Generalized Similarity Transformation Model for Power-law Laminar Boundary Layer Fluids with non-Linear Dynamic Viscosity

JACOB NAGLER
Faculty of Aerospace Engineering, Technion, Haifa 32000,
ISRAEL
syankitx@Gmail.com syanki@tx.technion.ac.il

Abstract:- In this paper, mathematical representation of similarity transformation model has been obtained for the steady laminar boundary layer of non-Newtonian flow with variable dynamic viscosity over a flat plate. The power-law fluid model has been adopted for the non-Newtonian fluid representation. The governing non-dimensional boundary layer equations have been transformed into ordinary differential equations using general similarity transformation. Comparison of different studies transformation models has been made for useful and common special similarity transformation. While excellent compatibility was found. This developed general model can be used as a proof to the current models and basic for future transformation models in a large variety of applications. Moreover, numerical solution procedure using quasi-linearization method for flow equation based on this latter transformation model was developed and presented.

Key-words:- Boundary layer, non-Newtonian fluid, Similarity solution transformation, Quasi-linearization.

1 Introduction

Boundary layer flow has been discussed for many decades. Boundary layer fluids can be divided into two types - Newtonian and non-Newtonian; while most common and useful is the power-law model. In 1955 Metzner [1] presented applications of non-Newtonian flow behavior in chemical engineering. Five years later, Schowalter [2] discussed boundary layer theory for power law fluids.

Some years later, works on non-Newtonian fluid types in different geometries have been published more frequently. For example, Acrivos *et al*. [3] have analyzed fluids flow along a flat plate. Moreover, in 1958 Strivastava [4] investigated the flow of non-Newtonian liquid near stagnation point. Five years later, Kapur and Gupta [5] published their study on 2D non-Newtonian flow in a channel. Three decades later, Garg and Rajagopal [6] have developed model in wedge geometry type for non-Newtonian flow.

Another important issue in boundary layer flow is the boundary condition on the wall. Innovative study about magnetic influence on non-Newtonian laminar flow was done by Shashidar Reddy *et al*. [7]. Additionally, Moallemi *et al.* [8] used homotopy perturbation technique for analyzing non-Newtonian flow in collector. During their research they investigated flow behavior while boundary condition imposed on.

This essay concentrates on general similarity solution transformation of boundary layer equations. Former studies on the subject were done by Sanyal [9], Schlichting [10] and others. During the last four decades the subject was in gradually push; beginning with studies on various similarity solutions for 2D power-law fluids which was done by Kapur *et al*.[11] and also by Hansen [12] for three-dimensional case. Last decade is characterized with various studies on wedge-geometry and other special cases which were done by Pakdemirli [13] and others. Also, similarity solution with compressible flow effect was done by Ludlow *et al*. [14].

This paper continues the author previous study on Newtonian and non-Newtonian fluids. While the former study [15] discussed non-linear dynamic viscosity effect on fluid, this paper object is to formalize generalized similarity transformation model for power-law laminar boundary layer fluids with non- linear dynamic viscosity compared with other studies. Moreover, numerical procedure based on quasi-linearization method has been developed.

2 Boundary Layer Formulation

The power-law rheological law for non-Newtonian shear stress τ_{xy} is described by:

$$\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{1}$$

while (x, y) are the Cartesian coordinates of any point in the flow domain, where x -axis is along the plate and y -axis is normal to it. Flow consistency parameter $K(x, y)$ is considered to be differentiable function of (x, y) coordinates. u represents the velocity component in the positive x direction, and n is the power law index.

The boundary layer continuity and momentum equations in case of 2D laminar flow of incompressible fluid with constant density ρ are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = C(x)\frac{\partial C}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \tag{3}$$

while body forces are neglected. u and v represent the components of the fluid velocity in positive x, y direction and τ_{xy} denotes non-Newtonian shear stress. C is assumed to be known function from the outer inviscid-flow analysis, which derived by Bernoulli's equation. In this study C is assumed to be constant.

The boundary conditions for Eq. (2-3) are:

$$u(x,0) = \frac{U_w}{U_\infty} \tag{4}$$

$$v(x,0) = \frac{V_w}{U_\infty} \tag{5}$$

$$\lim_{u\to\infty} u(x, y) = C \tag{6}$$

While U_w, U_∞, V_w and C are all constants that are not necessarily zero.

Using transformation with stream function $\psi(x, y)$ leads to:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \tag{7}$$

Substituting (7) into Eq. (2-3) yields:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\nu(x, y)\left|\frac{\partial^2 \psi}{\partial y^2}\right|^{n-1}\frac{\partial^2 \psi}{\partial y^2}\right) \tag{8}$$

while dynamic viscosity is defined by $\nu(x, y) = \dfrac{K(x, y)}{\rho}$.

The transformed boundary conditions relations are:

$$u(x,0) = \frac{\partial \psi}{\partial y}(x,0) = \frac{U_w}{U_\infty} \tag{9}$$

$$v(x,0) = \frac{\partial \psi}{\partial x}(x,0) = -\frac{V_w}{U_\infty} \tag{10}$$

$$\lim_{u\to\infty} u(x, y) = C \tag{11}$$

General similarity solution will be defined by the following parameters:

$$\psi(x, y) = H(x) f(\eta), \eta = M(x)G(y) \tag{12}$$

where ψ is the stream function and η is the similarity variable. Also, $H(x), M(x)$ and $G(y)$ are differentiable functions of (x, y), respectively. ψ derivatives would behave according to the following form using (12), by:

$$\begin{cases} \dfrac{\partial \psi}{\partial y} = H(x)M(x)G'(y)f' \\ \dfrac{\partial^2 \psi}{\partial y^2} = H(x)M(x)G''(y)f' + \\ H(x)\left[M(x)G'(y)\right]^2 f'' \\ \dfrac{\partial \psi}{\partial x} = H'(x)f + M'(x)G(y)H(x)f' \\ \dfrac{\partial^2 \psi}{\partial x \partial y} = H(x)M(x)M'(x)G(y)G'(y)f'' + \\ \left[H'(x)M(x) + M'(x)H(x)\right]G'(y)f' \end{cases} \tag{13}$$

Substituting relations (13) into Eq. (8) and dividing by product expression $H(x)M(x)$ leads to the following differential equation:

$$\begin{gathered}
\left[G'(y)f'\right]^2\left[M(x)H(x)\right]' - H'(x)G''(y)ff' \\
- M(x)\left[G'(y)\right]^2 H'(x)ff'' \\
- M'(x)G(y)H(x)G''(y)f'^2 = \\
\frac{\partial}{\partial y}\left\{\begin{matrix} \nu(x,y)\left|\begin{matrix} H(x)M(x)G''(y)f' \\ + H(x)\left[M(x)G'(y)\right]^2 f'' \end{matrix}\right|^{n-1} \\ \left[G''(y)f' + M(x)\left[G'(y)\right]^2 f''\right] \end{matrix}\right\}
\end{gathered} \tag{14}$$

while $H(x)M(x) \neq 0$ for any x value.

Substituting (12) in B.C. (9-11) leads to the following relations:

$$\begin{cases}
f'\left[M(x)G(0)\right]H(x)M(x)G'(0) = \dfrac{U_w}{U_\infty} \\
f\left[M(x)G(0)\right]H'(x) + \\
f'\left[M(x)G(0)\right]M'(x)H(x)G(0) = -\dfrac{V_w}{U_\infty} \\
f'\left[M(x)G(y \to \infty)\right] \\
= \dfrac{C}{H(x)M(x)G'(y \to \infty)}
\end{cases} \tag{15}$$

In order to define specific solution for physical problem, one should define the following functions: $\left[H(x), M(x), G(y)\right]$ according to its physical and mathematical nature. In the next section $G(y)$ function and viscosity $\nu(x,y)$ will be evaluated using B.C. (15).

3 $G(y)$ function evaluation using B.C.

Specific argument value input for f and f' will be achieved if $G(0)$ and $G(y \to \infty)$ would be converged to zero or diverged to infinity, respectively. Examination of these conditions together with other B.C. simultaneously is presented in Table 1. Realization of Table 1 leads to the following polynomial function:

$$G(y) = y \tag{16}$$

Viscosity function representation is formulated generally by:

$$\nu(x,y) = \varphi(\eta) \tag{17}$$

while $\eta = yM(x)$. Dealing with $H(x)M(x)$ functions evaluation will be the topic of next section.

4 $H(x), M(x)$ function evaluation using Eq. (14) coefficients analysis

Firstly, Simplifying Eq. (14) will be done by (19-20) relations substitution into (14), which yields:

$$\begin{gathered}
\left(f'\right)^2\left[M(x)H(x)\right]' - M(x)H'(x)ff'' = \\
M(x)\frac{\partial}{\partial \eta}\left\{\begin{matrix} \varphi(\eta)\left|H(x)\left[M(x)\right]^2 f''\right|^{n-1} \\ \left[M(x)f''\right] \end{matrix}\right\}
\end{gathered} \tag{18}$$

Development Eq. (18) derivatives and dividing the obtained equation by $\left[H(x)\right]^{n-1}$, leads to:

$$\begin{gathered}
\left[\frac{M'(x)H(x) + M(x)H'(x)}{\left[H(x)\right]^{n-1}\left[M(x)\right]^{2n}}\right]\left(f'\right)^2 - \frac{M(x)H'(x)}{\left[H(x)\right]^{n-1}\left[M(x)\right]^{2n}}ff'' \\
= \frac{\partial}{\partial \eta}\left[\varphi(\eta)\left|f''\right|^{n-1} f''\right]
\end{gathered} \tag{19}$$

while $y = \eta / M(x)$.

At this point, coefficients evaluation analysis will be done using Table 2 below.

Table 1: G **function behavior examination**

B.C.	Necessary conditions for constant B.C.
$f'\left[M(x)G(0)\right]G'(0)=\frac{U_w}{U_\infty}=\text{constant}$	$G(0)=0$ or $G(y\to 0)\to\infty$ and $G'(0)=\text{constant}\neq 0$
$f\left[M(x)G(0)\right]H'(x)+f'\left[M(x)G(0)\right]H(x)G(0)=-\frac{V_w}{U_\infty}=\text{constant}$	$G(0)=0$ & $H'(x)=\text{constant}$
$f'\left[M(x)G(y\to\infty)\right]G'(y\to\infty)=C=\text{constant}$	$G(y\to\infty)\to\infty$ and $G'(y\to\infty)=\text{constant}\neq 0$
G limitations can be summarized by: $G(0)=0$ and $G'(0)\neq 0$, $G(y\to\infty)\to\infty$ and $G'(y\to\infty)\neq 0$.	

Table 2: $H(x)$, $M(x)$ **functions evaluation using Eq. (14) coefficients analysis**

Coefficients relations	$\left(f'\right)^2$ coefficient zero equalization	ff'' coefficient zero equalization	Equalization between coefficients
Obtained equations	$\frac{M'(x)H(x)+M(x)H'(x)}{\left[H(x)\right]^{n-1}\left[M(x)\right]^{2n}}=0$	$\frac{\left[M(x)\right]^{1-2n}H'(x)}{\left[H(x)\right]^{n-1}}=0$	$\frac{M'(x)H(x)+M(x)H'(x)}{\left[H(x)\right]^{n-1}\left[M(x)\right]^{2n}}=\frac{M(x)H'(x)}{\left[H(x)\right]^{n-1}\left[M(x)\right]^{2n}}$
Relation between $H(x)$ and $M(x)$	$H(x)=\frac{\alpha}{M(x)},\ \alpha\neq 0$.	$H(x)=\text{constant}\neq 0$ while $M(x)\neq 0$. Impossible to solve since B.C. are non-existed.	$M(x)=\text{constant}\neq 0$ while $H(x)\neq 0$. Impossible to solve since B.C. are non-existed.
Flow equation – Step 1	$-\alpha^{1-2n}\left[H(x)\right]^n H'(x)ff''=\frac{\partial}{\partial\eta}\left[\varphi(\eta)\left\|f''\right\|^{n-1}f''\right]$		
$H(x)$ and $M(x)$ functions evaluation	$H'(x)\left[H(x)\right]^n=\text{constant}=\alpha_1 \to H(x)=\left[Q(x)\right]^{z(n)}$ while $z(n)$ is dependent on n power only.		
Flow equation – Step 2	$\alpha^{1-2n}\alpha_1 ff''+\frac{\partial}{\partial\eta}\left[\varphi(\eta)\left\|f''\right\|^{n-1}f''\right]=0$		

It can be inferred from Table 2 analysis that $H(x)$ form is:

$$H(x)=\left[Q(x)\right]^{z(n)}=\frac{\alpha}{M(x)},\ \alpha\neq 0 \quad (20)$$

3 $H(x)$ function examination

According to the previous section, it was proved that similarity solution does exist only for specific ratio between $H(x)$ and $M(x)$ function. In this section, evaluation of the appropriate $H(x)$ function that fulfills those specific B.C. will be done.

In many applications of fluid mechanics $M(x)$ and $H(x)$ are given by inverse ratio multiplied by constant [16-22]. Application of this case involved with adequate comparison between partial representative references will be done in section 4. General flow differential equation (see also Table 2) and adequate B.C. are obtained by substituting relations (16-17) and (20) into Eq. (14), by:

$$\alpha^{1-2n}\underbrace{H'(x)\left[H(x)\right]^{n}}_{\alpha_1}ff''+\frac{\partial}{\partial\eta}\left[\varphi(\eta)\left|f''\right|^{n-1}f''\right]=0 \quad (21)$$

while α,α_1 are constants.

$$f'(0)=\frac{U_w}{U_\infty}\frac{1}{\alpha}$$
$$f(0)=-\frac{V_w}{U_\infty}\frac{1}{H'(x)} \quad (22)$$
$$f'(\eta\to\infty)=\frac{C}{\alpha}$$

Eq. (21) should be dependent on one variable only (η). For this purpose, adequate $H(x)$ function should be defined using the following x coordinate's coefficients equalization:

$$H'(x)\left[H(x)\right]^{n}=\alpha_1 \quad (23)$$

Solution of Eq. (23) is obtained by substituting the following expression:

$$H(x)=\left[Q(x)\right]^{z(n)} \quad (24)$$

while $z(n)$ is dependent on n power only and $n>0\in\Re$. Substituting (24) into (23) together with coefficients equalization yields the following solution:

$$H(x)=\left(Ax+b_1\right)^{\frac{1}{n+1}} \quad (25)$$

while $Q(x)=x+b_1,\ z(n)=\frac{1}{n+1}$ and $A=\alpha_1\left(n+1\right)$.

Applying this procedure on Eq. (21) leads to the following flow equation:

$$\alpha^{1-2n}\alpha_1 ff''+\frac{\partial}{\partial\eta}\left[\varphi(\eta)\left|f''\right|^{n-1}f''\right]=0 \quad (26)$$

Analyzing second boundary condition (22) reveals that $f(0)$ is dependent on $H'(x)$ instead of being constant. Accordingly, there are two possible options for solution representation:

Option 1:
Assuming $V_w=0$.
Using this assumption doesn't obligate $H(x)$ behavior. Nevertheless, $H(x)$ will be defined by relation (25).
Option 2:
Forcing $H'(x)=\text{constant}$, leads to the following form of $H(x)=a_2x+b_2$, while $a_2,b_2\in\Re$. This expression coincides with former relation (25) for Newtonian case only ($n=1$). For non-Newtonian case only transformation (25) is valid.

In conclusion, the final similarity transformation has the following form for $M(x)=\frac{\alpha}{H(x)}$:

$$\psi=\left(Ax+b_1\right)^{\frac{1}{n+1}}f(\eta)$$
$$\eta=\alpha\frac{y}{\left(Ax+b_1\right)^{\frac{1}{n+1}}} \quad (27)$$

With the following B.C.:

$$f'(0)=\frac{U_w}{U_\infty}\frac{1}{\alpha}$$
$$f(0)=0 \tag{28}$$
$$f'(\eta\to\infty)=\frac{C}{\alpha}$$

Comparisons to other studies similarity transformation model will be made in the next section.

5 Applications of general similarity transformation (24) with study comparisons

Comparison between similarity transformation models is presented in Table 3. On the one hand, all similarity models obey to the general formulation (26). On the other hand, there are differences (usually slight) in model parameters in each study reference.

Table 3: Comparisons between Ref. [16-22] and general model $\psi=H(x)f(\eta),\ \eta=\alpha\dfrac{G(y)}{H(x)}$

Ref. /Functions	$H(x)$	$G(y)$	Constants comparison with relations (25)
M. Benlahsen *et al.* (2008) and G. BOGNÁR (2011)	$A_1x^{-\beta_1}$	y	$\eta=A_2x^{-\beta_2}y$ $A_1A_2=U_\infty$ and $\beta_2=-\beta_1$ which sustains: $H(x)=\dfrac{1}{M(x)}$
K. B. Pavlov and A.P. Shakhorin (2007), G. V. Zhizhin (1987) and K. B. Pavlov *et al.*(1982)	$x^{\frac{1}{n+1}}$	y	$\mathrm{Re}_x=\dfrac{U_\infty^{2-n}x^n}{k/\rho}$ $\alpha=\left[\dfrac{n(n+1)}{\rho/k}U_\infty^{2n-1}\right]^{-\frac{1}{n+1}},\ A=1,\ b_1=0$
Teipel (1974) and C. C. Hsu (1969)	$x^{\frac{1}{n+1}}$	y	$\alpha=\left[\dfrac{n(n+1)}{\rho/m}U_\infty^{2n-1}\right]^{-\frac{1}{n+1}},\ A=1,\ b_1=0$

Note that η power denominator expression appears with $\frac{1}{n}$ while in the table references above it appears with $\frac{1}{n+1}$. The distinction derives from the absolute value states development as was done here (24). Variable viscosity function $\varphi(\eta)$ is discussed broadly in previous study [15]. In the next section numerical approach solution will be examined.

6 Numerical approach solution for flow Eq. (26) with B.C. (28)

Recently Numerical procedure studies on this field were developed by Jhankal [23] and also Bognár & Csáti [24]. Their researches were involved with laminar boundary layer flow of a power law fluid on a moving plate in various applications (presence of magnetic field and non-Newtonian media, respectively). During their studies they used numerical iterative methods. Jhanakal used Runge-Kutta-Fehlberg Forth-Fifth order method while Bognár & Csáti used iterative transformation method (ITM). Also, Puttkammer [26] was solved the problem of classical boundary layer by using the shooting method and explicit discrete method. Moreover, quasi-linearization approach was applied by Lee [27], Huang [28] and Naikoti & Borra [29] on boundary layer with thermal effect in different studies.

Similarly to these models, numerical solution procedure will be developed in this section for flow Eq. (26) together with B.C (28).

Firstly, Eq. (26) full development presentation (for $f''>0$) would be described by:

$$\alpha^{1-2n}\alpha_1 ff''+\varphi'\left(f''\right)^n+n\varphi\left(f''\right)^{n-1}f'''=0 \quad (29)$$

Isolation of f''' from Eq. (29) would be helpful for quasi-linearization method numerical representation.

Hence, f''' would be performed by:

$$f'''=-\frac{\alpha^{1-2n}\alpha_1 f\left(f''\right)^{2-n}+\varphi' f''}{n\varphi} \quad (30)$$

Note that for $n=1/2$, α constant is non-exist. Also, $f''\neq 0$.

From here, Eq. (29-30) will be divided into two main cases of power-law index range with appropriate numerical solution using Table 4. According to Table 4, $x_3(0)$ should be guessed until solution is converged (Shooting Method). Solution can be done for example by using BVP4C function in MATLAB program or instead, using directly the following Euler forward numerical method of order $O(h^2)$ as follows [25-26]:

$$\begin{aligned}
f'(\eta_i)&=\frac{f(\eta_{i+1})-f(\eta_i)}{h}\\
f''(\eta_i)&=\frac{f'(\eta_{i+1})-f'(\eta_i)}{h} \qquad (31)\\
f'''(\eta_i)&=\frac{f''(\eta_{i+1})-f''(\eta_{i+1})}{h}=\\
&-\frac{\alpha^{1-2n}\alpha_1 f(\eta_i)\left[f''(\eta_i)\right]^{2-n}+\varphi' f''(\eta_i)}{n\varphi(\eta_i)}
\end{aligned}$$

while i is the grid index. h is grid space-gap and N is the grid length. In order to solve this equation one should guess $f''(\eta_i=0)$. More approximate or numerical discrete methods are presented by Bognár [16, 24], Puttkammer [26] and others.

Table 4: Quasi –linearization numerical procedure application of Eq. (29) & B.C.(28)

Quasi – linearization numerical procedure	B.C.
$\left[x_1,x_2,x_3\right]=\left[f,f',f''\right]$	$f(0)=x_1(0)=0$
$x_1=f$	$f'(0)=x_2(0)=\frac{U_w}{U_\infty}\frac{1}{\alpha}$
$x_1'=x_2=f'$	
$x_2'=x_3=f''$	$\lim_{\eta\to\infty}f'(\eta)=\lim_{\eta\to\infty}x_2(\eta)=\frac{C}{\alpha}$
$x_3'=-\frac{\alpha^{1-2n}\alpha_1 x_1\left(x_3\right)^{2-n}+\varphi' x_3}{n\varphi}=f'''$	$f'''(0)=x_3(0)=\text{Guess}$

7 Conclusion

Generalized similarity transformation model for power-law laminar boundary layer fluid with non-linear dynamic viscosity was developed. Excellent compatibility was found with other similarity transformation models.

Moreover, numerical solution was developed to general flow equation using quasi – linearization numerical procedure with appropriate Euler forward numerical relations.

This current model can be used as a proof to current problems and a basic model for future transformation models in a large variety of applications by examining other $H(x)$ function parameters (α_1, b_1) and variety of B.C. parameters $(U_w, U_\infty, C, \alpha)$.

References

[1] A. B. Metzner, Non-Newtonian Technology in Advances in Chemical Engineering, *Academic Press - New York*, Vol. 7, 1955.

[2] W.R. Schowalter, The application of boundary-layer theory to power law pseudoplastic fluids: similar solutions, *A.I.Ch.E. J.*, Vol. 24, No. 6, 1960.

[3] A. Acrivos, M.J. Shah, and E.E. Petersen, Momentum and Heat Transfer in Laminar Boundary-Layer Flows of Non-Newtonian Fluids past External Surface, *A. I. Ch. E. J.* Vol. 6, No. 2, 1960, pp. 312–317.

[4] A. C. Srivastava, The flow of non-Newtonian liquid near a stagnation point. *Z.A.M.P.*, Vol. 80, No. 9, 1958.

[5] J. N. Kapor and R. C. Gupta, *ZAMM* , Vol.135, No. 43, 1963.

[6] V. K. Garg and K. R. Rajagopal, Flow of a non-Newtonian fluid past a wedge, *Acta Mech*. 88, 1991, pp. 113.

[7] B. Shashidar Reddy, N. Kishan, and M. N. Rajasekhar, MHD boundary layer flow of a non-newtonian power-law fluid on a moving flat plate, *Pelagia Research Library*, *Advances in Applied Science Research* 3, 2012, pp. 1472-1481.

[8] N. Moallemi, I. Shafieenejad, M. Jadidi, and H. Davari, Homotopy Perturbation Technique to Analysis non-Newtonian Fluid Flow in Collector, *J. Basic. Appl. Sci. Res.*, 2(1), 2012, pp. 625-633.

[9] L. Sanyal, On Similarity Solutions of Prandtl's Boundary Layer Equations, Vol. 21, No.1, *Calcutta University*, 1954 ,pp. 7-9.

[10] H. Schlichting, Boundary Layer Theory, *Pergamon Press*, New York, 1955.

[11] J.N. Kapur and R. C. Srivastava, Similar Solutions of the Boundary Layer Equations for Power Law Fluids, *Kurze Mitteilungen - Brief Reports - Communications bre'ves* ,Vol. 14, 1963.

[12] T.Y. Na and A.G. Hansen, Similarity Solutions of a Class of Laminar Three-Dimensional Boundary Layer Equations of Power Law Fluids, *Int. J. Non-Linear Mechanics. Pergamon Press*, Vol. 2, 1967, pp. 373-385.

[13] M. Pakdemirli, Similarity Analysis of Boundary Layer Equations of a Class of non-Newtonian Fluids, *Int. J. Non-Linear Mechanics*. Vol. 29, No. 2, 1994, pp. 187-196.

[14] D. K. Ludlow, P. A. Clarkson, A. P. Bassom, New Similarity Solutions of the Unsteady Incompressible Boundary-Layer

Equations, *Q.J Mech. App. Math*. Vol. 53, 2000, pp. 175-206.

[15] J. Nagler, Laminar boundary layer model for power-law fluids with non-linear viscosity, WSEAS Transactions on Fluid Mechanics, 2014, Vol. 9.

[16] G. BOGNÁR , Analytic Solutions to a Boundary Layer Problem for Non-Newtonian Fluid Flow Driven by Power Law Velocity Profile, *WSEAS TRANSACTIONS on FLUID MECHANICS*, Issue 1, Vol. 6, 2011, pp. 22-31.

[17] M. Benlahsen, M. Guedda, and R. Kersner, The Generalized Blasius Equation Revisited, *Elsevier - Mathematical and Computer Modeling. J.*, Vol. 47, 2008, pp. 1063-1076.

[18] K. B. Pavlov and A.P. Shakhorin, On the Spatial Localization of the Laminar Boundary Layer in a non-Newtonian Dilatant Fluid, *Fluid Dynamics. J.*, Vol. 42, No. 1, 2007, pp. 44-49.

[19] G. V. Zhizhin, Laminar Boundary Layer In A non-Newtonian Fluid – Qualitative Discussion, Plenum, No. 3,1987, pp. 383-391.

[20] K. B. Pavlov, I. A. Fedotov, and A.P. Shakhorin, Structure Of Laminar Boundary Layer In non- Newtonian Dilatant Fluids, *Springer - Fluid Dynamics. J.*, Vol. 16 ,No. 4, 1981, pp. 605-608.

[21] I. Teipel, Discontinuities In Boundary Layer Problems of Power Law Fluids, *Pergamon Press*, Vol. 1, 1974, pp. 269-273.

[22] C. C. Hsu, A Simple Solution for Boundary Layer Flow of Power Law Fluids Past a Semi-infinite Flat Plate, *A.I.Ch.E. J.*, Vol. 15, No. 3, 1969, pp. 367-370.

[23] A. K. Jhankal, A Similarity Solution for Laminar Boundary Layer Flow of A Power-Law Fluid On A Moving Flat Plate in The Presence Of Transverse Magnetic Field. *International Journal of Mathematical Archive*, Vol. 4, No. 4, 2013, pp. 80-83.

[24] G. Bognár, and Z. Csáti, Numerical Solution to Boundary Layer Problems over Moving Flat Plate in Non-Newtonian Media. *Journal of Applied Mathematics and Physics*, Vol. 2, 2014, pp. 8-13.

[25] Numerical Analysis Notes – Numerical Differentiation, *University of Sulaimani –Faculty of Science – Physics Department*, Ch. 5., 2011.

[26] P. P. Puttkammer, *Boundary Layer over a Flat Plate*, B.Sc. Report, *University of Twente*, 2013.

[27] E. S. Lee , Quasi-linearization, non-linear boundary value problems and optimization, *Chemical Engineering Science*, *Pergamon Press*, Vol. 21, 1966, pp. 183-194.

[28] C. L. Huang, Application of Quasi-Linearization Technique to the Vertical Channel Flow and Heat Convection, *International Journal of Non-Linear Mechanics*, Vol. 13, No. 2, 1978, pp. 55-60.

[29] K. Naikoti, S. R. Borra Quasi-linearization Approach to MHD Effects on

Boundary Layer Flow of Power-Law Fluids Past A Semi Infinite Flat Plate with Thermal Dispersion, *International Journal of Nonlinear Science*, Vol.11, No.3, 2011, pp.301-311.