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Generalized Paranormal Operators

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***Abstract:** AN operator T ∈B (H) is said to be generalized P-paranormal if*

$$\| |T|^p U\, |T|^p x\ \| \|x\| \geq \frac{1}{M^p} \| |T|^p x\|^2$$

For all $x \in H$, $p > 0$, and $M > 0$, where U is the partial isometry appeared in the polar decomposition $T = U|T|$ of T. The aim of this note is to obtain some structure theorem for a class of generalized P-paranormal operators. Exactly we will give some conditions which are generalization of concepts of generalized paranormal operators.

***Keywords:** Paranormal Operators, Hilbert Space, Hypo Normal Operator, Log-Hypo Normal Operators, and Bounded Linear Operator.*

INTRODUCTION

Let H be an infinite dimensional complex Hilbert and B (H) denote the algebra of all bounded linear operators acting on H. Every operator T can be decomposed into T = U$|T|$with a partial isometry U, where $|T|$ is the square root of T $*$ T. If U is determined uniquely by the kernel condition ker (U) = ker$|T|$, then this decomposition is called the polar decomposition, which is one of the most important results in operator theory. In this paper, T = U$|T|$ denotes the polar decomposition satisfying the kernel condition ker (U) = ker($|T|$).

Recall that an operator T ∈B (H) is positive, T ≥ 0, if $(Tx, x) \geq 0$ for all $x \in H$. An operator T ∈ B(H) is said to be hyponormal if T*T ≥ TT*. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators. An operator T is said to be p-hyponormal if $(T*T)^p \geq (TT*)^p$ for p ∈(0, 1] and an operator T is said to be log-hyponormal if T is invertible and log$|T|$ ≥ log |T*|. P-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. An operator T ∈B (H) is said to be paranormal if it satisfies the following norm inequality

$$\|T^2 x\| \geq \| |Tx\|^2$$

For every unit vector $x \in H$. Ando [3] proved that every log-hyponormal operators is paranormal. It was originally introduced as an intermediate class between hyponormal operators and normaloid. It has been studied by many authors, so there are many to cite their references, for instance [3, 9, 22]. We say that an operator T belong to class A if $|T^2| \geq |T|^2$. Class A was first introduced by Furuta-Ito-Yamazaki as a subclass of paranormal which include the class of p-hyponormal and log-hyponormal operators.

DEFINITION 1.1

A bounded linear operator T on H is called generalized n-paranormal operator if for every unit vector $x \in H$, M > 0 and a positive integer n such that $n \geq 2$, T satisfies $\|T^n x\| \geq \frac{1}{M^{\frac{n}{2}}} \|Tx\|^n$

DEFINITION 1.2

Let T ∈B(H), An operator T belongs to generalized class A operator if for M> $0, T$ Satisfies

$$|T^2| \geq \frac{1}{M} |T|^2$$

Rai [18] has defined a bounded operator T on a Hilbert space H as generalized paranormal if for every unit vector $x \in H$ and M> $0, T$ Satisfies
$\|T^2 x\| \geq \frac{1}{M} \|Tx\|^2$. He also proved a result for every unit vector x,

$$\|T^{k+1} x\|^2 \geq \frac{1}{M^{2k-1}} \|T^k x\|^2 \|T^2 x\|$$

Where T is a bounded linear operator H, M>0 and k≥1

On the basis of the above result, we define the generalized n-paranormal operator as follows.

THEOREM 1.3

If T satisfies $|T^n|^{\frac{2}{n}} \geq \frac{1}{M}|T|^2$ for some positive integer n such that n≥2 and m>0, then T is a generalized n-paranormal operator.

LEMMA 1.4 Holder – Mccarthy inequality

Let T be a positive operator. Then the following inequalities hold for all $x \in H.$

(i) $< T^r x, x > \leq < Tx, x >^r \|x\|^{2(1-r)}$ $for\ 0 < r \leq 1$

(ii) $< T^r x, x > \geq < Tx, x >^r \|x\|^{2(1-r)}$ $for\ r \geq 1$

Proof:

Suppose T satisfies $|T^n|^{\frac{2}{n}} \geq \frac{1}{M}|T|^2$ -------- (B-1)

for some positive integer n such that n≥2 and m>0, Then for every unit vector $x \in H.$

$$\|T^n\|^2 = < |T^n|^2 x, x >$$
$$\geq < |T^n|^{\frac{2}{n}} x, x >^n$$
$$\geq < \frac{1}{M}|T|^2 x, x >^n \quad \text{---- (B-1)}$$
$$\geq < \frac{1}{M^n}|T|^2 x, x >^n$$
$$\geq \frac{1}{M^n}\|Tx\|^{2n}$$

Hence we have $\|T^n\| \geq \frac{1}{M^{\frac{2}{n}}}\|Tx\|^n$ for every unit vector $x \in H$

THEOREM 1.5

Let T ∈B(H). If T is generalized k -quasi – hyponormal then T is generalized (k+1) -paranormal.

Proof:

If T is a generalized k – quasi hyponormal then the following relation holds for every unit vector $x \in H$

$$\|T^{k+1}x\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\|T^*T^k x\| \quad \text{---- (B-1)}$$

To prove T is generalized (K+1) -paranormal it suffices to prove that

$$\|T^{k+1}x\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\|Tx\|^{k+1}$$

We know that for any bounded linear operator T on a Hilbert space H.

$$\|Tx\|^{k+1} \leq \|T^*T^k x\| \quad \text{---------- (B-3)}$$

Therefore from (B-2) and (B-3) we get

$$\|T^{k+1}x\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\|Tx\|^{k+1}$$

Hence T is a generalized (k+1) -paranormal operator.

THEOREM 1.6

Let 0<p<1. Every Generalized p-paranormal operator is generalized paranormal.

Proof:

We note that the Holder inequality by Mccarthy (i) of Lemma 1.4 has the following form.

$$\|S^p y\| \leq \|Sy\|^p \|y\|^{1-p}$$

For all $y \in H.$ Putting $S = |T|$ and $y = U|T|^p$in part (ii) of Lemma 1.4

We have

$$\||T|^p U\ |T|^p x\| \leq \||T|\ U|T|^p x\|^p \||T|^p x\ \|^{1-p}$$

Since the left hand side of the above inequality is greater than

$\||T|^p x\|^2$ / $M^p\|x\|$by the generalized absolute p-paranormality. It follows that

$$\||T|^p x\|^{1+p} \leq \||T|\ U|T|^p x\|^p \|x\| \quad \text{--------- (B.4)}$$

Hence, if we replace x by $|T|^{1-p}x$ in (B.4) then

$$\frac{1}{M^p}\|Tx\|^{p+1} \leq \||T|^{1-p}x\|\|T^2 x\|^p$$

Applying part (ii) of lemma 1.4 again it follows that

$$\||T|^{1-p}x\| \leq \|Tx\|^{1-p}\|x\|^p$$

Therefore it implies that

$$\frac{1}{M^p}\|Tx\|^{p+1} \leq \||T|^{1-p}x\|\|T^2 x\|^p$$
$$\leq \|Tx\|^{1-p}\|x\|^p\|T^2 x\|^p$$

so that

$$\frac{1}{M}\|Tx\|^2 \leq \|T^2 x\|\|x\|$$

This completes the proof.

THEOREM 1.7

Let T be a generalized p-paranormal operator, then

$\|T^3x\| \geq \frac{1}{M^2}\|T^2x\|\|Tx\|$For every unit vector $x \in H$

Proof:

For a unit vector x in H. We may assume that $\|Tx\| \neq 0$, we have

$$\|T^3x\| = \|Tx\| \left\|T^2\frac{Tx}{\|Tx\|}\right\|$$
$$\geq \frac{1}{M}\|Tx\| \left\|T\frac{Tx}{\|Tx\|}\right\|^2 \quad \text{(By theorem 1.6)}$$
$$\geq \frac{1}{M}\|Tx\| \left\|\frac{T^2x}{\|Tx\|}\right\|^2$$
$$\geq \frac{1}{M}\frac{\|Tx\|\|T^2x\|^2}{\|Tx\|^2}$$
$$\geq \frac{1}{M^2}\frac{\|Tx\|\|T^2x\| \ \ \|Tx\|^2}{\|Tx\|^2}$$
$$\|T^3x\| \geq \frac{1}{M^2}\|T^2x\|\|Tx\|$$

Hence the theorem.

THEOREM 1.8

Let T be a generalized p-paranormal operator, then

$\|T^4x\| \geq \frac{1}{M^5}\|T^2x\|^2\|Tx\|$For every unit vector $x \in H$

Proof:

For a unit vector x in H. We may assume that $\|Tx\| \neq 0$, we have

$$\|T^4x\| = \|Tx\| \left\|T^2\frac{T^2x}{\|Tx\|}\right\|$$
$$\geq \frac{1}{M}\|Tx\| \left\|T\frac{T^2x}{\|Tx\|}\right\|^2 \quad \text{(By theorem 1.6)}$$
$$\geq \frac{1}{M}\frac{\|Tx\|\|T^3x\|^2}{\|Tx\|^2}$$
$$\geq \frac{1}{M}\frac{\|Tx\|\|T^3x\| \ \ \|T^3x\|}{\|Tx\|^2}$$
$$\geq \frac{1}{M^5}\frac{\|T^2x\|^2\|Tx\|^2}{\|Tx\|} \quad \text{(By Theorem 1.7)}$$
$$\|T^4x\| \geq \frac{1}{M^5}\|T^2x\|^2\|Tx\|$$

THEOREM 1.9

Let T be a generalized P-Paranormal operator, then

$\|T^{k+1}x\|^2 \geq \frac{1}{M^{2k-1}}\|T^kx\|^2\|T^2x\|$ For a positive integer k≥ 1 and every unit vector x in H.

Proof:

We will use induction to establish the inequality.

$$\|T^{k+1}x\|^2 \geq \frac{1}{M^{2k-1}}\|T^kx\|^2\|T^2x\| \quad \text{For a positive integer } k \geq 1 \quad \text{...... (B.5)}$$

In case K=1

$$\|T^2x\|^2 = \|T^2x\| \ \ \|T^2x\| \quad \geq \frac{1}{M^2}\|Tx\|^4 \text{........... (B.6)}$$

hold by theorem (2.4). Now suppose that (B.5) holds for some k≥ 1 and we assume that $\|Tx\| \neq 0$ then

$$\|T^{k+2}x\|^2 = \|Tx\|^2 \left\|T^{k+1}\frac{Tx}{\|Tx\|}\right\|^2$$
$$\geq \|Tx\|^2 \frac{1}{M^{2k-1}} \left\|T^k\frac{Tx}{\|Tx\|}\right\|^2 \left\|T^2\frac{Tx}{\|Tx\|}\right\| \quad \text{(By B.5)}$$
$$\geq \|Tx\|^2 \frac{1}{M^{2k-1}} \left\|\frac{T^{k+1}x}{\|Tx\|}\right\| \ \ \frac{\|T^3x\|}{\|Tx\|}$$
$$\geq \frac{1}{M^{2k-1}}\|T^{k+1}x\|^2 \frac{1}{M^2}\frac{\|T^2x\|\|Tx\|}{\|Tx\|}$$
$$\geq \frac{1}{M^{2k+1}}\|T^{k+1}x\|^2\|T^2x\|$$

That is

$$\|T^{k+2}x\|^2 \quad \geq \frac{1}{M^{2k+1}}\|T^{k+1}x\|^2\|T^2x\| \quad \text{......... (B.7)}$$

For k+3

$$\|T^{k+3}x\|^2 = \|Tx\|^2 \left\|T^{k+2}\frac{Tx}{\|Tx\|}\right\|^2$$
$$\geq \|Tx\|^2 \frac{1}{M^{2k+1}} \left\|T^{k+1}\frac{Tx}{\|Tx\|}\right\|^2 \left\|T^2\frac{Tx}{\|Tx\|}\right\| \text{by (B.7)}$$

$$\geq \|Tx\|^2 \frac{1}{M^{2k+1}} \left\|\frac{T^{k+2}x}{\|Tx\|}\right\|^2 \left\|\frac{T^3x}{\|Tx\|}\right\|$$

$$\geq \|Tx\|^2 \frac{1}{M^{2k+1}} \frac{\|T^{k+2}x\|^2}{\|Tx\|^2} \frac{\|T^3x\|}{\|Tx\|}$$

$$\geq \frac{1}{M^{2k+1}} \|T^{k+2}x\|^2 \frac{1}{M^2} \frac{\|T^2x\|\|Tx\|}{\|Tx\|} \text{ (By theorem 1.7)}$$

$$\|T^{k+3}x\|^2 \geq \frac{1}{M^{2k+3}} \|T^{k+2}x\|^2 \|T^2x\| \text{........ (B.8)}$$

For k+4

$$\|T^{k+4}x\|^2 = \|Tx\|^2 \left\|T^{k+3} \frac{Tx}{\|Tx\|}\right\|^2$$

$$\geq \|Tx\|^2 \frac{1}{M^{2k+3}} \left\|T^{k+2} \frac{Tx}{\|Tx\|}\right\|^2 \left\|T^2 \frac{Tx}{\|Tx\|}\right\| \text{ (By B.8)}$$

$$\geq \|Tx\|^2 \frac{1}{M^{2k+3}} \left\|\frac{T^{k+3}x}{\|Tx\|}\right\|^2 \left\|\frac{T^3x}{\|Tx\|}\right\|$$

$$\geq \|Tx\|^2 \frac{1}{M^{2k+3}} \frac{\|T^{k+3}x\|^2}{\|Tx\|^2} \frac{\|T^3x\|}{\|Tx\|}$$

$$\geq \frac{1}{M^{2k+3}} \|T^{k+3}x\|^2 \frac{1}{M^2} \frac{\|T^2x\|\|Tx\|}{\|Tx\|} \text{ (By Theorem 1.7)}$$

$$\|T^{k+4}x\|^2 \geq \frac{1}{M^{2k+5}} \|T^{k+3}x\|^2 \|T^2x\|$$

The proof is complete.

THEOREM 1.10

Let$T \in B(H)$, If T is a generalized p-paranormal operator T, then T satisfies

$$\|T^{n+1}x\| \geq \frac{1}{M^{\frac{n(n+1)}{2}}} \|Tx\|^{n+1}$$

For any unit vector $x \in H, M > 0$ and positive integer n such that $n \geq 1$.

Proof:

For n=1 the statement is trivial. If the statement is true for n-1 then we have

$$\|T^{n+1}\| = \|Tx\| \left\|T^n \frac{Tx}{\|Tx\|}\right\|$$

$$\geq \frac{1}{M^{\frac{n(n-1)}{2}}} \|Tx\| \left\|\frac{T^2x}{\|Tx\|}\right\|^n$$

$$\geq \frac{1}{M^{\frac{n(n-1)}{2}}} \|Tx\| \frac{1}{M^n} \frac{\|Tx\|^{2n}}{\|Tx\|^n}$$

$$\geq \frac{1}{M^{\frac{n(n-1)}{2}}} \|Tx\|^{n+1}$$

Hence the Lemma.

CONCLUSION

Operator theory plays a vital role in the development of applied science and other related areas. This process can be extended in different dimensions.

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