

GEOMETRIC HYDRODYNAMICS OF COMPRESSIBLE FLUIDS

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ABSTRACT. We develop a geometric framework for Newton's equations on infinite-dimensional configuration spaces to describe numerous fluid dynamical equations. According to V. Arnold, the Euler equations of an incompressible fluid define a geodesic flow on the group of volume-preserving diffeomorphisms of a compact manifold. It turns out that a much greater variety of hydrodynamical systems can be viewed as Newton's equations (adding a potential energy to the kinetic energy Lagrangian) on the group of all diffeomorphisms and the space of smooth probability densities.

This framework encompasses compressible fluid dynamics, shallow water equations, Fisher information geometry, compressible and incompressible magnetohydrodynamics and can be adapted to include relativistic fluids and the infinite-dimensional Neumann problem. Relations between these diverse systems are described using the Madelung transform and the formalism of Hamiltonian and Poisson reduction.

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1. INTRODUCTION

The Euler equations of hydrodynamics describe the motion of an inviscid and incompressible fluid. In the 1960s V. Arnold proposed to regard these equations as equations of a geodesic flow on the group of volume-preserving diffeomorphisms. This observation led to further insights and many developments such as the formulation of new stability criteria for fluid flows, the construction of configuration spaces for a large number of PDEs of hydrodynamic origin, as well as explicit calculations of first integrals and Hamiltonian structures for such equations, the development of differential geometry of diffeomorphism groups, etc. In this paper we continue this line of research and describe a general framework for Newton's equations on diffeomorphism groups. This approach has a wider scope of applicability and includes a much larger class of equations. We also present a survey of other approaches, in particular, those based on the Madelung transform and semidirect product groups and describe their interrelations.

1.1. Zoo of non-dissipative hydrodynamical equations. We begin with an overview of several motivating examples. Groups of diffeomorphisms arise naturally as configuration spaces for flows of compressible and incompressible fluids. Consider a compact connected n -dimensional Riemannian manifold M (for our purposes M can be a domain in \mathbb{R}^n) and assume that it is filled with a non-viscous fluid (either a gas or a liquid). Once the group of smooth diffeomorphisms of M is equipped with the L^2 metric (corresponding essentially to fluid's kinetic energy) its geodesics can be shown to describe the motions of noninteracting particles in M whose velocity field v satisfies the inviscid Burgers' equation

$$\dot{v} + \nabla_v v = 0.$$

Wasserstein-Otto geometry	Fisher-Rao geometry
<i>Newton's equations on Diff(M)</i>	
<ul style="list-style-type: none"> • Inviscid Burgers' equation (§ 4.1) • Classical mechanics (§ 4.2) • Barotropic inviscid fluid (§ 4.4) • Fully compressible fluid (§ 6.1) • Magnetohydrodynamics (§ 6.2) 	<ul style="list-style-type: none"> • μ-Camassa-Holm equation (§ 8.1) • Optimal information transport (§ 7)
<i>Newton's equations on Dens(M)</i>	
<ul style="list-style-type: none"> • Hamilton-Jacobi equation (§ 4.2) • Linear Schrödinger equation (§ 9.2) • Non-linear Schrödinger (§ 9.2) 	<ul style="list-style-type: none"> • ∞-dim Neumann problem (§ 8.2) • Klein-Gordon equation (§ 8.3) • 2-component Hunter-Saxton (§ 9.5)

TABLE 1. Examples of Newton's equations.

Furthermore, the geodesics of the L^2 metric restricted to the subgroup of diffeomorphisms that preserve the Riemannian volume form μ describe the motions of an ideal fluid (that is, inviscid and incompressible) in M whose velocity field satisfies the incompressible Euler equations

$$\begin{cases} \dot{v} + \nabla_v v + \nabla P = 0 \\ \operatorname{div} v = 0. \end{cases} \quad (1.1)$$

The gradient term ∇P in (1.1) is defined uniquely by the divergence-free condition on the velocity field v and can be viewed as a force (pressure) constraining the fluid. (If M has a nonempty boundary then v is also required to be tangent to ∂M).

Similarly, the equations of a compressible (barotropic) fluid describing the evolution of the fluid velocity v and its density ρ , namely

$$\begin{cases} \dot{v} + \nabla_v v + \frac{1}{\rho} \nabla P(\rho) = 0 \\ \dot{\rho} + \operatorname{div}(\rho v) = 0, \end{cases}$$

can be interpreted as Newton's equations on the full diffeomorphism group of the manifold M . In this case the pressure function $P = P(\rho)$ depends on the density ρ .

In what follows we will argue that one can conveniently study these and other equations of mathematical physics including all the examples listed in Table 1 from a unified point of view. We begin by recalling some necessary background.

1.2. Newton's equations on spaces of diffeomorphisms and densities. Given a configuration space Q of some physical system (a Riemannian manifold) and a potential energy $V: Q \rightarrow \mathbb{R}$ (a differentiable function) *Newton's equations* take the form

$$\nabla_{\dot{q}} \dot{q} = -\nabla V(q). \quad (1.2)$$

An infinite-dimensional version of (1.2) on diffeomorphism groups was proposed by Smolentsev [51, 52] who used it to describe the motions of a barotropic fluid. In his study of slightly compressible fluids, Ebin [13] used the same framework to describe the corresponding incompressible limit. Another example was given by von Renesse [57] who showed that the transform introduced by Madelung [35, 36] maps (linear and non-linear) Schrödinger equations to Newton's equations on the space of probability measures (see § 9 below for details).

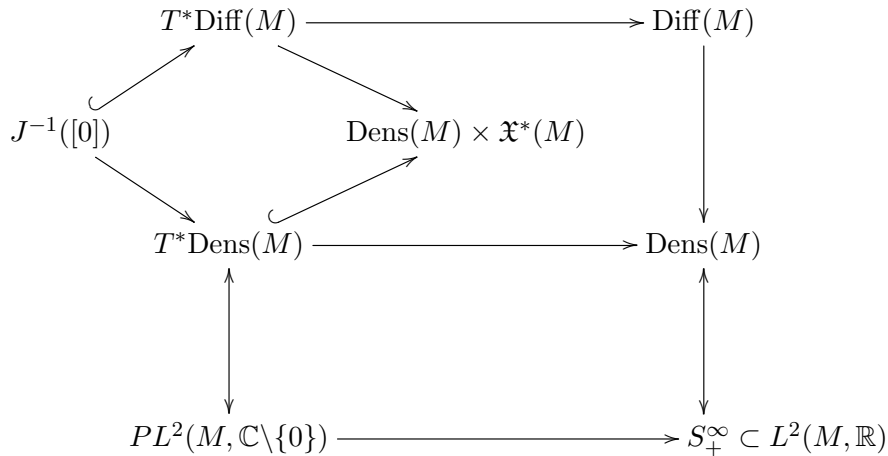


FIGURE 1. Relation between various phase space representations of Newton's equations on $\text{Diff}(M)$ and $\text{Dens}(M)$.

Our goal is to develop a rigorous geometric framework for Newton's equations (1.2) which can be unified with Arnold's differential geometric approach to the incompressible Euler equations to provide a very general setting for systems of hydrodynamical origin on diffeomorphism groups and spaces of densities. More precisely, given a compact n -dimensional manifold M we will equip the group of diffeomorphisms $\text{Diff}(M)$ and the space of non-vanishing probability densities $\text{Dens}(M)$ with the structures of smooth infinite-dimensional tame Fréchet manifolds (see § 3.3 for details) and study Newton's equations on these manifolds viewed as the associated configuration spaces. Using Hamiltonian reduction we will establish a correspondence between various representations of these equations, as summarized by the commutative diagram in Figure 1.

Observe that the right column of Figure 1 describes a natural projection $\pi: \text{Diff}(M) \rightarrow \text{Dens}(M)$ from the full diffeomorphism group to the space of normalized smooth densities on M with fibers that consist of all those diffeomorphisms which push a given reference density to any other density. It was shown by Otto [45] that π is a Riemannian submersion provided that $\text{Diff}(M)$ is equipped with a (non-invariant) L^2 -metric and $\text{Dens}(M)$ with the (Kantorovich-)Wasserstein metric used in the optimal mass transport. Furthermore, in [26] it was also shown that a different Riemannian submersion structure arises when $\text{Diff}(M)$ is equipped with a right-invariant homogeneous Sobolev \dot{H}^1 -metric and $\text{Dens}(M)$ with the Fisher-Rao (information) metric which plays an important role in geometric statistics. This turns out to be closely related to the Madelung transform (cf. the two-sided vertical arrows between two last rows in Figure 1) as we will discuss later in the paper.

Remark 1.1. To describe these settings in more detail let $\text{Diff}_\mu(M)$ be the subgroup of diffeomorphisms preserving the Riemannian volume form μ of M . Consider the fibration of the group of all diffeomorphisms over the space of densities

$$\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \text{Dens}(M),$$

discussed by Moser [41], whose cotangent bundles $T^*\text{Diff}(M)$ and $T^*\text{Dens}(M)$ are related by a symplectic reduction, cf. § 3 below. Moser's construction can be used to introduce two different

algebraic objects: the first is obtained by identifying $\text{Dens}(M)$ with the *left cosets*

$$\text{Diff}(M)/\text{Diff}_\mu(M) = \{\varphi \circ \text{Diff}_\mu(M) \mid \varphi \in \text{Diff}(M)\} \quad (1.3)$$

and the second by identifying it with the *right cosets*

$$\text{Diff}_\mu(M)\backslash\text{Diff}(M) = \{\text{Diff}_\mu(M) \circ \varphi \mid \varphi \in \text{Diff}(M)\}. \quad (1.4)$$

In this paper we will make use of both identifications.

As mentioned above, in order to define Newton's equations on $\text{Diff}(M)$ and $\text{Dens}(M)$ and to investigate their mutual relations we will choose Riemannian metrics on both spaces so that the natural projections π corresponding to (1.3) or (1.4) become (infinite-dimensional) Riemannian submersions. We will consider two such pairs of metrics. In §2, using left cosets, we will study a non-invariant L^2 -metric on $\text{Diff}(M)$ together with the Wasserstein-Otto metric on $\text{Dens}(M)$. In §7, using right cosets, we will focus on a right-invariant H^1 metric on $\text{Diff}(M)$ and the Fisher-Rao information metric on $\text{Dens}(M)$. Extending the results of [57], we will then derive in §9 various geometric properties of the Madelung transform. This will allow us to represent Newton's equations on $\text{Dens}(M)$ as Schrödinger-type equations for wave functions.

Here are some highlights and new contributions of this paper.

- (1) Following [51] and [13] we first revisit the case of the compressible barotropic Euler equations as a Poisson reduction of Newton's equations on $\text{Diff}(M)$ with the symmetry group $\text{Diff}_\mu(M)$ and show that the Hamilton-Jacobi equation of fluid mechanics corresponds to its horizontal solutions §4.4. We then describe the framework of Newton's equations for fully compressible (non-barotropic) fluids §6.1 and magnetohydrodynamics §6.2.
- (2) After reviewing the semidirect product approach to these equations we relate it to our approach in §5.3. We point out that the Lie-Poisson semidirect product algebra associated with the compressible Euler equations appears naturally in the Poisson reduction setting $T^*\text{Diff}(M) \rightarrow T^*\text{Diff}(M)/\text{Diff}_\mu(M)$. We then show that the semidirect product structure is consistent with the symplectic reduction at zero momentum for $T^*\text{Diff}(M)//\text{Diff}_\mu(M) \simeq T^*\text{Dens}(M)$, see §5 and Appendix B.
- (3) We develop a reduction framework for relativistic fluids in §6.3 and show how the relativistic Burgers' equation arises in this context. We relate it to the relativistic approaches in optimal transport in [9] and ideal hydrodynamics in [21].
- (4) Beside the L^2 and the Wasserstein-Otto metrics we also describe the geometry associated with the Sobolev H^1 and the Fisher-Rao metrics, see §7. We show that infinite-dimensional Neumann systems are (up to time rescaling) Newton's equations for quadratic potentials in these metrics (in suitable coordinates the Fisher information functional is an example of such a potential), see §8.2.
- (5) Using the approach presented in this paper we derive stationary solutions of the Klein-Gordon equation and show that they satisfy a stationary infinite-dimensional Neumann problem, see §8.3. We also show that the generalized two-component Hunter-Saxton equation is a Newton's equation in the Fisher-Rao setting, see §9.5.
- (6) We review the properties of the Madelung transform which relates linear and nonlinear Schrödinger equations to Newton's equations on $\text{Dens}(M)$ and can be used to describe horizontal solutions to Newton's equations on $\text{Diff}(M)$ with Diff_μ -invariant potentials, see §9 and [57, 28] as well as the so-called Schrödinger smoke [10].
- (7) Finally, we describe the Casimirs for compressible barotropic fluids, compressible and incompressible magnetohydrodynamics, see §10.

Remark 1.2. We point out that, in addition to Newton’s equations, there is another class of natural evolution equations on Riemannian manifolds given by the *gradient flows*

$$\dot{q} = -\nabla V(q),$$

where a given potential function V determines velocity rather than acceleration. An interesting example can be found in [45] where the heat flow on $\text{Dens}(M)$ is described as the gradient flow of the relative entropy functional providing a geometric interpretation of the second law of thermodynamics, cf. Remark 9.8 on its relation to a Hamiltonian setting.

Remark 1.3. Newton’s equations for fluids discussed in the present paper are assumed to be conservative systems with a potential force. However, the subject concerning Newton’s equations is broader and we mention briefly two topics related to non-conservative Newton’s equations for compressible and incompressible fluids that are beyond the scope of this paper.

For the first we observe that the dissipative term Δv in the viscous Burgers equation

$$\dot{v} + \nabla_v v = \gamma \Delta v$$

can be viewed as a (linear) friction force while the equation itself can be seen as Newton’s equation on $\text{Diff}(M)$ with a non-potential force. Similarly, the Navier-Stokes equations of a viscous incompressible fluid

$$\dot{v} + \nabla_v v + \nabla P = \gamma \Delta v, \quad \text{div } v = 0$$

can be understood as Newton’s equations on $\text{Diff}_\mu(M)$ with a non-potential friction force. There is a large literature treating the Navier-Stokes equations within a stochastic framework where the geodesic setting of the Euler equations is modified by adding a random force which acts on the fluid, see [17, 20].

The second topic is related to a recently discovered flexibility and non-uniqueness of weak solutions of the Euler equations. The constructions in [49, 50, 11] exhibit compactly supported weak solutions describing a moving fluid that comes to rest as $t \rightarrow \pm\infty$. Such constructions can be understood by introducing a special forcing term F (sometimes called “black noise”) into the equations $\dot{v} + \nabla_v v + \nabla P = F$, which is required to be “ L^2 -orthogonal to all smooth functions.” (More precisely, one constructs a family of solutions with more and more singular and oscillating force and “black noise” is a residual forcing observed in the limit). Using the standard definition of a weak solution this force is thus not detectable upon multiplication by smooth test functions and hence the existence of such solutions to the Euler equations becomes less surprising. Constructions of similar weak solutions to other PDEs rely on intricate limiting procedures involving possibly more singular and less detectable forces. The study of the geometry of Newton’s equations with “black noise” on diffeomorphism groups seems to be a promising direction of future research.

Notation. Unless mentioned otherwise M stands for a compact oriented Riemannian manifold. The spaces of smooth k -forms on M are denoted by $\Omega^k(M)$, smooth vector fields by $\mathfrak{X}(M)$ and smooth functions by $C^\infty(M)$. Given a Riemannian metric \mathbf{g} on M the symbol ∇ stands for the gradient as well as for the covariant derivative of \mathbf{g} . The Riemannian volume form is denoted by μ and is assumed to be normalized: $\int_M \mu = 1$. To simplify notation, we will often use typical vector calculus conventions $|v|^2 = \mathbf{g}(v, v)$ and $u \cdot v = \mathbf{g}(u, v)$. Throughout the text we will implicitly associate with a smooth density form $\varrho \in \Omega^n(M)$ the corresponding density function $\rho \in C^\infty(M)$ defined by $\varrho = \rho \mu$. The Lie derivative along a vector field v will be denoted by \mathcal{L}_v .

A Riemannian metric \mathbf{g} on M defines an isomorphism between the tangent and cotangent bundles. For a vector field v on M we will denote by v^b the corresponding 1-form on M , namely

$v^\flat = \mathbf{g}(v, \cdot)$. As usual, the inverse map will be denoted by \sharp . The pullback and pushforward of a tensor field β by a diffeomorphism φ is denoted $\varphi^*\beta$ and $\varphi_*\beta$ respectively.

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2. WASSERSTEIN-OTTO GEOMETRY

2.1. Newton's equations on $\text{Diff}(M)$. In this section we describe Newton's equations on the full diffeomorphism group. Following [3, 14] we first introduce a (weak) Riemannian structure¹.

Definition 2.1. The L^2 -metric on $\text{Diff}(M)$ is given by

$$\mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) = \int_M |\dot{\varphi}(x)|^2 d\mu(x) = \int_M |\dot{\varphi}|^2 \mu \quad (2.1)$$

or, equivalently, after a change of variables

$$\mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) = \int_M |v|^2 \varphi_* \mu, \quad (2.2)$$

where $\varphi \in \text{Diff}(M)$, $\dot{\varphi} = v \circ \varphi \in T_\varphi \text{Diff}(M)$ and v is a vector field on M .

Newton's equation on $\text{Diff}(M)$ is a second order differential equation of the form

$$\nabla_{\dot{\varphi}}^{\mathbf{G}} \dot{\varphi} = -\nabla^{\mathbf{G}} U(\varphi), \quad (2.3)$$

where $U: \text{Diff}(M) \rightarrow \mathbb{R}$ is a potential energy function and $\nabla^{\mathbf{G}}$ is the covariant derivative of the L^2 -metric. We are interested in the case in which potential energy depends on φ implicitly via the associated density, i.e.,

$$U(\varphi) = \bar{U}(\varrho), \quad (2.4)$$

where $\varrho = \varphi_* \mu$ and $\bar{U}: \Omega^n(M) \rightarrow \mathbb{R}$ is a given function.

Remark 2.2. In local coordinates the pushforward map reads

$$\varphi_* \mu = \varphi_*(f dx_1 \wedge \dots \wedge dx_n) = \det(D\varphi^{-1}) f \circ \varphi^{-1} dx_1 \wedge \dots \wedge dx_n.$$

A more explicit form of (2.3) is given by the following theorem.

Theorem 2.3 ([51, 52]). *Newton's equations on $\text{Diff}(M)$ for the metric (2.1) and a potential function (2.4) can be written as*

$$\nabla_{\dot{\varphi}} \dot{\varphi} = -\nabla \frac{\delta \bar{U}}{\delta \varrho} \circ \varphi. \quad (2.5)$$

In reduced variables $v = \dot{\varphi} \circ \varphi^{-1}$ and $\rho = \det(D\varphi^{-1})$ the equations assume the form

$$\begin{cases} \dot{v} + \nabla_v v + \nabla \frac{\delta \bar{U}}{\delta \varrho} = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0. \end{cases} \quad (2.6)$$

¹A rigorous infinite-dimensional setting for diffeomorphisms and densities will be given in § 3.3 below. Here, for simplicity, we emphasize only the underlying geometric structure, leaving aside technical issues.

The right-hand side of the equations in (2.5) is a result of a direct calculation which we state in a separate lemma.

Lemma 2.4. *If U is of the form (2.4) then*

$$\nabla^G U(\varphi) = \nabla \left(\frac{\delta \bar{U}}{\delta \varrho}(\varrho) \right) \circ \varphi,$$

where $\varrho = \varphi_* \mu$.

Proof. Since $\nabla^G U$ stands here for the gradient of U in the L^2 -metric (2.1) and $\varphi(t)$ is the flow of the vector field v , we have

$$\begin{aligned} \mathbb{G}_\varphi(\nabla^G U(\varphi), \dot{\varphi}) &= \frac{d}{dt} \bar{U}(\varphi_* \mu) = \left\langle \frac{\delta \bar{U}}{\delta \varrho}(\varphi_* \mu), -\mathcal{L}_v \varphi_* \mu \right\rangle \\ &= \left\langle \iota_v d \frac{\delta \bar{U}}{\delta \varrho}(\varphi_* \mu), \varphi_* \mu \right\rangle = \int_M \mathbf{g} \left(\nabla \left(\frac{\delta \bar{U}}{\delta \varrho}(\varphi_* \mu) \right), \dot{\varphi} \circ \varphi^{-1} \right) \varphi_* \mu \end{aligned}$$

where the brackets denote the natural L^2 pairing between functions and n -forms. The result now follows from (2.2). \square

Proof of Theorem 2.3. The equations in (2.5) follow directly from Lemma 2.4 and the fact that the covariant derivative with respect to the L^2 -metric is just the pointwise covariant derivative on M .

The reduced equations in (2.6) are derived in the Hamiltonian setting in §3.1 below. \square

The following special class of solutions to Newton's equations is of particular interest.

Proposition 2.5. *The gradient fields $v = \nabla \theta$ form an invariant set of solutions of the reduced equations (2.6) which can be viewed as the Hamilton-Jacobi equations*

$$\begin{cases} \dot{\theta} + \frac{1}{2} |\nabla \theta|^2 = -\frac{\delta \bar{U}}{\delta \varrho} \\ \dot{\rho} + \operatorname{div}(\rho \nabla \theta) = 0. \end{cases}$$

Proof. This follows from a direct computation using the identity $\nabla_{\nabla \theta} \nabla \theta = \frac{1}{2} \nabla |\nabla \theta|^2$. A geometric explanation for the appearance of the Hamilton-Jacobi equation will be given in the next section. \square

An important point we want to emphasize in this paper is that a large number of interesting systems in mathematical physics originate as Newton's equations on $\operatorname{Diff}(M)$ corresponding to different choices of potential functions. A partial list of examples discussed here is given in Table 2. We will also describe systems defined on various extensions of $\operatorname{Diff}(M)$ including the MHD equations or the relativistic as well as the fully compressible Euler equations.

We have already seen two different formulations of Newton's equations: the second order (Lagrangian) representation in (2.5) and the reduced first order (Eulerian) representation in (2.6). In order to obtain all the equations listed in Table 2 we will need two further formulations: one defined on the space of densities and another defined on the space of wave functions. We begin with the former.

2.2. Riemannian submersion over densities. The space of smooth probability densities on M , namely

$$\operatorname{Dens}(M) = \left\{ \varrho \in \Omega^n(M) \mid \varrho > 0, \int_M \varrho = 1 \right\}$$

Equation on $\text{Diff}(M)$	Potential $\bar{U}(\rho)$	Section
inviscid Burgers'	0	§ 4.1
Hamilton-Jacobi	$\int_M V \rho \mu, \quad V \in C^\infty(M)$	§ 4.2
shallow-water	$\int_M \ln(\rho) \rho \mu$	§ 4.3
barotropic compressible Euler	$\int_M e(\rho) \rho \mu, \quad e \in C^\infty(\mathbb{R})$	§ 4.4
linear Schrödinger	$\int_M (\nabla \sqrt{\rho} ^2 + V \rho) \mu$	§ 9.2
nonlinear Schrödinger (NLS)	$\int_M (\nabla \sqrt{\rho} ^2 + \frac{\kappa \rho^2}{2}) \mu$	§ 9.2

 TABLE 2. Various PDEs as Newton's equations on $\text{Diff}(M)$.

is an open subset of codimension one of an affine subspace of $\Omega^n(M)$. It can be given the structure of an infinite-dimensional manifold whose tangent bundle is trivial

$$T\text{Dens}(M) = \text{Dens}(M) \times \Omega_0^n(M)$$

where $\Omega_0^n(M) = \{\alpha \in \Omega^n \mid \int_M \alpha = 0\}$.

Definition 2.6. The *left coset projection* $\pi: \text{Diff}(M) \rightarrow \text{Dens}(M)$ between the space of diffeomorphisms and the space of probability densities is given by the pushforward map

$$\pi(\varphi) = \varphi_* \mu. \quad (2.7)$$

This projection relates the L^2 -metric (2.1) and the following metric on the space of densities.

Definition 2.7. The *Wasserstein-Otto metric* is a Riemannian metric on $\text{Dens}(M)$ given by

$$\bar{G}_\varrho(\dot{\varrho}, \dot{\varrho}) = \int_M \theta \dot{\varrho}, \quad (2.8)$$

where

$$-\text{div}(\rho \nabla \theta) = \dot{\rho}$$

and $\dot{\varrho} \in \Omega_0^n(M)$ is a tangent vector at $\varrho = \rho \mu \in \text{Dens}(M)$.

The Riemannian distance defined by the metric (2.8) on $\text{Dens}(M)$ is precisely the L^2 Kantorovich-Wasserstein distance of optimal transport, see Benamou and Brenier [6], Otto [45], Lott [34] or Villani [55].

Theorem 2.8 ([45]). *The projection (2.7) is an (infinite-dimensional) Riemannian submersion with respect to the L^2 -metric \mathbf{G} on $\text{Diff}(M)$ and the Wasserstein-Otto metric $\bar{\mathbf{G}}$ on $\text{Dens}(M)$. Namely, given a horizontal² vector $\dot{\varphi} \in T_\varphi \text{Diff}(M)$ one has*

$$\mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) = \bar{\mathbf{G}}_{\pi(\varphi)}(\dot{\varrho}, \dot{\varrho}),$$

where $\dot{\varrho} = T_\varphi \pi(\dot{\varphi})$.

²That is, such that $\mathbf{G}_\varphi(\dot{\varphi}, V) = 0$ for all $V \in \ker(T_\varphi \pi)$.

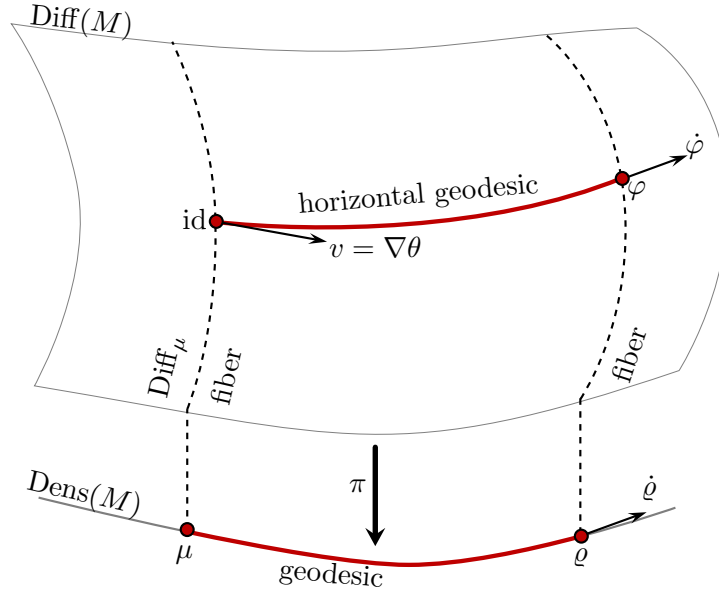


FIGURE 2. Illustration of the Riemannian submersion in [Theorem 2.8](#). Horizontal geodesics on $\text{Diff}(M)$ (potential solutions) are transversal to the fibres and project to geodesics on $\text{Dens}(M)$.

An illustration of this theorem is given in [Figure 2](#). The proof is based on two lemmas. Recall that the left coset projection is the pushforward action of $\text{Diff}(M)$ on μ . The corresponding isotropy group is the subgroup of volume-preserving diffeomorphisms

$$\text{Diff}_\mu(M) = \{\varphi \in \text{Diff}(M) \mid \varphi_*\mu = \mu\}$$

so that, if $[\varphi]$ is a left coset in $\text{Diff}(M)/\text{Diff}_\mu(M)$ then $\varphi' \in [\varphi]$ if and only if there exists $\eta \in \text{Diff}_\mu(M)$ such that $\varphi \circ \eta = \varphi'$.

The first lemma states in particular that the action of $\text{Diff}(M)$ on $\text{Dens}(M)$ is transitive.

Lemma 2.9. *Let $\pi: \text{Diff}(M) \rightarrow \text{Dens}(M)$ be the left coset projection (2.7). Then*

$$\begin{array}{ccc} \text{Diff}(M) & \longleftarrow & \text{Diff}_\mu(M) \\ \downarrow \pi & & \\ \text{Dens}(M) & & \end{array}$$

is a principal bundle. Consequently, the quotient space $\text{Diff}(M)/\text{Diff}_\mu(M)$ of left cosets is isomorphic to $\text{Dens}(M)$.

Proof. Surjectivity of π is a consequence of Moser's lemma [41]. The fact that π defines an infinite-dimensional principal bundle in the category of tame Fréchet manifolds (cf. §3.3 below) follows from a standard argument using the Nash-Moser-Hamilton theorem, cf. [19]. \square

The second lemma states that the L^2 -metric on $\text{Diff}(M)$ is compatible with the principal bundle structure above.

Lemma 2.10. *The L^2 -metric (2.1) is right-invariant with respect to the $\text{Diff}_\mu(M)$ action, namely*

$$\mathbf{G}_\varphi(u, v) = \mathbf{G}_{\varphi \circ \eta}(u \circ \eta, v \circ \eta)$$

for any $u, v \in T_\varphi \text{Diff}(M)$ and $\eta \in \text{Diff}_\mu(M)$.

Proof. Since $\eta_*\mu = \mu$ the result follows at once from (2.2). \square

In [3] Arnold used the L^2 -metric (2.1) to show that its geodesic equation on $\text{Diff}_\mu(M)$, when expressed in Eulerian coordinates, yields the classical Euler equations of an ideal fluid. This marked the beginning of *topological hydrodynamics*, cf. [5] or Appendix A.

The Riemannian submersion framework described above concerns objects that are extrinsic to Arnold's (intrinsic) point of view. More precisely, rather than restricting to the vertical directions tangent to the fibre $\text{Diff}_\mu(M)$, we consider the horizontal directions in the total space $\text{Diff}(M)$ and use the fact that any structure on $\text{Diff}(M)$ which is invariant under the right action of $\text{Diff}_\mu(M)$ induces a corresponding structure on $\text{Dens}(M)$ by Lemma 2.9.

We are now ready to prove the main result of this subsection.

Proof of Theorem 2.8. Given $\xi \in T_\varphi \text{Diff}(M)$ let $u = \xi \circ \varphi^{-1}$ and $\varrho = \pi(\varphi)$ with $\varrho = \rho\mu$ as before. Then

$$\begin{aligned} T_\varphi \pi \cdot \xi &= -\mathcal{L}_u(\varphi_*\mu) = -\mathcal{L}_u\varrho \\ &= -(\rho \operatorname{div} u + \mathcal{L}_u\rho)\mu \\ &= -\operatorname{div}(\rho u)\mu. \end{aligned}$$

The kernel of $T_\varphi \pi$ is $T_\varphi \text{Diff}_\varrho(M)$ and defines a vertical distribution. On the other hand, the horizontal distribution is

$$\mathcal{H}_\varphi = \{\xi \in T_\varphi \text{Diff}(M) \mid \xi \circ \varphi^{-1} = \nabla p \text{ for } p \in C^\infty(M)\}.$$

Indeed, if $\operatorname{div}(\rho v) = 0$ then from (2.1) we have

$$\mathbf{G}_\varphi(\nabla p \circ \varphi, v \circ \varphi) = \int_M \mathbf{g}(\nabla p, v)\varrho = \int_M (\mathcal{L}_v p)\varrho = - \int_M p \mathcal{L}_v \varrho = 0$$

and it follows that $T_\varphi \pi: \mathcal{H}_\varphi \rightarrow T_\varrho \text{Dens}$ is an isometry. Its inverse is

$$T_\varrho \text{Dens}(M) \ni \dot{\varrho} \mapsto \nabla(-\Delta_\rho^{-1}\dot{\rho}) \circ \varphi \in \mathcal{H}_\varphi$$

where $\Delta_\rho = \operatorname{div} \rho \nabla$. From (2.1) we now compute

$$\begin{aligned} \mathbf{G}_\varphi(\nabla(\Delta_\rho^{-1}\dot{\rho}) \circ \varphi, \Delta_\rho^{-1}\dot{\rho}) &= \int_M \mathbf{g}(\nabla(\Delta_\rho^{-1}\dot{\rho}), \nabla(\Delta_\rho^{-1}\dot{\rho}))\rho\mu \\ &= \int_M -\operatorname{div}(\rho \nabla(\Delta_\rho^{-1}\dot{\rho}))\Delta_\rho^{-1}\dot{\rho}\mu \\ &= \int_M -\dot{\rho}\Delta_\rho^{-1}\dot{\rho}\mu = \bar{\mathbf{G}}_\varrho(\dot{\varrho}, \dot{\varrho}) \end{aligned}$$

where the last equality follows from the definition of $\bar{\mathbf{G}}$. Thus, the projection π is a Riemannian submersion. \square

Remark 2.11. If E is a smooth function on $\text{Dens}(M)$ then from the above expression we have

$$\bar{\mathbf{G}}_\varrho(\nabla \bar{\mathbf{G}} E(\varrho), \dot{\varrho}) = \left\langle \frac{\delta E}{\delta \varrho}, \dot{\varrho} \right\rangle = \int_M \Delta_\rho^{-1} \Delta_\rho \frac{\delta E}{\delta \varrho} \dot{\varrho} = \bar{\mathbf{G}}_\varrho((-\Delta_\rho \frac{\delta E}{\delta \varrho})\mu, \dot{\varrho})$$

which gives the following formula for the gradient of E in the Wasserstein-Otto metric

$$\nabla^{\bar{G}}E(\varrho) = \left(-\Delta_\rho \frac{\delta E}{\delta \varrho} + \int_M \Delta_\rho \frac{\delta E}{\delta \varrho} \mu \right) \mu$$

since every vector tangent to $\text{Dens}(M)$ has zero mean. In particular, if E is the relative entropy $S(\varrho) = \int_M \ln(\varrho/\mu) \varrho$, then $\delta S/\delta \varrho = \ln \rho$ and since

$$\Delta_\rho \ln \rho = \text{div}(\rho \nabla(\ln \rho)) = \text{div} \nabla \rho = \Delta \rho$$

we recover the formula $\nabla^{\bar{G}}S(\varrho) = (\Delta \rho)\mu$, i.e., the Wasserstein gradient flow of entropy corresponds to the heat flow on the space of densities, cf. [45].

3. HAMILTONIAN SETTING

The point of view of incompressible hydrodynamics as a Hamiltonian system on the cotangent bundle of $\text{Diff}_\mu(M)$ described by Arnold [3] turned out to be remarkably useful in applications involving invariants and stability (this is reviewed in [Appendix A](#)). In the next sections we develop the framework for Newton's equations (adding a potential energy term to the kinetic energy which yields geodesics) on the group $\text{Diff}(M)$ of all diffeomorphisms (rather than volume-preserving ones).

3.1. Hamiltonian form of the equations and Poisson reduction. Newton's equations (2.5) can be viewed as a canonical Hamiltonian system on $T^*\text{Diff}(M)$. To write down this system it is convenient to identify each $T_\varphi^*\text{Diff}(M)$ with the dual of the space of vector fields $\mathfrak{X}^*(M)$ consisting of differential 1-forms with values in the space of densities

$$\mathfrak{X}^*(M) = \Omega^1(M) \otimes \text{Dens}(M)$$

where the tensor product is taken over the ring $C^\infty(M)$. The pairing between $\dot{\varphi} \in T_\varphi\text{Diff}(M)$ and $m \in T_\varphi^*\text{Diff}(M)$ is then given by

$$\langle m, \dot{\varphi} \rangle_\varphi = \int_M \iota_{\dot{\varphi} \circ \varphi^{-1}} m \quad (3.1)$$

(when $\varphi = \text{id}$ we will sometimes omit the subscript). This pairing does not depend on the Riemannian metric \mathbf{g} on M .

Consider the standard Lagrangian on $T\text{Diff}(M)$ in the kinetic-minus-potential energy form

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) - \bar{U}(\varphi_* \mu).$$

As usual, the passage to the Hamiltonian formulation on $T^*\text{Diff}(M)$ is obtained through the Legendre transform which in this case is given by

$$m = v^\flat \otimes \varrho \quad \text{where} \quad v = \dot{\varphi} \circ \varphi^{-1} \quad \text{and} \quad \varrho = \varphi_* \mu.$$

Lemma 3.1. *The Hamiltonian corresponding to the Lagrangian L is*

$$H(\varphi, m) = \frac{1}{2} \langle m, v \rangle + \bar{U}(\varphi_* \mu). \quad (3.2)$$

Proof. In the above notation for $\dot{\varphi} \in T_\varphi\text{Diff}(M)$ we have

$$\mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) = \int_M |v|^2 \varrho = \int_M (\iota_v v^\flat) \varrho = \langle v^\flat \otimes \varrho, \dot{\varphi} \rangle_\varphi.$$

The result follows since \mathbf{G}_φ is quadratic and \bar{U} is independent of $\dot{\varphi}$. \square

We can now turn to Newton's equations on $\text{Diff}(M)$.

Theorem 3.2. *The Hamiltonian form of the equations (2.5) is*

$$\begin{cases} \dot{m} = -\mathcal{L}_v m + d\left(\frac{1}{2}|v|^2 - \frac{\delta\bar{U}}{\delta\varrho}(\varphi_*\mu)\right) \otimes \varphi_*\mu \\ \dot{\varphi} = v \circ \varphi \end{cases} \quad (3.3)$$

where $m = v^\flat \otimes \varphi_*\mu$.

Proof. In canonical coordinates (φ, m_φ) on $T^*\text{Diff}(M)$ the Hamiltonian equations take the form

$$\dot{m}_\varphi = -\frac{\delta H}{\delta \varphi} \quad \text{and} \quad \dot{\varphi} = \frac{\delta H}{\delta m_\varphi}$$

where m_φ are the canonical momenta satisfying $m = m_\varphi \circ \varphi^{-1}$.

Given a Hamiltonian H on $T^*\text{Diff}(M) \simeq \text{Diff}(M) \times \mathfrak{X}^*(M)$ and a variation $\epsilon \rightarrow m_{\varphi, \epsilon}$ we have

$$\frac{d}{d\epsilon} H(\varphi, m_{\varphi, \epsilon} \circ \varphi^{-1}) = \left\langle \frac{\delta H}{\delta m}, \frac{d}{d\epsilon} m_{\varphi, \epsilon} \circ \varphi^{-1} \right\rangle_{\text{id}} = \left\langle \frac{\delta H}{\delta m} \circ \varphi, \frac{d}{d\epsilon} m_{\varphi, \epsilon} \right\rangle_\varphi$$

and thus $\dot{\varphi} = \frac{\delta H}{\delta m_\varphi} = v \circ \varphi$ where $v = \frac{\delta H}{\delta m}$. Differentiating $m_\varphi = m \circ \varphi$ with respect to the t variable we obtain

$$\dot{m} = -\mathcal{L}_v m + \dot{m}_\varphi \circ \varphi^{-1} = -\mathcal{L}_v m - \frac{\delta H}{\delta \varphi} \circ \varphi^{-1}.$$

As before, writing the Hamiltonian in (3.2) as $H(\varphi, m) = \bar{H}(\varrho, m)$ where $\varrho = \varphi_*\mu$ and letting $\epsilon \rightarrow \varphi_\epsilon$ be a variation of $\varphi_0 = \varphi$ generated by the field v we find

$$\frac{d}{d\epsilon} H(\varphi_\epsilon, m) = \left\langle d\frac{\delta \bar{H}}{\delta \varrho}, v \right\rangle_{\text{id}} = \left\langle d\frac{\delta \bar{H}}{\delta \varrho} \circ \varphi, \dot{\varphi} \right\rangle_\varphi.$$

Thus $\frac{\delta H}{\delta \varphi} = d\frac{\delta \bar{H}}{\delta \varrho} \circ \varphi$ and the equation for \dot{m} becomes

$$\dot{m} = -\mathcal{L}_v m - d\frac{\delta \bar{H}}{\delta \varrho}.$$

Finally, a straightforward computation using (3.2) gives

$$\frac{\delta \bar{H}}{\delta \varrho} = -\frac{|v|^2}{2} + \frac{\delta \bar{U}}{\delta \varrho},$$

which concludes the proof. \square

Rewriting the system (3.3) in terms of $\varrho = \varphi_*\mu$ and m provides an example of Poisson reduction with respect to $\text{Diff}_\mu(M)$ as the symmetry group. From (3.1) we obtain a formula for the cotangent action of this group on $T^*\text{Diff}(M)$, namely

$$\eta \cdot (\varphi, m) = (\varphi \circ \eta^{-1}, m). \quad (3.4)$$

Theorem 3.3 (Poisson reduction). *The quotient space $T^*\text{Diff}(M)/\text{Diff}_\mu(M)$ is isomorphic to $\text{Dens}(M) \times \mathfrak{X}^*(M)$. The isomorphism is given by the projection $\Pi(\varphi, m) = (\varphi_*\mu, m)$. Furthermore, Π is a Poisson map with respect to the canonical Poisson structure on $T^*\text{Diff}(M)$ and the Poisson structure on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ given by*

$$\{F, G\}(\varrho, m) = \left\langle \varrho, \mathcal{L}_{\frac{\delta F}{\delta m}} \frac{\delta G}{\delta \varrho} - \mathcal{L}_{\frac{\delta G}{\delta m}} \frac{\delta F}{\delta \varrho} \right\rangle + \left\langle m, \mathcal{L}_{\frac{\delta F}{\delta m}} \frac{\delta G}{\delta m} \right\rangle. \quad (3.5)$$

In the above statement both F and G are assumed to be differentiable with variational derivatives belonging to the smooth dual. In §3.3 below we provide an alternative construction in the setting of Fréchet spaces.

Proof. From (3.4) and Lemma 2.9 it follows that

$$T^*\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \text{Diff}(M)/\text{Diff}_\mu(M) \times \mathfrak{X}^*(M) \simeq \text{Dens}(M) \times \mathfrak{X}^*(M)$$

with the projection given by Π . The fact that Π is a Poisson map (in fact, a Poisson submersion) follows from the calculation in the proof of Theorem 3.2. \square

Remark 3.4. The bracket (3.5) is the classical Lie-Poisson structure on the dual of the semidirect product $\mathfrak{X}(M) \ltimes C^\infty(M)$.

Corollary 3.5. *Let H be a Hamiltonian function on $T^*\text{Diff}(M)$ satisfying*

$$H(\varphi, m) = H(\varphi \circ \eta, m) \quad \text{for all } \eta \in \text{Diff}_\mu(M).$$

Then $H = \bar{H} \circ \Pi$ for some function $\bar{H}: \text{Dens}(M) \times \mathfrak{X}^(M) \rightarrow \mathbb{R}$. In reduced variables $\varrho = \varphi_*\mu$ and m , the Hamiltonian equations assume the form*

$$\begin{cases} \dot{m} = -\mathcal{L}_v m - d\frac{\delta \bar{H}}{\delta \varrho} \otimes \varrho \\ \dot{\varrho} = -\mathcal{L}_v \varrho \end{cases} \quad (3.6)$$

where $v = \frac{\delta \bar{H}}{\delta m}$.

Proof. Using the Poisson form of the Hamiltonian equations $\dot{F} = \{H, F\}$ with $F(\rho, m) = \langle m, u \rangle + \langle \varrho, \theta \rangle$ we obtain from (3.5) the weak form of the equations

$$\langle \dot{m}, u \rangle + \langle \dot{\varrho}, \theta \rangle = \left\langle \varrho, \mathcal{L}_{\frac{\delta \bar{H}}{\delta m}} \theta - \mathcal{L}_u \frac{\delta \bar{H}}{\delta \varrho} \right\rangle + \left\langle m, \mathcal{L}_{\frac{\delta \bar{H}}{\delta m}} u \right\rangle$$

for any $u \in \mathfrak{X}(M)$ and $\theta \in C^\infty(M)/\mathbb{R}$. Rewriting the right-hand side as

$$\left\langle -\mathcal{L}_{\frac{\delta \bar{H}}{\delta m}} \varrho, \theta \right\rangle + \left\langle -\mathcal{L}_{\frac{\delta \bar{H}}{\delta m}} m - d\frac{\delta \bar{H}}{\delta \varrho} \otimes \varrho, u \right\rangle$$

completes the proof. \square

The following is the Hamiltonian analogue of Proposition 2.5.

Proposition 3.6. *The product manifold*

$$\text{Dens}(M) \times (\text{d}C^\infty(M) \otimes \varrho) = \{(\varrho, \text{d}\theta \otimes \varrho) \mid \theta \in C^\infty(M)\}$$

is a Poisson submanifold of $\text{Dens}(M) \times \mathfrak{X}^(M)$.*

Proof. From (3.6) we find that the momenta $m = \text{d}C^\infty(M) \otimes \varrho$ form an invariant set in $\Omega^1(M) \otimes \varrho$ for any choice of Hamiltonian \bar{H} . \square

It turns out that the submanifold in Proposition 3.6 is symplectic, as we shall discuss below.

3.2. Newton's equations on $\text{Dens}(M)$. Poisson reduction with respect to the cotangent action of $\text{Diff}_\mu(M)$ on $T^*\text{Diff}(M)$ leads to reduced dynamics on the Poisson manifold $T^*\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \text{Dens}(M) \times \mathfrak{X}^*(M)$ (cf. Theorem 3.3). This Poisson manifold is a union of symplectic leaves one of which can be identified with $T^*\text{Dens}(M)$ equipped with the canonical symplectic structure. Indeed, the latter turns out to be the symplectic quotient $T^*\text{Diff}(M)//\text{Diff}_\mu(M)$ corresponding to the zero-momentum leaf, see Appendix B. Here we identify $T^*\text{Dens}(M)$ as a symplectic submanifold of $\text{Dens}(M) \times \mathfrak{X}^*(M)$.

Lemma 3.7. *The (smooth part of the) cotangent bundle of $\text{Dens}(M)$ is*

$$T^*\text{Dens}(M) = \text{Dens}(M) \times C^\infty(M)/\mathbb{R} \simeq \text{Dens}(M) \times (\text{d}C^\infty(M) \otimes \varrho).$$

Proof. The smooth dual of $\Omega^n(M)$ is $C^\infty(M)$ with the pairing given by

$$\langle \theta, \varrho \rangle = \int_M \theta \varrho.$$

Since the space $T_\varrho \text{Dens}(M) = \Omega_0^n(M)$ of zero-mean forms is a subspace of $\Omega^n(M)$ it follows that

$$\begin{aligned} T^* \text{Dens}(M) &= \text{Dens}(M) \times \Omega^n(M)^* / \ker(\langle \cdot, \Omega_0^n(M) \rangle) \\ &= \text{Dens}(M) \times C^\infty(M) / \mathbb{R}. \end{aligned}$$

Taking the differential of the second factor we obtain the identification

$$\text{Dens}(M) \times C^\infty(M) / \mathbb{R} \simeq \text{Dens}(M) \times (\text{d}C^\infty(M) \otimes \varrho).$$

□

Next, we turn to Newton's equations on $\text{Dens}(M)$ for the Wasserstein-Otto metric (2.8).

Corollary 3.8. *The Hamiltonian on $T^* \text{Dens}(M)$ corresponding to Newton's equations on $\text{Dens}(M)$ with respect to the Wasserstein-Otto metric (2.8) is*

$$\tilde{H}(\varrho, \theta) = \frac{1}{2} \int_M |\nabla \theta|^2 \varrho + \bar{U}(\varrho)$$

and the Hamiltonian equations are

$$\begin{cases} \dot{\theta} + \frac{1}{2} |\nabla \theta|^2 + \frac{\delta \bar{U}}{\delta \varrho} = 0, \\ \dot{\varrho} + \mathcal{L}_{\nabla \theta} \varrho = 0. \end{cases} \quad (3.7)$$

Solutions of (3.7) correspond to horizontal solutions of Newton's equations (2.5) on $\text{Diff}(M)$, or, equivalently, to zero-momentum solutions of the reduced equations (3.6) with Hamiltonian

$$\bar{H}(\varrho, m) = \frac{1}{2} \langle m, v \rangle + \bar{U}(\varrho), \quad m = v^\flat \otimes \varrho.$$

Proof. Given the Hamiltonian (3.2) on $T^* \text{Diff}(M)$, the result follows directly from [Theorem 3.3](#) and the zero-momentum reduction result [Theorem B.4](#) in [Appendix B](#). □

3.3. Tame Fréchet manifolds. A natural functional-analytic setting for the results presented here is that of tame Fréchet spaces, cf. [Hamilton \[19\]](#). Alternatively, one could work with Sobolev H^s completions (or any reasonably strong Banach topology) of the function spaces that appear in this paper. If $s > \dim M/2 + 1$ then the Sobolev completions of the diffeomorphism groups $\text{Diff}^s(M)$ and $\text{Diff}_\mu^s(M)$ are smooth Hilbert manifolds but not Banach Lie groups since, e.g., the left multiplication and the inversion maps are not even uniformly continuous in the H^s topology.

On the other hand, both $\text{Diff}(M)$ and $\text{Diff}_\mu(M)$ can be equipped with the structure of tame Fréchet Lie groups. In this setting $\text{Diff}_\mu(M)$ becomes a closed tame Lie subgroup of $\text{Diff}(M)$ which can be viewed as a tame principal bundle over the quotient space $\text{Dens}(M) = \text{Diff}(M) / \text{Diff}_\mu(M)$ of either left or right cosets. Furthermore, the tangent bundle $T \text{Diff}(M)$ over $\text{Diff}(M)$ is also a tame manifold. However, since the dual of a Fréchet space, which itself is not a Banach space, is never a Fréchet space, to avoid working with currents on M it is expedient to restrict to a suitable subset of the (full) cotangent bundle over $\text{Diff}(M)$.

More precisely, consider the tensor product $T^*M \otimes \Lambda^n M$ of the cotangent bundle and the vector bundle of n -forms on M and define another bundle over $\text{Diff}(M)$ whose fibre over $\varphi \in \text{Diff}(M)$ is the space of smooth sections of the pullback bundle $\varphi^{-1}(T^*M \otimes \Lambda^n M)$ over M . We will refer to this object as (the smooth part of) the cotangent bundle of $\text{Diff}(M)$ and denote it also by $T^* \text{Diff}(M)$. We will write $\mathfrak{X}^*(M) = T_{\text{id}}^* \text{Diff}(M)$ and $\mathfrak{X}^{**}(M) = \mathfrak{X}(M)$. Throughout the

paper we will assume that derivatives of various Hamiltonian functions can be viewed as maps to the smooth cotangent bundle of the phase space.

Lemma 3.9. *$T^*\text{Diff}(M)$ is a tame Fréchet manifold and the map*

$$\text{Diff}(M) \times \mathfrak{X}^*(M) \ni (\varphi, m) \longmapsto (\varphi, m \circ \varphi) \in T^*\text{Diff}(M)$$

is an isomorphism of tame Fréchet manifolds.

Proof. Recall that $\text{Diff}(M)$ is an open subset of $C^\infty(M, M)$ and observe that $T^*\text{Diff}(M)$ is the inverse image of $\text{Diff}(M)$ under the smooth tame projection $m \rightarrow \pi \circ m$ between tame Fréchet manifolds $C^\infty(M, T^*M \otimes \Lambda^n M)$ and $C^\infty(M, M)$. The argument is routine: the space $T^*\text{Diff}(M)$ is trivialized by the fiber mapping since φ is a diffeomorphism, while the fact that the fiber mapping is smooth and tame with a smooth tame inverse follows since $\text{Diff}(M)$ is a tame Fréchet Lie group (all group operations are smooth tame maps). \square

Let $v_\varphi \in T_\varphi\text{Diff}(M)$ and $m_\varphi \in T_\varphi^*\text{Diff}(M)$. As before in (3.1) we have the pairing

$$(v_\varphi, m_\varphi) \mapsto \langle v_\varphi, m_\varphi \rangle_\varphi = \int_M \iota_{v_\varphi \circ \varphi^{-1}} m_\varphi \circ \varphi^{-1}$$

between the fibers $T_\varphi\text{Diff}(M)$ and $T_\varphi^*\text{Diff}(M)$.

Our goal in this section is to describe Poisson reduction of $T^*\text{Diff}(M)$ with respect to the right action of $\text{Diff}_\mu(M)$ as a smooth tame principal bundle. We will use the Poisson bivector for the canonical symplectic structure on $T^*\text{Diff}(M)$, which we identify with its right trivialization $\text{Diff}(M) \times \mathfrak{X}^*(M)$ as in Lemma 3.9. By construction, each element of $\mathfrak{X}^*(M)$ can be viewed as a tensor product $m = \alpha \otimes \varrho$ of a 1-form and a volume form on M . Choose $\varrho = \mu$ and note that for each $(\varphi, m) \in \text{Diff}(M) \times \mathfrak{X}^*(M)$ the Poisson bivector Λ on $\text{Diff}(M) \times \mathfrak{X}^*(M)$ is a bilinear form on $T_\varphi^*\text{Diff}(M) \times \mathfrak{X}(M)$ defined by

$$\begin{aligned} \Lambda_{(\varphi, m)}((n_{1\varphi}, v_1), (n_{2\varphi}, v_2)) &= \langle m, [v_1, v_2] \rangle_{\text{id}} \\ &\quad - \langle n_{1\varphi}, v_2 \circ \varphi \rangle_\varphi + \langle n_{2\varphi}, v_1 \circ \varphi \rangle_\varphi. \end{aligned}$$

Lemma 3.10. *The bivector Λ induces a smooth tame vector bundle isomorphism*

$$\Gamma: T^*(\text{Diff}(M) \times \mathfrak{X}^*(M)) \rightarrow T(\text{Diff}(M) \times \mathfrak{X}^*(M))$$

which at any point (φ, m) is given by

$$\Gamma_{(\varphi, m)}(n_\varphi, v) = (v \circ \varphi, -\mathcal{L}_v m - n_\varphi \circ \varphi^{-1}) \quad (3.8)$$

for any $n_\varphi \in T_\varphi^\text{Diff}(M)$ and $v \in \mathfrak{X}(M)$.*

Proof. First, observe that one can identify the tangent and cotangent bundles of $\text{Diff}(M) \times \mathfrak{X}^*(M)$ with $T\text{Diff}(M) \times \mathfrak{X}^*(M) \times \mathfrak{X}^*(M)$ and $T^*\text{Diff}(M) \times \mathfrak{X}^*(M) \times \mathfrak{X}(M)$ respectively. The formula in (3.8) can be verified by a direct calculation from

$$\langle \Gamma_{(\varphi, m)}(m_{1\varphi}, v_1), (m_{2\varphi}, v_2) \rangle_{(\varphi, m)} = \Lambda_{(\varphi, m)}((n_{1\varphi}, v_1), (n_{2\varphi}, v_2))$$

for any $n_{1\varphi}, n_{2\varphi} \in T_\varphi^*\text{Diff}(M)$ and $v_1, v_2 \in \mathfrak{X}(M)$ using integration by parts and the assumption that M has no boundary. Smoothness of Γ follows from the fact that all the operations in (3.9) are smooth tame maps. The inverse of Γ is given by

$$\Gamma_{(\varphi, m)}^{-1}(\dot{\varphi}, \dot{m}) = (\dot{\varphi} \circ \varphi^{-1}, -(\dot{m} + \mathcal{L}_{\dot{\varphi} \circ \varphi^{-1}} m) \circ \varphi).$$

Again, all the operations are smooth tame maps, which concludes smoothness also of the inverse. \square

Remark 3.11. That Γ is a symplectomorphism corresponds to the fact that $T^*\text{Diff}(M) \simeq \text{Diff}(M) \times \mathfrak{X}^*(M)$ is a symplectic manifold with canonical symplectic structure $\Omega_{(\varphi, m)}(\cdot, \cdot) = \langle \Gamma_{(\varphi, m)}^{-1}(\cdot, \cdot) \rangle$. The space $T^*\text{Dens}(M) = \text{Dens}(M) \times C^\infty(M)/\mathbb{R}$ is a tame Fréchet manifold, since so are both $\text{Dens}(M)$ and $C^\infty(M)/\mathbb{R}$.

Next, consider the Poisson bivector $\bar{\Lambda}$ defined on the tame Fréchet manifold $\text{Dens}(M) \times \mathfrak{X}^*(M)$ by

$$\begin{aligned} \bar{\Lambda}_{(\varrho, m)}((\theta_1, v_1), (\theta_2, v_2)) = & \langle m, [v_1, v_2] \rangle_{\text{id}} \\ & + \langle \theta_1, \mathcal{L}_{v_2} \varrho \rangle - \langle \theta_2, \mathcal{L}_{v_1} \varrho \rangle. \end{aligned}$$

Lemma 3.12. *The bivector $\bar{\Lambda}$ induces a smooth tame vector bundle homomorphism*

$$\bar{\Gamma}: T^*(\text{Dens}(M) \times \mathfrak{X}^*(M)) \rightarrow T(\text{Dens}(M) \times \mathfrak{X}^*(M))$$

which at any point (ϱ, m) is given by

$$\bar{\Gamma}_{(\varrho, m)}(\theta, v) = (-\mathcal{L}_v \varrho, -\mathcal{L}_v m - d\theta \otimes \varrho) \quad (3.9)$$

for any $\theta \in T^*_\varrho \text{Dens}(M)$ and $v \in \mathfrak{X}(M)$.

Proof. The proof follows the same steps as the proof of Lemma 3.10 with the adjustment that now $\bar{\Gamma}_{(\varrho, m)}$ is only a homomorphism, rather than an isomorphism, of vector bundles. \square

Remark 3.13. The Hamiltonian equations on $\text{Diff}(M) \times \mathfrak{X}^*(M) \simeq T^*\text{Diff}(M)$ and on $\text{Dens}(M) \times \mathfrak{X}^*(M)$, as discussed in the previous sections, can now be written as

$$(\dot{\varphi}, \dot{m}) = \Gamma(DH(\varphi, m)) \quad \text{and} \quad (\dot{\varrho}, \dot{m}) = \bar{\Gamma}(D\bar{H}(\varrho, m)).$$

Notice that $\bar{\Gamma}$ corresponds to a Poisson structure, but not to a symplectic structure as Γ does; $\bar{\Gamma}$ is not invertible whereas Γ is.

The next theorem is the main result of this section.

Theorem 3.14. *The following diagram*

$$\begin{array}{ccc} \text{Diff}_\mu(M) & \hookrightarrow & T^*\text{Diff}(M) \\ & & \downarrow \Pi: (\varphi, m_\varphi) \mapsto (\varphi_* \mu, m_\varphi \circ \varphi^{-1}) \\ & & \text{Dens}(M) \times \mathfrak{X}^*(M) \end{array}$$

is a smooth tame principal bundle. The projection Π is a Poisson submersion with respect to the Poisson structure on $\text{Dens}(M) \times \mathfrak{X}^*(M)$. Solutions to the Hamiltonian equations for a Diff_μ -invariant Hamiltonian on $T^*\text{Diff}(M)$ project to solutions of the Hamiltonian equations for the (unique) Hamiltonian \bar{H} on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ satisfying $H(\varphi, m_\varphi) = \bar{H}(\varphi_* \mu, m_\varphi \circ \varphi^{-1})$.

Proof. First, consider the map $(\varphi, m_\varphi) \rightarrow (\varphi, m)$ from $T^*\text{Diff}(M)$ to the product $\text{Diff}(M) \times \mathfrak{X}^*(M)$ and observe that it is a smooth tame vector bundle isomorphism, as in Lemma 3.9. The cotangent action of $\eta \in \text{Diff}_\mu(M)$ on (φ, m) acts on the first component by composition $(\varphi \circ \eta, m)$ and is clearly also a smooth tame map. Furthermore, we have

$$(\text{Diff}(M) \times \mathfrak{X}^*(M))/\text{Diff}_\mu(M) \simeq (\text{Diff}(M)/\text{Diff}_\mu(M)) \times \mathfrak{X}^*(M).$$

The fact that $\text{Diff}(M)$ is a smooth tame principal bundle over $\text{Dens}(M)$ with fiber $\text{Diff}_\mu(M)$ follows from the Nash-Moser-Hamilton theorem, cf. e.g., [19, Thm. III.2.5.3]. Consequently, $T^*\text{Diff}(M)$ is a smooth tame principal bundle over $\text{Dens}(M) \times \mathfrak{X}^*(M)$ with fiber $\text{Diff}_\mu(M)$.

The projection is a Poisson submersion and smooth solutions are mapped to smooth solutions: this follows from [Lemma 3.10](#) and [Lemma 3.12](#) together with a straightforward calculation showing that $T\Pi \circ \Lambda(DH) = \bar{\Lambda}(D\bar{H})$ whenever $H = \bar{H} \circ \Pi$. \square

Remark 3.15. We point out that the situation is more complicated if one works with Banach spaces such as Sobolev H^s or Hölder $C^{k,\alpha}$. In those settings the results in [Lemma 3.9](#), [Lemma 3.10](#), [Lemma 3.12](#) and [Theorem 3.14](#) need not hold. For example, the bundle projection in [Theorem 3.14](#) typically fails to be Lipschitz continuous in the H^s topology.

4. WASSERSTEIN-OTTO EXAMPLES

In this section we provide and study examples of Newton's equations on $\text{Diff}(M)$ with respect to the L^2 metric [\(2.1\)](#) and $\text{Diff}_\mu(M)$ -invariant potentials. We also derive the corresponding Poisson reduced equations on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ (cf. [§ 3.1](#)) and symplectic reduced equations on $T^*\text{Dens}(M)$ corresponding to Newton's equations for the Wasserstein-Otto metric [\(2.8\)](#) (cf. [§ 3.2](#) above).

4.1. Inviscid Burgers' equation. We start with the simplest case when the potential function is zero. The corresponding Newton's equations are the geodesic equations on $\text{Diff}(M)$.

Proposition 4.1. *Newton's equations with respect to the L^2 -metric [\(2.1\)](#) and with zero potential $\bar{U} = 0$ admit the following formulations:*

- the L^2 geodesic equations on $\text{Diff}(M)$

$$\nabla_{\dot{\varphi}} \dot{\varphi} = 0,$$

- the inviscid Burgers' equations on $\mathfrak{X}(M)$

$$\dot{v} + \nabla_v v = 0$$

where $v = \dot{\varphi} \circ \varphi^{-1}$,

- the Poisson reduced equations on $\text{Dens}(M) \times \mathfrak{X}^*(M)$

$$\begin{cases} \dot{m} + \mathcal{L}_v m - d(\frac{1}{2}|v|^2) \otimes \varrho = 0 \\ \dot{\varrho} + \mathcal{L}_v \varrho = 0 \end{cases}$$

where $m = v^\flat \otimes \varrho$,

- the symplectically reduced equations on $T^*\text{Dens}(M)$

$$\begin{cases} \dot{\theta} + \frac{1}{2}|\nabla\theta|^2 = 0 \\ \dot{\varrho} + \mathcal{L}_{\nabla\theta}\varrho = 0 \end{cases} \tag{4.1}$$

corresponding to the Hamiltonian form of the geodesic equations for the Wasserstein-Otto metric [\(2.8\)](#).

Observe that the system in [\(4.1\)](#) consists of the Hamilton-Jacobi equation for the kinetic energy Hamiltonian $H(x, p) = \frac{1}{2}g_x(p^\sharp, p^\sharp)$ on M and the transport equation for ϱ .

Proof. The results follow directly from [Theorem 2.3](#), [Theorem 3.3](#) and [Corollary 3.8](#) after setting $\bar{U} = 0$. \square

4.2. Classical mechanics and Hamilton-Jacobi equations. Let V be a smooth potential function on M and consider the corresponding potential function on the space of densities

$$\bar{U}(\varrho) = \int_M V \varrho \quad (4.2)$$

where $\varrho \in \text{Dens}(M)$.

Proposition 4.2. *Newton's equations with respect to the L^2 -metric (2.1) and the potential \bar{U} in (4.2) admit the following formulations:*

- the L^2 geodesic equations with potential on $\text{Diff}(M)$

$$\nabla_{\dot{\varphi}} \dot{\varphi} + \nabla V \circ \varphi = 0,$$

- the inviscid Burgers' equations with potential on $\mathfrak{X}(M)$

$$\dot{v} + \nabla_v v + \nabla V = 0$$

where $v = \dot{\varphi} \circ \varphi^{-1}$,

- the Poisson reduced equations on $\text{Dens}(M) \times \mathfrak{X}^*(M)$

$$\begin{cases} \dot{m} + \mathcal{L}_v m - d(\frac{1}{2}|v|^2 - V) \otimes \varrho = 0 \\ \dot{\varrho} + \mathcal{L}_v \varrho = 0 \end{cases}$$

where $m = v^\flat \otimes \varrho$,

- the symplectically reduced equations on $T^*\text{Dens}(M)$

$$\begin{cases} \dot{\theta} + \frac{1}{2}|\nabla\theta|^2 + V = 0 \\ \dot{\varrho} + \mathcal{L}_{\nabla\theta}\varrho = 0. \end{cases} \quad (4.3)$$

Observe that the system (4.3) consists of the Hamilton-Jacobi equation for the classical Hamiltonian $H(x, p) = \frac{1}{2}\mathbf{g}_x(p^\sharp, p^\sharp) + V(x)$ and the transport equation for ϱ .

Proof. Since $\frac{\delta}{\delta\varrho}\bar{U} = V$ the proposition follows by combining [Theorem 2.3](#), [Theorem 3.3](#) and [Corollary 3.8](#). \square

Remark 4.3. The corresponding problem in a sub-Riemannian setting was considered in [\[24\]](#).

4.3. Shallow water equations. Consider the case of a quadratic potential on the space of densities

$$\bar{U}(\varrho) = \frac{1}{2} \int_M \rho^2 \mu \quad (4.4)$$

where $\varrho = \rho\mu \in \text{Dens}(M)$.

Proposition 4.4. *Newton's equations with respect to the L^2 -metric (2.1) and the potential (4.4) take the following forms:*

- on $\text{Diff}(M)$

$$\nabla_{\dot{\varphi}} \dot{\varphi} + \nabla \rho \circ \varphi = 0$$

where $\varrho = \varphi_* \mu$,

- the shallow water equations on $\mathfrak{X}(M)$

$$\begin{cases} \dot{v} + \nabla_v v + \nabla \rho = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0 \end{cases} \quad (4.5)$$

where $v = \dot{\varphi} \circ \varphi^{-1}$ is the horizontal velocity field and ρ is the water depth,

- the Poisson reduced equations on $\text{Dens}(M) \times \mathfrak{X}^*(M)$

$$\begin{cases} \dot{m} + \mathcal{L}_v m - d(\frac{1}{2}|v|^2 - \rho) \otimes \varrho = 0 \\ \dot{\varrho} + \mathcal{L}_v \varrho = 0 \end{cases}$$

where $m = v^\flat \otimes \varrho$,

- the Hamiltonian form on $T^*\text{Dens}(M)$

$$\begin{cases} \dot{\theta} + \frac{1}{2}|\nabla\theta|^2 + \rho = 0 \\ \dot{\varrho} + \mathcal{L}_{\nabla\theta}\varrho = 0. \end{cases} \quad (4.6)$$

Since \bar{U} is a quadratic function the equations in (4.6) can be interpreted as the Hamiltonian form of an infinite-dimensional harmonic oscillator with respect to the Wasserstein-Otto metric (2.8).

Proof. The proof again follows directly from Theorem 2.3, Theorem 3.3, Corollary 3.8 and $\frac{\delta}{\delta\varrho}\bar{U} = \rho$. \square

4.4. Barotropic fluid equations. The motion of *barotropic fluids* is characterized by the fact that the pressure is a function of fluid's density. The corresponding equations on a Riemannian manifold expressed in terms of the velocity field v and the density function ρ have the form

$$\begin{cases} \dot{v} + \nabla_v v + \rho^{-1}\nabla P(\rho) = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0. \end{cases} \quad (4.7)$$

The function $P \in C^\infty(\mathbb{R})$ relates ρ and the pressure function $p = P(\rho)$. This relation depends on the properties of the fluid and is called the barotropic equation of state. Note that the equations of *barotropic gas dynamics* are usually specified by a particular choice $P(\rho) = \text{const} \cdot \rho^a$ (where, e.g., $a = 7/5$ corresponds to the standard approximation for atmospheric air.)

To connect these objects with our framework we let $e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function describing the internal energy $e(\rho)$ of a barotropic fluid per unit mass and consider a general potential

$$\bar{U}(\varrho) = \int_M \Phi(\rho) \mu \quad (4.8)$$

where $\Phi(\rho) = e(\rho)\rho$ and $\varrho = \rho\mu$. The relation between pressure and the internal energy is given by

$$P(\rho) = e'(\rho)\rho^2.$$

We also define the *thermodynamical work function* as

$$W(\rho) = \frac{\partial\Phi}{\partial\rho} = e'(\rho)\rho + e(\rho) = \rho^{-1}P(\rho) + e(\rho). \quad (4.9)$$

We have $\rho^{-1}\nabla P(\rho) = \nabla W(\rho)$ which helps explain the idea of introducing the work function W in that the force in (4.7) becomes a pure gradient (here $\nabla W(\rho)$ is understood as the gradient $\nabla(W \circ \rho)$ of a function on M). This can be arranged if the internal energy e depends functionally on ρ . As we have seen in the general form of (2.6) the internal work function W is more fundamental than the pressure function P in the following sense: when the internal energy depends on the derivatives of ρ it may not be possible to find the pressure function but one can always determine the work function.

Proposition 4.5. *Newton's equations for the L^2 -metric (2.1) and the potential (4.8) admit the following formulations:*

- on $\text{Diff}(M)$

$$\nabla_{\dot{\varphi}}\dot{\varphi} + \left(\rho^{-1}\nabla P(\rho)\right) \circ \varphi = 0$$

- the barotropic compressible fluid equations (4.7) on $\mathfrak{X}(M)$ for the velocity field $v = \dot{\varphi} \circ \varphi^{-1}$ and the density function ρ ,
- the Poisson reduced equations on $\text{Dens}(M) \times \mathfrak{X}^*(M)$

$$\begin{cases} \dot{m} + \mathcal{L}_v m - d\left(\frac{1}{2}|v|^2 - W(\rho)\right) \otimes \varrho = 0 \\ \dot{\varrho} + \mathcal{L}_v \varrho = 0 \end{cases}$$

where $m = v^\flat \otimes \varrho$ and $W(\rho)$ is the work function (4.9),

- the symplectically reduced form of the barotropic compressible fluid equations on $T^*\text{Dens}(M)$

$$\begin{cases} \dot{\theta} + \frac{1}{2}|\nabla\theta|^2 + W(\rho) = 0 \\ \dot{\varrho} + \mathcal{L}_{\nabla\theta}\varrho = 0. \end{cases}$$

Proof. The energy function of a compressible barotropic fluid with velocity v and density ρ is

$$E = \frac{1}{2} \int_M |v|^2 \rho \mu + \int_M e(\rho) \rho \mu,$$

where the first term corresponds to fluid's kinetic energy and the second is the potential energy under the barotropic assumption. Introducing the momentum variable $m = v^\flat \otimes \varrho$ we obtain a Hamiltonian on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ of the form

$$\bar{H}(\varrho, m) = \frac{1}{2} \langle m, v \rangle + \langle \varrho, e(\rho) \rangle \quad m = v^\flat \otimes \varrho. \quad (4.10)$$

It is clear that $\frac{\delta \bar{H}}{\delta m} \bar{H} = v$. Furthermore, we have

$$\left\langle \frac{\delta \bar{H}}{\delta \varrho}, \dot{\varrho} \right\rangle = \left\langle -\frac{1}{2}|v|^2, \dot{\varrho} \right\rangle + \langle \Phi'(\rho), \dot{\varrho} \rangle = \left\langle -\frac{1}{2}|v|^2 + \Phi'(\rho), \dot{\varrho} \right\rangle.$$

Substituting into (3.6) we arrive at the system

$$\begin{cases} \dot{m} = -\mathcal{L}_v m - d\left(-\frac{1}{2}|v|^2 + \Phi'(\rho)\right) \otimes \varrho \\ \dot{\varrho} = -\mathcal{L}_v \varrho. \end{cases}$$

To obtain the compressible Euler equations we rewrite the second term in the first equation using (4.9) as

$$\begin{aligned} d\Phi'(\rho) \otimes \rho \mu &= \left(d(\rho \Phi'(\rho)) - \Phi'(\rho) d\rho \right) \otimes \mu \\ &= d(\rho \Phi'(\rho) - \Phi(\rho)) \otimes \mu \\ &= dP(\rho) \otimes \mu \end{aligned}$$

to get

$$\begin{cases} \dot{m} = -\mathcal{L}_v m + \frac{1}{2} d\iota_v v^\flat \otimes \varrho - dP \otimes \mu \\ \dot{\varrho} = -\mathcal{L}_v \varrho. \end{cases} \quad (4.11)$$

Differentiating $m = \rho v^\flat \otimes \mu$ in the time variable

$$\dot{m} = (\rho \dot{v}^\flat + \dot{\rho} v^\flat) \otimes \mu = \dot{v}^\flat \otimes \varrho - v^\flat \otimes \mathcal{L}_v \varrho$$

and substituting into (4.11) we obtain

$$\begin{cases} \dot{v}^b \otimes \varrho = (-\mathcal{L}_v v^b + \frac{1}{2} d\iota_v v^b - \rho^{-1} dP) \otimes \varrho \\ \dot{\varrho} = -\mathcal{L}_v \varrho. \end{cases}$$

Using the identities $\mathcal{L}_v \varrho = \operatorname{div}(\rho v)\mu$ and $(\nabla_v v)^b = \mathcal{L}_v v^b - \frac{1}{2} d\iota_v v^b$ we now recover the compressible Euler equations (4.7), that is

$$\begin{cases} \dot{v} + \nabla_v v = -\rho^{-1} \nabla P(\rho) \\ \dot{\rho} + \operatorname{div}(\rho v) = 0. \end{cases}$$

To describe these equations as Newton's equations (3.3) on $\operatorname{Diff}(M)$ we introduce a potential $\bar{U}: \operatorname{Dens}(M) \rightarrow \mathbb{R}$ of the form (2.4), i.e.,

$$\bar{U}(\varrho) = \int_M \Phi(\rho)\mu$$

with $\frac{\delta}{\delta \varrho} \bar{U}(\varrho) = \Phi'(\rho) = e'(\rho)\rho + e(\rho)$. From Theorem 2.3 we find Newton's equations corresponding to the compressible Euler equations

$$\nabla_{\dot{\varphi}} \dot{\varphi} = -\nabla(\Phi'(\varrho)) \circ \varphi.$$

From Corollary 3.8 we get the symplectically reduced form on $T^*\operatorname{Dens}(M)$. \square

4.5. Short-time existence of compressible Euler equations. We include here a local existence result that applies to all the examples in this section. To this end consider the compressible Euler equations on a compact manifold M in the form

$$\begin{cases} \dot{v} + \nabla_v v + \nabla(W \circ \rho) + \nabla V = 0 \\ \dot{\rho} + \operatorname{div}(\rho v) = 0 \end{cases} \quad (4.12)$$

where W is the thermodynamical work function defined in (4.9). The equations discussed previously can be captured by different choices of the functions W and V . If W is strictly increasing then short-time solutions of these equations can be obtained using standard techniques (see e.g. [19, Thm. III.2.1.2] for a result for the shallow water equations (4.5) corresponding to $W(\rho) = \rho$).

Theorem 4.6. *For any $v_0 \in \mathfrak{X}(M)$, $\rho_0 \in \operatorname{Dens}(M)$ and any smooth function $W: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $W' > 0$ there exists a unique smooth solution (v, ρ) of the equations (4.12) satisfying $v(t_0) = v_0$, $\rho(t_0) = \rho_0$ and defined in some open neighbourhood of $t = t_0$.*

Proof. The basic idea is to transform (4.12) so that its linearization becomes a symmetric linear system. This can be achieved by a substitution $\rho = f \circ \sigma$ where σ is a new density function and $f: \mathbb{R} \rightarrow \mathbb{R}$ is the solution of the following scalar initial value problem

$$f' = \sqrt{\frac{f}{W'(f)}}, \quad f(0) = \min_{x \in M} \rho_0(x). \quad (4.13)$$

Compactness of M together with the assumption $W' > 0$ assure that the right-hand side of (4.13) is Lipschitz continuous in the interval given by the range of ρ_0 . Thus, there is a smooth solution f whose range covers the range of ρ_0 . Since $f(0) > 0$ and $W' > 0$ this solution is strictly increasing.

It follows that the corresponding linearized equations form a symmetric linear system in a neighborhood of the density $\rho = \rho_0$ and thus admit a unique tame solution by the general theory of symmetric systems. Applying the Nash-Moser-Hamilton theorem completes the proof. \square

Using the results in §3.3 it is possible to deduce from the above theorem short-time existence results for each of the equations considered above: the Newton systems on $\text{Diff}(M)$, the Poisson systems on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ or the canonical Hamiltonian systems on $T^*\text{Dens}(M)$.

5. SEMIDIRECT PRODUCT REDUCTION

In this section we recall one standard approach to the equations of compressible fluid dynamics using semidirect products, see [56, 37]. Recall from the earlier sections that the barotropic Euler equations can be viewed as a mechanical system on the configuration space $\text{Diff}(M)$ with the symmetry group $\text{Diff}_\mu(M)$. On the other hand, such a system can be also obtained by a semidirect product construction as a so-called Lie-Poisson system provided that the configuration space is extended so that it coincides with the given symmetry group. We unify these approaches in §5.3: any Lie-Poisson system on a semidirect product can be viewed as a Newton system with a smaller symmetry group. We begin with two standard examples.

5.1. Barotropic fluids via semidirect products. In order to describe a barotropic fluid (4.7) it is convenient to introduce the semidirect product group $S = \text{Diff}(M) \ltimes C^\infty(M)$ as a space of pairs (φ, f) equipped with the group structure given by

$$(\varphi, f) \cdot (\psi, g) = (\varphi \circ \psi, \varphi_*g + f), \quad \varphi_*g = g \circ \varphi^{-1} \quad (5.1)$$

which is smooth in the Fréchet topology.

The Lie algebra $\mathfrak{s} = \mathfrak{X}(M) \ltimes C^\infty(M)$ is also a semidirect product with a commutator given by

$$\text{ad}_{(v,b)}(u, a) = (-\mathcal{L}_v u, \mathcal{L}_u b - \mathcal{L}_v a). \quad (5.2)$$

The corresponding (smooth) dual space is $\mathfrak{s}^* = \mathfrak{X}^*(M) \times \Omega^n(M)$ whose elements are pairs (m, ϱ) with $m = \alpha \otimes \mu \in \mathfrak{X}^*(M)$ and $\varrho \in \Omega^n$, where μ is a fixed volume form and α is a 1-form on M . The pairing between \mathfrak{s} and \mathfrak{s}^* is given by

$$\langle (v, b), (m, \varrho) \rangle = \int_M (\iota_v \alpha) \mu + \int_M b \varrho.$$

The Lie algebra structure of \mathfrak{s} determines the Lie-Poisson structure on \mathfrak{s}^* and the corresponding Poisson bracket at $(m, \varrho) \in \mathfrak{X}^*(M) \times \Omega^n(M)$ is given by the formula (3.5). It is sometimes called the *compressible fluid bracket*. (We refer to §5.3 for a general setting of semidirect products and explicit formulas.)

In order to define a dynamical system on S we put a Riemannian metric on M and given a smooth function P (relating pressure to fluid's density ρ , as in §4.4) of the form $P(\rho) = \rho^2 \Phi'(\rho)$ define the energy function on \mathfrak{s}

$$E(v, \varrho) = \int_M \left(\frac{1}{2} |v|^2 \rho + \rho \Phi(\rho) \right) \mu.$$

Lifting E to the dual \mathfrak{s}^* with the help of the inertia operator of the Riemannian metric we obtain the following Hamiltonian on \mathfrak{s}^*

$$H(m, \varrho) = \int_M \left(\frac{1}{2\rho} |m|^2 + \rho \Phi(\rho) \right) \mu. \quad (5.3)$$

Observe that, by construction, the associated Hamiltonian system on the cotangent bundle T^*S is right-invariant with respect to the action of S .

Theorem 5.1 ([56, 37]). *The barotropic fluid equations (4.7) correspond to the Lie-Poisson system on \mathfrak{s}^* with the Poisson bracket of type (3.5) and the Hamiltonian (5.3).*

While the general barotropic equations described above are valid for any smooth initial velocity field, one is often interested only in *potential* solutions of the system. These are obtained from initial conditions of the form $v_0 = \nabla\theta_0$ where θ_0 is a smooth function on M . As we have already seen, such solutions retain their gradient form for all times and the equations can be viewed as the Hamilton-Jacobi equations, see (4.6). Potential solutions of this type arise naturally in the context of the Madelung transform, see §9 below.

Remark 5.2. The semidirect product framework is a natural setting whenever the physical model contains a quantity transported by the flow, e.g., the continuity equation (4.7). However, while the Hamiltonian point of view works similarly to the case of incompressible fluids, the Lagrangian approach encounters certain drawbacks, cf. [22]. These are mostly related to the fact that the Lagrangian is not quadratic and cannot be directly interpreted as a kinetic energy yielding geodesics on the group (for some attempts to bypass this problem using the Maupertuis principle see Smolentsev [51]; for a geodesic formulation in an extended phase space see Preston [47]). Furthermore, there is no physical interpretation of the action of the full semidirect product on its dual space: the particle reparametrization symmetry is related only to the action of the first (diffeomorphisms) but not of the second (functions) factor in the product $S = \text{Diff}(M) \ltimes C^\infty(M)$. One advantage of our point of view using Newton's equations is that it resolves such issues.

5.2. Incompressible magnetohydrodynamics. An approach based on semidirect products is also possible in the case of the equations of self-consistent magnetohydrodynamics (MHD). We start with the incompressible case and discuss the compressible case in detail in §6.2. The underlying system describes an ideal fluid whose divergence-free velocity v is governed by the Euler equations (see Appendix A for Lagrangian and Hamiltonian formulations). Assume next that the fluid has infinite conductivity and carries a (divergence-free) magnetic field \mathbf{B} . Transported by the flow (i.e., frozen in the fluid) \mathbf{B} acts reciprocally (via the Lorenz force) on the velocity field and the resulting MHD system on a three-dimensional Riemannian manifold M takes the form

$$\begin{cases} \dot{v} + \nabla_v v + \mathbf{B} \times \text{curl } \mathbf{B} + \nabla P = 0 \\ \dot{\mathbf{B}} + \mathcal{L}_v \mathbf{B} = 0 \\ \text{div } v = 0 \\ \text{div } \mathbf{B} = 0. \end{cases} \quad (5.4)$$

A natural configuration space for the system (5.4) is the semidirect product of the group of volume-preserving diffeomorphisms and the dual of the space $\mathfrak{X}_\mu(M)$ of divergence-free vector fields on a three-fold M . The corresponding Lie algebra is the semidirect product of $\mathfrak{X}_\mu(M)$ and its dual. The group product and the algebra commutator are given by the formulas (5.1) and (5.2), respectively.

More generally, the configuration space of incompressible magnetohydrodynamics on a manifold M of arbitrary dimension n is the semidirect product group $\text{IMH} = \text{Diff}_\mu(M) \ltimes \Omega^{n-2}(M)/d\Omega^{n-3}(M)$ (which for $n = 3$ reduces to $\text{Diff}_\mu(M) \ltimes \mathfrak{X}_\mu^*(M)$) with its Lie algebra $\text{imh} = \mathfrak{X}_\mu(M) \ltimes \Omega^{n-2}(M)/d\Omega^{n-3}(M)$. Since the dual of $\Omega^{n-2}(M)/d\Omega^{n-3}(M)$ is the space $\Omega_{cl}^2(M)$ of closed 2-forms on M we have $\text{imh}^* = \mathfrak{X}_\mu^*(M) \oplus \Omega_{cl}^2(M)$. Magnetic fields in M can be viewed as either closed 2-forms $\beta \in \Omega_{cl}^2(M)$ or $(n-2)$ fields \mathbf{B} that are related to β by $\beta = \iota_{\mathbf{B}}\mu$. This latter point of view will be useful also for the description of compressible magnetohydrodynamics.

The corresponding Poisson bracket on imh^* is given by the formula (3.5) interpreted accordingly.

Finally, as the Hamiltonian function we take the sum of the kinetic and magnetic energies of the fluid, i.e.

$$E(v, \mathbf{B}) = \frac{1}{2} \int_M (|v|^2 + |\mathbf{B}|^2) \mu$$

(here the Riemannian metric defines the inertia operator and hence the L^2 quadratic form on all spaces $\mathfrak{X}_\mu(M)$, $\mathfrak{X}_\mu^*(M)$ and $\Omega_{cl}^2(M)$, see, e.g., [5]). The Hamiltonian on $\mathfrak{im}\mathfrak{h}^*$ is

$$H(m, \mathbf{B}) = \frac{1}{2} \int_M (|m|^2 + |\mathbf{B}|^2) \mu. \quad (5.5)$$

Theorem 5.3 ([56, 5]). *The incompressible MHD equations (5.4) correspond to the Lie-Poisson system on $\mathfrak{im}\mathfrak{h}^*$ for the Hamiltonian (5.5).*

An analogue of this equation for compressible fluids in an n -dimensional manifold will be discussed in Section 6.2.

5.3. Reduction and momentum map for semidirect product groups. We exhibit here geometric structures behind the semidirect product reduction generalizing the considerations of Sections 5.1-5.2. Our main point is that the semidirect product approach is just a convenient way of presenting various Newton's systems on $\text{Diff}(M)$ for which the symmetry group is a proper subset of $\text{Diff}(M)$.

Let \mathcal{N} be a subgroup of $\text{Diff}(M)$. Suppose that $\text{Diff}(M)$ acts from the left on a linear space V (a left representation of $\text{Diff}(M)$). In the two previous sections V was taken to be the space of functions $C^\infty(M)$ or the dual of the space of divergence-free vector fields $\Omega^1(M)/dC^\infty(M)$. In particular, \mathcal{N} can be a subgroup of volume-preserving diffeomorphisms $\text{Diff}_\mu(M)$ however the consideration below is more general.

The quotient space of left cosets $\text{Diff}(M)/\mathcal{N}$ is acted upon from the left by $\text{Diff}(M)$. Assume now that $\text{Diff}(M)/\mathcal{N}$ can be embedded as an orbit in V and let $\gamma: \text{Diff}(M)/\mathcal{N} \rightarrow V$ denote the embedding. Since the action of $\text{Diff}(M)$ on V induces a linear left dual action on V^* we can construct the semidirect product $S = \text{Diff}(M) \ltimes V^*$. Let \mathfrak{s}^* be the dual of the corresponding semidirect product algebra \mathfrak{s} .

Proposition 5.4. *The quotient $T^*\text{Diff}(M)/\mathcal{N}$ is naturally embedded via a Poisson map in the Lie-Poisson space \mathfrak{s}^* .*

Proof. The Poisson embedding is given by

$$([\varphi], m) \mapsto (m, \gamma([\varphi])) \quad (5.6)$$

where we used the identifications

$$T^*\text{Diff}(M)/\mathcal{N} \simeq \text{Diff}(M)/\mathcal{N} \times \mathfrak{g}^* = \text{Diff}(M)/\mathcal{N} \times (\Omega^1 \otimes \text{Dens}(M))$$

and $\mathfrak{s}^* \simeq \mathfrak{g}^* \times V = (\Omega^1 \otimes \text{Dens}(M)) \times V$. Recall that the Lie algebra of $\text{Diff}(M)$ is the space $\mathfrak{X}(M)$ of vector fields on M whose dual is $\mathfrak{X}^*(M) = \Omega^1 \otimes \text{Dens}(M)$. The action of S on \mathfrak{s}^* is given by

$$(\varphi, a) \cdot (m, b) = \text{Ad}_{(\varphi, a)}^*(m, b) = (\varphi^*m - \mathcal{M}(a, b), \varphi^*b),$$

where $\varphi \in \text{Diff}(M)$ and $\mathcal{M}: V^* \times V \rightarrow \mathfrak{X}^*(M)$ is the momentum map associated with the cotangent lifted action of $\text{Diff}(M)$ on V^* . The corresponding infinitesimal action of \mathfrak{s} is

$$(v, \dot{a}) \cdot (m, b) = \text{ad}_{(v, \dot{a})}^*(m, b) = (\mathcal{L}_v m - \mathcal{M}(\dot{a}, b), \mathcal{L}_v b). \quad (5.7)$$

Since the second component is only acted upon by φ (or v) but not a (or \dot{a}), it follows from the embedding of $\text{Diff}(M)/\mathcal{N}$ as an orbit in V that we have a natural Poisson action of S (or \mathfrak{s}) on $T^*\text{Diff}(M)/\mathcal{N}$ via the Poisson embedding (5.6). Notice that the momentum map of S (or

\mathfrak{s}) acting on \mathfrak{s}^* is tautological, i.e. the identity: this follows from the fact that the Hamiltonian vector field on \mathfrak{s}^* for $H(m, b) = \langle m, v \rangle + \langle b, \dot{a} \rangle$ is given by (5.7). \square

We now return to the standard symplectic reduction (without semidirect products). The dual \mathfrak{n}^* of the subalgebra $\mathfrak{n} \subset \mathfrak{X}(M)$ is naturally identified with the affine cosets of $\mathfrak{X}^*(M)$ such that

$$m \in [m_0] \iff \langle m - m_0, v \rangle = 0 \quad \text{for any } v \in \mathfrak{n}.$$

The momentum map of the subgroup \mathcal{N} acting on $\mathfrak{X}^*(M)$ by φ^* is then given by $m \mapsto [m]$, since the momentum map of $\text{Diff}(M)$ acting on $\mathfrak{X}^*(M)$ is the identity. If $\langle m, \mathfrak{n} \rangle = 0$, i.e., $m \in (\mathfrak{X}(M)/\mathfrak{n})^*$, then $m \in [0]$ is in the zero momentum coset. Since we also have $T^*(\text{Diff}(M)/\mathcal{N}) \simeq \text{Diff}(M)/\mathcal{N} \times (\mathfrak{X}(M)/\mathfrak{n})^*$ this gives us an embedding as a symplectic leaf in $T^*\text{Diff}(M)/\mathcal{N} \simeq \text{Diff}(M)/\mathcal{N} \times \mathfrak{X}^*(M)$. The restriction to this leaf is called the *zero-momentum symplectic reduction*.

Turning next to the semidirect product reduction, we now have Poisson embeddings of $T^*(\text{Diff}(M)/\mathcal{N})$ in $T^*\text{Diff}(M)/\mathcal{N}$ and of $T^*\text{Diff}(M)/\mathcal{N}$ in \mathfrak{s}^* . The combined embedding of $T^*(\text{Diff}(M)/\mathcal{N})$ as a symplectic leaf in \mathfrak{s}^* is given by the map

$$([\varphi], a) \mapsto (\mathcal{M}(a, \gamma([\varphi])), \gamma([\varphi])). \quad (5.8)$$

This implies that we have a Hamiltonian action of S (or \mathfrak{s}) on the zero-momentum symplectic leaf $T^*(\text{Diff}(M)/\mathcal{N})$ inside $T^*\text{Diff}(M)/\mathcal{N}$, which in turn lies inside \mathfrak{s}^* .

Since S provides a natural symplectic action on \mathfrak{s}^* and since $\text{Diff}(M)/\mathcal{N}$ is an orbit in $V \simeq V^{**}$ we have, by restriction, a natural action of S on $T^*\text{Diff}(M)/\mathcal{N}$. Furthermore, since the momentum map associated with the group S acting on \mathfrak{s}^* is the identity, the Poisson embedding map (5.6) is the momentum map for S acting on $T^*\text{Diff}(M)/\mathcal{N}$. Thus, the momentum map of S acting on $T^*(\text{Diff}(M)/\mathcal{N})$ is given by (5.8).

The above considerations lead to the Madelung transform.

Theorem 5.5 ([28]). *Semidirect product reduction and Poisson embedding $T^*(\text{Diff}(M)/\mathcal{N}) \rightarrow \mathfrak{s}^*$ for the subgroup $\mathcal{N} = \text{Diff}_\mu(M)$ coincides with the inverse of the Madelung transform defined in § 9.*

6. MORE GENERAL LAGRANGIANS

6.1. Fully compressible fluids. For general compressible (non-barotropic) inviscid fluids the equation of state includes pressure $P = P(\rho, \sigma)$ as a function of both density ρ and specific entropy σ (defined as a smooth function on M representing entropy per unit mass, cf. Dolzhansky [12, Sect. 3.2]). Thus, the equations of motion describe the evolution of three quantities: the velocity of the fluid v , its density ρ and the specific entropy σ , namely

$$\begin{cases} \dot{v} + \nabla_v v + \rho^{-1} \nabla P(\rho, \sigma) = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0 \\ \dot{\sigma} + \text{div}(\sigma v) = 0. \end{cases} \quad (6.1)$$

The purpose of this section is to show that under natural assumptions this system also describes Newton's equations on $\text{Diff}(M)$ but with potential function of more general form than (2.4). Introducing the entropy density $\varsigma = \sigma \mu$ one can regard both density and entropy as n -forms. In view of the results in § 5.3 the full compressible Euler equations are a semidirect product representation of a Newton system on $\text{Diff}(M)$ whose symmetry group is a proper subgroup of $\text{Diff}_\mu(M)$.

Theorem 6.1. *The fully compressible system (6.1) is obtained using an embedding into the Lie-Poisson space $\mathfrak{s}_{(2)}^*$ where $\mathfrak{s}_{(2)} = \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}^2)$ (cf. Proposition 5.4) from Newton's equations on $\text{Diff}(M)$ with Lagrangian*

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \int_M |\dot{\varphi}|^2 \mu - \bar{U}(\varphi_* \varrho_0, \varphi_* \varsigma_0) \quad (6.2)$$

where $\bar{U}: \text{Dens}(M) \times \Omega^n(M) \rightarrow \mathbb{R}$ is a potential function (of density $\varrho = \rho\mu$ and entropy density $\varsigma = \sigma\mu$) of the form

$$U(\varrho, \varsigma) = \int_M e(\rho, \sigma) \varrho$$

and where the internal energy e and pressure P are related by

$$P(\rho, \sigma) = \rho^2 \frac{\partial e}{\partial \rho}(\rho, \sigma) + \sigma \rho \frac{\partial e}{\partial \sigma}(\rho, \sigma).$$

From the point of view of §5.3 the symmetry subgroup \mathcal{N} is given by $\text{Diff}_{\varrho_0}(M) \cap \text{Diff}_{\varsigma_0}(M)$. Our aim is to embed $T^*\text{Diff}(M)/\mathcal{N}$ in $\mathfrak{s}_{(2)}^* = \mathfrak{X}^*(M) \times (\Omega^n(M))^2$. To accomplish this we need to compute the momentum map for the cotangent lifted action of $\text{Diff}(M)$ on $T^*(\Omega^n(M))^2$.

Lemma 6.2. *The momentum map for the cotangent action of $\text{Diff}(M)$ on $T^*\Omega^n(M) \times T^*\Omega^n(M) = \Omega^n(M) \times C^\infty(M) \times \Omega^n(M) \times C^\infty(M)$ is*

$$J(\varrho, \theta, \varsigma, \kappa) = \varrho \otimes d\theta + \varsigma \otimes d\kappa.$$

Proof. From §3 we already know the momentum map for the action on $T^*\text{Dens}(M)$. This is the same as the action on $T^*\Omega^n(M)$. For diagonal actions we then just get a sum as stated in the lemma. \square

Proof of Theorem 6.1. Any Hamiltonian system on the Poisson space $\mathfrak{s}_{(2)}^*$ has the form

$$\dot{m} + \mathcal{L}_v m + J\left(\varrho, \frac{\delta H}{\delta \varrho}, \varsigma, \frac{\delta H}{\delta \varsigma}\right) = 0, \quad \dot{\varrho} + \mathcal{L}_v \varrho = 0, \quad \dot{\varsigma} + \mathcal{L}_v \varsigma = 0$$

where $v = \frac{\delta H}{\delta m}$. The Hamiltonian corresponding to the Lagrangian (6.2) is the same as in (4.10) except that the potential energy \bar{U} depends now also on ς . By Lemma 6.2 the first equation then becomes

$$\dot{m} + \mathcal{L}_v m + \varrho \otimes d\left(\frac{\delta \bar{U}}{\delta \varrho} - \frac{1}{2}|v|^2\right) + \varsigma \otimes d\left(\frac{\delta \bar{U}}{\delta \varsigma}\right) = 0. \quad (6.3)$$

The variational derivatives are given by

$$\frac{\delta \bar{U}}{\delta \varrho} = e(\rho, \sigma) + \rho \frac{\partial e}{\partial \rho}(\rho, \sigma) \quad \text{and} \quad \frac{\delta \bar{U}}{\delta \varsigma} = \rho \frac{\partial e}{\partial \sigma}(\rho, \sigma).$$

Using

$$\begin{aligned} dP(\rho, \sigma) &= d\left(\rho^2 \frac{\partial e}{\partial \rho}(\rho, \sigma) + \sigma \rho \frac{\partial e}{\partial \sigma}(\rho, \sigma)\right) \\ &= \rho d\left(e(\rho, \sigma) + \rho \frac{\partial e}{\partial \rho}(\rho, \sigma)\right) + \sigma d\left(\rho \frac{\partial e}{\partial \sigma}(\rho, \sigma)\right) \\ &= \rho d\frac{\delta \bar{U}}{\delta \varrho} + \sigma d\frac{\delta \bar{U}}{\delta \varsigma} \end{aligned}$$

we then recover from (6.3) the fully compressible Euler equations (6.1). \square

Observe that an invariant subset of solutions is given by those solutions with momenta $m = \varrho \otimes d\theta + \varsigma \otimes d\kappa$, where $\theta, \kappa \in C^\infty(M)$. They can be regarded as analogues of potential solutions of the barotropic fluid equations. We thereby obtain a canonical set of equations on $T^*(\Omega^n(M))^2$

$$\begin{cases} \dot{\varrho} = \frac{\delta \bar{H}}{\delta \theta} & \dot{\varsigma} = \frac{\delta \bar{H}}{\delta \kappa} \\ \dot{\theta} = -\frac{\delta \bar{H}}{\delta \varrho} & \dot{\kappa} = -\frac{\delta \bar{H}}{\delta \varsigma} \end{cases}$$

with the restricted Hamiltonian

$$\bar{H}(\varrho, \varsigma, \theta, \kappa) = H(\varrho \otimes d\theta + \varsigma \otimes d\kappa, \varrho, \varsigma).$$

We point out that the group $S_{(2)} = \text{Diff}(M) \ltimes C^\infty(M, \mathbb{R}^2)$ corresponding to $\mathfrak{s}_{(2)}$ is associated with a multicomponent version of the Madelung transform relating compressible fluids and the NLS-type equations, cf. the details in § 9 and see also [28]. Applying the multicomponent Madelung transform $\mathbf{M}^{(2)}$ one can also rewrite the fully compressible system on the space of rank-1 spinors $H^s(M, \mathbb{C}^2)$.

Remark 6.3. Solutions of barotropic fluid equations are contained in the solution space of the fully compressible Euler equations as “horizontal-within-horizontal” solutions in the following sense. Let the initial entropy function have the form $\sigma = s(\rho)$ for some function $s \in C^\infty(\mathbb{R}_+, \mathbb{R})$. Then

$$\dot{\sigma} = s'(\rho)\dot{\rho} = -s'(\rho) \text{div}(\rho u),$$

where the last equality follows from the evolution equation for ρ . From the equation for σ we obtain

$$\dot{\sigma} = -\text{div}(\sigma u) = -s'(\rho) \text{div}(\rho u).$$

Thus, the entropy remains in the form $\sigma = s(\rho)$ so that we obtain a barotropic flow with the pressure function $P(\rho, s(\rho))$. From a geometric point of view these solutions correspond to a special symplectic leaf in $\mathfrak{s}^* = \mathfrak{X}^*(M) \times \Omega^n(M) \times \Omega^n(M)$.

6.2. Compressible magnetohydrodynamics. Next, we turn to a description of compressible inviscid magnetohydrodynamics. A compressible fluid of infinite conductivity carries a magnetic field acting reciprocally on the fluid. The corresponding equations on a Riemannian 3-manifold M have the form

$$\begin{cases} \dot{v} + \nabla_v v + \rho^{-1} \mathbf{B} \times \text{curl} \mathbf{B} + \rho^{-1} \nabla P(\rho) = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0 \\ \dot{\mathbf{B}} + \text{curl} \mathbf{E} = 0, \quad \mathbf{E} = \mathbf{B} \times v, \end{cases} \quad (6.4)$$

where v is the velocity and ρ is density of the fluid, while \mathbf{B} is the magnetic vector field. Note that these equations reduce to the incompressible MHD equations (5.4) when density ρ is a constant.

As mentioned before, it is more natural to think of magnetic fields as closed 2-forms. This becomes apparent when the equations are generalized to a compressible setting or to other dimensions. (For instance, a non-volume-preserving diffeomorphism violates the divergence-free constraint of a magnetic vector field but preserves closedness of differential forms.) In fact, let $\Omega_{cl}^2(M)$ denote the space of smooth closed differential 2-forms on an n -manifold M . The diffeomorphism group acts on $\Omega_{cl}^2(M)$ by push-forward and the (smooth) dual of $\Omega_{cl}^2(M)$ is the quotient $\Omega^{n-2}(M)/d\Omega^{n-3}(M)$.

The cotangent lift of the left action of $\text{Diff}(M)$ to $T^*\Omega_{cl}^2(M) \simeq \Omega_{cl}^2(M) \times \Omega^{n-2}(M)/d\Omega^{n-3}(M)$ is given by

$$\varphi \cdot (\beta, [P]) = (\varphi_*\beta, \varphi_*[P]). \quad (6.5)$$

Observe that this is well-defined since push-forward commutes with the exterior differential.

Lemma 6.4. *The momentum map $I: T^*\Omega_{cl}^2(M) \rightarrow \mathfrak{X}^*(M)$ associated with the cotangent action in (6.5) is given by*

$$I(\beta, [P]) = \iota_u\beta \otimes \mu,$$

where the vector field u is uniquely defined by $\iota_u\mu = dP$.

As expected, the map J is independent of the choice of μ and a representative P . In what follows it will be convenient to replace μ by ϱ - resulting in a different vector field u but without affecting the momentum map.

Proof. The infinitesimal action of a vector field v on β is $-\mathcal{L}_v\beta$. Since it is a cotangent lifted action, the momentum map is given by

$$\begin{aligned} \langle I(\beta, [P]), v \rangle &= \langle -\mathcal{L}_v\beta, [P] \rangle = \int_M -\mathcal{L}_v\beta \wedge P \\ &= \int_M -d\iota_v\beta \wedge P = \int_M -\iota_v\beta \wedge dP. \end{aligned}$$

Now, if $\iota_u\mu = dP$, then

$$\int_M -\iota_v\beta \wedge dP = \int_M (\iota_v\iota_u\beta)\mu = \langle v, \iota_u\beta \otimes \mu \rangle.$$

□

Consider a Lagrangian on $T\text{Diff}(M)$ given by the fluid's kinetic and potential energies with an additional term involving the action on the magnetic field $\beta_0 \in \Omega_{cl}^2(M)$, namely

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \int_M |v|^2 \varrho - \int_M e(\rho) \varrho - \frac{1}{2} \int_M \beta \wedge \star\beta,$$

where $v = \dot{\varphi} \circ \varphi^{-1}$, $\varrho = \varphi_*\mu$ and $\beta = \varphi_*\beta_0$. As in Lemma 3.1 the corresponding Hamiltonian is

$$H(\varphi, m) = \frac{1}{2} \langle m, v \rangle + \int_M e(\rho) \varrho + \frac{1}{2} \int_M \beta \wedge \star\beta \quad (6.6)$$

where $m = v^\flat \otimes \varrho$. Letting $\text{Diff}_{\beta_0}(M)$ denote the isotropy subgroup for the action of $\text{Diff}(M)$, the (right) symmetry group of the Hamiltonian (6.6) is

$$\mathcal{G} = \text{Diff}_\mu(M) \cap \text{Diff}_{\beta_0}(M).$$

The corresponding Lie algebra consists of vector fields such that

$$\text{div } v = 0 \quad \text{and} \quad \mathcal{L}_v\beta_0 = 0.$$

If M is even-dimensional and β_0 is non-degenerate then the pair (M, β_0) is a symplectic manifold and the Lie algebra consists of symplectic vector fields that also preserve the first integral β_0^n/μ .

Next, we proceed to carry out Poisson reduction, i.e., to compute the reduced equations on $T^*\text{Diff}(M)/\mathcal{G} \simeq \text{Diff}(M)/\mathcal{G} \times \mathfrak{X}^*(M)$. In contrast to the case $\mathcal{G} = \text{Diff}_\mu(M)$ studied in § 3 there is no simple way to identify $\text{Diff}(M)/\mathcal{G}$ and so it will be convenient to use the semidirect product reduction framework developed in § 5.3 above. To this end, consider the semidirect product algebra $\mathfrak{cmh} = \mathfrak{X}(M) \times (C^\infty(M) \oplus \Omega^{n-2}(M)/d\Omega^{n-3}(M))$ and its dual

$$\mathfrak{cmh}^* = \mathfrak{X}^*(M) \times (\Omega^n(M) \oplus \Omega_{cl}^2(M)).$$

We have a natural embedding of $T^*\text{Diff}(M)/\mathcal{G}$ in \mathfrak{cmh}^* via the map $([\varphi], m) \mapsto (m, \varphi_*\mu, \varphi_*\beta_0)$ and the corresponding Hamiltonian on \mathfrak{cmh}^* is

$$\bar{H}(\varrho, \beta, m) = \frac{1}{2} \langle m, v \rangle + \int_M e(\rho)\varrho + \frac{1}{2} \int_M \beta \wedge \star\beta.$$

Theorem 6.5. *The Poisson reduced form on*

$$T^*\text{Diff}(M)/\mathcal{G} \simeq \text{Diff}(M)/\mathcal{G} \times \mathfrak{X}^*(M) \subset \mathfrak{cmh}^*$$

of the Euler-Lagrange equations for the Hamiltonian (6.6) is

$$\begin{cases} \dot{m} + \mathcal{L}_v m + \iota_u \beta \otimes \varrho + d \left(\frac{\delta H}{\delta \varrho} \right) \otimes \varrho = 0, \\ \dot{\rho} + \text{div}(\rho v) = 0, \\ \dot{\beta} + \mathcal{L}_v \beta = 0, \end{cases} \quad (6.7)$$

where the field u is defined by $\iota_u \varrho = d \left(\frac{\delta H}{\delta \beta} \right)$ and the momentum variable is $m = v^\flat \otimes \varrho$. For a three-fold M these equations correspond to the equations of the compressible inviscid magneto-hydrodynamics (6.4) where the magnetic field \mathbf{B} is related to the closed 2-form β by $\iota_{\mathbf{B}}\mu = \beta$.

Proof. In general, if $\text{Diff}(M)$ acts on a space S from the left with the momentum map $I: T^*S \rightarrow \mathfrak{X}^*(M)$ then the Poisson reduced system is

$$\begin{cases} \dot{m} + \mathcal{L}_v m - I \left(s, \frac{\delta L}{\delta s} \right) = 0, \\ \dot{s} + \mathcal{L}_v s = 0. \end{cases}$$

In our case, $S = \text{Dens}(M) \times \Omega_{cl}^2(M)$ and the momentum map is

$$(\rho, \theta, \beta, [P]) \mapsto (d\theta + \iota_u \beta) \otimes \varrho, \quad \iota_u \varrho = dP.$$

The rest of the proof follows from direct calculations. \square

Corollary 6.6. *The equations (6.7) admit special ‘horizontal’ solutions corresponding to momenta of the form*

$$m = d\theta \otimes \varrho + \iota_u \beta \otimes \varrho, \quad \iota_u \varrho = dP.$$

These solutions can be expressed in the variables $(\rho, \beta, \theta, [P]) \in T^(\text{Dens}(M) \times \Omega_{cl}^2(M))$ as a canonical Hamiltonian system for the Hamiltonian*

$$\tilde{H}(\rho, \beta, \theta, [P]) = \int_M \left(\frac{1}{2} \iota_v (d\theta + \iota_u \beta) \varrho + e(\rho)\rho + \frac{1}{2} \beta \wedge \star\beta \right) \quad (6.8)$$

where $v^\flat = d\theta + \iota_u \beta$ and $\iota_u \varrho = dP$.

Proof. The horizontal solutions correspond to the submanifold $J^{-1}(0)$, where J is the corresponding momentum map. We refer to [Appendix B](#) for details on symplectic reduction. The Hamiltonian (6.8) is just the restriction of \bar{H} to the special momenta. \square

6.3. Relativistic inviscid Burgers’ equation. In this section we present a relativistic version of the Otto calculus, motivated by the treatment in Brenier [9]. We show that it leads to a relativistic Lagrangian on $\text{Diff}(M)$ and employ Poisson reduction of §3.1 to obtain the relativistic hydrodynamics equations.

As in the classical case, we consider a path in the space of diffeomorphisms as a family of free relativistic particles. Given $\varphi: [0, 1] \times M \rightarrow M$ the action is then given by

$$S(\varphi) = - \int_0^1 \int_M c^2 \sqrt{1 - \frac{1}{c^2} \mathfrak{g} \left(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t} \right)} \mu dt. \quad (6.9)$$

It is natural to think of this action as the restriction to a fixed reference frame of the corresponding action functional $\mathbb{S}: \text{Diff}(\bar{M}) \rightarrow \mathbb{R}$ on the Lorentzian manifold $\bar{M} = [0, 1] \times M$ equipped with the Lorentzian metric

$$\bar{\mathbf{g}}((\dot{t}, \dot{x}), (\dot{t}, \dot{x})) = c^2 \dot{t}^2 - \mathbf{g}(\dot{x}, \dot{x}).$$

More explicitly, this extended action is given by

$$\mathbb{S}(\bar{\varphi}) = \int_{\bar{M}} \sqrt{\bar{\mathbf{g}}\left(\frac{\partial \bar{\varphi}}{\partial t}, \frac{\partial \bar{\varphi}}{\partial t}\right)} \bar{\mu} \quad (6.10)$$

where $\bar{\mu} = -c dt \wedge \mu$ is the volume form associated with $\bar{\mathbf{g}}$.³ In contrast with the classical case, the action (6.10) is left-invariant under the subgroup of Lorentz transformations $\text{Diff}_{\bar{\mathbf{g}}}(\bar{M}) = \{\bar{\varphi} \in \text{Diff}(\bar{M}) \mid \bar{\varphi}^* \bar{\mathbf{g}} = \bar{\mathbf{g}}\}$ in the following sense: if $\bar{\eta} = (\tau, \eta) \in \text{Diff}_{\bar{\mathbf{g}}}(\bar{M})$ then

$$\mathbb{S}(\bar{\eta} \circ \bar{\varphi}) = \int_{\bar{M}} \sqrt{\bar{\mathbf{g}}(T\bar{\eta} \cdot \frac{\partial \bar{\varphi}}{\partial t}, T\bar{\eta} \cdot \frac{\partial \bar{\varphi}}{\partial t})} \bar{\mu} = \int_{\bar{M}} \sqrt{\bar{\eta}^* \bar{\mathbf{g}}\left(\frac{\partial \bar{\varphi}}{\partial t}, \frac{\partial \bar{\varphi}}{\partial t}\right)} \bar{\mu} = \mathbb{S}(\bar{\varphi}).$$

Returning to (6.9), the associated Lagrangian on $\text{Diff}(M)$ is

$$L(\varphi, \dot{\varphi}) = - \int_M c^2 \sqrt{1 - \frac{1}{c^2} \mathbf{g}(\dot{\varphi}, \dot{\varphi})} \mu. \quad (6.11)$$

Since the Lagrangian is right-invariant with respect to $\text{Diff}_{\mu}(M)$, we can carry out Poisson reduction of the corresponding Hamiltonian system on $T^*\text{Diff}(M)$ as described above.

Brenier [9] used such an approach to derive a relativistic heat equation. We are now in a position to use it for relativistic hydrodynamics.

Theorem 6.7. *The relativistic Lagrangian (6.11) on $\text{Diff}(M)$ induces a Poisson reduced system on $\text{Dens}(M) \times \mathfrak{X}^*(M)$. The Hamiltonian is given by*

$$\bar{H}(\rho, m) = \int_M \sqrt{\rho^2 + \frac{1}{c^2} \mathbf{g}^b(m, m)} \mu$$

and the governing equations are

$$\begin{cases} \dot{m} = -\mathcal{L}_v m - d\left(\frac{c^2 \rho}{\sqrt{\rho^2 + c^{-2} \mathbf{g}^b(m, m)}}\right) \otimes \varrho \\ \dot{\varrho} + \mathcal{L}_v \varrho = 0 \end{cases} \quad (6.12)$$

where $v = m / \sqrt{\rho^2 + c^{-2} \mathbf{g}^b(m, m)}$.

Proof. The reduced Lagrangian for (6.11) is given by

$$\ell(\rho, v) = \int_M -c^2 \sqrt{1 - \frac{1}{c^2} \mathbf{g}(v, v)} \rho \mu.$$

The momentum variable is given by the Legendre transformation

$$m = \frac{\delta L}{\delta v} = \gamma \rho v^b \quad \text{for} \quad \gamma = \frac{1}{\sqrt{1 - c^{-2} \mathbf{g}(v, v)}}$$

with the inverse

$$v = \frac{m}{\sqrt{\rho^2 + c^{-2} \mathbf{g}^b(m, m)}}.$$

³While in classical mechanics the action stands for the length square, note that in the classical limit, i.e. for small velocities, $\sqrt{1 - \frac{1}{c^2} \mathbf{g}(\dot{\varphi}, \dot{\varphi})} \approx \left(1 - \frac{1}{2c^2} \mathbf{g}(\dot{\varphi}, \dot{\varphi})\right)$, so that formula (6.10) leads to the classical action.

The corresponding Hamiltonian is

$$\bar{H}(\rho, m) = \langle m, v \rangle - \ell(\rho, v) = \int_M c^2 \sqrt{\rho^2 + \frac{\mathbf{g}^b(m, m)}{c^2}} \mu,$$

so that

$$\frac{\delta \bar{H}}{\delta \rho} = \frac{c^2 \rho}{\sqrt{\rho^2 + c^{-2} \mathbf{g}^b(m, m)}},$$

and the result follows from [Corollary 3.5](#). \square

Remark 6.8. As $c \rightarrow \infty$ we formally recover the classical inviscid Burgers' equation in [§ 4.1](#). Indeed, assuming $\mathbf{g}^b(m, m)$ is small in comparison with c^2 , a Taylor expansion of the right-hand side of [\(6.12\)](#) gives

$$\begin{aligned} -d \left(\frac{c^2 \rho}{\sqrt{\rho^2 + c^{-2} \mathbf{g}^b(m, m)}} \right) &= -d \left(c^2 - \frac{1}{2\rho^2} \mathbf{g}^b(m, m) + O(c^{-2} \mathbf{g}^b(m, m)) \right) \\ &\rightarrow d(\mathbf{g}^b(m, m)/2\rho^2) \quad \text{as } c \rightarrow \infty. \end{aligned}$$

As we also have $v \rightarrow m/\rho$ as $c \rightarrow \infty$ we recover the classical inviscid Burgers' equation.

Remark 6.9. In order to obtain the equations of relativistic hydrodynamics one needs to incorporate internal energy via the reduced Hamiltonian on $\text{Dens}(M) \times \mathfrak{X}^*(M)$ given by

$$\bar{H}(\rho, m) = \int_M (c^2 + e(\rho)) \sqrt{\rho^2 + \frac{\mathbf{g}^b(m, m)}{c^2}} \mu,$$

where e is the internal energy function, cf. Landau and Lifshitz [\[30\]](#) and Holm and Kupershmidt [\[21\]](#). This gives a relativistic version of the classical barotropic equations in [§ 4.4](#).

7. FISHER-RAO GEOMETRY

7.1. Newton's equations on $\text{Diff}(M)$. We now focus on another important Riemannian structure on $\text{Diff}(M)$. This structure is induced by the Sobolev H^1 -inner product on vector fields and has the same relation to the Fisher-Rao metric on $\text{Dens}(M)$ as the L^2 -metric on $\text{Diff}(M)$ to the Wasserstein-Otto metric on $\text{Dens}(M)$.

Definition 7.1. Let (M, \mathbf{g}) be a compact Riemannian manifold with volume form μ . For any $\varphi \in \text{Diff}(M)$ and $v \in T_e \text{Diff}(M)$ we set

$$\mathbf{G}_\varphi(v \circ \varphi, v \circ \varphi) = \int_M \mathbf{g}(-\Delta v, v) \mu + F(v, v), \quad (7.1)$$

where Δ is the Laplacian on vector fields and F is a quadratic form depending only on the vertical (divergence-free) component of v .

Remark 7.2. From the point of view of the geometry of $\text{Dens}(M)$ (and for most of our applications) only the first term on the right-hand side of [\(7.1\)](#) is relevant. However, it is convenient to work with the above metric on $\text{Diff}(M)$, in particular, because of its relation to a number of familiar equations, cf. [\[26, 39\]](#) and below. Note also the following analogy between the Wasserstein and the Fisher-Rao structures: while the non-invariant L^2 -metric induces a factorization of $\text{Diff}(M)$ where one of the factors solves the optimal mass transport problem, the invariant metric [\(7.1\)](#) induces a different factorization of $\text{Diff}(M)$ which solves an *optimal information transport* problem; cf. [\[39\]](#).

Consider a potential function of the form

$$U(\varphi) = \bar{U}(\varphi^*\mu), \quad \varphi \in \text{Diff}(M), \quad (7.2)$$

where \bar{U} is a potential functional on $\text{Dens}(M)$ as before. It is interesting to compare the present setting with that of Section 2.1, where the potential function on $\text{Diff}(M)$ was defined using pushforwards rather than pullbacks. As a result one works with the left cosets rather than with the right cosets, cf. Remark 7.10 below.

Theorem 7.3. *Newton's equations of the metric (7.1) on $\text{Diff}(M)$ with a potential function (7.2) have the form*

$$\begin{cases} \mathcal{A}\dot{v} + \mathcal{L}_v \mathcal{A}v + d\left(\frac{\delta \bar{U}}{\delta \varrho}(\varphi^*\mu) \circ \varphi^{-1}\right) \otimes \mu = 0 \\ \dot{\varphi} = v \circ \varphi, \end{cases} \quad (7.3)$$

where the inertia operator $\mathcal{A}: \mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$ is given by

$$\mathcal{A}v = (-\Delta v^b + F(v, \cdot)) \otimes \mu. \quad (7.4)$$

Proof. The derivation of the equation in the case of zero potential can be found in [39]. Modifications needed here follow from the calculation

$$\left. \frac{d}{ds} \right|_{s=0} \bar{U}(\varphi_s^*\mu) = \int_M \frac{\delta \bar{U}}{\delta \varrho} \varphi^* \mathcal{L}_u \mu = \left\langle d\left(\frac{\delta \bar{U}}{\delta \varrho} \circ \varphi^{-1}\right) \otimes \mu, u \right\rangle,$$

where $s \rightarrow \varphi_s \in \text{Diff}(M)$ is the flow of the vector field u in $\mathfrak{X}(M)$. \square

We proceed with a Hamiltonian formulation. As in § 3.1 we will identify cotangent spaces $T_\varphi^* \text{Diff}(M)$ with $\mathfrak{X}^*(M)$.

Proposition 7.4. *The Hamiltonian form of Newton's equations (7.3) on $T^* \text{Diff}(M)$ is*

$$\begin{cases} \frac{d}{dt} \varphi^* m + d\left(\frac{\delta \bar{U}}{\delta \varrho}(\varphi^*\mu)\right) \otimes \varphi^* \mu = 0 \\ \dot{\varphi} = u \circ \varphi \end{cases} \quad (7.5)$$

where $m = \mathcal{A}u \in \mathfrak{X}^*(M)$.

Proof. This follows simply by pulling back by φ the equations in (7.3) and applying the identity

$$\frac{d}{dt} \varphi^* m = \varphi^* \dot{m} + \varphi^* \mathcal{L}_u m.$$

\square

Remark 7.5. Observe that if the potential function is zero, then the equation in (7.5) expresses conservation of the momentum $\varphi^* m$ associated with the right invariance of the metric.

7.2. Riemannian submersion over densities. We turn to the geometry of the fibration of $\text{Diff}(M)$ with respect to the metric (7.1).

Definition 7.6. The *right coset projection* $\pi: \text{Diff}(M) \rightarrow \text{Dens}(M)$ between diffeomorphisms and smooth probability densities is given by

$$\pi(\varphi) = \varphi^* \mu. \quad (7.6)$$

As before, it turns out that the projection (7.6) is a Riemannian submersion if the base space is equipped with a suitable metric.

Definition 7.7. The *Fisher-Rao metric* is the Riemannian metric on $\text{Dens}(M)$ given by

$$\mathbf{G}_\varrho(\dot{\varrho}, \dot{\varrho}) = \int_M \frac{\dot{\varrho}}{\varrho} \frac{\dot{\varrho}}{\varrho} \varrho, \quad (7.7)$$

where $\dot{\varrho} \in \Omega_0^n(M)$ represents a tangent vector at $\varrho \in \text{Dens}(M)$.

Theorem 7.8. *The right coset projection (7.6) is a Riemannian submersion with respect to the metric (7.1) on $\text{Diff}(M)$ and the Fisher-Rao metric on $\text{Dens}(M)$. In particular, if $\dot{\varphi} \in T_\varphi \text{Diff}(M)$ is horizontal, i.e.,*

$$\mathbf{G}_\varphi(\dot{\varphi}, \dot{\eta}) = 0, \quad \forall \dot{\eta} \in \ker(T_\varphi \pi),$$

then $\mathbf{G}_\varphi(\dot{\varphi}, \dot{\varphi}) = \mathbf{G}_{\pi(\varphi)}(\dot{\varrho}, \dot{\varrho})$ where $\dot{\varrho} = \pi_* \dot{\varphi}$.

Proof. See [39, Thm. 4.9]. □

Remark 7.9. It follows from the Hodge decomposition that the horizontal distribution on $\text{Diff}(M)$ consists of elements of the form $\nabla p \circ \varphi$, cf. [39] for details.

Remark 7.10. The setting of Theorem 7.8 is quite different from that of Theorem 2.8. In the latter, the Riemannian metric on $\text{Diff}(M)$ is right-invariant with respect to $\text{Diff}_\mu(M)$ and automatically descends to the quotient from the right, namely $\text{Diff}(M)/\text{Diff}_\mu(M)$. In the former, the metric is right-invariant with respect to $\text{Diff}(M)$ and descends to the quotient from the left, namely $\text{Diff}_\mu(M)\backslash\text{Diff}(M)$. Thus, in Theorem 7.8 the right-invariance property is retained after taking the quotient and therefore the Fisher-Rao metric on $\text{Dens}(M)$ remains right-invariant with respect to the action of $\text{Diff}(M)$ (corresponding to right translation of the fibers), which is easy to verify.

Proposition 7.11. *The gradient of a smooth function $\bar{U}: \text{Dens}(M) \rightarrow \mathbb{R}$ with respect to the Fisher-Rao metric is*

$$\nabla^{\mathbf{G}} \bar{U}(\varrho) = \frac{\delta \bar{U}}{\delta \varrho} \varrho - \lambda \varrho,$$

where λ is a Lagrange multiplier such that $\nabla^{\mathbf{G}} \bar{U}(\varrho) \in T_\rho \text{Dens}(M)$.

Proof. Let $\dot{\varrho} \in \Omega_0^n(M)$ and let $\delta \bar{U}/\delta \varrho$ be a representative of the variational derivative in $C^\infty(M)/\mathbb{R}$. We have

$$\mathbf{G}_\varrho(\nabla^{\mathbf{G}} \bar{U}(\varrho), \dot{\varrho}) = \left\langle \frac{\delta \bar{U}}{\delta \varrho}, \dot{\varrho} \right\rangle = \int_M \frac{\delta \bar{U}}{\delta \varrho} \dot{\varrho} = \mathbf{G}_\varrho\left(\frac{\delta \bar{U}}{\delta \varrho} \varrho, \dot{\varrho}\right),$$

which yields the explicit form of the gradient. □

We end this subsection by recalling a particularly remarkable property of the Fisher-Rao metric. Let $S^\infty(M) = \{f \in C^\infty(M) \mid \int_M f^2 \mu = 1\}$ be the unit sphere in the pre-Hilbert space $C^\infty(M) \subset L^2(M, \mathbb{R})$.

Theorem 7.12. *The square root map*

$$\phi: \text{Dens}(M) \rightarrow S^\infty(M), \quad \rho \mu \mapsto \sqrt{\rho} \quad (7.8)$$

is a Riemannian isometry between $\text{Dens}(M)$ equipped with the metric \mathbf{G} in (7.7) and the (geodesically convex) subset

$$S_+^\infty(M) = \{f \in S^\infty(M) \mid f > 0\}$$

of the sphere $S^\infty(M)$.

This result was first obtained by Friedrich [16] and later independently in [26] in the Euler-Arnold framework of diffeomorphism groups.

7.3. Newton's equations on $\text{Dens}(M)$. Recall that in §3.2 the Hamilton equations on $T^*\text{Dens}(M)$ were obtained by symplectic reduction of a $\text{Diff}_\mu(M)$ -invariant system on $T^*\text{Diff}(M)$. In the setting with the right coset projection (7.6) and the metric (7.1) the situation is quite different, since the Riemannian metric is not left-invariant with respect to $\text{Diff}_\mu(M)$ (otherwise, interchanging push-forwards and pull-backs would give a completely 'dual' theory). Nevertheless, there is a *zero momentum reduction* on the Hamiltonian side corresponding to the Riemannian submersion structure described in §7.2.

Proposition 7.13. *The exact momenta, i.e. tensor products of the form*

$$\{\text{d}f \otimes \mu \mid f \in C^\infty(M)\}, \quad (7.9)$$

form an invariant set for the system (7.5).

Proof. Substituting (7.9) in (7.5) we get

$$\text{d}\left(\frac{\text{d}}{\text{d}t}\varphi^*f\right) \otimes \varphi^*\mu + \text{d}(\varphi^*f) \otimes \frac{\text{d}}{\text{d}t}\varphi^*\mu + \text{d}\left(\frac{\delta\bar{U}}{\delta\varrho}(\varphi^*\mu)\right) \otimes \varphi^*\mu = 0,$$

where $\mathcal{A}u = \text{d}f \otimes \mu$ and

$$\frac{\text{d}}{\text{d}t}\varphi^*\mu = \varphi^*\mathcal{L}_u\mu = \varphi^*(\text{div } u)\varphi^*\mu.$$

From (7.4) we find that solutions of the form $u = \nabla p$ define (up to a constant) $f = \Delta p = \text{div } u$, so that

$$\text{d}\left(\frac{\text{d}}{\text{d}t}\varphi^*f\right) \otimes \varphi^*\mu + \text{d}(\varphi^*f)^2 \otimes \varphi^*\mu + \text{d}\left(\frac{\delta\bar{U}}{\delta\varrho}(\varphi^*\mu)\right) \otimes \varphi^*\mu = 0.$$

Using $\frac{\text{d}}{\text{d}t}(\varphi^*f) = \varphi^*\dot{f} + \varphi^*\mathcal{L}_u f$ we then obtain

$$\varphi^*\left(\text{d}\left(\dot{f} + \mathcal{L}_u f + f^2 + \frac{\delta\bar{U}}{\delta\varrho}(\varphi^*\mu) \circ \varphi^{-1}\right) \otimes \mu\right) = 0, \quad (7.10)$$

which proves the assertion. \square

Remark 7.14. The momenta of the form (7.9) are preserved because they belong to the preimage of zero of the momentum map for the left action of $\text{Diff}_\mu(M)$ on $\text{Diff}(M)$. Thus, if the Hamiltonian is invariant under $\text{Diff}_\mu(M)$ acting on this preimage (though not necessarily on the entire phase space) then we obtain the zero momentum symplectic leaf.

Theorem 7.15. *Newton's equations with respect to the Fisher-Rao metric (7.7) on $\text{Dens}(M)$ and a potential $\bar{U}: \text{Dens}(M) \rightarrow \mathbb{R}$ have the form*

$$\ddot{\rho} - \frac{\dot{\rho}^2}{2\rho} + \frac{\delta\bar{U}}{\delta\varrho}\rho = \lambda\rho \quad (7.11)$$

where λ is a multiplier subject to $\int_M \varrho = 1$. Furthermore, the Lagrangian and Hamiltonian are $L(\varrho, \dot{\varrho}) = \frac{1}{2}\mathbf{G}_\varrho(\dot{\varrho}, \dot{\varrho}) - \bar{U}(\varrho)$ and $H(\varrho, \theta) = \frac{1}{2}\langle \theta^2, \varrho \rangle + \bar{U}(\varrho)$, respectively. The corresponding Hamiltonian equations have the form

$$\begin{cases} \dot{\varrho} - \theta\varrho = 0 \\ \dot{\theta} + \frac{1}{2}\theta^2 + \frac{\delta\bar{U}}{\delta\varrho}(\varrho) = \lambda. \end{cases} \quad (7.12)$$

Solutions of (7.12) correspond to potential solutions (cf. Proposition 7.13) of Newton's equations (7.5) on $\text{Diff}(M)$.

Proof. The result follows directly from the proof of Proposition 7.13 by setting $\theta = \varphi^*f$ and $\varrho = \varphi^*\mu$. \square

8. FISHER-RAO EXAMPLES

8.1. The μ CH equation and Fisher-Rao geodesics. The periodic μ CH equation (also known in the literature as the μ HS equation) is a nonlinear evolution equation of the form

$$\mu(u_t) - u_{xxt} - 2u_x u_{xx} - u u_{xxx} + 2\mu(u)u_x = 0 \quad (8.1)$$

where $\mu(u) = \int_{S^1} u dx$. It was derived in [25] as an Euler-Arnold equation on the group of diffeomorphisms of the circle equipped with the right-invariant Sobolev metric given at the identity by the inner product

$$\langle u, v \rangle_{H^1} = \mu(u)\mu(v) + \int_{S^1} u_x v_x dx.$$

The μ CH equation is known to be bihamiltonian and admit smooth, as well as cusped, soliton-type solutions. It may be viewed as describing a director field in the presence of an external (e.g., magnetic) force. The associated Cauchy problem has been studied extensively in the literature, cf. [25, 18, 48]. Many of its geometric properties can also be found in [54]. The following result was proved in [39]

Proposition 8.1. *The μ CH equation (8.1) is a (right-reduced) Newton's equation (7.3) with vanishing potential on S^1 . Geodesics of the Fisher-Rao metric (7.7) on $\text{Dens}(S^1)$ correspond to horizontal solutions of the μ CH equation described by the equations*

$$\begin{cases} \dot{\rho} - \theta\rho = 0, \\ \dot{\theta} + \frac{1}{2}\theta^2 - \frac{1}{2}\int_{S^1} \theta^2 \rho dx = 0. \end{cases}$$

As in Theorem 7.15, the relation between u , ρ and θ is given by $\rho = \varphi_x$ and $\theta = u_x \circ \varphi$, where φ is the Lagrangian flow of u .

Observe that the Euler-Arnold equation of the metric (7.1) can be naturally viewed as a higher-dimensional generalization of the equation (8.1), see [39]. Furthermore, in the one-dimensional case horizontal solutions of this equation can be written in terms of the derivative u_x . In higher dimensions we similarly have

Proposition 8.2. *The geodesic equations of the Fisher-Rao metric (7.7) on $\text{Dens}(M)$ reduce to the following equations on $T_\mu^* \text{Dens}(M)$*

$$\dot{f} + \mathcal{L}_{\nabla p} f + \frac{1}{2}f^2 = \frac{1}{2}\int_M f^2 \mu, \quad \Delta p = f$$

where $f = \text{div } u$ and $\theta = f \circ \varphi$.

Proof. The equations follow directly from (7.10) with $\bar{U} \equiv 0$. □

8.2. The infinite-dimensional Neumann problem. In 1856 Neumann showed that the geodesic equations on an ellipsoid in \mathbb{R}^{n+1} with the induced metric describe (up to a change of the time parameter) the motion of a point on the n -dimensional sphere S^n under the influence of a quadratic potential, see e.g., [44, 42, 43].

Let us describe a natural infinite-dimensional generalization of the Neumann problem. Consider the unit sphere

$$S^\infty(M) = \left\{ f \in C^\infty(M) \mid \int_M f^2 \mu = 1 \right\}$$

in the pre-Hilbert space $C^\infty(M) \cap L^2(M, \mu)$ and the quadratic potential function

$$V(f) = \frac{1}{2} \langle \nabla f, \nabla f \rangle_{L^2} = \frac{1}{2} \int_M |\nabla f|^2 \mu. \quad (8.2)$$

We seek a curve $f: [0, 1] \rightarrow S^\infty(M)$ that minimizes the action functional for the Lagrangian

$$L(f, \dot{f}) = \frac{1}{2} \langle \dot{f}, \dot{f} \rangle_{L^2} - \frac{1}{2} \langle \nabla f, \nabla f \rangle_{L^2} = \frac{1}{2} \int_M (\dot{f}^2 + f \Delta f) \mu.$$

Proposition 8.3. *Newton's equations associated with the infinite-dimensional Neumann problem with potential (8.2) have the form*

$$\ddot{f} - \Delta f = -\lambda f, \quad (8.3)$$

where λ is a Lagrange multiplier subject to the constraint $\int_M f^2 \mu = 1$. In fact, we have $\lambda = 2L(f, \dot{f}) = \int_M (\dot{f}^2 + f \Delta f) \mu$.

Proof. This is a simple consequence of the integration by parts formula. \square

Remark 8.4. The classical (finite-dimensional) Neumann problem is a system on the tangent bundle TS^n with the Lagrangian given by

$$L(q, \dot{q}) = \frac{|\dot{q}|^2}{2} - q \cdot Aq, \quad \text{where } q \in S^n \subset \mathbb{R}^{n+1}$$

and where A is a symmetric positive definite $(n+1) \times (n+1)$ matrix. This system is related to the geodesic flow on the ellipsoid $x \cdot Ax = 1$, see e.g., [42, Sec. 3]. The corresponding Hamiltonian system on T^*S^n is integrable and if the eigenvalues $\alpha_1, \dots, \alpha_{n+1}$ of A are all different then first integrals are given by

$$F_k(q, p) = q_k^2 + \sum_{j \neq k} \frac{p_j q_k - p_k q_j}{\alpha_k - \alpha_j},$$

where q_k and p_k are the components of q and p with respect to the eigenbasis of A . An interesting question, not pursued here, is to study the problem of integrability of the infinite-dimensional Neumann problem described in this section.

Our next objective is to show that the infinite-dimensional Neumann problem on $S^\infty(M)$ corresponds to Newton's equations on $\text{Dens}(M)$ with respect to the Fisher-Rao metric and a natural choice of the potential function. The latter is given by the Fisher information functional

$$I(\varrho) = \frac{1}{2} \int_M \frac{|\nabla \rho|^2}{\rho} \mu, \quad \text{where } \varrho = \rho \mu. \quad (8.4)$$

Lemma 8.5. *The gradient of $I(\varrho)$ with respect to the Fisher-Rao metric can be computed from either of the two expressions*

$$\begin{aligned} \nabla^G I(\varrho) &= \left(\frac{1}{2} \frac{|\nabla \rho|^2}{\rho} - \Delta \rho \right) \mu - \lambda \varrho \\ &= -2 \left(\sqrt{\rho} \Delta \sqrt{\rho} \right) \mu - \lambda \varrho. \end{aligned}$$

Proof. Using the identities $\nabla \log \rho = \nabla \rho / \rho$ and $\nabla \sqrt{\rho} = \frac{1}{2} \rho^{-\frac{1}{2}} \nabla \rho$ we can rewrite the Fisher information functional as

$$I(\varrho) = \frac{1}{2} \int_M |\nabla \log \rho|^2 \varrho = 2 \int_M |\nabla \sqrt{\rho}|^2 \mu.$$

Differentiating the first of these expressions in the direction of the vector $\dot{\varrho} = \dot{\rho} \mu$ yields

$$\begin{aligned} \left\langle \frac{\delta I}{\delta \varrho}, \dot{\varrho} \right\rangle &= \int_M \left(\frac{1}{2} |\nabla \log \rho|^2 \dot{\varrho} + \mathbf{g}(\nabla \log \rho, \nabla(\dot{\rho}/\rho)) \varrho \right) \\ &= \int_M \left(\frac{1}{2} |\nabla \log \rho|^2 \dot{\varrho} - \rho^{-1} \Delta \rho \dot{\varrho} \right) = \left\langle \frac{1}{2} \frac{|\nabla \rho|^2}{\rho^2} - \frac{\Delta \rho}{\rho}, \dot{\varrho} \right\rangle. \end{aligned}$$

Similarly, differentiating the second yields

$$\left\langle \frac{\delta I}{\delta \varrho}, \dot{\varrho} \right\rangle = 2 \int_M \mathbf{g} \left(\nabla \sqrt{\rho}, \nabla (\dot{\rho} / \sqrt{\rho}) \right) \mu = \left\langle -2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}, \dot{\varrho} \right\rangle.$$

The result now follows from [Proposition 7.11](#). \square

Proposition 8.6. *Newton's equations (7.11) on $\text{Dens}(M)$ with respect the Fisher-Rao metric and the Fisher-Rao potential (8.4) are*

$$\ddot{\rho} - \Delta \rho - \frac{1}{2\rho} (\dot{\rho}^2 - |\nabla \rho|^2) = \lambda \rho,$$

where λ is a Lagrange multiplier for the constraint $\int_M \varrho = 1$. The map $\rho \mapsto f = \sqrt{\rho}$ establishes an isomorphism with the infinite-dimensional Neumann problem (8.3).

Proof. The form of the equation on $\text{Dens}(M)$ follows from [Theorem 7.15](#). It is straightforward to check that $V(\sqrt{\rho}) = I(\rho)/4$. The result then follows from isometric properties of the square root map (7.8). \square

Remark 8.7. Of particular interest are the stationary solutions to the Neumann problem (8.3), i.e., those with $\nabla^{S^\infty} V(f) = \Delta f - \lambda f = 0$, in which case f is a normalized eigenvector of the Laplacian with eigenvalue λ . If $\dot{f} = 0$ then $\lambda = \int_M f \Delta f \mu = -2V(f)$. Consequently, the stationary solutions correspond to the principal axes of the corresponding infinite-dimensional ellipsoid $\langle f, \Delta f \rangle_{L^2} = 1$.

It is also possible to obtain quasi-stationary solutions this way. Indeed, assume that the eigenspace of λ is at least two-dimensional (for example, when $M = S^n$). If $f_1, f_2 \in S^\infty(M)$ are two orthogonal eigenvectors with eigenvalue λ then it is straightforward to check that a solution originating from f_1 with initial velocity $a f_2$ for $a \in \mathbb{R}$ is given by

$$f(t, x) = \cos(at) f_1 + \sin(at) f_2.$$

8.3. The Klein-Gordon equation. The *Klein-Gordon equation*

$$\ddot{f} - \Delta f = -m^2 f, \quad m \in \mathbb{R} \tag{8.5}$$

describes spin-less scalar particles of mass m . It is invariant under Lorentz transformations and can be viewed as a relativistic quantum equation. To see how it relates to the Neumann problem of the previous subsection let $M \times S^1$ denote the space-time manifold equipped with the Minkowski metric of signature $(+ + + -)$ and consider a quadratic functional

$$\bar{V}(f) = \frac{1}{2} \int_{M \times S^1} (|\nabla f|^2 - f^2) \mu \wedge dt$$

which is the L^2 -norm of the the Minkowski gradient $\bar{\nabla} f = (\nabla f, -f)$.

Proposition 8.8. *The stationary solutions of the infinite-dimensional Neumann problem with potential \bar{V} on the hypersurface*

$$S^\infty(M \times S^1) = \left\{ f \in C^\infty(M \times S^1) \mid \int_{M \times S^1} f^2 \mu \wedge dt = 1 \right\}$$

satisfy the Klein-Gordon equation (8.5) with mass parameter $m^2 = 2\bar{V}(f)$.

Proof. This is a calculation analogous to that in [Remark 8.7](#). \square

9. GEOMETRIC PROPERTIES OF THE MADELUNG TRANSFORM

In this section we recall several results concerning the Madelung transform which provides a link between geometric hydrodynamics and quantum mechanics, see [27, 28]. It was introduced in the 1920's by E. Madelung in an attempt to give a hydrodynamical formulation of the Schrödinger equation. Using the setting developed in previous sections we can now present a number of surprising geometric properties of this transform.

Definition 9.1. Let ρ and θ be real-valued functions on M with $\rho > 0$. The *Madelung transform* is defined by

$$\Phi(\rho, \theta) = \sqrt{\rho e^{2i\theta/\hbar}}, \quad (9.1)$$

where \hbar is a parameter (Planck's constant).⁴

Observe that Φ is a complex extension of the square root map described in [Theorem 7.8](#). Heuristically, the functions $\sqrt{\rho}$ and θ/\hbar can be interpreted as the absolute value and argument of the complex-valued function $\psi := \sqrt{\rho} e^{2i\theta/\hbar}$ as in polar coordinates.

9.1. Madelung transform as a symplectomorphism. Let $PC^\infty(M, \mathbb{C})$ denote the complex projective space of smooth complex-valued functions on M . Its elements can be represented as cosets $[\psi]$ of the L^2 -sphere of smooth functions, where $\tilde{\psi} \in [\psi]$ if and only if $\tilde{\psi} = e^{i\alpha}\psi$ for some $\alpha \in \mathbb{R}$. A tangent vector at a coset $[\psi]$ is a linear coset of the form $[\dot{\psi}] = \{\dot{\psi} + c\psi \mid c \in \mathbb{R}\}$. Following the geometrization of quantum mechanics by Kibble [29], a natural symplectic form on $TPC^\infty(M, \mathbb{C})$ is

$$\Omega_{[\psi]}^{PC^\infty(M, \mathbb{C})}([\dot{\psi}_1], [\dot{\psi}_2]) = 2\hbar \int_M \text{Im}(\dot{\psi}_1 \overline{\dot{\psi}_2}) \mu. \quad (9.2)$$

The projective space $PC^\infty(M, \mathbb{C} \setminus \{0\})$ of non-vanishing complex functions is a submanifold of $PC^\infty(M, \mathbb{C})$. It turns out that the Madelung transform induces a symplectomorphism between $PC^\infty(M, \mathbb{C} \setminus \{0\})$ and the cotangent bundle of probability densities $T^*\text{Dens}(M)$. Namely, we have

Theorem 9.2 ([28]). *The Madelung transform (9.1) induces a map*

$$\Phi: T^*\text{Dens}(M) \rightarrow PC^\infty(M, \mathbb{C} \setminus \{0\}) \quad (9.3)$$

which is a symplectomorphism (in the Fréchet topology of smooth functions) with respect to the canonical symplectic structure of $T^\text{Dens}(M)$ and the symplectic structure (9.2) of $PC^\infty(M, \mathbb{C})$.*

The Madelung transform was shown to be a symplectic submersion from $T^*\text{Dens}(M)$ to the unit sphere of non-vanishing wave functions by von Renesse [57]. The stronger symplectomorphism property stated in [Theorem 9.2](#) is deduced using the projectivization $PC^\infty(M, \mathbb{C} \setminus \{0\})$.

9.2. Examples: linear and nonlinear Schrödinger equations. Let ψ be a wavefunction and consider the family of Schrödinger equations (or Gross-Pitaevsky equations) with Planck's constant \hbar and mass m of the form

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + f(|\psi|^2)\psi \quad (9.4)$$

where $V: M \rightarrow \mathbb{R}$ and $f: \mathbb{R}_+ \rightarrow \mathbb{R}$. Setting $f \equiv 0$ we obtain the linear Schrödinger equation with potential V , while setting $V \equiv 0$ yields a family of nonlinear Schrödinger equations (NLS); typical choices are $f(a) = \kappa a$ or $f(a) = \frac{1}{2}(a - 1)^2$.

⁴In the publications [27, 28] the convention $\hbar = 2$ is used.

From the point of view of geometric quantum mechanics (cf. Kibble [29]), equation (9.4) is Hamiltonian with respect to the symplectic structure (9.2), which is compatible with the complex structure of $PL^2(M, \mathbb{C})$. The Hamiltonian associated with (9.4) is

$$H(\psi) = \frac{\hbar^2}{2m} \|\nabla\psi\|_{L^2}^2 + \int_M (V|\psi|^2 + F(|\psi|^2)) \mu, \quad (9.5)$$

where $F: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a primitive of f .

Observe that the L^2 norm of a wave function satisfying the Schrödinger equation (9.4) is conserved in time. Furthermore, the equation is equivariant with respect to phase change $\psi(x) \mapsto e^{i\alpha}\psi(x)$ and hence it descends to the projective space $PC^\infty(M, \mathbb{C})$.

Proposition 9.3 (cf. [35, 57]). *The Madelung transform Φ maps the family of Schrödinger Hamiltonians (9.5) to a family of Hamiltonians on $T^*\text{Dens}(M)$ given by*

$$\tilde{H}(\varrho, \theta) = H(\Phi(\varrho, \theta)) = \frac{1}{2m} \int_M |\nabla\theta|^2 \varrho + \frac{\hbar^2}{8m} \int_M \frac{|\nabla\rho|^2}{\rho} \mu + \int_M (V\varrho + F(\rho)\mu).$$

In particular, if $m = 1$ we recover Newton's equations (3.7) on $\text{Dens}(M)$ for the potential function

$$\bar{U}(\varrho) = \frac{\hbar^2 I(\varrho)}{4} + \int_M (V\varrho + F(\rho)\mu),$$

where I is Fisher's information functional (8.4). The extension (2.6) to a fluid equation on $T\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \mathfrak{X}(M) \times \text{Dens}(M)$ is

$$\begin{cases} \dot{v} + \nabla_v v + \nabla \left(V + f(\rho) - \frac{\hbar^2}{2} \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0. \end{cases} \quad (9.6)$$

Remark 9.4. For the linear Schrödinger equation (9.4), where $f \equiv 0$, notice that the ‘‘classical limit’’ immediately follows from (9.6): as $\hbar \rightarrow 0$ we recover classical mechanics and the Hamilton-Jacobi equation as presented in §4.2.

Remark 9.5. In this section we have seen how the Schrödinger equation can be expressed as a compressible fluid equation via Madelung's transform. Conversely, the classical equations of hydrodynamics can be formulated as nonlinear Schrödinger equations (since the Madelung transform is a symplectomorphism, so any Hamiltonian on $T^*\text{Dens}(M)$ induces a corresponding Hamiltonian on $PC^\infty(M, \mathbb{C})$). In particular, potential solutions of the compressible Euler equations of a barotropic fluid (4.7) can be expressed as solutions to an NLS equation with Hamiltonian

$$H(\psi) = \frac{\hbar^2}{2} \|\nabla\psi\|_{L^2}^2 - \frac{\hbar^2}{2} \|\nabla|\psi|\|_{L^2}^2 + \int_M e(|\psi|^2) |\psi|^2 \mu$$

where $e = e(\rho)$ is the specific internal energy of the fluid. The choice $e = 0$ gives a Schrödinger-type formulation for potential solutions of Burgers' equation describing geodesics of the Wasserstein-Otto metric (2.8) on $\text{Dens}(M)$. We thus have a geometric framework that connects optimal transport for cost functions with potentials, the Euler equations of compressible hydrodynamics, and the NLS-type equations described above.

Remark 9.6. Another relevant development is the Schrödinger Bridge problem, which seeks the most likely probability law for a diffusion process in the probability space, that matches marginals at two end-points in time, as we discuss in the next section: one can interpret it as a stochastic perturbation of Wasserstein-Otto geodesics on the density space for given end-points.

The Madelung transform allows one to translate questions about the Schrödinger equation to questions about probability laws, cf. Zambrini [60].

9.3. The Madelung and Hopf-Cole transforms. There is a real version of the complex Madelung transform.

Definition 9.7. Let ρ and θ be real-valued functions on M with $\rho > 0$ and let γ be a positive constant. The (symmetrised) *Hopf-Cole transform* is the mapping $HC: (\rho, \theta) \mapsto (\eta^+, \eta^-) \in C^\infty(M, \mathbb{R}^2)$ defined by

$$\eta^\pm = \sqrt{\rho e^{\pm\theta/\gamma}}. \quad (9.7)$$

In [31] it is shown that this map, along with its generalizations, has the property that its inverse HC^{-1} takes the constant symplectic structure $d\eta^- \wedge d\eta^+$ on $C^\infty(M, \mathbb{R}^2)$ to (a multiple of) the standard symplectic structure on $T^*\text{Dens}(M)$. Note that the choice $\gamma = -i\hbar/2$ correspond to the standard Madelung transform (9.1): the function η^+ becomes the complex-valued wave function ψ and thus the symplectic properties of HC can be viewed as an extension of those of the Madelung map Φ .

Consider the viscous Burgers equation

$$\dot{v} + \nabla_v v = \gamma \Delta v.$$

It is known that a simple variant $\eta = \sqrt{e^{-\theta/\gamma}}$ of the Hopf-Cole map (9.7) takes potential solutions $v = \nabla\theta$, which satisfy the Hamilton-Jacobi equation

$$\dot{\theta} + \frac{1}{2}|\nabla\theta|^2 = \gamma\Delta\theta,$$

to solutions of the heat equation $\dot{\eta} = \gamma\Delta\eta$.

Similarly, the Hopf-Cole map can be used to transform certain barotropic-type systems to forward and backward heat equations. Indeed, this can be verified directly from the Schrödinger example in § 9.2: choosing Planck's constant to be imaginary $\hbar = \mp 2i\gamma$ in the Schrödinger equation (9.4) with $V \equiv 0$, $f \equiv 0$, and $m = 1$ gives the forward and backward heat equations

$$\dot{\eta}^\pm = \pm\gamma\Delta\eta^\pm.$$

The corresponding barotropic fluid system, readily obtained from (9.6) with $\hbar = \pm 2i\gamma$, is

$$\begin{cases} \dot{v} + \nabla_v v + 2\gamma^2 \nabla \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0 \\ \dot{\rho} + \text{div}(\rho v) = 0. \end{cases} \quad (9.8)$$

This is again a Newton system on $\text{Dens}(M)$, but where the potential function is corrected by *minus* the Fisher functional (instead of *plus* the Fisher functional as in Proposition 9.3). Equipped with two-point boundary conditions $\rho|_{t=0} = \rho_0$ and $\rho|_{t=1} = \rho_1$, horizontal solutions $v = \nabla\theta$ of (9.8) correspond to solutions of a dynamical formulation of the Schrödinger bridge problem, as surveyed by Leonard [33]. In this way one can study non-conservative systems, allowing viscosity, in a symplectic setting. It is interesting to incorporate the incompressible Navier-Stokes equation into this framework. This would require a two-component version of the map in [31] related to the two-component Madelung transform in the Schrödinger's smoke example below.

Remark 9.8. Equation (9.8) also has another relation to the heat flow connected to an invariant submanifold of $T^*\text{Dens}(M)$. Consider the submanifold

$$\Gamma = \{(\nabla\theta, \rho) \mid \rho = e^{-\theta/\gamma}\}.$$

A straightforward calculation shows that Γ is an invariant submanifold for (9.8) and the evolution on Γ is given by a system of decoupled equations

$$\begin{cases} \dot{\rho} = \gamma \Delta \rho \\ \dot{\theta} + |\nabla \theta|^2 = \gamma \Delta \theta. \end{cases}$$

Furthermore, since $\log \rho$ is the variational derivative of the entropy functional $S(\rho) = \int_M (\log \rho) \varrho$, it follows that the substitution $\theta = -\gamma \log \rho$ (or $\rho = e^{-\theta/\gamma}$) corresponds to the momentum in the direction of negative entropy. This embeds into a Newton system Otto's observation [45] that the heat flow is the L^2 -Wasserstein gradient flow of the entropy functional.

9.4. Example: Schrödinger's smoke. We have shown that the Madelung transform provides a link between quantum mechanics and *compressible* hydrodynamics. In this section we describe how *incompressible* hydrodynamics is related to the so-called incompressible Schrödinger equation. The approach described here was developed in computer graphics by Chern et al. [10] to obtain a fast algorithm that could be used to visualize realistic smoke motion.

It is clear that the standard Maduelung transform is not adequate to describe incompressible hydrodynamics since the group of volume-preserving diffeomorphisms lies in the kernel of the Madelung projection (any flow along $\text{Diff}_\mu(M)$ projects to the constant wave function $\psi = 1$). Instead, one has to consider the multi-component Madelung transform, cf. [28]. For simplicity, we use two components, although one can easily extend the constructions to the case of several components.

Consider the diagonal action of $\text{Diff}(M)$ on $T^*\text{Dens}(M) \times T^*\text{Dens}(M)$ and the associated momentum map given by

$$J(\varrho_1, \theta_1, \varrho_2, \theta_2) = d\theta_1 \otimes \varrho_1 + d\theta_2 \otimes \varrho_2.$$

Fix $\mu_1, \mu_2 \in \text{Dens}(M)$. As in § 5.3 the Poisson manifold

$$T^*\text{Diff}(M)/(\text{Diff}_{\mu_1}(M) \cap \text{Diff}_{\mu_2}(M))$$

can be embedded as a Poisson submanifold of the dual \mathfrak{s}^* of the semidirect product algebra $\mathfrak{s} = \mathfrak{X}(M) \ltimes C^\infty(M, \mathbb{R}^2)$. Given a Hamiltonian $\bar{H}(\varrho_1, \varrho_2, m)$ on \mathfrak{s}^* the governing equations are

$$\begin{cases} \dot{m} = -\mathcal{L}_v m - J(\varrho_1, \frac{\delta \bar{H}}{\delta \varrho_1}, \varrho_2, \frac{\delta \bar{H}}{\delta \varrho_2}) \\ \dot{\varrho}_1 = -\mathcal{L}_v \varrho_1 \\ \dot{\varrho}_2 = -\mathcal{L}_v \varrho_2 \end{cases}$$

where $v = \frac{\delta \bar{H}}{\delta m}$. The zero-momentum symplectic reduction, corresponding to momenta of the form $m = J(\varrho_1, \theta_1, \varrho_2, \theta_2)$, yields a canonical system

$$\begin{cases} \dot{\varrho}_1 = \frac{\delta \tilde{H}}{\delta \theta_1} = -\mathcal{L}_v \varrho_1 \\ \dot{\varrho}_2 = \frac{\delta \tilde{H}}{\delta \theta_2} = -\mathcal{L}_v \varrho_2 \\ \dot{\theta}_1 = -\frac{\delta \tilde{H}}{\delta \varrho_1} \\ \dot{\theta}_2 = -\frac{\delta \tilde{H}}{\delta \varrho_2} \end{cases}$$

on $T^*\text{Dens}(M) \times T^*\text{Dens}(M)$ for the Hamiltonian

$$\tilde{H}(\varrho_1, \varrho_2, \theta_1, \theta_2) = \bar{H}(\varrho_1, \varrho_2, J(\varrho_1, \theta_1, \varrho_2, \theta_2)).$$

Next, we turn to the incompressible case. Imposing the holonomic constraint $\varrho_1 + \varrho_2 = 1$ for the equations on $T^*(\text{Dens}(M) \times \text{Dens}(M))$ leads to a constrained Hamiltonian system

$$\begin{cases} \dot{\varrho}_1 = \frac{\delta \tilde{H}}{\delta \theta_1} \\ \dot{\varrho}_2 = \frac{\delta \tilde{H}}{\delta \theta_2} \\ \dot{\theta}_1 = -\frac{\delta \tilde{H}}{\delta \varrho_1} - p \\ \dot{\theta}_2 = -\frac{\delta \tilde{H}}{\delta \varrho_2} - p \\ \varrho_1 + \varrho_2 - 1 = 0 \end{cases} \quad (9.9)$$

where $p \in C^\infty(M)$ is a Lagrange multiplier.

The induced cotangent constraint on (θ_1, θ_2) is obtained by

$$0 = \frac{d}{dt}(\varrho_1 + \varrho_2 - 1) = \dot{\varrho}_1 + \dot{\varrho}_2 = \frac{\delta \tilde{H}}{\delta \theta_1} + \frac{\delta \tilde{H}}{\delta \theta_2} = -\mathcal{L}_v(\varrho_1 + \varrho_2) = -\mathcal{L}_v \mu,$$

which implies that the vector field v is divergence-free. Therefore, solutions of (9.9) correspond to zero-momentum solutions of the incompressible fluid equations on $T^*\text{Diff}_\mu(M) \simeq \text{Diff}_\mu(M) \times \mathfrak{X}_\mu^*(M)$ with the Hamiltonian

$$H(\varphi, u^b) = \bar{H}(\varphi_*\mu_1, \varphi_*\mu_2, u^b \otimes \mu).$$

In particular, the choice

$$\bar{H}(\varrho_1, \varrho_2, m) = \frac{1}{2} \langle m, u \rangle, \quad m = u^b \otimes (\varrho_1 + \varrho_2) \quad (9.10)$$

yields special solutions to the incompressible Euler equations, see [Appendix A](#) (and also, if the constraints are dropped, special solutions to the inviscid Burgers equation in [§ 4.1](#)).

Schrödinger's smoke is an approximation to the zero-momentum incompressible Euler solutions, where the Hamiltonian \tilde{H} corresponding to (9.10) is replaced by the sum of two independent Hamiltonian systems

$$\tilde{H}(\varrho_1, \varrho_2, \theta_1, \theta_2) = \frac{1}{2} \langle d\theta_1 \otimes \varrho_1, \nabla \theta_1 \rangle + \frac{1}{2} \langle d\theta_2 \otimes \varrho_2, \nabla \theta_2 \rangle + \hbar^2 I(\varrho_1) + \hbar^2 I(\varrho_2).$$

This approximation corresponds to dropping the cross-terms between θ_1 and θ_2 in the original kinetic energy and adding the Fisher information functionals as potentials for ϱ_1 and ϱ_2 . Applying the two-component Madelung transform

$$\Psi = (\psi_1, \psi_2) := \left(\sqrt{\rho_1 e^{2i\theta_1/\hbar}}, \sqrt{\rho_2 e^{2i\theta_2/\hbar}} \right)$$

and setting $\hbar = 1$ gives the *incompressible Schrödinger equation*

$$i\dot{\Psi} = -\Delta \Psi + p\Psi,$$

where, as before, the pressure function $p \in C^\infty(M)$ is a Lagrange multiplier for the pointwise constraint $|\Psi|^2 = 1$. Notice that the resulting equation is a wave-map equation on $S^3 \subset \mathbb{C}^2$ [\[53\]](#).

Remark 9.9. It has been observed that numerical solutions to the incompressible Schrödinger equations (ISE) yield realistic visualization of the dynamics of smoke [\[10\]](#). However, it is an open question in what sense (or, in which regime) these solutions are approximations to solutions of the incompressible Euler equations.

9.5. Madelung transform as a Kähler morphism. We now assume that both spaces $T^*\text{Dens}(M)$ and $PC^\infty(M, \mathbb{C})$ are equipped with suitable Riemannian structures. Consider first the tangent bundle $TT^*\text{Dens}(M)$. Its elements can be described as 4-tuples $(\varrho, \theta, \dot{\varrho}, \dot{\theta})$ with $\varrho \in \text{Dens}(M)$, $[\theta] \in C^\infty(M)/\mathbb{R}$, $\dot{\varrho} \in \Omega_0^n(M)$ and $\dot{\theta} \in C^\infty(M)$ subject to the constraint

$$\int_M \dot{\theta} \varrho = 0.$$

Definition 9.10. The *Sasaki* (or *Sasaki-Fisher-Rao*) metric on $T^*\text{Dens}(M)$ is the lift of the Fisher-Rao metric (7.7)

$$\mathbb{G}_{(\varrho, [\theta])}^*((\dot{\varrho}, \dot{\theta}), (\dot{\varrho}, \dot{\theta})) = \frac{1}{4} \int_M \left(\left(\frac{\dot{\varrho}}{\varrho} \right)^2 + \dot{\theta}^2 \right) \varrho. \quad (9.11)$$

On the projective space $PC^\infty(M, \mathbb{C})$ we define the infinite-dimensional *Fubini-Study metric*

$$\mathbb{G}_\psi^*(\dot{\psi}, \dot{\psi}) = \frac{\langle \dot{\psi}, \dot{\psi} \rangle_{L^2}}{\langle \psi, \psi \rangle_{L^2}} - \frac{\langle \psi, \dot{\psi} \rangle_{L^2} \langle \dot{\psi}, \psi \rangle_{L^2}}{\langle \psi, \psi \rangle_{L^2}^2}. \quad (9.12)$$

Theorem 9.11 ([28]). *The Madelung transform (9.3) with $\hbar = 2$ is an isometry between the spaces $T^*\text{Dens}(M)$ equipped with (9.11) and $PC^\infty(M, \mathbb{C} \setminus \{0\})$ equipped with (9.12).*

Since the Fubini-Study metric together with the complex structure of $PC^\infty(M, \mathbb{C})$ defines a Kähler structure, it follows that $T^*\text{Dens}(M)$ also admits a natural Kähler structure which corresponds to the canonical symplectic structure scaled by 1/4. Note that an almost complex structure on $T^*\text{Dens}(M)$ corresponding via the Madelung transform to the Wasserstein-Otto metric does not integrate to a complex structure [40]. In contrast, the corresponding complex structure becomes integrable (and simple) when the Fisher-Rao metric is used in place of the Wasserstein-Otto metric, as shown in [28]. It would be interesting to write down Kähler potentials for all metrics compatible with the corresponding complex structure on $T^*\text{Dens}(M)$ and identify those invariant under the action of the diffeomorphism group.

Example 9.12. The 2-component Hunter-Saxton (2HS) equation is a system

$$\begin{cases} \dot{u}_{xx} = -2u_x u_{xx} - u u_{xxx} + \sigma \sigma_x \\ \dot{\sigma} = -(\sigma u)_x, \end{cases} \quad (9.13)$$

where u and σ are time-dependent periodic functions on the real line. This system can be viewed as a high-frequency limit of the two-component Camassa-Holm equation, cf. [59].

It turns out that (9.13) describes the geodesic flow of a right-invariant H^1 -type metric on the semidirect product $\mathcal{G} = \text{Diff}_0(S^1) \ltimes C^\infty(S^1, S^1)$ of the group of circle diffeomorphisms that fix a prescribed point and the space of S^1 -valued maps of a circle. Furthermore, there is an isometry between subsets of the group \mathcal{G} and the unit sphere in the space of wave functions $\{\psi \in C^\infty(S^1, \mathbb{C}) \mid \|\psi\|_{L^2} = 1\}$, see [32]. In [28] it is proved that the 2HS equation (9.13) with initial data satisfying $\int_{S^1} \sigma dx = 0$ is equivalent to the geodesic equation of the Sasaki-Fisher-Rao metric (9.11) on $T^*\text{Dens}(S^1)$ and the Madelung transformation induces a Kähler map to geodesics in $PC^\infty(S^1, \mathbb{C})$ equipped with the Fubini-Study metric.

Note also that (subject to the t -invariant condition $\sigma = 0$) the 2-component Hunter-Saxton equation (9.13) reduces to the standard Hunter-Saxton equation. This is a consequence of the fact that horizontal geodesics on $T^*\text{Dens}(M)$ with the Sasaki-Fisher-Rao metric descend to geodesics on $\text{Dens}(M)$ with the Fisher-Rao metric.

10. CASIMIRS OF COMPRESSIBLE FLUIDS

10.1. Casimirs of barotropic fluids. In many respects the behaviour of barotropic compressible fluids is similar to that of incompressible fluids (while the fully compressible fluids resemble thermodynamical rather than mechanical systems). In particular, their Hamiltonian description suggests similar sets of Casimir invariants of motion. Recall that the incompressible Euler equations on a manifold M are geodesic equations on the group $\text{Diff}_\mu(M)$ and hence a Hamiltonian system on the corresponding dual space $\mathfrak{X}_\mu^*(M)$, see [Appendix A](#). The equations of compressible barotropic fluids (4.7) are known to be related to the semidirect product group $S = \text{Diff}(M) \ltimes C^\infty(M)$, see Section 5.1. Its Lie algebra is $\mathfrak{s} = \mathfrak{X}(M) \ltimes C^\infty(M)$ and the corresponding dual space $\mathfrak{s}^* = \mathfrak{X}^*(M) \oplus \Omega^n(M)$ was described in §3.1.

The equations of barotropic fluids are Hamiltonian equations on \mathfrak{s}^* with the Lie-Poisson bracket given by the formula (3.5) and the invariants of the corresponding coadjoint action, i.e. the Casimir functions, are the first integrals of the equations of motion.

Recall that the smooth part of the dual of the semidirect product algebra $\mathfrak{s} = \mathfrak{X}(M) \ltimes C^\infty(M)$ can be identified with $\mathfrak{s}^* = \Omega^1(M) \otimes \Omega^n(M) \oplus \Omega^n(M)$ via the pairing

$$\langle (v, f), (\alpha \otimes \varrho, \varrho) \rangle = \int_M (\iota_v \alpha) \varrho + \int_M f \varrho.$$

In what follows we restrict to the subset $\Omega_+^n(M)$ of $\Omega^n(M)$ corresponding to everywhere positive densities on M . It turns out that the equations of incompressible fluid also have an infinite number of conservation laws in the even-dimensional case and possess at least one first integral in the odd-dimensional case, see §A.2 and [5, 46].

The following proposition shows that Casimir functions for a barotropic fluid are similar to the ones for an incompressible fluid.

Proposition 10.1 ([46]). *Let $\alpha \in \Omega^1(M)$ and $\varrho \in \Omega_+^n(M)$. If $\dim M = 2m + 1$ then the functional*

$$I(\alpha \otimes \varrho, \varrho) = \int_M \alpha \wedge (d\alpha)^m$$

is a Casimir function on $\mathfrak{s}^ = \mathfrak{X}^*(M) \oplus \Omega^n(M)$ (i.e., it is invariant under the coadjoint action of $S = \text{Diff}(M) \ltimes C^\infty(M)$).*

If $\dim M = 2m$ then for any measurable function $h: \mathbb{R} \rightarrow \mathbb{R}$ the functional

$$I_h(\alpha \otimes \varrho, \varrho) = \int_M h\left(\frac{(d\alpha)^m}{\varrho}\right) \varrho$$

is a Casimir function on $\mathfrak{s}^ = \mathfrak{X}^*(M) \oplus \Omega^n(M)$.*

Proof. The proof is based on the fact that the coadjoint action of the group $\text{Diff}(M) \ltimes C^\infty(M)$ on the dual space $\mathfrak{s}^* = \mathfrak{X}^*(M) \oplus \Omega^n(M)$ is given by

$$\text{Ad}_{(\varphi, f)}^*(\alpha \otimes \varrho, \varrho) = ((\varphi^* \alpha + \varphi^* df) \otimes \varphi^* \varrho, \varphi^* \varrho).$$

Thus, α and ϱ transform according to the rules $\alpha \mapsto \varphi^* \alpha + d\varphi^* f$ and $\varrho \mapsto \varphi^* \varrho$ and it is now straightforward to check that the functionals I and I_h are invariant under such transformations. Indeed, up to the change of coordinates by a diffeomorphism φ , the 1-form α changes within its coset $[\alpha]$ and the functionals I and I_h are well defined on the cosets. \square

The above argument shows that, in a certain sense, a barotropic fluid “becomes incompressible” when viewed in a coordinate system which “moves with the flow.” The Hamiltonian approach makes it possible to apply Casimir functions to study stability of barotropic fluids

and inviscid MHD systems: their dynamics is confined to coadjoint orbits of the corresponding groups and Casimir functions can be used to describe the corresponding conditional extrema of the Hamiltonians.

10.2. Casimirs of magnetohydrodynamics. We start with the 3D incompressible magnetohydrodynamics described in §5.2, cf. equations (5.4). In this case the configuration space of a magnetic fluid is the semidirect product $\text{IMH} = \text{Diff}_\mu(M) \ltimes \mathfrak{X}_\mu^*(M)$ of the volume preserving diffeomorphism group and the dual space $\mathfrak{X}_\mu^*(M) = \Omega^1(M)/d\Omega^0(M)$ of the Lie algebra of divergence free vector fields on a 3-manifold M . The semidirect product algebra is $\text{imh} = \mathfrak{X}_\mu(M) \ltimes \mathfrak{X}_\mu^*(M)$ and its action is given by formula (5.2). The corresponding dual space is

$$\text{imh}^* = \mathfrak{X}_\mu^*(M) \oplus \mathfrak{X}_\mu(M) = \Omega^1(M)/d\Omega^0(M) \oplus \mathfrak{X}_\mu(M)$$

and the Poisson brackets on imh^* are given by (3.5), interpreted accordingly.

Proposition 10.2 ([5]). *Let M be a manifold with $H_1(M) = 0$, and let $[\alpha] \in \Omega^1(M)/d\Omega^0(M)$ and $\mathbf{B} \in \mathfrak{X}_\mu(M)$. Then the magnetic helicity*

$$I(\mathbf{B}) = \int_M (\mathbf{B}, \text{curl}^{-1} \mathbf{B}) \mu$$

and the cross-helicity

$$J(\alpha, \mathbf{B}) = \int_M \iota_{\mathbf{B}} \alpha \mu$$

are Casimir functions on imh^* (i.e., are invariant under the coadjoint action of $\text{IMH} = \text{Diff}_\mu(M) \ltimes \mathfrak{X}_\mu^*(M)$).

The condition $H_1(M) = 0$ ensures that any magnetic field \mathbf{B} has a vector potential $\text{curl}^{-1} \mathbf{B}$. It turns out that these are the only Casimirs for incompressible magnetohydrodynamics: any other sufficiently smooth Casimir is a function of these two, cf. [15].

Consider now the setting of compressible magnetohydrodynamics on a Riemannian manifold of arbitrary dimension, see (6.7) above. Recall also from §6.2 that the semidirect product group associated with the compressible MHD equations is

$$\text{CMH} = \text{Diff}(M) \ltimes (C^\infty(M) \oplus \Omega^{n-2}(M)/d\Omega^{n-3}(M)).$$

The corresponding Lie algebra is

$$\text{cmh} = \mathfrak{X}(M) \ltimes (C^\infty(M) \oplus \Omega^{n-2}(M)/d\Omega^{n-3}(M))$$

with dual

$$\text{cmh}^* = \mathfrak{X}^*(M) \oplus \Omega^n(M) \oplus \Omega_{cl}^2(M),$$

where $\Omega_{cl}^2(M)$ is the space of closed 2-forms referred to as “magnetic 2-forms.” Recall that if M is a three-fold then a magnetic vector field \mathbf{B} and a magnetic 2-form $\beta \in \Omega_{cl}^2(M)$ are related by $\iota_{\mathbf{B}} \mu = \beta$. We again confine our constructions to positive densities $\Omega_+^n(M)$.

Proposition 10.3. *Let $\alpha \in \mathfrak{X}^*(M)$, $\varrho \in \Omega^n(M)$ and $\beta \in \Omega_{cl}^2(M)$. If $\dim M = 2n + 1$ then the generalized cross-helicity functional*

$$J(\alpha, \varrho, \beta) = \int_M \alpha \wedge \beta^n$$

is a Casimir function on cmh^* .

If $\dim M = 2n + 1$ and $H_2(M) = 0$, so that $d\gamma = \beta$ for some 1-form γ , then

$$I(\beta) = \int_M \gamma \wedge \beta^n$$

is a Casimir function on \mathfrak{cmh}^* .

If $\dim M = 2n$ then for any measurable function $h: \mathbb{R} \rightarrow \mathbb{R}$ the functional

$$I_h(\varrho, \beta) = \int_M h\left(\frac{\beta^n}{\varrho}\right) \varrho$$

is a Casimir function on \mathfrak{cmh}^* .

If \mathbf{B} is a vector field on M defined by $\iota_{\mathbf{B}}\varrho = \beta^n$ then the functional J can be equivalently written as

$$J(\alpha, \varrho, \beta) = \int_M \alpha \wedge \iota_{\mathbf{B}}\varrho = \int_M \iota_{\mathbf{B}}\alpha \varrho.$$

In the three-dimensional ($n = 1$) and incompressible ($\varrho = \mu$) case it reduces to the cross-helicity functional $J(\alpha, \mathbf{B})$ of Proposition 10.2.

Proof. The coadjoint action is

$$\text{Ad}_{(\varphi, f, [P])^{-1}}^*(\alpha \otimes \varrho, \varrho, \beta) = ((\varphi^*\alpha + \varphi^*\iota_u\beta + \varphi^*df) \otimes \varphi^*\varrho, \varphi^*\varrho, \varphi^*\beta)$$

where the vector field u is defined by the condition $\iota_u\varrho = dP$. Since both ϱ and β are transported by φ , the only non-trivial functional to check is the generalized cross-helicity J .

For this purpose we first note that since β is closed then so is β^n . Hence, the change of variables formula gives

$$\begin{aligned} J(\varphi^*\alpha + \varphi^*\iota_u\beta + \varphi^*df, \varphi^*\varrho, \varphi^*\beta) &= \int_M (\alpha + \iota_u\beta + df) \wedge \beta^n \\ &= J(\alpha, \varrho, \beta) + \int_M \iota_u\beta \wedge \beta^n + \int_M d(f\beta^n), \end{aligned}$$

where the last term on the right-hand side vanishes by Stokes' theorem while the $(2n+1)$ -form $\iota_u\beta \wedge \beta^n$ vanishes pointwise on M . The latter holds since evaluating this form on any $2n+1$ linearly independent vectors tangent to M is equivalent to evaluating β^{n+1} on any linearly dependent set of $2n+2$ tangent vectors containing u , which is evidently zero. \square

Remark 10.4. Other differential-geometric invariants of hydrodynamical equations include Ertel-type invariants [58], local invariants [1, 2], invariants of Lagrangian type [7], and many others.

APPENDIX A. GEOMETRIC FRAMEWORK FOR THE INCOMPRESSIBLE EULER EQUATIONS

A.1. Geodesic and Hamiltonian formulations. In [3] V. Arnold suggested the following general framework for the Euler equation on an arbitrary group describing a geodesic flow with respect to a suitable one-sided invariant Riemannian metric on this group. Let a (possibly infinite-dimensional) Lie group G be the configuration space of some physical system. The tangent space at the identity element e is the corresponding Lie algebra $\mathfrak{g} = T_eG$. Fix a positive definite quadratic form (the “energy”) $E(v) = \frac{1}{2}\langle v, Av \rangle$ on \mathfrak{g} and right translate it to the tangent space T_aG at any point $a \in G$ (this is “translational symmetry” of the energy). In this way the energy defines a right-invariant Riemannian metric on the group. The geodesic flow on G with respect to this energy metric represents extremals of the least action principle, i.e., actual motions of the physical system.

Applied to the group $G = \text{Diff}_\mu(M)$ of diffeomorphisms preserving the volume form μ of an n -dimensional manifold M , this framework provides an infinite-dimensional Riemannian setting for the Euler equations (1.1) of an ideal fluid in M . Namely, the right-invariant energy metric is given here by the L^2 -inner product on divergence-free vector fields on M that constitute the Lie

algebra $\mathfrak{g} = \mathfrak{X}_\mu(M) = \{v \in \mathfrak{X}(M) \mid \mathcal{L}_v \mu = 0\}$. The geodesic flow of this metric is then governed by the incompressible Euler equations.

This approach also provides the following Hamiltonian framework for classical hydrodynamics.

Theorem A.1. (see e.g. [5])

a) *The dual space to the Lie algebra $\mathfrak{X}_\mu(M)$ is $\mathfrak{X}_\mu^*(M) = \Omega^1(M)/dC^\infty(M)$, the space of cosets of 1-forms on M modulo exact 1-forms. The coadjoint action of $\text{Diff}_\mu(M)$ is given by change of coordinates in a 1-form, while the coadjoint action of $\mathfrak{X}_\mu(M)$ is given by the Lie derivative along a vector field $\text{ad}_v^* = \mathcal{L}_v$; it is well-defined on the cosets in $\Omega^1(M)/dC^\infty(M)$.*

b) *The inertia operator $A : \mathfrak{X}_\mu(M) \rightarrow \mathfrak{X}_\mu^*(M)$ is defined by assigning to a given divergence-free vector field v the coset $\alpha = [v^\flat]$ in $\Omega^1(M)/dC^\infty(M)$.*

c) *The incompressible Euler equations (1.1) on the dual space $\mathfrak{X}_\mu^*(M)$ have the form*

$$\partial_t[\alpha] = -\mathcal{L}_v[\alpha], \quad (\text{A.1})$$

where $[\alpha] \in \Omega^1(M)/dC^\infty(M)$ and $\alpha = v^\flat$.

The proof follows from the fact that the map $v \mapsto \iota_v \mu$ provides an isomorphism of the space of divergence-free vector fields and the space of closed $(n-1)$ -forms on M , i.e., $\mathfrak{g} = \mathfrak{X}_\mu(M) \simeq \Omega_{cl}^{n-1}(M)$, since $d(\iota_v \mu) = \mathcal{L}_v \mu = 0$. The dual space is $\mathfrak{g}^* = (\Omega_{cl}^{n-1}(M))^* = \Omega^1(M)/dC^\infty(M)$ and the pairing is given by

$$\langle v, [\alpha] \rangle = \int_M (\iota_v \alpha) \mu.$$

For more details we refer to [5].

Remark A.2. Equation (A.1) can be rewritten in terms of a representative 1-form and a differential of a (pressure) function

$$\partial_t \alpha + \mathcal{L}_v \alpha = -dP$$

which is a more familiar form of the Euler equations of an ideal fluid.

Note that each coset $[\alpha]$ contains a unique coclosed 1-form $\bar{\alpha} \in [\alpha]$ which is related to a divergence-free vector field v by means of the metric on M , namely $\bar{\alpha} = v^\flat$. Such a choice of a representative $\bar{\alpha}$ defines the (pressure) function P uniquely modulo a constant since $\Delta P = \delta dP$ is prescribed for each time t .

The analysis in Sobolev for the incompressible Euler equations was provided by Ebin and Marsden [14]. The approach via generalized flows for this equation was proposed by Brenier [8].

A.2. Casimirs for the equations of ideal fluids. The Hamiltonian description of the dynamics of an ideal fluid gives some insight into the nature of its first integrals. Recall that the Euler equation is a Hamiltonian system on the dual space $\mathfrak{X}_\mu^*(M)$ with respect to the Poisson-Lie structure and with the fluid energy as the Hamiltonian. In this setting we have

Proposition A.3 ([46, 5]). *For the group $\text{Diff}_\mu(M)$ the following functionals are Casimirs on the dual space $\mathfrak{X}_\mu^*(M) = \Omega^1(M)/dC^\infty(M)$ (the space of cosets $[u] \in \Omega^1(M)/dC^\infty(M)$).*

If $\dim(M) = 2m + 1$, then the functional

$$I([u]) = \int_M u \wedge (du)^m$$

is a Casimir function on $\mathfrak{X}_\mu^(M)$, i.e. it is invariant under the coadjoint action of the group $\text{Diff}_\mu(M)$.*

If $\dim(M) = 2m$, then the functionals

$$I_h([u]) = \int_M h\left(\frac{(du)^m}{\mu}\right) \mu$$

are Casimir functions on $\mathfrak{X}_\mu^*(M)$ for any measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$.

Here, the quotient $(du)^m/\mu$ of a $2m$ -form and the volume form is a function, which being composed with h can be integrated against the volume form μ over M .

Proof. First, we have to check that I and I_h are well-defined functionals on $\Omega^1(M)/dC^\infty(M)$. Note that for any exact 1-form df we have $I(df) = 0$ and $I_h(df) = 0$. Similarly, we find that each of the functionals I and I_h depends on a coset but not on a representative, e.g., $I(u) = I(u + df) = I([u])$. Furthermore, the group $\text{Diff}_\mu(M)$ acts on $\Omega^1(M)/dC^\infty(M)$ by change of coordinates $[u] \mapsto \varphi^*[u]$ for any $\varphi \in \text{Diff}_\mu(M)$. Since both I and I_h are defined in a coordinate-free way, they are invariant under this action. \square

Corollary A.4. *The functionals I and, respectively, I_h on $\Omega^1(M)/dC^\infty(M)$ are first integrals of the incompressible Euler equations in M in odd and even dimension, respectively.*

Proof. Since the Euler equations are Hamiltonian with respect to the standard Lie-Poisson bracket on the dual space of the Lie algebra $\mathfrak{X}_\mu(M)$, the flow lines remain always tangent to coadjoint orbits of $\text{Diff}_\mu(M)$. By Proposition A.3, the functions I and I_h are constant on coadjoint orbits and hence are constant along the flow lines. \square

Remark A.5. The functionals I and I_h are Casimirs of the Lie-Poisson bracket on $\mathfrak{X}_\mu^*(M) = \Omega^1(M)/dC^\infty(M)$, i.e. they yield conservation laws for *any* Hamiltonian equation on this space. In particular, both I and I_h are first integrals of the Euler equations for an *arbitrary metric* on M . They express “kinematic symmetries” of the hydrodynamical system, while the energy is an invariant related to the system’s “dynamics.”

Example A.6. If M is a domain in \mathbb{R}^3 then the function

$$I(v) = \int_M u \wedge du = \int_M (v, \text{curl } v) d^3x$$

is a first integral of the Euler equations, where the 1-form $u = v^\flat$ is related to the velocity field v by means of the Euclidean metric. The last integral has a natural geometric meaning of the *helicity* of the vector field $\xi = \text{curl } v$ defined by $\iota_\xi \mu = du$.

Example A.7. Similarly, if M is a domain in \mathbb{R}^2 we find infinitely many first integrals of the Euler equations, namely

$$I_h(v) = \int_M h(\text{curl } v) d^2x,$$

where $\text{curl } v = \partial v_1/\partial x_2 - \partial v_2/\partial x_1$ is the *vorticity function* on $M \subset \mathbb{R}^2$.

Remark A.8. While the functions I and I_h on $\Omega^1(M)/d\Omega^0(M)$ are Casimirs, generally speaking, they do *not* form a *complete* set of invariants of the coadjoint representation.

In the 2D case the complete set of invariants includes a measured Reeb graph of the vorticity function $\text{curl } v$ and circulation data of the field v on the surface M , see [23]. In the 3D case the invariant I is shown to be unique among C^1 -Casimirs [15], while there are more invariants of ergodic nature (such as pairwise linkings of the trajectories of the vorticity field) that are not continuous functionals [4].

APPENDIX B. SYMPLECTIC REDUCTION

In §3.1 and §3.2 we described Poisson reduction on $T^*\text{Diff}(M)$ with respect to the cotangent action of $\text{Diff}_\mu(M)$. This lead to reduced dynamics on the Poisson manifold $T^*\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \text{Dens}(M) \times \mathfrak{X}^*(M)$ (Theorem 3.3). Furthermore, any Hamiltonian system descends to symplectic leaves and $T^*\text{Dens}(M)$ with the canonical symplectic structure is one of the symplectic leaves of $T^*\text{Diff}(M)/\text{Diff}_\mu(M)$. In this appendix we shall describe *symplectic reduction* which leads to the same manifold $T^*\text{Dens}(M)$ - the symplectic quotient $T^*\text{Diff}(M)//\text{Diff}_\mu(M)$ corresponding to the cotangent bundle $T^*\text{Dens}(M)$ equipped with the canonical symplectic structure.

As before, let $\mathfrak{X}_\mu(M) = \{u \in \mathfrak{X}(M) \mid \mathcal{L}_u\mu = 0\}$ be the Lie algebra of $\text{Diff}_\mu(M)$. Recall that the dual space is naturally isomorphic to $\mathfrak{X}_\mu^*(M) = \Omega^1(M)/dC^\infty(M)$, see Theorem A.1.

Lemma B.1. *The (smooth) dual $\mathfrak{X}_\mu(M)^*$ can be identified with the quotient space*

$$\mathfrak{X}^*(M)/(dC^\infty(M) \otimes \mu) = (\Omega^1(M) \otimes \text{Dens}(M))/(dC^\infty(M) \otimes \mu), \quad (\text{B.1})$$

where \otimes is taken over smooth functions on M . The cotangent left action of $\text{Diff}_\mu(M)$ on $T^*\text{Diff}(M)$ is Hamiltonian. The associated momentum map $J: T^*\text{Diff}(M) \rightarrow \mathfrak{X}_\mu(M)^*$ is given by

$$J(\varphi, m) = \varphi^*m + dC^\infty(M) \otimes \mu \quad (\text{B.2})$$

where $\varphi \in \text{Diff}(M)$ and $m \in \mathfrak{X}(M) \simeq T_\varphi^*\text{Diff}(M)$. The momentum map is equivariant, i.e.,

$$J(\eta \cdot (\varphi, m)) = \eta_*J(\varphi, m)$$

for all $\eta \in \text{Diff}_\mu(M)$.

Proof. From the Hodge decomposition it follows that $\langle m, v \rangle = 0$ for all $v \in \mathfrak{X}_\mu(M)$ if and only if $m = d\theta \otimes \mu$ for some $\theta \in C^\infty(M)$. This proves (B.1).

From the standard Lie-Poisson theory we find that the momentum map for $\text{Diff}(M)$ acting on $T^*\text{Diff}(M)$ is given by $(\varphi, m) \mapsto \varphi^*m$. Since $\text{Diff}_\mu(M)$ is a subgroup of $\text{Diff}(M)$, it follows from (B.1) that the momentum map must be (B.2).

Regarding the equivariance statement, we have

$$\begin{aligned} \eta_*J(\varphi, m) &= \eta_*\varphi^*m + \eta_*(dC^\infty(M) \otimes \mu) \\ &= (\varphi \circ \eta^{-1})^*m + d\eta_*C^\infty(M) \otimes \eta_*\mu \\ &= (\varphi \circ \eta^{-1})^*m + dC^\infty(M) \otimes \mu \\ &= J(\varphi \circ \eta^{-1}, m) = J(\eta \cdot (\varphi, m)), \end{aligned}$$

as required. □

Lemma B.2. *The zero momentum level set*

$$J^{-1}([0]) = \{(\varphi, d\theta \otimes \varphi_*\mu) \mid \varphi \in \text{Diff}(M), \theta \in C^\infty(M)\}$$

is invariant under the action of $\text{Diff}_\mu(M)$, i.e., for any $\eta \in \text{Diff}_\mu(M)$ and $(\varphi, m) \in J^{-1}([0])$ one has $\eta \cdot (\varphi, m) \in J^{-1}([0])$.

Proof. We have $[0] = dC^\infty(M) \otimes \mu$ so that if $m = d\theta \otimes \varphi_*\mu$ then

$$\begin{aligned} J(\varphi, m) &= \varphi^*(d\theta \otimes \varphi_*\mu) + dC^\infty(M) \otimes \mu \\ &= d\varphi^*\theta \otimes \mu + dC^\infty(M) \otimes \mu = dC^\infty(M) \otimes \mu = [0]. \end{aligned}$$

Next, assume that $J(\varphi, m) = [0]$ and write $m = \alpha \otimes \varphi_*\mu$ for some $\alpha \in \Omega^1(M)$. Since $\varphi^*m \in [0]$ it follows that $\varphi^*\alpha$ must be exact, i.e., $\varphi^*\alpha = d\theta$. Thus, $\alpha = \varphi_*d\theta = d\varphi_*\theta$, and so α is exact.

The fact that $J^{-1}([0])$ is invariant under $\text{Diff}_\mu(M)$ follows from the equivariance property in [Lemma B.1](#), since $\eta_*[0] = [0]$ for all $\eta \in \text{Diff}_\mu(M)$. This concludes the proof. \square

To identify the symplectic structure of the quotient we shall first identify the momentum map associated with the action of $\text{Diff}(M)$ on $T^*\text{Dens}(M)$.

Lemma B.3. *The associated momentum map $I: T^*\text{Dens}(M) \rightarrow \mathfrak{X}^*(M)$ for the left cotangent action of $\text{Diff}(M)$ on $T^*\text{Dens}(M)$ is given by*

$$I(\varrho, \theta) = d\theta \otimes \varrho.$$

Proof. The smooth dual of $\Omega^n(M)$ is $C^\infty(M)$ with the natural pairing

$$\langle \theta, \dot{\varrho} \rangle = \int_M \theta \dot{\varrho}.$$

Since $T_\varrho \text{Dens}(M) = \Omega_0^n(M)$ is a subspace of $\Omega^n(M)$, it follows that

$$\begin{aligned} T^*\text{Dens}(M) &= \text{Dens}(M) \times \Omega^n(M)^* / \ker(\langle \cdot, \Omega_0^n(M) \rangle) \\ &= \text{Dens}(M) \times C^\infty(M) / \mathbb{R}. \end{aligned}$$

The infinitesimal left action of $\mathfrak{X}(M)$ is $u \cdot \varrho = -\mathcal{L}_u \varrho$ and the momentum map $I: (\varrho, \theta) \rightarrow \mathfrak{X}^*(M)$ is then given by

$$\langle I(\varrho, \theta), u \rangle = \langle \theta, -\mathcal{L}_u \varrho \rangle \quad \text{for all } u \in \mathfrak{X}(M).$$

By Cartan's formula we obtain

$$\langle I(\varrho, \theta), u \rangle = \langle \mathcal{L}_u \theta, \varrho \rangle = \langle \iota_u d\theta, \varrho \rangle = \langle d\theta \otimes \varrho, u \rangle,$$

which proves the lemma. \square

The main result of this section is

Theorem B.4. *The zero momentum symplectic quotient*

$$T^*\text{Diff}(M) // \text{Diff}_\mu(M) = J^{-1}([0]) / \text{Diff}_\mu(M)$$

is isomorphic, as a symplectic manifold, to $T^\text{Dens}(M)$ and the symplectomorphism $T^*\text{Dens}(M) \rightarrow T^*\text{Diff}(M) // \text{Diff}_\mu(M)$ is given by*

$$(\varrho, \theta) \mapsto (\varrho, I(\varrho, \theta)). \tag{B.3}$$

Thus $T^*\text{Dens}(M)$ can be viewed as a symplectic leaf of the Poisson manifold $T^*\text{Diff}(M) / \text{Diff}_\mu(M)$. [Theorem B.4](#) is an infinite-dimensional variant of the following general result: For a homogeneous space $B = G/H$ the zero momentum reduction space $T^*G // H$ is symplectomorphic to T^*B through the mapping

$$(q, p) \mapsto (q, I(q, p)),$$

where I is the momentum map for the natural action of G on $T^*B \ni (q, p)$, see Marsden and Ratiu [38]

Proof. Let $m = d\theta \otimes \varphi_* \mu$ so that $(\varphi, m) \in J^{-1}([0])$. If $\eta \in \text{Diff}_\mu(M)$ then

$$\eta \cdot (\varphi, m) = (\varphi \circ \eta^{-1}, d\theta \otimes \varphi_* \mu).$$

By Moser-Hamilton's result in [Lemma 2.9](#) it follows that in the Fréchet category we have

$$J^{-1}([0]) / \text{Diff}_\mu(M) \simeq \{(\varrho, m) \in \text{Dens}(M) \times \mathfrak{X}^*(M) \mid m = d\theta \otimes \varrho\}.$$

Thus, the symplectic quotient $T^*\text{Diff}(M)//\text{Diff}_\mu(M)$ is naturally identified with a subbundle of the Poisson manifold $T^*\text{Diff}(M)/\text{Diff}_\mu(M) \simeq \text{Dens}(M) \times \mathfrak{X}^*(M)$ in [Theorem 3.3](#). By conservation of momentum this subbundle is invariant under the flow of any Hamiltonian. To prove that it is a symplectic leaf it suffices to show that the map corresponding to [\(B.3\)](#)

$$\Phi: (\varrho, \theta) \mapsto (\varrho, I(\varrho, \theta))$$

is a diffeomorphism and Poisson. The former follows from the fact that the kernel of d on $C^\infty(M)/\mathbb{R}$ is trivial. It thus remains to show that

$$\{F \circ \Phi, G \circ \Phi\} = \{F, G\} \circ \Phi$$

for any $F, G \in C^\infty(\text{Dens}(M) \times \mathfrak{X}^*(M))$.

We have

$$\begin{aligned} \left\langle \frac{\delta F \circ \Phi}{\delta \varrho}(\varrho, \theta), \dot{\varrho} \right\rangle &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\varrho + \epsilon \dot{\varrho}, d\theta \otimes (\varrho + \epsilon \dot{\varrho})) \\ &= \left\langle d\theta \otimes \dot{\varrho}, \underbrace{\frac{\delta F}{\delta m}}_{v_F} \right\rangle + \left\langle \dot{\varrho}, \frac{\delta F}{\delta \varrho} \right\rangle \\ &= \left\langle \dot{\varrho}, \mathcal{L}_{v_F} \theta + \frac{\delta F}{\delta \varrho} \right\rangle \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} \left\langle \frac{\delta F \circ \Phi}{\delta \theta}(\varrho, \theta), \dot{\theta} \right\rangle &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\varrho, d(\theta + \epsilon \dot{\theta}) \otimes \varrho) \\ &= \left\langle d\dot{\theta} \otimes \varrho, v_F \right\rangle \\ &= - \left\langle \mathcal{L}_{v_F} \varrho, \dot{\theta} \right\rangle. \end{aligned} \tag{B.5}$$

Combining [\(B.4\)](#) and [\(B.5\)](#) we get

$$\begin{aligned} \{F \circ \Phi, G \circ \Phi\}(\varrho, \theta) &= \\ &= \left\langle -\mathcal{L}_{v_G} \varrho, \mathcal{L}_{v_F} \theta + \frac{\delta F}{\delta \varrho} \right\rangle - \left\langle -\mathcal{L}_{v_F} \varrho, \mathcal{L}_{v_G} \theta + \frac{\delta G}{\delta \varrho} \right\rangle \\ &= \langle \varrho, (\mathcal{L}_{v_F} \mathcal{L}_{v_G} - \mathcal{L}_{v_G} \mathcal{L}_{v_F}) \theta \rangle + \left\langle \varrho, \mathcal{L}_{v_G} \frac{\delta F}{\delta \varrho} - \mathcal{L}_{v_F} \frac{\delta G}{\delta \varrho} \right\rangle \\ &= \left\langle \varrho, \iota_{\mathcal{L}_{v_F} v_G} d\theta \right\rangle - \left\langle \varrho, \mathcal{L}_{v_F} \frac{\delta G}{\delta \varrho} - \mathcal{L}_{v_G} \frac{\delta F}{\delta \varrho} \right\rangle \\ &= \langle d\theta \otimes \varrho, \mathcal{L}_{v_F} v_G \rangle - \left\langle \varrho, \mathcal{L}_{v_F} \frac{\delta G}{\delta \varrho} - \mathcal{L}_{v_G} \frac{\delta F}{\delta \varrho} \right\rangle \\ &= \{F, G\} \circ \Phi(\varrho, \theta). \end{aligned}$$

This concludes the proof. \square

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