General Plasma Physics I Notes
AST 551

Nick McGreivy
Princeton University

Fall 2017

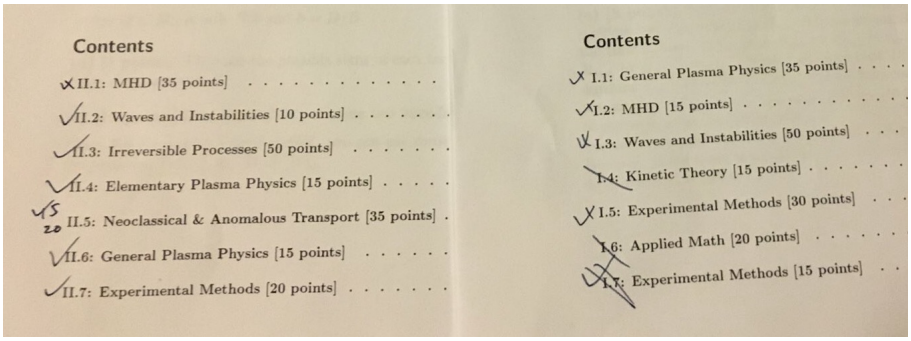
Contents

0 Introduction

These notes are intended to summarize and explain the topics discussed during class in the Fall 2017 section of AST551, General Plasma Physics I. I am writing these notes primarily as a learning experience. I find that in order to learn physics topics, I need to examine that topic in detail, going through any derivations myself step by step. Thus, you will see that I try not to skip steps as much as possible. I also find that to *really* understand something, I need to explain it to someone else. Thus, I will do my best to not only include the math, but do my best to explain the physics behind the math, opting for wordiness over brevity. If I do not fully explain a concept, it is probably because I do not understand the physics behind that concept yet. I've found that with many of the plasma physics books I've looked at, I understand the math and the derivations, but walk away without an understanding of the physics. My goal with these notes is for that not to be the case.

Obviously, these notes are a work in progress. I have yet to write on many important sections from the course, and many sections need to be expanded or explained in more depth. However, my hope is that when these notes are completed, they might be useful for me and possibly future students as they prepare for generals or take AST551. If you are reading these notes and find a typo or an error, please shoot me an email at mcgreivy@princeton.edu so I can fix it.

Speaking of generals, I've attached a picture with the cover of the a previous written section of the generals exam. Of the 360 points in the written section of this generals exam, 190 of the points are from a topic which is covered in this class. Many of these topics are covered again in more advanced courses, so one might argue that learning this material well after GPP1 isn't essential. I'm not convinced.



These notes are divided into 6 chapters, not necessarily correlated with the order the topics were covered in class. The first chapter covers the most basic topics in plasma physics, including plasma oscillations, Debye shielding, space-time scales, and a bit on collisions. The second chapter covers single particle

motion, including particle drifts, adiabatic invariants, mirror machines, and the iso-rotation theorem. The third chapter will introduce kinetic theory, the Vlasov equation and discuss collision operators. The fourth chapter will cover fluid equations and MHD, including 1D MHD equilibrium. Chapter 5 will cover some fundamental waves in plasmas, from kinetic, fluid, and MHD perspectives. Chapter 6 will cover Landau damping, to the extent it was covered in class.

So far, the best resource I have found for learning the fundamentals of plasma physics is Paul Bellan's book, *Fundamentals of Plasma Physics*. Every derivation is done step-by-step in great detail, so that the reader is not lost, and each concept is explained thoroughly and often with good physical insight. The downside of the book is that it is quite long. Everything Professor Bhatacharjee does is exceptional, and his textbook *Introduction to Plasma Physics with Space, Laboratory, and Astrophysical Applications* is no exception. His book contains many of the same topics covered in these notes, plus many more. It would be a great reference book for this course, and less time and algebra intensive than Bellan. *Physics of Fully Ionized Plasmas* by Lyman Spitzer is a really old, fairly short book, with an old-fashioned take to the subject. Sam Cohen once told me it's the only book I need to read to understand plasma physics. I don't believe him. *Introduction to Plasma Physics and Controlled Fusion* by Francis Chen is often referenced as a good book for beginning students - however, I think the level is appropriate for an undergraduate starting a summer of research into plasma physics, not for a graduate student concentrating in plasma physics.

Greg Hammett imbued us first-year students with three pieces of wisdom during the first lecture for GPP1 way back in September. I figure I should pass that advice on. The first piece of advice is to remember how fortunate we are to be at this wonderful university, and to make the most of this experience. The second piece of advice is to find meaning and purpose in our lives outside of work. The third piece of advice is to get some sleep.

1 Basics

> It's unbelievable how much you
> don't know about the game
> you've been playing all your life.
>
> Mickey Mantle

Plasma physics, as you may or may not have been told (once you are immersed in the field for long enough, you will inevitably be told this at some point), is a rich, varied subject. This richness comes mathematically, experimentally, as well as through the numerous applications of plasma physics research.

Research in plasma physics draws knowledge from of a huge number of areas of physics, including electromagnetics, thermodynamics, statistical mechanics, nuclear physics, and atomic physics. Experiments in plasma physics often involve vacuum systems, superconducting coils, cryogenic systems, complex optical instruments, advanced materials for plasma-facing components, waveguides, and much much more. Computational plasma physics involves developing and implementing numerical algorithms, linking computational work to physical models, theory, and experiment, and often uses some of the most powerful supercomputers in existence.

There are lots of applications of plasma physics. A few of the numerous applications of plasma physics include astrophysics (where over 99% of the visible universe is in the plasma state), plasma thrusters, water processing, and fusion energy. Fusion energy, which is easily one of the most challenging scientific endeavors today, also holds one of the greatest rewards. The long-term promise and allure of fusion energy comes from the immense energy bound up in the atomic nucleus and the readily available fuel sources[1] which release that energy which could power humanity for many millions of years. In addition, fusion power is carbon-dioxide free, does not have the risk of nuclear meltdown, doesn't require large land usage, and is a steady power source.

Throughout these notes, we will start to see some of this mathematical and physical richness come to play. However, as we did in class, these notes will focus on the theoretical foundations of the subject rather than concentrate heavily on any particular application of plasma physics.

1.1 Finals words before the onslaught of equations

One important question has not been answered so far - what is a plasma? Most briefly, a plasma is an ionized gas. But of course this response leaves much to the imagination. How ionized does it need to be to be a plasma? A gas of what?

[1] Deuterium is readily available in seawater. It should be emphasized that tritium, while theoretically capable of being generated from lithium, does not exist in significant quantities naturally and the process of creating tritium has not been demonstrated on a large scale. This is one of the most challenging tasks facing developers of future D-T reactor.

As Nat Fisch points out, states of matter are really approximations of reality. Take, for example, a closed box stuffed chock full of gravel. Each individual rock in that gravel certainly behaves like a solid when we observe it. If we were to take that box and throw it in the air, it would rotate approximately like a solid body. But when we open that box and pour that gravel into a funnel, the behavior of the gravel is better described with a fluid approximation. Similarly, the tectonic plates which makes up the earth's continents are certainly solid when we look at them over the course of a day or a month or a year. But when we look at them over a timescale of millions of years, the plates travel, flow, and merge, certainly unlike a solid.

Thus, whether some real physical system can be treated as one of the idealized states of matter depends on how we are observing that system. Alternatively, in the language of plasma physics, the state of matter some system is in depends on the the timescales and length scales which we are observing the system over. For example, in gas clouds in the interstellar medium, the degree of ionization is very low and the magnetic fields are very small, but over large enough scales and over long enough times, their evolution is apparently well-described by the equations of plasma physics.

In some sense, plasmas fit somewhere along an energy spectrum, where the spectrum ranges over the energy per particle (i.e. temperature). At one end of the spectrum is condensed matter physics, i.e. solids. These are at the lowest temperature. As we increase the temperature, eventually the solids become fluids, fluids become gases, and at some point they become plasma-like. In the temperature range where the gas becomes fully ionized, we have an ideal classical plasma ($\sim$10eV to $\sim$100KeV). If we were to turn up the temperature even further, then at $\sim$1MeV positrons start to become produced, and we have a relativistic QED plasma (something I know nothing about!), and we have to develop other equations to understand this system. In this energy range, we are already out of the realm of classical plasma physics. If we really crank up the energy dial, up to $\sim$100MeV, then we'll have a quark-gluon plasma, which is confined by the strong force rather than the Electromagnetic force. What we see from this illustration is that plasma physics is the physics of matter within a certain restricted temperature range.

This still doesn't answer our question of "what is a plasma"! It turns out that this definition is a bit technical, but I'll state is here. Some system is a plasma if the number of plasma particles in a Debye sphere is much greater than 1, or $n_0 \frac{4}{3}\pi\lambda_D^3 \gg 1$. In effect, this means that the plasma is electrically neutral on scales larger than the Debye length. We will explore these ideas more in section 1.4.

1.1.1 Logical framework of Plasma Physics

Here is a half-truth: Plasma physics has been fully solved. Suppose we have a large number of particles, each of which has charge q_i and mass m_i. These

particles interact via the Lorentz force,

$$m_i \frac{d^2\vec{x}_i}{dt^2} = q_i(\vec{E} + \frac{d\vec{x}_i}{dt} \times \vec{B}) \tag{1.1}$$

The initial conditions for the electric and magnetic fields are given by two of Maxwell's equations,

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.2}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1.3}$$

while the time-evolution of the electric and magnetic fields are determined by the other two Maxwell equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.4}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{1.5}$$

Given a set of initial conditions for the particles, we can solve for the time evolution of these particles. If we really wanted to be precise, we could even use the Lorentz-invariant Lorentz force law, and calculate the time-evolution of the plasma particles to arbitrary precision. Unfortunately, this simplistic approach doesn't work for a myriad of reasons, both physical and practical.

Practically, such an approach is not solvable analytically, and requires so much computing power that even on the most powerful supercomputers, it would take something on the order of the age of the universe to simulate even the most basic plasma configurations. And this is only the simplest model of a plasma possible! To make any practical progress in plasma physics, we obviously need a description of a plasma which can be practically solved. Thus, we will need to *approximate* somehow in order to get a tractable solution.

Physically, this simple model described in equations 1.1 through 1.5 is wrong. Firstly, there is no consideration of boundary conditions. In any terrestrial plasma, the plasma will be confined to some region by a solid[2] boundary, and the plasma particles will interact with the boundary in some complex way. Much of plasma physics research involves understanding the effects of plasmas as they interact with materials. These notes will not cover plasma-material interactions. In astrophysical plasmas, the boundaries are either ignored, not well-defined, or do not exist. In practice, periodic or open boundary conditions are often used to understand astrophysical plasmas.

Secondly, not every particle is ionized, and neutral particles would (rigorously) need to be treated with quantum mechanics. Indeed, a proper treatment of collisions between even ionized particles in plasmas would (rigorously) involve quantum mechanics. Atomic physics, including collision cross sections and reaction rates, needs to be included to understand collisions as well as the ionization

[2]Or, in some applications, a fluid boundary

and recombination of particles. We also need to account for the radiation emitted through these atomic processes and through particle acceleration.

As you can imagine, a rigorous, complete description of a plasma would get extremely complicated extremely quickly. Thus, approximation will be our friend as we study this subject. With the approximation schemes we make in these notes and throughout our study of plasmas, we will need to keep track of when the approximations we make are valid, so as not to apply some equation to a physical situation where it is not applicable.

There are a number of theoretical models for plasma physics. These include considering only single-particle dynamics (particle drifts, the subject of section 2), many particle dynamics (particle-in-cell computing, not covered in these notes), and using various statistical models and fluid models.

One such statistical model involves averaging over all possible ensembles[3], to get a 6-dimensional[4] time-evolving distribution function f, which tells us the number of particles at a given position with a given velocity. This distribution function f is called the Vlasov distribution, and this model is known as kinetic theory, which will be further discussed in section 3.

Another model involves treating each species[5] in the plasma as a fluid. This requires taking moments of f, and replacing an arbitrary velocity distribution with, at each position in space, a mean velocity, temperature, and a pressure tensor. Alternatively, instead of treating the each species as a fluid, we can treat the plasma as a single fluid, and calculate an overall mean velocity, a total current, and a single temperature and pressure tensor. This approximation is called Magnetohydrodynamics, or MHD. These fluid models of a plasma will be further discussed in chapter 4.

1.2 Plasma Oscillations

We will start our investigation of plasma physics by looking at plasma oscillations. We start here for a couple reasons. Firstly, plasma oscillations illustrate many of the equations and techniques used throughout our study of this field. Secondly, plasma oscillations are the most simple example of what is called collective dynamics. Dynamics is the study of how a system evolves over time. Collective dynamics simply means that when interacting, many plasma particles can conspire to create macroscopic behavior which is different than what would be observed if the particles were not interacting. Our study of waves and Landau damping are other examples of collective behavior.

Intuitively, plasma oscillations arise due to the electrostatic force which arises when electrons are displaced from an equilibrium. Suppose some number of electrons are displaced to the right, as in figure 1. Since there is now a positive charge density to the left and a negative charge density to the right, an electric

[3] What is an ensemble? We will explain this again in chapter 3, but I will include a definition here. Suppose we know the macrostate of a system but nothing about any given particle. An ensemble is a microstate of a system consistent with the known macrostate.

[4] 3 spatial dimensions, 3 velocity dimensions

[5] Species means the types of ions, electrons, and various neutral atoms

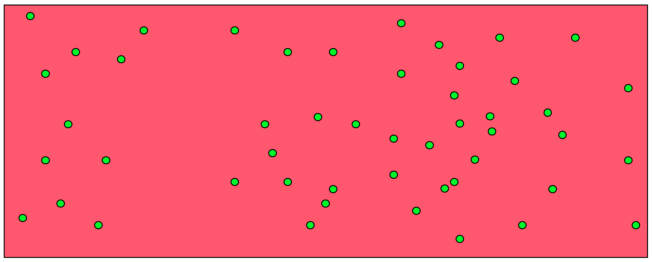


Figure 1: An initial electron density configuration. Perturbation is exaggerated for illustration.

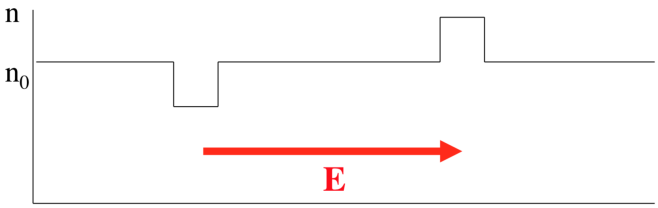


Figure 2: Electric field due to density perturbation.

field is setup which points towards the right, as in figure 2. Thus, the electrons on the right side will feel a force to the left, and will be accelerated leftwards. Thus, eventually the higher electron density will change from right to left. This process will repeat itself, and the net effect is that the density perturbations will oscillate in time (with zero group and phase velocity). These oscillations are called plasma oscillations.

To derive these plasma oscillations, we have to start somewhere. For simplicity, we will assume that a multi-species fluid approximation is valid. Instead of looking at the density individual particles, we will treat the density of particles as a continuous smooth field. Thus, for each species we have a continuity equation and a momentum equation. We will also use Poisson's equation and assume that any electric fields are curl-free ($\vec{E} = -\vec{\nabla}\phi$), and that the magnetic field is zero. We're looking for oscillations of the electrons, which we expect to be much faster than any oscillations of the ions because the electrons are much lighter[6]. We will therefore assume that the ions are stationary ($\vec{v_i} = 0$) and have a constant and static density n_0. Thus, we have

$$\vec{\nabla} \cdot \vec{E} = e(n_0 - n_e) \tag{1.6}$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{v_e}) = 0 \tag{1.7}$$

[6]This differentiation of timescales between electrons and ions due to their different masses will be a recurring theme throughout our study of plasma physics

$$m_e n_e \frac{\partial \vec{v_e}}{\partial t} + m_e(\vec{v_e} \cdot \vec{\nabla})(n_e \vec{v_e}) = -e n_e \vec{E} \tag{1.8}$$

This is our first experience with fluid equations, which we have obviously not derived. Equation 1.7 is a continuity equation for electron density, similar to the charge conservation equation in electrodynamics. It just means that if the electron density inside a fixed infinitesimal volume changes in time, it is because there are electrons flowing across the boundary of that infinitesimal volume. Equation 1.8 is a momentum equation for the electrons. Essentially, it means that the mass times acceleration of electrons is equal to the force they feel due to the electric field.

For those who have seen fluid equations before, note that there is no pressure term in the momentum equation. Pressure, as you will remember from elementary kinetic theory of gases, is an effect which comes about due to the motion of molecules. Thus, whatever results we will derive are technically applicable only in the approximation of a zero-temperature plasma, where the molecules do not have thermal velocities. For a plasma with a non-zero temperature, we will see in chapter 5 that the wave dispersion relation changes, such that the wave has a non-zero group and phase velocity.

From these equations, we will introduce a method, called linearization, which will prove useful throughout our understanding of plasma waves. With this method, we take some equilibrium ($\frac{\partial}{\partial t} \to 0$) solution to the equations, and call the values of the relevant variables the 0th order solution to the equations. From there, we will assume there is some perturbation to equilibrium solution, and call the perturbations to the relevant variables the first-order quantities. We plug the linearized quantities into the equations we have, ignore any terms which are second-order or higher, and then look for oscillatory solutions.

Let's see linearization in action. For plasma oscillations, we start with the most basic equilibrium possible: a zero-velocity plasma ($\vec{v_0} = 0$), with a uniform density of electrons and ions ($n_0(\vec{x}) = n_0$) and zero electric field ($\phi_0 = \text{constant}$). Then, we apply a small perturbation to all relevant quantities, except ion density which is assumed to be constant over the timescales we are interested in. Thus, $\vec{v_e} = v_1$, $n_e = n_0 + n_{e,1}$, $\phi = \phi_1$. By ignoring all terms second-order or higher, we have

$$\vec{\nabla}^2 \phi_1 = -\frac{e}{\epsilon_0}(n_0 - n_0 - n_{e,1}) = e n_{e,1}/\epsilon_0 \tag{1.9}$$

Thus,

$$\frac{\partial n_{e,1}}{\partial t} = -\vec{\nabla} \cdot (n_0 \vec{v_1}) = -n_0 \vec{\nabla} \cdot \vec{v_1} \tag{1.10}$$

where

$$m_e n_0 \frac{\partial \vec{v_1}}{\partial t} = e n_0 \vec{\nabla} \phi_1 \tag{1.11}$$

Now taking the divergence of the linearized momentum equation, we have

$$m_e n_0 \frac{\partial \vec{\nabla} \cdot \vec{v_1}}{\partial t} = -m_e \frac{\partial^2 n_{e,1}}{\partial t^2} = e n_0 \vec{\nabla}^2 \phi_1 = \frac{e^2 n_0}{\epsilon_0} n_{e,1} \tag{1.12}$$

$$\frac{\partial^2 n_{e,1}}{\partial t^2} = -\omega_p^2 n_{e,1}(\vec{x}, t) \tag{1.13}$$

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e} \tag{1.14}$$

Because the derivative is a partial derivative with respect to time, this equation gives a solution for the density perturbation which oscillates in time, but not in space. The way we visualize this is as follows: imagine we take some electrons from one point in space and displacing them slightly to another position. This electron density is shown graphically in Figure 3. At each point in space, the perturbation will oscillate sinusoidally, so at some later time $t = 2\pi/4\omega_p$, the density will be instantaneously constant, and another quarter-period later the leftmost electron density perturbation will have a higher electron density. Remember: physically, we can think of plasma oscillations as arising due to

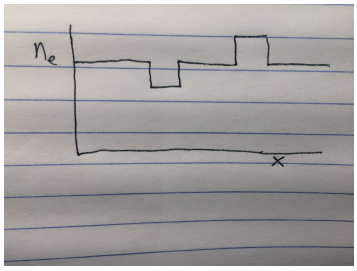


Figure 3: Initial electron density perturbation. Over time, these bumps will rise and fall but stay at constant position in space in a cold plasma.

electrostatic forces which cause the electrons to accelerate back and forth. In between the leftward and rightward bumps in Figure 3, there is an electrostatic electric field which pushes the electrons between the two bumps back and forth, creating the density oscillation. We will see later that in a warm plasma, the temperature allows for plasma waves with some non-zero group velocity.

Although we are focused on the density here, note that the electron fluid velocity oscillates in time, as does ϕ.

1.3 Debye Shielding

As we remember from electromagnetism, the electric field inside a conductor is 0. Otherwise, charges would move around, causing the electric field to change, until the electric field eventually became 0.

Plasmas, in general, are highly conducting. Thus, we should expect that the electric field inside a plasma is 0, right? Well, not exactly. Indeed plasmas, like conductors, screen external electric fields quite well. However, the electric field inside a plasma is not necessarily zero. If we place a charge Ze in a warm plasma and make it stay there, then the electric potential a distance r away

from the charge is

$$\frac{Ze}{4\pi\epsilon_0 r}e^{-r/\lambda_D} \tag{1.15}$$

where λ_D is a constant called the Debye length which depends on, among other things, temperature. Note that the potential falls off in a plasma faster than $1/r$, due to the exponential dependence. This faster-than-exponential falloff of the plasma potential is what is called Debye shielding or Debye screening. Over distances significantly longer than a couple Debye lengths, the plasma potential due to a charge in the plasma is very small. Loosely speaking, plasmas are net neutral over distances longer than a Debye length.

Let's derive this. Imagine inserting a test particle of infinitesimal charge Q into a plasma. Assume that each species (represented by σ) in the plasma is in thermal equilibrium with temperature T_σ, and that each species can be treated as a fluid with density n_σ. Now, it is true that

$$n_\sigma = n_0 e^{\frac{-q_\sigma\phi}{k_B T_\sigma}} \tag{1.16}$$

This was argued to be true in class based on statistical mechanics, but I am having trouble deriving it using the Grand Canonical Ensemble. The simplest way to derive this follows Bellan section 1.6. We start with a fluid equation for each species.

$$m_\sigma \frac{d\vec{v_\sigma}}{dt} = q_\sigma \vec{E} - \frac{1}{n_\sigma}\vec{\nabla}P_\sigma \tag{1.17}$$

Assuming the inertial term is negligible (which physically means the changes in the plasma are slow), the electric field is electrostatic ($\vec{E} = -\vec{\nabla}\phi$), the temperature is spatially uniform, and the ideal gas law $P_\sigma = n_\sigma k_B T_\sigma$ holds, then this reduces to

$$0 = -n_\sigma q_\sigma \vec{\nabla}\phi - k_B T_\sigma \vec{\nabla} n_\sigma \tag{1.18}$$

which has the solution

$$n_\sigma = n_0 e^{-q_\sigma\phi/k_B T_\sigma} \tag{1.19}$$

These assumptions are all consistent with the plasma being in it's maximum-entropy state, which is the assumption used to derive it from statistical mechanics. Now, we assume that $k_B T_\sigma \gg q_\sigma\phi$, which is true assuming the test particle is of infinitesimal charge so that ϕ is small as well. Thus

$$n_\sigma \approx n_0(1 - \frac{q_\sigma\phi}{k_B T_\sigma}) \tag{1.20}$$

.

The second equation we will use is Poisson's equation. Assuming the test charge is at the origin, we have that

$$-\vec{\nabla}^2\phi = \frac{1}{\epsilon_0}(Q\delta^{(3)}(\vec{r}) + \sum_\sigma n_\sigma(\vec{r})q_\sigma) \tag{1.21}$$

Using equation 1.20 and the fact that the plasma is net neutral to zeroth order, this simplifies to

$$-\vec{\nabla}^2\phi + \frac{\phi}{\lambda_D^2} = \frac{1}{\epsilon_0} Q\delta^{(3)}(\vec{r}) \tag{1.22}$$

where

$$\frac{1}{\lambda_D^2} = \sum_\sigma \frac{1}{\lambda_\sigma^2} \tag{1.23}$$

$$\lambda_\sigma = \sqrt{\frac{\epsilon_0 k_B T_\sigma}{q_\sigma^2 n_0}} \tag{1.24}$$

Now, $\vec{\nabla}^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r})$. We look for a solution of the form $\phi = \frac{f(r)Q}{4\pi\epsilon_0 r}$ where $f(0) = 1$. Remembering that $\vec{\nabla}^2(\frac{1}{r}) = -4\pi\delta^{(3)}(\vec{r})$, we find that $f'' = \frac{f}{\lambda_D^2}$ so $f(r) = e^{-r/\lambda_D}$, where the positive exponential solution is ruled out due to boundary conditions at infinity - the potential at infinity can't be infinity. Thus, we have our Debye shielding equation for a test charge Q in a plasma,

$$\frac{Q}{4\pi\epsilon_0 r} e^{-r/\lambda_D} \tag{1.25}$$

Note that the Debye length is larger for larger values of T_σ, and smaller for larger values of q_σ. This makes sense, since Debye screening is an effect we see due to the thermal motion of charged particles in tandem with the electrostatic forces they feel. Loosely speaking, a species' charge causes it to want to stay close to any test charge Q in the plasma, thus large q_σ should decrease the Debye length, by increasing the electrostatic force on these particles. On the other hand, a species' thermal motion causes it to zoom around randomly, thus large T_σ should increase the Debye length by increasing these random speeds. A zero temperature plasma has zero Debye length, because (in equilibrium) the particles will have no thermal velocity and thus exactly cancel the potential due to any test charges.

Note also the following nifty little relation, neglecting a factor of $\sqrt{3}$ in the thermal velocity $v_{T,\sigma}$:

$$\lambda_\sigma = \frac{v_{T,\sigma}}{\omega_{p,\sigma}} = \sqrt{\frac{k_B T_\sigma}{m_\sigma}}\sqrt{\frac{m_\sigma \epsilon_0}{q_\sigma^2 n_0}} \tag{1.26}$$

This is something you'll want to remember!

Here's something seemingly contradictory that confused me: the electric field inside a plasma is not always 0! But we learned in freshman physics that the electric field inside a conductor is 0. We also know that plasmas are highly conducting. So what is it about a plasma which is different than a typical conductor, such as a metal? Actually, in terms of shielding of electric fields, nothing! In an idealized metal, the electrons are at a temperature T and are free to move around as they please. Their behavior obeys Poisson's equation and the Boltzmann relation. Therefore, we must see Debye shielding in a metal! In fact, if we were to put a test charge in a metal and hold it there, we would see

a potential quite like the Debye potential. At the edge of a charged conductor, where there is a surface charge, the electric field will not go to 0 immediately inside the conductor! Instead, it will fall to from σ/ϵ_0 to 0 over a couple debye lengths.

1.4 Collisions in Plasmas

We discussed an extremely simple model of collisions in a plasma, to get a rough estimate for the mean free path, the distance of closest approach, and the collision cross-section between two plasma particles.

The distance of closest approach is approximately the distance at which the average kinetic energy equals the electrostatic potential energy. This would occur if we had a particle with energy $\frac{3}{2}k_BT$ moving directly towards a stationary particle, until the electrostatic potential energy is $\frac{1}{4\pi\epsilon_0}\frac{q^2}{b}$, where b is the distance of closest approach. Solving for b, we get

$$b = \frac{q^2}{6\pi\epsilon_0 k_B T} \tag{1.27}$$

.

The collision cross-section is roughly $\sigma = \pi b^2$, so

$$\sigma = \frac{q^4}{36\pi\epsilon_0^2 (k_B T)^2} \tag{1.28}$$

The mean free path is defined as

$$l = \frac{1}{\sigma n} \tag{1.29}$$

. This is a basic result from statistical mechanics.

Until we get into the collision operator in chapter 3, this is all that we say in this course about collisions, which is a bit unfortunate. Hopefully future courses will cover collisions in depth, as they are certainly important.

1.5 Plasma Length and Time Scales

There are numerous length scales in plasmas:

- Distance of closest approach, $b = \frac{e^2}{6\pi\epsilon_0 k_B T}$
- Interparticle spacing, $n^{-1/3}$
- Mean free path, $\lambda_{mfp} = \frac{1}{n\pi b^2}$
- Electron gyroradius, $\rho_e = \frac{m_e v_{T,e}}{eB} = \frac{\sqrt{k_B T_e m_e}}{eB}$
- Ion gyroradius, $\rho_i = \frac{\sqrt{k_B T_i m_i}}{ZeB}$

- Debye Length, $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n_0}}$

The electron and ion gyroradius size depends on the local magnetic field, which can vary dramatically between different plasmas. We can, however, say that the ion gyroradius is nearly always significantly higher than the electron gyroradius, as long as the electron temperature is not dramatically smaller than the ion temperature, which almost never is the case. However, in a plasma (where the number of particles in a debye sphere is much greater than 1), we have the following ordering of scale lengths:

- $b \ll n^{-1/3} \ll \lambda_D \ll \lambda_{mfp}$

We can prove this as follows: Suppose we define Λ as the number of particles in a debye sphere, $\Lambda = \frac{4\pi}{3} n \lambda_D^3 \gg 1$. Then if we define our scale length in units of b, such that $b = 1$, we have that

$$\lambda_D^2 = \frac{\epsilon_0 k_B T}{e^2 n_0} = \frac{1}{6\pi n_0 b} \tag{1.30}$$

and thus

$$\frac{b}{\lambda_D} = \frac{1}{6\pi n_0 \lambda_D^3} = O(\frac{1}{\Lambda}) \tag{1.31}$$

We also have that

$$\frac{n^{-1/3}}{b} = \frac{n^{-1/3}}{\lambda_D} \frac{\lambda_D}{b} = O(\Lambda^{-1/3}) O(\Lambda) = O(\Lambda^{2/3}) \tag{1.32}$$

Finally, we have that

$$\frac{\lambda_m fp}{\lambda_D} = \frac{1}{n\pi b^2 \lambda_D} = \frac{\lambda_D^2}{\pi b^2} \frac{1}{n\lambda_D^3} = O(\Lambda^2) O(1/\Lambda) = O(\Lambda) \tag{1.33}$$

Thus, we have $b : 1$, $n^{-1/3} : \Lambda^{2/3}$, $\lambda_D : \Lambda$, $\lambda_{mfp} : \Lambda^2$, which gives our ordering of scale lengths in a plasma where the number of particles in a debye sphere is much greater than 1.

We can see now why we've chosen our definition of a plasma to be where the number of particles in a Debye sphere, Λ, is much greater than 1. If we choose this definition, then we have a definite ordering of scale lengths, meaning we can use the same equations to treat a wide variety of different plasmas.

This condition, the number of particles in a Debye sphere, also implies that the plasma is quasineutral over length scales larger than a Debye length. How do we know this? Well, there's an ingenious calculation we can do which shows that this is the case. Imagine that all of the electrons in some region of space were to all move radially outwards from a point until their velocity becomes zero, as in figure 4. How large of a spherical region could the electrons evacuate, such that we are quasineutral over that region? Well, the trick is to set the thermal energy of the electrons inside the volume (which all end up at the surface of the sphere) equal to the energy stored in the electromagnetic field created by

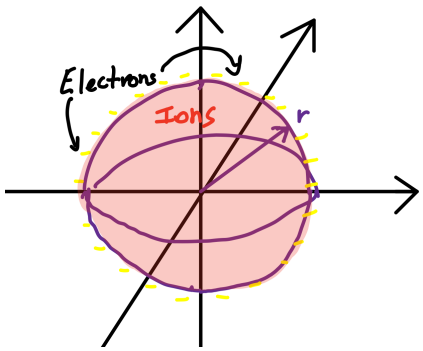


Figure 4: Illustration of the electrons evacuating a region of space with radius r, leaving ions behind. This calculation is used to show why the number of particles in a Debye sphere being much greater than 1 implies overall quasineutrality over distances larger than a Debye length.

ions left behind in the absence of the electrons. The electric field is created by the ions left behind, and from Gauss's law $E_r 4\pi r^2 = \frac{ne4\pi r^3}{3\epsilon_0}$, so $E_r = \frac{ner}{3\epsilon_0}$. The electromagnetic field energy is $\int \frac{\epsilon_0}{2} E^2 dV = \frac{2\pi n^2 e^2}{9\epsilon_0} \int r'^4 dr' = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0}$. The thermal energy per particle is $\frac{3}{2}k_B T$, so the total thermal energy of the electrons in that volume is $2\pi n k_B T r_{max}^3$. Setting these equal, we have

$$2\pi n k_B T r_{max}^3 = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0} \tag{1.34}$$

so the maximum radius r_{max} that the thermal energy of the electrons could evacuate is

$$r_{max} = \sqrt{\frac{45\epsilon_0 k_B T}{ne^2}} \approx 7\lambda_D \tag{1.35}$$

Thus, the largest region of space that can be evacuated of electrons is a few Debye length. Now, the logic goes like this. If the number of particles in a Debye sphere is much greater than 1, then the situation required to make this happen (where all of the electrons are moving radially outwards) would be extremely unlikely, as the number of particles is large. Thus, if there are many particles in a Debye sphere, our plasma is extremely likely to be quasineutral. If there is less than 1 particle in a Debye sphere, then it is possible that the plasma will be non-neutral over length scales larger than a Debye sphere.

What about timescales? Here are some of the most important frequencies:

- Electron gyrofrequency, $\frac{eB}{m_e}$
- Ion gyrofrequency, $\frac{q_i B}{m_i}$
- Plasma frequency, $\sqrt{\frac{e^2 n_0}{\epsilon_0 m_e}}$

- Momentum collision frequency
- Energy collision frequency

2 Single Particle Motion

> The human mind is more attuned to thinking geometrically rather than thinking analytically. We may be able to solve an equation describing a process, but only when we are able to make a mental picture of how the process proceeds, do we feel that we have understood the process.
>
> Arnab Rai Choudhuri

As we have seen, the mean free path of particles in a plasma is significantly longer than any of the other scale lengths, assuming the number of particles in a debye sphere is much greater than 1 (which is how we are defining a plasma). Thus, particles often travel a long distance before colliding with other particles. For many plasmas, the collision timescale $(1/\nu)$ is much longer than other relevant timescales. In these plasmas, an ion or electron might $\vec{v} \times \vec{B}$ rotate (sometimes called gyromotion or Larmor motion) many times before it collides with another plasma particle, changing it's trajectory.

Thus, for many plasmas, analyzing the motion of individual charged particles gives valuable insight into the behavior of the plasma as a whole. We will first investigate the motion of particles in prescribed electric and magnetic fields. We will also see that periodic motion in the absence of collisions leads to the existence of conserved quantities for individual particles, which can be helpful for analyzing the motion of particles in complicated electromagnetic fields. We will then analyze the magnetic mirror machine, the classic example of single-particle motion. Lastly, we'll discuss the isorotation theorem, an example of single-particle motion not typically found in textbooks which Nat covered in class.

2.1 Guiding Center Drifts

Imagine we have a constant, static magnetic field in the z-direction, $\vec{B} = B_0\hat{z}$. If we put a charged particle of charge q and mass m in that magnetic field, then the particle will spiral around the magnetic field, while it's velocity in the z-direction will remain constant. Let's see this. The force on the particle will be $\vec{v} \times \vec{B}$, which always points perpendicular to the motion. This is the condition for uniform circular motion. Thus, we have a centripetal acceleration $v_\perp^2/R = qv_\perp B_0/m$. This is easily solved, as in freshman physics, to give a frequency $\Omega = qB_0/m$ and a gyroradius $\frac{mv_\perp}{qB_0}$.

In many plasmas, there exists some sort of uniform background magnetic field[7]. Thus, the most basic, ubiquitous behavior of single particles in a plasma

[7] Usually due to some external magnetic coils, internal current, or a background field in outer space.

is gyromotion around this background magnetic field. However, the behavior of particles in spatially, time-varying fields is much more complicated. We will see that the guiding center (center of the gyromotion) motion involves various drifts in addition to the gyromotion.

Suppose there exists a charged particle of mass m and charge q in arbitrary electric and magnetic fields, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$[8]. The equation of motion for the charged particle is

$$\ddot{\vec{r}} = \frac{q}{m}(\vec{E}(\vec{r},t) + \dot{\vec{r}} \times \vec{B}(\vec{r},t)) \tag{2.1}$$

Let us assume that $\vec{E}$ and $\vec{B}$ are known. In general, this expression cannot be integrated exactly to solve for the motion. However, we will make a few approximations for this problem to become solvable. Firstly, we will assume that the particle gyro-orbits around the magnetic field, and that the gyroradius of the particle is small relative to the length scales ($\frac{B}{\vec{\nabla}B}$) over which the electric and magnetic fields change. Thus, $\vec{r}(t) = \vec{r}_{gc}(t) + \vec{r}_c(t)$ where $\vec{r}_{gc}$ is the position of the guiding center of the particle, and $\vec{r}_c$ is the vector from the guiding center to the particles position. We will see that there are a number of drifts of $\vec{r}_{gc}$ which add to each other in the limit that the gyroradius is much smaller than the relevant lengthscales of the magnetic and electric fields.

We define $\vec{r}_c$ as $\frac{m\hat{b}\times\dot{\vec{r}}}{qB}$, where the magnetic field is evaluated at the position of the guiding center. Note that this definition makes sense intuitively[9]. Why? Well, remember when we had a constant magnetic field, such that the gyroradius was $\frac{mv_\perp}{qB}$? Well, notice that our definition is essentially the same - it points in the direction we want it to (perpendicular to both $\vec{B}$ and $\vec{v}$) and reduces to our previous expression in the limit that the magnetic field is constant in space.

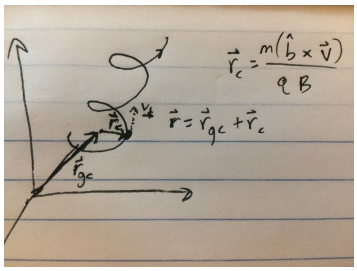


Figure 5: A negatively charged particle moving in a magnetic field. The gyroradius, $\vec{r}_c$, and guiding center position, $\vec{r}_{gc}$, are shown.

[8] Bellan's book does a wonderfully rigorous, although not particularly physically enlightening derivation of the drift equations. Spitzer and Chen, on the other hand, give wonderfully intuitive but less rigorous explanations of these drifts. These notes aim for somewhere in the middle.

[9] How we decide to split $\vec{r}$ between the guiding center drift and the cyclotron orbit is somewhat arbitrary, but this definition makes things easier mathematically.

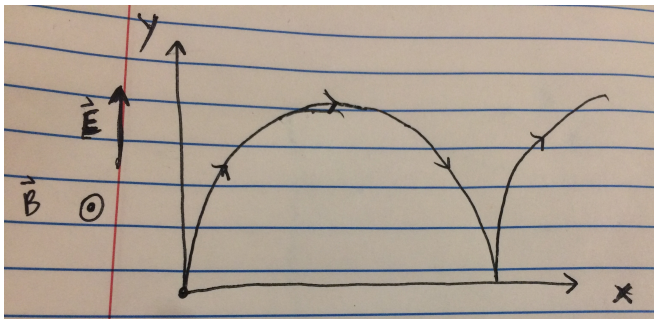


Figure 6: Illustration of the $\vec{E} \times \vec{B}$ drift for a particle which starts at rest at the origin. For particles with other initial velocities, the motion will have a different trajectory but the same net guiding center velocity.

2.1.1 $\vec{E} \times \vec{B}$ Drift

The first guiding center drift we will examine is called the E cross B drift.

$$\vec{v}_{E\times B} = \frac{\vec{E} \times \vec{B}}{B^2} \tag{2.2}$$

First off, we know that if there is an electric field in the same direction as the local magnetic field, our particle will accelerate in that direction without feeling any magnetic force. What about for electric fields perpendicular to the local magnetic field direction? Well, this gives rise to the $\vec{E} \times \vec{B}$ drift which is under study at the moment. How do we understand this intuitively? Well, imagine we have a static constant B field in the z-direction, and a static constant E field in the y-direction. Now imagine at $t = 0$ putting a charge $+q$ at the origin with zero initial velocity. What will happen? The electric field will cause the charge to initially accelerate in the y-direction. As the charge picks up speed in the y-direction, the magnetic field puts a force in the x-direction on the charge, causing it to turn in the positive x-direction. As the particle turns, eventually it's velocity is entirely in the x-direction. Now, the magnetic force will be in the negative y-direction, and it turns out that this force will be stronger than the electric force in the positive y-direction. Thus, the particle starts to curve downwards, in the negative y-direction. At some point, the particle will come to rest at $y = 0$,[10] and then the process will repeat itself. However, the particle will have been displaced in the x-direction, which is also the $\vec{E} \times \vec{B}$ direction. This process is illustrated in figure 6.

For negatively charged particles (i.e. electrons), they will initially accelerate in the opposite direction, but the magnetic force will cause them to curve towards the right in figure 6, again creating a $\vec{E} \times \vec{B}$ drift.

Mathematically, we derive this as follows. For simplicity, we will assume

[10] It must come to rest at $y = 0$, by conservation of energy

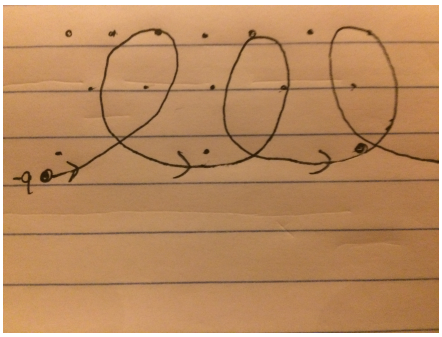


Figure 7: Illustration of the motion of a negatively charged particle in a magnetic field gradient.

that the magnetic field is constant over the gyroorbit of the particle. We have

$$\vec{r}_{gc}(t) = \vec{r} - \vec{r}_c = \vec{r} - \frac{m\hat{b} \times \dot{\vec{r}}}{qB} \tag{2.3}$$

Taking the time-derivative, we have

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}} - \frac{m\hat{b}}{qB} \times \ddot{\vec{r}} \tag{2.4}$$

Inserting the equation of motion into $\ddot{\vec{r}}$, we have

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}} - \frac{\hat{b} \times \vec{E}}{B} - \hat{b} \times (\dot{\vec{r}} \times \hat{b}) \tag{2.5}$$

Now, we can recognize that the rightmost term is $\dot{\vec{r}}_\perp$, the velocity perpendicular to the local magnetic field. We also know that $\dot{\vec{r}} = \dot{\vec{r}}_\parallel + \dot{\vec{r}}_\perp$. Thus, we have

$$\dot{\vec{r}}_{gc} = \dot{\vec{r}}_\parallel + \frac{\vec{E} \times \vec{B}}{B^2} \tag{2.6}$$

This is the E cross B drift we described earlier.

Notice that if we replaced $q\vec{E}$ with an arbitrary force F, we would get a drift velocity $v_F = \frac{\vec{F} \times \vec{B}}{qB^2}$. For example, this force could be the force of gravity, $F = -mg\hat{z}$. In laboratory plasmas, gravity causes positive and negative particles to drift in opposite directions, until a small electric field arises which cancels this force. In general, the force of gravity is ignored in laboratory plasmas, as it is essentially negligible.

2.1.2 Grad-B drift

The grad-B drift is an effect that arises due to changes in the magnetic field strength perpendicular to the magnetic field direction. The grad-B drift is equal

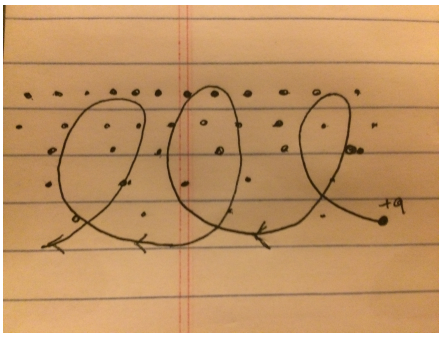


Figure 8: Illustration of the motion of a positively charged particle in a magnetic field gradient.

to

$$\vec{v}_{\vec{\nabla}B} = \frac{v_\perp}{2}\frac{r_c\hat{b}\times\vec{\nabla}B}{B^2} = \frac{mv_\perp^2}{2}\frac{\vec{B}\times\vec{\nabla}B}{qB^3} \tag{2.7}$$

The grad-B drift arises due to the decreased radius of curvature in regions of stronger magnetic field. This effect is shown in Figures 8 and 7.

The grad-B drift was derived in class by getting an equation for the particle drift to 1st order, then plugging in the 0th order motion to that equation. This is a classic technique from classical mechanics, which I remember using multiple times in my classical mechanics course.

Suppose we integrate the vector gyroradius $\vec{v}_c$ over one orbit. To 0th order in $\epsilon = r_c/L$, where L is the length scale over which the magnetic field changes, we have the relation

$$\int_0^{2\pi} \vec{r}_c d\theta = \hat{b}\times\Delta\vec{r} \tag{2.8}$$

where $\Delta\vec{r}$ is the distance the guiding center travels in one rotation along the magnetic field. You could argue this on geometric grounds, as the sum over the gyroradius vectors tells us how much the gyrocenter has shifted. However, this geometric interpretation wasn't immediately obvious to me, so it's probably easier to justify it algebraically. Equation 2.8 is true because $\vec{v}_c = \frac{m\hat{b}\times\dot{\vec{r}}}{qB}$, and $\int \vec{v}_c dt = \frac{m\hat{b}\times\Delta\vec{r}}{qB}$, since $\hat{b}$ and B are constant over a gyroorbit to 0th order in ϵ. But $\frac{d\theta}{dt} = \Omega = \frac{qB}{m}$, so $dt = md\theta/qB$. Changing variables to θ gives equation 2.8.

Now, to 1st order in ϵ, we allow the possibility that the magnetic field is not constant over a gyroperiod. Thus, we have $B(\vec{r}_{gc}+\vec{r}_c) \approx B(\vec{r}_{gc}) + (\vec{r_c}\cdot\vec{\nabla})B$. Assuming the magnetic field direction is constant over a gyroperiod, to 1st order in ϵ, we get

$$\vec{v}_c(\vec{r}) = \frac{m}{q}\frac{\hat{b}\times\dot{\vec{r}}_c}{B(\vec{r}_{gc}+\vec{r}_c)} \approx \frac{m}{qB(\vec{r}_{gc})}(\hat{b}\times\dot{\vec{r}}_c)(1-\frac{(\vec{r}_c\cdot\vec{\nabla})B}{B(\vec{r}_{gc})}) \tag{2.9}$$

Equations 2.8 and 2.9 together give equations for our drift motion to 1st

order in ϵ. Now, we plug in our 0th order solution to the 1st order equations to solve for the perturbed motion.

Assume we have a positive particle and we set our coordinate system to point along the local magnetic field, such that $\vec{B} = B_z(\vec{r})\hat{z}$. To 0th order, $\vec{r}_c(\theta) = r_c(\cos\theta\hat{x} - \sin\theta\hat{y})$. Similarly, $\dot{\vec{r}}_c = v_\perp(-\sin\theta\hat{x} - \cos\theta\hat{y})$. Integrating $\vec{v}_c$ from 0 to 2π, the first term in equation 2.9 integrates to 0. From our 0th order cyclotron motion, $(\vec{r}_c \cdot \vec{\nabla})B = r_c\cos\theta\frac{\partial B_z}{\partial x} - r_c\sin\theta\frac{\partial B_z}{\partial y}$. Thus, the second term in $\vec{r}_c$, integrated over 2π, becomes

$$\frac{mr_cv_\perp}{qB^2(\vec{r}_{gc})}\hat{b}\times\int_0^{2\pi} d\theta[(\sin\theta\cos\theta\frac{\partial B_z}{\partial x} - \sin^2\theta\frac{\partial B_z}{\partial y})\hat{x} + (\cos^2\theta\frac{\partial B_z}{\partial x} - \sin\theta\cos\theta\frac{\partial B_z}{\partial y})\hat{y}] \tag{2.10}$$

The first and fourth terms integrate to 0, and the second and third terms integrate to $-\pi\frac{\partial B_z}{\partial y}\hat{x} + \pi\frac{\partial B_z}{\partial x}\hat{y}$ which equals $\pi\hat{b}\times\vec{\nabla}B$. Using equations 2.8 and 2.10, we get

$$\Delta\vec{r} = \frac{\pi mr_cv_\perp}{qB^2(\vec{r}_{gc})}\hat{b}\times\vec{\nabla}B \tag{2.11}$$

. Now, $\vec{v}_{\vec{\nabla}B} = \Delta\vec{r}/T = \frac{\Delta\vec{r}qB}{2\pi m}$. This gives

$$\vec{v}_{\vec{\nabla}B} = \frac{r_cv_\perp}{2B}\hat{b}\times\vec{\nabla}B = \frac{mv_\perp^2}{2}\frac{\hat{b}\times\vec{\nabla}B}{qB^2} \tag{2.12}$$

This is the promised result.

2.1.3 Curvature Drift

In a magnetic field which changes direction slowly, charged particles (approximately) follow the field lines[11]. Although this statement is fundamental to plasma physics, it really is a remarkable fact when you think about it! Why should particles trajectories curve and twist with a magnetic field?

Frankly, I don't have a compelling answer for this, and if someone does please let me know! Here is the best answer I could come up with: Imagine that in some region in space, the local magnetic field curves. If a charged particle travels along the field line with components parallel and perpendicular to the magnetic field, any motion perpendicular to the field will get washed out by the gyromotion, while any motion parallel to the field will not be affected. Try a particle might to move perpendicular to the magnetic field line, it can't get very far away from the field line it was originally on because any motion in one of the directions perpendicular to the field will be transferred into motion in the other perpendicular direction by the Lorentz force. Thus, the vast majority of the motion by our particle in a curved magnetic field will be in the parallel direction.

Now, the curvature drift is a drift which arises whenever the magnetic field lines are not straight. Unlike for the $\vec{E}\times\vec{B}$ drift or the grad-B drift, as far

[11] Technically, their guiding centers follow the field lines to 0th order in $\epsilon = r_c/L$.

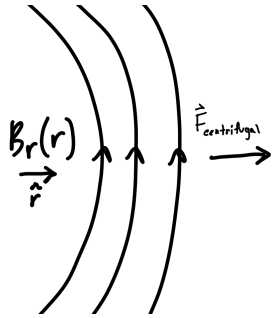


Figure 9: Curved magnetic field lines. Particles drifting along the field lines will drift upwards or downwards depending on their charge due to the curvature drift.

as I know textbooks don't paint a clear physical picture for what the particle does as it drifts due to the curvature drift. However, there is simple, physically enlightening way of deriving the curvature drift, based on the derivation of the curvature drift in Spitzer's book. Imagine a particle spiraling around field lines in a curved, constant-strength magnetic field, as in figure 9. In the rotating frame of the particle, there is some centrifugal pseudo-force in the radial direction, equal to $\vec{F} = m\frac{v_\parallel^2}{R}\hat{r}$. Plugging this force into the $\frac{\vec{F}\times\vec{B}}{qB^2}$ drift, we get the drift

$$\vec{v}_D = \frac{mv_\parallel^2}{qB}\frac{\hat{r}}{R}\times\hat{b} \tag{2.13}$$

This is the curvature drift for a magnetic field which has a radius of curvature R! For a more general magnetic field, the curvature drift is

$$\vec{v}_c = \frac{mv_\parallel^2}{qB}\hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b} \tag{2.14}$$

So what is the physical picture we should have in mind? I suggest that the physical picture we should have is that the particles feel a force outwards due to the centrifugal force in the particles reference frame, and because of that outwards force they initially move 'outwards' slightly (in the same way that the electric field in the $\vec{E}\times\vec{B}$ drift causes particles to initially move slightly in the direction of the electric field). However, the magnetic field causes particle then to drift in a direction perpendicular to both B and the 'outwards' direction, with the sign depending on the charge. In figure 9, the particles would first have some radial velocity outwards, before the magnetic field causes them to drift upwards in the same way that it does for the $\vec{E}\times\vec{E}$ drift.

Let's derive the curvature drift more formally, as was done in class. Suppose we split the particles position between the guiding center drift $\vec{r}_{gc}$ and the gyroradius, $\vec{r}_c$, and we expand B around the gyrocenter, such that $\vec{B}(\vec{r}) =$

$\vec{B}(\vec{r_{gc}}) + (\vec{r}_c \cdot \vec{\nabla})\vec{B}$. Also suppose that the $\vec{E} = 0$. Thus, our equation of motion, equation 2.1, becomes

$$\ddot{\vec{r}}_{gc} + \ddot{\vec{r}}_c = \frac{q}{m}(\dot{\vec{r}}_{gc} + \dot{\vec{r}}_c) \times (\vec{B}(\vec{r}_{gc}) + (\vec{r}_c \cdot \vec{\nabla})\vec{B}(\vec{r}_{gc})) \tag{2.15}$$

We can, in the limit $\epsilon \ll 1$, define the gyrotron motion to be the solution to the equation

$$\ddot{r}_c = \dot{\vec{r}}_c \times \vec{B}(\vec{r}_{gc}) \tag{2.16}$$

Thus, our equation for the drift of the guiding center becomes

$$\ddot{\vec{r}}_{gc} = \frac{q}{m}(\dot{\vec{r}}_{gc} \times \vec{B}(\vec{r}_{gc}) + \dot{\vec{r}}_c \times (\vec{r}_c \cdot \vec{\nabla})\vec{B}(\vec{r}_{gc}) + \dot{\vec{r}}_{gc} \times (\vec{r}_c \cdot \vec{\nabla})\vec{B}(\vec{r}_{gc})) \tag{2.17}$$

Now, let us average this equation over one gyroperiod. The third term, because it is linear in $\vec{r}_c$, will integrate to 0 to first order in ϵ. Thus we have

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{q}{m}(\langle\dot{\vec{r}}_{gc}\rangle \times \vec{B}(\vec{r}_{gc}) + \langle\dot{\vec{r}}_c \times (\vec{r}_c \cdot \vec{\nabla})\vec{B}(\vec{r}_{gc})\rangle) \tag{2.18}$$

The rightmost term will end up contributing to the $\vec{\nabla}B$ drift. As we have calculated this drift, we will not do so again. If we were to calculate this term, we could show it equals $\frac{-\mu\vec{\nabla}B}{m}$ where $\mu = \frac{mv_\perp^2}{2B}$.

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{q}{m}(\langle\dot{\vec{r}}_{gc}\rangle \times \vec{B}(\vec{r}_{gc})) + \frac{-\mu\vec{\nabla}B}{m} \tag{2.19}$$

The first term on the right will end up contributing to the curvature drift. Let us calculate that now. Crossing equation 2.19 with $\hat{b}$, we have

$$\langle\ddot{\vec{r}}_{gc}\rangle \times \hat{b} = \frac{\mu\hat{b} \times \vec{\nabla}B}{m} + \frac{qB}{m}(\langle\dot{\vec{r}}_{gc}\rangle \times \hat{b}) \times \hat{b} \tag{2.20}$$

Now, this last term equals $-\frac{qB}{m}\langle\dot{\vec{r}}_{gc,\perp}\rangle$. This result is actually easy to see - use the geometric interpretation of the cross product to convince yourself of this. Solving for $\langle\dot{\vec{r}}_{gc,\perp}\rangle$, we have

$$\langle\dot{\vec{r}}_{gc,\perp}\rangle = \frac{\mu\hat{b} \times \vec{\nabla}B}{qB} - \frac{m\langle\ddot{\vec{r}}_{gc}\rangle \times \hat{b}}{qB} \tag{2.21}$$

Hey look, our first term is the $\vec{\nabla}B$ drift, as promised! The second term simply requires solving for $\ddot{\vec{r}}_{gc}$. Well, to first-order in $\epsilon = \vec{r}_c/L$, we have

$$\langle\ddot{\vec{r}}_{gc}\rangle = \frac{d}{dt}\dot{\vec{r}}_{gc} = \frac{d}{dt}(v_\parallel\hat{b} + \vec{v}_{drift}) + O(\epsilon^2) = \frac{dv_\parallel}{dt}\hat{b} + v_\parallel\frac{d\hat{b}}{dt} + O(\epsilon^2) \tag{2.22}$$

Note that since there is no 0th order electric field, the drift velocity is 1st order in ϵ. Thus, it's time derivative will be second order in ϵ, so it can be

ignored. We can also conclude that $\frac{dv_\parallel}{dt} = 0$, averaged over a cyclotron period, is 0, because there is no 0th order electric field. So $\langle \ddot{\vec{r}}_{gc} \rangle = v_\parallel \frac{d\hat{b}}{dt} + O(\epsilon^2)$.

Now, $\frac{d\hat{b}}{dt} = v_\parallel (\hat{b} \cdot \vec{\nabla})\hat{b}$. Why is this true? Well, $\frac{d\hat{b}}{dt} = \frac{\partial \hat{b}}{\partial s}\frac{\partial s}{\partial t} = v_\parallel \frac{\partial \hat{b}}{\partial s}$, where s is the distance along a magnetic field line. Now, $\frac{\partial \hat{b}}{\partial s} = (\hat{b} \cdot \vec{\nabla})\hat{b}$. This is not proved directly in Bellan or in the class notes, but it makes sense geometrically. We could also convince ourselves of this by looking at a point in space where the magnetic field is instantaneously in the z-direction. Then, $\frac{\partial \hat{b}}{\partial s} = \frac{\partial \hat{b}}{\partial z}$, and $(\hat{b} \cdot \vec{\nabla})\hat{b} = (\frac{\partial}{\partial z})\hat{b}$. Thus, the two expressions are equivalent, and we have our result $\frac{d\hat{b}}{dt} = v_\parallel (\hat{b} \cdot \vec{\nabla})\hat{b}$. Thus to 1st order in ϵ

$$\langle \ddot{\vec{r}}_{gc} \rangle = v_\parallel^2 (\hat{b} \cdot \vec{\nabla})\hat{b} \tag{2.23}$$

Plugging this into equation 2.21,[12] we get

$$\dot{\vec{r}}_{gc,\perp} = \frac{\mu \hat{b} \times \vec{\nabla} B}{qB} + \frac{m v_\parallel^2}{qB} \hat{b} \times (\hat{b} \cdot \vec{\nabla})\hat{b} \tag{2.24}$$

The second term is the curvature drift, as promised! Why is it called the curvature drift? Well, imagine we had a magnetic field which at some point in space, was in the $\hat{\phi}$ direction with a radius of curvature R. Then $(\hat{b} \cdot \vec{\nabla})\hat{b} = (\frac{1}{R}\frac{d}{d\phi})\hat{\phi} = -\hat{r}/R$. Thus, the curvature drift here is $\frac{m v_\parallel^2}{qBR}\hat{r} \times \hat{b}$. Thus, the curvature drift causes a drift in curved magnetic fields, perpendicular to both $\hat{b}$ and the vector to the center of curvature.

2.1.4 Polarization Drift

The polarization drift is a drift that arises due to a time-dependent v_{drift}. However, if our time-variation in v_{drift} is mostly due to a time-dependent $v_{\vec{E}\times\vec{B}}$, then we get a polarization drift due to a time-dependent electric field. This is

$$v_p = \frac{d\vec{E}}{dt}\frac{m}{qB^2} \tag{2.25}$$

Chen explains the polarization drift physically. He writes "The physical reason for the polarization current is simple. Consider an ion at rest in a magnetic field. If a field E is suddenly applied, the first thing the ion does is to move in the direction of E. Only after picking up a velocity v does the ion feel a Lorentz force $ev \times B$ and begin to move downward. If E is now kept constant, there is no further v_p drift but only a v_E drift. However, if E is reversed, there is again a momentary drift, this time to the left. Thus v_p is a startup drift due to inertia and occurs only in the first half-cycle of each gyration during which E changes. Consequently, v_p goes to zero with ω/ω_c." This is a reasonable explanation, but

[12] Here we remove the brackets because it is understood that this velocity is a drift velocity, which is by definition a time-averaged quantity.

there is a curious 180 degree phase difference here. Question: Why is there a phase difference?

In class, this was derived as follows: Imagine we have a time-dependent electric field, and a static, constant magnetic field. Our equation of motion for a single particle, equation 2.1, becomes

$$\ddot{\vec{r}}_{gc} + \ddot{\vec{r}}_c = \frac{q}{m}(\vec{E}(\vec{r},t) + (\dot{\vec{r}}_{gc} + \dot{\vec{r}}_c) \times \vec{B}(\vec{r})) \tag{2.26}$$

Keeping only the terms involving the guiding center motion, and ignoring any 1st order spatial variation in $\vec{E}$ relative to the 0th order electric field, we have

$$\ddot{\vec{r}}_{gc} = \frac{q}{m}(\vec{E}(\vec{r}_{gc},t) + \dot{\vec{r}}_{gc} \times \vec{B}(\vec{r}_{gc})) \tag{2.27}$$

Crossing this with $\hat{b}$ gives, following the same steps as in the derivation of the curvature drift,

$$\dot{\vec{r}}_{gc,\perp} = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{m}{qB}\ddot{\vec{r}}_{gc} \times \hat{b} \tag{2.28}$$

This equation is only true for an electric field which doesn't vary much over the course of a gyroorbit. Mathematically, this is equivalent to $\frac{1}{E\Omega}\frac{\partial \vec{E}}{\partial t} \ll 1$, $\frac{r_c}{E}\vec{\nabla}E \ll 1$.

Now, let's solve this equation iteratively. To 0th order,

$$\dot{\vec{r}}_{gc} = v_{gc,\parallel}\hat{b} + \frac{\vec{E} \times \vec{B}}{B^2} \tag{2.29}$$

Taking the time derivative of this 0th order solution, because our magnetic field is constant and static, we get $\ddot{\vec{r}}_{gc} = \frac{dv_{gc,\parallel}}{dt}\hat{b} + \frac{\frac{d\vec{E}}{dt} \times \vec{B}}{B^2}$. Plugging this into equation 2.28 gives

$$\dot{\vec{r}}_{gc,\perp} = \frac{\vec{E} \times \vec{B}}{B^2} + \frac{m}{qB^2}\frac{d\vec{E}}{dt} \tag{2.30}$$

The second term is the polarization drift. Note that if we had not assumed the magnetic field was constant, the curvature and $\vec{\nabla}B$ drift terms would show up in this equation as well.

2.1.5 Drift Currents

So far, we've derived four main classes of drifts of particles in magnetic fields:

- $\vec{v}_{E\times B} = \frac{\vec{E}\times\vec{B}}{B^2}$
- $\vec{v}_{\vec{\nabla}B} = \frac{mv_\perp^2}{2}\frac{\hat{b}\times\vec{\nabla}B}{qB^2}$
- $\vec{v}_c = \frac{mv_\parallel^2}{qB}\hat{b} \times (\hat{b}\cdot\vec{\nabla})\hat{b}$
- $\vec{v}_p = \frac{m}{qB^2}\frac{d\vec{E}}{dt}$

With the exception of the $\vec{E} \times \vec{B}$ drift, each of these drifts are linear in q. Thus, particles of opposite charge will go in different directions due to each of these three drifts. When particles of opposite charge don't travel at the same velocity, we have a current! So these three drifts contribute to currents, $\vec{J} = \sum_\sigma n_\sigma q_\sigma \vec{v}_\sigma$.

It turns out, as we will see in chapter 4, that pressure for the fluid description of a plasma is a tensor defined as $\vec{\vec{P}} = \sum_\sigma m_\sigma \int \vec{v}' \vec{v}' f_\sigma d^3\vec{v}$ where $\vec{v}'$ is defined as $\vec{v} - \vec{u}$ where $\vec{u}$ is the mean fluid velocity and f is the distribution function in the Vlasov treatment, to be introduced in chapter 3. Now, if we have a Maxwellian distribution, such that we have a well-defined temperature in the direction parallel and perpendicular to the local magnetic field, our pressure tensor becomes diagonal

$$\vec{\vec{P}} = \begin{bmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{bmatrix} \tag{2.31}$$

where $P_\parallel = \sum_\sigma n_\sigma \kappa T_{\sigma,\parallel} = \sum_\sigma n_\sigma m_\sigma \langle v^2_{\sigma,\parallel} \rangle$ and $P_\perp = \sum_\sigma n_\sigma \kappa T_{\sigma,\perp} = \sum_\sigma \frac{1}{2} n_\sigma m_\sigma \langle v^2_{\sigma,\perp} \rangle$ where the factor of $1/2$ comes from the fact that there are two perpendicular directions.

So we have

- $\vec{J}_{E\times B} = 0$
- $\vec{J}_{\vec{\nabla}B} = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma \langle v^2_{\perp,\sigma} \rangle}{2q_\sigma} \frac{\hat{b} \times \vec{\nabla} B}{qB^2} = \frac{\vec{B} \times \vec{\nabla} B}{B^3} \sum_\sigma \frac{1}{2} n_\sigma m_\sigma \langle v^2_{\perp,\sigma} \rangle = \frac{\vec{B} \times \vec{\nabla} B}{B^3} P_\perp$
- $\vec{J}_c = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma v^2_\parallel}{q_\sigma B} \hat{b} \times (\hat{b} \cdot \vec{\nabla}) \hat{b} = \frac{\hat{b} \times (\hat{b} \cdot \vec{\nabla}) \hat{b}}{B} P_\parallel$
- $\vec{J}_p = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma}{q_\sigma B^2} \frac{d\vec{E}}{dt} = (\frac{\rho}{B^2}) \frac{d\vec{E}}{dt}$

There is another current which arises in a plasma due to the effects of multiple particles, and therefore which we can't account for just considering single-particle motion. This is the magnetization current[13], $\vec{\nabla} \times \vec{M}$, where $\vec{M} = -(\frac{P_\perp \hat{b}}{B})$. Bellan has a nice physical explanation of the magnetization current.

Why do we care about the drift currents? We can show that $\vec{J}_{total} \times \vec{B}$ gives us, after a bunch of algebra, the MHD equations. Bellan goes through this algebra, and I won't. It's a good exercise, but too long. Thus, the single particle drift picture (accounting for magnetization current) contains enough information to get us to the MHD equations.

2.2 Adiabatic Invariants

There are lots of invariants we know of. Energy and momentum are the simplest examples - in any closed system, the total energy and total momentum are

[13] This magnetization current has the same physical origin as the magnetization current in magnetized materials in electromagnetics. Here, the current which gives rise to a maagnetic moment is the Larmor orbits of particles around a magnetic field, instead of the intrinsic magnetic moment of molecules.

constant. It turns out that for collisionless plasma particles, there are a couple *adiabatic* invariants which are enormously useful in understanding the motion of plasma particles in complicated electromagnetic fields. Before we discuss these adiabatic invariants, we must ask ourselves the obvious question: *What even is an adiabatic invariant*?

Suppose we have a system with some canonical[14] coordinate Q and it's canonical momentum P, and a Hamiltonian H for that system such that $\frac{\partial H}{\partial P} = \dot{Q}$, and $-\frac{\partial H}{\partial Q} = \dot{P}$. This is what we call a Hamiltonian system - a dynamical system governed by Hamilton's equations.[15] The most obvious example of a Hamiltonian system relevant to plasma physics is that of a charged particle in some electric and magnetic fields.

Now, suppose we have some slowly changing parameter, $\lambda(t)$, in the Hamiltonian, so that $H(Q, P, t, \lambda(t))$. Also suppose that the canonical coordinates of the system undergo some nearly periodic motion. Then, the integral

$$I = \oint P dQ \tag{2.32}$$

is constant over any one period of motion.[16] This integral is the general form of any adiabatic invariant. This explanation, while brief and to the point, overlooks many of the details (which are certainly important!). Let's try to understand these details, before we derive this result.

What does it mean for a system to be nearly periodic? We don't have a good mathematical definition of this. Intuitively, however, we have some idea of what this might mean. As an example, consider the simple pendulum with no energy losses. It's frequency is $\sqrt{g/l}$, and it certainly undergoes periodic motion. Now imagine we slowly change the length of the pendulum, $l(t)$. Although the canonical $P - Q$ coordinates of the pendulum will not be exactly the same after each oscillation, the pendulum will nearly return to it's starting point after each oscillation. If the length of the pendulum changes slowly, it's period is $\sqrt{g/l(t)}$ Thus, we say that the motion is nearly periodic.

When we say that $\lambda(t)$ changes slowly, how slow is slowly? Well, the derivation we will do depends on $\lambda(t)$ being differentiable from one period to the next. So the result that the integral in equation 2.32 is constant will be exact in the limit that the change in $\lambda(t)$ over any one period is infinitesimal. In the more plausible limit that $\frac{T}{\lambda}\frac{d\lambda}{dt} \ll 1$, where T is the period of the system's periodic motion, then the change in I at any time never becomes greater than some small value, $O(\epsilon)$. Mathematically, this is $\frac{I(t)-I(0)}{I(0)} < O(\epsilon)$ for all t where $\epsilon \ll 1$.

[14] Don't worry too much about this word 'Canonical'. It basically just means a set of coordinates which we can use the Lagrangian or Hamiltonian formalism with.

[15] Really, I should consider the case of multiple coordinates, such that Q and P are vectors. However, I'm not sure how to prove adiabatic invariance in this case, since defining the beginning and end of the nearly periodic motion is trickier. Thus, I've kept them as non-vectors.

[16] This was not proven in class, unless I missed it in my notes, which is possible. It is proven in Hong's supplemental notes. I find Bellan's derivation easier to understand, thus these notes will prove this result using his method.

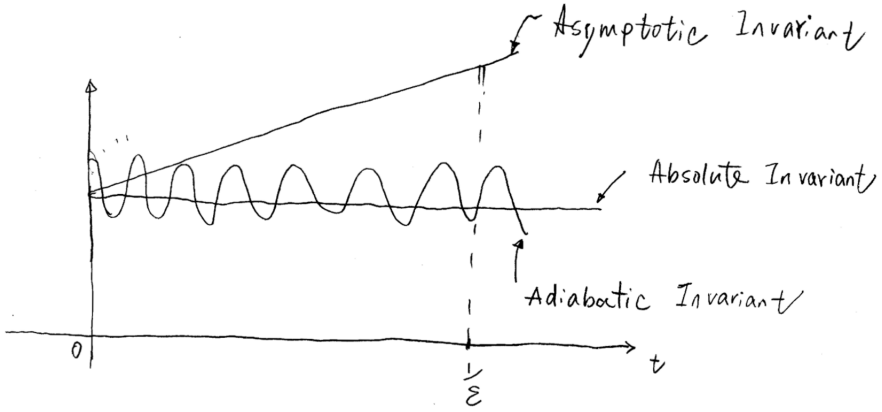


Figure 10: The change in time of an adiabatic invariant, compared with an absolute invariant (such as energy) and an asymptotic invariant.

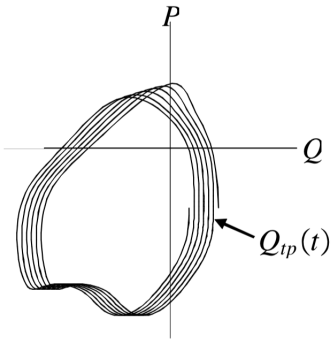


Figure 11: Nearly periodic motion in the $P - Q$ plane.

Hong has a nice visualization of this in Figure 10.

If we are integrating over a period, how do we define the beginning and end of a period if the endpoint is not the same the starting point? Well, in the $P - Q$ plane the motion will have some periodic behavior like that in figure 11. We integrate from one turning point Q_{tp} to the next, where Q_{tp} is defined as the location during the cycle where $\frac{dQ}{dt} = 0$ and Q has it's maximum value during the period. In addition to being physically reasonable, this definition is mathematically convenient, as we will see now. Let's derive the invariance of I. We want to show that $\frac{dI}{dt} = 0$ - this will require us to be very careful with our partial derivatives. So pay attention.

$$\frac{dI}{dt} = \frac{d}{dt} \oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} P dQ \tag{2.33}$$

Now, since $E(t) = H(P, Q, \lambda(t), t)$, we can in principle invert this to write $P(E(t), Q, \lambda(t))$. So

$$\frac{dI}{dt} = \oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} P(E(t), Q, \lambda(t)) dQ \tag{2.34}$$

Using the chain rule, we have

$$\frac{dI}{dt} = \oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \left[\left(\frac{\partial P}{\partial Q} \right)_{E,\lambda} \frac{dQ}{dt} + \left(\frac{\partial P}{\partial t} \right)_{Q} \right] dQ \tag{2.35}$$

In the first term, we can replace $\frac{dQ}{dt}(\frac{\partial P}{\partial Q})dQ$ with $\frac{d}{dQ}(P\frac{dQ}{dt})dQ$, as $\frac{d}{dQ}\frac{dQ}{dt} = 0$ because we can exchange the order of integration to get $\frac{d}{dt}\frac{dQ}{dQ} = 0$. We can now integrate by parts, and this term becomes $[P\frac{dQ}{dt}]_{Q_{tp}(t)}^{Q_{tp}(t+\tau)}$, which goes to 0 because $\frac{dQ}{dt} = 0$ at the turning points[17]. Equation 2.35 becomes

Thus,

$$\frac{dI}{dt} = \int \left(\frac{\partial P}{\partial t} \right)_{Q} dQ \tag{2.36}$$

Now, let's attack this remaining term.

$$\left(\frac{\partial P}{\partial t} \right)_{Q} = \left(\frac{\partial P}{\partial \lambda} \right)_{Q,E} \frac{d\lambda}{dt} + \left(\frac{\partial P}{\partial E} \right)_{Q,\lambda} \frac{dE}{dt} \tag{2.37}$$

Now, we can use some tricks to simplify these two terms. Since $E(t) = H(P, \lambda(t), t)$, then we can say that $\frac{dH}{dE} = 1 = (\frac{\partial H}{\partial P})_{Q,\lambda}(\frac{\partial P}{\partial E})_{Q,\lambda}$, so $\frac{\partial P}{\partial E} = (\frac{\partial H}{\partial P})^{-1}$. Since E is a function only of time, $(\frac{dE}{d\lambda})_Q = 0 = (\frac{\partial H}{\partial \lambda})_{Q,P} + (\frac{\partial H}{\partial P})_{Q,\lambda}(\frac{\partial P}{\partial \lambda})_{Q,E}$. Thus, $(\frac{\partial P}{\partial \lambda})_{Q,E} = -(\frac{\partial H}{\partial \lambda})_{Q,P}/(\frac{\partial H}{\partial P})_{Q,\lambda}$. Plugging these results into equation 2.37 and then into equation 2.36, we get

$$\frac{dI}{dt} = \int \frac{1}{(\frac{\partial H}{\partial P})_{Q,\lambda}} \left[\frac{dE}{dt} - \frac{d\lambda}{dt} \left(\frac{\partial H}{\partial \lambda} \right)_{Q,P} \right] \tag{2.38}$$

Hey, this looks like something nice! Let's solve for $\frac{dE}{dt}$, using $E(t) = H(P, Q, \lambda(t))$ and Hamilton's equations $\frac{\partial H}{\partial P} = \frac{dQ}{dt}$ and $\frac{\partial H}{\partial Q} = -\frac{dP}{dt}$.

$$\frac{dE}{dt} = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial Q}\frac{dQ}{dt} + \frac{\partial H}{\partial P}\frac{dP}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} \tag{2.39}$$

The second and third teams cancel from Hamilton's equations. Plugging this into equation 2.38 gives $\frac{dI}{dt} = 0$. This completes our proof of the adiabatic invariance of I.

A lot just happened - it might be helpful to recap. Our Hamiltonian depends

[17] This is where the derivation breaks down when we treat P and Q as vectors, rather than in one dimension, because we can't expect that all the Q's will have their derivatives equal to 0 at the same time. Any ideas on how to solve this?

on some slowly varying parameter, $\lambda(t)$. Some particle executes nearly periodic motion in the $P-Q$ plane. We've shown, by carefully keeping track of partial derivatives, that the integral $I = \int PdQ$ is invariant, as long as $\lambda(t)$ is differentiable from one period to the next.

Question: Where does the differentiability of λ show up in the proof? I can't find it.

2.2.1 First Adiabatic Invariant μ

The quantity μ,

$$\mu = \frac{mv_\perp^2}{2B} = \frac{KE_\perp}{B} \tag{2.40}$$

is an adiabatic invariant for single plasma particles. This is the quantity that is adiabatically conserved due to the periodic motion of particles gyrating around magnetic field lines. TODO: add intuition - chen has a nice derivation for a particle in converging field lines I think.

The proof of μ-conservation in class was fairly non-rigorous, but I will reproduce it here. TODO: Perhaps add in multiple derivations of μ-conservation? Imagine we have a particle in a magnetic field, $\vec{B}(\vec{r},t)$ which changes slowly in space and time. If this particle does not collide with other particles, then we have conservation of energy.

$$0 = \frac{d}{dt}(mv_\perp^2/2 + mv_\parallel^2/2) = \frac{d}{dt}(\mu B + \frac{1}{2}mv_\parallel^2) \tag{2.41}$$

Expanding this, we have

$$\frac{d\mu}{dt}B + \mu\frac{dB}{dt} + mv_\parallel\frac{dv_\parallel}{dt} + O(\epsilon^2) = 0 \tag{2.42}$$

As we argued for in the curvature drift derivation, $\frac{dB}{dt} = v_\parallel(\hat{b}\cdot\vec{\nabla})B$. Now, we can take the dot product of $\hat{b}$ with equation 2.19 to get

$$\hat{b}\cdot\frac{d}{dt}(\dot{\vec{r}}_{gc}) = \frac{dv_\parallel}{dt} = \frac{-\mu\hat{b}\cdot\vec{\nabla}B}{m} \tag{2.43}$$

Plugging these results into equation 2.42, we get

$$0 = \frac{d\mu}{dt}B + \mu v_\parallel\hat{b}\cdot\vec{\nabla}B - mv_\parallel\frac{\mu\hat{b}\cdot\vec{\nabla}B}{m} = \frac{d\mu}{dt}B \tag{2.44}$$

Thus, μ does not change in time for a single particle moving in a slowly varying magnetic field. It turns out that μ is also conserved for particles in slowly varying electromagnetic fields.

2.2.2 Second Adiabatic Invariant $\mathcal{J}$

Imagine we have a particle in arbitrary electromagnetic fields which vary slowly in time[18] whose guiding center undergoes some approximately periodic motion.

[18] The concept of "slowly" will be defined in a moment.

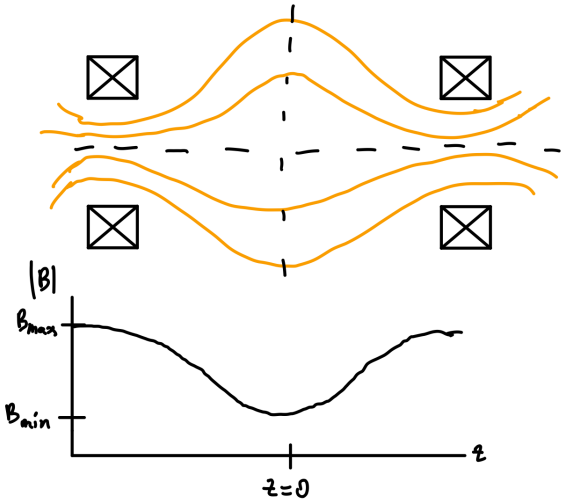


Figure 12: Illustration of the magnetic mirror (top) and the magnetic field magnitude as a function of z (bottom). The magnetic field is cylindrically symmetric.

This periodic motion is most often a particle bouncing back and forth between two regions. Here, we define "slowly" to mean that the timescale over which the electromagnetic fields change is much longer than the particle's bounce period, $\frac{\tau_{bounce}}{B}\frac{d\vec{B}}{dt} \ll 1$. In such a circumstance, then the quantity $\mathcal{J}$ is constant in time for each particle.

$$\mathcal{J} = \oint v_{\parallel} dt \tag{2.45}$$

This was not derived in class. However, the conservation of this quantity should not be surprising, as it fits the bill in terms of our general adiabatic invariant discussed earlier.

2.3 Mirror Machine

The classic illustration of single-particle motion is the magnetic mirror. The most basic magnetic mirror consists of two cylindrically symmetric current-carrying coils which set up a cylindrically symmetric magnetic field. This is shown in Figure 12.

The crucial thing to realize when it comes to magnetic mirrors, is the following. Solving for the motion of particles in the mirror machine relies on invariants of motion of collisionless single particles. For the classic mirror machine, the invariants are energy $E = \frac{1}{2}mv_{\perp}^2 + \frac{1}{2}mv_{\parallel}^2 + q\phi$ and $\mu = \frac{mv_{\perp}^2}{2B}$. The second adiabatic invariant $\mathcal{J}$ is sometimes used as well. Specifically, conservation of $\mathcal{J}$ is used when the fields in the magnetic mirror change slowly in time relative to

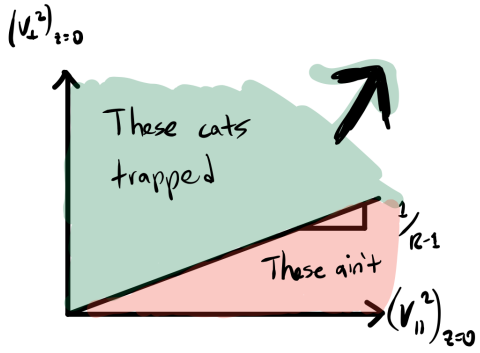


Figure 13: Plot of equation 2.48 showing the particles which are trapped and which aren't. As we can see, articles with larger perpendicular velocities are trapped.

the bounce time between the ends of the mirror.

The strategy is to equate the invariants E and μ at the midplane ($z = 0$) of the magnetic mirror to the motion at the maximum z, where $v_\parallel^2 = 0$. If $\phi = 0$, and B is minimum at the midplane, then from

$$E = (\frac{1}{2}mv_\perp^2)_{z=0} + (\frac{1}{2}mv_\parallel^2)_{z=0} = (\frac{1}{2}mv_\perp^2)_{B=B_{max}} \tag{2.46}$$

and

$$\mu = \frac{1}{B_{min}}(\frac{1}{2}mv_\perp^2)_{z=0} = \frac{1}{B_{max}}(\frac{1}{2}mv_\perp^2)_{B=B_{max}} \tag{2.47}$$

we can easily (I promise) solve (try it!) for the condition on the trapped particles,

$$\left(\frac{v_\perp^2}{v_\parallel^2}\right)_{z=0} \geq \frac{1}{R-1} \tag{2.48}$$

where $R = \frac{B_{max}}{B_{min}}$. Make sure you know how to do this calculation, as you will be asked to do it multiple times over this course, and you will have to do slightly more complex versions on the homework. They might, for example, introduce an electric field, or prescribe a magnetic field shape that changes in time, etc etc.

Our intuition with the equation for the condition on trapped particles, equation 2.48 is relatively simple: particles with high perpendicular velocities are trapped in the mirror, while particles with high parallel velocities are lost from the mirror. The physical picture to have in your head is the following: as a particle goes into a region with higher magnetic field, the parallel velocity it has gets converted into larger perpendicular velocity, increasing the gyroradius. If the particle has enough parallel velocity, then the perpendicular velocity will increase, but not enough to bring the parallel velocity to zero. Note also: if the mirror ratio $R = 1$, then the ratio of perpendicular velocity to parallel velocity

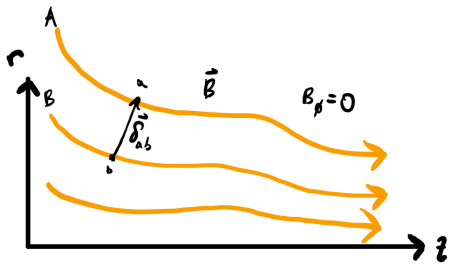


Figure 14: An illustration of the geometry considered for the isorotation theorem. Here, $B_\phi = 0$, while B_r and B_z are cylindrically symmetric. A and B are two magnetic surfaces, and a and b are two points on these surfaces separated by a vector $\vec{\delta}_{ab}$ with magnitude δ_{ab}. a and b are chosen such that $\vec{\delta}_{ab}$ lies in the r-z plane and is perpendicular to the local magnetic field.

required for trapping goes to infinity, and we don't have any trapped particles. Which makes sense, because we don't have any magnetic field.

2.4 Isorotation Theorem

The iso-rotation theorem is not usually seen in introductory textbooks, but Nat covers it because it is a relatively simple application of single-particle motion which has a simple result.

The statement of the isorotation theorem is as follows: in a cylindrically symmetric region of magnetic fields where $B_\phi = 0$, where $\vec{E} \times \vec{B}$ motion dominates the perpendicular particle motion and magnetic surfaces are equipotential surfaces, then for all the particles on a given magnetic surface, the rotation rate is constant.

There is a corrolary of the isorotation theorem which is proved in class as well. The corrolary says that under the same set of assumptions, then as particles drift from one surface to another, they gain in potential energy equal to exactly twice the energy lost in azimuthal drift energy, so as to climb up the potential. I found this to be a rather confusing statement, so we'll unpack this corrolary more as we go on.

Why would we expect magnetic surfaces to be equipotential surfaces? Here is Nat's answer: It comes down to the ability of particles to stream along field lines, while their motion is confined perpendicular to the field lines. If $E_\parallel \neq 0$, then the ions and electrons will quickly move in opposite directions to get rid of that $E_\parallel$. So we would more or less expect $E_\parallel$ to be 0, so $\vec{E}$ would be perpendicular to magnetic field lines.

The importance and application of the isorotation theorem is unclear. I suspect Nat had been thinking about it for his 'electric tokamak' idea, which a couple grad students have worked on recently, and he thought it would be fun to discuss in the course. If we have a 'straight' electric stellerator (where $\frac{\partial}{\partial z}$ is

not necessarily 0) in a cylindrical geometry, and the poloidal magnetic field is 0, then the poloidal rotation rate is constant if the particles rotate primarily due to $\vec{E} \times \vec{B}$ rotation.

2.4.1 Magnetic Surfaces

Before we prove the isorotation theorem, we should discuss magnetic surfaces, since they are introduced in class along with the isorotation theorem and are referred to in the theorem. The most general definition of a magnetic surface is **a surface in space where all the magnetic field lines on the surface stay on the surface**.

It is also important to note that magnetic field lines are not in any way guarenteed to form magnetic surfaces. The most general behavior of magnetic field lines is stochastic (i.e. random) behavior, meaning a given magnetic field line, if followed forever, will fill a volume in space.

We should also note that magnetic field lines do not necessarily close in on themselves, even in the special case where we have magnetic surfaces. A magnetic field line on a magnetic surface might go around the surface forever, never closing on itself. In principle, magnetic field lines can close on themselves after some finite distance.

When would we expect to see magnetic surfaces in the first place? Good question! Magnetic surfaces are, in ideal MHD, predicted to arise in equilibrium in magnetized plasmas. Starting with the MHD equilibrium equation $\vec{J} \times \vec{B} = \vec{\nabla} P$ (which does a pretty decent job when studying overall plasma stability), we can dot this equation with $\vec{B}$ to get that $\vec{B} \cdot \vec{\nabla} P = 0$. Let's think about this equation for a second. The change in P is always perpendicular to $\vec{B}$, which implies that as we follow $\vec{B}$, our field line will have constant P. If the gradient of P does not vanish anywhere, this implies that $\vec{B}$ field lines lie on surfaces of constant P. We can perform the same procedure with $\vec{J}$ instead of $\vec{B}$, to conclude that the vector field $\vec{J}$ lies on surfaces of constant P as well.

Now, here is a fun result, which is covered in GPP2 but I thought I'd include in these notes since it's relatively straightforward mathematically but conceptually fun. A theorem from topology says that the simplest topological form for a non-vanishing vector field which lies on a smooth surface is a torus. I'm not exactly sure what the word 'simplest' means in this context, but for our purposes, that isn't what is important. What is important is that if we have an MHD equilibrium where $\vec{J} \times \vec{B} = \vec{\nabla} P$, then we have our $\vec{B}$ field on a surface of constant P, meaning we have a magnetic surface. And if we have magnetic surfaces, this theorem says the simplest surface we can have is a toroidal one! In other words, if we want to create a plasma in an MHD equilibrium, we're basically stuck using a torus.

In cylindrically symmetric systems (where ϕ is ignorable), a magnetic surface is defined as the surface defined by constant rA_ϕ. This comes from the result that $\vec{B} \cdot \vec{\nabla}(rA_\phi) = 0$, implying (using the same logic we used earlier with P) that $\vec{B}$ lies on a surface of constant rA_ϕ. This result is easily proved, as shown now.

We can arbitrarily write $\vec{A}$ for cylindrically symmetric B as $\vec{A} = A_r(r,z)\hat{r} + A_\phi(r,z)\hat{\phi} + A_z(r,z)\hat{z}$. Thus, we have for $\vec{B}$,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \big(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\big)\hat{\phi} - \frac{\partial A_\phi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\hat{z} \tag{2.49}$$

Thus,

$$\vec{B} \cdot \vec{\nabla}(rA_\phi(r,z)) = -\frac{\partial A_\phi}{\partial z}\frac{\partial}{\partial r}(rA_\phi) + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\frac{\partial (rA_\phi)}{\partial z} = 0 \tag{2.50}$$

Note that the ϕ-component of $\vec{B}$ doesn't show up in the dot product because of cylindrical symmetry. This proves that for cylindrically symmetric systems, surfaces of constant rA_ϕ are magnetic surfaces.

2.4.2 Proof of Iso-rotation Theorem

Remember our assumptions here: we've assumed that we have a cylindrically symmetric region of electromagnetic fields where $B_\phi = 0$, that $\vec{E} \times \vec{B}$ motion dominates the drift motion and that we have magnetic surfaces which are also equipotential surfaces. We'll first prove the isorotation theorem, before proving it's corrolary.

The isorotation theorem says that all particles on a given magnetic surface isorotate, i.e. they all rotate at the same frequency. The rotation rate $\Omega = \frac{v_\phi}{r}$, so we'll want to find v_ϕ. We have $\vec{B} = \vec{\nabla} \times \vec{A}$, and we can write $\vec{A}$ as $\vec{A} = A_\phi(r,z)\hat{\phi}$, so that $B_\phi = 0$. Because we assume that the drift velocity is dominated by the $\vec{E} \times \vec{B}$ velocity, we have $v_\phi = \frac{E}{B}$, where these are just the magnitudes since we assume our flux surface is also an equipotential surface and hence $\vec{E}$ is perpendicular to $\vec{B}$. This is the point in the derivation where we require that $B_\phi = 0$, for if B_ϕ were not 0, then a particle's parallel velocity would give it some component in the ϕ-direction, so v_ϕ would not be $\frac{E}{B}$.

Now imagine, as in figure 14, that there are two nearby magnetic surfaces A and B and two points a and b on these surfaces separated by a vector $\vec{\delta}_{ab}$. Suppose that the distance δ_{ab} is small. Here, a and b are chosen such that the vector between the two points lies in the r-z plane and is perpendicular to the local B-field. This choice will make our integration easier in a moment. In this case, $E = \frac{-\Delta V_{ba}}{\delta}$ where ΔV_{ba} is the difference in electric potential between points b and a. Now imagine taking the vector $\vec{\delta}_{ab}$ and rotating it around the z-axis to form a surface which resembles a bent annulus (i.e. a hollow disk). If we integrate $\vec{B} \cdot d\vec{S}$ over this surface, we get

$$\int_S \vec{B} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l} = \Delta\Big[2\pi A_\phi r\Big] \approx B 2\pi r\delta \tag{2.51}$$

Note that this is the magnitude of $\vec{B}$, because we chose points a and b such that the vector $\vec{\delta}_{ab}$ points perpendicular to B, meaning the area integral picks out

the magnitude of $\vec{B}$. Solving for B, we have $= \frac{\Delta[2\pi A_\phi r]}{2\pi r\delta}$. Solving for Ω, we get

$$\Omega = \frac{v_\phi}{r} = \frac{E}{Br} = \frac{\frac{-\Delta V_{ba}}{\delta}}{r\frac{\Delta[2\pi A_\phi r]}{2\pi r\delta}} = \frac{-\Delta V_{ba}}{\Delta[A_\phi r]} \tag{2.52}$$

Since magnetic surfaces are, for cylindrically symmetric systems, constant-$(A_\phi r)$ surfaces, then the denominator is going to be the same regardless of which points on A and B we choose. Since we are assuming that A and B are equipotential surfaces, then the numerator is going to be the same regardless of which points on A and B we choose. Thus, Ω will be approximately the same for all particles between surfaces A and B, assuming of course that the distance between A and B is small.

Note that the isorotation theorem does *not* say that the rotation rate is the same for all of the particles *everywhere* in the system. The electric field might be very strong on one magnetic surface, but very weak on another magnetic surface. In this case, the rotation rate Ω will not necessarily be the same for both surfaces.

TODO: Corrolary doesn't make sense

Now it is time to prove the corollary of the isorotation theorem. Remember, the statement of the corrolary is the following: as particles drift from one surface to another, they gain in potential energy equal to exactly twice the energy lost in azimuthal drift energy, so as to climb up the potential.

We start our proof with conservation of rotational angular momentum, $p_\phi = mrv_\phi + qrA_\phi$. This is a result of Lagrange's equations in cylindrically symmetric systems. Since $\Delta p_\phi = 0$ as a particle drifts between surfaces A and B as in figure 14, we have that $m\Delta(rv_\phi) = -q\Delta(rA_\phi)$. Now, from the constancy of Ω between any two magnetic surfaces, we get that $r = \frac{v_\phi}{\Omega}$, so between surfaces A and B we find that

$$\frac{1}{\Omega}(mv_{\phi,B}^2 - mv_{\phi,A}^2) = -q\Delta(rA_\phi) = -q(\frac{V_A - V_B}{\Omega}) \tag{2.53}$$

$$W_{\phi,A} - W_{\phi,B} = \frac{q}{2}(V_A - V_B) \tag{2.54}$$

I didn't make a fuss over the minus signs while going through this derivation, but it turns out they are important in deriving the corrolary and you need to keep track of them! Here, I picked a convention for positive angular velocity Ω, and made sure all my signs were consistent with that.

If q is positive, then the energy lost in going from surface A to surface B is half the energy gained in electric potential energy in changing surfaces. If q is negative, then huh?

Question: What is going on with charge here? The first theorem has no reference to charge, while for the second theorem the change in the ϕ-velocity depends on the charge.

3 Kinetic Theory

> It is only the plasma itself which does not understand how beautiful the theories are and absolutely refuses to obey them.
>
> Hannes Alfvén

Let's forget about plasma physics for a second, and think about the field of classical fluid mechanics. Fluids, like all states of matter, are made up of individual molecules or atoms. The most fundamental assumption made in fluid mechanics[19] is called the continuum assumption. Under this assumption, we treat all quantities as continuous and well-defined at each point in space. Strictly speaking, this requires for each quantity that we set the value of that quantity at each point in space equal to the average value of that quantity over a volume large enough to contain many molecules but much smaller than the relevant macroscopic lengths of the fluid.

Wikipedia phrases this similarly: "The continuum assumption is an idealization of continuum mechanics under which fluids can be treated as continuous, even though, on a microscopic scale, they are composed of molecules. Under the continuum assumption, macroscopic (observed/measurable) properties such as density, pressure, temperature, and bulk velocity are taken to be well-defined at "infinitesimal" volume elements – small in comparison to the characteristic length scale of the system, but large in comparison to molecular length scale." Also, the Navier-Stokes equations for fluids "are based on the assumption that the fluid, at the scale of interest, is a continuum, in other words is not made up of discrete particles but rather a continuous substance." Thus, in an ideal fluid, there is a well-defined, smooth mass distribution at each point in space, $\rho(\vec{r}, t)$, as well as a well-defined, smooth field which represents the mean velocity, $\vec{v}(\vec{r}, t)$.

However, there are cases in classical fluid mechanics where the continuum assumption is not valid. Wikipedia has this to say: "Those problems for which the continuum hypothesis fails, can be solved using statistical mechanics. To determine whether or not the continuum hypothesis applies, the Knudsen number, defined as the ratio of the molecular mean free path to the characteristic length scale, is evaluated. Problems with Knudsen numbers below 0.1 can be evaluated using the continuum hypothesis, but (sic) molecular approach (statistical mechanics) can be applied for all ranges of Knudsen numbers." Well, as we showed in section 1, in a plasma the mean free path is significantly longer than the Debye length, which is the scale length over which a plasma is electrically neutral. In fact, for a fusion-relevant plasma with number density $n \approx 10^{20}/m^3$ and temperature 1KeV, we have a mean free path of roughly 3km, much longer than the relevant scale lengths. Thus, our Knudsen number is very large in most plasmas, which motivates us to abondon the simple approach used in fluid

[19] Really, this is the unifying assumption for all of continuum mechanics.

mechanics.[20] Instead, we will use an approach called kinetic theory.

Here is a preview of where we are going with kinetic theory. We will examine the time-evolution of particles in 6-D phase space[21], describing the evolution of particles with a function called N. We will write down an equation for the time-evolution of N, called the Klimontovich equation. Combined with the Lorentz force law and Maxwell's equations, this set of equations is exactly equivalent to a bunch of charged particles interacting through electromagnetic forces.

At this point, we go from an approach which tracks each individual particle to a smooth distribution function which tracks the density of particles in phase space. We replace N, which is a non-continuous function of delta functions in phase space (see figure 15), with a smooth, continuous function called f. To get from N to f, we average N over the ensemble corresponding to N. An ensemble is defined as all of the possible microstates corresponding to a given macrostate. *Kinetic theory involves the study of f.* The equation describing the evolution of f is called the Vlasov equation. What separates the Vlasov approach from those used for simple classical fluids is that we are accounting for the distribution of velocities. By accounting for the distribution of velocities, we are still accounting for the fact that plasmas are made of discrete particles. We are not, however, removing the continuum assumption.

In a classical fluid the velocity distribution function is replaced by 3 components representing the mean velocity (the mean velocity vector $\vec{u}(\vec{r},t)$) of the velocity distribution at each point in space and time and nine components of the pressure tensor (actually six, since it's an antisymmetric tensor), which are found by averaging over the microscopic velocity distribution. By averaging over the velocity distribution, we remove potentially important information about the plasma (by going from infinity degrees of freedom regarding the velocity distribution to $3 + 6$ degrees of freedom), and limit ourselves to the range of behaviors we can study.

The Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f + \vec{a} \cdot \vec{\nabla}_v f = C(f) \tag{3.1}$$

where $C(f)$ is the collision operator, representing the effects of collisions between particles. First, we will ensemble-average the Klimontovich equation to get the Vlasov equation. Next, we will examine some of the properties of the Vlasov equation and examples of the collision operator.

[20] Actually, the picture is more complex than this. In some plasmas, a fluid approximation is justified. When a fluid approximation is justified will be explained in chapter 4.

[21] Phase space simply means that each particle is labeled by it's 3 spatial components and 3 velocity components, if you remember your undergraduate classical mechanics course.

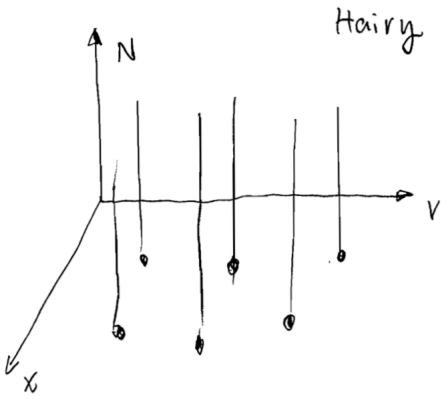


Figure 15: Visualization of N. Each delta function represents the trajectory of a particle in phase space.

3.1 Klimantovich Equation

Suppose we have N_0 particles in some region of space. Suppose $N(\vec{x}, \vec{v}, t)$ describes the evolution of those N_0 particles in phase space. Thus,

$$N(\vec{x}, \vec{v}, t) = \sum_{i=1}^{N_0} \delta^{(3)}(\vec{x} - \vec{x}_i(t))\delta^{(3)}(\vec{v} - \vec{v}_i(t)) \tag{3.2}$$

where $\vec{x}_i(t)$ and $\vec{v}_i(t)$ represent the position and velocity of the ith particle. Note that the units of N are $\frac{1}{L^3 v^3}$.

Taking the partial derivative with respect to time of N and using the chain rule we get

$$\frac{\partial N(\vec{x}, \vec{v}, t)}{\partial t} = \sum_{i=1}^{N_0} \frac{\partial \vec{x}_i}{\partial t} \cdot \frac{\partial N}{\partial \vec{x}_i} + \frac{\partial \vec{v}_i}{\partial t} \cdot \frac{\partial N}{\partial \vec{v}_i} \tag{3.3}$$

Using our definition for N (equation 3.2), we get

$$\frac{\partial N}{\partial \vec{x}_i} = \frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}_i}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) = -\frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) \tag{3.4}$$

Similarly,

$$\frac{\partial N}{\partial \vec{v}_i} = -\delta^{(3)}(\vec{x} - \vec{x}_i(t))\frac{\partial \delta^{(3)}(\vec{v} - \vec{v}_i(t))}{\partial \vec{v}} \tag{3.5}$$

In equation 3.3, we can replace $\frac{\partial \vec{x}_i}{\partial t}$ with $\vec{v}_i$, and $\frac{\partial \vec{v}_i}{\partial t}$ with $\vec{a}_i$. Plugging in 3.4 and 3.5 to 3.3, we get

$$\frac{\partial N}{\partial t} = -\sum_{i=1}^{N_0} \vec{v}_i \cdot \frac{\partial \delta^{(3)}(\vec{x} - \vec{x}_i(t))}{\partial \vec{x}}\delta^{(3)}(\vec{v} - \vec{v}_i(t)) + \vec{a}_i \cdot \delta^{(3)}(\vec{x} - \vec{x}_i(t))\frac{\partial \delta^{(3)}(\vec{v} - \vec{v}_i(t))}{\partial \vec{v}} \tag{3.6}$$

We pull the gradients out of the equation first. This is a legal move, because $\vec{v}_i$ and $\vec{a}_i$ are coordinates representing the position of a single particle in phase space, and thus commute just fine with the derivatives in equation 3.6.

After we do this, we can simplify the $\vec{v}_i$ and $\vec{a}_i$. Because of the delta functions, $\vec{v}_i(t)$ will become $\vec{v}$ and $\vec{a}_i(t)$ will become $\vec{a}$. Making these replacements we can next pull the dot products out of the equation and replace the delta functions with N.[22] Moving everything to the left hand side, we have the Klimontovich equation

$$\frac{\partial N}{\partial t} + \vec{\nabla}_x \cdot (\vec{v}N) + \vec{\nabla}_v \cdot (\vec{a}N) = 0 \tag{3.7}$$

Physically, this equation actually has a fairly simple meaning. It comes from conservation of particles, in the same way that the continuity equation comes from the conservation of charge. Geometrically, the Klimontovich equation is equivalent to the idea that the number of particles leaving a region in phase space is the number of particles flowing across the border of that region in phase space. Figure 16 shows one such region in phase space. Mathematically, this is

$$\frac{\partial}{\partial t}\left[\int_V N(\vec{x},\vec{v},t)d^3\vec{x}d^3\vec{v}\right] = -\int_S N\vec{v}_6 \cdot d\vec{A} = -\int_V \vec{\nabla}_6 \cdot (N\vec{v}_6)d^3\vec{x}d^3\vec{v} \tag{3.8}$$

where the 6 represents the 6 dimension of phase space, V is a volume of interest and S is the surface of that volume. Moving the right hand side over to the left gives the Klimontovich equation, as promised.

Since $\vec{x}$ and $\vec{v}$ are independent variables, $\vec{\nabla}_x \cdot (\vec{v}N) = \vec{v} \cdot \vec{\nabla}_x N$. If $\vec{\nabla}_v \cdot \vec{a} = 0$, which is sometimes true, we can therefore write the Klimontovich equation as

$$\frac{\partial N}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N + \vec{a} \cdot \vec{\nabla}_v N = 0 \tag{3.9}$$

Perhaps the most important acceleration example to consider is the Lorentz force, $\vec{a} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$. Here, $\vec{\nabla}_v \vec{a} = 0$. We can prove this as follows: $\vec{\nabla}_v = \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} + \frac{\partial}{\partial v_z}\hat{z}$, so $\vec{\nabla}_v \cdot \vec{a} = \frac{\partial a_x}{\partial v_x} + \frac{\partial a_y}{\partial v_y} + \frac{\partial a_z}{\partial v_z}$. Now, $\frac{\partial a_i}{\partial v_i} = \frac{q}{m}(\frac{\partial E_i}{\partial v_i} + \frac{\partial}{\partial v_i}(\vec{v} \times \vec{B})_i) = 0$ where the last step is because the ith component of $\vec{v} \times \vec{B}$ does not include v_i, but rather the other two components of $\vec{v}$. Thus, equation 3.10 can be used for the Lorentz force.

One last comment before we average over the ensemble to get the Vlasov equation. Using the chain rule, the total derivative of N with respect to time, $\frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial N}{\partial \vec{x}} + \frac{\partial \vec{v}}{\partial t} \cdot \frac{\partial N}{\partial \vec{v}}$. By inspection, we see that setting this equal to

[22] Why can't we apply the delta function first to equation 3.6, pull the dot products out of the equation, and then pull the gradients out of the sum next to get $\frac{\partial N}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N + \vec{a} \cdot (\vec{\nabla}_v N) = 0$? This is a bit subtle. The derivative with respect to $\vec{x}$ or $\vec{v}$ on the delta function means that there is no longer a delta function which sets all the $\vec{x}_i$ to $\vec{x}$ or the $\vec{v}_i$ to $\vec{v}$. Basically, the derivative of a delta function is not a delta function. It is only when we remove the derivative that we again have a delta function which makes $\vec{a}_i(\vec{x}_i, \vec{v}_i, t) = \vec{a}(\vec{x}, \vec{v}, t)$. So we cannot change the order we perform these operations.

Figure 16: 3-D Visualization of the 6-D volume V (labeled C) in phase space which we consider.

0 gives us equation 3.10. What does this imply? Well, the observant reader[23] might notice that $\frac{dN}{dt}$ is the same as the convective derivative in 6 dimensions. Thus, the total derivative follows the motion of a plasma particle in phase space. Thus, the phase space density is constant following a particles trajectory. Therefore, if $\vec{\nabla}_v \vec{a} = 0$, then $\frac{dN}{dt} = 0$ and we can say that the phase space density is constant as individual elements of N move around in phase space. While this doesn't have much consequence in the delta-function representation of N, it will be more meaningful and less obvious when we replace N with f, a smooth function.

We said earlier that the Klimontovich equation combined with the Lorentz Force Law and Maxwell's equations is exactly equivalent to a number of charged particles interacting through electromagnetic forces. Let us see now how that works for a fully-ionized plasma.

Because our plasma is fully-ionized, we know that for each species of plasma particles σ with charge q and mass m, the acceleration is $\vec{a} = \frac{q_\sigma}{m_\sigma}(\vec{E}_m + \vec{v} \times \vec{B}_m)$. Ignoring gravity, this is the only other force which can act outside of the nucleus. In this form, we label $\vec{E}_m$ and $\vec{B}_m$ with the subscript m (which stands for microscopic) to represent the fact that on a microscopic level, $\vec{E}$ and $\vec{B}$ fluctuate significantly from place to place. These fluctuations account for the effect of collisions in the plasma. Thus, our Klimontovich equation becomes

$$\frac{\partial N_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla}_x N_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v N_\sigma = 0 \tag{3.10}$$

Now, in order to solve this partial differential equation, we obviously need an initial condition and boundary conditions on N_σ. However, we also need to know $\vec{E}_m$ and $\vec{B}_m$ and how they evolve in time. Thus, we need microscopic

[23] Alternatively, a fluid mechanics geek.

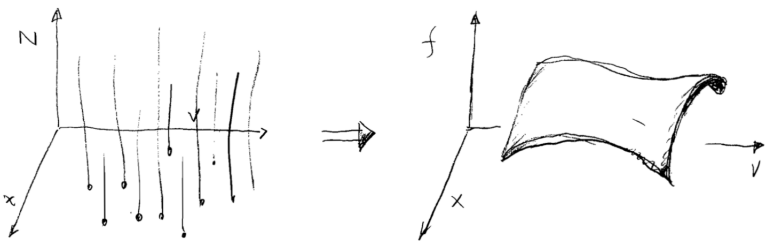


Figure 17: Visualization of ensemble-averaging N to get f.

formulations of Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E}_m = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int d^3\vec{v} N_\sigma(\vec{x}, \vec{v}, t) \tag{3.11}$$

$$\vec{\nabla} \cdot \vec{B}_m = 0 \tag{3.12}$$

$$\vec{\nabla} \times \vec{E}_m = -\frac{d\vec{B}_m}{dt} \tag{3.13}$$

$$\vec{\nabla} \times \vec{B}_m = \mu_0 \sum_\sigma q_\sigma \int d^3\vec{v}(\vec{v} N_\sigma(\vec{x}, \vec{v}, t)) + \mu_0\epsilon_0 \frac{\partial \vec{E}_m}{\partial t} \tag{3.14}$$

This system of equations is equivalent to a number of charged particles interacting through electromagnetic forces.

3.2 Vlasov Equation

Now, instead of tracking each individual particle, we want to replace N with a smooth function f accounting for the number of particles at a given position with a given velocity. An example of this is shown in Figure 17. We also want to replace our microscopically varying electric and magnetic fields $\vec{E}_m$ and $\vec{B}_m$ with smooth vector fields $\vec{E}$ and $\vec{B}$. Let us define, where brackets represent an average over the ensembles, $f(\vec{x}, \vec{v}, t) \equiv \langle N(\vec{x}, \vec{v}, t) \rangle$, $\vec{B} \equiv \langle \vec{B}_m \rangle$, $\vec{E} \equiv \langle \vec{E}_m \rangle$. Let us also define $\delta N \equiv f - N$, $\delta\vec{E} \equiv \vec{E}_m - \vec{E}$, and $\delta\vec{B} \equiv \vec{B}_m - \vec{B}$.[24]

When we perform these averages, Maxwell's equations are the same as our normal Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int d^3\vec{v} f_\sigma(\vec{x}, \vec{v}, t) \tag{3.15}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.16}$$

[24]Note that these definitions are slightly different from the definitions used by Hong in his notes.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.17}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \sum_\sigma q_\sigma \int d^3\vec{v} f_\sigma(\vec{x}, \vec{v}, t) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{3.18}$$

These are the same as Maxwell's equations before ensemble averaging, just with m removed. Why is this the case? Well, it comes from the fact that Maxwell's equations are linear. We can set $\vec{B}_m = \vec{B} + \delta\vec{B}$, do the same with $\vec{E}_m$, and subtract off the microscopic portions of Maxwell's equations from both sides without making any approximations.

Ensemble-averaging the Klimontovich equation, we get the Vlasov-Maxwell equation. This would be the same as our Klimontovich equation with f replacing N, except we have an additional non-linear correlation term which comes to the right side.

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma = -\Big\langle \frac{q_\sigma}{m_\sigma}(\delta\vec{E} + \vec{v} \times \delta\vec{B}) \cdot \vec{\nabla}_v \delta N_\sigma \Big\rangle \tag{3.19}$$

Note that terms first-order in the fluctuations average to 0 by definition, but terms second-order in fluctuations, i.e. correlation terms, do not automatically average to 0. The correlation term on the right hand side accounts for the effect of particle-particle interactions, i.e. collisions. How can we see this intuitively? Without having any rigor, we can say roughly that the ensemble-average of $q_\sigma \delta N \delta\vec{E}$ will not go to 0, as where there is some δN there will also be a correlated $\delta\vec{E}$ as there are more particles at that microscopic position.

As far as I know, we are not able to calculate the right-hand side exactly. In most applications, we simply set this term to be equal to some collision operator, $C(f)$. Solving the Vlasov-Maxwell equation in practice requires choosing a collision operator which hopefully is approximately equal to the right-hand side of equation 3.19 and is practically solvable.

3.2.1 Some facts about f

- The total number of particles for species σ in our plasma, N_σ, is $\int d^3\vec{x} d^3\vec{v} f_\sigma(\vec{x}, \vec{v}, t)$
- The particle density of the species σ, $n_\sigma(\vec{x}, t) = \int f_\sigma(\vec{x}, \vec{v}, t) d^3\vec{v}$. The number density at $\vec{x}$ is equal to the integral of f over all possible velocities.
- The mean velocity of the species σ, $\vec{u}_\sigma = \frac{1}{n_\sigma} \int \vec{v} f_\sigma(\vec{x}, \vec{v}, t) d^3\vec{v}$. In other words, the mean velocity $\vec{u}$ is thus the first moment of f with respect to velocity, divided by the density.
- The plasma energy per volume in the particle kinetic energy for species σ is $\int d^3\vec{v} \frac{1}{2} m_\sigma v^2 f_\sigma$. This accounts for both the thermal energy of the plasma as well as the kinetic energy of the mean plasma flow.

3.2.2 Properties of Collisionless Vlasov-Maxwell Equations

Suppose that $C(f) = 0$, such that our plasma is collisionless. We expect our Vlasov-Maxwell equation, $\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla} f = 0$, to have a few basic properties, such as particle, energy, and momentum conservation.

Particle conservation comes about automatically because f is conserved as we follow a section of f around in phase space. $\frac{df}{dt}$ is the convective derivative in phase space, and since it equals zero then the value of f is constant as a particle travels around in phase space. Thus, N, the total integral of f over velocity space and real space, doesn't change with time.

The total energy (plasma plus electromagnetic) is conserved through the evolution of f under the Vlasov-Maxwell equation. The total energy

$$\mathcal{E} = \frac{1}{2} \int d^3\vec{x} \left[\epsilon_0 E^2 + \frac{B^2}{\mu_0} + \sum_\sigma \int d^3\vec{v} m_\sigma v^2 f_\sigma \right] \tag{3.20}$$

is constant in time - we prove this on a homework assignment. The same is true with total momentum,

$$\vec{\mathcal{P}} = \int d^3\vec{x} \left[\epsilon_0 \vec{E} \times \vec{B} + \sum_\sigma \int d^3\vec{v} m_\sigma \vec{v} f_\sigma \right] \tag{3.21}$$

This is also proved in the homework.

Now, it turns out that if $c_i(\vec{x}, \vec{v}, t, \vec{B}(\vec{x}), \vec{E}(\vec{x})$ is a constant of motion for a single particle, such that

$$\frac{dc_i}{dt} = \frac{\partial c_i}{\partial t} + \vec{v} \cdot \vec{\nabla} c_i + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v c_i = 0 \tag{3.22}$$

then any function $f(c_1, c_2, ...,)$ which is a function of c_i's is a solution of the Vlasov equation. This is easily shown, as

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial c_i} \frac{dc_i}{dt} = 0 \tag{3.23}$$

3.2.3 Entropy of a distribution function

This isn't covered in class, but I wanted to introduce the concept of entropy in plasmas briefly.

The definition of the entropy of a distribution function f is

$$S = -\int d^3\vec{v} \int d^3\vec{x} f(\vec{x}, \vec{v}) \ln f(\vec{x}, \vec{v}) \tag{3.24}$$

Why is this true? Forget about distribution functions for a second, and imagine we have N distinguishable pegs we can put into N holes, so that exactly 1 peg goes into each hole. Since the pegs are distinguishable, we have $N!$ ways of ordering the N pegs into the N holes (N options for the first peg, $N-1$ options for the second peg, etc until there is 1 option for the last peg).

Now suppose that we group together the N holes into M groups, such that the ith group has $f(i)$ holes in that group, and $\sum_i^M f(i) = N$. Now suppose we want to place the pegs in the holes again, such that we care about the ordering of the pegs within each group of holes and any peg can go into any hole. In that case, then the number of ways of arranging the pegs into these M groups must also equal $N!$ Why? If any peg can be in any group and the order matters within the group, then there are N ways to place the first peg, $N-1$ ways to place the second, etc etc. Therefore the number of ways of arranging these N pegs is the same as if we had no groups in the first place.

Now suppose we don't care about the internal arrangement of the pegs within each of the M groups, but we do care about which pegs go into which group. How many ways can we arrange the pegs into these M groups such that we don't care about the internal arrangement of the pegs within each group? Well, we don't know it yet, but let us call this result C. We know that since there are $f(i)!$ ways of arranging the pegs within group i, then the number of ways of arranging the pegs into these M groups such that we do care about the ordering of the pegs within each group is

$$C \times f(1)! \times f(2)! \times ... \times f(M)! \tag{3.25}$$

But from our previous paragraph, we know this equals $N!$ Solving for C, we get

$$C = \frac{N!}{f(1)! \times f(2)! \times ... \times f(M)!} \tag{3.26}$$

How does this relate to entropy? An important sentence is about to come up, so buckle up and pay attention. C is the number of microscopic states corresponding to the macrostate given by the $f(i)$'s. So using our definition of entropy from statistical mechanics, $S = k_B \ln \Omega$, we have

$$\begin{aligned} S &= k_B \ln C \\ &= \ln\left(\frac{N!}{f(1)! \times f(2)! \times ... \times f(M)!}\right) \\ &= \ln N! - \ln f(1)! - \ln f(2)! - ... - \ln f(M)! \end{aligned} \tag{3.27}$$

Now suppose $f(i) \gg 1$ for all i, such that we can use Stirling's formula $\ln N! \approx N \ln N - N$ to simplify the entropy. Using $\sum_i^M f(i) = N$, we cancel the N term to get

$$S = N \ln N - \sum_i^M f(i) \ln f(i) \tag{3.28}$$

Since N is a constant, we can drop it from the entropy to get

$$S = -\sum_i^M f(i) \ln f(i) \tag{3.29}$$

What does this have to do with distribution functions? Well, suppose we have a known distribution function $f(\vec{x}, \vec{v}, t)$. This is our macrostate. Each

point $(\vec{x}, \vec{v})$ in phase space can be thought of as a group of holes, and each particle can be thought of as a peg. We have a known number of particles in each point in phase space, analogous to having a known number of holes in each group. The microstate is the particular arrangement of particles (pegs) which gives us our macrostate $f(\vec{x}, \vec{v}, t)$ (the number of holes in each group of holes).

If we have a known number of pegs $f(i)$ in each group of holes, then the entropy is given by equation 3.29. Therefore, if we have a known number of particles in each point in phase space, $f(\vec{x}, \vec{v}) d^3\vec{x} d^3\vec{v}$, then the entropy (turning the sum over i into an integral over $\vec{x}$ and $\vec{v}$) is

$$S = -\int d^3\vec{v} \int d^3\vec{x} f(\vec{x}, \vec{v}) \ln f(\vec{x}, \vec{v}) \tag{3.30}$$

the same as equation 3.24. This is the entropy of a distribution function for a plasma.

If I wanted to get fancy[25] I could show that the maximum entropy distribution function is a Maxwellian. It's a bit outside the scope of these notes though, since I'm already on a topic which isn't covered in class.

3.3 Collisions in the Vlasov Description

3.3.1 Heuristic Estimate of Collision Operator

When we go from the Klimontovich equation to the Vlasov equation by ensemble-averaging, we get a collision operator term

$$C(f_\sigma) = -\langle \frac{q_\sigma}{m_\sigma} (\delta\vec{E} + \vec{v} \times \delta\vec{B}) \cdot \vec{\nabla}_v \delta N_\sigma \rangle \tag{3.31}$$

In general, we can't solve exactly for this term. For now, let's try to get a heuristic estimate of what this might be. Let's look only at the $\delta\vec{E} \cdot \vec{\nabla}_v \delta N_\sigma$ term, for simplicity.

We can write N heuristically as $\frac{\overline{N}}{L^3 v_T^3}$, where $\overline{N}$ is the total number of particles in the system and L is the length scale of the system. Now, from the law of large numbers, $\delta\overline{N} \sim \sqrt{\overline{N}}$ on average for a given microstate.

From Gauss's equation $\vec{\nabla} \cdot \delta\vec{E} = \sum_\sigma \frac{q_\sigma}{\epsilon_0} \int \delta N_\sigma d^3\vec{v}$. Thus, $\delta\vec{E} \sim \frac{\lambda_D q (\delta N) v_T^3}{\epsilon_0} \sim \frac{q \lambda_D \delta\overline{N}}{\epsilon_0 L^3}$, where the λ_D arises because the distance scale over which $\vec{E}$ changes is the Debye length.

Thus, plugging in our numbers to get a heuristic estimate of $C(f)$, we find

$$C(f) \sim \delta\vec{E} \frac{q\delta\overline{N}}{m v_T^4 L^3} \sim \frac{q^2 \lambda_D \delta\overline{N}^2}{\epsilon_0 m v_T^4 L^6} \sim \frac{q^2 \overline{N}}{\epsilon_0 m \omega_P L^3} \frac{1}{L^3 v_T^3} \sim \frac{\omega_P}{L^3 v_T^3} \tag{3.32}$$

where $\omega_P^2 = \frac{q^2 n}{\epsilon_0 m}$ and $\frac{v_T}{\lambda_D} = \omega_P$ and $\frac{\overline{N}}{L^3} = n$ have been used.

Question: This ω_P seems too high, no? Shouldn't the collision frequency be way less than the plasma frequency?

[25] Bellan does this in his textbook.

3.3.2 Properties of Collision Operator

So far, we've been pretty mute about the collision operator, $C(f)$. Let's start discussing this now.

When we have more than one plasma species, we need to account for the possibility of collisions between different plasma species as well as collisions between the same species. Thus, we should instead define our collision operator for f_σ as the sum of collision operators between all plasma species, $C(f_\sigma) = \sum_{\sigma'} C(f_\sigma, f_{\sigma'})$. If we were being rigorous, we should technically account for the possibility of collisions between 3 or more different particles from potentially different species. However, it is not a bad approximation (and much simpler) to consider only consider binary collisions (i.e. collisions between two particles), which we will do.

There are a couple properties we would hope that our collision operator might have:

- Particles conservation: For all σ, $\int C(f_\sigma, f'_\sigma) d^3\vec{v} = 0$. Physically, this means that collisions between particles of species σ and σ' at some position $\vec{x}$ only change the velocity of the particles of species σ at $\vec{x}$, and maintain the number of particles of species σ at $\vec{x}$.

- Momentum conservation: $\sum_{\sigma,\sigma'} \int m_\sigma \vec{v} C(f_\sigma, f_{\sigma'}) d^3\vec{v} = 0$. Physically, this means that while particles can exchange momentum between different species, the total momentum at each point $\vec{x}$ remains constant.

- Energy conservation: $\sum_{\sigma,\sigma'} \int \frac{m_\sigma v_\sigma^2}{2} C(f_\sigma, f_{\sigma'}) d^3\vec{v} = 0$ Physically, this means that while particles can exchange energy between different species, the total energy at each point $\vec{x}$ remains constant. If particles were to fuse, releasing atomic energy, this would no longer be strictly true.

- We would hope that $C(f_\sigma, f_{\sigma'})$ be bilinear in f, meaning for some constants α and β, we have $C(\alpha f_\sigma, \beta f_{\sigma'}) = \alpha\beta C(f_\sigma, f_{\sigma'})$ Physically, this means that the number of collisions at each point in space is proportional to the number of particles at that point in space. While this is strictly not necessary, it is certainly a reasonable assumption.

- In general, we want C to be local, meaning that C at $\vec{x} = \vec{x}_0$ depends only on $f_\sigma(\vec{x}_0, \vec{v}, t)$ and $f_{\sigma'}(\vec{x}_0, \vec{v}, t)$, and not on any other $\vec{x}$. It also means that C doesn't depend on any derivatives of f with respect to position, but possibly derivatives with respect to velocity.

- We want, as $t \to \infty$, f to go to a Maxwellian. Otherwise, our collision operator is not reaching a maximum entropy state.

- $C(f)$ should ensure that $f \geq 0$. f cannot be negative.

3.3.3 Examples of Collision Operators

Here, we investigate some of the collisions operators introduced in class.

One such operator is the Krook collision operator,

$$C(f) = -\nu(f - f_m) \tag{3.33}$$

This is a simple way of writing the collision operator, and it clearly gives a maxwellian distribution as t goes to infinity. Whether it has the required conservation properties was given in class as an exercise for students to do at home. Well, we are students at home, so now it is time to complete that exercise! We can see that if we choose f_m such that the density of the maxwellian distribution (the factor in front of the gaussian) at each point in space corresponds to the density of f, then this collision operator conserves particles. If the maxwellian is a drifting maxwellian at each point in space, such that the mean velocity of f at that point in space is the same as the drift velocity of the maxwellian, then it conserves momentum as well. If the temperature of the maxwellian f_m depends spatially on the local energy in f at that point, then sure, energy is conserved. However, with these three constraints on $n(\vec{x})$, $T(\vec{x})$ and $\vec{u}(\vec{x})$, our plasma will not reach a uniform maxwellian at infinite time! Instead, it will reach a local maxwellian at each point in space, which is not what we want! To get the correct behavior as t goes to infinity, we will have to make f_m be constant in space and time, which means our collision operator no longer conserves the quantities we want it to (at least locally).

Another operator is the collision operator $C(f) = 0$, which is true in the "mush limit". In this limit, we take $e \to 0$, $m_e \to 0$, $ne \to$ constant, $n \to \infty$, $\frac{e}{m} \to$constant, and therefore $\lambda_D \to$ constant, $v_T \to$ constant, and $\omega_p \to$ constant. In this limit, the collision frequency is much much less than the plasma frequency, and the collision operator can be ignored. Apparently, most plasmas are in this limit, where the collision operator term is small.

Question: Why is collision frequency being less than plasma frequency measure of it being ignored?

A third operator we discussed is related to diffusion in velocity space, presumably simulating the effect of some wave. This looks like

$$C(f) = \frac{\partial}{\partial v_\parallel} D(v_\parallel) \frac{\partial}{\partial v_\parallel} f \tag{3.34}$$

This can be thought of as some diffusion in parallel velocity space, and so our vlasov equation becomes analogous to the diffusion equation $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$. This operator could represent, artificially, the effect of some waves being launched in the plasma and creating diffusion of particles in velocity space.

3.4 Lorentz Collision Operator

A fourth collision operator we discussed was the Lorentz Collision Operator, which is so important it deserves it's own subsection. The Lorentz collision

operator can be written in various forms, but only one form is covered in GPP1. This form is

$$\mathcal{L}(f) = \nu(v)\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}f\right] \tag{3.35}$$

where $\mu = \frac{v_\parallel}{v} = \cos\theta$, the angle parallel to the magnetic field. μ ranges from -1 to 1. Note this μ is *not* the adiabatic invariant! The frequency $\nu(v) \sim \frac{1}{v^3}$. This collision operator is not derived in GPP1, but it comes from the Focker-Planck operator (also not derived in class) and an assumption that the ions are a cold drifting population.

The Lorentz collision operator represents pitch-angle (the angle with respect to the magnetic field) scattering of electrons due to collisions with ions, in a system where the azimuthal angle with respect to the magnetic field ϕ is negligible. Note that the Lorentz collision operator, like the diffusion in velocity space due to waves, is qualitatively similar to the diffusion equation. Here, the particles diffuse in velocity space due to collisions with the ions. The Lorentz collision operator is valid under the assumption that $Z_i \gg 1$, so the electron-electron collisions are negligible and only electron-ion collisions are important. It also relies on the assumption that $v_{T,i} \ll v_{T,e}$, as in the derivation the ions are assumed to be a drifting delta function population.

A helpful property of the Lorentz collision operator is that it is self-adjoint. This means that

$$\int d^3\vec{v} g\mathcal{L}(f) = \int d^3\vec{v} f_m f\mathcal{L}(g) \tag{3.36}$$

We can prove this property by integrating by parts. First, we need to remember that $\mathcal{L}(f) = \nu(v)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}$ and $\int d^3\vec{v} = \int v^2 dv \int d_\phi \int_{-1}^{1} d\mu$. Thus,

$$\int v^2 dv \int d\phi \int_{-1}^{1} \nu(v) g\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}d\mu = -\int v^2 dv \int d\phi \int_{-1}^{1} \nu(v)\frac{\partial g}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}d\mu \tag{3.37}$$

But this would give us the same result for $\int f\mathcal{F}(g)d^3\vec{v}$, which proves the self-adjointness described by equation 3.36.

There are other nice mathematical properties of the Lorentz operator - it turns out that if P_n is the nth Legendre polynomial, then

$$\mathcal{L}(P_n(\mu)) = -n(n+1)P_n(\mu) \tag{3.38}$$

Since the Legendre polynomials are complete, we can write any f in terms of them. $f(\mu, v, t) = \sum_n P_n(\mu)a_n(v,t)$. Now, we can see that the larger-n, smaller-v components of f pitch-angle scatter faster. Let's see this. If we have a spatially-homogenous, zero-field plasma, then our Vlasov equation is $\frac{\partial f}{\partial t} = \nu(v)\mathcal{L}(\mu)f$. Expanding f in terms of the Legendre polynomials, we have $\frac{\partial a_n}{\partial t} = -\nu(v)n(n+1)a_n$, so $a_n(v,t) = a_n(v,0)e^{-\nu(v)n(n+1)t}$. Thus, the larger-$n$, smaller-$v$ particles pitch-angle scatter more quickly.

3.4.1 Lorentz Conductivity

What is the conductivity of a plasma? In other words, if we put some electric field $\vec{E}$ in a plasma (never mind how it got there, or the fact that a plasma tends to shield large-scale electric fields), then there should be some current $\vec{J}$, where the constant of proportionality between the two is σ, the conductivity. In solids, this is typically written $\vec{J} = \sigma\vec{E}$. While currents in plasma can arise even if there is no electric field (for example, due to the single-particle drifts or magnetization current), it is helpful to get a sense of how much current we will get for a given electric field.

The cross-field conductivity, $\sigma_\perp$, is in general different than the parallel conductivity $\sigma_\parallel$. Actually, if there is an electric field perpendicular to a magnetic field in a plasma, there is a net plasma fluid $\vec{E} \times \vec{B}$ drift which arises when particles are collisionless and drift. In addition, there will be some current perpendicular to the field, which arises due to collisions causing particles to drift perpendicular to the field. I don't yet understand the full picture for the current which arises in a plasma, and it wasn't discussed in class, so it won't be discuss it in these notes.

Question: Is this explanation for cross-field current correct?

Here, we will look at plasma conductivity in an unmagnetized plasma, so we can ignore all the complications of particle drifts and larmor orbits and whatnot. We'll use our favorite collision operator, the Lorentz collision operator, in a spatially-homogeneous plasma with a net electric field $\vec{E}$. Remember, the Lorentz collision operator assumes the ions are a cold, drifting population such that $f_i = n_i(\vec{x})\delta^3(\vec{v} - \vec{v}_i)$, and the significant collisions for the electrons are with the ions. Thus, we are interested f for the electrons. Our Vlasov-Maxwell equation for the electrons is

$$\frac{\partial f}{\partial t} - \frac{e}{m}\vec{E} \cdot \frac{\partial f}{\partial \vec{v}} = \mathcal{L}(f) = \nu(v)\frac{\partial}{\partial \mu}((1-\mu^2)\frac{\partial f}{\partial \mu}) \tag{3.39}$$

Now, we assume a steady-state solution, such that $\frac{\partial}{\partial t} \to 0$. This equilibrium is an equilibrium between collisions and the electric field which is pushing particles. If the electric field is not unreasonably strong, then we expect our equilibrium distribution to be similar to a Maxwellian, with some small departure. Thus, we can write f as $f_m(1+g)$ where g is some arbitrary function and $g \ll 1$ everywhere in phase space. We'll also put $\vec{E}$ in the z-direction. So our Vlasov-Maxwell equation becomes

$$-\frac{e}{m}E_z\frac{\partial f_m(1+g)}{\partial v_z} = \mathcal{L}(f_m(1+g)) \tag{3.40}$$

Since g is small and it's derivatives are also small, then $\frac{\partial f_m(1+g)}{\partial v_z}$ can be approximated as $\frac{\partial f_m}{\partial v_z}$, which is $-\frac{mv_z}{k_BT}f_m$. Since $C(f_m) = 0$, and the collision operator is linear, then $C(f_m(1+g)) = C(f_m) + C(f_m g) = f_m C(g)$. Thus, we have

$$-\frac{e}{k_B T} E_z v_z f_m = f_m \mathcal{L}(g) \tag{3.41}$$

We can cancel the f_m and use the definition of μ, $\mu = v_\parallel / v$, to write

$$\frac{-eE_z v\mu}{k_B T} = \nu(v) \frac{\partial}{\partial \mu}((1-\mu^2)\frac{\partial g}{\partial \mu}) \tag{3.42}$$

Now, if we can solve this equation for g, we can solve for J! Since the electrons will move much more quickly than the ions, we can ignore the ion motion and just look at the current due to the electrons. The integral of $f_m v_z$ goes to zero, so only the second term contributes.

$$J_z = -e \int f_m (1+g) v_z d^3\vec{v} = -e \int f_m g v_z d^3\vec{v} \tag{3.43}$$

So let's set out to solve for g! Unfortunately, we can't do this in general. But suppose we expand g in terms of the Legendre polynomials, such that $g = \sum_n a_n(v,t) P_n(\mu)$. Remember when we introduced the Lorentz collision operator, we showed that for a homogenous, zero-field plasma, the larger-n components of f pitch-angle scatter (i.e. equilibrate) faster. This motivates us to look at only the $n = 1$ component of g, since the higher-n components we expect to equilibrate faster. Also, the $n = 0$ Legendre polynomial is just a constant, so this is not interesting. Thus, we'll approximate g to be $a_1 P_1(\mu) = a_1 \mu$. Plugging this into equation 3.42, we get

$$-\frac{eE_z v\mu}{k_B T} = \nu(v) \frac{\partial}{\partial \mu}[(1-\mu^2)a_1] = -2\nu(v)\mu a_1 \tag{3.44}$$

Thus, $a_1 = \frac{eE_z v}{2\nu(v) k_B T}$, so $g = \frac{eE_z v\mu}{2\nu k_B T}$. This allows us to solve for J_z using equation 3.43:

$$J_z = -\frac{e^2 E_z}{2k_B T} \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int v^2 dv f_m v^2 \mu^2 = -\frac{2\pi e^2 E_z}{3k_B T} \int_0^\infty \frac{v^4}{\nu(v)} f_m(v) dv \tag{3.45}$$

From $\vec{J} = \sigma \vec{E}$, we have $\sigma = \frac{2\pi e^2}{3k_B T} \int_0^\infty \frac{v^4}{\nu(v)} f_m(v) dv$. I won't carry out the integral, but it isn't hard to do, as $\nu(v) \sim \frac{1}{v^3}$.

TODO: Find the minus sign error in current expression.

4 Fluid Equations and MHD

> The small, clean fusion reactor I am considering is NOT describable by MHD. Thank goodness!
>
> Samuel Cohen

So earlier in these notes, I cited Wikipedia's article on the Knudsen number, the ratio of the mean free path to the characteristic length scale of the system. For high-temperature plasmas, the mean free path is often enormous compared to the system size, meaning our fluid (here a plasma) cannot be treated with a fluid description. However, I mentioned in a footnote that the fluid description for a plasma is often a good approximation. Let's discuss that apparent contradiction in more depth now.

4.1 Fluid Equations

4.2 MHD Equations

4.3 1D MHD Equilibrium

5 Waves in Plasmas

> Then (Fermi) said "Of course such waves could exist." Fermi had such authority that if he said "of course" today, every physicist said "of course" tomorrow.
>
> Hannes Alfvén, on MHD waves

There are lots of waves in plasmas. As a first-year student, I find keeping track of all the different waves tends to be confusing. However, it is also a very important topic, worthy of an entire course during the second year. Thus, understanding the topic in depth seems rather important.

As far as I can tell, there are three main things we need to keep track of when thinking about plasma waves. Firstly, we need to remember the name of the wave. Unfortunately, many of the waves have multiple names, so this becomes rather inconvenient. Secondly, we need to remember the dispersion relation of the wave. This tells us about the group and phase velocity of the wave, and sometimes whether it can propogate at all. Thirdly, we need to remember the assumptions made in deriving the dispersion relation, so we can determine when we expect that wave to arise in physical situations. At the end of this chapter, hopefully we'll have a better understanding of these three things for a number of fundamental plasma waves, or at minimum a reference we can go to if our memory fails us. Hopefully we can also gain a physical picture for these waves, although that is not always possible given the vigorous algebraic manipulations required to derive dispersion relations for many of these waves.

In Chapter 1, we started by deriving the most basic of waves, plasma oscillations. These oscillations were derived assuming stationary ions, zero temperature, zero magnetic field, and using a fluid description for the electrons. By linearizing the equations and rearranging, we obtained a characteristic frequency of $\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e}$.

We will start this chapter by again looking at electrostatic plasma oscillations, but this time using the Vlasov-Maxwell equation as opposed to the electron fluid equation. We will derive a dispersion relation for electrostatic plasma oscillations which will take us to the world of complex functions. We'll save the pain of that subject until chapter 6, and in this chapter just find a dispersion relation for the first order correction to plasma oscillations due to temperature effects. We call this plasma oscillation with finite temperature Langmuir Waves. We'll then look at lower-frequency electrostatic waves, where the electrons move faster than the wave. In this limit, the waves are called ion acoustic waves.

As we move to the fluid description of waves, we'll again look at the Langmuir waves and ion acoustic waves, but from a fluid description rather than a kinetic description. We'll see that both these waves arise from the fluid description, just with different assumptions about the frequency of the wave relative to the thermal velocity of each species. Using the fluid model, we'll derive the

dispersion relation for electromagnetic waves in plasmas, which amazingly have a phase velocity faster than the speed of light! We'll also see that a low-frequency electrostatic wave cannot exist.

In the MHD description of waves, we'll look at the fundamental MHD waves which arise in a uniform background plasma with a magnetic field. There are three waves which arise: the Alfvén wave, and the fast and slow magnetosonic waves. Each of these waves will be discussed in detail.

5.1 Kinetic Description of Waves

As promised, we'll start by looking at electrostatic plasma oscillations using the Vlasov-Maxwell equation. Here, we allow the possibility that the particles have some non-zero temperature, i.e. the velocity distribution function is not simply a delta function at each point.

Our starting point is the collisionless unmagnetized Vlasov-Maxwell equation,

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma - \frac{q_\sigma}{m_\sigma} \vec{\nabla}\phi \cdot \vec{\nabla}_v f_\sigma = 0 \tag{5.1}$$

Now, we will apply the method of linearization to the Vlasov-Maxwell equation. We assume our 0th order solution is not time-dependent or spatially dependent, and also assume that the 0th order electric and magnetic fields are 0, so we only get a ϕ_1 term. The induced-B term, $\vec{B}_1$, is small relative to $\vec{E}_1$, which we can see through the following argument: we expect $\vec{E}_1 \parallel \vec{k}$, and the induced $\vec{B}$ will be due to magnetic induction, so $\vec{\nabla} \times \vec{E}_1 = \vec{k} \times \vec{E}_1 = 0 = -\frac{\partial \vec{B}_1}{\partial t}$.

Our goal is going to be to solve for the first-order perturbation to f, $f_{\sigma,1}$. Once we have this, we can integrate it to solve for the density $n_{\sigma,1}$. This is sometimes called the σ *response function* (electron response function or ion response function). Our next step is to plug the response functions for the various species into Gauss's law, to solve for our dispersion relation. Let's see how this works. Linearizing the Vlasov-Maxwell equation, we get

$$\frac{\partial f_{\sigma,1}}{\partial t} + \vec{v} \cdot \vec{\nabla} f_{\sigma,1} - \frac{q_\sigma}{m_\sigma} \vec{\nabla}\phi \cdot \vec{\nabla}_v f_\sigma = 0 \tag{5.2}$$

Notice all of the quantities are, 1st order in total. In order words, each term is either 1st order in f, or 0th order in f and 1st order in some other quantity (here, this quantity is ϕ). When we linearize, typically the quantities which are 0th order in total either go to zero or cancel each other. This is indeed the case here, as there is no time-dependence or spatial dependence of the 0th order solutions and there are no 0th order fields.

Now, we are looking for wave solutions, so we assume an exponential dependence $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ for each of the first-order quantities. This gives

$$-i\omega f_{\sigma,1} + \vec{v} \cdot i\vec{k} f_{\sigma,1} - \frac{q_\sigma}{m_\sigma} i\phi_1 \vec{k} \cdot \vec{\nabla}_v f_{\sigma,0} = 0 \tag{5.3}$$

Cancelling i and solving for $f_{\sigma,1}$ (which we want so we can solve for the response functions), we get

$$f_{\sigma,1} = \frac{q_\sigma}{m_\sigma} \frac{\phi_1 \frac{\partial f_{\sigma,0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} \tag{5.4}$$

where the parallel means parallel to $\vec{k}$.

We can solve for the perturbed density, $n_{\sigma,1}$, by integrating over velocity.

$$n_{\sigma,1} = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial f_{\sigma,0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} d^3\vec{v} = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} dv_\parallel \tag{5.5}$$

where $g_\sigma = \int f_{\sigma,0} d^2\vec{v}_\perp$. If we know our $f_{\sigma,0}$, then we just perform that integral, plug it into Gauss's law, and we have our dispersion relation. So we're good, right? Unfortunately, there's a problem: we've got a $v_\parallel - \frac{\omega}{k}$ term in the denominator. When we integrate, this term will go to zero at some $v_\parallel$, and if there is some finite $f_{\sigma,0}$ at that $v_\parallel$, then this integral will blow up and our integral will become infinite, which is obviously not what we want. There isn't a simple solution to this problem, and you'll have to stay tuned until Chapter 6 to see what Landau damping is and how it resolves this issue. For now, we'll look at limiting cases of this integral, and use these limiting cases to investigate various types of electrostatic plasma waves.

Now, there are two limiting cases of this integral we are interested in. The first is called the adiabatic case, where the changes due to the wave occur so quickly that the particles don't have time to react. Mathematically, this means $\frac{\omega}{k} \gg v_{th,\sigma}$. The second case is called the isothermal case, where the changes due to the wave are so slow that the particles have plenty of time to react to the wave's behavior. Mathematically, this means $\frac{\omega}{k} \ll v_{th,\sigma}$.

The first limiting case is the adiavatic case, where $\frac{\omega}{k} \gg v_{th,\sigma}$. In this limit, $f_{\sigma,0}$ is essentially zero at the phase velocity of the wave, because the thermal velocity is so much lower and most of the particles are similar in velocity to the thermal velocity. Thus, the integral doesn't blow up because the portion of the integral which would otherwise blow up has a zero numerator.

To solve for $n_{\sigma,1}$ in this limiting case, the first thing we'll do is integrate the integrand by parts. This gives $\frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} = \frac{g_\sigma}{(v_\parallel - \frac{\omega}{k})^2} + \left[\frac{g_\sigma}{v_\parallel - \frac{\omega}{k}}\right]_{-\infty}^{\infty}$, and the last term will go to zero because f_σ and hence g_σ is zero at $v_\parallel \to \pm\infty$. Now, we can rewrite $\frac{1}{(v_\parallel - \frac{\omega}{k})^2}$ as $\frac{k^2}{\omega^2} \frac{1}{(1 - \frac{kv_\parallel}{\omega})^2}$. In this limit we can Taylor expand to get $\frac{k^2}{\omega^2}(1 + 2\frac{kv_\parallel}{\omega} + 3\frac{v_\parallel^2 k^2}{\omega^2})$. With these manipulations, our integral for $n_{\sigma,1}$ becomes

$$n_{\sigma,1} = \frac{q_\sigma k^2 \phi_1}{\omega^2 m_\sigma} \int g_\sigma (1 + 2\frac{kv_\parallel}{\omega} + 3\frac{k^2 v_\parallel^2}{\omega^2}) dv_\parallel \tag{5.6}$$

Remember the definition of g_σ, $\int f_{\sigma,0} d^2\vec{v}_\perp$. From this definition, we can see that the first term integrates to $n_{\sigma,0}$. If the mean velocity in the parallel velocity is zero, the second term integrates to zero. The third term is, approximately, $3v_{th,\sigma}^2 \frac{k^2}{\omega^2} n_{\sigma,0}$. Finally, after all that work, we have $n_{\sigma,1}$ in the adiabatic limit where $\frac{\omega}{k} \gg v_{th,\sigma}$.

Table 1: Response Functions, $n_{\sigma,1}$

Adiabatic, $\frac{\omega}{k} \gg v_{th,\sigma}$	Isothermal, $\frac{\omega}{k} \ll v_{th,\sigma}$
$\frac{q_\sigma n_{\sigma,0} k^2}{m_\sigma \omega^2}\phi_1(1+3\frac{k^2 v_{th,\sigma}^2}{\omega^2})$	$-\frac{q_\sigma n_{\sigma,0}}{m_\sigma v_{th,\sigma}^2}\phi_1$

$$n_{\sigma,1} = \frac{q_\sigma n_{\sigma,0}}{m_\sigma}\frac{k^2\phi_1}{\omega^2}(1+3\frac{k^2 v_{th,\sigma}^2}{\omega^2}) \tag{5.7}$$

What about in the isothermal limit where $\frac{\omega}{k} \ll v_{th,\sigma}$? We can't solve for this exactly. However, if our zeroth order distribution function is Maxwellian, then we can solve the integral in equation 5.5 to get something nice. Let's do this now. If we have a Maxwellian, $f_{\sigma,0} = n_{\sigma,0}(\frac{m_\sigma}{2\pi k_B T_\sigma})^{3/2}\exp(-\frac{m_\sigma v^2}{2k_B T_\sigma})$, and so integrating over the perpendicular directions gives $g_\sigma = n_{\sigma,0}(\frac{m_\sigma}{2\pi k_B T_\sigma})^{1/2}\exp(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma})$. Taking the derivative, $\frac{\partial g_\sigma}{\partial v_\parallel} = -n_{\sigma,0}\frac{m_\sigma^{3/2}}{(2\pi)^{1/2}(k_B T_\sigma)^{3/2}} v_\parallel \exp(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma})$. Now, we ignore the $v_\parallel - \frac{\omega}{k}$ and replace that with just $v_\parallel$. Why do we do this? Well, the numerator goes as $v_\parallel$ for small $v_\parallel$, so if $\frac{\omega}{k}$ is really small, then the numerator is essentially zero when the denominator goes to zero. We can see this geometrically as well, if we visualize a Maxwellian distribution. Near the peak of a Maxwellian distribution, the derivative of the distribution is about zero because the maxwellian has a local maximum at it's peak, so this part of the integral can be ignored. This assumption is pretty dodgy because the integral technically blows up, but it's one we need to make to solve for the electron response in this isothermal limit. If we make this dodgy assumption, then the $v_\parallel$ on top and in the bottom cancel. We have

$$n_{\sigma,1} = -\frac{q_\sigma m_\sigma^{1/2} n_{\sigma,0}\phi_1}{(k_B T_\sigma)^{3/2}(2\pi)^{1/2}}\int \exp(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma})dv_\parallel \tag{5.8}$$

Using $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, our integral becomes $\sqrt{\frac{2\pi k_B T_\sigma}{m_\sigma}}$, so our response function for σ is

$$n_{\sigma,1} = -\frac{q_\sigma n_\sigma}{m_\sigma}\frac{m_\sigma}{k_B T_\sigma}\phi_1 = -\frac{q_\sigma n_\sigma}{m_\sigma v_{th,\sigma}^2}\phi_1 \tag{5.9}$$

We've done a lot of algebra so far, but the process has been pretty simple: we linearize the Vlasov-Maxwell equation, look for electrostatic wave solutions, solve for $f_{\sigma,1}$, and integrate to get $n_{\sigma,1}$ (the response function) in either an adiabatic or isothermal limit. Now comes the fruit of our labor: we can plug our response function into Gauss's Law in various limits. Each of these limits corresponds to a new plasma wave.

5.1.1 Langmuir Wave

The Langmuir wave is the finite-temperature version of the plasma oscillation. This is a fast oscillation, such that the phase velocity of the wave is faster than

the thermal velocity of both the ions and the electrons. Thus, both species have an adiabatic response function. The Langmuir wave goes by many names, including the electron plasma wave, the Bohm-Gross wave, or just the Bohm wave. Typically, though, the Langmuir wave is what it's called.

Gauss's law, $\vec{\nabla}^2\phi_1 = -\frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma,1}$, becomes

$$-k^2\phi_1 = -\frac{e^2 n_{e,0}k^2}{\epsilon_0 m_e\omega^2}\phi_1(1+3\frac{k^2v_{th,e}^2}{\omega^2}) - \frac{q_i^2 n_{i,0}k^2}{\epsilon_0 m_i\omega^2}\phi_1(1+3\frac{k^2v_{th,i}^2}{\omega^2}) \tag{5.10}$$

Notice that we can replace the $\frac{q_\sigma^2 n_{\sigma,0}}{\epsilon_0 m_\sigma}$ with $\omega_{p,\sigma}^2$. We expect to see the plasma frequency show up, since the Langmuir wave is a finite-temperature version of the plasma oscillation. We can also cancel the ϕ_1 and the k^2, and multiple by ω^2 to get

$$\omega^2 = \omega_{p,e}^2(1+3\frac{k^2v_{th,e}^2}{\omega^2}) + \omega_{p,i}^2(1+3\frac{k^2v_{th,i}^2}{\omega^2}) \tag{5.11}$$

Hey look! We've got a dispersion relation, i.e. an equation for ω in terms of k. Our dispersion relation is 4th-degree polynomial equation for ω. Let's solve this perturbatively. Since the Langmuir wave is the finite-temperature version of the plasma oscillation, we expect our solution to be close to the plasma frequency. Thus, we can approximate the $\frac{1}{\omega^2}$ terms on the right of this equation as $\frac{1}{\omega_{p,e}^2}$. This gives

$$\omega^2 = \omega_{p,e}^2 + \omega_{p,i}^2 + 3k^2v_{th,e}^2 + 3\frac{\omega_{p,i}^2}{\omega_{p,e}^2}k^2v_{th,i}^2 \tag{5.12}$$

This is the dispersion relation for Langmuir waves! Often, it is just written as $\omega^2 = \omega_{p,e}^2 + 3k^2v_{th,e}^2$ since the ion terms are much smaller than the electron terms.

5.1.2 Ion Acoustic Wave

The ion acoustic wave is an electrostatic wave where the ions are still adiabatic, but the electrons are isothermal. More explicitly, we have the relation $v_{th,i} \ll \frac{\omega}{k} \ll v_{th,e}$. It is called the ion acoustic wave because the wave is a modified sound wave, where the electrons create the sound wave and the electrostatic forces on the ions modify the sound wave.

TODO: is this the right explanation?

To solve for the dispersion relation, we again plug our ion and electron response functions into Gauss's law. This gives us (substituting in the plasma frequency and cancelling ϕ_1 and dividing by k^2 as before)

$$-1 = \frac{\omega_{p,e}^2}{k^2v_{th,e}^2} - \frac{\omega_{p,i}^2}{\omega^2}(1+3\frac{k^2v_{th,i}^2}{\omega^2}) \tag{5.13}$$

Now, we can solve this equation perturbatively as well. To lowest order, since $\frac{\omega}{k} \gg v_{th,i}$, we can drop the $3\frac{k^2v_{th,i}^2}{\omega^2}$ relative to 1. We'll also drop the -1

completely, because the other two terms which remain are much bigger than 1. Why? Well, we can rewrite $\frac{\omega_{p,e}^2}{k^2 v_{th,e}^2} = \frac{1}{k^2 \lambda_{D,e}^2}$. Since $k \sim \frac{1}{\lambda}$, then this term is $\frac{\lambda^2}{\lambda_{D,e}^2}$, which will be much greater than 1 because the wavelength of the ion acoustic wave is assumed to be larger than the electron debye length. Thus, the first term is much greater than 1. The second term is also much great than 1, because the frequency is assumed to be much less than the ion plasma frequency. Dropping these two terms, to lowest order we have

$$0 = \frac{\omega_{p,e}^2}{k^2 v_{th,e}^2} - \frac{\omega_{p,i}^2}{\omega^2} \tag{5.14}$$

so to lowest order,

$$\frac{\omega^2}{k^2} = \frac{\omega_{p,i}^2}{\omega_{p,e}^2} v_{th,e}^2 \tag{5.15}$$

To second order, we still ignore the -1 term, we include the $3\frac{k^2 v_{th,i}^2}{\omega^2}$ term and plug in our first-order ω^2/k^2 solution to this term. This gives

$$\frac{\omega_{p,e}^2}{k^2 v_{th,e}^2} - \frac{\omega_{p,i}^2}{\omega^2}(1 + 3\frac{v_{th,i}^2 \omega_{p,e}^2}{v_{th,e}^2 \omega_{p,i}^2}) = 0 \tag{5.16}$$

so to the second order,

$$\frac{\omega^2}{k^2} = \frac{\omega_{p,i}^2}{\omega_{p,e}^2} v_{th,e}^2 + 3v_{th,i}^2 \tag{5.17}$$

This dispersion relation looks a bit rough, but we can simplify it to see the physical meaning using our definition $\omega_{p,\sigma}^2 = \frac{n_\sigma q_\sigma^2}{\epsilon_0 m_\sigma}$. Thus, $\frac{\omega_{p,i}^2}{\omega_{p,e}^2} = \frac{q_i^2 m_e}{e^2 m_i}$. We also have $v_{th,\sigma}^2 = \frac{k_B T_\sigma}{m_\sigma}$ so this first term becomes $\frac{q_i^2}{e^2}\frac{k_B T_e}{m_i} = \frac{q_i^2}{e^2} c_s^2$, where c_s is the sound speed, also called the acoustic speed.

Note that this is consistent with the physical picture we described earlier for the Ion Acoustic Wave.

$$\frac{\omega^2}{k^2} = \frac{q_i^2}{e^2} c_s^2 + v_{th,i}^2 \tag{5.18}$$

5.1.3 Isothermal Electrostatic Waves Don't Exist

Imagine we tried to find an electrostatic wave where both the ions and electrons were isothermal. Plugging our electron and ion response functions into Gauss's law, we get

$$0 = 1 + \frac{\omega_{p,e}^2}{k^2 v_{th,e}^2} - \frac{\omega_{p,i}^2}{k^2 v_{th,i}^2} \tag{5.19}$$

There is no frequency dependence! Actually, this is an equation for Debye shielding, and is the same as equation 1.22 way back from chapter 1 but without

the charge Q. Physically, this tells us that an electrostatic oscillation where the ions and the electrons are both isothermal is not possible, because both the electrons and the ions will Debye shield and cancel out any oscillation at that slow frequency.

5.2 Plasma Waves in the Fluid Description

5.2.1 Langmuir Waves and Ion Acoustic Waves with the Fluid Model

5.2.2 Electromagnetic Waves

5.3 MHD Waves

5.4 Magnetized Plasma Waves (Advanced)

5.5 Two-Stream Instability (Advanced)

6 Landau Damping

> The Landau paper takes many days, if not years to appreciate. Go slowly, and enjoy it. It is the foundation of plasma physics.
>
> George Morales

TODO: Talk about Landau Damping

6.1 Fundamentals of Complex Analysis

Before we dive into some of the math behind Landau Damping, we should take a few minutes to understand some basic facts about complex analysis, since we'll be working with integrals over the complex plane in this chapter as well as Laplace Transforms.

6.1.1 Integrals of Analytic Functions in the Complex Plane

One fundamental result of complex analysis is that the closed integral of an analytic function in the complex plane is zero. Now I don't know about you, but when I hear statements like that, I'm usually pretty confused. So let's unpack that statement some most.

When we perform real-valued integrals of a single-valued function $f(x)$, we are used to integrating a function along the real axis. This simple case is illustrated in figure 18. Now, it would seem obvious that if we integrate $f(x)$ from x_1 to x_2 and then back to x_1, the integral will equal 0. And this is indeed true. However, nothing is stopping us from plugging complex values into $f(x)$ instead of only real values. Additionally, nothing is stopping us from integrating over not just the real axis, but integrating into the complex plane as well.

Imagine we wanted to integrate $f(x)$ in the complex plane, such as the integral in figure 19. Because this integral is now in the complex plane, we can throw whatever intuition we had about integrals out the window. To get some intuition for this sort of integral, let's solve for this integral by hand and see what we get, supposing that $f(x) = x^n$.

Starting at the origin, the first part of the integral is just integrating from 0 to 1 along the real axis as we are used to.

$$\int_0^1 x^n dx = \frac{1}{n+1}\left[x^{n+1}\right]_0^1 = \frac{1}{n+1} \tag{6.1}$$

The second leg of the integral involves holding the real component of x at 1 and integrating over the imaginary component of x.

$$\int_0^i (1+y)^n dy = \frac{1}{n+1}\left[(1+y)^{n+1}\right]_0^i = \frac{1}{n+1}(1+i)^{n+1} - \frac{1}{n+1} \tag{6.2}$$

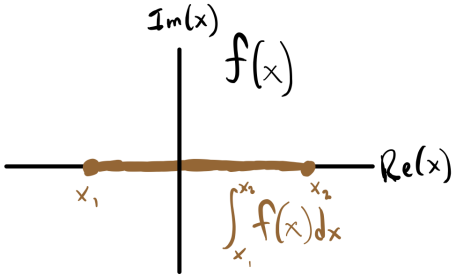


Figure 18: Integration along the real axis of a function $f(x)$.

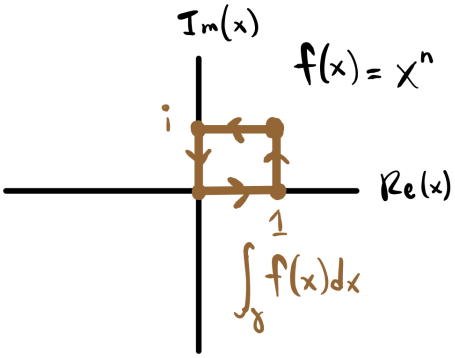


Figure 19: Complex integral of a function $f(x) = x^n$ in the complex plane.

The third leg of the integral involves holding the imaginary component of x at i and integrating the real component from 1 to 0.

$$\int_1^0 (i+x)^n dx = \frac{1}{n+1}\left[(i+x)^{n+1}\right]_1^0 = \frac{i^{n+1}}{n+1} - \frac{(1+i)^{n+1}}{n+1} \tag{6.3}$$

The fourth leg of the integral is done in similar fashion, fixing the real component to 0.

$$\int_0^i y^n dy = -\frac{i^{n+1}}{n+1} \tag{6.4}$$

Adding these 4 parts up, we can see that they add to 0. Our integral of $f(x) = x^n$, around a specific closed path in the complex plane, integrates to 0! Now it turns out that we could have integrated x^n around any closed path in the complex plane, and our integral would have still come out to 0. I haven't proven anything, but you can probably imagine that our result might hold independent of the shape of the integration path in the complex plane.

An analytic function is, for our purposes, a function which can be written as a Taylor series. For some complex function $f(x)$, it is analytic if it can be written $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ for any point x_0 in the complex plane. I typically think of analytic functions as functions which don't blow up at any single points in space, and are smooth everywhere.

Now, if x^n integrates to 0 over any closed path in the complex plane, then any analytic function which can be written as an infinite sum of x^n's will also integrate to 0 over the complex plane. This is what I was referring to earlier when I wrote "the closed integral of ananalytic function in the complex plane is zero". Not so bad!

6.1.2 Integrals of Non-Analytic Functions in the Complex Plane

Things get a little bit more complicated when functions are not analytic. What this whole 'non-analytic' business usually means is that our function $f(x)$ has points in the complex plane where the function 'blows up'. These points are typically called 'poles'. If we want to integrate over a path in the complex plane[26] which encloses a pole, then our integral will no longer necessarily be 0. We'll show in a second that an integral which encloses a pole is equal to the 'residue' of that pole, in a theorem known as the residue theorem.

Imagine, as in figure 20, that we wanted to perform an integral in the complex plane which enclosed a pole. Well, from the result we discussed earlier that the closed integral of any analytic function in the complex plane is 0, we can distort our integral so it looks like a perfect circle around our pole, as in figure 20. Make sure you understand why we can distort our integral like that - basically, we're just adding a bunch of closed integrals which all sum to 0, so that the path of the integral changes.

[26] These integrals are often referred to as 'contour integrals'.

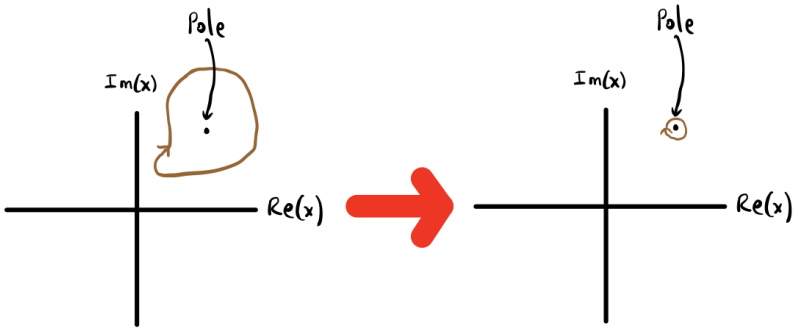


Figure 20: Integrating around a pole in the complex plane. The integral can be changed to a circle around the pole without changing the result of the integral.

Now, any function (whether analytic or not) in the vicinity of a pole a can we written as a Taylor series $\sum_{-\infty}^{\infty} c_n(x-a)^n$. We can use this to perform a circular integral around a pole. Our path for $x(t)$ is a circle around the pole at $x = a$, so $x(t) = a + re^{it}$. This gives $dx = rie^{it}$. Our integral around the pole in figure 20 then becomes

$$\int_\gamma f(x)dx = \int_{t=0}^{t=2\pi} \sum_{-\infty}^{\infty} c_n(re^{it})^n rie^{it}dt \tag{6.5}$$

Because $\int_0^{2\pi} e^{imt}dt$ is 0 for all integers m except $m = 0$, then only $n = -1$ contributes to the integral in equation 6.5. Thus, our integral becomes

$$\int_{t=0}^{t=2\pi} c_{(-1)}idt = 2\pi i c_{(-1)} \tag{6.6}$$

This is the 'residue' of the integral around the pole: the coefficient $c_{(-1)}$ of the expansion of the function $f(x)$, times 2 πi.

In summary, there are two key points we should keep in mind when doing complex integration. Firstly, because the integral around a closed path in the complex plane of an analytic function is zero, we can deform the path of our integral arbitrarily in the complex plane so long as we don't cross over any poles. And if our integral encloses a pole, we can solve for the integral using the residue theorem.

6.1.3 Laplace Transforms

Let's take some time to make sure we understand Laplace transforms. The Laplace transform of a function $\psi(t)$ is defined as

$$\widetilde{\psi}(p) = \int_0^\infty \psi(t)e^{-pt}dt, \text{ Re}(p) > \gamma \tag{6.7}$$

where γ is the fastest-growing exponential term in $\psi(t)$. The inequality here simply means that $\widetilde{\psi}(p)$ is only defined for the specified values of p. The inverse Laplace transform of $\widetilde{\psi}(p)$ is

$$\psi(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} dp\widetilde{\psi}(p)e^{pt},\ \beta > \gamma \tag{6.8}$$

where β is a real number. Actually, in class the Laplace transform was defined a bit differently. Here, I'm using the definition used in Bellan, whose In class, we replaced p with $i\omega$, which means that the inverse Laplace transform will require an integration along the real axis, not the imaginary axis. This difference isn't so important, but I wanted to point that out so as to avoid confusion as much as possible when comparing with class notes. Hopefully, by the end of this chapter you'll see why we used $i\omega$ in class as opposed to p, and you'll be able to understand Laplace Transforms regardless of whether p or $i\omega$ is used.

Having the mathematical definition of a Laplace transform is great, but it doesn't mean we understand what a Laplace transform is or what it does or how it works. Before we understand Laplace transforms, we need to recognize that p is a complex number. This is pretty important. It's also important that the t-integration is from $t = 0$ to infinity, as opposed to $-\infty$ to ∞ as in Fourier transforms. Take a moment to think about those facts, and what they imply about Laplace transforms. What do you think $\widetilde{\psi}(p)$ represents? Once you think about that for a bit, you should reach a conclusion along these lines: the Laplace transform takes a function which starts at $t = 0$ and goes to infinity, and instead of breaking it up into oscillatory components which are real valued (as in a Fourier Transforms), the components of the Laplace transform are both oscillatory *and* exponentially growing or decaying. In other words, instead of it telling us how much of each real frequency is in a function, it tells us (more or less) how much of each complex frequency is in a function.

Why do we require, in the Laplace transform, that $\mathrm{Re}(p) > \gamma$, where γ is the fastest growing exponential term in $\psi(t)$? Well, suppose $f(t)$ goes as $e^{\gamma t}$ as $t \to \infty$. Then if $\mathrm{Re}(p) < \gamma$, then $\widetilde{\psi}(p)$ will blow up (as we can see from the definition of the Laplace transform). The same is true if $f(t)$ is exponentially decaying as $t \to \infty$ - we still require that $\mathrm{Re}(p) > \gamma$, where γ is the slowest-decaying exponential, for otherwise the integral blows up.[27]

Imagine we had some function $f(t)$, and we wanted to know what $\widetilde{f}(p)$ was. Well, if p is purely real, then we have a simple idea of what $\widetilde{f}(p)$ is - it's the integral of the function $f(t)$, integrated to infinity with a weight function e^{-pt} applied to the integration. While there isn't a physical interpretation for this, we can more or less understand what $\widetilde{f}(p)$ is giving us. On the other hand, if p is purely imaginary, then (assuming $f(t)$ decays at infinity sufficiently fast that

[27] If the function neither exponentially grows nor decays as time goes to infinity, then I'm not quite sure how to handle issues of where in the complex plane the transform is defined. I'll have to think about that one. TODO: think about it

the integral converges) $\widetilde{\psi}(p)$ tells us how much of the frequency $\mathrm{Im}(p)$ is in the function $f(t)$.

Let's do one more example to make sure we understand intuitively what a Laplace transform is. Imagine $f(t) = e^{-\alpha t}\cos(\omega t)$. This function is exponentially decaying and oscillating at the same time, of course. Now imagine we took $\widetilde{f}(-\alpha + i\omega)$. What will this give us? Well, for this particular value of p ($p = -\alpha + i\omega$), $\widetilde{f}(p)$ will go to infinity, because the exponentials cancel and we'll be integrating $\cos^2 \omega t$ from 0 to infinity. As $\mathrm{Re}(p)$ increases above $-\alpha$, $\widetilde{f}(p)$ becomes some finite number, and gradually decreases towards zero as $\mathrm{Re}(p)$ increases. If we imagine varying $\mathrm{Im}(p)$, it turns out that $\widetilde{f}(p)$ will be maximum around $\mathrm{Im}(p) = \omega$ and fall off as $\mathrm{Im}(p)$ changes. So for $f(t) = e^{-\alpha t}\cos(\omega t)$, we get a pole at $p = -\alpha + i\omega$. This also illustrates an important point about Laplace transforms: the fastest growing exponential term in $f(t)$ (in this example, the term is $-\alpha$) is related to the pole of $\widetilde{f}(p)$ with the largest real component in that the real components of each are the same.

We can actually derive the inverse transform, equation 6.8 from the definition of the Laplace transform, equation 6.7. Bellan goes through this, and it's an illustrative exercise, so I'll go through it here now. Let's start by considering the integral

$$g(t) = \int_C \widetilde{\psi}(p) e^{pt} dp \tag{6.9}$$

This is a reasonable guess for the inverse transform, since we expect the inverse transform to have a e^{pt} factor hanging out in the integral. The contour C over which we integrate in p-space is undefined at the moment - we'll define it in a bit. Plugging in the definition of $\widetilde{\psi}(p)$, we get

$$g(t) = \int_{t'=0}^{\infty} dt' \psi(t') \int_C e^{p(t-t')} dp \tag{6.10}$$

We'll have to be careful though: $\widetilde{\psi}(p)$ isn't defined for $\mathrm{Re}(p) < \gamma$, so we'll have to make sure our contour C doesn't venture into that region in p-space. To evaluate $g(t)$, we'll have to choose a contour C. But here is a fact: we can write the delta function $\delta(t)$ as

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \tag{6.11}$$

Hey wait a minute! That looks an awful lot like what we've got going on in equation 6.10. If we define the right integration contour C, then we might be able to get out a delta function, and thus write $g(t)$ in terms of $\psi(t)$. The trick will be to hold the real part of p constant, and greater than γ ($p_r = \beta > \gamma$) and vary the imaginary part of p from negative infinity to positive infinity. If we do this, then $dp = i dp_i$, and $e^{p(t-t')} = e^{p_r(t-t')} e^{p_i(t-t')}$. Thus, equation 6.10 becomes

$$g(t) = \int_{t=0}^{\infty} dt' \psi(t') \int_{\beta - i\infty}^{\beta + i\infty} e^{p_r(t-t')} e^{p_i(t-t')} dp \tag{6.12}$$

$$g(t) = i \int_{t=0}^{\infty} dt' \psi(t') e^{\beta(t-t')} \int_{-\infty}^{\infty} e^{p_i(t-t')} dp_i \tag{6.13}$$

where we turned $e^{p_r(t-t')}$ into $e^{\beta(t-t')}$ because p_r was held constant over the entire integral. Now, this last equation is in the form of $\delta(t)$ as desired, so it becomes $2\pi\delta(t-t')$. Thus,

$$g(t) = 2\pi i \int_{0}^{\infty} dt' \psi(t') e^{\beta(t-t')} \delta(t-t') = 2\pi i \psi(t) \tag{6.14}$$

This completes our proof: $g(t)$ is $2\pi i$ times $\psi(t)$, assuming we integrate from $\beta - i\infty$ to $\beta + i\infty$ and $\beta > \gamma$. And since the inverse transform of $\widetilde{\psi}(t)$ should give us $\psi(t)$, then equation 6.8 must be the inverse Laplace transform.

I've got one more fact (and short proof) related to Laplace transforms which we'll have to use. It turns out that

$$\int_{0}^{\infty} dt \frac{d\psi}{dt} e^{-pt} = p\widetilde{\psi}(p) - \psi(0) \tag{6.15}$$

We can prove this simply by integrating by parts.

$$\int_{0}^{\infty} dt \frac{d\psi}{dt} e^{-pt} = [\psi(t) e^{-pt}]_{0}^{\infty} - \int_{0}^{\infty} dt \psi(t) \frac{d}{dt}(e^{-pt}) = -\psi(0) + p \int_{0}^{\infty} dt \psi(t) e^{-pt} \tag{6.16}$$

Since this last integral is the definition of the Laplace transform $\widetilde{\psi}(p)$, we've proved our result for the Laplace transform of a derivative.

6.1.4 Analytic Continuation

Imagine we have a function $f(t) = e^{qt}$ where q is some complex number, and we want to take the Laplace transform of $f(t)$. What do we get? Well, using the definition of the Laplace transform, we have

$$\widetilde{f}(p) = \int_{0}^{\infty} e^{(q-p)t} dt = \Big[\frac{1}{q-p} e^{(q-p)t}\Big]_{t=0}^{t=\infty} = \begin{cases} \infty \text{ for } \mathrm{Re}(p) > \mathrm{Re}(q) \\ \frac{1}{p-q} \text{ for } \mathrm{Re}(q) > \mathrm{Re}(p) \end{cases} \tag{6.17}$$

Rhe transform is only defined for Re(p) >Re(q), but I've included the . Note that if we tried to calculate the transform for Re(p) < Re(q), then we would have gotten infinity as our answer, which isn't analytic of course. So we shouldn't attempt to take Laplace transforms for a value of p in the region where it isn't defined, as this will give us infinity.

Now, let's get from $\widetilde{f}(p)$ back to $f(t)$ by taking the inverse laplace transform. Using the definition of the inverse Laplace transform, we have

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \widetilde{f}(p) e^{pt} dp_i \tag{6.18}$$

It would be great if we could use method of residues to evaluate this integral, as we would just need to calculate the residue of the pole at $p = q$, and viola we have the integral. However, we have two problems: the integral isn't a closed contour in p-space, but rather a straight line. And the method of residues requires a closed contour over an analytic function. The second problem is that the inverse laplace transform $\widetilde{f}(p)$ isn't defined to the left of the pole, so even if we wanted to we couldn't form a closed path which enclosed the pole.

However, it turns out that both these problems can be fixed and hence the integral can be solved using method of residues, by use of an ingenious little trick called analytic continuation. Watch carefully. The inverse Laplace transform, $\widetilde{f}(p)$, isn't defined for $\mathrm{Re}(p) < \mathrm{Re}(q)$. However, the analytic expression for the Laplace transform in the region where it is defined, $\frac{1}{p-q}$, is only non-analytic at the point $p = q$. What we do - and this is the key step when performing analytic continuation - is *redefine* the inverse Laplace transform to be $\frac{1}{p-q}$ everywhere that $\frac{1}{p-q}$ is analytic, even in the region where the Laplace transform was previously undefined. In this example, that means that our redefined Laplace transform is now defined everywhere except the pole at $p = q$. Once we do this, then we can actually close our integral infinitely far to the left in the left half of p-space, as illustrated in figure 21. In this leftwards section of the contour integral, the infinitely negative real component of p causes the exponential term e^{pt} to go to 0, and this other section of the integral which closes the integral evaluates to 0. However, now (as in figure 21) we have a closed contour in p-space, so we can use the method of residues to evaluate the integral.

When we use method of residues, we expect that the residue gives us back our original function, $f(t) = e^{qt}$. Well, this is indeed the case. Remember, the residue is $2\pi i$ times the $c_{(-1)}$ term in the expansion of the function $f(x) = \sum_{-\infty}^{\infty} c_n (x-a)^n$ around $x = a$. Well, here our function $f(x)$ is the function being integrated in equation 6.18, which is $\frac{1}{2\pi i(p-q)} e^{pt}$. The $2\pi i$'s cancel, and if we expand the function around the point $p = q$, the $n = -1$ coefficient in the expansion of $(p-q)^n$ is (not surprisingly) e^{qt}. This makes sense, as we're looking for the coefficient for the $\frac{1}{p-q}$ term in the expansion of $\frac{1}{p-q} e^{pt}$, which already contains a $\frac{1}{p-q}$ in it. Bellan writes this out explicitly as[28]

$$f(t) = \frac{1}{2\pi i} \oint \frac{1}{p-q} e^{pt} dp = \lim_{p \to q} 2\pi i (p-q) \big[\frac{1}{2\pi i (p-q)} e^{pt} \big] = e^{qt} \tag{6.19}$$

Let's recap what just happened: we wanted to take the inverse Laplace transform of a function, but we couldn't actually carry out the integral. So

[28] I don't really understand why this expression is the residue, but I'm not by any means experienced in actually calculating residues of poles. This is the only residue we explicitly calculate in this class, and I'm only calculating it to illustrate how analytic continuation works. So if you want to understand this expression and actually get practice calculating residues of poles, open up a book on complex analysis.

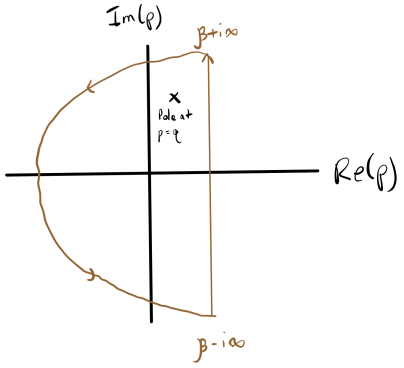


Figure 21: The integration contour used to close the integral in equation 6.18. These contours are called Bromwich contours. Although the inverse Laplace transform is not defined for Re(p) ¡ Re(q), we have used analytical continuation to be able to extend our integration path into the left half of p-space. We can then solve this integral using method of residues.

we extended the realm of validity of the inverse transform, which allowed us to close the integral in the left half of p-space. Because of the exponential term in the integral, closing the integral added nothing to the original integral, and we were able to use the method of residues to calculate an integral which we previously couldn't calculate.

In summary, analytic continuation involves making a new $\widetilde{\psi}(p)$ which

- Equals the old $\widetilde{\psi}(p)$ in the region $\mathrm{Re}(p) > \gamma$,
- is *also* defined in the region $\mathrm{Re}(p) < \gamma$,
- is analytic over the integration path.

As long as we follow these constraints, then analytic continuation is a useful means of evaluating inverse Laplace integrals.

QUESTION: Bellan page 160, requirement 3. Is analytic where? What if there are multiple poles? This is important - we can't just say 'must be analytic' because there are still poles to be evaluated.

6.2 Fourier Transform in Space, Laplace Transform in Time

Okay, enough math. Let's get back to the physics of Landau damping. Of course, we'll be using all the math I just introduced. Otherwise I wouldn't have bothered to introduce it.

To begin to understand Landau damping, we're going to derive a pseudo-dispersion relation for oscillations which perturb a homogenous, zero-field plasma equilibrium. Why do I say pseudo? Well, typically a dispersion relation has an

exact relation between wavenumber and frequency. Here we don't have that, we solve for the time-evolution of f based on the initial conditions. What motivates us to look for a dispersion relation? Actually, the reason is pretty simple. If our frequency ω in the dispersion relation has some imaginary component, then the oscillations will be exponentially damped or exponentially growing. If the frequency is purely real, then the oscillations will continue forever without being damped. So if we are trying to understand damping, we want to figure out any complex frequencies which might arise. Crucially, we will do this using a kinetic treatment, with the Vlasov-Maxwell equation. For simplicity, we'll use the collisionless Vlasov-Maxwell equation. Even using a collisionless equation, we still get damping. This is often called 'collisionless damping'. If we were to look for oscillations using collisionless *fluid* equations, we get waves which aren't damped, as in chapter 5.

Actually, we've gotten a preview of Landau damping in chapter 5 already. Take a look back at equation 5.5. Remember how we got this equation for the perturbed density $n_{\sigma,1}$ - we started with the collisionless Vlasov-Maxwell equation, linearized, and then Fourier-transformed in space and time to get $f_{\sigma,1}$[29]. We then integrated $f_{\sigma,1}$ over velocity space to get $n_{\sigma,1}$. However, this equation blows up in the denominator, when $v_\parallel = \frac{w}{k}$. Density going to infinity isn't good. In Chapter 5, we ignored this by expanding the denominator in isothermal and adiabatic limits. It turns out that what we did in Chapter 5 was *wrong*, and we'll need to take a different approach. We erred when we Fourier transformed. TODO: Why did we err?

Let's start, as we did in chapter 5, with the collisionless Vlasov Maxwell equation.

$$\frac{\partial f_\sigma}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_\sigma = 0 \tag{6.20}$$

Now, let's linearize this equation around an equilibrium. Here, the equilibrium is a spatially homogenous, zero-field equilibrium, such that $\vec{E}_0$ and $\vec{B}_0$ equal zero. This gives

$$\frac{\partial f_{\sigma,1}}{\partial t} + \vec{v} \cdot \vec{\nabla} f_{\sigma,1} + \frac{q_\sigma}{m_\sigma}(\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \vec{\nabla}_v f_{\sigma,0} = 0 \tag{6.21}$$

Now, as before, we can Fourier transform in space, so all the first-order quantities go as $e^{i\vec{k}\cdot\vec{x}}$. Thus, the $\vec{\nabla}$ becomes $i\vec{k}$. We can also ignore $\vec{v} \times \vec{B}_1$ relative to $\vec{E}_1$, using $\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$. Since there is no zeroth-order field, we can write $kB_1 \sim \omega \frac{1}{c^2} E_1$, so $B_1 \sim \frac{\omega}{k}\frac{1}{c^2}E_1$ and $B_1 \sim \frac{v_g}{c^2}E_1$. Since $\frac{v_g v}{c^2} \ll 1$, we can ignore $\vec{v} \times \vec{B}_1$. TODO: Why doesn't $\vec{J}$ come in? Now, we have

[29] What we really did was assume an exponential dependence of all the first-order quantities $e^{i\vec{k}\cdot\vec{x} - i\omega t}$. However, this is equivalent to Fourier transforming in space and time, as Fourier transforming in time picks out a particular ω and Fourier transforming in space picks out a particular $\vec{k}$, which we are doing. So when I say 'we Fourier transform' what I mean is that we assume an exponential dependence in the linearized quantities. Sorry if that is confusing.

$$\frac{\partial f_{\sigma,1}}{\partial t} + i\vec{v}\cdot\vec{k}f_{\sigma,1} + \frac{q_\sigma}{m_\sigma}\vec{E}_1\cdot\vec{\nabla}_v f_{\sigma,0} = 0 \tag{6.22}$$

Now, here comes the crucial point. We're going to Laplace transform in time instead of Fourier transform. When you read about this, you'll see people write this is treating the problem as an "initial value problem". I'm no mathematician, but here's what I understand this statement to mean: when we Fourier transform, each wavenumber k has a particular frequency ω which determines the time-evolution of the function of interest, in this case $f_{\sigma,1}$. However, when we Laplace transform, there is not a particular single frequency (neither real nor imaginary) which gives us the behavior of a component with wavenumber k in $f_{\sigma,1}$. Instead, the behavior of the wavenumber component k in $f_{\sigma,1}(t=0)$ has some complicated behavior described by many (complex) frequencies, and Laplace transforming allows us to solve for that complicated behavior. Using equation 6.15, we'll see that solving for the complicated time-evolving behavior requires knowing the initial condition on $f_{\sigma,1}$. Now, taking the Laplace transform gives

$$\int_0^\infty \big[\frac{\partial f_{\sigma,1}}{\partial t}e^{-pt} + i\vec{v}\cdot\vec{k}f_{\sigma,1}e^{-pt} + \frac{q_\sigma}{m_\sigma}\vec{E}_1\cdot\vec{\nabla}_v f_{\sigma,0}e^{-pt}\big]dt = 0 \tag{6.23}$$

On the first term, we can use equation 6.15 to simplify, to get $p\widetilde{f}_{\sigma,1}(\vec{x},\vec{v},p) - f_{\sigma,1}(\vec{x},\vec{v},t=0)$. In the second and third terms, we replace the first-order quantities with their Laplace transforms. Thus, we have

$$p\widetilde{f}_{\sigma,1}(\vec{x},\vec{v},p) - f_{\sigma,1}(\vec{x},\vec{v},t=0) + i\vec{v}\cdot\vec{k}\widetilde{f}_{\sigma,1}(\vec{x},\vec{v},p) + \frac{q_\sigma}{m_\sigma}\widetilde{\vec{E}_1}(\vec{x},p)\cdot\vec{\nabla}_v f_{\sigma,0} = 0 \tag{6.24}$$

This equation is a bit messy, and it's only going to get worse from here unfortunately. However, for now we can clean things up a bit by solving for $\widetilde{f}_{\sigma,1}$.

$$\widetilde{f}_{\sigma,1}(\vec{x},\vec{v},p) = \frac{f_{\sigma,1}(\vec{x},\vec{v},t=0) - \frac{q_\sigma}{m_\sigma}\widetilde{\vec{E}}_1(\vec{x},p)\cdot\vec{\nabla}_v f_{\sigma,0}}{p + i\vec{v}\cdot\vec{k}} \tag{6.25}$$

We'll assume $f_{\sigma,1}$ is known at $t=0$, which is equivalent to saying we know what our initial perturbation is. Thus, we have an equation for $\widetilde{f_{\sigma,1}}$ in terms of things we know, and $\widetilde{\vec{E}_1}$, which we don't know. Thus, if we can get one more equation with both $\widetilde{f_{\sigma,1}}$ and $\widetilde{\vec{E}_1}$, we can solve our dispersion relation. Fortunately, there is one equation which describes electrostatic plasma oscillations we haven't used yet: Gauss's Law. And just like we Laplace transformed the linearized Vlasov-Maxwell equation, we can also Laplace transform the linearized Gauss's law. Remember how we did this - we multiplied the equation by e^{-pt} and then integrated over t from 0 to ∞. We then used the definition of

the Laplace transform the replace the first order quantities with their Laplace transforms, which have a squiggle above them. For Gauss's law, this leaves

$$\vec{\nabla} \cdot \widetilde{\vec{E}}_1(p, \vec{x}) = i\vec{k} \cdot \widetilde{\vec{E}}_1(p, \vec{x}) = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int d^3\vec{v} \widetilde{f}_{\sigma,1}(p, \vec{x}, \vec{v}) \tag{6.26}$$

We have an expression for $\widetilde{f}_{\sigma,1}$ in equation 6.25, which we can plug into the above equation. We'll then isolate for $\widetilde{\vec{E}}_1$.

$$i\vec{k} \cdot \widetilde{\vec{E}}_1(p, \vec{x}) = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int \left(\frac{f_{\sigma,1}(\vec{x}, \vec{v}, t=0)}{p + i\vec{v} \cdot \vec{k}}\right) d^3\vec{v} - \frac{1}{\epsilon_0} \sum_\sigma \frac{q_\sigma^2}{m_\sigma} \int \left(\frac{\widetilde{\vec{E}}_1(\vec{x}, p) \cdot \vec{\nabla}_v f_{\sigma,0}}{p + i\vec{v} \cdot \vec{k}}\right) d^3\vec{v} \tag{6.27}$$

Now, if we have purely electrostatic perturbations, then $\vec{E}_1 = -\vec{\nabla}\phi_1 = -i\vec{k}\phi_1$, which implies $\vec{E} \parallel \vec{k}$. Thus, $\widetilde{\vec{E}}_1 = \widetilde{E}_1 \hat{k}$, so the dot product $\vec{k} \cdot \widetilde{\vec{E}_1}$ simplifies to a scalar and we can solve equation 6.27 for $\widetilde{E_1}$.

$$\widetilde{E}_1 \left[ik + \frac{1}{\epsilon_0} \sum_\sigma \frac{q_\sigma^2}{m_\sigma} \int \frac{\hat{k} \cdot \vec{\nabla}_v f_{\sigma,0}}{p + i\vec{v} \cdot \vec{k}} d^3\vec{v} \right] = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int \left(\frac{f_{\sigma,1}(\vec{x}, \vec{v}, t=0)}{p + i\vec{v} \cdot \vec{k}}\right) d^3\vec{v} \tag{6.28}$$

Solving for $\widetilde{E}_1$, we get

$$\widetilde{E}_1 = \frac{N(p)}{D(p)} \tag{6.29}$$

where

$$N(p) = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int \left(\frac{f_{\sigma,1}(\vec{x}, \vec{v}, t=0)}{p + i\vec{v} \cdot \vec{k}}\right) d^3\vec{v} \tag{6.30}$$

$$D(p) = ik + \frac{1}{\epsilon_0} \sum_\sigma \frac{q_\sigma^2}{m_\sigma} \int \frac{\hat{k} \cdot \vec{\nabla}_v f_{\sigma,0}}{p + i\vec{v} \cdot \vec{k}} d^3\vec{v} \tag{6.31}$$

The N stands for numerator, and D stands for denominator. Actually, my $N(p)$ and $D(p)$ differ slightly from those derived in class and in Bellan's book. This is because I chose p as my Laplace variable, as opposed to $i\omega$. I also solved for $\widetilde{\vec{E}}_1$, as opposed to $\widetilde{\phi_1}$ as is done in Bellan.

I should emphasize that $\widetilde{E}_1$ is now known! Or at least, in principle it is known for a given p, since we known $f_{\sigma,0}$ and $f_{\sigma,1}(t=0)$.

A lot has happened, so we're going to take a break here and recap what we've done. We started with the Vlasov-Maxwell equation. We linearized around a homogenous, zero-field equilibrium assuming that whatever perturbations were created would be electrostatic in nature. The equation which remains can't be solved purely with Fourier transforms - that is the approach used in chapter 5, when we found that density blows up when $v_\parallel = \frac{\omega}{k}$. Instead, we've taken a

different approach, by Fourier transforming in space but Laplace transforming in time. This allows us to solve for the first-order electric field in terms of the *initial value* of the perturbed f. Lastly, we Laplace transform Gauss's law and combine that with the linearized Laplace transformed Vlasov-Maxwell equation to solve for E_1.

6.3 Landau Contours and All That Jazz

We're going to attempt to solve for $\vec{E}_1$. If we can solve for $\vec{E}_1$, then in principle we have everything we need to solve for the time-evolution of f.

Well, it turns out that solving for $\vec{E}_1$ is easy, at least in principle. All we have to do is take the inverse Laplace transform of $\widetilde{E_1}$. From equation 6.8, we have

$$\vec{E}_1 = \frac{\hat{k}}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{N(p)}{D(p)}e^{pt}dp \tag{6.32}$$

Take a look at figure 22. This represents the integration in p-space taken to solve for $\vec{E}_1$. The x's represent the poles of $\frac{N(p)}{D(p)}$. $\frac{N(p)}{D(p)}$ isn't defined in the shaded red area, as the real part of p in this area is less than the real part of the fastest growing pole of $\widetilde{E_1} = \frac{N(p)}{D(p)}$. β, as you might have realized, is greater than the real part of any poles of $\frac{N(p)}{D(p)}$.[30]

Now, the integral in equation 6.32 can be completed using the method of analytic continuation. As a reminder, analytic continuation will define $\frac{N(p)}{D(p)}$ in the red region in figure 22, so that the integral can be extended into the left half of p-space, and the integral can be solved using method of residues. If we want $\frac{N(p)}{D(p)}$ to remain analytic over an integration path like that of the Bromwich contour in figure 21, we'll need to make sure that $N(p)$ and $D(p)$ both remain analytic as we extend these functions into the left half of p-space. It turns out that keeping these functions analytic is a bit tricky, and requires the use of these funny integration paths called Landau contours.

Remember our definitions of $N(p)$ and $D(p)$ from before.

$$N(p) = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \int \Big(\frac{f_{\sigma,1}(\vec{x},\vec{v},t=0)}{p+i\vec{v}\cdot\vec{k}}\Big)d^3\vec{v} \tag{6.33}$$

$$D(p) = ik + \frac{1}{\epsilon_0}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int \frac{\hat{k}\cdot\vec{\nabla}_v f_{\sigma,0}}{p+i\vec{v}\cdot\vec{k}}d^3\vec{v} \tag{6.34}$$

[30] Bellan actually writes this incorrectly in his book - he says "β is chosen to be larger than the fastest growing exponential term in $N(p)/D(p)$." However, this is wrong - there are two correct, equivalent ways of writing this. The first is as I've written it here - that β is greater than the real part of any poles of $\frac{N(p)}{D(p)}$. The second correct way of writing it would be that β is greater than the fastest growing exponential in $E_1(t)$, γ. Note that there are two ways of writing it because the fastest growing exponential term in $E_1(t)$ is the same as the real part of the largest pole in $\widetilde{E}_1(p)$, as I discussed in an example earlier in this chapter.

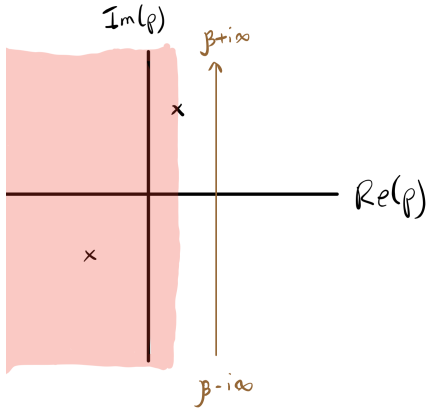


Figure 22: The integration path taken in equation 6.32. The x's represent the poles of $N(p)/D(p)$, which is being integrated here. Here, β is larger than the largest real component of the poles of $N(p)/D(p)$. The red area represents the region in p-space where the Laplace transform of E_1 is not defined, and thus where the integration path cannot go.

We can integrate over the two directions perpendicular to $\vec{k}$, and rewrite these expressions as follows:

$$N(p) = \frac{1}{ik\epsilon_0} \sum_\sigma q_\sigma \int_{-\infty}^{\infty} \left(\frac{F_{\sigma,1}(\vec{x}, \vec{v}, t=0)}{v_\parallel - \frac{ip}{k}}\right) dv_\parallel \tag{6.35}$$

$$D(p) = ik + \frac{1}{ik\epsilon_0} \sum_\sigma \frac{q_\sigma^2}{m_\sigma} \int_{-\infty}^{\infty} \frac{\frac{\partial F_{\sigma,0}}{\partial v_\parallel}}{v_\parallel - \frac{ip}{k}} dv_\parallel \tag{6.36}$$

where $F_{\sigma,1} = \int f_{\sigma,1} d^2\vec{v}_\perp$ and $F_{\sigma,0} = \int \frac{\partial f_{\sigma,0}}{\partial v_\parallel} d^2\vec{v}_\perp$. As these integrals are constructed, we integrate along the real axis. Each of the integrals has a pole at $v_\parallel = \frac{ip}{k}$, but as long as $\mathrm{Re}(p)$ doesn't equal zero, this isn't a problem for us because the integration path doesn't go over the pole.[31]

I spoke too soon - this is a problem for us. Remember what we want to do - we want to extend the definition of $N(p)$ and $D(p)$ into the left half of p-space, such that $N(p)$ and $D(p)$ remain analytic. At $\mathrm{Re}(p) = 0$, then $\mathrm{Im}(p) \neq 0$, so our integral is now integrating over a pole! This will create a discontinuity in $N(p)$ and $D(p)$ at $\mathrm{Re}(p) = 0$ - there will be a jump in the value of the integral between $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(p) < 0$. And a discontinuity in our analytic continuation at $\mathrm{Re}(p) = 0$ means our analytic continuation is no longer analytic over the Bromwich contour, which means we can't use the method of residues to evaluate the integral.

[31] Let's assume $k > 0$ for simplicity. The sign of k doesn't change the result or the interpretation of course.

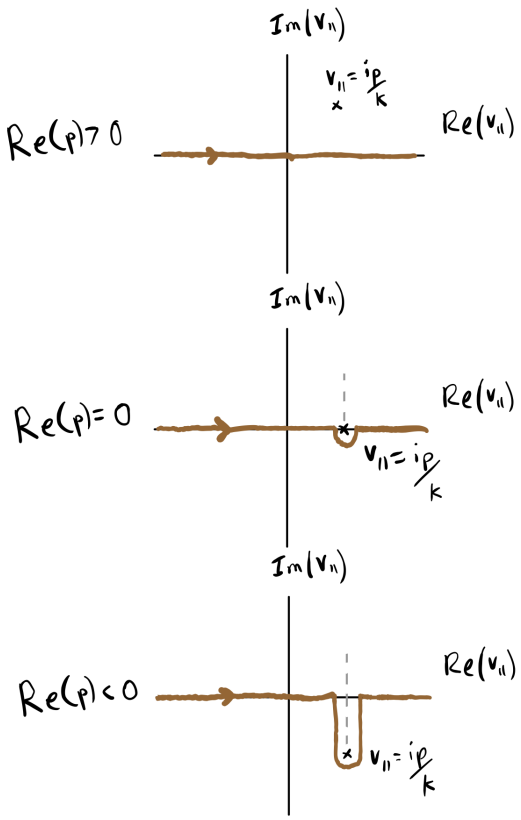


Figure 23: As Re(p) drops below 0, the integration paths in $N(p)$ and $D(p)$ are deformed to prevent the pole from crossing the integration path, so that $N(p)$ and $D(p)$ each remain analytic.

As Landau does, Landau found a clever solution. We can make $N(p)$ and $D(p)$ analytic if we are willing to wander off the real axis during our integration. Take a look at figure 23. Once $\text{Re}(p) = 0$, the integration path drops below the real axis as shown in the figure. As Re(p) decreases below 0, we deform the integration path further below the imaginary axis to prevent the pole from crossing over the contour integral and creating a discontinuity in $N(p)$ or $D(p)$. These deformed contours are called Landau contours.

Now, this is as far as we got in class in regards to Landau damping. Unfortunately, this is a very unsatisfying point to stop in regards to Landau damping. All we've done is some complex analysis and gotten an expressions for $\vec{E}_1$, which supposedly we know how to evaluate in principle. We haven't gotten any understanding of the physics behind Landau damping, or reaped the benefits of the calculations we've done. Fortunately, you (and I!) will have the pleasant expe-

rience of taking AST553, Plasma Waves and Instabilities, where we will revisit Landau damping in great depth. Hopefully, at the conclusion of that course, we'll get to the important stuff: the physics. But for now, have patience. There is much still for us to learn about this game we call Plasma Physics.