.5), $\lambda_{mfp} \gg \lambda_D \gg n^{-1/3} \gg b$.

On the other hand, if $\Lambda \ll 1$, then our strongly-coupled plasma has the following properties:

[41] We haven't done this calculation yet, but it's a fairly straightforward calculation. We can write $\frac{KE}{PE} \sim \frac{4\pi\epsilon_0 k_B T_\sigma \lambda_D}{q_\sigma^2} \sim n_\sigma \lambda_D^3 = \Lambda \gg 1$.

- The collision operator in the Vlasov equation dominates relative to the terms on the LHS. This means that we can't use the Vlasov equation to solve for the behavior of our plasma, because we don't have an exact expression for the form of the operator. We have to solve the Klimontovich equation or some other single-particle equation instead.
- Large-angle collisions between particles dominate.
- The plasma is non-neutral.
- The electric potential energy per particle is much larger than the kinetic energy.
- There is a definite ordering of scale length, opposite to that of the weakly-coupled plasmas.

3.3.3 Properties of Collision Operator

Let's discuss the collision operator in more depth. When we have more than one plasma species, we need to account for the possibility of collisions between different plasma species as well as collisions between the same species. Thus, we should instead define our collision operator for collisions affecting species σ as the sum of collisions between all plasma species, $C(f_\sigma) = \sum_\alpha C(f_\sigma, f_\alpha)$. If we were being rigorous, we should technically account for the possibility of collisions between 3 or more different particles from potentially different species. However, it is not a bad approximation (and much simpler) to consider only consider binary collisions (i.e. collisions between two particles), which we will do.

There are a couple properties we would hope that our collision operator might have:

- Particle conservation: For all σ and α,

$$\int C(f_\sigma, f_\alpha) d^3\vec{v} = 0$$

  Physically, this means that collisions between particles of species σ and α at some position $\vec{x}$ only change the velocity of the particles of species σ at $\vec{x}$, and maintain the number of particles of species σ at $\vec{x}$.

- Momentum conservation:

$$\sum_{\sigma,\alpha} \int m_\sigma \vec{v} C(f_\sigma, f_\alpha) d^3\vec{v} = 0$$

  Physically, this means that while particles can exchange momentum between different species, the total momentum at each point $\vec{x}$ remains constant. In particular, species cannot impart momentum to themselves, which implies that

$$\sum_{\sigma,\sigma} \int m_\sigma \vec{v} C(f_\sigma, f_\sigma) d^3\vec{v} = 0$$

- Energy conservation:

$$\sum_{\sigma,\alpha}\int \frac{m_\sigma v^2}{2} C(f_\sigma, f_\alpha) d^3\vec{v} = 0$$

  Physically, this means that while particles can exchange energy between different species, the total energy at each point $\vec{x}$ remains constant. If particles were to fuse, releasing atomic energy, this would no longer be strictly true. In particular, species cannot give energy to themselves, which implies that

$$\sum_{\sigma,\sigma}\int \frac{m_\sigma v^2}{2} C(f_\sigma, f_\sigma) d^3\vec{v} = 0$$

- We would hope that $C(f_\sigma, f_\alpha)$ be bilinear in f, meaning for some constants a and b, we have $C(af_\sigma, bf_\alpha) = abC(f_\sigma, f_\alpha)$ Physically, this means that the frequency of collisions at each point in phase space is proportional to the number of particles at that point in phase space. For example, if we double the number density n in some region, then for electron-electron collisions we have $C(2f_e, 2f_e) = 4C(f_e, f_e)$. The physical explanation for why the electron-electron collision operator will be four times as large is because the frequency of collisions is twice as high, and the number of particles is twice as high, so there is 4 times as much of a change in f due to collisions as when our density was halved. While this bilinearity assumption is strictly not necessary, it certainly is a reasonable assumption.

- In general, we want C to be local, meaning that C at $\vec{x} = \vec{x}_0$ depends only on $f_\sigma(\vec{x}_0, \vec{v}, t)$ and $f_\alpha(\vec{x}_0, \vec{v}, t)$, and not on any other $\vec{x}$. It also means that C doesn't depend on any derivatives of f with respect to position. C can, however, depend on derivatives with respect to velocity.

- We want, as $t \to \infty$, f to approach a Maxwellian velocity distribution. Otherwise, our collision operator is not bringing us to the maximum entropy state.

- $C(f)$ should ensure that $f \geq 0$. f cannot be negative.

3.3.4 Examples of Collision Operators

Here, we investigate some of the collisions operators introduced in class. One such operator is the Krook collision operator,

$$C(f) = -\nu(f - f_m) \tag{3.35}$$

This is a simple way of writing the collision operator, and it clearly gives a Maxwellian distribution as t goes to infinity. Whether it has the required conservation properties was given in class as an exercise for students to do at home. Well, we are students at home, so now it is time to complete that exercise! We can see[42] that if we choose f_m such that the density of the Maxwellian distribution (the factor in front of the Gaussian) at each point in space corresponds to the density of f, then this collision operator conserves particles. If the Maxwellian is a drifting Maxwellian at each point in space, such that the mean velocity of f at that point in space is the same as the drift velocity of the Maxwellian, then it conserves momentum as well. If the temperature of the Maxwellian f_m depends spatially on the local energy in f at that point, then sure, energy is conserved. However, with these three constraints on $n(\vec{x})$, $T(\vec{x})$ and $\vec{u}(\vec{x})$, our plasma will not reach a uniform Maxwellian at infinite time! Instead, it will reach a local Maxwellian at each point in space, which is not what we want! To get the correct behavior as t goes to infinity, we will have to make f_m be constant in space and time, which means our collision operator no longer conserves the quantities we want it to (at least locally).

Another operator is the collision operator $C(f) = 0$, which is true in the "mush limit". In this limit, we take $e \to 0$, $m_e \to 0$, $ne \to$ constant, $n \to \infty$, $\frac{e}{m} \to$constant, and therefore $\lambda_D \to$ constant, $V_T \to$ constant, and $\omega_p \to$ constant. In this limit, the collision frequency is much much less than the plasma frequency, and the collision operator can be ignored relative to the terms on the LHS of the Vlasov equation.

A third operator we discussed is related to diffusion in velocity space, presumably simulating the effect of some wave. This looks like

$$C(f) = \frac{\partial}{\partial v_\parallel} D(v_\parallel) \frac{\partial}{\partial v_\parallel} f \tag{3.36}$$

This can be thought of as some diffusion in parallel velocity space, and so our vlasov equation becomes analogous to the diffusion equation $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$. This operator could represent, artificially, the effect of some waves being launched in the plasma and creating diffusion of particles in velocity space.

Question: I don't understand this operator. Please explain.

3.4 Lorentz Collision Operator

A fourth collision operator we discussed was the Lorentz Collision Operator, which is so important it deserves it's own subsection. The Lorentz collision operator can be written in various forms, but only one form is covered in GPP1. This form is

$$\mathcal{L}(f) = \nu(v) \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial}{\partial \mu} f \right] \tag{3.37}$$

where $\mu = \frac{v\|}{v} = \cos\theta$, the angle parallel to the magnetic field. μ ranges from -1 to 1. Note this μ is *not* the adiabatic invariant! The frequency $\nu(v) \sim \frac{1}{v^3}$. This collision operator is not derived in GPP1, but it comes from the Focker-Planck operator (also not derived in class) and an assumption that the ions

[42] Sorry that I don't explain this in more detail. Also not sorry.

are a cold drifting population. The Lorentz collision operator represents pitch-angle (the angle with respect to the magnetic field) scattering of electrons due to collisions with ions, in a system where the azimuthal angle with respect to the magnetic field ϕ is negligible. Note that the Lorentz collision operator, like the diffusion in velocity space due to waves, is qualitatively similar to the diffusion equation. Here, the particles diffuse in velocity space due to collisions with the ions. The Lorentz collision operator is valid under the assumption that $Z_i \gg 1$, so the electron-electron collisions are negligible and only electron-ion collisions are important. It also relies on the assumption that $V_{Ti} \ll V_{Te}$, as in the derivation the ions are assumed to be a drifting delta function population. A helpful property of the Lorentz collision operator is that it is self-adjoint. This means that

$$\int d^3\vec{v} g \mathcal{L}(f) = \int d^3\vec{v} f \mathcal{L}(g) \tag{3.38}$$

We can prove this property by integrating by parts. First, we need to remember that $\mathcal{L}(f) = \nu(v)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}$ and $\int d^3\vec{v} = \int v^2 dv \int d\phi \int_{-1}^{1} d\mu$. Integrating by parts,

$$\int d^3\vec{v} g \mathcal{L}(f) = \int v^2 dv \int d\phi \int_{-1}^{1} \nu(v) g \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} d\mu =$$

$$-\int v^2 dv \int d\phi \int_{-1}^{1} \nu(v) \frac{\partial g}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} d\mu \tag{3.39}$$

The boundary term goes to zero because $\mu = \pm 1$ on the boundary, so $1-\mu^2 = 0$ making the boundary term zero. But our final expression is manifestly (clearly) self-adjoint! In other words, we would get the same result if we took the integral $\int f\mathcal{F}(g) d^3\vec{v}$, which proves the self-adjointness described by equation 3.38.

There are other nice mathematical properties of the Lorentz operator - it turns out that if P_n is the nth Legendre polynomial, then

$$\mathcal{L}(P_n(\mu)) = -n(n+1)P_n(\mu) \tag{3.40}$$

Since the Legendre polynomials are complete, we can write any f in terms of them. This means that

$$f(\mu, v, t) = \sum_n P_n(\mu) a_n(v, t)$$

Now, it turns out that the larger-n, smaller-v components of f pitch-angle scatter faster. Let's show this. If we have a spatially-homogenous, zero-field plasma, then our Vlasov equation is

$$\frac{\partial f}{\partial t} = \mathcal{L}(f)$$

Expanding f in terms of the Legendre polynomials, we have

$$\sum_n P_n(\mu)\frac{\partial a_n(v,t)}{\partial t} = \sum_n L(P_n(\mu)) a_n(v,t) = -\sum_n n(n+1)P_n(\mu) a_n(v,t)$$

$$\frac{\partial a_n}{\partial t} = -\nu(v)n(n+1)a_n$$

which has the solution

$$a_n(v,t) = a_n(v,0)e^{-\nu(v)n(n+1)t}$$

Since $\nu(v) \sim \frac{1}{v^3}$, the larger-$n$, smaller-$v$ particles pitch-angle scatter more quickly.

3.4.1 Lorentz Conductivity

What is the conductivity of a plasma? In other words, if we put some electric field $\vec{E}$ in a plasma (never mind how it got there, or the fact that a plasma tends to shield large-scale electric fields), then there should be some current $\vec{J}$, where the constant of proportionality between the two is σ, the conductivity. In solids, this is typically written $\vec{J} = \sigma\vec{E}$. While currents in plasma can arise even if there is no electric field (for example, due to the single-particle drifts or magnetization current), it is helpful to get a sense of how much current we will get for a given electric field.

The cross-field conductivity, $\sigma_\perp$, is in general different than the parallel conductivity $\sigma_\parallel$. Actually, if there is an electric field perpendicular to a magnetic field in a plasma, there is a net plasma fluid $\vec{E} \times \vec{B}$ drift. In addition, there will be some current in the direction of $\vec{E}$, which we get due to the deconfining effects of collisions. I don't really understand how cross-field currents work in a plasma, and it wasn't discussed in class, so it won't be discuss it more in these notes.[43]

Here, we will look at plasma conductivity in an unmagnetized plasma, so we can ignore all the complications of particle drifts and larmor orbits and whatnot. We'll use our favorite collision operator, the Lorentz collision operator, in a spatially-homogeneous plasma with a net electric field $\vec{E}$. Remember, the Lorentz collision operator assumes the ions are a cold, drifting population such that $f_i = n_i(\vec{x})\delta^3(\vec{v} - \vec{u}_i)$, and that the significant collisions for the electrons are with the ions. Thus, we are interested in f_e, which carry most of the current anyways. Our Vlasov-Maxwell equation for the electrons is

$$\frac{\partial f_e}{\partial t} - \frac{e}{m}\vec{E} \cdot \frac{\partial f_e}{\partial \vec{v}} = \mathcal{L}(f_e) = \nu(v)\frac{\partial}{\partial \mu}\Big[(1-\mu^2)\frac{\partial f_e}{\partial \mu}\Big] \tag{3.41}$$

Now, we assume a steady-state solution, such that $\frac{\partial}{\partial t} \to 0$. This equilibrium is an equilibrium between collisions and the electric field which is pushing particles. If the electric field is not unreasonably strong, then we expect our equilibrium distribution to be similar to a Maxwellian, with some small departure from the maximum-entropy state. Thus, we can write $f = f_m(1+g)$ where g is some

[43]Spitzer has a nice discussion of currents in a plasma. Note to future Nick: understand that.

arbitrary function and $g \ll 1$ everywhere in phase space. We'll also put $\vec{E}$ in the z-direction. So our Vlasov-Maxwell equation becomes

$$-\frac{e}{m}E_z\frac{\partial[f_m(1+g)]}{\partial v_z} = \mathcal{L}\big(f_m(1+g)\big) \tag{3.42}$$

Since g is small, it's derivatives are also small, so $\frac{\partial f_m(1+g)}{\partial v_z}$ can be approximated as $\frac{\partial f_m}{\partial v_z}$, which is $-\frac{mv_z}{k_BT}f_m$. Since $C(f_m) = 0$, then $C(f_m(1+g)) = C(f_m) + C(f_m g) = f_m C(g)$. Equation 3.42 becomes

$$\frac{e}{k_BT}E_z v_z f_m = f_m\mathcal{L}(g) \tag{3.43}$$

We can cancel the f_m and use the definition of μ, $\mu = v_\parallel/v$, to write

$$\frac{eE_z v\mu}{k_BT} = \nu(v)\frac{\partial}{\partial\mu}\Big[(1-\mu^2)\frac{\partial g}{\partial\mu}\Big] \tag{3.44}$$

Now, if we can solve this equation for g, we can figure out J. How do we figure out J? Firstly, since the electrons move much more quickly than the ions, we'll ignore the ion motion and just look at the current due to the electrons. The integral of $f_m v_z$ in the current integral goes to zero, so only the second term contributes.

$$J_z = -e\int f_m(1+g)v_z d^3\vec{v} = -e\int f_m g v_z d^3\vec{v} \tag{3.45}$$

We know how to get our current, if we can just solve for g. So let's set out to solve for g! Unfortunately, we can't do this exactly. But suppose we expand g in terms of the Legendre polynomials, such that $g = \sum_n a_n(v,t)P_n(\mu)$. Remember that when we introduced the Lorentz collision operator, we showed that for a homogenous, zero-field plasma, the larger-n components of f pitch-angle scatter (i.e. equilibrate) faster. If you've forgotten this result, go back to equation 3.4. This motivates us to look at only the $n = 1$ component of g, since the higher-n components will equilibrate faster. Also, the $n = 0$ Legendre polynomial is just a constant, so $n = 0$ is not interesting. Thus, we'll approximate g to be $a_1P_1(\mu) = a_1\mu$. Plugging this into equation 3.44, we get

$$\frac{eE_z v\mu}{k_BT} = \nu(v)\frac{\partial}{\partial\mu}\Big[(1-\mu^2)a_1\Big] = -2\nu(v)\mu a_1 \tag{3.46}$$

Thus, $a_1 = -\frac{eE_z v}{2\nu(v)k_BT}$, so $g = -\frac{eE_z v\mu}{2\nu k_BT}$. This allows us to solve for J_z using equation 3.45:

$$J_z = \frac{e^2E_z}{2k_BT}\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu\int v^2dv f_m v^2\mu^2 = \frac{2\pi e^2E_z}{3k_BT}\int_0^\infty\frac{v^4}{\nu(v)}f_m(v)dv \tag{3.47}$$

From $\vec{J} = \sigma\vec{E}$, we have

$$\sigma = \frac{2\pi e^2}{3k_BT}\int_0^\infty\frac{v^4}{\nu(v)}f_m(v)dv \tag{3.48}$$

I won't carry out the integral, but it isn't hard to do in principle, as $\nu(v) \sim \frac{1}{v^3}$. This is the Lorentz conductivity of a plasma.

4 Fluid and MHD Equations

The small, clean fusion reactor I am considering is NOT describable by MHD. Thank goodness!

Samuel Cohen

The workhorse of modern plasma physics is computer simulation.[44] If we want to do plasma science, at some point or another we're going to either use a computer simulation or learn about the work of someone who did use a computer simulation. Simulations are generally cheaper than experiments and more versatile than analytic models, with complete diagnostic knowledge about the plasma being studied. Experiments also tend to move slowly or break, while a computer code, once written, doesn't break or require significant funding to maintain.

There are a huge variety of physical models used in computer codes to study plasmas. The choice of physical model usually depends on the characteristics of the plasma being studied. However, a great number of these physical models can be classified in one of four general categories. These categories are Particle-In-Cell (PIC), Multi-fluid (Fluid), Gyrokinetic, and Magnetohydrodynamic (MHD). Understanding these models, the physical assumptions they rely on, and where they fail are important pieces of knowledge for a plasma physicist to have. In order to evaluate the validity and physical accuracy of a simulation or a calculation, we have to be able to recall the assumptions made in the model used to get that result. Anyone can run a simulation and get a result - but to properly analyze that simulation, we need to figure out the validity of the model used in that simulation. And that requires knowledge of the physics of the model. Unfortunately, remembering and keeping track of all of the assumptions made in each of the various models of plasmas is tricky. I once heard someone say that it's the trickiest part of plasma physics. There is a lot to remember, and it's hard to wrap your head around all of it. Let's get started.

We don't dwell on PIC or gyrokinetic models in this class.[45] We did a brief derivation of the fluid equations and MHD in GPP1, glossing over most of the details. The same is true of GPP2, a class almost entirely dedicated to MHD. In GPP2, we focusing on solving problems using MHD and left the full derivation of the MHD model as an optional assigned reading.[46] I believe that learning the process of deriving these fluid models is important so that we can recall

[44] This might be a controversial statement. I don't care. I'm being dramatic on purpose.

[45] For a wonderful intuitive discussion of PIC models and how they relate to fluid models, watch the beginning of Antoine Cerfon's SULI lecture on MHD from either 2016 or 2015. (https://suli.pppl.gov/2016/course/) His lecture from 2018 might even be better, although I haven't watched it. If someone starts an Antoine Cerfon fan club I'll be the first to sign up. The dude is great.

[46] Chapters 2 and 9 in Ideal MHD by Freidberg, along with Braginskii's 1965 paper were assigned for us to learn this material. Chapter 2 of Bellan also derives the multi-fluid and MHD models with more algebraic detail, although sometimes with less conceptual detail than is done by Freidberg and without reference to Braginskii. I have referenced these chapters heavily in writing this section of the notes. However, each of us derives the fluid equations in

the assumptions of each model. In this chapter, we'll derive the multi-fluid and MHD models, starting from the Vlasov equation. In doing so, I'll do my best to emphasize the physical assumptions made by each model.

I haven't taken the diagnostics class yet, so I can't tell you much about diagnostics and how they relate to the fluid model. But what I can tell you is this - many diagnostics assume the plasma can be described as a fluid (i.e. Maxwellian distribution) in order to extract variables like density and temperature. Sometimes these are good assumptions, and sometimes they aren't. If they aren't good assumptions, then the numbers we measure for density and temperature can be wrong or inconsistent with other diagnostics.[47] If we want to fully understand diagonistic techniques, we should also understand the models used by those diagnostic techniques.[48]

Just so there isn't any confusion, I'm going to tell you exactly what we're going to do in this chapter. First, we'll derive the fluid equations from the Vlasov equation by taking moments of the Vlasov equation. We'll see that we don't have to make many assumptions in order to get fluid equations. The problem is that these equations can't be closed (i.e. more unknowns than equations) unless we make additional assumptions about the collisionality of the plasma (high collisionality) and the heat flux (isothermal or adiabatic). Second, we're going to derive MHD equations, which are basically just the multi-fluid equations written in terms of single-fluid variables $\vec{J}$ and $\vec{u}$. The only additional assumptions we'll make for MHD are that MHD describes the plasma over large length scales and slow frequencies. MHD in it's most basic form does not form a closed set of equations. Lastly, we'll derive the ideal MHD equations. This is one of the many ways of closing the MHD equations. Ideal MHD makes three key assumptions about the plasma, related to the collisionality of the plasma and the size of the phenomenon of interest. We could discuss other MHD models which are less restrictive, but for the sake of time we'll restrict ourselves to the ideal MHD model.

4.1 Deriving Fluid Equations

When deriving fluid equations, our starting place is the Vlasov-Maxwell equation,

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma = \sum_\alpha C(f_\sigma, f_\alpha) \tag{4.1}$$

This is a pretty general starting point, in fact. Let's remember the assumptions we made in deriving the Vlasov-Maxwell equation. We had a totally general

a slightly different way. The biggest difference between the derivations is *at what point* in the derivation we make certain assumptions. I found it was helpful to understand the differences between Bellan's and Freidberg's derivations, so that I could structure this section in the way that made the most sense to me.

[47]For example, see the abstract of this excellent paper by my friend Olivier Izacard.

[48]Hopefully that provides some motivation for the experimentalists out there to learn this stuff.

model including a bunch of charges interacting via electromagnetic forces, took the ensemble average, and hence replaced a discrete distribution function N with a smooth distribution function for species σ, f_σ. Any effects due to the discrete nature of particles (i.e., collisions) are contained in the collision operator $\sum_\alpha C(f_\sigma, f_\alpha)$. So as a reminder, the 'correctness' so to speak of the Vlasov approach ultimately boils down to choosing the right collision operator.[49] Additionally, the ensemble-average in the Vlasov-Maxwell equation removes any knowledge about the microstate of the system and only gives us information about the macrostate of the system. This would be important if, for example, our system had extreme sensitivity to initial conditions (i.e. was chaotic).

From our rather general starting point, the key is to take *moments* of the distribution function. The word *moment* you may have learned in your multivariable calculus class, and if you were like me you weren't sure exactly what it meant. Well, here the word moment means the same thing as it does in *moment of inertia* way back from first-semester physics:

$$I = \int \rho(r) r^2 dV \tag{4.2}$$

where ρ is mass density and r is the distance from the axis of rotation to the point being integrated. Here, we take mass density (i.e. inertia), multiply by r to the *second* power and integrate over all space. The name moment of inertia is totally ambiguous, for it doesn't tell us *which* moment of inertia we are taking. Are we taking the zeroth moment of r (total mass)? The first moment of r (center of mass)? A more precise and impossible to remember name for 'moment of inertia' might be '*second radial moment of mass density*'.

The procedure is to take the 0th moment of velocity, 1st moment of velocity, and 2nd moment of velocity of the Vlasov-Maxwell equation. In doing so, we will derive the continuity equation, the momentum equation, and the energy equation for the multi-fluid model. The momentum equation and energy equation will be the most tricky equations in the multi-fluid model, because of the closure problem.

What is the closure problem? Let's give a preview of this problem now. f_σ is a 6-dimensional distribution function. At each point in space, f_σ gives us infinite degrees of freedom with regards to the velocity distribution. The velocity distribution at each point in space can be anything.[50] However, when take a moment of velocity of f_σ by multiplying by some power of $\vec{v}$ and integrating over velocity, we are left with a scalar (or a vector or a matrix of scalars) field, with some finite number of degrees of freedom in the field. Because the number of degrees of freedom has been decreased from infinity to some finite number, we must have lost information from the Vlasov equation if we take a finite number of moments of the Vlasov equation. The fluid model doesn't have an infinite number of moments of the Vlasov equation, to the contrary it only

[49]We've alaredy seen a few examples of collision operators in this course. Deriving collision operators is a major part of Irreversibles, and to a much lesser extent Transport.

[50]So long as it goes to zero at infinity sufficiently quickly and it integrates to the desired density.

has 3 (continuity, momentum, energy). Thus, we have lost information from the Vlasov equation. In order to make the equations solvable, i.e. close the equations, some assumption about higher-order moments of f_σ must be made.

4.1.1 Continuity Equation

Taking the 0th moment of velocity of f_σ, we multiply the Vlasov equation by 1 (since $(\vec{v})^0 = 1$) and integrate over all velocity. The below equation is going to become the continuity equation, once all is said and done.

$$\int \frac{\partial f_\sigma}{\partial t} d^3\vec{v} + \int \vec{v} \cdot \vec{\nabla} f_\sigma d^3\vec{v} + \int \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma d^3\vec{v} = \int \sum_\alpha C(f_\sigma, f_\alpha) d^3\vec{v} \tag{4.3}$$

We can take the time-derivative out of the first term, which gives $\frac{\partial}{\partial t} \int f_\sigma d^3\vec{v} = \frac{\partial n_\sigma}{\partial t}$. This last step should be obvious, if it isn't then review section 3.2.1, and make sure you understand what f is.

The second term looks like it's going to be tricky to simplify, because of the gradient. However, remember that $\vec{x}$ and $\vec{v}$ are independent variables in the Vlasov description. Thus, we can put the $\vec{v}$ inside the derivative and pull the gradient with respect to space outside of the velocity integral. Writing this out explicitly, we get

$$\int \vec{v} \cdot \vec{\nabla} f_\sigma d^3\vec{v} = \int \vec{\nabla} \cdot (f_\sigma \vec{v}) d^3\vec{v} = \vec{\nabla} \cdot \int f_\sigma \vec{v} d^3\vec{v} = \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma)$$

Don't get confused by the variable names: remember that $\vec{u}_\sigma$ is the mean fluid velocity of the species σ, while $\vec{v}$ is the velocity in phase space such that f_σ is a function of $\vec{x}$ and $\vec{v}$.

The third term goes to zero, as can be shown by integrating by parts. Forgetting about the constant $\frac{q_\sigma}{m_\sigma}$ for a moment, we have

$$\int (\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma d^3\vec{v} = \int \vec{\nabla}_v \cdot (f_\sigma(\vec{E} + \vec{v} \times \vec{B})) d^3\vec{v} - \int f_\sigma \vec{\nabla}_v \cdot (\vec{E} + \vec{v} \times \vec{B}) d^3\vec{v}$$

The first term becomes a boundary term, which integrates to zero at the boundary at infinity in velocity space because f_σ is zero there. The second term, the one with the fields inside the dot product with respect to velocity, is zero because $\vec{E}$ doesn't depend on $\vec{v}$ and $(\vec{v} \times \vec{B})_i$ is perpendicular to v_i, so taking the derivative with respect to v_i gives 0. If that doesn't make sense, work it out yourself to see that this divergence with respect to velocity goes to 0. Thus, the third term in equation 4.3 is 0. By the way, I often like to write $\vec{\nabla}_v$ as $\frac{\partial}{\partial \vec{v}}$, so that $\vec{\nabla}_v \cdot \vec{F} = \frac{\partial}{\partial \vec{v}} \cdot \vec{F} = \sum_i \frac{\partial F_i}{\partial v_i}$. You might see this notation elsewhere, don't let it confuse you.

The RHS term is also zero, from the particle conservation required of the collision operator C. If you remember from section 3.3.3, the physical reason for this is because C shouldn't change the number of particles at a particular $\vec{x}$, only the velocity of the particles at this position.

With these simplifications, equation 4.3 becomes the multi-fliuid continuity equation,

$$\frac{\partial n_\sigma}{\partial t} + \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma) = 0 \tag{4.4}$$

Note that we've already started to see the closure problem in action: we took the zeroth moment of velocity of the Vlasov equation, to get an equation for the evolution of the zeroth moment of velocity of f_σ, n_σ. However, the continuity equation has a term which includes the first moment of velocity of f_σ, $\vec{u}_\sigma$. We have one equation so far, and two unknowns. We will, in the next section, try to close our equations by taking more moments. We will end up with two equations, but we will pay the price of introducing a third unknown, the pressure tensor. Taking moments of the Vlasov equation will always give more unknowns than equations. This is the essence of the closure problem.

The continuity equation is entirely general - it doesn't depend on any particular acceleration term in the Vlasov equation. Really, it's a geometrical statement more than it is an equation related to any physical law or equation. It just tells us that if the density in some region changes, particles must be flowing out of that region.

4.1.2 Momentum Equation

Taking the first moment of velocity of f_σ, we multiply the Vlasov equation by $\vec{v}$ and integrate over all velocity. This will become the momentum equation, sometimes called the equation of motion.

$$\int \frac{\partial f_\sigma}{\partial t} \vec{v} d^3\vec{v} + \int \vec{v}(\vec{v} \cdot \vec{\nabla} f_\sigma) d^3\vec{v} + \int \frac{q_\sigma}{m_\sigma} \vec{v}(\vec{E} + \tag{4.5}$$

$$\vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma d^3\vec{v} = \int \sum_\alpha C(f_\sigma, f_\alpha) \vec{v} d^3\vec{v}$$

The first term simplifies in the same way as with the continuity equation: we pull the time derivative out of the integral, and replace $\int f_\sigma \vec{v} d^3\vec{v}$ with $n_\sigma \vec{u}_\sigma$.

Let's worry about the second term later, and try to simplify the third and fourth terms first. The third term can be integrated by parts, to get

$$\frac{q_\sigma}{m_\sigma} \int \vec{\nabla}_v \cdot \left[(\vec{E} + \vec{v} \times \vec{B}) f_\sigma \vec{v} \right] d^3\vec{v} - \frac{q_\sigma}{m_\sigma} \int f_\sigma \vec{\nabla} \cdot (\vec{v}(\vec{E} + \vec{v} \times \vec{B})) d^3\vec{v}$$

This first term becomes a boundary term at infinity in velocity-space which goes to 0 because f_σ goes to zero at infinite velocity. The $\vec{E} + \vec{v} \times \vec{B}$ can be taken outside the gradient because (as we established before) it's gradient with respect to velocity is zero. This leaves us with

$$-\frac{q_\sigma}{m_\sigma} \int f_\sigma (\vec{E} + \vec{v} \times \vec{B}) \cdot (\vec{\nabla}_v \vec{v}) d^3\vec{v}$$

Now, $\frac{\partial v_j}{\partial v_i} = \delta_{ij}$, so the third term becomes

$$-\frac{q_\sigma}{m_\sigma}\int f_\sigma(\vec{E}+\vec{v}\times\vec{B})d^3\vec{v}$$

We can pull the $\vec{E}$ and $\vec{B}$ out of the integral, because they don't depend on $\vec{v}$. Thus, we have

$$-\frac{q_\sigma}{m_\sigma}\vec{E}\int f_\sigma d^3\vec{v} - \frac{q_\sigma}{m_\sigma}\int f_\sigma \vec{v} d^3\vec{v}\times\vec{B} = -\frac{q_\sigma n_\sigma \vec{E}}{m_\sigma} - \frac{q_\sigma n_\sigma}{m_\sigma}\vec{u}\times\vec{B}$$

Great, we've figured out what the third term in equation 4.5 is. Now let's figure out the fourth term, the term on the RHS. The RHS term *isn't* zero, as we know from section 3.3.3. $\sum_{\sigma,\alpha}\int \vec{v}C(f_\sigma, f_\alpha)d^3\vec{v} = 0$ because the total momentum at each point in space is conserved. We also have that $\int \vec{v}C(f_\sigma, f_\sigma)d^3\vec{v}$ is zero, because a species cannot transfer momentum to itself due to collisions. However, one species *can* transfer net momentum to a different species, if the two species have some different net velocity. Because of momentum conservation requirements on C, we can simplify the RHS slightly by removing the $\sigma = \sigma$ term in the collision operator.

$$RHS = \sum_{\alpha\neq\sigma}\int C(f_\sigma, f_\alpha)\vec{v}d^3\vec{v}$$

Technically, we can't simplify this further until we have a specific collision operator to work with. However, we show on a homework in the Transport class that if we use the Lorentz collision operator, then

$$\int C(f_\sigma, f_\alpha)\vec{v}d^3\vec{v} = -\nu_{\sigma\alpha}n_\sigma(\vec{u}_\sigma - \vec{u}_\alpha) = \frac{1}{m_\sigma}\vec{R}_{\sigma\alpha}$$

Often, we are concerned with the collisions between electrons and ions. If this is the case, then we write this term as $\vec{R}_{ie}$ or $\vec{R}_{ei}$. If our only two species in the plasma are electrons and ions, then we can say that $\vec{R}_{ei} = -\vec{R}_{ie}$, which must be true by overall momentum conservation. We can interpret $\vec{R}_{\sigma\alpha}$ as the force per unit volume imparted to species σ due to collisions with species α.[51] So the RHS of equation 4.5 becomes

$$\sum_{\alpha\neq\sigma}\frac{1}{m_\sigma}\vec{R}_{\sigma\alpha}$$

Now we turn our attention to the second term, $\int \vec{v}(\vec{v}\cdot\vec{\nabla}f_\sigma)d^3\vec{v}$. The first thing we can do is pull the gradient out of the integral, since $\vec{v}$ doesn't depend on $\vec{x}$. This gives us $\int \vec{\nabla}\cdot(f_\sigma\vec{v}\vec{v})$. There is a nice little trick we use to simplify this

[51] Bellan has a minus sign convention which is different than mine here. Many people seem to follow that convention. I'm using a different convention, which Bill Tang uses. I like Tang's convention, because I find it a bit easier to think of $\vec{R}_{\sigma\alpha}$ as a force rather than as a drag force.

term: we can write $\vec{v}$ as the sum of two terms, $\vec{u}_\sigma(\vec{x})$ (the mean fluid velocity) and $\vec{v}'$ (the fluctuation of the velocity relative to the mean. Since $\vec{v} = \vec{u}_\sigma(\vec{x}) + \vec{v}'$ and $\int f_\sigma \vec{v} d^3\vec{v} = \int f_\sigma \vec{u}_\sigma d^3\vec{v}$, then $\int f_\sigma \vec{v}' d^3\vec{v} = 0$. Note the $d^3\vec{v}$ could equally be written $d^3\vec{v}'$, because at a given point in space $\vec{u}_\sigma$ is constant. Replacing $\vec{v}$ with $\vec{u}_\sigma + \vec{v}'$, we have 4 terms, each of which is a divergence of a rank-2 tensor (which gives a vector).

$$\vec{\nabla} \cdot \int \vec{u}_\sigma \vec{u}_\sigma f_\sigma d^3\vec{v}' + \vec{\nabla} \cdot \int \vec{u}_\sigma \vec{v}' f_\sigma d^3\vec{v}' + \vec{\nabla} \cdot \int \vec{v}' \vec{u}_\sigma f_\sigma d^3\vec{v}' + \vec{\nabla} \cdot \int \vec{v}' \vec{v}' f_\sigma d^3\vec{v}'$$

We can pull the mean velocity $\vec{u}_\sigma$ out of the velocity integrals. The second and third terms in this expression go to zero because they are linear in $\vec{v}'$, and $\vec{u}_\sigma \int \vec{v}' f_\sigma d^3\vec{v}' = 0$. The first term now simply becomes $\vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma \vec{u}_\sigma)$. To simplify the fourth term, we define our pressure tensor, $\overleftrightarrow{P}_\sigma$. We define the ij component of the pressure tensor $\overleftrightarrow{P}_\sigma$, $P_{\sigma ij}$, to be $\int m_\sigma f_\sigma v_i' v_j' d^3\vec{v}'$. Thus, this fourth term becomes $\frac{1}{m_\sigma}$ times the divergence of the pressure tensor.

Having simplified all the terms in equation 4.5, we can now write our momentum equation. It makes things simpler if we multiply by m_σ first.

$$m_\sigma \frac{\partial (n_\sigma \vec{u}_\sigma)}{\partial t} + m_\sigma \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma \vec{u}_\sigma) + \vec{\nabla} \cdot \overleftrightarrow{P}_\sigma - q_\sigma n_\sigma \vec{E} - q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} = \sum_{\alpha \neq \sigma} \vec{R}_{\sigma\alpha}$$

or as it is often written,

$$m_\sigma \frac{\partial (n_\sigma \vec{u}_\sigma)}{\partial t} + m_\sigma \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma \vec{u}_\sigma) = q_\sigma n_\sigma \vec{E} + q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} - \vec{\nabla} \cdot \overleftrightarrow{P}_\sigma + \sum_{\alpha \neq \sigma} \vec{R}_{\sigma\alpha} \quad (4.6)$$

We can actually simplify the LHS of equation 4.6 considerably. It turns out there is a hidden $\frac{\partial n_\sigma}{\partial t} + \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma)$ in the LHS, which equals 0 by the continuity equation.[52] Simplifying the LHS, we get

$$m_\sigma \vec{u}_\sigma \frac{\partial n_\sigma}{\partial t} + m_\sigma n_\sigma \frac{\partial \vec{u}_\sigma}{\partial t} + m_\sigma \vec{u}_\sigma \vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma) + m_\sigma n_\sigma (\vec{u}_\sigma \cdot \vec{\nabla}) \vec{u}_\sigma$$

The first and third terms combine to give $m_\sigma \vec{u}_\sigma$ times the LHS of the continuity equation, which is zero. Thus, we can rewrite our momentum equation once more as

$$m_\sigma n_\sigma \frac{\partial \vec{u}_\sigma}{\partial t} + m_\sigma n_\sigma (\vec{u}_\sigma \cdot \vec{\nabla}) \vec{u}_\sigma = q_\sigma n_\sigma \vec{E} + q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} - \vec{\nabla} \cdot \overleftrightarrow{P}_\sigma + \sum_{\alpha \neq \sigma} \vec{R}_{\sigma\alpha} \quad (4.7)$$

This is our momentum equation for species σ. It's a bit messy, but each of these terms has a distinct physical meaning. The LHS comes from the inertia of the fluid, and can be seen as the $m\vec{a}$ term in $\vec{F} = m\vec{a}$. The RHS has all the forces

[52] We call this an embedded continuity equation. It turns out that higher moments of the Vlasov equation have embedded lower-moment equations in them which we can use to simplify.

on the fluid. These forces include electromagnetic, internal pressure forces, and collisions with other species. Many people will write the LHS of equation 4.7 as

$$m_\sigma n_\sigma \frac{\partial \vec{u}_\sigma}{\partial t} + m_\sigma n_\sigma (\vec{u}_\sigma \cdot \vec{\nabla})\vec{u}_\sigma = m_\sigma n_\sigma \frac{d\vec{u}_\sigma}{dt}$$

where the time derivative $\frac{d}{dt}$ is understood to represent two terms, the partial time derivative and what is sometimes called the convective derivative term,[53] $(\vec{u}_\sigma \cdot \vec{\nabla})$. If you're familiar with the Lagrangian and Eulerian specification of flow fields, then you'll recognize that $\frac{d\vec{u}_\sigma}{dt}$ is a Lagrangian derivative - this just means it is the derivative as we follow a fluid particle around. There are lots of different ways of understanding the physical meaning of the LHS of equation 4.7. The partial derivative term has to do with the time-evolution of the velocity field (holding position constant). This is the same as taking the time derivative in the Eulerian description. The second term, the convective derivative, can be understood by thinking about a flow which is constant in time but not in space. If the velocity $\vec{u}_\sigma$ increases in some direction, then a fluid particle moving in the flow will accelerate as $\vec{u}_\sigma$ changes, even though there is no time dependence of the flow field. This is what the convective derivative term represents.

Once again, we see the closure problem rearing it's ugly head: we took the first moment of velocity of the Vlasov equation, to get an equation for the time-evolution of the first moment of velocity of f_σ. This was successful, but it introduced a new term, $\overleftrightarrow{P}_\sigma$, which is related to the second moment of velocity of f_σ. We'll have to take the second moment of velocity of the Vlasov equation to get an equation for $\overleftrightarrow{P}_\sigma$. We'll do this in the next section, and it will give us our energy equation. If you understand the closure problem by now, then you'll see what is going to happen: the energy equation will introduce a term which is a third moment of velocity of f_σ, which won't allow us to have as many equations as unknowns once again.

I glossed over some of the details related to the rank-2 tensors I needed to derive these equations. If the details of the tensors in the derivation don't make total sense, I recommend writing out the equations component by component (for example, $\vec{\nabla} \cdot (n_\sigma \vec{u}_\sigma \vec{u}_\sigma) = \frac{\partial}{\partial x_j}(n_\sigma u_{\sigma i} u_{\sigma j})$), so that each step makes total sense.[54]

4.1.3 Energy Equation

We'll solve for the energy equation following a similar procedure as before: we'll take the second moment of velocity of the Vlasov equation, and simplify as much as possible. Actually, instead of multiplying by v^2, we'll multiply by $\frac{1}{2}m_\sigma v^2$ and integrate over velocity so that the connection with energy becomes

[53] Confusingly, sometimes the convective derivative refers to both terms, and sometimes it refers to just one term. Here I'll use convective derivative to just refer to the second term.

[54] This is what I forced myself to do in my graduate E+M class during undergrad to gradually learn how to manipulate rank-2 tensors in calculations such as this one. Once you do these things enough, working with tensors in calculations like this becomes second nature.

clear. Doing this, we have

$$\frac{m_\sigma}{2}\int v^2\frac{\partial f_\sigma}{\partial t}d^3\vec{v}+\frac{m_\sigma}{2}\int v^2\vec{v}\cdot\vec{\nabla}f_\sigma d^3\vec{v}+$$
$$\frac{1}{2}\int q_\sigma v^2(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{\nabla}_v f_\sigma d^3\vec{v}=\frac{m_\sigma}{2}\int\sum_\alpha v^2C(f_\sigma,f_\alpha)d^3\vec{v} \tag{4.8}$$

Once again, we have a bunch of terms we need to write in terms of macroscopic variables. The strategy will be similar to what we used in deriving the momentum equation: we'll set $\vec{v}=\vec{u}_\sigma(\vec{x})+\vec{v}'$, and immediately eliminate all terms linear in $\vec{v}'$, because they integrate to zero. Using this strategy, let's work on the first term.

$$\frac{m_\sigma}{2}\int v^2\frac{\partial f_\sigma}{\partial t}d^3\vec{v}=\frac{m_\sigma}{2}\frac{\partial}{\partial t}\int u_\sigma^2 f_\sigma d^3\vec{v}+\frac{\partial}{\partial t}\int\frac{m_\sigma}{2}v'^2 f_\sigma d^3\vec{v}'=$$
$$\frac{1}{2}\frac{\partial}{\partial t}\left(m_\sigma n_\sigma u_\sigma^2+\text{Tr}(\overleftrightarrow{P}_\sigma)\right)$$

Not too bad. Now let's work on the second term. We can pull the gradient out right away, before plugging in $\vec{v}=\vec{u}_\sigma+\vec{v}'$. Eliminating terms linear in $\vec{v}'$, we have

$$\frac{m_\sigma}{2}\int v^2\vec{v}\cdot\vec{\nabla}f_\sigma d^3\vec{v}=\frac{m_\sigma}{2}\frac{\partial}{\partial\vec{x}}\cdot\int\left[u_\sigma^2\vec{u}_\sigma+v'^2\vec{v}'+v'^2\vec{u}_\sigma+2(\vec{u}_\sigma\cdot\vec{v}')\vec{v}'\right]f_\sigma d^3\vec{v}'$$

Of the terms in parentheses, only the second can't be defined in terms of variables we've seen before. This is a third moment of f_σ, which relates to the heat flux of the particles. Actually, we can go ahead and define the heat flux $\vec{Q}_\sigma$ as $\int\frac{m_\sigma v'^2}{2}\vec{v}'f_\sigma d^3\vec{v}'$. For the fourth term, we can just take the $(\vec{u}_\sigma\cdot)$ out of the integral, so that the second term in equation 4.8 becomes

$$\vec{\nabla}\cdot\left[\frac{m_\sigma n_\sigma u_\sigma^2}{2}\vec{u}_\sigma+\vec{Q}_\sigma+\frac{1}{2}\text{Tr}(\overleftrightarrow{P}_\sigma)\vec{u}_\sigma+\vec{u}_\sigma\cdot\overleftrightarrow{P}_\sigma\right]$$

Objectively, this 'simplified' second term of equation 4.8 is pretty nasty. The third term in equation 4.8 can first be rewritten as

$$\frac{1}{2}\int q_\sigma v^2(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{\nabla}_v f_\sigma d^3\vec{v}=\frac{q_\sigma}{2}\int v^2\vec{\nabla}_v\cdot(f_\sigma(\vec{E}+\vec{v}\times\vec{B}))d^3\vec{v}$$

because the derivative doesn't operate on (i.e. commutes with) $\vec{E}+\vec{v}\times\vec{B}$, as we've discussed a few times already. Now, we can integrate by parts on this term, as we've done before. The gradient with respect to velocity on $\frac{v^2}{2}$ becomes $\vec{v}$, as $\sum_i\frac{\partial v_i^2}{\partial v_j}\hat{e}_j=2v_i\delta_{ij}\hat{e}_j=2v_j\hat{e}_j$. Also, when we integrate by parts, the boundary term goes to 0 as usual because f_σ is zero at infinity in velocity space. With all these manipulations, the third term in equation 4.8 becomes

$$-\frac{q_\sigma}{2}\int 2\vec{v}\cdot(\vec{E}+\vec{v}\times\vec{B})f_\sigma d^3\vec{v}=-q_\sigma n_\sigma\vec{u}_\sigma\cdot\vec{E}$$

where we've used $\vec{v}\cdot(\vec{v}\times\vec{B})=0$. The only term left to simplify is the collision term on the RHS of 4.8. Let's do that now.

$$\frac{m_\sigma}{2}\int\sum_\alpha v^2 C(f_\sigma,f_\alpha)d^3\vec{v}=\sum_{\alpha\neq\sigma}\int(u_\sigma^2+v'^2+2\vec{u}_\sigma\cdot\vec{v}')C(f_\sigma,f_\alpha)d^3\vec{v}'=$$

$$0+\sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha+\sum_{\alpha\neq\sigma}\vec{u}_\sigma\cdot\vec{R}_{\sigma\alpha}$$

where

$$\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha\equiv\frac{m_\sigma}{2}\int v'^2C(f_\sigma,f_\alpha)d^3\vec{v}'$$

and we've used our usual definition for $\vec{R}_{\sigma\alpha}$. The first term went to zero because of particle conservation, while the $\alpha=\sigma$ terms were removed from the summation because a species can't impart momentum or energy to itself due to collisions. Physically, W_σ is the thermal energy per volume of species σ and $(\frac{\partial W_\sigma}{\partial t})_\alpha$ is the rate of change of the thermal energy per volume of species σ due to collisions with species α.[55] Putting our simplifications to each of the terms in equations 4.8 together, our energy equation becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(m_\sigma n_\sigma u_\sigma^2+\mathrm{Tr}(\overleftrightarrow{P}_\sigma)\Big)+\vec{\nabla}\cdot\Big[\frac{m_\sigma n_\sigma u_\sigma^2}{2}\vec{u}_\sigma+\vec{Q}_\sigma+\frac{1}{2}\mathrm{Tr}(\overleftrightarrow{P}_\sigma)\vec{u}_\sigma+\vec{u}_\sigma\cdot\overleftrightarrow{P}_\sigma\Big]$$
$$-q_\sigma n_\sigma\vec{u}_\sigma\cdot\vec{E}=\sum_{\alpha\neq\sigma}\big(\frac{\partial W_\sigma}{\partial t}\big)_\alpha+\sum_{\alpha\neq\sigma}\vec{u}_\sigma\cdot\vec{R}_{\sigma\alpha}\tag{4.9}$$

In principle, we are done solving for the energy equation. However, we can simplify this equation significantly, using the continuity equation and our momentum equation. We will see that there are embedded continuity and momentum equations hidden in this equation. This allows us to eliminate $\vec{E}$ from the energy equation, and reduce the overall number of terms. Let's start by looking at the first and third terms of equation 4.9.

$$\frac{\partial}{\partial t}(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2)+\vec{\nabla}\cdot(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2\vec{u}_\sigma)=$$

$$\frac{m_\sigma u_\sigma^2}{2}\big(\frac{\partial n_\sigma}{\partial t}+\vec{\nabla}\cdot(n_\sigma\vec{u}_\sigma)\big)+\frac{m_\sigma n_\sigma}{2}\big(\frac{\partial u_\sigma^2}{\partial t}+(\vec{u}_\sigma\cdot\vec{\nabla})u_\sigma^2\big)$$

Setting the embedded continuity equation to zero, these terms become

$$\frac{m_\sigma n_\sigma}{2}\big(\frac{\partial u_\sigma^2}{\partial t}+(\vec{u}_\sigma\cdot\vec{\nabla})u_\sigma^2\big)$$

[55] Here I'm using notation and conventions which are slightly different from both Bellan and Freidberg. I've tried to use notation which combines physical intuition with simplicity, without deviating too far from convention.

This is a nice simplification. However, this can be further simplified into the LHS of the momentum equation dotted with $\vec{u}_\sigma$. To do so, let's write everything in terms of components, using the Einstein summation convention. We have

$$\frac{\partial u_\sigma^2}{\partial t} = \sum_i \frac{\partial u_{\sigma i}^2}{\partial t} = 2u_{\sigma i}\frac{\partial u_{\sigma i}}{\partial t} = 2\vec{u}_\sigma \cdot \frac{\partial \vec{u}_\sigma}{\partial t}$$

$$(\vec{u}_\sigma \cdot \vec{\nabla})u_\sigma^2 = \sum_j (u_{\sigma i}\frac{\partial}{\partial x_i})u_{\sigma j}^2 = 2u_{\sigma j}u_{\sigma i}\frac{\partial u_{\sigma j}}{\partial x_i} = 2\vec{u}_\sigma \cdot [(\vec{u}_\sigma \cdot \vec{\nabla})\vec{u}_\sigma]$$

Putting it all together, we can rewrite the first and third terms of equation 4.9 as

$$\frac{m_\sigma n_\sigma}{2}\Big(2\vec{u}_\sigma \cdot \frac{\partial \vec{u}_\sigma}{\partial t} + 2\vec{u}_\sigma \cdot [(\vec{u}_\sigma \cdot \vec{\nabla})\vec{u}_\sigma]\Big) = \vec{u}_\sigma \cdot \Big(m_\sigma n_\sigma \frac{d\vec{u}_\sigma}{dt}\Big)$$

As promised, this is the LHS of the momentum equation dotted with $\vec{u}_\sigma$.[56] Now, we replace the LHS of the momentum equation with the RHS. This means that we're replaced the first and third terms in equation 4.9 with

$$\vec{u}_\sigma \cdot \Big[q_\sigma n_\sigma \vec{E} + q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} - \vec{\nabla} \cdot \overleftrightarrow{P}_\sigma + \sum_{\alpha \neq \sigma} \vec{R}_{\sigma\alpha}\Big]$$

The term with $\vec{B}$ is zero because of the definition of the cross product. The term with $\vec{E}$ cancels the term with $\vec{E}$ in equation 4.9. The term with $\vec{R}_{\sigma\alpha}$ cancels with the same term on the RHS of equation 4.9. This allows us to rewrite our energy equation as

$$\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Tr}(\overleftrightarrow{P}_\sigma) + \vec{\nabla} \cdot \Big[\vec{Q}_\sigma + \frac{1}{2}\mathrm{Tr}(\overleftrightarrow{P}_\sigma)\vec{u}_\sigma + \vec{u}_\sigma \cdot \overleftrightarrow{P}_\sigma\Big] - \vec{u}_\sigma \cdot (\vec{\nabla} \cdot \overleftrightarrow{P}_\sigma) = \sum_{\alpha \neq \sigma} (\frac{\partial W_\sigma}{\partial t})_\alpha \tag{4.10}$$

This is a nice simplification. But we're not done. We can simplify yet again, so that eventually we'll have massaged the multi-fluid energy equation into the form in Freidberg equation 2.22. We'll simplify the pressure terms. If we remember of definition of the pressure tensor in the multi-fluid approach, we have $P_{\sigma ij} = \int m_\sigma f_\sigma v_i' v_j' d^3\vec{v}'$. $\overleftrightarrow{P}_\sigma$ can be written as $P_\sigma \overleftrightarrow{I} + \overleftrightarrow{\Pi}_\sigma$, where $\overleftrightarrow{\Pi}_\sigma$ is the non-isotropic portion of $\overleftrightarrow{P}_\sigma$, and $\overleftrightarrow{\Pi}_\sigma$ has zero trace. Thus, the trace of $\overleftrightarrow{P}_\sigma$ is a scalar equal to $3P_\sigma$ which, even if our distribution function isn't isotropic or Maxwellian, allows us to define a pseudo-temperature which represents the thermal energy in the plasma in a similar way. If the distribution function were isotropic and Maxwellian, then $\mathrm{Tr}(\overleftrightarrow{P}_\sigma)$ would equal $3n_\sigma T_\sigma$. So we define our temperature to be the trace of $\overleftrightarrow{P}_\sigma$ (or $3P_\sigma$) divided by $3n_\sigma$. With this definition, we can write

$$\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Tr}(\overleftrightarrow{P}_\sigma) + \vec{\nabla} \cdot (\frac{1}{2}\mathrm{Tr}(\overleftrightarrow{P}_\sigma) = \frac{1}{2}\frac{\partial}{\partial t}3n_\sigma T_\sigma + \frac{1}{2}\vec{\nabla} \cdot (3n_\sigma \vec{u}_\sigma T_\sigma)) =$$

[56] This is an embedded momentum equation.

$$\frac{3}{2}\frac{\partial n_\sigma}{\partial t}T_\sigma + \frac{3}{2}\vec{\nabla}\cdot(n_\sigma\vec{u}_\sigma)T_\sigma + \frac{3}{2}n_\sigma\frac{\partial T_\sigma}{\partial t} + \frac{3}{2}n_\sigma\vec{u}_\sigma\cdot\vec{\nabla}T_\sigma = \frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}$$

In the last step, we replaced the first two terms (out of four) with the continuity equation times T_σ,[57] and the last two terms with $\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}$. What about the other terms with $\overleftrightarrow{P}_\sigma$ in equation 4.10? It helps to look at these terms component by component. We use the Einstein summation convention.

$$\vec{\nabla}\cdot(\vec{u}_\sigma\cdot\overleftrightarrow{P}_\sigma) - \vec{u}_\sigma\cdot(\vec{\nabla}\cdot\overleftrightarrow{P}_\sigma) = \frac{\partial}{\partial x_i}(u_{\sigma j}P_{\sigma ij}) - u_{\sigma i}\frac{\partial}{\partial x_j}P_{\sigma ij} =$$

$$P_{\sigma ij}\frac{u_{\sigma j}}{\partial x_i} + u_{\sigma j}\frac{\partial}{\partial x_i}P_{\sigma ij} - u_{\sigma i}\frac{\partial}{\partial x_j}P_{\sigma ij}$$

From the symmetry of $P_{\sigma ij}$ and the fact that i and j are dummy indices, we can see that the two terms cancel to give

$$P_{\sigma ij}\frac{\partial}{\partial x_i}u_{\sigma j} = \overleftrightarrow{P}_\sigma : \vec{\nabla}\vec{u}_\sigma$$

where the : symbol means taking the tensor inner product, i.e. summing over both indices. With these simplifications, we can write our multi-fluid energy equation as

$$\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt} + \overleftrightarrow{P}_\sigma : \vec{\nabla}\vec{u}_\sigma + \vec{\nabla}\cdot\vec{Q}_\sigma = \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha \tag{4.11}$$

This is the final, most simplified form of the energy equation. We see that the closure problem pops up once more with our energy equation - we've got a term which is the third moment of velocity of f_σ, $\vec{Q}_\sigma$. We could continue to take moments forever if we wanted, but we're going to stop doing so here. Instead, we'll look for an artificial way to close the fluid equations. That will be the subject of the next section.

4.1.4 Closure of Fluid Equations

Let's briefly recap what we've done so far. We started with the Vlasov-Maxwell equation, and took moments of velocity of that equation. The 0th moment gave us the continuity equation. The 1st moment gave us a $\vec{F} = m\vec{a}$ -type equation, which we called the momentum equation (or the equation of motion). The second moment gave us an equation for the energy of our fluid. The only problem is we have an unknown variable, $\vec{Q}_\sigma$. We don't have any equations for $\vec{Q}_\sigma$, and taking further moments of the Vlasov equation only delays the suffering. We're going to have to bite the bullet eventually, and make some sort of approximation so we can close the fluid equations.

Oh, I should say one more thing before we talk about the different ways of closing our system of equations. Our plasma is made up of multiple species σ -

[57] Here is another embedded continuity equation.

these species include electrons, ions, neutrals, etc. Thus, the multi-fluid model keeps track of a density, velocity, pressure tensor, etc, for *every* species in the plasma. So to be clear, there *is* a continuity equation, momentum equation, and energy equation for every single species in the plasma.

So far, we haven't made any assumptions, other than that the Vlasov model is a valid description of the Plasma (see section 3.2 for a discussion of to what extent this model is valid), and that our only forces are Electromagnetic (so we're ignoring gravity, etc).

As we will see throughout the rest of this chapter, sufficiently high plasma collisionality is often the key element needed to close fluid-based equations. Why? Because collisions tend to drive the distribution function f_σ towards isotropy and also towards a Maxwellian distribution. If f_σ is isotropic (uniform in all directions), then the off-diagonal terms in $\overleftrightarrow{P}$ are zero, and we can replace the pressure tensor $\overleftrightarrow{P}_\sigma$ with a scalar pressure P_σ. If f_σ is Maxwellian, then we can use the ideal gas law to write P_σ in terms of T_σ. In fact, isotropy is the most common assumption we make when solving the fluid equations - *that our plasma has an isotropic velocity distribution*, and we can replace $\overleftrightarrow{P}_\sigma$ with a scalar P_σ. We want the divergence of the pressure tensor to be replaced by the gradient of a pressure scalar - thus, we'll define our scalar pressure (for isotropic systems) as

$$P_\sigma = \int m_\sigma f_\sigma v_i'^2 d^3\vec{v}' = \frac{1}{3}\int m_\sigma f_\sigma v'^2 d^3\vec{v}' \tag{4.12}$$

We'll still need an equation for the evolution of P_σ, but this assumption makes solving the system so much easier. With this assumption, our momentum equation becomes

$$m_\sigma n_\sigma \frac{d\vec{u}_\sigma}{dt} = q_\sigma n_\sigma \vec{E} + q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} - \vec{\nabla} P_\sigma + \sum_{\alpha\neq\sigma} \vec{R}_{\sigma\alpha} \tag{4.13}$$

If we assume isotropy, we can rewrite our energy equation in a much simpler form as well. Let's start with equation 4.10, which I've reproduced below.

$$\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Tr}(\overleftrightarrow{P}_\sigma) + \vec{\nabla}\cdot\left[\vec{Q}_\sigma + \frac{1}{2}\mathrm{Tr}(\overleftrightarrow{P}_\sigma)\vec{u}_\sigma + \vec{u}_\sigma\cdot\overleftrightarrow{P}_\sigma\right] - \vec{u}_\sigma\cdot(\vec{\nabla}\cdot\overleftrightarrow{P}_\sigma) = \sum_{\alpha\neq\sigma}\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha$$

With the assumptions of isotropy, we have $Tr(\overleftrightarrow{P}_\sigma) = 3P_\sigma$, $\vec{\nabla}\cdot(\vec{u}_\sigma\cdot\overleftrightarrow{P}_\sigma) = \vec{u}_\sigma\cdot\vec{\nabla}P_\sigma + P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma$, and $\vec{\nabla}\cdot\overleftrightarrow{P}_\sigma = \vec{\nabla}P_\sigma$. Thus, our energy equation becomes

$$\frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}(\vec{u}_\sigma\cdot\vec{\nabla})P_\sigma + \frac{5}{2}P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma = -\vec{\nabla}\cdot\vec{Q}_\sigma + \sum_{\alpha\neq\sigma}\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha \tag{4.14}$$

The assumption of isotropy alone is not enough to close our fluid equations, because we still have that pesky $\vec{Q}_\sigma$ term. There are two main limits we're going to consider: the isothermal limit and the adiabatic limit. In the isothermal limit, the temperature is the same everywhere. This corresponds to slow changes in the state of the plasma, so that the plasma has enough time to equilibrate and reach a constant temperature. In the adiabatic limit, no heat flows, which means $Q = 0$.[58] This corresponds to fast changes in the plasma, so there is no time for heat to flow.

Before we discuss these limits in a bit more depth, we should say something about this concept of temperature that we like to use. In a thermodynamics sense, the definition of temperature is $\frac{1}{T} = \frac{\partial S}{\partial U}$. However, the temperature of two systems sharing energy is only well-defined if the entropy is at it's maximum. Any distribution function which is not Maxwellian is not at a maximum-entropy state, so temperature is not well-defined unless f is Maxwellian. If f is Maxwellian *and* isotropic, then the ideal gas law $P_\sigma = \frac{3}{2} n_\sigma k_B T_\sigma$ is true. If f is Maxwellian *but not* isotropic, then we have a different temperature in each direction.[59] Let's look at the adiabatic and isothermal limits in more depth.

Adiabatic Limit: In the adiabatic limit, no heat flows, so that $Q = 0$ in thermodynamic terms. In plasma physics terms, this means that the RHS (heat flux term and collisional energy transfer term) is negligible relative to the terms on the LHS of the energy equation. The LHS terms are of order $v_{ph} P/L$ where L is a length scale of the variation of the quantities in the plasma and v_{ph} is the phase velocity of motion of disturbances in the plasma. The heat flux term goes like $\vec{\nabla} \cdot \int m_\sigma v'^2 \vec{v}' f_\sigma d^3\vec{v}' \sim P V_{T\sigma}/L$. Thus, the heat flux term is smaller than the LHS terms if $V_{ph} \gg v_{T\sigma}$. In words, the phase velocity of adiabatic changes in the plasma is fast relative to the thermal velocity. Fast changes are what we expect in the isothermal limit, so this makes sense. The collision terms on the RHS are more complicated to estimate, but in the adiabatic limit we want these to be negligible as well.

So in this adiabatic limit, we assume $v_{ph} \gg V_{T\sigma}$, and are left with only the terms on the LHS of the isotropic energy equation. These can be simplified as follows:

$$3 \frac{dP_\sigma}{dt} = -5 P_\sigma \vec{\nabla} \cdot \vec{u}_\sigma$$

From the continuity equation,

$$\frac{\partial n_\sigma}{\partial t} + \vec{u} \cdot \vec{\nabla} n_\sigma + n_\sigma \vec{\nabla} \cdot \vec{u}_\sigma = 0$$

$$\frac{1}{n_\sigma} \frac{dn_\sigma}{dt} = -\vec{\nabla} \cdot \vec{u}_\sigma \tag{4.15}$$

so we have

$$\frac{1}{P_\sigma} \frac{dP_\sigma}{dt} = \frac{5}{3} \frac{1}{n_\sigma} \frac{dn_\sigma}{dt} \tag{4.16}$$

[58] Does this remind you of an adiabatic process in thermodynamics, where $Q = 0$? That was a rhetorical question, of course it does. Now you see why we write the heat flux using the variable $\vec{Q}$.

[59] If this is the case, typically we have a parallel temperature in the direction parallel to the magnetic field and a perpendicular temperature in the two directions perpendicular to the magnetic field.

Actually, we can replace $\frac{5}{3}$ by γ, the ratio of specific heats.[60] This equation has the solution (check it by taking the time derivative)

$$\frac{P_\sigma}{n_\sigma^\gamma} = \text{constant} \tag{4.17}$$

where the constant doesn't change in time, but could potentially change in space. This is the last equation we need to close the multi-fluid equations in the adiabatic limit: an energy equation for the evolution of P_σ.

There is one additional subtlety with the adiabatic limit that we should keep in mind: to solve the energy equation, we had to assume that collisions were negligible. However, we need collisions to maintain isotropy so that a scalar pressure is well-defined. Thus, in the adiabatic limit we need collisions to be small, but not so small that our plasma loses isotropy which allows us to calculate a scalar pressure in the first place.

Isothermal Limit: In the isothermal limit, the temperature is the same everywhere. This requires very strong collisionality, so that not only is temperature well-defined at each point in space, but also heat can easily flow so that each point remains in thermal equilibrium with every other point. Since the distribution function is assumed to be Maxwellian and isotropic (this high-collisionality assumption is the assumption we use to close the fluid equations), then P_σ is a scalar equal to $n_\sigma k_B T_\sigma$. Since we're in the isothermal limit, the pressure gradient in the momentum equation becomes $k_B T_\sigma \vec{\nabla} n_\sigma$. The temperature T_σ is treated as an initial condition on the fluid behavior. Thus, we don't need to solve the energy equation to close the system of equations.

For the isothermal limit, it turns out that we can use the same equation of state as with the adiabatic limit, but with $\gamma = 1$. Let's show how that is indeed the case now. If $\gamma = 1$, then taking the gradient of the equation of state we have

$$\vec{\nabla} P_\sigma \frac{1}{n_\sigma} = P_\sigma \vec{\nabla} n_\sigma \frac{1}{n_\sigma^2}$$

so that

$$\vec{\nabla} P_\sigma = \frac{P_\sigma}{n_\sigma} \vec{\nabla} n_\sigma = k_B T_\sigma \vec{\nabla} n_\sigma$$

which is the equation of state in the isothermal limit. So the isothermal and adiabatic limits have the same equation of state, just with different values of γ.

You might be wondering what happened to the energy equation in the isothermal limit. Specifically, you might be wondering what happens to the $\vec{Q}$ term in this limit, and how it conspires to give us our equation of state. If you remember from the adiabatic limit, we were able to show that as long as $v_{ph} \gg V_{T\sigma}$, then we could neglect the $\vec{\nabla} \cdot \vec{Q}$ term relative to the LHS terms. In the isothermal limit, we have the limit $V_{T\sigma} \gg v_{ph}$, which naively implies that $\vec{\nabla} \cdot \vec{Q}$ is much larger than the terms on the LHS and we can neglect the LHS. This isn't quite true. What actually happens is that, if the plasma is initially

[60] The fact that we got a factor of $5/3$ here is because we're working in three dimensions. If we were working in N dimensions, this factor would be $N + 2/N = \gamma$.

in some state where $\vec{\nabla} T_\sigma \neq 0$, then the heat flux $\vec{Q}$ is initially very large so as to force the temperature to quickly become uniform. We still ignore collisions between species. Once that happens, then $\vec{\nabla} \cdot \vec{Q}$ no longer dominates the other terms. Instead, $\vec{\nabla} \cdot \vec{Q}$ *must be* equal to $-P_\sigma \vec{\nabla} \cdot \vec{u}_\sigma$, so that our energy equation (equation 4.14) becomes

$$\frac{3}{2}\frac{dP_\sigma}{dt} + \frac{3}{2}P_\sigma \vec{\nabla} \cdot \vec{u}_\sigma = 0$$

which gives us the energy equation with $\gamma = 1$. So in the isothermal limit, once the temperature is uniform then $\vec{Q}_\sigma$ is known. We can think of this in the thermodynamics way: $\Delta U = -W$, so if the internal energy of the plasma doesn't change then the heat added equals the work done. The $-P_\sigma \vec{\nabla} \cdot \vec{u}_\sigma$ is the work done by the plasma as it expands against a pressure force.

Actually, there is something important I have been glossing over so far when discussing the multi-fluid equations. We haven't yet stated our equations for the fields $\vec{E}$ and $\vec{B}$! Of course, these are Maxwell's equations, adapted for the multi-fluid model

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma n_\sigma \tag{4.18}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4.19}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.20}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \sum_\sigma q_\sigma n_\sigma \vec{u}_\sigma + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4.21}$$

To a good approximation, most plasmas are net neutral over scales larger than the debye length. Thus, often the divergence of $\vec{E}$ will simply be assumed to be zero for computational simplicity. The displacement current will often be neglected as well for computational reasons, so long as speeds are slow relative to the speed of light. However, the fluid model works perfectly well without any modifications to Maxwell's equations.

4.1.5 Summary of Assumptions Made in Fluid Model

Technically, our multi-fluid model as described by equations 4.4, 4.7, and 4.11 is completely general (ignoring boundary conditions and atomic physics). However, the third-order moment, $\vec{Q}$, can't be solved for unless we make some approximations to close the system, so the completely general fluid model by itself is useless. We needed to make some approximations for this model to become useful. When we make these approximations, we are implicitly assuming that we *can* represent our plasma only in terms of variables like density, mean velocity, pressure, etc.[61] Many plasmas have distribution functions where describing the plasma *only* in terms of these variables doesn't really make sense, or at least isn't helpful.

[61] Actually, density is pretty much always a meaningful variable. It's the other ones which can be pretty meaningless for certain distribution functions.

To get a useful, closed system of equations with the multi-fluid model, we didn't actually have to make too many assumptions:

- Our distribution function f_σ is isotropic, so $\overleftrightarrow{P}_\sigma$ is a scalar of 1 component rather than a tensor with 6 linearly independent components. We expect our plasma to be isotropic if it is sufficiently collisional.
- Either Isothermal or Adiabatic heat flow, so that we have an equation of state $\frac{P_\sigma}{n_\sigma^\gamma}$ =constant. These required assuming either slow or fast changes in the plasma relative to the thermal velocity.

These multi-fluid models are sometimes useful for *analytic* calculations, especially the more simplified forms of the energy equation. However, often the multi-fluid model is applied to regions where neither the adiabatic or isothermal limits properly capture the physics of the system. As an example, one of the more common areas where multi-fluid codes are used is in studying the scrape-off layer (SOL) of Tokamaks. These codes include UEDGE, SOLPS, BoRiS, and many more. To investigate the SOL, a means of closing the fluid equations without assuming adiabatic or isothermal limits is required. I won't get into it too much, but essentially Braginskii's classic 1965 paper is used to close the equations. In the paper, each of the terms in the full energy equation is estimated for a *collisional, magnetized* plasma using physical reasoning and heuristic arguments. The result is that $\vec{Q}_\sigma$ and $\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha$ are replaced by what are called Braginskii coefficients, coefficients calculated based on the other variables in the plasma. I won't derive the Braginskii coefficients in these notes,[62] However, this is an important result to remember: the Braginskii model (and hence the SOL multi-fluid codes) requires a *collisional*, magnetized, and isotropic plasma for the Braginskii coefficients and hence the multi-fluid model to accurately model the plasma behavior. Often, it is the collisional assumption which is most easily violated in these sorts of systems.

I'll say one more thing before we move on to MHD. There is a lot of physics ignored by the multi-fluid model as it is currently formulated. Radiation, excited electron states, ionization, recombination, molecular dissociation and association, fusion, and plasma-boundary interactions are *completely* ignored by the multi-fluid model as it is currently formulated. However, this isn't to say that these effects *can't* be modeled effectively with a multi-fluid model. UEDGE, for example, pulls from well-respected atomic databases to determine ionizing cross-sections, molecular dissociation rates, radiation rates at various temperatures, etc. The plasma-boundary interactions must be set by the user of the code, so to the extent the user understands the plasma-material interactions in their system, the code will handle these interactions more or less correctly. So a computer code using the multi-fluid models needs to do a lot more than just solve the multi-fluid equations for the geometry of interest.

[62] The derivation of Braginskii coefficients is done at the very end of Irreversibles.

4.2 Deriving MHD Equations

The MHD equations are derived from the multi-fluid equations. Instead of the single-fluid variables n_σ and $\vec{u}_\sigma$, we'll assume our plasma has a single mass density ρ, a single mean fluid velocity $\vec{u}$, and a current density $\vec{J}$. We'll make two simplifying 'asymptotic' assumptions: that the characteristic lengths in the equations are large relative to the Debye length, and the characteristic frequencies are small relative to the electron plasma frequency and cyclotron frequency. In other words, with MHD we are interested in low-frequency phenomenon with large spacial scales. Once we make these assumptions, we'll modify and simplify the multi-fluid equations. To get the MHD continuity equation, we sum the multi-fluid momentum equation over species. To get the MHD momentum equation, we also sum the multi-fluid momentum equation over species. To get MHD ohm's law, we subtract the ion momentum equation from the electron momentum equation. To get the MHD energy equations, we rewrite the multi-fluid energy equation in terms of single-fluid variables ρ, $\vec{u}$, and $\vec{J}$. At this point, our system of MHD equations will still not be closed. There are various ways of closing the MHD equations, including Ideal MHD, Kinetic MHD and Doubly Adiabatic MHD. In section 4.3, I'll take the more general MHD equations we derive in this section and go through the steps to derive ideal MHD.[63]

4.2.1 Asymptotic Assumptions in MHD

Any MHD model relies on two key assumptions. First, that the length scales are much larger than the Debye length in a plasma. Formally, we can express this either by taking $\epsilon_0 \to 0$ so that $\lambda_D \to 0$, or saying that $\lambda_D/L \ll 1$. Second, that the frequencies in MHD are much less than both the electron plasma frequency and the electron cyclotron frequency. Formally, we can express this either by taking $m_e \to 0$ so that $\Omega_e \to \infty$ and $\omega_{pe} \to \infty$, or saying that $\frac{\omega}{\omega_{pe}} \ll 1$ and $\frac{\omega}{\Omega_e} \ll 1$.

Let's see how these assumptions start to play out. Of course, if we take the purely mathematical approach of setting $\epsilon_0 \to 0$, then the charge density has to be zero for the divergence of $\vec{E}$ to not be infinite. Similarly, the displacement current term can be completely neglected in Ampere's law. If we instead use the more physically intuitive assumption that $\lambda_D/L \ll 1$ then as we saw in chapter 1, on length scales much larger than the debye length plasmas are net neutral. So if $\frac{\lambda_D}{L} \ll 1$ then the sum $\sum_\sigma n_\sigma q_\sigma \approx 0$. If the frequencies are small, then, as we'll show in a moment, we can also drop the displacement current term in Ampere's law relative to the other terms. This means that regardless of whether we set $\epsilon_0 \to 0$ or assume $\lambda_D/L \ll 1$, in either case we can simplify Gauss's law and Ampere's law in the same way.

We can justify neglecting the displacement current using Faraday's law and

[63] This subsection and the next rely greatly on Freidberg for guidance. This is my attempt at rewriting that information as clearly as I can.

Ampere's law. From Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $E \sim \omega B/k$, so

$$\frac{\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}{\vec{\nabla} \times \vec{B}} \approx \frac{\omega^2}{k^2 c^2} \approx \frac{v_{ph}^2}{c^2}$$

So if ω is sufficiently small, or equivalently v_{ph} is much less than the speed of light, then the displacement current can be neglected.

We've seen how these two assumptions simplify some of Maxwell's equations in MHD. As we derive the other equations in MHD, we'll see how these assumptions help us get simpler forms of the relevant MHD equations.

4.2.2 MHD Continuity Equation

Multiplying each of the multi-fluid continuity equations by mass and summing over species, we have

$$\sum_\sigma \frac{\partial m_\sigma n_\sigma}{\partial t} + \vec{\nabla} \cdot (m_\sigma n_\sigma \vec{u}_\sigma) = \frac{\partial}{\partial t}(\sum_\sigma m_\sigma n_\sigma) + \vec{\nabla} \cdot (\sum_\sigma m_\sigma n_\sigma \vec{u}_\sigma)$$

$$\frac{\partial}{\partial t}(\sum_\sigma m_\sigma n_\sigma) + \vec{\nabla} \cdot (\sum_\sigma m_\sigma n_\sigma \vec{u}_\sigma) = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u})$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{4.22}$$

where we have defined the mass density $\rho = \sum_\sigma m_\sigma n_\sigma$ and the mean fluid velocity as $\vec{u} = \frac{1}{\rho} \sum_\sigma n_\sigma m_\sigma \vec{u}_\sigma$. Let's notice something right away with the MHD variables ρ and $\vec{u}$: since they are weighted by mass, then the ion's dominate the mass density and they dominate the fluid velocity, simply because $m_e/m_i \ll 1$. Essentially, these variables are measures of the ion mass density and ion velocity.

4.2.3 MHD Momentum Equation

Summing the LHS of the momentum equation (using 4.6 instead of 4.7 so that we can put the summation inside the derivatives) over species, we have

$$\frac{\partial}{\partial t}(\sum_\sigma m_\sigma n_\sigma \vec{u}_\sigma) + \vec{\nabla} \cdot (\sum_\sigma m_\sigma n_\sigma \vec{u}_\sigma \vec{u}_\sigma)$$

Now, if we define $\rho = \sum_\sigma m_\sigma n_\sigma$, then we can't replace this equation with single-fluid variables because of the $\vec{u}_\sigma \vec{u}_\sigma$ term which doesn't have a nice simplification. However, we can use the asymptotic assumption that $m_e \to 0$. We also have to assume that our plasma is fully ionized and made up of only two species, ions of charge $+e$ and mass m_i, and electrons of charge $-e$ and mass m_e. If we use these two assumptions, then $\rho \approx m_i n_i$ and $\vec{u} = \vec{u}_i$, so that summing of the LHS of the momentum equations gives

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u})$$

Summing the RHS of the momentum equation (4.6) over species, and recognizing that

$$\vec{J} = \sum_\sigma q_\sigma n_\sigma \vec{u}_\sigma$$

we have

$$\vec{J} \times \vec{B} - \vec{\nabla} \cdot (\overleftrightarrow{P}_i + \overleftrightarrow{P}_e)$$

where the terms with $\vec{E}$ canceled and $\vec{R}_{ei} = -\vec{R}_{ie}$ from momentum conservation. Setting the LHS and RHS equal, we get

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = \vec{J} \times \vec{B} - \vec{\nabla} \cdot (\overleftrightarrow{P}_i + \overleftrightarrow{P}_e) \tag{4.23}$$

Once again, we have an embedded continuity equation on the LHS. Working this out explicitly, the LHS equals

$$\vec{u}\frac{\partial \rho}{\partial t} + \vec{u}\vec{\nabla} \cdot (\rho \vec{u}) + \rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \vec{\nabla})\vec{u} = \rho \frac{d\vec{u}}{dt}$$

Thus we have

$$\rho \frac{d\vec{u}}{dt} = \vec{J} \times \vec{B} - \vec{\nabla} \cdot (\overleftrightarrow{P}_i + \overleftrightarrow{P}_e) \tag{4.24}$$

This is our momentum equation in MHD. It's only in more simplified forms of MHD that the ion and electron pressures get replaced by a single pressure, $\overleftrightarrow{P}_{MHD}$.

4.2.4 MHD Ohm's Law

Thought Ohm's law was $V = IR$? Well, sure, maybe for circuits. But for plasma physics, that would just be too easy. Instead, we've got a whole plethora of terms which will make up Ohm's law. When we make a bunch of approximations later on to simplify this equation, a lot of these terms will go away and you'll see why we call this funky equation Ohm's law.

To get the full version of the equation, let's take our electron momentum equation, divide it by n_e and write it in terms of the single-fluid variables.

$$m_e \frac{d\vec{u}_e}{dt} = -e\vec{E} - e\vec{u}_e \times \vec{B} - \frac{1}{n_e}\vec{\nabla} \cdot \overleftrightarrow{P}_e - \frac{1}{n_e}\sum_{\alpha \neq \sigma} \nu_{e\alpha} m_e (\vec{u}_e - \vec{u}_\alpha)$$

Now, we'd like to write this in terms of $\vec{J}$ and $\vec{u}$. From the definition of $\vec{J}$ and $\vec{u}$, we have $\vec{J} = \sum_\sigma q_\sigma n_\sigma \vec{u}_\sigma = -en_e\vec{u}_e + \sum_{\sigma \neq e} q_\sigma n_\sigma \vec{u}_\sigma$ and $\rho\vec{u} = \sum_\sigma n_\sigma m_\sigma \vec{u}_\sigma = n_e m_e \vec{u}_e + \sum_{\sigma \neq e} m_\sigma n_\sigma \vec{u}_\sigma$. Looking at these definitions, we aren't able to write this only in terms of single-fluid variables, unless we assume that our plasma is fully ionized and made up of only 2 species, ions of charge $+e$ and mass m_i and electrons of charge $-e$ and mass m_e. This is the assumption that Bellan and Freidberg have to make to write Ohm's law in a friendly form. If we make this key assumption, then we can simplify our definition for $\vec{J}$ and $\vec{u}$ and hence simplify the above expression. $\vec{J}$ becomes $-en_e\vec{u}_e + en_i\vec{u}_i$ and

$\rho\vec{u} = m_e n_e \vec{u}_e + m_i n_i \vec{u}_i$. We'll also assume that $n_e = n_i = n$, which we have from our asymptotic assumption of large length scales. Thus, our expression simplifies to

$$m_e \frac{d\vec{u}_e}{dt} = -e\vec{E} - \frac{1}{n}\vec{\nabla}\cdot\overleftrightarrow{P}_e + \frac{1}{n}\vec{J}\times\vec{B} - e\vec{u}_i\times\vec{B} - \frac{1}{n}\nu_{ei}m_e(\vec{u}_e - \vec{u}_i)$$

Now, in a two-component plasma, $-(\vec{u}_e - \vec{u}_i) = \frac{1}{en_e}\vec{J}$. Additionally, since $m_i \gg m_e$, then $\vec{u}_i \approx \vec{u}$ to a good approximation. We can also drop the electron inertia term relative to the magnetic force term $e\vec{u}_e \times \vec{B}$, which we now justify using the asymptotic assumption of low frequencies relative to the electron cyclotron frequency. Since $m_e\frac{d\vec{u}_e}{dt} \sim \omega m_e u_e$ and $e\vec{u}_e \times \vec{B} \sim e u_e B$, then

$$\frac{m_e \frac{d\vec{u}_e}{dt}}{e\vec{u}_e \times \vec{B}} \sim \frac{\omega m_e}{eB} = \frac{\omega}{\Omega_e} \ll 1$$

Thus, the inertia term is negligible.[64] With these simplifications, we have our generalized MHD ohm's law,[65]

$$\vec{E} + \vec{u}\times\vec{B} + \frac{1}{en}\vec{\nabla}\cdot\overleftrightarrow{P}_e - \frac{1}{en}\vec{J}\times\vec{B} = \eta\vec{J} \tag{4.25}$$

where $\eta = \frac{\nu_{ei}m_e}{en_e}$. In writing Ohm's law in this relatively simple form, we had to make four assumptions: firstly, that our plasma is fully ionized and made up of a single ion species with charge $+e$. Secondly, we set $n_e \approx n_i$, such that our plasma is net neutral (which is true based on our first asymptotic assumption of large length scales). Thirdly, that $m_e \ll m_i$, so we can write $\vec{u}$ as $\vec{u}_i$. Fourth, we used the assumption that frequencies are much lower than the electron gyrofrequency, which is our second asymptotic assumption, to remove the electron inertia term.

4.2.5 MHD Energy Equation

When we get to ideal MHD, our energy equation is going to be $\frac{d}{dt}(\frac{P}{\rho^\gamma}) = 0$. However, we haven't gone from regular MHD to ideal MHD yet. When we do so, we're going to see that there are a bunch of terms in the energy equation which we can ignore relative to other terms, which allows us to get our ideal MHD energy equation. Thus, we'd love to turn our energy equation for the multi-fluid model into an equation of the form $\frac{d}{dt}(\frac{P_\sigma}{\rho^\gamma}) = \text{RHS}$, where the RHS has some number of terms which we will eventually neglect. This will be the goal of all the algebra I'm about to do: get the LHS into the form $\frac{d}{dt}(\frac{P_\sigma}{\rho^\gamma})$ equals

[64] Technically, this justification doesn't work in the direction parallel to the magnetic field. Really, we should treat the parallel direction and the perpendicular directions separately in MHD.

[65] Compare this with equation 2.31 in Ideal MHD by Freidberg or 2.74 in Bellan. These are all almost exactly equivalent equations, with perhaps some minor differences in what assumptions have been made at that point in time and which assumptions have not yet been made.

something. To start, I've rewritten the multi-fluid energy equation (equation 4.11) below.

$$\frac{3}{2}n_\sigma \frac{dT_\sigma}{dt} + \overleftrightarrow{P}_\sigma : \vec{\nabla}\vec{u}_\sigma + \vec{\nabla}\cdot\vec{Q}_\sigma = \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha \tag{4.26}$$

If you remember from our derivation of this equation, our first term $\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}$ was a simplified version of $\frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}\vec{\nabla}\cdot(P_\sigma\vec{u}_\sigma)$, which we simplified using the continuity equation. Let's play with this term a bit.

$$\frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}\vec{\nabla}\cdot(P_\sigma\vec{u}_\sigma) = \frac{3}{2}\frac{dP_\sigma}{dt} + \frac{3}{2}P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma$$

Alright, we've got our first term in a form we like. Now let's go to town on the second term, $\overleftrightarrow{P}_\sigma : \vec{\nabla}\vec{u}_\sigma$. Remember, we can write $\overleftrightarrow{P}_\sigma = P_\sigma\overleftrightarrow{I} + \overleftrightarrow{\Pi}_\sigma$, so

$$\overleftrightarrow{P}_\sigma : \vec{\nabla}\vec{u}_\sigma = P_\sigma\overleftrightarrow{I} : \vec{\nabla}\vec{u}_\sigma + \overleftrightarrow{\Pi}_\sigma : \vec{\nabla}\vec{u}_\sigma$$

By components,

$$P_\sigma\overleftrightarrow{I} : \vec{\nabla}\vec{u}_\sigma = P_\sigma\delta_{ij}\frac{\partial}{\partial x_i}\vec{u}_{\sigma j} = P_\sigma\frac{\partial}{\partial x_i}\vec{u}_{\sigma i} = P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma$$

so

$$P_\sigma\overleftrightarrow{I} : \vec{\nabla}\vec{u}_\sigma = P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma + \overleftrightarrow{\Pi}_\sigma : \vec{\nabla}\vec{u}_\sigma$$

Alright, let's again write down the multi-fluid energy equation, using the manipulations we just did on the first and second terms.

$$\frac{3}{2}\frac{dP_\sigma}{dt} + \frac{3}{2}P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma + P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma + \overleftrightarrow{\Pi}_\sigma : \vec{\nabla}\vec{u}_\sigma = \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha - \vec{\nabla}\cdot\vec{Q}_\sigma \tag{4.27}$$

The $P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma$ terms combine to get $\frac{5}{2}P_\sigma\vec{\nabla}\cdot\vec{u}_\sigma$. From our continuity equation, we have equation 4.15,

$$\vec{\nabla}\cdot\vec{u}_\sigma = -\frac{1}{n_\sigma}\frac{dn_\sigma}{dt}$$

and so this term becomes

$$-\frac{5}{2}P_\sigma\frac{1}{n_\sigma}\frac{dn_\sigma}{dt}$$

Let's rewrite our multi-fluid energy equation once more, moving terms from the LHS to the RHS and then dividing by $\frac{3}{2}\rho^\gamma$. I'll also put some strategic m_i terms in, and replace $\frac{5}{3}$ with the more general variable γ (which works for any number of dimensions, not just 3).

$$\frac{1}{\rho^\gamma}\frac{dP_\sigma}{dt} - \frac{1}{\rho^\gamma}\gamma P_\sigma\frac{1}{m_i n_\sigma}\frac{d(m_i n_\sigma)}{dt} = \frac{2}{3\rho^\gamma}\Big[\sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha - \vec{\nabla}\cdot\vec{Q}_\sigma - \overleftrightarrow{\Pi}_\sigma : \vec{\nabla}\vec{u}_\sigma\Big] \tag{4.28}$$

Actually, we're really close to having this in the form we want it. The LHS looks a little bit like $\frac{d}{dt}\big(\frac{P_\sigma}{\rho^\gamma}\big)$, except we don't have the ρ term we need. Well, here is where we use our favorite asymptotic assumptions to make things right: in the limit that $m_e \to 0$ (our second asymptotic assumption), and assuming $n_e = n_i$ (our first asymptotic assumption), and assuming we have only electrons and one singly charged ion species, then $\rho \approx m_i n$. We can also say that $\vec{u} = \frac{m_i n_i \vec{u}_i + m_e n_e \vec{u}_e}{\rho} \approx \vec{u}_i$. Thus, our LHS becomes

$$\frac{1}{\rho^\gamma}\frac{dP_\sigma}{dt} - \frac{1}{\rho^{\gamma+1}}\gamma P_\sigma \frac{d\rho}{dt} = \frac{d}{dt}\Big(\frac{P_\sigma}{\rho^\gamma}\Big) \tag{4.29}$$

However, there is a minor subtlety here: the $\frac{d}{dt}$ operator is still operating with respect to the multi-fluid variables, meaning $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}_\sigma \cdot \vec{\nabla}$. We want to change this to the operator $\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$. For ions, this isn't a problem: we've approximated $\vec{u}_i$ as $\vec{u}$, so the two operators are equal in our approximation. However, for electrons we have to be a bit more careful. Since $\vec{J} = en\vec{u}_i - en\vec{u}_e$, then $\vec{u}_e = \vec{u}_i - \frac{\vec{J}}{ne} \approx \vec{u} - \frac{\vec{J}}{ne}$. Thus, $\frac{\partial}{\partial t} + \vec{u}_e \cdot \vec{\nabla} \approx \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} - \frac{\vec{J}}{ne} \cdot \vec{\nabla}$.

We can see the light at the end of the tunnel now. Substituting equation 4.29 into the LHS of equation 4.28, and replacing the operator $\frac{d}{dt}$ with the correct one as discussed in the previous paragraph, for the ions we have

$$\frac{d}{dt}\Big(\frac{P_i}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_i}{\partial t}\Big)_e - \vec{\nabla} \cdot \vec{Q}_i - \overleftrightarrow{\Pi}_i : \vec{\nabla}\vec{u}\Big] \tag{4.30}$$

and for the electrons we have (replacing $\vec{u}_e$ with $\vec{u} - \frac{\vec{J}}{ne}$)

$$\frac{d}{dt}\Big(\frac{P_e}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_e}{\partial t}\Big)_i - \vec{\nabla} \cdot \vec{Q}_e - \overleftrightarrow{\Pi}_e : \vec{\nabla}(\vec{u} - \frac{\vec{J}}{en})) + \frac{1}{en}\vec{J} \cdot \vec{\nabla}\Big(\frac{P_e}{\rho^\gamma}\Big)\Big] \tag{4.31}$$

These are our MHD energy equations! We have them in the form we want them: a LHS which is in the form $\frac{d}{dt}(\frac{P}{\rho^\gamma})$, and a RHS which is a bunch of terms we can examine one by one (and for ideal MHD, eventually neglect).

4.2.6 Information Content of the MHD Equations

To get to the MHD equations, we have to make assumptions about the length-scales and time-scales described by MHD. Besides those assumptions, however, the MHD equations carry the same information as the fluid equations, except the equations are written in terms of single-fluid variables $\vec{J}$, ρ, and $\vec{u}$ instead of multi-fluid variables n_σ and $\vec{u}_\sigma$. For a two-species fully-ionized plasma, there are 4 multi-fluid variables (n_e, n_i, $\vec{u}_e$ and $\vec{u}_i$) while there are 3 MHD variables (ρ, $\vec{J}$, and $\vec{u}$) and the equation $n_e = n_i$ (which is like having a fourth MHD variable). Since the information contained in the two models is the same (except for the asymptotic assumptions), then we would expect that there be the same number of equations in the multi-fluid model as there are in the MHD model. This is indeed the case, although we have to carefully keep track of where the information contained in each of the multi-fluid equations goes in the MHD model. I'm going to explain in detail what I mean below, so hopefully if this statement is confusing it will make more sense after seeing the details worked out.

For a two-species fully-ionized plasma, we have two continuity equations in the multi-fluid model.

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{u}_i) = 0$$

For the information content of the equations in the multi-fluid model to match the information content of the MHD equations, we must have the same number of continuity equations on both models. Well, taking the limit $m_e \to 0$, the continuity equation for the ions (muliplied by m_i) becomes the MHD continuity equation.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

What about the information contained in the electron continuity equation? This needs to make it into the MHD model somehow. Well, if we multiply the electron continuity equation by e and subtract it from the ion continuity equation, we have

$$\frac{\partial e n_i}{\partial t} - \frac{\partial e n_e}{\partial t} + \vec{\nabla} \cdot (e n_i \vec{u}_i - e n_e \vec{u}_e) = 0$$

Using the MHD approximation $n_e = n_i$, this becomes

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{4.32}$$

This relationship might be considered, in a sense, the 'second continuity equation' for MHD, just like there are two continuity equations in the multi-fluid model. What about the information contained in the multi-fluid momentum equations for the electrons and the ions? Well, summing the two equations gives the MHD momentum equation, and the electron equation gives Ohm's law for MHD. Thus, we have 2 equations in the multi-fluid model, and 2 equations in the MHD model, as we want.

We also have 2 energy equations in MHD, corresponding to the 2 energy equations in the multi-fluid model. Thus, the information in the multi-fluid picture is the same as the information in the MHD model, with two additional assumptions in the MHD model which allow the MHD equations to be slightly simplified.

So far I've neglected Maxwell's equations in this discussion. Maxwell's equations are the same in the multi-fluid model as in MHD, except with $\epsilon \to 0$ and

4.2.7 Summary of Assumptions Made in MHD

MHD is less general than the multi-fluid model. The multi-fluid model technically makes no assumptions about the plasma, at least until we attempt to close the equations somehow. The MHD model, like the multi-fluid model, represents a set of equations which aren't closed. However, we *do* make a number of assumptions about the plasma which restrict the realm of validity of MHD. In other words, we make certain assumptions which limit the physics phenomenon we can study with MHD. These assumptions are:

- The first asymptotic assumption: The length scales we are considering are large, relative to the Debye length. The primary physical consequence of this assumption is that our plasma is neutral on the scales we are interested in, and is equivalent to taking the limit that $\epsilon_0 \to 0$.
- The second asymptotic assumption: The frequencies we are interested in are slow, relative to both the electron gyrofrequency and the electron plasma frequency. The primary physical consequence of this assumption is that the electrons respond infinitely fast to any forces on them. This is equivalent to taking $m_e \to 0$.
- The mean fluid velocity is the same as the ion velocity. This comes from $m_e \to 0$, so it's not really a new assumption, but it has a different physical consequence.
- There are only two species in the plasma, a singly ionized charge $+1e$ ion species and electrons with charge $-e$. This allows us to write much simpler equations. When we make this assumption, we are assuming as well our plasma is fully ionized.

MHD works just fine even if we have multiple ion species in the plasma. However, the equations get a bit more complicated and ugly to write, so for simplicity the books typically assume there is only a single ion species. When we eventually derive ideal MHD the equations don't depend on how many ion species there are, only that the plasma is fully ionized and that the ions are much heavier than the electrons.

So in summary, the MHD model can be derived from the fluid model in a relatively straightforward but algebra-intensive way. The MHD model describes long-wavelength, slow-frequency behavior.

4.3 Deriving Ideal MHD

I've rewritten the MHD equations below.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{4.33}$$

$$\rho \frac{d\vec{u}}{dt} - \vec{J} \times \vec{B} + \vec{\nabla}(P_i + P_e) = -\vec{\nabla} \cdot (\overleftrightarrow{\Pi}_i + \overleftrightarrow{\Pi}_e) \tag{4.34}$$

$$\vec{E} + \vec{u} \times \vec{B} = \frac{1}{en}\vec{J} \times \vec{B} + \eta \vec{J} - \frac{1}{en}\vec{\nabla} P_e - \frac{1}{en}\vec{\nabla} \cdot \overleftrightarrow{\Pi}_e \tag{4.35}$$

$$\frac{d}{dt}\Big(\frac{P_i}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_i}{\partial t}\Big)_e - \vec{\nabla}\cdot\vec{Q}_i - \overleftrightarrow{\Pi}_i : \vec{\nabla}\vec{u}\Big] \tag{4.36}$$

$$\frac{d}{dt}\Big(\frac{P_e}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_e}{\partial t}\Big)_i - \vec{\nabla}\cdot\vec{Q}_e - \overleftrightarrow{\Pi}_e : \vec{\nabla}(\vec{u} - \frac{\vec{J}}{en})) + \frac{1}{en}\vec{J}\cdot\vec{\nabla}\Big(\frac{P_e}{\rho^\gamma}\Big)\Big] \tag{4.37}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4.38}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{4.39}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \tag{4.40}$$

For equations 4.33-4.37, we have terms on the LHS and terms on the RHS. Ideal MHD is a model where those terms on the RHS are negligible, and $P_e = P_i$. Let's go through the steps to derive this model.

There are three additional assumptions of the ideal MHD model. Let's go through those assumptions now, and examine how they simplify the equations of MHD.

4.3.1 High Collisionality

The most basic assumption required to close fluid equations is that the plasma is highly collisional, so that the distribution function reaches an maximum-entropy state (isotropic Maxwellian) and the pressure tensor can be written as a scalar, so the off-diagonal components $\overleftrightarrow{\Pi}_\sigma$ are negligible.

Physically, this means that the collision time for both the ions and the electrons is short compared to the characteristic timescale of changes in the plasma in the MHD description. The characteristic timescale of changes in the plasma is $\frac{1}{\omega}$, where $\omega \sim \frac{\partial}{\partial t}$ is the characteristic frequency associated with oscillations described by MHD (MHD frequency). For the ions, collisions are dominated by collisions with other ions, with a timescale τ_{ii}. For the electrons, both collisions with the electrons and the ions are important, with timescales $\tau_{ee} \sim \tau_{ei}$. Since the ions are much heavier and move slower, it takes longer for them to feel the effects of collisions with other particles, so $\tau_{ii} \sim (\frac{m_i}{m_e})^{\frac{1}{2}}\tau_{ee}$.

Since $\frac{\partial}{\partial t} \sim \omega$, we also have $|\vec{\nabla}| \sim k \sim \frac{1}{a}$ where a is the scale length of the variations in the plasma and k is the characteristic wavenumber of MHD. Putting this together, we can say that

$$\frac{\omega}{k} \sim |\vec{v}| \sim V_{Ti}$$

where V_{Ti} is the thermal ion velocity. Actually, we don't really know how fast we would expect characteristic velocities in MHD to be. However, from physical intuition we know that the fastest these velocities might be is the ion thermal velocity, because the electrons have no mass and hence respond infinitely fast to electromagnetic fields in the plasma, while the ions carry all of the mass and take time to propogate. Since the ions will tell us about the characteristic velocity of MHD, the fastest they can carry disturbances is their thermal speed.

The condition that the collision time is short can be written as, for the ions and electrons respectively

$$\omega\tau_{ii} \sim V_{Ti}\tau_{ii}/a \ll 1$$

$$\omega\tau_{ee} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} V_{Ti}\tau_{ii}/a \ll 1$$

Since $\frac{m_e}{m_i} \approx \frac{1}{40}$, then the condition on the ions being collisional is much more restrictive than the condition on the electrons being collisional.

We also require that the mean free path of both the electrons and the ions be smaller than the scale length of the plasma. Remembering that the mean free path $\lambda_{mfp,i} = V_{Ti}\tau_{ii}$ and $\lambda_{mfp,e} = V_{Te}\tau_{ee}$, then for the ions

$$\lambda_{mfp,i}/a \sim V_{Ti}\tau ii/a \ll 1$$

and for the electrons

$$\lambda_{mfp,e}/a \sim V_{Te}\tau_{ee}/a \sim V_{Ti}\tau_{ii}$$

Thus, the condition for the mean free path is the same for the ions and electrons, and is mathematically equivalent to the assumption that ions be highly collisional. Thus, we can summarize these requirements with a single inequality, the high collisionality requirement.

$$\frac{V_{Ti}\tau_{ii}}{a} \ll 1 \tag{4.41}$$

This requirement is enough to eliminate all of the RHS terms, except the energy transfer term $(\frac{\partial W_\sigma}{\partial t})_\alpha$ in the energy equation. To get rid of this term, we actually need an even more stringent assumption, which I am calling the *very* high collisionality assumption.

$$\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{V_{Ti}\tau_{ii}}{a} \ll 1 \tag{4.42}$$

The *very* high collisionality assumption is more stringent than the high collisionality assumption by a factor of the square root of the ion to electron mass. Technically, ideal MHD requires the *very* high collisionality assumption to be true. In fusion plasmas, even the high collisionality assumption is *never* satisfied. There is lots of subtle and interesting plasma physics to discuss regarding the fact that this requirement is not satisfied in fusion plasmas yet ideal MHD is still used, but I won't get into it in these notes.[66]

[66] I lied, this is just too juicy a topic to not try to say something. Here is what Hong and Ian had to say in regards to the question "I'm trying to figure out why the fluid description can be valid for a plasma. For most plasmas, the Knudsen number is much much greater than 1, and therefore (naively applying results for fluid mechanics to plasma physics) we would think that the continuum assumption is not valid. Yet, we often use MHD models to describe plasmas, and apparently these models often do quite well. Why do they work even when the mean free path is so long? Does it have to do with the particles being confined to Larmor orbits along

4.3.2 Small Gyroradius

We also require that the ion gyroradius be much smaller than the characteristic length scale in the plasma. Of course, since the electron gyroradius is much smaller than the ion gyroradius, that will automatically be true as well. Mathematically, this is

$$\frac{\rho_i}{a} \ll 1 \tag{4.43}$$

This requirement is well satisfied in fusion experiments, as the gyroradius is typically quite small and the experiments are typically much bigger. Notice that equation 4.43 is equivalent to the statement that MHD frequencies are much lower than the ion cyclotron frequency, as you can see below

$$\frac{\rho_i}{a} = \frac{mV_{Ti}}{eBa} = \frac{V_{Ti}}{\Omega_i a} = \frac{\omega}{\Omega_i} \ll 1$$

Thus, the second assumption we make is that the ion gyroradius is small, or equivalently that the MHD frequencies are slow compared to the ion cyclotron frequency.

4.3.3 Low Resistivity

The third and final assumption of ideal MHD is that resistive diffusion is negligible. I found this statement to be rather confusing when written this way. What it really means is that the magnetic field lines are frozen into the plasma, and don't diffuse due to resistivity in the plasma. This intuitively means that the resistivity is very low, or equivalently that the conductivity is very high.

We make this requirement because we want the $\eta\vec{J}$ term in the MHD Ohm's law to be negligible relative to the LHS terms, such as $\vec{u} \times \vec{B}$. Thus, we want

$$\frac{|\eta\vec{J}|}{|\vec{u} \times \vec{B}|} \ll 1$$

Well, from Braginskii we know that the electrical resistivity $\eta \approx \frac{m_e}{ne^2\tau_{ei}}$. Thus,

the field lines?" Hong wrote "There are different ways to look at this problem. 1) If you look from the perspective of maintaining a fluid element, in the transverse direction, small Larmor radius indeed plays a role. Freidberg's book on ideal MHD discusses this in the first two chapters. 2) If you look at the moments of kinetic equation, it turns out that the continuity equation and momentum equation are pretty much always valid. It the energy equation that we have to put in a lot of assumptions." Ian wrote "In a toroidal device, the periodicity along the toroidal dimension also helps. There are many cases where the fluid model fails rather dramatically however; for instance, the MHD fluid model predicts much more rapid damping of poloidal flows, since it enforces equipartition between $v_{\parallel}'$ (or equivalently, $T_{\parallel}$) and $'v_{\perp}$ (or equivalently, $T_{\perp}$), which naturally diverge from each other over the course of a poloidal orbit (as you can check from the conservation of adiabatic invariants p_ϕ and mu). However, in solving for static equilibria and small perturbations around these equilibria, it often works quite well. So it really depends on the application as well."

our inequality can be written approximately as[67]

$$\frac{m_e}{ne^2\tau_{ei}}\frac{|\vec{J}|}{V_{Ti}B} \ll 1 \tag{4.44}$$

Now, from the momentum balance equation we have $|\vec{J}| \sim |\vec{\nabla}P_i|/|\vec{B}|$. From $P_i \sim nm_iV_{Ti}^2$, we get $|\vec{J}| \sim \frac{m_inV_{Ti}^2}{aB}$. So

$$\frac{m_e}{ne^2\tau_{ei}}\frac{|\vec{J}|}{V_{Ti}B} = \frac{m_e}{ne^2\tau_{ei}}\frac{m_inV_{Ti}^2}{aV_{Ti}B^2} = \frac{m_em_iV_{Ti}}{e^2\tau_{ei}aB^2} \tag{4.45}$$

We also know that $\tau_{ei} \sim \tau_{ee} \sim (\frac{m_e}{m_i})^{\frac{1}{2}}\tau_{ii}$. Putting these together into equation 4.45, we have

$$\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{m_i^2V_{Ti}^2}{e^2B^2}\frac{1}{aV_{Ti}\tau_{ii}} = \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\left(\frac{\rho_i}{a}\right)^2\left(\frac{V_{Ti}\tau_{ii}}{a}\right)^{-1} \ll 1 \tag{4.46}$$

This is the third assumption we are going to make in Ideal MHD - that the resistive term is negligible. This puts an additional requirement on the collisionality of the plasma: the collision time must be small to satisfy the first assumption of Ideal MHD, but this equation says that it can't be so small that the $(\frac{V_{Ti}\tau_{ii}}{a})^{-1}$ term becomes too large.

As a side note, there is a version of MHD where the resistive term is *not* negligible, called resistive MHD. In this form of MHD, the first two assumptions of Ideal MHD are satisfied, but the third is not. In this case, the resistive MHD Ohm's law becomes $\vec{E}+\vec{u}\times\vec{B} = \eta\vec{J}$. Restistive MHD is most likely to be useful in describing astrophysical plasmas, which often have very small magnetic fields and hence large Larmor radii ρ_i.

4.3.4 Ideal MHD Momentum Equation

Let's look at the terms on the RHS of equation 4.34,

$$-\vec{\nabla}\cdot(\overleftrightarrow{\Pi}_i + \overleftrightarrow{\Pi}_e)$$

Using the three assumptions made in Ideal MHD, hopefully we can show that these terms are both zero. Well, from Braginskii it can be shown that the leading-order effect on the matrix elements of $\overleftrightarrow{\Pi}$ is ion viscosity (the electron viscosity is smaller by a factor of $(\frac{m_e}{m_i})^{\frac{1}{2}}$), which Braginskii calculates to be

$$\Pi_{jj} \sim \mu\Big(2\vec{\nabla}_\parallel\cdot\vec{v}_\parallel - \frac{2}{3}\vec{\nabla}\cdot\vec{v}\Big) \sim \mu\frac{V_{Ti}}{a} \tag{4.47}$$

where $\mu \sim nT_i\tau_{ii} \sim nm_iV_{Ti}^2\tau_{ii}$. Using $P_i \sim nk_BT_i \sim nm_iV_{Ti}^2$, can compare the magnitude of these terms with the scalar pressure.

$$\frac{|\vec{\nabla}\cdot\overleftrightarrow{\Pi}_i|}{|\vec{\nabla}P_i|} \sim \mu\frac{V_{Ti}}{a^2}\frac{a}{nm_iV_{Ti}^2} \sim (\frac{\tau_{ii}V_{Ti}}{a}) \ll 1$$

[67] We're getting a bit ahead of ourselves. I was hoping not to look at the specific terms in the MHD equations quite yet, but I can't think of another way to understand resistive diffusion.

This is much less than 1 by the high collisionality assumption. Since the ion viscosity is larger than the electron viscosity, $\Pi_i \gg \Pi_e$, and $\vec{\nabla} P_i \gg \vec{\nabla} \cdot \overleftrightarrow{\Pi}_i$, then the terms on the RHS of the momentum equation are both negligible. We'll see later that in ideal MHD, $P_i = P_e$ and $P = P_i + P_e$, so we can write our ideal MHD momentum equation as

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} P + \vec{J} \times \vec{B} \tag{4.48}$$

This is one of the most important equations in all of plasma physics.

4.3.5 Ideal MHD Ohm's Law

Let's look at the terms on the RHS of equation 4.35

$$\frac{1}{en} \vec{J} \times \vec{B} + \eta \vec{J} - \frac{1}{en} \vec{\nabla} P_e - \frac{1}{en} \vec{\nabla} \cdot \overleftrightarrow{\Pi}_e$$

Each of these terms we will show is negligible, based on the assumptions of ideal MHD. The $\eta \vec{J}$ term we've already shown to be negligible relative to the $\vec{u} \times \vec{B}$ term on the LHS, based on the low resistivity assumption. We now go through the other terms one by one and show that they are negligible as well.

As we've shown for the momentum equation using Braginskii-based arguments, $\frac{|\vec{\nabla} \cdot \overleftrightarrow{\Pi}_e|}{|\vec{\nabla} P_e|} \ll 1$. Thus, the fourth term in the above expression is negligible relative to the third term. From the momentum equation, $\vec{J} \times \vec{B} \sim \vec{\nabla} P_e$, so the first and third terms are of similar magnitude. This means that all we have to do is show that the *first* term is negligible relative to $\vec{u} \times \vec{B}$, and the third term must be as well.

$$\frac{|\vec{\nabla} P_e|}{|en\vec{u} \times \vec{B}|} \sim \frac{m_i V_{Ti}^2}{e V_{Ti} B a} \sim \frac{\rho_i}{a} \ll 1$$

where we have used the small gyroradius assumption. Thus, all of the terms on the RHS of Ohm's law are negligible, and we are left with the ideal MHD Ohm's law,

$$\vec{E} + \vec{u} \times \vec{B} = 0 \tag{4.49}$$

As I mentioned before, if the low resistivity assumption is not satisfied, but the other two assumptions are satisfied, then this becomes the resistive MHD ohm's law,

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{J} \tag{4.50}$$

4.3.6 Ideal MHD Energy Equation

Let's look at the terms on the RHS of equation 4.36 and 4.37,

$$\frac{2}{3\rho^\gamma} \Big[\Big(\frac{\partial W_i}{\partial t} \Big)_e - \vec{\nabla} \cdot \vec{Q}_i - \overleftrightarrow{\Pi}_i : \vec{\nabla} \vec{u} \Big]$$

$$\frac{2}{3\rho^\gamma} \Big[\Big(\frac{\partial W_e}{\partial t} \Big)_i - \vec{\nabla} \cdot \vec{Q}_e - \overleftrightarrow{\Pi}_e : \vec{\nabla} (\vec{u} - \frac{\vec{J}}{en})) + \frac{1}{en} \vec{J} \cdot \vec{\nabla} \Big(\frac{P_e}{\rho^\gamma} \Big) \Big]$$

The terms with $\overleftrightarrow{\Pi}_\sigma$ we'll be able to set to zero relative to $\frac{\partial P_i}{\partial t}$,[68] as we'll show below. Remember that the leading order contribution to $\overleftrightarrow{\Pi}_i$ is viscosity, which (as we showed above) goes like $\mu\frac{V_{Ti}}{a}$, where $\mu \sim nT_i\tau_{ii}$. Remember also that the ion viscosity is larger than electron viscosity by a factor $(\frac{m_e}{m_i})^{\frac{1}{2}}$, so $|\overleftrightarrow{\Pi}_e| \sim (\frac{m_e}{m_i})^{\frac{1}{2}}|\overleftrightarrow{\Pi}_i|$. Thus,

$$\frac{|\overleftrightarrow{\Pi}_e : \vec{\nabla}(\vec{J}/en)|}{|\frac{\partial P_e}{\partial t}|} \sim (\frac{m_e}{m_i})^{\frac{1}{2}}\mu\frac{V_{Ti}}{a}\frac{m_i n V_{Ti}^2}{Bena^2}\frac{1}{\omega m_i n V_{Ti}^2} \sim \Big(\frac{m_e}{m_i}\Big)^{\frac{1}{2}}\Big(\frac{\tau_{ii}V_{Ti}}{a}\Big)\Big(\frac{m_i V_{Ti}^2}{Bea^2\omega}\Big)$$

Using $V_{Ti}/a \sim \omega$ and $\rho_i \sim \frac{mV_{Ti}}{eB}$, this becomes

$$\Big(\frac{m_e}{m_i}\Big)^{\frac{1}{2}}\Big(\frac{\tau_{ii}V_{Ti}}{a}\Big)\Big(\frac{\rho_i}{a}\Big) \ll 1$$

where the terms in parentheses are all much less than 1, due to the high collisionality and small gyroradius assumptions. Performing a similar analysis on the other terms, we have

$$\frac{|\overleftrightarrow{\Pi}_i : \vec{\nabla}\vec{u}|}{|\frac{\partial P_e}{\partial t}|} \sim \mu\frac{V_{Ti}}{a}\frac{V_{Ti}}{a}\frac{1}{\omega m_i n V_{Ti}^2} \sim \frac{\tau_{ii}V_{Ti}}{a} \ll 1$$

Because the ion viscosity is larger than the electron viscosity by a factor of $(\frac{m_i}{m_e})^{\frac{1}{2}}$, the following term must give us the same result except smaller by that ratio.

$$\frac{|\overleftrightarrow{\Pi}_e : \vec{\nabla}\vec{u}|}{|\frac{\partial P_e}{\partial t}|} \sim \Big(\frac{m_e}{m_i}\Big)^{\frac{1}{2}}\frac{\tau_{ii}V_{Ti}}{a} \ll 1$$

and also

$$\frac{|(\vec{J}\cdot\vec{\nabla}P_e)/en|}{|\frac{\partial P_e}{\partial t}|} \sim \frac{m_i n V_{Ti}^2}{a^2 Ben}\frac{1}{\omega} \sim \Big(\frac{\rho_i}{a}\Big) \ll 1$$

With these four simplifications, the only remaining terms on the RHS of the energy equations are the $\vec{\nabla}\cdot\overleftrightarrow{Q}_\sigma$ and $(\frac{\partial W_\sigma}{\partial t})_\alpha$ terms. We'll now argue that these terms are negligible as well. Let's think about the physical significance of the $(\frac{\partial W_\sigma}{\partial t})_\alpha$ term: this tells us the rate per volume of thermal energy transfer to species σ due to collisions with species α. If the temperature of each species is equal, we would expect that no energy is transferred between species. The timescale over which energy equilibrates is τ_{eq}, the energy equilibration time. Generally the energy equilibration time is quite long, larger than τ_{ii} by a factor $(\frac{m_i}{m_e})^{\frac{1}{2}}$. Based on these physical arguments, it makes sense that we can write this term[69] as

$$\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha \sim \frac{n(T_\alpha - T_\sigma)}{\tau_{eq}}$$

[68]Why do we compare these terms to $\frac{\partial P_i}{\partial t}$? Well, on the LHS one of the terms is $\frac{1}{\rho^\gamma}\frac{\partial P_i}{\partial t}$, but there is also a leading $\frac{1}{\rho^\gamma}$ term on the RHS which cancels when we take the ratio.

[69]Presumably Braginskii derives this as well, although I'm not sure. Make sure it makes sense to you why this term would have this form.

Now, this term is *not* small compared to the LHS[70], *unless* we assume that $T_i \approx T_e$[71]. For this to be the case, the energy equilibrium time has to be short relative to the characteristic timescale over which the plasma varies ($\frac{1}{\omega}$). If the energy equilibrium time is short, the temperature of the two species will be the same. Mathematically, we can write this as $\tau_{eq}\omega \ll 1$, to ensure that $T_e \approx T_i$. But since $\omega \sim V_{Ti}/a$ and $\tau_{eq} \sim (\frac{m_i}{m_e})^{\frac{1}{2}}\tau_{ii}$, then we can write this requirement as

$$\Big(\frac{m_i}{m_e}\Big)^{\frac{1}{2}}\Big(\frac{V_{Ti}\tau_{ii}}{a}\Big) \ll 1$$

But this is exactly the *very* high collisionality assumption of ideal MHD made in equation 4.42. To get temperature equilibration, we require a factor $(\frac{m_i}{m_e})^{\frac{1}{2}}$ better than the high collisionality assumption from equation 4.41. This is the first time in the ideal MHD derivation where we have require the *very* high collisionality assumption and not just the high collisionality assumption.[72] However, we really do have to satisfy the *very* high collisionality assumption and not just the high collisionality assumption, for us to be able to set the ion and electron temperatures equal and thereby neglect this term for both ions and electrons.

Having neglected all these terms, only one term is remaining on the RHS of the electron and ion energy equations.

$$\frac{d}{dt}\Big(\frac{P_i}{\rho^\gamma}\Big) = -\frac{2}{3\rho^\gamma}\Big[\vec{\nabla}\cdot\vec{Q}_i\Big]$$

$$\frac{d}{dt}\Big(\frac{P_e}{\rho^\gamma}\Big) = -\frac{2}{3\rho^\gamma}\Big[\vec{\nabla}\cdot\vec{Q}_e\Big]$$

Since we've argued that the electron and ion temperatures must be equal in ideal MHD, and MHD requires that the electron and ion densities are equal, then the electron and ion pressures are equal, and the total pressure $P = P_{MHD} = P_e + P_i$. Thus, we can add these two energy equations together to get

$$\frac{d}{dt}\Big(\frac{P}{\rho^\gamma}\Big) = -\frac{2}{3\rho^\gamma}\Big[\vec{\nabla}\cdot(\vec{Q}_i + \vec{Q}_e)\Big]$$

So what about the heat flux terms remaining on the RHS? Well, Braginskii shows that for a collisional, magnetized plasma, the heat flux is proportional to the temperature gradient.[73] In fact, the heat flux is strongest *parallel* to the local magnetic field.[74] Thus, we can ignore the heat flux in the perpendicular

[70] You can convince yourself of this by comparing it with the $\frac{\partial P_\sigma}{\partial t}$ term, and setting $T_\alpha \approx 0$.

[71] Of course, all of this assumes that a temperature even *exists* in the first place. However, since we've assumed high collisionality in ideal MHD, we have a Maxwellian plasma distribution and thus it makes sense to talk about our plasma having a definite temperature.

[72] Actually, this is a lie. In the MHD momentum equation we set $P_e = P_i$, which requires us using the very high collisionality assumption so that temperatures are equal.

[73] This makes sense. It's just like Fourier's law of thermal conduction used to model most materials.

[74] This makes sense, because perpendicular to the magnetic field the particles are restricted, while particles stream freely parallel to the field. Thus, the particles can convect their heat more easily in the parallel direciton

direction relative to that in the parallel direction. We can write the heat flux in the parallel direction as $Q_{\sigma\parallel} \approx -\kappa_\parallel \vec{\nabla}_\parallel T$. Thus, this term becomes

$$\frac{d}{dt}\Big(\frac{P}{\rho^\gamma}\Big) = -\frac{2}{3\rho^\gamma}\Big[\vec{\nabla}_\parallel\Bigg((\kappa_{\parallel i} + \kappa_{\parallel e})\vec{\nabla}_\parallel T\Big)\Big]$$

Braginskii also shows that the electron parallel thermal conductivity is larger than the ion parallel thermal conductivity by the ratio $(\frac{m_i}{m_e})^{\frac{1}{2}}$. Braginskii also shows that $\kappa_{\parallel e} \sim nT_e\tau_{ee}/m_e$. Thus, comparing this term with $\frac{\partial P}{\partial t}$, we have

$$\frac{\vec{\nabla}_\parallel(\kappa_{\parallel e}\vec{\nabla}_\parallel T)}{\frac{\partial P}{\partial t}} \sim \frac{1}{a^2}\frac{nT^2\tau_{ee}}{m_e}\frac{1}{\omega nT} \sim \frac{m_i V_{Ti}^2}{m_e a^2 \omega}\frac{m_e^{\frac{1}{2}}\tau_{ii}}{m_i^{\frac{1}{2}}} \sim \Big(\frac{m_i}{m_e}\Big)^{\frac{1}{2}}\Big(\frac{V_{Ti}\tau_{ii}}{a}\Big) \ll 1$$

which is much less than 1 by the *very* high collisionality assumption of equation 4.42. Thus, the heat flux term is negligible too, and we are left with our ideal MHD energy equation (also called the ideal MHD equation of state)

$$\frac{d}{dt}\Big(\frac{P}{\rho^\gamma}\Big) = 0 \tag{4.51}$$

This concludes the derivation of ideal MHD.

4.3.7 Summary of Assumptions Made in Ideal MHD

I've compiled the equations of ideal MHD below.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \tag{4.52}$$

$$\rho\frac{d\vec{u}}{dt} - \vec{J}\times\vec{B} + \vec{\nabla}P = 0 \tag{4.53}$$

$$\vec{E} + \vec{u}\times\vec{B} = 0 \tag{4.54}$$

$$\frac{d}{dt}\Big(\frac{P}{\rho^\gamma}\Big) = 0 \tag{4.55}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4.56}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{4.57}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \tag{4.58}$$

Wow, what a ride this chapter has been. We started with the Vlasov-Maxwell equation, and after 30 pages of algebra and explanations we've finally arrived at our destination. I've already recapped how we derived the multi-fluid equations from the Vlasov equation, and how we derived the MHD equations from the multi-fluid equations. Now let's recap how we derived the ideal MHD equations from the MHD equations.

Starting with the MHD equations, we required the following assumptions to derive the ideal MHD model:

- The characteristic velocity of MHD is much smaller than the ion thermal velocity V_{Ti}. This is a reasonable assumption, as the electrons are assumed to respond infinitely quickly to any applied field so the velocity can be no faster than the ion thermal velocity. This allows us to approximate $\vec{u}$ as V_{Ti} when estimating the size of the terms we eventually want to neglect. This also allows us to introduce a characteristic frequency $\omega \sim \frac{V_{Ti}}{a}$ which tells us the maximum frequency (and timescale) of disturbances in the MHD model.

- *Very* high collisionality, described by equation 4.42. Physically, this requirement is that the energy equilibration time be short compared to the ion thermal velocity divided by the system size.

- Small gyroradius relative to the system size, as described by equation 4.43.

- Low resistivity, as described by equation 4.46. This ensures that the plasma is frozen into the field lines, and it is sometimes said that resistive diffusion is negligible.

- The plasma is magnetized, so that the Braginskii coefficients can be used to approximate the RHS terms. Braginskii assumes a collisional,[75] magnetized plasma, which is consistent with the high collisionality and small gyroradius assumptions of ideal MHD.

Making each of these assumptions, we compare the terms on the RHS of equations 4.33-4.37 to the terms of the LHS and show that each of the RHS terms is negligible relative to the LHS terms. To justify the neglect of these terms, we refer to Braginskii's transport coefficients, which are cited but not derived in these notes. Once we've neglected the RHS terms, we have ideal MHD.

4.3.8 The Electric Field in Ideal MHD

Most of this entire chapter covers material not covered in GPP1 and covered quickly in GPP2. However, I think the details of these derivations are extremely important, which is why I've chosen to include this chapter in these notes. There is one more detail about ideal MHD which I think is important, but is tricky to understand. That detail is what happens with the $\vec{E}$ field in ideal MHD.

In ideal MHD, the electric field $\vec{E}$ is not an independent variable. Rather, the electric field can be calculated from the velocity field and magnetic field through Ohm's law $\vec{E} = -\vec{u} \times \vec{B}$. In other words, $\vec{u}$ and $\vec{B}$ are dynamical variables, while $\vec{E}$ is not a dynamical variable but rather a calculated variable which 'comes along for the ride'. As a consequence of this, $\vec{E}_\parallel$ is always zero in ideal MHD, while $\vec{E}_\perp$ isn't necessarily zero.

[75] Since Braginskii presumably allows for non-equal ion and electron temperatures, Braginskii only requires the high collisionality requirement and not the *very* high collisionality requirement of ideal MHD. This requirement is that the ion collision time small compared with the ion thermal velocity divided by the system size.

What about the divergence of $\vec{E}$? I've always been confused why we don't include $\vec{\nabla} \cdot \vec{E} = 0$ as one of the equations of MHD. After all, we've made the assumption that $n_e = n_i$, so $\vec{\nabla} \cdot \vec{E}$ must be 0, right? Well, take the divergence of the ideal MHD ohm's law. You'll see that $\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{u} \times \vec{B})$, which is not necessarily zero since $\vec{u}$ and $\vec{B}$ are dynamical variables which can take any value. In ideal MHD we have that $n_e = n_i$, AND that $\vec{\nabla} \cdot \vec{E} \neq 0$. How is that possible?

This all gets very confusing very quickly. Here is how I think about it.[76] From Ohm's law, we can calculate $\vec{E}$. This ensures that $E_{\parallel}$ is zero, and we only have $\vec{E}_{\perp}$. There are two components of $\vec{E}_{\perp}$ - the curl-free electrostatic component and the divergence-free electromagnetic component. The divergence-free electromagnetic component is not zero when we plug $\vec{E}$ into Ampere's law to solve for $-\frac{\partial \vec{B}}{\partial t}$, so this is the component that is important. The curl-free electrostatic component *is* zero when we plug it into Faraday's law, so this component doesn't matter in solving for the dynamical variables of MHD, $\vec{u}$ and $\vec{B}$. This is why Freidberg says that ideal MHD treats the electrostatic component of the electric field incorrectly and the electromagnetic component correctly.

Here is a second, perhaps easier way to think about the electric field in ideal MHD: we don't even need to solve for it in the first place. With Ohm's law and Faraday's law, we can eliminate the electric field entirely. We replace these two equations with a single equation

$$\vec{\nabla} \times (\vec{u} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}$$

which means we can, if we so desire, completely forget about the existence of the electric field in ideal MHD and just focus on solving for $\vec{u}$ and $\vec{B}$.

4.4 MHD Equilibrium

You'll hear a lot about MHD equilibrium in GPP2, and you'll also solve some problems related to it in GPP1. They're important topics, but relatively straightforward, at least at the simple level we see in GPP1. For that reason, I'm not going to bother writing about MHD equilibrium in these notes, except for the most basic concept. We start with the ideal MHD equation of motion,

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} P + \vec{J} \times \vec{B} \tag{4.59}$$

Often, we want to know if some zero-velocity plasma configuration is in equilibrium and whether it is stable. The question of whether it is stable is trickier than the question of whether it is an equilibrium, and we'll wait until GPP2 to tackle that problem. However, the question of finding a zero-velocity plasma equilibrium is easy: all we need to do is set the LHS of the ideal MHD momentum equation to zero, because in equilibrium the time-derivatives and velocity

[76] Much thanks to Amitava for some helpful discussions, as well as the books by Freidberg and Choudhuri for helping me figure this subtlety out.

are zero. Thus, some plasma geometry is in an MHD equilibrium if the following equation is satisfied:

$$\vec{\nabla} P = \vec{J} \times \vec{B} \tag{4.60}$$

5 Waves in Plasmas

Then (Fermi) said "Of course such waves could exist." Fermi had such authority that if he said "of course" today, every physicist said "of course" tomorrow.

HANNES ALFVÉN, ON MHD WAVES

There are lots of waves in plasmas. As a first-year student, I find keeping track of the the different waves we learn about tends to be confusing. However, it is also a very important topic, worthy of an entire course during the second year. Thus, understanding the topic in depth seems rather important.

As far as I can tell, there are three main things we need to keep track of when thinking about plasma waves. Firstly, we need to remember the name of the wave. Unfortunately, many of the waves have multiple names, so this becomes rather inconvenient. Secondly, we need to remember the dispersion relation of the wave. This tells us about the group and phase velocity of the wave, and sometimes whether it can propogate at all. Thirdly, we need to remember the assumptions made in deriving the dispersion relation, so we can determine when we might expect that wave to arise in physical situations. At the end of this chapter, we'll have a better understanding of these three things for a few of the most fundamental waves in plasmas. I'll do my best to help the reader gain a physical and geometrical understanding of each of the waves that we investigate, so that they can have some intuition for how and why each wave propogates. However, getting a physical picture is not going to be possible for all of the various waves we discuss in this chapter and beyond.[77]

In Chapter 1, we started by deriving the most basic of waves, plasma oscillations. These oscillations were derived assuming stationary ions, zero temperature, zero magnetic field, and using a fluid description for the electrons. By linearizing the equations and rearranging, we obtained a characteristic frequency of $\omega_P^2 = \frac{e^2 n_0}{\epsilon_0 m_e}$.

We will start this chapter by again looking at electrostatic plasma oscillations, but this time looking at the effect of a finite temperature on the oscillations. We'll use the Vlasov-Maxwell equation as opposed to the electron fluid equation. We will derive a dispersion relation for electrostatic plasma oscillations which will take us to the world of complex functions. We'll save the pain of that subject until chapter 6, and in this chapter just find a dispersion relation for the first order correction to plasma oscillations due to temperature effects. Whenever we take a first-order correction of the Vlasov equation due to finite-temperature effects, we say we are making a "warm-plasma approximation" or just that we have a "warm plasma". This warm-plasma plasma oscillation has a name: Langmuir Waves. We'll then look at lower-frequency electrostatic waves in a warm plasma,[78] where the electrons thermal velocity is faster than the wave

[77] "Beyond" means AST553, the plasma waves course taught by Ilya Dodin.

[78] I'm using the term warm plasma, where it is understood that by saying "warm" we mean

phase velocity. In this limit, the waves are called ion acoustic waves. We'll also see that we can't have an electrostatic wave which has a phase velocity slower than both the electron *and* ion thermal velocity, for if we did than this the electric field would just be Debye-shielded for both the ions and the electrons.

In the second part of the chapter, we'll look at the fluid description of waves. First, we'll revisit zero-temperature zero-velocity electrostatic plasma oscillations, allowing for the possibility that the ions move as well. There is a lot of interesting physics to uncover in these oscillations. After that, we'll again look at Langmuir waves and ion acoustic waves, but from a fluid description rather than a kinetic description. We'll see that both these waves can be derived from the fluid description, only with different assumptions about the frequency of the wave relative to the thermal velocity of each species. Lastly, we'll derive the dispersion relation for electromagnetic waves in plasmas, which amazingly have a phase velocity faster than the speed of light! We'll also learn why glass is transparent but metals are not.

In the MHD description of waves, we'll look at the fundamental MHD waves which arise in a uniform background plasma with a magnetic field. Unlike the waves from earlier in the chapter, these waves are due to the interaction of magnetic fields with plasma rather than due to electrostatic oscillations or the interaction of electromagnetic radiation with plasma. There are three waves which arise in magnetized plasmas: the Alfvén wave, and the fast and slow waves. Each of these waves will be discussed in detail.

Lastly, we'll investigate the streaming instability which arises when two plasma species have different net velocities.

5.1 Kinetic Description of Waves

As promised, we'll start by looking at electrostatic plasma oscillations using the Vlasov-Maxwell equation. Here, we allow the possibility that the particles have some non-zero temperature, i.e. the velocity distribution function is not simply a delta function at each point. We'll go on to see that this approach *fails*, because we get a singularity in the solution. To resolve these singularities, we'll need to use Landau damping. For now, we simply expand our solution to first order, which allows us to solve for the existence of two types of electrostatic waves in a warm plasma.

Our starting point is the collisionless unmagnetized Vlasov-Maxwell equation,

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma - \frac{q_\sigma}{m_\sigma} \vec{\nabla} \phi \cdot \vec{\nabla}_v f_\sigma = 0 \tag{5.1}$$

Now, we will apply the method of linearization to the Vlasov-Maxwell equation. We assume our 0th order solution is not time-dependent or spatially dependent, and also assume that the 0th order electric and magnetic fields are 0, so we only get a ϕ_1 term. The induced-B term, $\vec{B}_1$, is small relative to $\vec{E}_1$. We know

we look at the first-order correction of the Vlasov equation due to finite-temperature effects.

this because for electrostatic oscillations the first-order electric field should be curl-free, so $-\frac{\partial \vec{B}_1}{\partial t} = 0$, which implies that $\vec{B}_1$ stays at zero.

Our goal is going to be to solve for the first-order perturbation to f, $f_{\sigma 1}$. Once we have this, we can integrate it to solve for the first-order density perturbation $n_{\sigma 1}$. This first-order density is sometimes called the σ *response function* (electron response function or ion response function). Our next step is to plug the response functions for the various species into Gauss's law to solve for our dispersion relation. Let's see how this works. Linearizing the Vlasov-Maxwell equation, we get

$$\frac{\partial f_{\sigma 1}}{\partial t} + \vec{v} \cdot \vec{\nabla} f_{\sigma 1} - \frac{q_\sigma}{m_\sigma} \vec{\nabla} \phi_1 \cdot \vec{\nabla}_v f_{\sigma 0} = 0 \tag{5.2}$$

Notice all of the quantities are, 1st order in total. In order words, each term is either 1st order in f, or 0th order in f and 1st order in some other quantity (here, this quantity is ϕ). When we linearize around a known solution, typically the quantities which contain no first-order terms either go to zero or cancel each other. This is indeed the case here, as there is no time-dependence or spatial dependence of the 0th order solutions and there are no 0th order fields.

Now, we are looking for wave solutions, so we assume an exponential dependence $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ for each of the first-order quantities. This gives

$$-i\omega f_{\sigma 1} + \vec{v} \cdot i\vec{k} f_{\sigma 1} - \frac{q_\sigma}{m_\sigma} i\phi_1 \vec{k} \cdot \vec{\nabla}_v f_{\sigma 0} = 0 \tag{5.3}$$

Cancelling i and solving for $f_{\sigma 1}$ (which we want so we can solve for the response functions), we get

$$f_{\sigma 1} = \frac{q_\sigma}{m_\sigma} \frac{\phi_1 \frac{\partial f_{\sigma 0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} \tag{5.4}$$

where the parallel means parallel to $\vec{k}$. We can solve for the perturbed density (i.e. the response function) $n_{\sigma 1}$, by integrating over velocity.

$$n_{\sigma 1} = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial f_{\sigma 0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} d^3\vec{v} = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} dv_\parallel \tag{5.5}$$

where $g_\sigma = \int f_{\sigma 0} d^2\vec{v}_\perp$. If we know our $f_{\sigma 0}$, then we just perform that integral, plug it into Gauss's law, and we have our dispersion relation. So we're gucci, right? Unfortunately, there's a problem: we've got a $v_\parallel - \frac{\omega}{k}$ term in the denominator. When we integrate, this term will go to zero at some $v_\parallel$, and if there is some finite $f_{\sigma 0}$ at that $v_\parallel$, then this integral will blow up and we have infinite density, which is not what we want. Unless of course we want to make fusion energy, in which case business is boomin'. There isn't a simple solution to this problem, and you'll have to stay tuned until Chapter 6 to see what Landau damping is and how it resolves this issue. For now, we'll look at limiting cases of this integral, and use these limiting cases to investigate various types of electrostatic plasma waves.

Now, there are two limiting cases of this integral we are interested in. The first is called the adiabatic case, where the changes due to the wave occur so quickly that the particles don't have time to react. Mathematically, this means $\frac{\omega}{k} \gg V_{T\sigma}$. The second case is called the isothermal case, where the changes due to the wave are so slow that the particles have plenty of time to react to the wave's behavior. Mathematically, this means $\frac{\omega}{k} \ll V_{T\sigma}$. Before we worry about solving for dispersion relations for various types of waves, let's worry about calculating the response functions for an arbitrary species σ in these two limiting cases. Once we have the response function in these limits solved for, then solving for the dispersion relation of Langmuir and Ion Acoustic waves will be much easier.

Let's first solve for the response function for species σ in the adiabatic limit, where $\frac{\omega}{k} \gg V_{T\sigma}$. In this limit, $f_{\sigma 0}$ is essentially zero at the phase velocity of the wave, because most of the particles are similar in velocity to the thermal velocity and because the thermal velocity is so much lower than $\frac{\omega}{k}$. Thus, the integral doesn't blow up because the portion of the integral which would otherwise blow up has an effectively zero numerator. To solve for $n_{\sigma 1}$ in this limiting case, the first thing we'll do is integrate the integrand by parts. This gives

$$\int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} dv_\parallel = \int \frac{g_\sigma}{(v_\parallel - \frac{\omega}{k})^2} dv_\parallel + \Big[\frac{g_\sigma}{v_\parallel - \frac{\omega}{k}}\Big]_{-\infty}^{\infty}$$

The last term will go to zero because f_σ and hence g_σ is zero at $v_\parallel \to \pm\infty$. Now, we can write

$$\frac{1}{(v_\parallel - \frac{\omega}{k})^2} = \frac{k^2}{\omega^2}\frac{1}{(1 - \frac{kv_\parallel}{\omega})^2}$$

In the adiabatic limit where $\frac{kV_{T\sigma}}{\omega} \ll 1$ we can Taylor expand using the small-x expansion

$$(1+ax)^b = 1 + abx + a^2(b)(b-1)\frac{x^2}{2!} + ...$$

to get

$$\frac{1}{(v_\parallel - \frac{\omega}{k})^2} \approx \frac{k^2}{\omega^2}\Big(1 + 2\frac{kv_\parallel}{\omega} + 3\frac{v_\parallel^2 k^2}{\omega^2}\Big)$$

With these manipulations, our integral for $n_{\sigma 1}$ becomes

$$n_{\sigma 1} = \frac{q_\sigma \phi_1}{m_\sigma}\frac{k^2}{\omega^2}\int g_\sigma\Big(1 + 2\frac{kv_\parallel}{\omega} + 3\frac{k^2 v_\parallel^2}{\omega^2}\Big)dv_\parallel \tag{5.6}$$

Remember the definition of g_σ, $\int f_{\sigma 0} d^2\vec{v}_\perp$. From this definition, we can see that the first term integrates to $n_{\sigma 0}$. If the mean velocity in the parallel direction is zero, the second term integrates to zero. The third term can't be calculated exactly unless we know $f_{\sigma 0}$ but is, approximately, $3V_{T\sigma}^2\frac{k^2}{\omega^2}n_{\sigma 0}$. Finally, after all that work, we have $n_{\sigma 1}$ in the adiabatic limit where $\frac{\omega}{k} \gg V_{T\sigma}$.

$$n_{\sigma 1} = \frac{q_\sigma n_{\sigma 0}}{m_\sigma}\frac{k^2\phi_1}{\omega^2}\Big(1 + 3\frac{k^2 V_{T\sigma}^2}{\omega^2}\Big) \tag{5.7}$$

Okay, so we've got the response function for σ in the adiabatic limit. What about the response function for σ in the isothermal limit where $\frac{\omega}{k} \ll V_{T\sigma}$? We can't solve for this exactly. However, if our zeroth order distribution function is Maxwellian, then we can solve the integral in equation 5.5 to get something nice. Let's do this now. If we have a Maxwellian[79]

$$f_{\sigma 0} = n_{\sigma 0}\Big(\frac{m_\sigma}{2\pi k_B T_\sigma}\Big)^{3/2} \exp\Big(-\frac{m_\sigma v^2}{2k_B T_\sigma}\Big)$$

and so integrating over the perpendicular directions gives

$$g_\sigma = n_{\sigma 0}\Big(\frac{m_\sigma}{2\pi k_B T_\sigma}\Big)^{1/2} \exp\Big(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}\Big)$$

Taking the derivative,

$$\frac{\partial g_\sigma}{\partial v_\parallel} = -n_{\sigma 0}\frac{m_\sigma^{3/2} v_\parallel}{(2\pi)^{1/2}(k_B T_\sigma)^{3/2}} \exp\big(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}\big)$$

Now, we ignore the $v_\parallel - \frac{\omega}{k}$ and replace that with just $v_\parallel$. Why do we do this? Well, the numerator goes as $v_\parallel$ for small $v_\parallel$, so if $\frac{\omega}{k}$ is really small as it would be in the isothermal limit, then the numerator is essentially still zero when the denominator goes to zero. We can see this geometrically as well, if we visualize a Maxwellian distribution. Near the peak of a Maxwellian distribution, the derivative of the distribution is about zero because the Maxwellian has a local maximum at it's peak, so this part of the integral can be ignored. Essentially, we're just ignoring the very beginning of our density integral. This assumption is pretty dodgy because the integral technically blows up, but it's one we need to make to solve for the electron response in this isothermal limit. Thus, we'll choose to completely ignore the $\frac{\omega}{k}$ in the denominator. If we make this dodgy assumption, then the $v_\parallel$ on top and in the bottom cancel. We have

$$n_{\sigma 1} = -\frac{q_\sigma m_\sigma^{1/2} n_{\sigma 0}\phi_1}{(k_B T_\sigma)^{3/2}(2\pi)^{1/2}} \int \exp\Big(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}\Big) dv_\parallel \tag{5.8}$$

Using $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, our integral becomes $\sqrt{\frac{2\pi k_B T_\sigma}{m_\sigma}}$, so our response function for σ is

$$n_{\sigma 1} = -\frac{q_\sigma n_{\sigma 0}}{m_\sigma}\frac{m_\sigma}{k_B T_\sigma}\phi_1 = -\frac{q_\sigma n_{\sigma 0}}{m_\sigma V_{T\sigma}^2}\phi_1 \tag{5.9}$$

We've done a lot of algebra so far, but the process has been pretty simple: we linearize the Vlasov-Maxwell equation, look for electrostatic wave solutions, solve for $f_{\sigma 1}$, and integrate to get $n_{\sigma 1}$ (the response function) in either an adiabatic or isothermal limit. Now comes the fruit of our labor: we can plug our response function into Gauss's Law in various limits. Each of these limits corresponds to a new plasma wave.

[79] This isn't a bad assumption in the isothermal limit, because isothermal implies the oscillations are slow and hence the plasma has more time between oscillations to approach a maximum-entropy state.

Table 1: Response Functions, $n_{\sigma 1}$

Adiabatic, $\frac{\omega}{k} \gg V_{T\sigma}$	Isothermal, $\frac{\omega}{k} \ll V_{T\sigma}$
$\frac{q_\sigma n_{\sigma 0} k^2}{m_\sigma \omega^2}\phi_1(1+3\frac{k^2 V_{T\sigma}^2}{\omega^2})$	$-\frac{q_\sigma n_{\sigma 0}}{m_\sigma V_{T\sigma}^2}\phi_1$

5.1.1 Langmuir Wave

The Langmuir wave is the finite-temperature version of the plasma oscillation. This is a fast oscillation, such that the phase velocity of the wave is faster than the thermal velocity of both the ions and the electrons. Thus, both species have an adiabatic response function. The Langmuir wave goes by many names, including the electron plasma wave, the Bohm-Gross wave, or just the Bohm wave. These are just names, and people use them, so we'll use them.

Using an adiabatic response function for both ions and electrons because our wave is *fast*, then Gauss's law becomes

$$-k^2\phi_1 = -\frac{e^2 n_{e0} k^2}{\epsilon_0 m_e \omega^2}\phi_1(1+3\frac{k^2 V_{Te}^2}{\omega^2}) - \frac{q_i^2 n_{i0} k^2}{\epsilon_0 m_i \omega^2}\phi_1(1+3\frac{k^2 V_{Ti}^2}{\omega^2}) \tag{5.10}$$

Notice that we can replace the $\frac{q_\sigma^2 n_{\sigma 0}}{\epsilon_0 m_\sigma}$ with $\omega_{p\sigma}^2$. We expect to see the plasma frequency show up, since the Langmuir wave is a finite-temperature version of the plasma oscillation. We can also cancel the ϕ_1 and the k^2, and multiple by ω^2 to get

$$\omega^2 = \omega_{pe}^2(1+3\frac{k^2 V_{Te}^2}{\omega^2}) + \omega_{pi}^2(1+3\frac{k^2 V_{Ti}^2}{\omega^2}) \tag{5.11}$$

Hey look! We've got a dispersion relation, i.e. an equation for ω in terms of k. Our dispersion relation is 4th-degree polynomial equation for ω. Let's solve this perturbatively. Since the Langmuir wave is the finite-temperature version of the plasma oscillation, we expect our solution to be close to the plasma frequency. Thus, we can approximate the $\frac{1}{\omega^2}$ terms on the right of this equation as $\frac{1}{\omega_{pe}^2}$. This gives

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 + 3k^2 V_{Te}^2 + 3\frac{\omega_{pi}^2}{\omega_{pe}^2} k^2 V_{Ti}^2 \tag{5.12}$$

This is the approximate dispersion relation for Langmuir waves! Actually, this fourth term is extremely small compared to the third term, since $\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{m_e}{m_i}$ and $V_{Ti}^2 = \frac{m_e}{m_i} V_{Te}^2$. Thus, our dispersion relation is approximately

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 + 3k^2 V_{Te}^2 \tag{5.13}$$

Often, it is just written as $\omega^2 = \omega_{pe}^2 + 3k^2 V_{Te}^2$ since the ion terms are much smaller than the electron terms.

Unfortunately, I don't have a good physical picture for the Langmuir wave. While we've seen a nice physical picture for electron plasma oscillations way back in chapter 1, the finite-temperature effects change the dispersion relation of this electrostatic oscillation in a way which is difficult to understand through some physical model. Overall, though, the physical picture is similar to that of plasma oscillations.

5.1.2 Ion Acoustic Wave

The ion acoustic wave is an electrostatic wave where the ions are still adiabatic, but the electrons are isothermal. More explicitly, we have the relation $V_{Ti} \ll \frac{\omega}{k} \ll V_{Te}$. It is called the ion acoustic wave because the wave is a modified sound wave, where the electrons create the pressure force which drives the sound wave and ions carry the mass of the plasma which provides the inertia.[80] Sound waves are in general slow-frequency and long-wavelength relative to the enormously high-frequency, short-wavelength plasma oscillations in a plasma. Thus, we'll assume ahead of time (on physical grounds) that the wavelengths are much longer than the Debye lengths and the frequencies are much lower than the Debye frequency. We'll find that ion acoustic waves are dispersionless, meaning $\frac{\omega}{k}$ does not depend on k.

To solve for the dispersion relation, we again plug our ion and electron response functions into Gauss's law. Because of the phase velocity of the wave, we use the adiabatic response function for the ions and the isothermal response function for the electrons. This gives us

$$-\vec{\nabla}^2\phi_1 = k^2\phi_1 = \frac{1}{\epsilon_0}\Big[-\frac{e^2 n_{e0}\phi_1}{m_e V_{Te}^2} + \frac{q_i^2 n_{i0} k^2 \phi_1}{m_i \omega^2}\Big(1 + 3\frac{k^2 V_{Ti}^2}{\omega^2}\Big)\Big]$$

Dividing by $\phi_1 k^2$ and putting in the plasma frequencies gives

$$1 = -\frac{\omega_{pe}^2}{k^2 V_{Te}^2} + \frac{\omega_{pi}^2}{\omega^2}(1 + 3\frac{k^2 V_{Ti}^2}{\omega^2}) \tag{5.14}$$

Now, we can solve this equation perturbatively as well. To lowest order, since $\frac{\omega}{k} \gg V_{Ti}$, we can drop the $3\frac{k^2 V_{Ti}^2}{\omega^2}$ relative to 1. We'll also drop the LHS completely, because the other two terms which remain are much bigger than 1. How do we know that? Well, we can rewrite $\frac{\omega_{pe}^2}{k^2 V_{Te}^2} = \frac{1}{k^2 \lambda_{De}^2}$. Since $k = \frac{1}{\lambda}$, then this term is $\frac{\lambda^2}{\lambda_{De}^2}$, which, since we're looking at the long-wavelength limit, will be much greater than 1. Thus, the first term on the RHS is much greater

[80] Wikipedia has this to say about the ion acoustic wave: "In contrast to [sound waves in a gas], the pressure and the density are provided by separate species, the pressure by the electrons and the [mass] density by the ions. The two are coupled through a fluctuating electric field." It also says "In plasma physics, an ion acoustic wave is one type of longitudinal oscillation of the ions and electrons in a plasma, much like acoustic waves traveling in neutral gas. However, because the waves propagate through positively charged ions, ion acoustic waves can interact with their electromagnetic fields, as well as simple collisions. In plasmas, ion acoustic waves are frequently referred to as acoustic waves or even just sound waves. They commonly govern the evolution of mass density, for instance due to pressure gradients, on time scales longer than the frequency corresponding to the relevant length scale. Ion acoustic waves can occur in an unmagnetized plasma or in a magnetized plasma parallel to the magnetic field."

than 1. The second term on the RHS is also much greater than 1, because the frequency is assumed to be much less than the ion plasma frequency. Having dropped these two terms, to lowest order we have

$$0 = \frac{\omega_{pe}^2}{k^2 V_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2} \tag{5.15}$$

so to lowest order,

$$\frac{\omega^2}{k^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2} V_{Te}^2 = \frac{m_e}{m_i} V_{Te}^2 = \frac{k_B T_e}{m_i} = c_s^2 \tag{5.16}$$

where we've defined the plasma sound speed (sometimes called the acoustic speed) to be $c_s^2 = \frac{k_B T_e}{m_i}$.

Now let's go back and get the next-order correction. To second order, we still ignore the -1 term, we include the $3\frac{k^2 V_{Ti}^2}{\omega^2}$ term and plug in our first-order ω^2/k^2 solution to this term. This gives

$$\frac{\omega_{pe}^2}{k^2 V_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{V_{Ti}^2 \omega_{pe}^2}{V_{Te}^2 \omega_{pi}^2}\right) = 0 \tag{5.17}$$

Rearranging and simplifying, to the second order we have

$$\frac{\omega^2}{k^2} = c_s^2 + 3V_{Ti}^2 \tag{5.18}$$

Note that the perturbative solution we used only works if $T_e \gg T_i$. Note that this is consistent with the physical picture we described earlier for the Ion Acoustic Wave and described by Wikipedia. The ion acoustic wave propagates at approximately the sound speed, with a modification for the ion thermal velocity. In a normal sound wave, the pressure provides a restoring force while the mass provides inertia resisting acceleration. In an ion acoustic wave, the electron pressure provides the restoring force, while the ion mass provides the inertia.

Let's develop a better physical picture of ion acoustic waves. You've seen a visualization of sound waves in air before, right? If not, check out figure 19. For a sound wave in air, there is a density perturbation (of the molecules in the air) which creates pressure gradients which cause the individual air molecules to vibrate in place. The net effect is that the density/pressure perturbation is propagated in some direction. Now let's think about ion acoustic waves. An ion acoustic wave is the equivalent of a sound wave. So we expect that if a perturbation of the plasma occurs, the pressure gradients will cause the plasma to return to it's original state. In the process, it creates a propogating wave just like a sound wave. However, in a plasma, things are a little more complicated. Pay attention. Let's assume for now that $T_i = 0$, so electrons carry the pressure in the plasma. Suppose there is some perturbation of the plasma mass density. Because ions carry most of the mass of a plasma, this means that the ions are

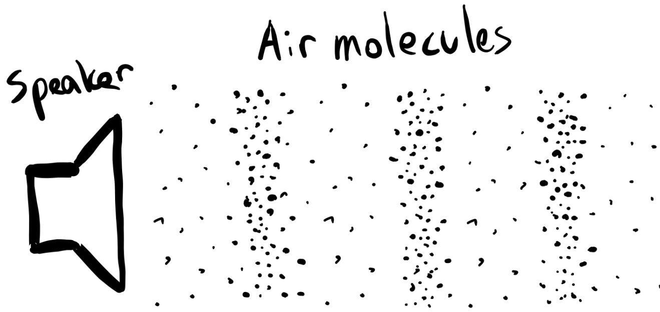

Figure 19: Sound waves in air. Compressions of the molecules in the air create the pressure disturbances which act as the restoring force.

being perturbed. The electrons, moving so much faster than the ions, will *Debye shield* the electric field created by the perturbed ion density. However, in doing so the thermalized electrons have created a *pressure variation* correlated with the variation in the ion density. That pressure variation is the force which drives the propogation of the wave. An electric field from the electrons acting on the ions along the direction of the wave propogation causes the cold ions to move. Note that if $T_i > 0$, this picture is modified slightly because the ions have some thermal velocity, increasing the phase velocity of the wave.[81]

Let's look at ion acoustic waves in a slightly different way. Let's start by plugging the isothermal electron and adiabatic ion response functions into Gauss's law, as before. This gives us

$$-\vec{\nabla}^2\phi_1 = k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1}$$

$$k^2 = \frac{1}{\epsilon_0}\Big[\frac{e^2 n_0}{m_e V_{Te}^2} + \frac{e^2 n_0 k^2}{m_i \omega^2}(1 + 3\frac{k^2 V_{Ti}^2}{\omega^2})\Big]$$

$$k^2 = \frac{e^2 n_0}{\epsilon_0 m_e V_{Te}^2}\Big[1 + \frac{m_e}{m_i}\frac{k^2 V_{Te}^2}{\omega^2}(1 + 3\frac{k^2 V_{Ti}^2}{\omega^2})\Big]$$

Since $\frac{e^2 n_0}{\epsilon_0 m_e} = \omega_{Pe}^2$ and $\frac{\omega_{Pe}^2}{V_{Te}^2} = \frac{1}{\lambda_{De}^2}$, we can multiply by λ_{De}^2 on both sides to get

$$k^2\lambda_{De}^2 = 1 + \frac{m_e}{m_i}\frac{k^2 V_{Te}^2}{\omega^2}(1 + 3\frac{k^2 V_{Ti}^2}{\omega^2}) \tag{5.19}$$

Ion acoustic waves are slow-frequency, long-wavelength oscillations in a plasma where $T_e \gg T_i$. This means that $k^2\lambda_{De}^2 \sim \frac{\lambda_{De}^2}{\lambda^2} \ll 1$. This means we drop

[81] One might say that non-zero ion thermal velocity increases the effective pressure of the plasma, but this isn't really right. That's because the ions are adiabatic over this time and so don't *have* a pressure.

the $k^2\lambda_{De}^2$ term in the above equation relative to 1. Since we haven't added or subtracted anything (only multiplied and divided), then this LHS term still represents the $-\vec{\nabla}^2\phi_1$ side of Gauss's law. So for ion acoustic waves, Gauss's law gives us $\sum_\sigma q_\sigma n_\sigma \approx 0$, or $n_{e1} = n_{i1}$. This is called the *quasineutrality condition* for ion acoustic waves. It tells us that for ion acoustic waves in a plasma, not only is the plasma neutral to zeroth order, but it is neutral to first order as well.

At this point, you might be wondering how it's possible that an ion acoustic wave can exist in the first place. After all, the ion acoustic wave is an electrostatic oscillation at long wavelengths and slow frequencies. But Debye shielding says that there can be no large-scale electric fields in a plasma on slow timescales. It seems like such an oscillation shouldn't be able to exist! To make things even more confusing, we have a non-zero electrostatic electric field ϕ_1,[82] but Gauss's law says that our plasma is totally neutral. To me, these seem like paradoxes. Fortunately, they aren't. The resolution is

Question: what is the resolution?

5.1.3 Isothermal Electrostatic Waves Don't Exist

Imagine that we were trying to find an electrostatic wave where both the ions and electrons were isothermal. Plugging our electron and ion response functions into Gauss's law, we get

$$0 = 1 + \frac{\omega_{pe}^2}{k^2 V_{Te}^2} + \frac{\omega_{pi}^2}{k^2 V_{Ti}^2} \tag{5.20}$$

There is no frequency dependence! Actually, this is an equation for Debye shielding, and is the same as equation 1.22 way back from chapter 1 but without the charge Q. Physically, this tells us that an electrostatic oscillation where the ions and the electrons are both isothermal is not possible, because both the electrons and the ions will Debye shield any disturbances in ϕ and cancel out any oscillation at that slow frequency.

5.2 Plasma Waves in the Fluid Description

5.2.1 Revisiting Plasma Oscillations

Way back in chapter 1, we derived electron plasma oscillations. To do this, we assumed that we had uniform density zero-temperature fluids of stationary ions and electrons, and allows the electron density to fluctuate due to elecrostatic oscillations. Using the linearized electron continuity equation, electron momentum equation and Poisson's equation, we derived a partial differential equation which told us that the electrons oscillated in space with a frequency

[82] To confirm this, look at the isothermal electron response function $n_{e1} = \frac{en_0\phi_1}{m_e V_{Te}^2}$. We know that in an ion acoustic there is a non-zero electron density perturbation, which means there must be a non-zero electric potential perturbation ϕ_1.

equal to the plasma frequency

$$\omega_{pe}^2 = \frac{n_e e^2}{\epsilon_0 m_e} \tag{5.21}$$

We're going to do something very similar now, except we're going to allow the ions to move as well. Although this exercise is a bit repetitive, we're going to conduct it nevertheless to illustrate some concepts about waves in plasmas in general and also some physical insights about plasma oscillations.

Just to be explicit, I'll explain more precisely what we're going to do. We're going to derive electrostatic oscillations for a zero-temperature, homogeneous, zero net velocity plasma with multiple species σ. To start, we linearize the continuity and momentum equations around a homogeneous zero-velocity equilibrium, assume that the system has oscillations of the form $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$[83], and plug the resulting equations into the linearized Poisson's equation to solve for the dispersion relation.

Thus, the perturbed variables can be written as

$$n_\sigma = n_{\sigma 0} + n_{\sigma 1} + ...$$

$$\vec{u}_\sigma = \vec{u}_{\sigma 1} + ...$$

$$\phi = \phi_0 + \phi_1 + ...$$

while the linearized equations are

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0}\vec{\nabla}\cdot\vec{u}_{\sigma 1} = 0 \tag{5.22}$$

$$m_\sigma n_{\sigma 0}\frac{\partial \vec{u}_{\sigma 1}}{\partial t} = -n_{\sigma 0} q_\sigma \vec{\nabla}\phi_1 \tag{5.23}$$

$$-\vec{\nabla}^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} \tag{5.24}$$

For each of the quantities which are first-order perturbations, we assume an oscillation of the form $e^{i\vec{k}\cdot\vec{x}-i\omega t}$.[84] This means that we take $\frac{\partial}{\partial t} \rightarrow -i\omega$ and $\vec{\nabla} \rightarrow i\vec{k}$. Our equations become

$$-i\omega n_{\sigma 1} + n_{\sigma 0} i\vec{k}\cdot\vec{u}_{\sigma 1} = 0 \tag{5.25}$$

$$-i\omega\vec{u}_{\sigma 1} = -\frac{q_\sigma}{m_\sigma} i\vec{k}\phi_1 \tag{5.26}$$

$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} \tag{5.27}$$

[83] This is equivalent to looking at a single fourier component of the oscillations.
[84] This is equivalent to Fourier transforming in space and time.

Cancelling the i's and solving for $n_{\sigma 1}$, we get

$$n_{\sigma 1} = \frac{n_{\sigma 0}\vec{k} \cdot \vec{u}_{\sigma 1}}{\omega} = \frac{q_\sigma k^2 \phi_1}{m_\sigma \omega^2} \tag{5.28}$$

Plugging this into the linearized Poisson's equation gives

$$k^2 \phi_1 = \sum_\sigma \frac{q_\sigma^2 n_{\sigma 0} k^2 \phi_1}{\epsilon_0 m_\sigma \omega^2}$$

$$\omega^2 = \sum_\sigma \omega_{p\sigma}^2 \tag{5.29}$$

where $\omega_{p\sigma}^2 = \frac{n_\sigma q_\sigma^2}{\epsilon_0 m_\sigma}$. This is our dispersion relation for plasma oscillations! This is only slightly different from the dispersion relation for electron plasma relations. Here, the frequency is higher than just the electron plasma frequency, because we have to add the ion plasma frequencies in quadrature. For a two-component plasma, this becomes

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2$$

If we remember from chapter 1, the physical origin of electron plasma oscillations is the electric force pulling back on electrons when the density is perturbed slightly. See, for example, figure 2. This is why if we take the limit $m_e \to 0$, $\omega_{pe} \to \infty$: because the electrons no longer have any inertia and so they respond to electrostatic perturbations infinitely quickly.

Does this physical intuition hold up now that we've allowed the ions to move? Well, yes, if we recognize that the ions and the electrons are going to be oscillating *out of phase* by 180 degrees. The electric field created by the ions *increases* the electric field created by the electrons, so that the restoring force on each species is larger and hence the oscillation frequency is larger. The only way the electric field will be larger (which allows the oscillation frequency to be larger than simply the electron plasma frequency, as we've calculated) is if the density perturbations for the electrons and ions are out of phase. Fortunately, this physical intuition agrees with the equations, as it must. Look at equation 5.28 - since the charge of the electrons and ions are opposite, then the sign of n_{e1} and n_{i1} must be opposites. This corresponds to an 180 degree phase difference between the two density perturbations. From this equation, we can also see that the electron density perturbation will be larger than the ion density perturbation by the ratio $\frac{m_i}{m_e} \approx 2000$. Thus, the ion density perturbation is small relative to the electron density perturbation.

How does figure 2 change when we add ions to the picture? Let's look at the initial condition shown in figure 20, with zero initial velocity. The initial ion perturbation is $-\frac{m_e}{m_i}$ times the initial electron density perturbation. The result is that the ion and electron densities at each point in space oscillate up and down in place, at the plasma frequency $\omega_p^2 = \sum_\sigma \omega_{p\sigma}^2$. This is just like figure 2. The difference with this initial condition is that (a) the ions are allowed to move and (b) the ion density perturbation is always 180 degrees out of phase with the electron density perturbation.

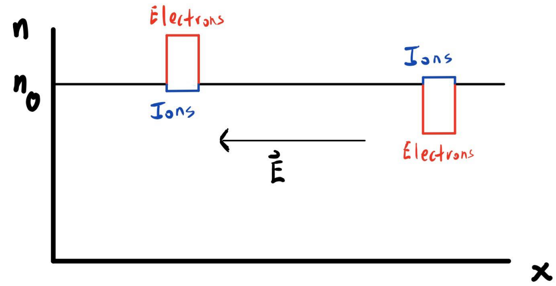

Figure 20: An initial density perturbation where the electron and ion density perturbations begin out of phase with zero initial velocity. The initial electron density perturbation is larger than the initial ion density perturbation by a factor of $\frac{m_i}{m_e}$. The ion and electron density perturbations will oscillate up and down in space, 180 degrees out of phase with each other.

What if instead we have the initial condition shown in figure 21? Here, both the ions *and* the electrons have an initial density perturbation which is positive. Thus, the density perturbations are initially in phase with each other. What does the time-evolution of this look like? This is quite a bit trickier actually. The solution is the following: the electrons and ions will still oscillate, still at the frequency $\omega^2 = \omega_{pe}^2 + \omega_{pi}^2$, *and* still be 180 degrees out of phase. What changes in this situation is the *background* density, $n_{\sigma 0}$. The new background density $n_{\sigma 0}$ is shown by the dashed black line in figure 21. Interestingly, the background density is no longer spatially homogeneous - instead, it's going to be somewhere between the initial density of the electrons and the ions. So long as $|n_{\sigma 1}| \ll |n_{\sigma 0}|$ and $\vec{\nabla} n_{\sigma 0}$ is small so that the $\vec{u}_{\sigma 1} \cdot \vec{\nabla} n_{\sigma 0}$ term we have ignored in the linearized continuity equation is negligible, then this analysis is correct.

Question: do I have this right? Is the density perturbation going to give us ion acoustic waves? Or does it depend on the initial velocity?

As we see from our equations, plasma oscillations allow oscillations at any k. If you're wondering how we can have small-k and hence large-λ electrostatic oscillations in a plasma, where large-scale electric fields are not allowed due to Debye shielding, it's because Debye shielding only takes place when a plasma is close to equilibrium. Plasma oscillations are extremely high-frequency phenomenon, so they aren't constrained by Debye shielding.

Why do we have two solutions? The reason is that we assumed an exponential dependence $e^{i\vec{k}\cdot\vec{x}-i\omega t}$. This assumptions allows for *two* solutions, a left-moving wave and a right-moving wave. For plasma oscillations, and indeed for all waves we look at in these notes, we will find that the left-moving and right-moving solutions have the same frequency. This is because the waves we consider in this class (once we choose a $\vec{k}$) propogate in an otherwise symmetric plasma, so we have no reason to think that waves propogating leftwards will have a different frequency than waves propogating rightwards. We *will* see that

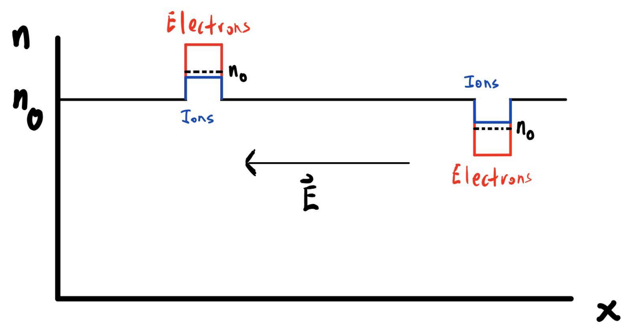

Figure 21: An initial density perturbation where the electron and ion density perturbations begin in phase with zero initial velocity. As in figure 20, the ion and electron density perturbations will oscillate up and down in space, 180 degrees out of phase with each other. The difference is that the zeroth-order density, n_0, is no longer spatially homogenous.

for MHD waves, where there is a magnetic field, the direction of $\vec{k}$ relative to the magnetic field determines the dispersion relation, because the $\vec{B}$-field breaks the symmetry of the plasma. However, because of the left/right symmetry of the magnetic field, we don't see a difference in frequency between left-moving and right-moving waves.

Question: what is significance of having three solutions? four solutions?

5.2.2 Langmuir Waves and Ion Acoustic Waves with the Fluid Model

In section 5.1, we derived electrostatic plasma waves in a warm plasma[85] using the Vlasov-Maxwell equation. We derived response functions in isothermal and adiabatic limits, and used these response functions to determine the dispersion relation of two types of waves. The first wave, called Langmuir waves, was valid for the situation where both ions and electrons are adiabatic, so the wave travels much faster than both of their thermal velocities. This Langmuir wave was similar to plasma oscillations, except taking into account the effect of temperature to first order. The second wave, called the ion acoustic wave, was valid for the situation where the ions were adiabatic and the electrons were isothermal. This wave is similar to a sound wave, except the electrons Debye

[85] What does a warm plasma mean? I've already explained this a bit, but it doesn't hurt to talk more in depth about this concept. A cold plasma is a plasma where the distribution function is a delta function in velocity space. Every particle of species σ at $\vec{x}$ has the same velocity, $\vec{u}_\sigma(\vec{x})$. Thus, in a cold plasma there is no pressure. The proper definitions of warm and hot plasmas are a bit tricky and should probably be saved for the plasma waves course, but here is what I will say for now: the hot-plasma description allows for any distribution function f. Kinetic effects, such as Landau damping, need to be taken into account. A warm plasma takes kinetic effects into account to first order in some parameter. In many plasmas which have non-zero temperature, this parameter is small, and we can make a taylor expansion of the kinetic effects around that parameter. Thus, one might say we are taking into account kinetic effects to first order.

shield the ions and provide the restoring pressure force for the wave, while the ions provide the inertia. Our solution only worked in the case where $T_e \gg T_i$. Lastly, we saw that completely isothermal waves in a plasma were not possible.

Here, we are going to derive these two waves in a different way. Rather than use a kinetic model, we're going to use a multi-fluid model. Once again, we're going to take into account the effects of finite temperature. As with the kinetic derivation, we're again going to first try to derive the response function for each species, $n_{\sigma 1}$. Once we have the response function for each species, we can solve for the dispersion relation. Let's do this now. The multi-fluid equation of motion is

$$m_\sigma n_\sigma \frac{d\vec{u}_\sigma}{dt} = -q_\sigma n_\sigma \vec{\nabla}\phi - \vec{\nabla} P_\sigma \tag{5.30}$$

The equation of state is, for the adiabatic case,

$$\frac{d}{dt}\left(\frac{P_\sigma}{n_\sigma^\gamma}\right) = 0 \tag{5.31}$$

For the isothermal case, the equation of state is simply

$$\vec{\nabla} T_\sigma = 0 \tag{5.32}$$

Linearizing the continuity, momentum, and adiabatic energy equations around a stationary, homogeneous equilibrium we have

$$\frac{\partial \vec{u}_{\sigma 1}}{\partial t} = -\frac{q_\sigma}{m_\sigma}\vec{\nabla}\phi_1 - \frac{1}{m_\sigma n_{\sigma 0}}\vec{\nabla} P_{\sigma 1} \tag{5.33}$$

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0}\vec{\nabla}\cdot\vec{u}_{\sigma 1} = 0 \tag{5.34}$$

$$\frac{dP_{\sigma 1}}{dt}\frac{1}{n_{\sigma 0}^\gamma} - \gamma \frac{P_{\sigma 0}}{n_{\sigma 0}^{\gamma+1}}\frac{dn_{\sigma 1}}{dt} = 0$$

$\frac{d}{dt}$ becomes, to first order in the perturbed quantities, $\frac{\partial}{\partial t}$ because $\vec{u}_{\sigma 0} = 0$ and the zeroth-order quantities are spatially homogenous. Thus, we can rewrite the adiabatic energy equation as

$$\frac{\partial P_{\sigma 1}}{\partial t} = \gamma \frac{P_{\sigma 0}}{n_{\sigma 0}}\frac{\partial n_{\sigma 1}}{\partial t} \tag{5.35}$$

In the isothermal case, the energy equation is simply $\vec{\nabla} T_\sigma = 0$, which gives us $-\vec{\nabla} P_{\sigma 1} = -k_B T_\sigma \vec{\nabla} n_{\sigma 1}$. Remember what we're trying to accomplish by linearizing: we want to solve for the electron response function $n_{\sigma 1}$. As before, we're going to assume all first-order quantities oscillate like $e^{i\vec{k}\cdot\vec{x}-i\omega t}$. Let's solve for the adiabatic response function first. Assuming an exponential oscillation, using the linearized equations we find

$$-i\omega \vec{u}_{\sigma 1} = -i\frac{q_\sigma}{m_\sigma}\vec{k}\phi_1 - \frac{iP_{\sigma 1}\vec{k}}{m_\sigma n_{\sigma 0}}$$

$$-i\omega n_{\sigma 1} + in_{\sigma 0}\vec{k}\cdot\vec{u}_{\sigma 1} = 0$$

$$-i\omega P_{\sigma 1} = -i\omega\gamma\frac{P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1}$$

Simplifying, we can write

$$\vec{u}_{\sigma 1} = (\frac{q_\sigma\phi_1}{m_\sigma\omega} + \frac{P_{\sigma 1}}{m_\sigma n_{\sigma 0}\omega})\vec{k}$$

$$n_{\sigma 1} = n_{\sigma 0}\frac{\vec{k}\cdot\vec{u}_{\sigma 1}}{\omega}$$

$$P_{\sigma 1} = \gamma\frac{P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1}$$

Plugging $P_{\sigma 1}$ into the first expression, we get

$$\vec{u}_{\sigma 1} = (\frac{q_\sigma\phi_1}{m_\sigma\omega} + \gamma\frac{P_{\sigma 0}n_{\sigma 1}}{m_\sigma n_{\sigma 0}^2\omega})\vec{k}$$

Now, plugging $\vec{u}_{\sigma 1}$ into the expression for $n_{\sigma 1}$ and using $P_{\sigma 0} = n_{\sigma 0}k_BT_{\sigma 0}$, we have

$$n_{\sigma 1} = \frac{k^2}{\omega^2}(\frac{n_{\sigma 0}q_\sigma\phi_1}{m_\sigma} + \gamma\frac{k_BT_\sigma}{m_\sigma}n_{\sigma 1})$$

We've got to rearrange this to isolate $n_{\sigma 1}$.

$$n_{\sigma 1}(\frac{\omega^2}{k^2} - \gamma V_{T\sigma}^2) = \frac{n_{\sigma 0}q_\sigma}{m_\sigma}\phi_1$$

$$n_{\sigma 1} = \frac{\frac{n_{\sigma 0}q_\sigma}{m_\sigma}\phi_1}{\frac{\omega^2}{k^2} - \gamma V_{T\sigma}^2} \tag{5.36}$$

This is our adiabatic response function for species σ. Let's find the isothermal response function. Assuming an exponential oscillation, using the linearized equations and the equation of state $\vec{\nabla}P_{\sigma 1} = k_BT_\sigma\vec{\nabla}n_{\sigma 1}$, we have

$$-i\omega\vec{u}_{\sigma 1} = -i\frac{q_\sigma}{m_\sigma}\vec{k}\phi_1 - i\vec{k}\frac{k_BT_\sigma n_{\sigma 1}}{m_\sigma n_{\sigma 0}}$$

$$-i\omega n_{\sigma 1} + in_{\sigma 0}\vec{k}\cdot\vec{u}_{\sigma 1} = 0$$

Simplifying, we can write

$$\vec{u}_{\sigma 1} = \Big(\frac{q_\sigma}{m_\sigma\omega}\phi_1 + V_{T\sigma}^2\frac{n_{\sigma 1}}{\omega n_{\sigma 0}}\Big)\vec{k}$$

$$n_{\sigma 1} = \frac{n_{\sigma 0}\vec{k}\cdot\vec{u}_{\sigma 1}}{\omega}$$

Plugging the $\vec{u}_{\sigma 1}$ expression into the $n_{\sigma 1}$ expression, we have

$$n_{\sigma 1} = \frac{k^2}{\omega^2}(\frac{q_\sigma n_{\sigma 0}}{m_\sigma}\phi_1 + V_{T\sigma}^2 n_{\sigma 1})$$

We've got to rearrange this to isolate $n_{\sigma 1}$.

$$n_{\sigma 1} = \frac{\frac{n_{\sigma 0} q_\sigma}{m_\sigma} \phi_1}{\frac{\omega^2}{k^2} - V_{T\sigma}^2} \tag{5.37}$$

This is our isothermal response function for species σ. Wait a second. This is the same as equation 5.36, except with $\gamma = 1$. Of course, thinking back to the equations we used to derive this result, this must be the case. The only difference between the adiabatic and isothermal equations is the equation of state, where in the isothermal case $\gamma = 1$. Thus, we're learned a valueable lesson: when doing analysis of fluid equations, we can solve the equations with general γ. We don't have to solve two different equations for the isothermal and adiabatic cases. If we want to use an isothermal instead of adiabatic equation of state, just set $\gamma = 1$, and we recover the isothermal results.

In the adiabatic case, we have that $\frac{\omega}{k} \gg V_{T\sigma}$. We can thus Taylor expand equation 5.36 to take into account the effects of temperature to lowest order.

$$n_{\sigma 1} = \frac{k^2 n_{\sigma 0} q_\sigma \phi_1}{m_\sigma \omega^2 \Big(1 - \frac{\gamma V_{T\sigma}^2 k^2}{\omega^2}\Big)}$$

$$n_{\sigma 1} \approx \frac{k^2 n_{\sigma 0} q_\sigma \phi_1}{m_\sigma \omega^2} (1 + \gamma \frac{V_{T\sigma}^2 k^2}{\omega^2}) \tag{5.38}$$

In the isothermal case, on the other hand, we have that $V_{T\sigma} \gg \frac{\omega}{k}$. Thus, we can ignore the $\frac{\omega}{k}$ term in the denominator relative to the $V_{T\sigma}$ term. Thus, our isothermal response function becomes

$$n_{\sigma 1} \approx -\frac{n_{\sigma 0} q_\sigma}{m_\sigma V_{T\sigma}^2} \phi_1 \tag{5.39}$$

Great, so we have our response functions.[86] Now to use these to derive the dispersion relation for Langmuir and Ion Acoustic waves. For Langmuir waves, both the ions and the electrons are adiabatic. Plugging the adiabatic response functions into Poisson's equation gives

$$k^2 \phi_1 = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma n_{\sigma 1} = \frac{k^2}{\omega^2} \phi_1 \Big[\omega_{pe}^2 (1 + \gamma \frac{V_{Te}^2 k^2}{\omega^2}) + \omega_{pi}^2 \Big(1 + \gamma \frac{V_{Ti}^2 k^2}{\omega^2}\Big) \Big]$$

Dividing by $k^2 \phi_1$ and multiplying by ω^2, we get

$$\omega^2 = \omega_{pe}^2 (1 + \gamma \frac{V_{Te}^2 k^2}{\omega^2}) + \omega_{pi}^2 (1 + \gamma \frac{V_{Ti}^2 k^2}{\omega^2}) \tag{5.40}$$

[86] Notice that these are almost exactly the same as equations 5.7 and 5.9, which we derived using the Vlasov equation. Based on that observation, we can conclude that we're going to get almost exactly the same dispersion relation as we did there. Regardless, I'm going to solve these equations again to give us practice in doing so.

As we did in the kinetic case, we can solve this equation iteratively. To a first approximation, $\omega^2 \approx \omega_{pe}^2$, so we can replace the $\frac{1}{\omega^2}$ terms inside the parentheses by $\frac{1}{\omega_{pe}^2}$. We can also ignore $\omega_{pi}^2 V_{Ti}^2$ relative to $\omega_{pe}^2 V_{Te}^2$. This gives us our dispersion relation for Langmuir waves,

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 + \gamma V_{Te}^2 k^2 \tag{5.41}$$

This is the same as the dispersion relation for Langmuir waves we derived using the kinetic description, equation 5.13, *except* we replace the 3 in the former equation with a γ. Why are our dispersion relations different? Did we do something wrong? Actually, no. Look back at equation 5.6 - we didn't know what $f_{\sigma 0}$ was, so we defined $V_{T\sigma}^2$ as $\int g_{\sigma 0} v_\parallel^2 dv_\parallel$. Here, we have a different definition of $V_{T\sigma}^2$, which means we get a different dispersion relation.

Okay, so we've found the dispersion relation for Langmuir waves using the fluid picture. We saw that it agreed with the dispersion relation derived using the kinetic picture. What about ion acoustic waves? Well, if we remember the deal with ion acoustic waves, we'll remember that the electrons are isothermal and the ions are adiabatic. Plugging in these response functions into Poisson's equation gives

$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} = \frac{k^2}{\omega^2}\phi_1\left[\omega_{pi^2}\left(1+\gamma\frac{V_{Ti}^2 k^2}{\omega^2}\right) - \frac{\omega^2\omega_{pe}^2}{k^2 V_{Te}^2}\right]$$

Dividing by $k^2\phi_1$, we have the dispersion relation for Ion Acoustic waves,

$$1 = -\frac{\omega_{pe}^2}{k^2 V_{Te}^2} + \frac{\omega_{pi}^2}{\omega^2}(1+\gamma\frac{V_{Ti}^2 k^2}{\omega^2}) \tag{5.42}$$

We'll solve this equation for ω^2 in the same way we did in the kinetic description. Since the ion acoustic wave is a low-frequency, long-wavelength wave, we'll drop the -1 term completely. We'll then solve the equation perturbatively, initially dropping the $\gamma\frac{V_{Ti}^2 k^2}{\omega^2}$ term relative to 1. This gives us

$$\frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2 V_{Te}^2} = 0$$

so to lowest order,

$$\omega^2 = k^2 V_{Te}^2 \frac{\omega_{pi}^2}{\omega_{pe}^2} = k^2\frac{k_B T_e}{m_i} = k^2 c_s^2$$

To the next lowest order, we plug this ω^2 into the $\frac{\gamma V_{Ti}^2 k^2}{\omega^2}$ term in equation 5.42 (while still ignoring the 1) to get

$$0 = -\frac{\omega_{pe}^2}{k^2 V_{Te}^2} + \frac{\omega_{pi}^2}{\omega^2}(1+\gamma\frac{V_{Ti}^2}{V_{Te}^2\frac{m_e}{m_i}})$$

$$\frac{\omega^2}{k^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2} V_{Te}^2 (1 + \gamma \frac{k_B T_i}{V_{Te}^2 m_e}) = c_s^2 + \gamma V_{Ti}^2 \quad (5.43)$$

This is our dispersion relation for Ion Acoustic waves. Once again, we've replaced a factor of 3 in the kinetic dispersion relation (equation 5.18) by a factor of γ because of the different definitions of $V_{T\sigma}$.

Great! We've been successful in deriving the dispersion relation for two types of electrostatic waves in warm plasmas (Langmuir and Ion Acoustic) using two different plasma models (kinetic and fluid). Deriving these waves two ways was a bit of overkill, but I think it's good to see both ways of deriving them. I certainly learned a lot from the experience!

Looking at the adiabatic and isothermal response functions for species σ (equations 5.36 and 5.37), we see that the response function goes to infinity when $\frac{\omega^2}{k^2} = \gamma V_{T\sigma}^2$. How can the first-order density perturbation be infinity? It can't, of course. The fluid model fails near $\frac{\omega}{k} = V_{T\sigma}$. We can't use the fluid model unless $\frac{\omega}{k}$ is much larger or much smaller than $V_{T\sigma}$. This is essentially the same issue we had in the kinetic model (equation 5.5), where the denonimator went to infinity at $v_\parallel = \frac{\omega}{k}$. When we aren't in the realm (so $\frac{\omega}{k}$ is either much larger or much smaller than $V_{T\sigma}$), we can say that our plasma is in the warm plasma limit. A hot plasma is where we need a fully kinetic model, including the effects of Landau damping, to correctly model the plasma. We aren't ready to fully understand kinetic theory yet, and that's okay.

5.2.3 Electromagnetic Plasma Waves

So far, all of the waves in plasmas which we've looked at have been electrostatic waves. We've assumed the equilibrium magnetic field is zero, and the induced magnetic and electric fields are zero because the phase velocities are slow relative to the speed of light. However, electrostatic waves in plasmas are far from the only waves which can exist. One of the most basic waves in a plasma is an electromagnetic wave, just like light, except propogating in a plasma. Amazingly, we will see that the phase velocity of this wave is faster than the speed of light! Of course, the group velocity is slower than the speed of light, as it must be. Let's solve for the dispersion relation of this wave.[87]

To start, we have our multi-fluid equations, and Maxwell's equations. We assume a cold plasma, and linearize around a homogenous stationary equilibrium. However, we no longer assume that $\vec{E} = -\vec{\nabla}\phi$ and $\vec{B}_1 = 0$. Thus, our equations are

$$m_\sigma n_{\sigma 0} \frac{\partial \vec{u}_{\sigma 1}}{\partial t} = q_\sigma n_{\sigma 0} \vec{E}_1 \quad (5.44)$$

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0} \vec{\nabla} \cdot \vec{u}_{\sigma 1} = 0 \quad (5.45)$$

$$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \quad (5.46)$$

[87]This is a classic prelims problem.

$$\vec{\nabla} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \tag{5.47}$$

Since electromagnetic plasma waves are essentially light waves, we're going to derive them in the same way as we derive light waves. When we derived light waves in undergrad, we took the curl of Ampere's law and Faraday's law. Let's start by doing that on Ampere's law, using Faraday's law to simplify as usual.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}_1) = -\vec{\nabla}^2 \vec{B}_1 = \mu_0 \vec{\nabla} \times \vec{J}_1 - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}_1}{\partial t^2} \tag{5.48}$$

Unlike in a vacuum, here we have a curl of $\vec{J}_1$ term. But we can use the multi-fluid equation of motion to solve for the curl of $\vec{J}$! How? I'll show you. We start by taking the curl of the momentum equation[88]

$$\vec{\nabla} \times (m_\sigma n_{\sigma 0} \frac{\partial \vec{u}_{\sigma 1}}{\partial t}) = q_\sigma n_{\sigma 0} \vec{\nabla} \times \vec{E}_1 = -q_\sigma n_{\sigma 0} \frac{\partial \vec{B}_1}{\partial t}$$

Since both the LHS and RHS terms have a time-derivative, then we can remove the time-derivative and the terms will still be equal. Removing the time-derivative, and multiplying by q_σ / m_σ, we have

$$\vec{\nabla} \times (n_{\sigma 0} q_\sigma \vec{u}_{\sigma 1}) = -\frac{q_\sigma^2 n_{\sigma 0}}{m_\sigma} \vec{B}_1 = -\epsilon_0 \omega_{p\sigma}^2 \vec{B}_1$$

Notice that $\vec{J}_1 = \sum_\sigma n_{\sigma 0} q_\sigma \vec{u}_{\sigma 1}$. So if we sum this equation over species, the LHS will become $\vec{\nabla} \times \vec{J}_1$. Thus,

$$\vec{\nabla} \times \vec{J}_1 = -\sum_\sigma \epsilon_0 \omega_{p\sigma}^2 \vec{B}_1$$

Plugging in $\vec{\nabla} \times \vec{J}_1$ into equation 5.48, we have

$$-\vec{\nabla}^2 \vec{B}_1 = -\mu_0 \epsilon_0 \omega_p^2 \vec{B}_1 - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}_1}{\partial t^2} \tag{5.49}$$

where $\omega_p^2 = \sum_\sigma \omega_{p\sigma}^2$. Great! If we assume an exponential dependence in first-order quantities as usual, and divide by $\vec{B}_1$ then this equation becomes

$$k^2 = -\frac{\omega_p^2}{c^2} + \frac{\omega^2}{c^2}$$

Solving for ω^2, we find

$$\omega^2 = k^2 c^2 + \omega_p^2 \tag{5.50}$$

This is our dispersion relation for electromagnetic waves. Note that if we had no plasma, it would reduce to the dispersion relation for light waves in a vacuum.

[88] We don't actually have to use the continuity equation to solve for the dispersion relation of electromagnetic waves. We only use Maxwell's equations and the equation of motion

Note also that $\frac{\omega^2}{k^2} = c^2 + \frac{\omega_p^2}{k^2}$ which is greater than c^2. Thus, the phase velocity is larger than the speed of light. As you can see by taking $\frac{d\omega}{dk}$, the group velocity is not greater than the speed of light.

$$v_{ph} = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}} \geq c \tag{5.51}$$

$$v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{c^2 k^2 + \omega_p^2}} = \frac{c}{\sqrt{1 + \frac{\omega_p^2}{c^2 k^2}}} \leq c \tag{5.52}$$

I don't have a great physical picture for how the plasma increases the phase velocity of the wave. However, the wave is similar to an electromagnetic wave, so we might imagine that there are crossed $\vec{E}$ and $\vec{B}$ fields which oscillate in time and propogate along the perpendicular direction. The electric field from the electromagnetic wave accelerates particles, as we can see from equation 5.44, along the direction of the electric field. This creates a current parallel to the electric field, which leads to some complicated set of electric and magnetic fields which increases the phase velocity of the wave.

From our dispersion relation, we can see that for $\omega < \omega_p$, electromagnetic waves can't propogate in a plasma. At $\omega = \omega_p$, then $k = 0$, so at any smaller ω there can't be a real k. Physically, if $\omega < \omega_p$, then the oscillation is slow enough that the plasma particles have enough time to Debye shield the electric field created by the light wave. Since the electric fields are debye shielded, the wave no longer propagates.

All if this discussion tells us something important: the index of refraction in a plasma is less than 1! If we shine some light[89] (above the plasma frequency) into a plasma, then that light wave simply becomes an electromagnetic wave in a plasma. The index of refraction

$$n = c/v_g = \frac{1}{\sqrt{1 + \frac{\omega_p^2}{c^2 k^2}}} \leq 1 \tag{5.53}$$

So the wavelength of the light we shine into the plasma actually increases and travels faster than the speed of light. Weird.

We can use this observation to understand why glass is transparent but metals aren't. Glass is an insulator, meaning the electrons in glass require a lot of energy to be promoted to a higher energy level. What this means is that when low-energy light, for example in the visible range, passes through glass, it doesn't get absorbed much by the electrons in the glass. Only at very high energies does glass begin to absorb light as the electrons absorb light of that energy. Metals, on the other hand, have a large number of free electrons, which act very much like a plasma. This means that for an electromagnetic wave to be able to propogate through a metal, the frequency of the wave needs to be above the plasma frequency of the metal. Since n_e is very high in a metal, the plasma frequency in a metal is also very high. Only very high-energy electromagnetic radiation can propogate through a metal.

[89] In plasma diagnostics, this is often some sort of laser.

5.3 MHD Waves

MHD waves are waves which propogate in a magnetized plasma. There are three types of MHD waves, the Shear Alfvén wave, the fast wave, and the slow wave. These waves involve the interaction of perturbations in the plasma with the perturbation of the magnetic field. Understanding MHD waves can be difficult. Each of the three waves has a different physical meaning, and the physical interpretation of the wave depends on the angle between the magnetic field and the direction of propogation of the wave. The shear Alfvén wave has to do with perturbations of the plasma perpendicular to the magnetic field. Because magnetic field lines are frozen-into the plasma in MHD,[90] then when the fluid is perturbed from an equilibrium, the magnetic field is perturbed with the fluid. A magnetic tension force pulls the fluid back, with the effect that this disturbance propogates along the field line. The fast and slow modes are more complicated. For propogation along the field line, these modes represent a sound wave and a second shear Alfvén wave (perpendicular to the other shear Alfén wave). For propogation perpendicular to the field line, the fast wave is what's called a *magnetosonic wave*, which is where plasma pressure combines with a magnetic pressure to propogate disturbances perpendicular to the field line. The slow wave propogating perpencicular to the magnetic field has zero frequency and doesn't propogate. Hopefully these ideas make more sense at the end of this subsection.

To derive MHD waves, we look at an infinite, homogeneous, stationary equilibrium with a constant magnetic field $\vec{B}_0$. We then linearize around this most basic equilibrium, and assume each of the variables oscillate with dependence $e^{i\vec{k}\cdot\vec{x}-i\omega t}$. This is similar to the path we took to derive the dispersion relation of electrostatic waves, except we didn't have an equilibrium magnetic field before. However, since our waves *aren't* electrostatic waves, then we *won't* be using Poisson's equation to solve for the dispersion relation. Instead, we'll be using our momentum equation to get 3 equations for ω^2 as a function of $\vec{k}$. We can solve these equations to get our dispersion relation. One way of solving these equations is to set the determinant of a matrix equation equal to zero.[91] For MHD waves, I prefer using a different method, which I think better illustrates the physics of the MHD waves.[92] The method involves introducing a known solution for the shear Alfvén wave, which simplifies the equations from a 3x3 matrix into a 2x2 matrix. We then take the determinant of this matrix to solve for the dispersion relation of the fast and slow waves.

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[90]This is an idea we haven't discussed in this course. In a perfectly conducting plasma, magnetic field lines are connected to the plasma which lies on those lines. If the plasma moves, the field lines move with it. A hand-wavy proof for this statement is using $\vec{J} = \sigma\vec{E}$ and $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$. In a perfectly conducting plasma, $\sigma \to \infty$, so $\frac{d}{dt}\int \vec{B}\cdot d\vec{A} = -\frac{1}{\sigma}\oint \vec{J}d\vec{l} = 0$ where the integral is over a loop moving with the plasma.

[91]This is how Hong solves for the dispersion relation of MHD waves in class. It's a good way of solving the equations, because for more complex waves we are going to have to solve for the dispersion relation in this manner. I use a different method here.

[92]This is the method used by Russell Kulsrud in chapter 5 of his book 'Plasma Physics for Astrophysics'. I still see Kulsrud, emeritus professor, in his office every now and then.

Let's get started. Applying the method of linearization, $\vec{B} = \vec{B}_0 + \vec{B}_1$, $\rho = \rho_0 + \rho_1$, $\vec{u} = \vec{u}_1$, $\vec{J} = \vec{J}_1$ etc. Our MHD equations are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} P + \vec{J} \times \vec{B}$$

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0$$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}$$

$$\vec{E} + \vec{u} \times \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Linearizing, these become

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u}_1 = 0 \tag{5.54}$$

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\vec{\nabla} P_1 + \vec{J}_1 \times \vec{B}_0 \tag{5.55}$$

$$\frac{dP_1}{dt} - \gamma \frac{d\rho_1}{dt} \frac{P_0}{\rho_0} = \frac{\partial P_1}{\partial t} - \gamma \frac{\partial \rho_1}{\partial t} \frac{P_0}{\rho_0} = 0 \tag{5.56}$$

$$\vec{J}_1 = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}_1 \tag{5.57}$$

$$\vec{E}_1 + \vec{u_1} \times \vec{B}_0 = 0 \tag{5.58}$$

$$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \tag{5.59}$$

Plugging equation 5.59 into equation 5.58, we get

$$\vec{\nabla} \times (\vec{u}_1 \times \vec{B}_0) = \frac{\partial \vec{B}_1}{\partial t} \tag{5.60}$$

Plugging equation 5.57 into equation 5.55, we have

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\vec{\nabla} P_1 + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}_1) \times \vec{B}_0 \tag{5.61}$$

Plugging equation 5.54 into equation 5.56, we have

$$\frac{\partial P_1}{\partial t} = -\gamma P_0 \vec{\nabla} \cdot \vec{u}_1 \tag{5.62}$$

Now, we're going to do something we haven't seen in these notes before: introduce the displacement vector $\vec{\xi}$, defined as $\frac{\partial \vec{\xi}}{\partial t} = \vec{u}_1$. What is $\vec{\xi}$? From it's definition, we can see that the partial time-derivative of $\vec{\xi}$ is the first-order velocity perturbation. This means that $\vec{\xi}$ is like position, or better yet displacement. $\vec{\xi}$ more or less tells us how far and in what direction the plasma has gotten displaced from equilibrium. $\vec{\xi}$, like $\vec{u}_1$, is a first-order quantity. Replacing $\vec{u}_1$ in equations 5.60-5.62 in favor of $\vec{\xi}$, we get

$$\frac{\partial}{\partial t}\Big(\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)\Big)=\frac{\partial \vec{B}_1}{\partial t}$$

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2}=-\vec{\nabla}P_1+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0$$

$$\frac{\partial P_1}{\partial t}=-\frac{\partial}{\partial t}(\gamma P_0\vec{\nabla}\cdot\vec{\xi})$$

Note how the first and third equations both have a time derivative in front of them. Thus, we can integrate[93] these two equations to remove the time derivative. This gives

$$\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)=\vec{B}_1 \tag{5.63}$$

$$P_1=-\gamma P_0\vec{\nabla}\cdot\vec{\xi} \tag{5.64}$$

Plugging these into the second equation, the linearized equation of motion, we have

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2}=\gamma P_0\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi})+\frac{1}{\mu_0}\bigg(\vec{\nabla}\times\Big(\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)\Big)\bigg)\times\vec{B}_0 \tag{5.65}$$

At this point, we have an equation for $\vec{\xi}$ in term of only zeroth order quantities. This is an equation for the evolution of the displacement vector $\vec{\xi}$. Assuming an exponential dependence in $\vec{\xi}$, we have

$$-\rho_0\omega^2\vec{\xi}=-\gamma P_0\vec{k}(\vec{k}\cdot\vec{\xi})-\frac{1}{\mu_0}\bigg(\vec{k}\times\Big(\vec{k}\times(\vec{\xi}\times\vec{B}_0)\Big)\bigg)\times\vec{B}_0 \tag{5.66}$$

This vector equation is really 3 equations for ω^2 in terms of $\vec{k}$ and $\vec{\xi}$. This is the equation we will use to solve for the dispersion relation of MHD waves. In principle, we could write this equation as an eigenvalue equation

$$\Big[\overleftrightarrow{\mathbf{M}}(\vec{B}_0,\vec{k})-\rho_0\omega^2\overleftrightarrow{\mathbf{I}}\Big]\vec{\xi}=0 \tag{5.67}$$

The eigenvalue is ω^2, while the eigenvectors are $\vec{\xi}$. To solve for the allowed ω^2, we could take the determinant of the LHS matrix, which would give us 3 equations for ω^2. This would work just fine - but we're going to take a slightly different approach instead. Let's start by defining a coordinate system where $\vec{B}_0$ is in the z-direction, and $\vec{k}$ is in the x-z plane and makes an angle θ with the z-axis, as in figure 22. This coordinate system is still fully general, because we can always just rotate our axes so that $\vec{k}$ lies in the x-z plane.

There will be 3 MHD waves, each with a different value of ω^2 and a $\vec{\xi}$. We'll solve for the intermediate wave, or shear-Alfvén wave, first, followed by the slow and fast waves.

[93] Assuming an initial condition which makes the integration constant go to zero.

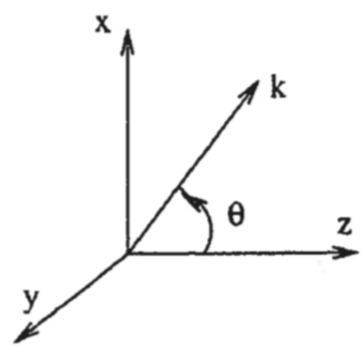

Figure 22: The geometry used to derive the MHD waves. $\vec{B}_0$ points in the z-direction, while the $\vec{k}$ vector points in the x-z plane.

5.3.1 Intermediate Wave

To solve for the intermediate mode, we'll assume that $\vec{\xi} = \xi_y$. Thus, the displacement of the plasma due to the wave is in the direction perpendicular to the magnetic field direction, and also perpendicular to $\vec{k}$. Since $\vec{k} = k_x\hat{x} + k_z\hat{z}$, then $\vec{k} \cdot \vec{\xi} = 0$. This eliminates the first term on the RHS of equation 5.66. The second term is **nasty**, but since $\vec{\xi}$ and $\vec{B}_0$ only have one component we can simplify it considerably. $\vec{\xi} \times \vec{B}_0 = \xi_y B_0 \hat{x}$ as we can see from figure 22, so $\vec{k} \times (\vec{\xi} \times \vec{B}_0) = k_z \xi_y B_0 \hat{y}$. Taking the cross product with $\vec{k}$ again, we have

$$\vec{k} \times \left(\vec{k} \times (\vec{\xi} \times \vec{B}_0)\right) = k_z^2 \xi_y B_0 \hat{x} - k_z k_x \xi_y B_0 \hat{z}$$

and finally taking the cross product with $\vec{B}_0$, we have

$$\left(\vec{k} \times \left(\vec{k} \times (\vec{\xi} \times \vec{B}_0)\right)\right) \times \vec{B}_0 = -k_z^2 \xi_y B_0^2 \hat{y}$$

Notice that $k_z^2 B_0^2 = (\vec{k} \cdot \vec{B}_0)^2$. Thus, for the intermediate mode equation 5.66 reduces to

$$-\rho_0 \omega^2 \xi_y = -\frac{(\vec{k} \cdot \vec{B}_0)^2}{\mu_0} \xi_y \tag{5.68}$$

Thus, our dispersion relation for the intermediate mode is

$$\omega^2 = \frac{(\vec{k} \cdot \vec{B}_0)^2}{\mu_0 \rho_0} \tag{5.69}$$

If you remember back to the beginning of this chapter, I said there are three main things we need to keep in mind when thinking about plasma waves. The name, the dispersion relation, and the assumptions used in deriving the wave. The intermediate mode is, as I've mentioned, also called the shear-Alfvén wave.

It has the dispersion relation listed above, and we derived it using ideal MHD in an infinite homogeneous stationary magnetized plasma. It is one of the three MHD waves we find in homogenous magnetized plasmas.

Great! So we have those three main things, so we understand everything we need to know about the intermediate wave, right? Wrong. I don't know about you, but right now I'm asking myself "what did I just learn?", and the answer which comes to mind is "I have no idea". Unfortunately, simply solving for the dispersion relation of a wave tells us almost nothing about the physics behind that wave. Let's talk about the physics of shear-Alfvén waves.

Suppose we start with an infinite, homogeneous, magnetized stationary plasma, where the $\vec{B}$ field is pointing in the z-direction as in figure 22. Now suppose we apply an initial perturbation to that equilibrium such that the displacement vector $\vec{\xi}$ has no y-dependence, has a dependence $e^{ik_z z}$ in the z-direction[94], and has arbitrary x-dependence. Thus,

$$\xi_y(x, z, t = 0) = f(x)e^{ik_z z} \tag{5.70}$$

This is our initial condition on the perturbation. How does the perturbation evolve in time? Well, before we had been assuming that our perturbation had dependence $e^{i\vec{k}\cdot\vec{x}-i\omega t}$. Here, however, we have an initial perturbation which is slightly more general. To determine how a more general initial perturbation evolves in time, we need to Fourier transform the initial perturbation, give each Fourier component a dependence $e^{i\vec{k}\cdot\vec{x}-i\omega t}$, and then take the inverse Fourier transform to get the time-evolution of the perturbation. Since we can write our initial condition as

$$\xi_y(x, z, t = 0) = e^{ik_z z} \int dk_x \tilde{f}(k_x)e^{ik_x x} \tag{5.71}$$

Time-evolving each Fourier component, we have

$$\xi_y(x, z, t) = \int dk_x \tilde{f}(k_x)e^{ik_x x + ik_z z - i\omega t} \tag{5.72}$$

Since $\omega = k_z v_A$ doesn't depend on k_x, we can pull the time-dependence out of the integral.

$$\xi_y(x, z, t) = e^{ik_z z - i\omega t} \int dk_x \tilde{f}(k_x)e^{ik_x x} = f(x)e^{ik_z z - i\omega t} \tag{5.73}$$

This is a pretty cool result. The result is that any initial x-dependence will, assuming our perturbation sets up a shear-Alfvén mode, simply travel in the z-direction with phase velocity $k_z v_A$[95] without changing it's shape. As Kulsrud nicely puts it: "Note also that all the displaced lines lie in the planes x = constant, and the force is also in these planes. Thus, the dynamics of the mode

[94] This is equivalent to looking at a single k_z-component of the mode, and allowing every k_x.

[95] The Alfvén velocity v_A is defined as $\frac{B}{\sqrt{\mu_0 \rho}}$.

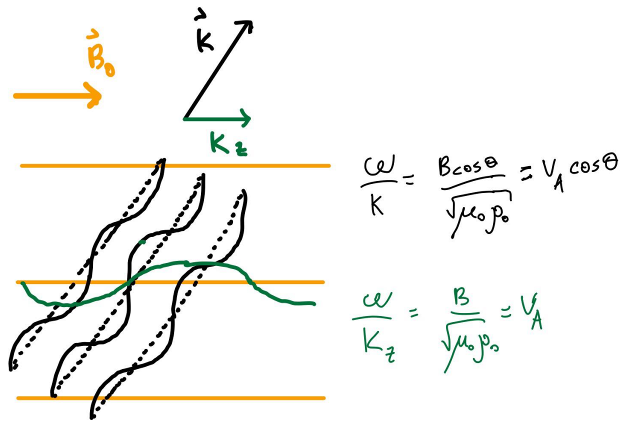

Figure 23: This figure tries to, with low to moderate success, illustrate the shear Alfvén wave. Suppose we have a straight equilibrium magnetic field $\vec{B}_0$ (yellow) and a perturbation to that magnetic field (black), with wavevector $\vec{k}$. The wave will propogate along the vector $\vec{k}$ with phase velocity $v_A \cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{B}_0$. We can also think of the shear Alfvén wave as a series of planes with $x =$ constant where the disturbance propogates in the z-direction with phase velocity v_A.

is such that each plane moves independently of the others. We assume that the mode has an x dependence given by the factor $e^{ik_x x}$, but this dependence is actually obtained by arranging the displacement of the lines in each place relative to the others to gain a pure sinusoidal dependence." This concept is illustrated roughly in figure 23.[96] This physical picture explains why the intermediate mode is also called the shear-Alfvén wave: shear-Alfvén waves are waves which shear the magnetic field lines and propogate along the magnetic field at the Alfvén velocity $\frac{B}{\sqrt{\rho}}$.

Note that the $\vec{B}_1$ is in the same direction as $\vec{\xi}$, as we can see from equation 5.63. This makes sense based on the frozen-in flux theorem of MHD: the displacement is in the same direction as the induced $\vec{B}$, so that plasma stays connected to a magnetic field line as required.

I think we have a nice physical picture of *how* the MHD intermediate wave oscillates in space. The next question to ask ourselves is *why* it arises. Just like

[96]This figure also illustrates something about the wavevector, $\vec{k}$, which nobody ever told me but I wish somebody had. If we have a wavevector $\vec{k}$, the wavelength of the wave is $2\pi/k$. But this wavelength is if we're looking along the $\hat{k}$ direction. If we're looking along the $\hat{z}$ direction, the wavelength is $\frac{2\pi}{k_z}$. You can sort of visualize this by looking at the green wavy thingy, which has a longer wavelength than $2\pi/k$ by a factor $1/\cos\theta$. Mathematically, this follows from $e^{i\vec{k}\cdot\vec{x}-i\omega t} = e^{ik_x x+ik_y y+ik_z z-i\omega t}$, because at fixed x and y then we need to travel $2\pi/k_Z$ in z before the exponential repeats itself. Naturally, the same is be true with k_x and k_y.

we could say the sound wave in air (which has phase velocity $\sqrt{\frac{\gamma P}{\rho}}$) is created due to the interplay of pressure and the mass density of the air, we can say that the shear-Alfvén wave (which has phase velocity $\frac{B}{\sqrt{\mu_0 \rho}}$) is created due to the interplay of the magnetic tension force and the mass density of the plasma. In both cases, the inertia is ρ. While the pressure force scales like $\sim kP$, the magnetic tension force scales like $\sim \frac{kB^2}{\mu_0}$, so replacing P with B^2/μ_0 gives the Alfvén velocity, which makes sense.

5.3.2 Slow and Fast Waves

There is a basic theorem from linear algebra which says the following: "The eigenvectors of a symmetric matrix A corresponding to different eigenvalues are orthogonal to each other." Looking back at equations 5.67 and 5.66, we can see that the matrix $\overleftrightarrow{\mathbf{M}}$ is symmetric. Thus, the eigenvectors $\vec{\xi}$ of this matrix are orthogonal. We've already solved for the first eigenvector, the intermediate mode. Since this eigenvector points in the $\hat{y}$-direction, then the other two eigenvectors must only have x and z components. Using this knowledge, let's write out the x and z-components of equation 5.66. We'll see that we have a 2x2 matrix equation for ω^2, which we will *then* take the determinant of to solve for the eigenvalues and eventually the eigenvectors.

The term $-\gamma P_0 \vec{k}(\vec{k}\cdot\vec{\xi})$ in equation 5.66 becomes $-\gamma P_0 \vec{k}(k_x \xi_x + k_z \xi_z)$. The second term is again nasty, but can be simplified: $\vec{\xi}\times\vec{B}_0 = -B_0 \xi_x \hat{y}$. Since $\vec{k}\times(\vec{k}\times\hat{y}) = -k^2\hat{y}$, then $\vec{k}\times(\vec{k}\times(\vec{\xi}\times\vec{B}_0)) = k^2 \xi_x B_0 \hat{y}$. Finally, we can write

$$-\frac{1}{\mu_0}\left(\vec{k}\times\left(\vec{k}\times(\vec{\xi}\times\vec{B}_0)\right)\right)\times\vec{B}_0 = -\frac{1}{\mu_0}k^2\xi_x B_0\hat{y}\times\vec{B}_0 = -\frac{1}{\mu_0}k^2 B_0^2 \xi_x \hat{x}$$

Thus, we can write the x and z components of equation 5.66 as

$$-\rho\omega^2\xi_x = -\gamma P_0 k_x(k_x\xi_x + k_z\xi_z) - \frac{1}{\mu_0}k^2 B_0^2 \xi_x$$

$$-\rho\omega^2\xi_z = -\gamma P_0 k_z(k_x\xi_x + k_z\xi_z)$$

Using $k_z = k\cos\theta$ and $k_x = k\sin(\theta)$ and dividing by ρk^2, these become

$$\frac{\omega^2}{k^2}\xi_x = (c_s^2\sin^2\theta + v_A^2)\xi_x + c_s^2\sin\theta\cos\theta\xi_z \tag{5.74}$$

$$\frac{\omega^2}{k^2}\xi_z = c_s^2(\cos\theta\sin\theta\xi_x + \cos^2\theta\xi_z) \tag{5.75}$$

where $c_s^2 = \frac{\gamma P_0}{\rho}$ and $v_A^2 = \frac{B^2}{\mu_0\rho}$. These two equations can be written as a 2x2 eigenvalue equation, where the matrix is symmetric so the eigenvalues are orthogonal. To solve this matrix equation, we take the determinant and set it to zero.

$$\begin{bmatrix} (c_s^2\sin^2\theta + v_A^2) - \frac{\omega^2}{k^2} & c_s^2\sin\theta\cos\theta \\ c_s^2\sin\theta\cos\theta & c_s^2\cos^2\theta - \frac{\omega^2}{k^2} \end{bmatrix}$$

Setting the determinant to zero, we have

$$(\frac{\omega}{k})^4 - c_s^2\cos^2\theta(\frac{\omega}{k})^2 - (c_s^2\sin^2\theta + v_A^2)(\frac{\omega}{k})^2 + (c_s^2\sin^2\theta + v_A^2)c_s^2\cos^2\theta - c_s^4\sin^2\theta\cos^2\theta = (\frac{\omega}{k})^4 - (c_s^2 + v_A^2)(\frac{\omega}{k})^2 + v_A^2 c_s^2\cos^2\theta = 0 \quad (5.76)$$

This last line is a quadratic equation for $(\frac{\omega}{k})^2$, which can be solved to get

$$(\frac{\omega}{k})^2 = \frac{(c_s^2 + v_A^2)}{2} \pm \frac{1}{2}\sqrt{(c_s^2 + v_A^2)^2 - 4v_A^2 c_s^2\cos^2\theta}$$

We can rewrite the term inside the parentheses as

$$(c_s^2 + v_A^2)^2 - 4v_A^2 c_s^2\cos^2\theta = c_s^4 + v_A^4 + 2v_A^2 c_s^2 - 4v_A^2 c_s^2\cos^2\theta = c_s^4 + v_A^4 - 2v_A^2 c_s^2 + 4c_s^2 v_A^2 - 4v_A^2 c_s^2\cos^2\theta = (v_A^2 - c_s^2)^2 + 4v_A^2 c_s^2\sin^2\theta$$

Replacing the term inside the parentheses, we have

$$(\frac{\omega}{k})^2 = \frac{(c_s^2 + v_A^2)}{2} \pm \frac{1}{2}\sqrt{(v_A^2 - c_s^2)^2 + 4v_A^2 c_s^2\sin^2\theta} \quad (5.77)$$

Great, so we have our dispersion relation for the fast and slow waves! The fast wave corresponds to the plus solution, while the slow wave corresponds to the minus solution. Unfortunately, the fast and slow waves are quite a bit more complicated than the intermediate wave. While the intermediate wave represents a disturbance being propogated along a field line, there isn't as much of a simple geometric picture for the fast and slow waves. To make things even more complicated, whether or not $v_A > c_s$ or $c_s > v_A$ determines the analytic form of the dispersion relation, and which wave is the fast or slow mode. To make things simpler, let's look only at the cases where $\theta = 0$ and $\theta = 90°$.

$\boldsymbol{\theta = 0}$: When $\theta = 0$, we are looking at propogation along the magnetic field. In this case, our dispersion relation reduces to

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + v_A^2}{2} \pm \frac{|c_s^2 - v_A^2|}{2} \quad (5.78)$$

The two solutions are

$$\frac{\omega^2}{k^2} = v_A^2$$

$$\frac{\omega^2}{k^2} = c_s^2$$

Whether the first or the second solution is the fast or slow mode depends on which of the two speeds is larger. However, the physics of the two modes don't depend on which is larger. The mode corresponding to $\omega^2 = k^2 c_s^2$ is a sound wave, propogating along the magnetic field direction. The magnetic field doesn't play a role because the magnetic field isn't changed in any way when the sound wave propogates along the field. The mode corresponding to $\omega^2 = k^2 v_A^2$ is the exact same as the shear Alfvén wave of the previous section - the wave propogates along the magnetic field line at the Alfvén speed, and corresponds to the magnetic field line shearing. Thus, for propogation along the magnetic field line, there are three orthogonal modes: a sound wave,[97] and two orthogonal shear-Alfvén waves.

$\boldsymbol{\theta = 90°}$**:** When $\theta = 90°$, we are looking at propogation perpendicular to the magnetic field. In this case, our dispersion relation reduces to

$$(\frac{\omega}{k})^2 = \frac{c_s^2 + v_A^2}{2} \pm \frac{c_s^2 + v_A^2}{2}$$

The two solutions are

$$\frac{\omega^2}{k^2} = v_A^2 + c_s^2$$

$$\frac{\omega^2}{k^2} = 0$$

For propogation perpendicular to the magnetic field, there is only one mode, which combines the magnetic pressure and pressure forces together. This mode, called a magnetosonic wave or compressional-Alfvén wave, involves simultaneous compression of the magnetic field and of plasma to create a wave which propogates perpendicular to the magnetic field. The magnetic field lines for the magnetosonic wave are illustrated in figure 24.

5.4 Streaming Instability

The streaming instability wasn't covered in class as far as I know, but it is covered in Hong's notes for the course. Since it seems like a pretty important topic and Hong clearly wants us to have learned it in GPP1, I'm going to discuss it in these notes.[98]

The streaming instability is an instability which arises in plasmas where one species is moving with a net velocity relative to another species. The origin of this instability is electrostatic - meaning, the existence of a magnetic field and a finite temperature don't change the existence of the instability. They might change the criteria for when the instability occurs, but they aren't required for the existence of the instability. In the rest of this section, we will study this instability using a linear analysis, looking at electrostatic oscillations using a multi-fluid model. We'll see that the growth rate of this instability is extremely fast, both for an electron-positron plasma (where each species has the same mass) and the more realistic situation where electrons are streaming through a stationary ion background.

[97] This is the MHD analog to the ion acoustic wave.

[98] As I often do, I'm going to pull heavily from Bellan for this section of the notes.

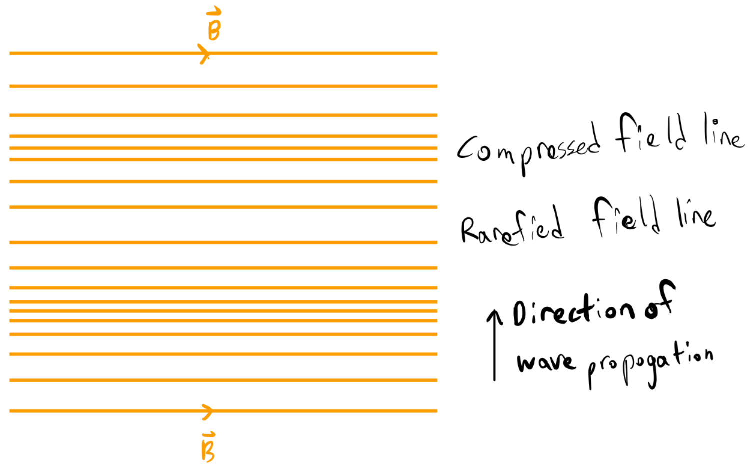

Figure 24: Illustration of the magnetic field lines of a compressional-Alfvén wave propogating perpendicular to the magnetic field ($\theta = 90$). Unlike the shear-Alfvén wave, the magnetic pressure ($B^2/2\mu_0$) contributes to the restoring force.

The thing about using linear analyses to understand instabilities is that if there is an instability, after a very short time the perturbed first-order quantities are no longer small relative to the zeroth-order quantities, and the linear analysis fails. What this implies is that we can't use linear analysis to understand the *dynamics* (i.e. time-evolution) of an instability. We can, however, use linear analysis to determine the *criteria* for the onset of instability. In plasma physics, whenever we use linear analysis to investigate some instability, the most we can determine is the onset condition of that instability.

Since the two-stream instability arises due to electrostatic effects at zero temperature, we're going to use the multi-fluid equations and Poisson's equation to solve for the instability. To start, we'll have the multi-fluid continuity equations, momentum equations, as well as Poisson's equation. We'll linearize these equations, assume an exponential dependence $e^{i\vec{k}\cdot\vec{x}-i\omega t}$, and plug everything into Poisson's equation. These are exactly the same steps used to derive plasma oscillations way back in chapter 1.[99] However, there is one crucial difference between the two-stream instability and plasma oscillations, which is that in the two-stream instability each species has a streaming velocity $\vec{u}_{\sigma 0}$. Of course, the velocity of species 1 might be different from the velocity of species 2, for if the velocities were the same then we could just go to the frame of reference where these velocities were the same and then we would have our electrostatic plasma oscillations. If there is an instability, we should find that the frequency

[99] In chapter 1, we ignored the ion dynamics and assumed the ions were at rest and only the electrons moved. We didn't have to do that. Here we allow the possibility that the ions move.

has some positive imaginary component.

Linearizing the continuity, momentum, and poisson's equation around a homogeneous equilibrium where each species is streaming past one another, we have $n_\sigma = n_{\sigma 0} + n_{\sigma 1}$, $\vec{u}_\sigma = \vec{u}_{\sigma 0} + \vec{u}_{\sigma 1}$ and $\phi = \phi_1$. Thus, the linearized continuity equation for each species becomes

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0}\vec{\nabla}\cdot\vec{u}_{\sigma 1} + \vec{u}_{\sigma 0}\cdot\vec{\nabla}n_{\sigma 1} = 0 \tag{5.79}$$

using the fact that the zeroth order quantities are homogeneous and constant in time. The linearized momentum equation for each species becomes

$$m_\sigma n_{\sigma 0}\frac{\partial \vec{u}_{\sigma 1}}{\partial t} + m_\sigma n_{\sigma 0}(\vec{u}_{\sigma 0}\cdot\vec{\nabla})\vec{u}_{\sigma 1} = -q_\sigma n_{\sigma 0}\vec{\nabla}\phi_1 \tag{5.80}$$

where we can immediately get rid of the $n_{\sigma 0}$ term. The linearized Poisson's equation becomes

$$\frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} = -\vec{\nabla}^2\phi_1 \tag{5.81}$$

When we assume exponential dependence of the first-order quantities, the $\frac{\partial}{\partial t}$ terms become $-i\omega$ and the $\vec{\nabla}$ terms become $i\vec{k}$. Thus, from the linearized continuity equation 5.79 we have

$$i n_{\sigma 1}(-\omega + \vec{u}_{\sigma 0}\cdot\vec{k}) = -i n_{\sigma 0}\vec{k}\cdot\vec{u}_{\sigma 1}$$

which can be written as

$$n_{\sigma 1} = \frac{n_{\sigma 0}\vec{k}\cdot\vec{u}_{\sigma 1}}{\omega - \vec{u}_{\sigma 0}\cdot\vec{k}} \tag{5.82}$$

From the linearized momentum equation, we have

$$i\vec{u}_{\sigma 1}(\vec{u}_{\sigma 0}\cdot\vec{k} - \omega) = -i\frac{q_\sigma}{m_\sigma}\vec{k}\phi_1$$

which can be written as

$$\vec{u}_{\sigma 1} = \frac{\frac{q_\sigma}{m_\sigma}\vec{k}\phi_1}{\omega - \vec{k}\cdot\vec{u}_{\sigma 0}} \tag{5.83}$$

Plugging equation 5.83 into equation 5.82, we get

$$n_{\sigma 1} = \frac{k^2 n_{\sigma 0}\frac{q_\sigma}{m_\sigma}\phi_1}{(\omega - \vec{k}\cdot\vec{u}_{\sigma 0})^2} \tag{5.84}$$

Inserting this into the linearized Poisson's equation, we get

$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \frac{k^2 n_{\sigma 0}\frac{q_\sigma}{m_\sigma}\phi_1}{(\omega - \vec{k}\cdot\vec{u}_{\sigma 0})^2}$$

which simplifies to

$$1 = \sum_\sigma \frac{q_\sigma^2 n_{\sigma 0}}{\epsilon_0 m_\sigma (\omega - \vec{k}\cdot\vec{u}_{\sigma 0})^2} = \sum_\sigma \frac{\omega_{p\sigma}^2}{(\omega - \vec{k}\cdot\vec{u}_{\sigma 0})^2} \tag{5.85}$$

This might seem familiar - can you think where you've seen something like this before? Spoiler alert - if you're paying attention, you'll realize that if we set $\vec{u}_{\sigma 0}$ to zero in this equation, and we take $m_i \to \infty$, then this reduces to the dispersion relation we derived for electron plasma oscillations way back in chapter 1, equation 1.13.[100]

Before we go any further, let's recap what we've done. We're investigating an instability which arises in zero-temperature unmagnetized plasmas, when one species has a different net velocity than the other (i.e. the species are streaming past one another). We started with two-fluid equations for each species, assumed the only forces acting were electrostatic, and linearized around an equilibrium where the zeroth order velocity isn't zero. We then assumed there was some oscillatory wave solution. With this approach, we've derived a dispersion relation which will allow us to solve for ω as a function of $\vec{k}$. We'll see that if the velocities of the two species are not equal, then for a certain range of k (small k), then ω has an imaginary component which corresponds to a quickly growing instability.

5.4.1 Electron-Positron Streaming Instability

An electron-positron plasma is a theoretical construct which could never exist for very long in reality, because the electrons and positrons would quickly annihilate each other. Nevertheless, we will examine this toy model as a means of illustrating the physics of the streaming instability.

Imagine, for simplicity, that the positrons have streaming velocity $+\vec{u}_0$ and the electrons have streaming velocity $-\vec{u}_0$. Since electrons and positrons have the same charge and mass, then we can write $\omega_{pe} = \omega_{pp}$, and thus we can rewrite our dispersion relation in equation 5.85 as

$$1 = \frac{1}{(\frac{\omega}{\omega_{pe}} - \frac{\vec{k}\cdot\vec{u}_0}{\omega_{pe}})^2} + \frac{1}{(\frac{\omega}{\omega_{pe}} + \frac{\vec{k}\cdot\vec{u}_0}{\omega_{pe}})^2} \tag{5.86}$$

Now, we can define some dimensionless variables to make our lives 'easier'. If we define $\frac{\omega}{\omega_{pe}}$ as z and $\frac{\vec{k}\cdot\vec{u}_0}{\omega_{pe}}$ as λ[101], we can write this as

$$1 = \frac{1}{(z-\lambda)^2} + \frac{1}{(z+\lambda)^2} \tag{5.87}$$

This can be rewritten as a fourth-order quadratic equation for z, if we multiply both sides by $(z-\lambda)^2$ and $(z+\lambda)^2$. Working through some algebra, we have

$$(z-\lambda)^2(z+\lambda)^2 - (z+\lambda)^2 - (z-\lambda)^2 = 0$$

[100] Actually, the equation I'm referencing isn't exactly the same. But if we Fourier transform in time to get an ω^2 on the LHS and then divide by ω^2, we get $1 = \omega_{pe}^2/\omega^2$ as I asserted.
[101] Following Bellan's convention.

$$(z^2 - 2\lambda z + \lambda^2)(z^2 + 2\lambda z + \lambda^2) - 2z^2 - 2\lambda^2 = 0$$
$$z^4 - 2z^2(\lambda^2 + 1) + \lambda^2(\lambda^2 - 2) = 0 \quad (5.88)$$

Solving for z^2, we get

$$z^2 = (\lambda^2 + 1) \pm \sqrt{(\lambda^2 + 1)^2 - \lambda^2(\lambda^2 - 2)} = (\lambda^2 + 1) \pm \sqrt{(4\lambda^2 + 1)} \quad (5.89)$$

Remembering that $z = \frac{\omega}{\omega_{pe}}$, then we know that negative z^2 corresponds to an imaginary ω, while positive z^2 corresponds to a real ω. Well, we get a negative z^2 solution if

$$\sqrt{4\lambda^2 + 1} > \lambda^2 + 1$$

Rather than trying to solve this quartic equation, which is doable but a lot of work, we can be clever and try to find the minimum and maximum values of λ where this inequality is true. Setting $\lambda = 0$, then we see this inequality becomes an equality. Additionally, setting $\lambda = \sqrt{2}$, we see this inequality again becomes an equality. Since for any intermediate value between 0 and $\sqrt{2}$ (try, for example, $\lambda = 1$) the LHS is indeed larger than the RHS, then we can see that indeed we have a solution corresponding to an instability for λ between 0 and $\sqrt{2}$. Remembering that $\lambda = \frac{\vec{k} \cdot \vec{u}_0}{\omega_{pe}}$, then the condition for instability corresponds to

$$0 < ku_0 < \sqrt{2}\omega_{pe} \quad (5.90)$$

Now that we have the condition for instability, can we figure out how fast the growth rate (in the linear regime) is. To do so, let's look at the negative solution to equation 5.89, which I've rewritten below.

$$z^2 = (\lambda^2 + 1) - \sqrt{(4\lambda^2 + 1)}$$

Since instability corresponds to negative z^2, to get the maximum growth rate we need to find the minimum z^2. Taking the derivative of the above equation with respect to λ so that we can find λ_{min} (a local minumum where $\frac{dz}{d\lambda} = 0$), we have

$$2z\frac{dz}{d\lambda} = 2\lambda - \frac{4\lambda}{\sqrt{4\lambda^2 + 1}} = 0 \quad (5.91)$$

Solving this equation for λ_{min} by setting $\frac{dz}{d\lambda} = 0$, we have

$$1 = \frac{2}{\sqrt{4\lambda_{min}^2 + 1}}$$
$$\lambda_{min}^2 = \frac{3}{4}$$
$$\lambda_{min} = \frac{\sqrt{3}}{2} \quad (5.92)$$

This is the value of λ where z^2 has a minimum, so ω will have the *largest* imaginary component at λ_{min}. At λ_{min}, we have that

$$z^2 = \frac{7}{4} - \sqrt{2} = -\frac{1}{4}$$

By taking the square root we see that $\omega = i\frac{\omega_{pe}}{2}$, which is imaginary and with a magnitude almost as large as the electron plasma frequency. At the k corresponding to the maximum growth rate of the instability, we have an imaginary growth rate almost as large in magnitude as the electron plasma frequency! This is a huge frequency.

The consequence of this is clear. When two species with the same mass stream past one another in a zero-temperature, unmagnetized plasma, this equilibrium *very* quickly becomes unstable due to the streaming instability.

5.4.2 Electron-Ion Streaming Instability

Here, we examine the more realistic situation where we have electrons moving with velocity $+\vec{u}_0$ past stationary ions. Unlike the previous example, the two species do not have the same mass. In this situation, we can write our dispersion relation as

$$1 = \frac{\omega_{pe}^2}{(\omega - \vec{k}\cdot\vec{u}_0)^2} + \frac{\omega_{pi}^2}{\omega^2}$$

Since $\omega_{pe}^2 = \frac{m_e}{m_i}\omega_{pi}^2$, we can rewrite this as

$$1 = \Big(\frac{\omega}{\omega_{pe}} - \frac{\vec{k}\cdot\vec{u}_0}{\omega_{pe}}\Big)^{-2} + \frac{\frac{m_e}{m_i}}{\frac{\omega^2}{\omega_{pe}^2}} \tag{5.93}$$

Using the dimensionless variable $\epsilon = \frac{m_e}{m_i}$ in addition to the dimensionless variables used in the previous section $z = \frac{\omega}{\omega_{pe}}$ and $\lambda = \frac{\vec{k}\cdot\vec{u}_0}{\omega_{pe}}$, we write this as

$$\frac{1}{(z-\lambda)^2} + \frac{\epsilon}{z^2} - 1 = 0 \tag{5.94}$$

This is harder to solve than the dispersion relation in equation 5.87, but we can do so nevertheless. Pay close attention, because there are a lot of details to keep track of. Once again, this expression is a fourth-order equation for z. In figure 25, I've plotted the LHS of this equation for $\epsilon = \frac{1}{2000}$ and both $\lambda = .9$ and $\lambda = 1.2$. You can see that at $z = 0$ and $z = \lambda$, this equation becomes positive infinity, and at $z = \pm\infty$, the LHS approaches -1. These two facts alone (as we can see from figure 25) ensure that there are at least 2 real roots to the equation and thus two real z solutions. However, between $z = 0$ and $z = \lambda$, there are either two more real roots (as in $\lambda = 1.2$) or no real roots (and thus two complex roots, as in $\lambda = 0.9$).

What's the point of this plotting nonsense? Well, complex roots mean that we might have a positive imaginary component to ω, which corresponds to the streaming instability we're trying to solve for. Actually, if there are two complex roots of equation 5.94, then these complex roots *must* be complex conjugates of each other. Why do we know this? Well, since the fourth-order equation for z described by equation 5.94 has only real coefficients, then any imaginary components of z must be complex conjugates of each other so that there are no

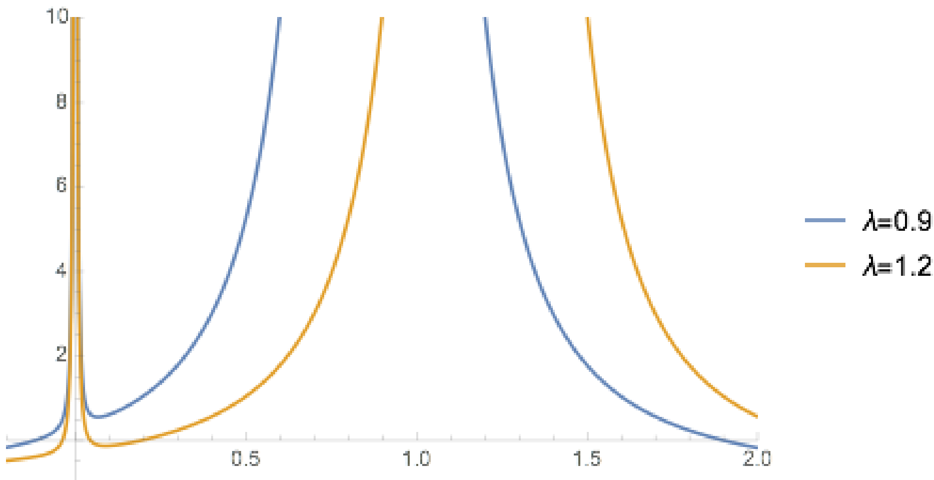

Figure 25: A plot of the LHS of equation 5.94 as a function of z, for two different values of λ. Notice that for $\lambda = 1.2$, there are 4 real roots, while for $\lambda = 0.9$ there are only 2 real roots.

imaginary coefficients in the fourth-order equation. In other words, the fourth-order equation can be written as

$$(z-a)(z-b)(z-c)(z-d) = 0$$

Since two of the roots are real (as we can see from figure 25), if two other two roots are imaginary then they must be complex conjugates of each other so that the equation has no imaginary coefficients when they are multiplied together. Since one of the imaginary solutions will have a negative imaginary component and the other will have a positive imaginary component, if there are any imaginary roots of the equation then there must be one solution which corresponds to an instability.

So what is the condition for the instability to onset? What I've just argued is that the condition for instability to onset is when equation 5.94 transitions from having 4 real roots to having 2 real roots. Based on figure 25, we know this transition happens somewhere between $\lambda = 0.9$ and $\lambda = 1.2$. Actually, it happens at

$$\lambda = (1 + \epsilon^{\frac{1}{3}})^{\frac{3}{2}} \tag{5.95}$$

which is only slightly larger than $\lambda = 1$ since ϵ is so small. In physical variables, the condition for instability is therefore

$$0 < \vec{k} \cdot \vec{u}_0 < \omega_{pe} \Big[1 + \Big(\frac{m_e}{m_1}\Big)^{\frac{1}{3}}\Big]^{\frac{3}{2}} \tag{5.96}$$

Let's solve for this condition now. To solve for this condition, we've established that we we need to determine when equation 5.94 transitions from having 4 real roots to having 2 real roots. This transition happens when we have a *repeated root*, which means we can write equation 5.94 as

$$(z-a)^2(z-b)(z-c)=0$$

$$(z^2-2az+a^2)(z^2-bz-cz+bc)=0$$

$$z^4-2az^3-bz^3-cz^3+a^2z^2+bcz^2+2abz^2+2acz^2-a^2bz-a^2cz-2abcz+a^2bc=0$$

$$z^4-(2a+b+c)z^3+(a^2+bc+2ab+2ac)z^2-(a^2b+a^2c+2abc)z+a^2bc=0 \quad (5.97)$$

Rewriting equation 5.94, we have

$$z^2+\epsilon(z-\lambda)^2-z^2(z-\lambda)^2=0$$

$$z^4-2\lambda z^3+\lambda^2z^2-z^2+\epsilon z^2+2\epsilon\lambda z-\epsilon\lambda^2=0$$

$$z^4+2\lambda z^3+(1+\lambda^2+\epsilon)z^2-(2\epsilon\lambda)z+\epsilon\lambda^2 \quad (5.98)$$

By setting equation 5.98 equal to equation 5.97, we're solving for the condition where equation 5.94 has a repeated root. Both have a factor of z^4, so we can cancel this factor. This gives us three equations for three unknowns,

$$2\lambda=-2a-b-c \quad (5.99)$$

$$1+\lambda^2+\epsilon=a^2+bc+2ab+2ac \quad (5.100)$$

$$2\epsilon\lambda=a^2b+a^2c+2abc \quad (5.101)$$

These equations are a pain to solve, but the good news is that they're quadratic instead of quartic, so we can in principle solve them. I won't solve the equations in these notes for sake of time, but at least the reader knows how it would be solved in principle.

5.4.3 An Apparent Contradiction

Now that we understand the two-stream instability, I have a fun question for you: how can a plasma even support a current? This is a fun question because the two-stream instability suggests that a plasma can't support a current. Think about it - a current means we have electrons and ions moving with a net velocity relative to each other. The two-stream instability tells us that if electrons and ions are moving with a velocity relative to each other, then in a zero-temperature, unmagnetized plasma an instability develops for sufficiently small k. So a zero-temperature, unmagnetized plasma shouldn't be able to support a current!

I asked Hong about this, and he agreed it was a fun question. He's actually written (at least) two papers on the topic (co-authored with, among other people the late Ronald Davidson, former PPPL director). One of the papers[102] showed

[102] Titled 'On the structure of the two-stream instability–complex G-Hamiltonian structure and Krein collisions between positive- and negative-action modes'. I have no idea what that means.

that when finite-temperature effects are taken into account, there is a lower limit to the onset of the instability. Algabraeically, this means that instead of the condition for instability being $ku_0 <$ some number, the have the condition for instability being some small number $< ku_0 <$ some larger number. Thus, we no longer see the instability at arbitrarily small k (and hence at arbitrarily long wavelength).[103] However, this doesn't seem to settle the problem of whether a plasma can carry a current, for if the instability exists for some range of k with finite temperature then we would expect it to arise in any plasma with finite temperature which carries a current. This is, from my conversation with Hong, actually an unresolved problem in plasma physics. Hong wrote to me "We still need to figure out if an experimental setup will allow an unstable kV to exist. For example, maybe the size of the device is too small to fit in one unstable wavelength. Also, when k is too large, viscosity may kick in to damp the instability."

Question: Is it possible to get a nice physical picture for why the two-stream instability arises?

[103] For theory enthusiasts only: at $k = 0$ in the zero-temperature case, whether or not an instability develops apparently depends on how you take the limit. I think Hong wrote a paper about this as well.

6 Landau Damping

The Landau paper takes many days, if not years to appreciate. Go slowly, and enjoy it. It is the foundation of plasma physics.

George Morales

A plasma isn't really a fluid. A plasma is a kinetic gas. A plasma is almost never in thermodynamic equilibrium. For that reason, we can't use a fluid model to study a plasma. Well, we *could*, if we wanted to, we just get the wrong answers. To get the *right answers*, we need to use kinetic theory. When we use kinetic theory, we find that the interaction of waves with particles traveling close to the phase velocity of that wave leads to some new and unintuitive results, which we call Landau damping. In GPP1, we don't really get into the physics of Landau damping. That's fine. Instead, we introduce the basics of complex analysis, and began to apply those concepts to electrostatic plasma waves. Let's get into it.

6.1 Fundamentals of Complex Analysis

Before we dive into the fundamentals of Landau Damping as they apply to electrostatic waves, we should take some time to understand some basic facts about complex analysis which we'll need to know.

6.1.1 Integrals of Analytic Functions in the Complex Plane

One fundamental result of complex analysis is that the closed integral of an analytic function in the complex plane is zero. Now I don't know about you, but when I hear statements like that, I'm usually pretty confused. So let's unpack that statement some more.

When we perform real-valued integrals of a single-valued function $f(x)$, we integrate that function along the real axis. This simple case is illustrated in figure 26. Now, it would seem obvious that if we integrate $f(x)$ from x_1 to x_2 and then back to x_1, the integral will equal 0. And this is indeed true. However, nothing is stopping us from plugging complex values of x into $f(x)$ instead of only real values. Additionally, nothing is stopping us from integrating over not just the real axis, but integrating into the complex plane as well.

Imagine we wanted to integrate $f(x)$ in the complex plane, such as the integral in figure 27. Because this integral is now in the complex plane, we can throw whatever intuition we have from ordinary real integrals out the window. To get a better idea of what's going on in this sort of integral, let's solve it by hand, supposing that $f(x) = x^n$. Starting at the origin, the first part of the integral is just integrating from 0 to 1 along the real axis.

$$\int_0^1 x^n dx = \frac{1}{n+1}\left[x^{n+1}\right]_0^1 = \frac{1}{n+1} \tag{6.1}$$

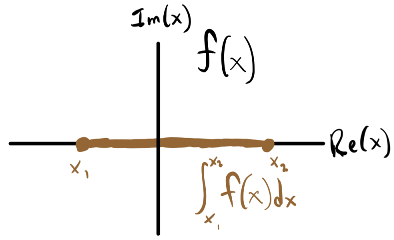

Figure 26: Integration along the real axis of a function $f(x)$.

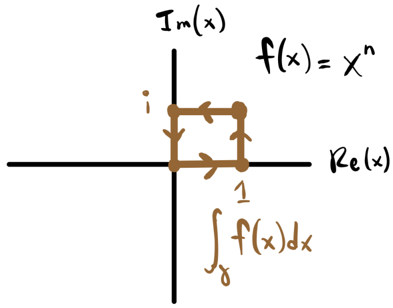

Figure 27: Complex integral of a function $f(x) = x^n$ in the complex plane.

The second leg of the integral involves holding the real component of x at 1 and integrating over the imaginary component of x.

$$\int_0^i (1+y)^n dy = \frac{1}{n+1}\left[(1+y)^{n+1}\right]_0^i = \frac{(1+i)^{n+1}}{n+1} - \frac{1}{n+1} \tag{6.2}$$

The third leg of the integral involves holding the imaginary component of x at i and integrating the real component from 1 to 0.

$$\int_1^0 (i+x)^n dx = \frac{1}{n+1}\left[(i+x)^{n+1}\right]_1^0 = \frac{i^{n+1}}{n+1} - \frac{(1+i)^{n+1}}{n+1} \tag{6.3}$$

The fourth leg of the integral is done in similar fashion, fixing the real component to 0.

$$\int_i^0 y^n dy = \frac{1}{n+1}[y^{n+1}]_i^0 = -\frac{i^{n+1}}{n+1} \tag{6.4}$$

Adding the four legs of the integral, we can see that they add to 0.

$$\frac{1}{n+1} + \frac{(1+i)^{n+1}}{n+1} - \frac{1}{n+1} + \frac{i^{n+1}}{n+1} - \frac{(1+i)^{n+1}}{n+1} - \frac{i^{n+1}}{n+1} = 0$$

Let's recap. Our integral of $f(x) = x^n$, around a specific closed path in the complex plane, integrates to 0. Now it turns out that we could have integrated

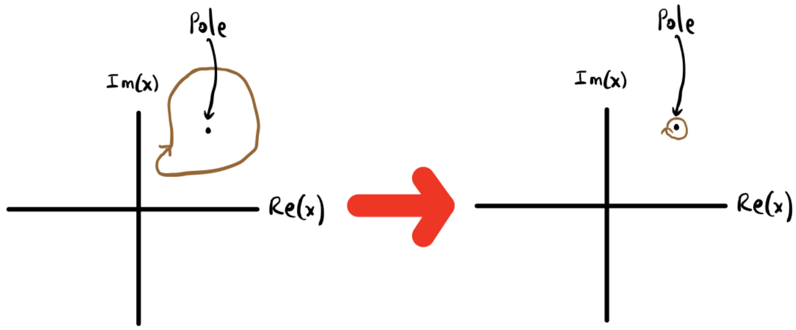

Figure 28: Integrating around a pole in the complex plane. The integral can be changed to a circle around the pole without changing the result of the integral.

x^n around any closed path in the complex plane, and our integral would have still come out to 0. I haven't proven anything, but you can probably imagine that our result might hold independent of the shape of the integration path in the complex plane. Just take a bunch of infinitely small squares and add them together to create any arbitrary path.

An analytic function is, for our purposes, a function which can be written as a Taylor series. For some complex function $f(x)$, it is analytic if it can be written $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for any point x_0 in the complex plane. I think of analytic functions as functions which don't blow up anywhere in space, and are smooth everywhere.

Why'd I look at x^n? Well, if x^n integrates to 0 over any closed path in the complex plane, then any analytic function which can be written as an infinite sum of x^n's will also integrate to 0 over the complex plane. This is what I was referring to earlier when I wrote "the closed integral of an analytic function in the complex plane is zero". Not so bad!

6.1.2 Integrals of Non-Analytic Functions in the Complex Plane

Things get a little bit more complicated when functions are not analytic. What this whole 'non-analytic' business usually means is that our function $f(x)$ has points in the complex plane where the function 'blows up'. These points are called 'poles'. If we want to integrate over a path in the complex plane[104] which encloses a pole, then our integral will no longer necessarily be 0. We'll show in a second that an integral which encloses a pole is equal to the 'residue' of that pole, in a theorem known as the residue theorem.

Imagine, as in figure 28, that we wanted to perform an integral in the complex plane which enclosed a pole. Well, from the result we discussed earlier that the closed integral of any analytic function in the complex plane is 0, we can distort

[104] These integrals are often referred to as 'contour integrals'.

our integral so it looks like a perfect circle around our pole, as in figure 28. Make sure you understand why we can distort our integral like that - basically, we're just adding together a bunch of closed integrals which all sum to 0, so that the path of the integral changes without changing the result of the integral.

Now, any function (whether analytic or not) in the vicinity of a pole a can we written as a Taylor series $\sum_{-\infty}^{\infty} c_n(x-a)^n$. Notice that now n ranges from $-\infty$ to ∞ instead of 0 to ∞. We can use this to perform a circular integral around a pole. Our path for $x(t)$ is a circle around the pole at $x = a$, so $x(t) = a + re^{it}$. This gives $dx = rie^{it}dt$. Our integral around the pole in figure 28 then becomes

$$\int_\gamma f(x)dx = \int_{t=0}^{t=2\pi} \sum_{-\infty}^{\infty} c_n(re^{it})^n rie^{it}dt \tag{6.5}$$

Because $\int_0^{2\pi} e^{imt}dt$ is 0 for all integers m except $m = 0$, then only $n = -1$ contributes to the integral in equation 6.5. Thus, our integral becomes

$$\int_{t=0}^{t=2\pi} c_{(-1)}idt = 2\pi i c_{(-1)} \tag{6.6}$$

This is the 'residue' of the integral around the pole: the coefficient $\mathrm{c}_{(-1)}$ of the expansion of the function $f(x)$, times $2\pi i$. In summary, there are two key points we should keep in mind when doing complex integration. Firstly, because the integral around a closed path in the complex plane of an analytic function is zero, we can arbitrarily deform the path of our integral in the complex plane so long as we don't cross over any poles. And if our integral encloses a pole, we can solve for the integral using the residue theorem.

6.1.3 Laplace Transforms

Let's take some time to make sure we understand Laplace transforms. The Laplace transform of a function $\psi(t)$ is defined as

$$\widetilde{\psi}(p) = \int_0^\infty \psi(t)e^{-pt}dt,\ \mathrm{Re}(p) > \gamma \tag{6.7}$$

where γ is the fastest-growing exponential term in $\psi(t)$. The inequality here simply means that $\widetilde{\psi}(p)$ is only defined for the specified values of p. The inverse Laplace transform of $\widetilde{\psi}(p)$ is

$$\psi(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} dp\widetilde{\psi}(p)e^{pt},\ \beta > \gamma \tag{6.8}$$

where β is a real number. Actually, in class the Laplace transform was defined a bit differently. Here, I'm using the definition used by Bellan. In class, instead of p we had $i\omega$, which means that the inverse Laplace transform will require an integration along the real axis, not the imaginary axis. This difference isn't so important, but I wanted to point that out so as to avoid confusion as much as possible when comparing with class notes. Hopefully, by the end of this chapter you'll see why we used $i\omega$ in class as opposed to p, and you'll be able to understand Laplace Transforms regardless of whether p or $i\omega$ is used.

Having the mathematical definition of a Laplace transform is great, but it doesn't mean we understand what a Laplace transform is or what it does or how it works. Before we understand Laplace transforms, we need to recognize that p is a complex number. This is pretty important. It's also important that the t-integration is from $t = 0$ to infinity, as opposed to $-\infty$ to ∞ as in Fourier transforms. Take a moment to think about those facts, and what they imply about Laplace transforms. What do you think $\widetilde{\psi}(p)$ represents? Once you think about that for a bit, you should reach a conclusion along these lines: the Laplace transform takes a function which starts at $t = 0$ and goes to infinity, and instead of breaking it up into oscillatory components which are real valued (as in a Fourier Transforms), the components of the Laplace transform are both oscillatory *and* exponentially growing or decaying. In other words, instead of it telling us how much of each real frequency is in a function, it tells us how much of each complex frequency is in a function.

Why do we require, in the Laplace transform, that $\text{Re}(p) > \gamma$, where γ is the fastest growing exponential term in $\psi(t)$? Well, suppose $f(t)$ goes as $e^{\gamma t}$ as $t \to \infty$. Then if $\text{Re}(p) < \gamma$, then $\widetilde{\psi}(p)$ will blow up (as we can see from the definition of the Laplace transform). The same is true if $f(t)$ is exponentially decaying as $t \to \infty$. In this case, we still require that $\text{Re}(p) > \gamma$, where γ is the slowest-decaying exponential, for otherwise the integral blows up. If the function neither exponentially grows nor decays as time goes to infinity, then we just require that $\text{Re}(p) > 0$.

Imagine we had some function $f(t)$, and we wanted to know what $\widetilde{f}(p)$ was. Well, if $p = p_r$ is purely real, then we have a simple idea of what $\widetilde{f}(p_r)$ is - it's the integral of the function $f(t)$, integrated to infinity with a weight function $e^{-p_r t}$ applied to the integration. While there isn't really physics interpretation for this, just from it's definition can more or less understand what $\widetilde{f}(p_r)$ is giving us. On the other hand, if $p = p_i$ is purely imaginary, then (assuming $f(t)$ decays at infinity sufficiently fast that the integral converges) $\widetilde{\psi}(p_i)$ tells us how much of the frequency p_i is in the function $f(t)$.

Let's do another example to make sure we understand intuitively what a Laplace transform is. Imagine $f(t) = e^{-\alpha t}\cos(\omega t)$. This function is exponentially decaying and oscillating at the same time, of course. Now imagine we took $\widetilde{f}(p = -\alpha + i\omega)$. What will this give us? Well, for this particular value of p, $\widetilde{f}(p)$ will go to infinity, because the exponentials cancel and we'll be integrating $e^{-i\omega t}\cos\omega t = \cos^2\omega t - i\cos\omega t\sin\omega t$ from 0 to infinity. As $\text{Re}(p)$ increases above $-\alpha$, $\widetilde{f}(p)$ becomes some finite number, and gradually decreases towards zero as $\text{Re}(p)$ increases. If we imagine varying $\text{Im}(p)$, it turns out that $\widetilde{f}(p)$ will be maximum around $\text{Im}(p) = \omega$ and fall off as $\text{Im}(p)$ changes. So for $f(t) = e^{-\alpha t}\cos(\omega t)$, we get a pole at $p = -\alpha + i\omega$. This also illustrates an important point about Laplace transforms: the fastest growing exponential term in $f(t)$ (or if there are no exponentially growing terms as in this example, the slowest decaying exponential term in $f(t)$, here this is $-\alpha$) is related to the pole of $\widetilde{f}(p)$ with the largest real component in that the real components of each are the same. In other words, our Laplace transform $\widetilde{f}(p)$ has a pole at $\text{Re}(p) = -\alpha$, because α is the slowest-decaying exponential term in $f(t)$.

We can actually derive the inverse Laplace transform (equation 6.8) from the definition of the Laplace transform (equation 6.7). Bellan goes through this, and it's an illustrative exercise, so I'll go through it here now. Let's start by considering the integral

$$g(t) = \int_C \widetilde{\psi}(p) e^{pt} dp \tag{6.9}$$

This is a reasonable guess for the inverse transform, since we expect the inverse transform to have a e^{pt} factor hanging out in the integral, the opposite of the e^{-pt} factor in the Laplace transform.[105] The contour C over which we integrate in p-space is undefined at the moment - we'll define it soon. Plugging in the definition of $\widetilde{\psi}(p)$, we get

$$g(t) = \int_{t'=0}^{\infty} dt' \psi(t') \int_C e^{p(t-t')} dp \tag{6.10}$$

We'll have to be careful though: $\widetilde{\psi}(p)$ isn't defined for $\text{Re}(p) < \gamma$, so we'll have to make sure our contour C doesn't venture into that region in p-space. To evaluate $g(t)$, we'll have to choose a contour C. But here is a fact: we can write the delta function $\delta(t)$ as

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tag{6.11}$$

Hey wait a minute! That looks an awful lot like what we've got going on in equation 6.10. If we define the right integration contour C, then we might be able to get out a delta function, and thus write $g(t)$ in terms of $\psi(t)$. The trick will be to hold the real part of p constant and greater than γ ($p_r = \beta > \gamma$), while varying the imaginary part of p from negative infinity to positive infinity. If we do this, then $dp = i dp_i$, and $e^{p(t-t')} = e^{p_r(t-t')} e^{ip_i(t-t')}$. With this trick, equation 6.10 becomes

$$g(t) = \int_{t'=0}^{\infty} dt' \psi(t') \int_{\beta - i\infty}^{\beta + i\infty} e^{p_r(t-t')} e^{p_i(t-t')} dp \tag{6.12}$$

$$g(t) = i \int_{t'=0}^{\infty} dt' \psi(t') e^{\beta(t-t')} \int_{-\infty}^{\infty} e^{ip_i(t-t')} dp_i \tag{6.13}$$

where we turned $e^{p_r(t-t')}$ into $e^{\beta(t-t')}$ because p_r was held constant over the entire integral. Now, this last integral is in the form of $\delta(t)$ in equation 6.11, so it becomes $2\pi\delta(t-t')$.

$$g(t) = 2\pi i \int_0^{\infty} dt' \psi(t') e^{\beta(t-t')} \delta(t-t') = 2\pi i \psi(t) \tag{6.14}$$

[105] If $g(t)$ were the inverse Laplace transform (its not), we would find that $g(t) = \psi(t)$.

This completes our proof: $g(t)$ is $2\pi i$ times $\psi(t)$, assuming we integrate from $\beta - i\infty$ to $\beta + i\infty$ and $\beta > \gamma$. And since the inverse transform of $\widetilde{\psi}(p)$ should give us $\psi(t)$, then equation 6.8 must be the inverse Laplace transform.

I've got one more fact (and short proof) related to Laplace transforms which we'll have to use. It turns out that

$$\int_0^\infty \frac{d\psi}{dt} e^{-pt} dt = p\widetilde{\psi}(p) - \psi(0) \tag{6.15}$$

We can prove this simply by integrating by parts.

$$\int_0^\infty \frac{d\psi}{dt} e^{-pt} dt = \Big[\psi(t)e^{-pt}\Big]_0^\infty - \int_0^\infty dt \psi(t) \frac{d}{dt}(e^{-pt}) = -\psi(0) + p\int_0^\infty \psi(t)e^{-pt} dt \tag{6.16}$$

Since this last integral is the definition of the Laplace transform $\widetilde{\psi}(p)$, we've proved our result for the Laplace transform of a derivative. What we've just proved is important. It tells us that when we Laplace transform a function, if that function can be written as a time-derivative of another function, then we solve for the Laplace transform in terms of the initial value of that other function. This is why you sometimes hear people call Laplace transforms initial value problems.

6.1.4 Analytic Continuation

Imagine we have a function $f(t) = e^{qt}$ where q is some complex number, and we want to take the Laplace transform of $f(t)$. What do we get? Well, using the definition of the Laplace transform, we have

$$\widetilde{f}(p) = \int_0^\infty e^{(q-p)t} dt = \Big[\frac{1}{q-p} e^{(q-p)t}\Big]_{t=0}^{t=\infty} = \begin{cases} \infty \text{ for } \mathrm{Re}(p) < \mathrm{Re}(q) \\ \frac{1}{p-q} \text{ for } \mathrm{Re}(q) < \mathrm{Re}(p) \end{cases} \tag{6.17}$$

Note that if we tried to calculate the transform for $\mathrm{Re}(p) < \mathrm{Re}(q)$, then we would have gotten infinity as our answer, which isn't analytic. So we shouldn't attempt to take Laplace transforms for a value of p in the region where it isn't defined, as this will give us infinity.

Now, let's get from $\widetilde{f}(p)$ back to $f(t)$ by taking the inverse laplace transform. Using the definition of the inverse Laplace transform, we have

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \widetilde{f}(p) e^{pt} dp_i \tag{6.18}$$

This integral is doable, but a bit of a pain. It would be easier if we could somehow use method of residues to evaluate this integral, as we would just need to calculate the residue of the pole at $p = q$, and viola we have the integral. However, we have two problems: the integral isn't a closed contour in p-space, but rather a straight line. And the method of residues requires a closed contour over an analytic function. The second problem is that the inverse laplace transform

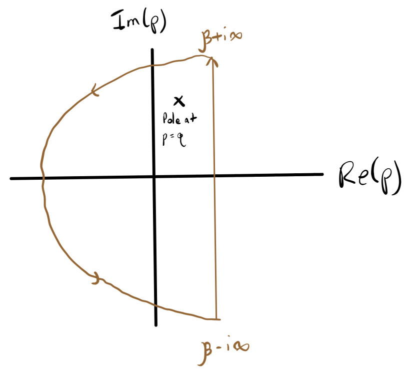

Figure 29: The integration contour used to close the integral in equation 6.18. These contours are called Bromwich contours. Although the inverse Laplace transform is not defined for Re(p) <Re(q), we have used analytical continuation to be able to extend our integration path into the left half of p-space. We can then solve this integral using method of residues.

$\widetilde{f}(p)$ isn't defined to the left of the pole, so even if we wanted to we couldn't form a closed path which enclosed the pole.

However, it turns out that both these problems can be fixed and hence the integral can be solved using method of residues, by use of an ingenious little trick called analytic continuation. Watch carefully. The inverse Laplace transform, $\widetilde{f}(p)$, isn't defined for $\text{Re}(p) < \text{Re}(q)$. However, the analytic expression for the Laplace transform in the region where it is defined, $\frac{1}{p-q}$, is only non-analytic at the point $p = q$. What we do - and this is the key step when performing analytic continuation - is *redefine* the inverse Laplace transform to be $\frac{1}{p-q}$ everywhere that $\frac{1}{p-q}$ is analytic, even in the region where the Laplace transform was previously undefined. In this example, that means that our redefined Laplace transform is now defined everywhere except the pole at $p = q$. Once we do this, then we can actually close our integral infinitely far to the left in the left half of p-space, as illustrated in figure 29. In this leftwards section of the contour integral, the infinitely negative real component of p causes the exponential term e^{pt} to go to 0, and this other section of the integral which closes the integral evaluates to 0. However, now (as in figure 29) we have a closed contour in p-space, so we can use the method of residues to evaluate the integral.

When we use method of residues, we would expect that the residue gives us back our original function, $f(t) = e^{qt}$. Well, this is indeed the case. Remember, the residue is $2\pi i$ times the $c_{(-1)}$ term in the expansion of the function $f(p) = \sum_{-\infty}^{\infty} c_n (p-a)^n$ around $p = a$. Well, here our function $f(p)$ is the function being integrated in equation 6.18, which is $\frac{1}{2\pi i(p-q)} e^{pt}$. The $2\pi i$'s cancel, and if we expand the function around the point $p = q$, the $n = -1$ coefficient in the expansion of $(p-q)^n$ is (not surprisingly) e^{qt}. This makes sense, as we're looking for the coefficient for the $\frac{1}{p-q}$ term in the expansion of $\frac{1}{p-q}e^{pt}$, which already contains a $\frac{1}{p-q}$ in it. Bellan writes this out explicitly as[106]

$$f(t) = \frac{1}{2\pi i}\oint \frac{1}{p-q}e^{pt}dp = \lim_{p\to q} 2\pi i(p-q)\big[\frac{1}{2\pi i(p-q)}e^{pt}\big] = e^{qt} \tag{6.19}$$

Let's recap what just happened: we wanted to take the inverse Laplace transform of a function, but we didn't know how to actually carry out the integral. So we extended the realm of validity of the inverse transform, which allowed us to close the integral in the left half of p-space. Because of the exponential term in the integral, closing the integral added nothing to the original integral, and we were able to use the method of residues to calculate an integral which we previously couldn't calculate. In summary, analytic continuation involves making a new $\widetilde{\psi}(p)$ which

- Equals the old $\widetilde{\psi}(p)$ in the region $\text{Re}(p) > \gamma$,
- is *also* defined in the region $\text{Re}(p) < \gamma$,
- is analytic over the integration path.

As long as we follow these constraints, then analytic continuation is a useful means of evaluating inverse Laplace integrals.

6.2 Fourier Transform in Space, Laplace Transform in Time

Okay, enough math. Let's get back to the physics of Landau damping. Of course, we'll be using all the math I just introduced. Otherwise I wouldn't have bothered to introduce it.

To begin to understand Landau damping, we're going to derive a pseudo-dispersion relation for oscillations which perturb a homogenous, zero-field plasma equilibrium. Why do I say pseudo? Well, typically a dispersion relation has an exact relation between wavenumber and frequency. Here we don't have that, we solve for the time-evolution of f based on the initial conditions. What motivates us to look for a dispersion relation? Actually, the reason is pretty simple. If our frequency ω in the dispersion relation has some imaginary component, then the oscillations will be exponentially damped or exponentially growing. If the frequency is purely real, then the oscillations will continue forever without being damped. So if we are trying to understand damping, we want to figure out any complex frequencies which might arise. Crucially, we will do this using a kinetic treatment, with the Vlasov-Maxwell equation. For simplicity, we'll use the collisionless Vlasov-Maxwell equation. Even using a collisionless equation, we still get damping. This is often called 'collisionless damping'. If we were

[106] I don't really understand why this expression is the residue, but I'm not by any means experienced in actually calculating residues of poles. This is the only residue we explicitly calculate in this class, and I'm only calculating it to illustrate how analytic continuation works. So if you want to understand this expression and actually get practice calculating residues of poles, open up a book on complex analysis.

to look for oscillations using collisionless *fluid* equations, we get waves which aren't damped, as in chapter 5.

Actually, we've gotten a preview of Landau damping in chapter 5 already. Take a look back at equation 5.5. Remember how we got this equation for the perturbed density $n_{\sigma 1}$ - we started with the collisionless Vlasov-Maxwell equation, linearized, and then Fourier-transformed in space and time to get $f_{\sigma 1}$.[107] We then integrated $f_{\sigma 1}$ over velocity space to get $n_{\sigma 1}$. However, this equation blows up in the denominator, when $v_\parallel = \frac{w}{k}$. Density going to infinity isn't good. In chapter 5, we ignored this by using a warm-plasma approximation and expanding the denominator in isothermal and adiabatic limits. In this chapter, we take a more general kinetic approach, using a Laplace transform instead of a Fourier transform. Let's start, as we did in chapter 5, with the collisionless Vlasov Maxwell equation.

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma = 0 \tag{6.20}$$

Now, let's linearize this equation around an equilibrium. Here, the equilibrium is a spatially homogenous, zero-field equilibrium, such that $\vec{E}_0$ and $\vec{B}_0$ equal zero. This gives

$$\frac{\partial f_{\sigma 1}}{\partial t} + \vec{v} \cdot \vec{\nabla} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma}(\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \vec{\nabla}_v f_{\sigma 0} = 0 \tag{6.21}$$

Now, as before, we can Fourier transform in space, meaning all the first-order quantities go as $e^{i\vec{k}\cdot\vec{x}}$. Thus, the $\vec{\nabla}$ becomes $i\vec{k}$. We can also ignore $\vec{v} \times \vec{B}_1$ relative to $\vec{E}_1$, using $\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$. Since there is no zeroth-order field, we can write $kB_1 \sim \omega \frac{1}{c^2} E_1$, so $B_1 \sim \frac{\omega}{k}\frac{1}{c^2} E_1$ and $B_1 \sim \frac{v_g}{c^2} E_1$. Since $\frac{|\vec{v} \times \vec{B}_1|}{|E_1|} \sim \frac{v_g v}{c^2} \ll 1$, we can ignore $\vec{v} \times \vec{B}_1$.[108] So ignoring $\vec{v} \times \vec{B}_1$, we have

$$\frac{\partial f_{\sigma 1}}{\partial t} + i\vec{v} \cdot \vec{k} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma}\vec{E}_1 \cdot \vec{\nabla}_v f_{\sigma 0} = 0 \tag{6.22}$$

Now, here comes the crucial point. We're going to Laplace transform in time instead of Fourier transform. When you read about this, you'll see people write that by taking the Laplace transform, we're treating the problem as an "initial value problem". I'm no mathematician, but here's what I understand this statement to mean: when we Fourier transform, each wavenumber k has a

[107] What we really did was assume an exponential dependence of all the first-order quantities $e^{i\vec{k}\cdot\vec{x} - i\omega t}$. However, this is equivalent to Fourier transforming in space and time, as Fourier transforming in time picks out a particular ω and Fourier transforming in space picks out a particular $\vec{k}$, which we are doing. So when I say 'we Fourier transform' what I mean is that we assume an exponential dependence in the linearized quantities. Sorry if that is confusing.

[108] Question: Here's something I don't understand. What about the $\mu_0 \vec{J}$ term in Ampere's law? Why are we ignoring that? Here's my best attempt at an answer: what we just showed is that B_1 is negligible relative to E_1. The conclusion is that $\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \approx 0$. What Ampere's law tells us is that any changes in E_1 with respect to time are accompanied by currents, not an induced magnetic field.

particular frequency ω which determines the time-evolution of the function of interest, in this case $f_{\sigma 1}$. However, when we Laplace transform, there is not a particular single frequency (neither real nor imaginary) which gives us the behavior of a component with wavenumber k in $f_{\sigma 1}$. Instead, the behavior of the wavenumber component k in $f_{\sigma 1}(t=0)$ has some complicated behavior described by many (complex) frequencies, and Laplace transforming allows us to solve for that complicated behavior. Using equation 6.15, we'll see that solving for the complicated time-evolving behavior requires knowing the initial condition on $f_{\sigma 1}$. Now, taking the Laplace transform gives

$$\int_0^\infty \Big[\frac{\partial f_{\sigma 1}}{\partial t}e^{-pt} + i\vec{v}\cdot\vec{k}f_{\sigma 1}e^{-pt} + \frac{q_\sigma}{m_\sigma}\vec{E}_1\cdot\vec{\nabla}_v f_{\sigma 0}e^{-pt}\Big]dt = 0 \tag{6.23}$$

On the first term, we can use equation 6.15 to simplify, to get $p\widetilde{f}_{\sigma 1}(\vec{x},\vec{v},p) - f_{\sigma 1}(\vec{x},\vec{v},t=0)$. In the second and third terms, we replace the first-order quantities with their Laplace transforms. Thus, we have

$$p\widetilde{f}_{\sigma 1}(\vec{x},\vec{v},p) - f_{\sigma 1}(\vec{x},\vec{v},t=0) + i\vec{v}\cdot\vec{k}\widetilde{f}_{\sigma 1}(\vec{x},\vec{v},p) + \frac{q_\sigma}{m_\sigma}\widetilde{\vec{E}}_1(\vec{x},p)\cdot\vec{\nabla}_v f_{\sigma 0} = 0 \tag{6.24}$$

This equation is a bit messy, and it's only going to get worse from here unfortunately. However, for now we can clean things up a bit by solving for $\widetilde{f}_{\sigma 1}$.

$$\widetilde{f}_{\sigma 1}(\vec{x},\vec{v},p) = \frac{f_{\sigma 1}(\vec{x},\vec{v},t=0) - \frac{q_\sigma}{m_\sigma}\widetilde{\vec{E}}_1(\vec{x},p)\cdot\vec{\nabla}_v f_{\sigma 0}}{p + i\vec{v}\cdot\vec{k}} \tag{6.25}$$

We'll assume $f_{\sigma 1}$ is known at $t=0$, which is equivalent to saying we know what our initial perturbation is. Thus, we have an equation for $\widetilde{f}_{\sigma 1}$ in terms of things we know, and $\widetilde{\vec{E}}_1$, which we don't know. Thus, if we can get one more equation with both $\widetilde{f}_{\sigma 1}$ and $\widetilde{\vec{E}}_1$, we can solve our dispersion relation. Fortunately, there is one equation which describes electrostatic plasma oscillations we haven't used yet: Gauss's Law. And just like we Laplace transformed the linearized Vlasov-Maxwell equation, we can also Laplace transform the linearized Gauss's law. Remember how we did this - we multiplied the equation by e^{-pt} and then integrated over t from 0 to ∞. We then used the definition of the Laplace transform the replace the first order quantities with their Laplace transforms, which have a squiggle above them. For Gauss's law, this leaves

$$\vec{\nabla}\cdot\widetilde{\vec{E}}_1(p,\vec{x}) = i\vec{k}\cdot\widetilde{\vec{E}}_1(p,\vec{x}) = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \widetilde{f}_{\sigma 1}(p,\vec{x},\vec{v})d^3\vec{v} \tag{6.26}$$

We have an expression for $\widetilde{f}_{\sigma 1}$ in equation 6.25, which we can plug into the above equation. We'll then isolate for $\widetilde{\vec{E}}_1$.

$$i\vec{k}\cdot\widetilde{\vec{E}}_1(p,\vec{x}) = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \Big(\frac{f_{\sigma 1}(\vec{x},\vec{v},t=0)}{p + i\vec{v}\cdot\vec{k}}\Big)d^3\vec{v}$$

$$-\frac{1}{\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int \Big(\frac{\widetilde{\vec{E}}_1(\vec{x},p)\cdot\vec{\nabla}_v f_{\sigma 0}}{p+i\vec{v}\cdot\vec{k}}\Big)d^3\vec{v} \tag{6.27}$$

Now, if we have purely electrostatic perturbations, then $\vec{E}_1 = -\vec{\nabla}\phi_1 = -i\vec{k}\phi_1$, which implies $\vec{E} \parallel \vec{k}$. Thus, $\widetilde{\vec{E}}_1 = \widetilde{E}_1\hat{k}$, so the dot product $\vec{k}\cdot\widetilde{\vec{E}}_1$ simplifies to a scalar and we can solve equation 6.27 for $\widetilde{E}_1$.

$$\widetilde{E}_1\Big[ik + \frac{1}{\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int \frac{\hat{k}\cdot\vec{\nabla}_v f_{\sigma 0}}{p+i\vec{v}\cdot\vec{k}}d^3\vec{v}\Big] =$$

$$\frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \Big(\frac{f_{\sigma 1}(\vec{x},\vec{v},t=0)}{p+i\vec{v}\cdot\vec{k}}\Big)d^3\vec{v} \tag{6.28}$$

Solving for $\widetilde{E}_1$, we get

$$\widetilde{E}_1 = \frac{N(p)}{D(p)} \tag{6.29}$$

where

$$N(p) = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \Big(\frac{f_{\sigma 1}(\vec{x},\vec{v},t=0)}{p+i\vec{v}\cdot\vec{k}}\Big)d^3\vec{v} \tag{6.30}$$

$$D(p) = ik + \frac{1}{\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int \frac{\hat{k}\cdot\vec{\nabla}_v f_{\sigma 0}}{p+i\vec{v}\cdot\vec{k}}d^3\vec{v} \tag{6.31}$$

The N stands for numerator, and D stands for denominator. Actually, my $N(p)$ and $D(p)$ differ slightly from those derived in class and in Bellan's book. This is because I chose p as my Laplace variable, as opposed to $i\omega$ as was done in class. I also solved for $\widetilde{\vec{E}}_1$, as opposed to $\widetilde{\phi}_1$ as is done in Bellan. I should point out that $\widetilde{E}_1$ is now known! Or at least, in principle it is known for a given p, since we know $f_{\sigma 0}$ and $f_{\sigma 1}(t=0)$.

A lot has happened, so we're going to take a break here and recap what we've done. We started with the Vlasov-Maxwell equation. We linearized around a homogenous, zero-field equilibrium assuming that whatever perturbations were created would be electrostatic in nature. The equation which remains can't be solved by Fourier transforming in space and time. We tried that approach in chapter 5, but found that density blows up when $v_\parallel = \frac{\omega}{k}$. Instead, we've taken a different approach, by Fourier transforming in space but Laplace transforming in time. This allows us to solve for the first-order electric field in terms of the *initial value* of the perturbed $f_{\sigma 1}$. Lastly, we Laplace transform Gauss's law and combine that with the linearized Laplace transformed Vlasov-Maxwell equation to solve for $\widetilde{E}_1$.

6.3 Landau Contours and All That Jazz

We're going to attempt to solve for $\vec{E}_1$. If we can solve for $\vec{E}_1$, then in principle we have everything we need to solve for the time-evolution of f_σ using

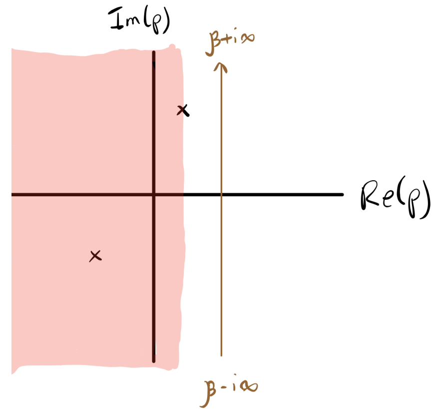

Figure 30: The integration path taken in equation 6.32. The x's represent the poles of $N(p)/D(p)$, which is being integrated here. Here, β is larger than the largest real component of the poles of $N(p)/D(p)$. The red area represents the region in p-space where the Laplace transform of E_1 is not defined, and thus where the integration path cannot go.

equation 6.22. Well, it turns out that solving for $\vec{E}_1$ is easy, at least in principle. All we have to do is take the inverse Laplace transform of $\widetilde{E}_1$. From equation 6.8, we have

$$\vec{E}_1 = \frac{\hat{k}}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{N(p)}{D(p)}e^{pt}dp \tag{6.32}$$

Take a look at figure 30. This represents the integration in p-space taken to solve for $\vec{E}_1$. The x's represent the poles of $\frac{N(p)}{D(p)}$. $\frac{N(p)}{D(p)}$ isn't defined in the shaded red area, as the real part of p in this area is less than the real part of the fastest growing pole of $\widetilde{E}_1 = \frac{N(p)}{D(p)}$. β, as you might have realized, is greater than the real part of any poles of $\frac{N(p)}{D(p)}$.[109]

Now, the integral in equation 6.32 can be completed using the method of analytic continuation. As a reminder, analytic continuation will define $\frac{N(p)}{D(p)}$ in the red region in figure 30, so that the integral can be extended into the left half of p-space, and the integral can be solved using method of residues. If we want $\frac{N(p)}{D(p)}$ to remain analytic over an integration path like that of the Bromwich contour in figure 29, we'll need to make sure that $N(p)$ and $D(p)$ both remain

[109]Bellan actually writes this incorrectly in his book - he says "β is chosen to be larger than the fastest growing exponential term in $N(p)/D(p)$." However, this is wrong - there are two correct, equivalent ways of writing this. The first is as I've written it here - that β is greater than the real part of any poles of $\frac{N(p)}{D(p)}$. The second correct way of writing it would be that β is greater than the fastest growing exponential in $E_1(t)$, γ. Note that there are two ways of writing it because the fastest growing exponential term in $E_1(t)$ corresponds to the pole in $\widetilde{E}_1(p)$ with the largest real component, as I discussed in an example earlier in this chapter.

analytic as we extend these functions into the left half of p-space. It turns out that keeping these functions analytic is a bit tricky, and requires the use of these funny integration paths called Landau contours. Remember our definitions of $N(p)$ and $D(p)$ from before.

$$N(p) = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \Big(\frac{f_{\sigma 1}(\vec{x},\vec{v},t=0)}{p + i\vec{v}\cdot\vec{k}}\Big) d^3\vec{v} \tag{6.33}$$

$$D(p) = ik + \frac{1}{\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int \frac{\hat{k}\cdot\vec{\nabla}_v f_{\sigma 0}}{p + i\vec{v}\cdot\vec{k}} d^3\vec{v} \tag{6.34}$$

We can integrate over the two directions perpendicular to $\vec{k}$, and rewrite these expressions as follows:

$$N(p) = \frac{1}{ik\epsilon_0}\sum_\sigma q_\sigma \int_{-\infty}^{\infty} \Big(\frac{F_{\sigma 1}(\vec{x},\vec{v},t=0)}{v_\parallel - \frac{ip}{k}}\Big) dv_\parallel \tag{6.35}$$

$$D(p) = ik + \frac{1}{ik\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int_{-\infty}^{\infty} \frac{\frac{\partial F_{\sigma 0}}{\partial v_\parallel}}{v_\parallel - \frac{ip}{k}} dv_\parallel \tag{6.36}$$

where $F_{\sigma 1} = \int f_{\sigma 1} d^2\vec{v}_\perp$ and $\frac{\partial F_{\sigma 0}}{\partial v_\parallel} = \int \frac{\partial f_{\sigma 0}}{\partial v_\parallel} d^2\vec{v}_\perp$. As these integrals are constructed, we integrate along the real axis. Each of the integrals has a pole at $v_\parallel = \frac{ip}{k}$, but as long as $\text{Re}(p)$ doesn't equal zero, this isn't a problem for us because the integration path doesn't go over the pole.[110] See figure 31, since this probably doesn't make sense on a first reading.

I spoke too soon - this is a problem for us. Remember what we want to do - we want to extend the definition of $N(p)$ and $D(p)$ into the left half of p-space, such that $N(p)$ and $D(p)$ remain analytic. At $\text{Re}(p) = 0$, if $\text{Im}(p) \neq 0$, then our integral is now integrating over a pole! This will create a discontinuity in $N(p)$ and $D(p)$ at $\text{Re}(p) = 0$, meaning there will be a jump in the value of the integral between $\text{Re}(p) > 0$ and $\text{Re}(p) < 0$. And a discontinuity in our analytic continuation at $\text{Re}(p) = 0$ means our analytic continuation is no longer analytic over the Bromwich contour, which means we can't use the method of residues to evaluate the integral.

As Landau does, Landau found a clever solution. We can make $N(p)$ and $D(p)$ analytic if we are willing to wander off the real axis during our integration. Take a look at figure 31. Once $\text{Re}(p) = 0$, the integration path drops below the real axis as shown in the figure. As $\text{Re}(p)$ decreases below 0, we deform the integration path further below the imaginary axis to prevent the pole from crossing over the contour integral and creating a discontinuity in $N(p)$ or $D(p)$. These deformed contours are called Landau contours.[111]

[110] Let's assume $k > 0$ for simplicity. The sign of k doesn't change the result or the interpretation of course.

[111] This is a tricky concept. You probably will want to read through this section a few times to let the ideas really soak in.

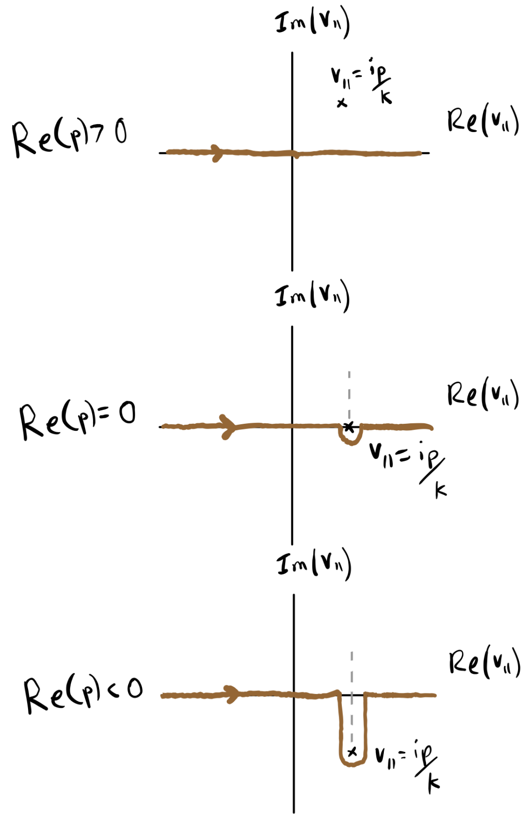

Figure 31: As Re(p) drops below 0, the integration paths in $N(p)$ and $D(p)$ are deformed to prevent the pole from crossing the integration path, so that $N(p)$ and $D(p)$ each remain analytic.

Now, this is as far as we got in class in regards to Landau damping. Unfortunately, this is a very unsatisfying point to stop in regards to Landau damping. All we've done is some complex analysis and gotten an expression for $\vec{E}_1$, which supposedly we know how to evaluate in principle. We haven't gotten any understanding of the physics behind Landau damping, or reaped the benefits of the calculations we've done. Fortunately, you (and I!) will have the pleasant experience of taking AST553, Plasma Waves and Instabilities, where we will revisit Landau damping in great depth. Hopefully, at the conclusion of that course, we'll get to the important stuff: the physics. But for now, let's have patience. We've got oh-so-much still to learn.