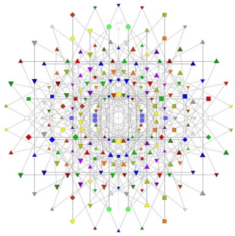

1 GALILEO

Contents

1.1 Principle of Galilean relativity

Galileo Galilei

Principles of relativity address the problem of how events that occur in one place or state of motion are observed from another. And if events occurring in one place or state of motion look different from those in another, how should one determine the laws of motion?

Galileo approached this problem via a thought experiment which imagined observations of motion made inside a ship by people who could not see outside. He showed that the people isolated inside a uniformly moving ship would be *unable to determine by measurements made inside it whether they were moving!*

> ...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
>
> – Galileo, *Dialogue Concerning the Two Chief World Systems* [Ga1632]

Galileo's thought experiment showed that a man who is below decks on a ship cannot tell whether the ship is docked or is moving uniformly through the water at constant velocity. He may observe water dripping from a bottle, fish swimming in a tank, butterflies flying, etc. Their behaviour will be just the same, whether the ship is moving or not.

Definition 1.1.1 (Galilean transformations) *Transformations of reference location, time, orientation or state of uniform translation at constant velocity are called* ***Galilean transformations****.*

Definition 1.1.2 (Uniform rectilinear motion) *Coordinate systems related by Galilean transformations are said to be in **uniform rectilinear motion** relative to each other.*

Galileo's thought experiment led him to the following principle.

Definition 1.1.3 (Principle of Galilean relativity) *The laws of motion are independent of reference location, time, orientation or state of uniform translation at constant velocity. Hence, these laws are invariant (i.e., they do not change their forms) under Galilean transformations.*

Remark 1.1.1 (Two tenets of Galilean relativity) Galilean relativity sets out two important tenets:

- It is impossible to determine who is actually at rest.
- Objects continue in uniform motion unless acted upon.

The second tenet is known as ***Galileo's law of inertia***. It is also the basis for ***Newton's first law of motion***. □

1.2 Galilean transformations

Definition 1.2.1 (Galilean transformations) *Galilean transformations of a coordinate frame consist of space-time translations, rotations and reflections of spatial coordinates, as well as Galilean "boosts" into uniform rectilinear motion.*

In three dimensions, the Galilean transformations depend smoothly on ten real parameters, as follows:

- *Space-time translations,*

$$g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{r}_0, t + t_0) .$$

  These possess four real parameters: $(\mathbf{r}_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$, *for the three dimensions of space, plus time.*

- *Spatial rotations and reflections,*

$$g_2(\mathbf{r}, t) = (O\mathbf{r}, t)\,,$$

*for any linear **orthogonal transformation** $O : \mathbb{R}^3 \mapsto \mathbb{R}^3$ with $O^T = O^{-1}$. These have three real parameters, for the three axes of rotation and reflection. Because the inverse of an orthogonal transformation is its transpose ($O^{-1} = O^T$) they preserve both the lengths and relative orientations of vectors. It has two connected components corresponding to the positive and negative values of the determinant,* $\det O = \pm 1$, *which changes sign under reflections.*

- *Galilean boosts into uniform rectilinear motion,*

$$g_3(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}_0 t, t)\,.$$

These have three real parameters: $\mathbf{v}_0 \in \mathbb{R}^3$, for the three components of the velocity boost vector.

Definition 1.2.2 (Group) *A **group** G is a set of elements that possesses a binary product (multiplication), $G \times G \to G$, such that the following properties hold:*

- *The product gh of g and h is associative, that is, $(gh)k = g(hk)$.*
- *An identity element exists, $e :\; eg = g$ and $ge = g$, for all $g \in G$.*
- *The inverse operation exists, $G \to G$, so that $gg^{-1} = g^{-1}g = e$.*

Definition 1.2.3 (Lie group) *A **Lie group** is a group that depends smoothly on a set of parameters. That is, a Lie group is both a group and a smooth manifold, for which the group operation is by composition of smooth invertible functions.*

Proposition 1.2.1 (Lie group property) *Galilean transformations form a Lie group, modulo reflections.*

Proof. Any Galilean transformation

$$g \in G(3) :\; \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$$

may be expressed uniquely as a composition of the three basic transformations $\{g_1, g_2, g_3\} \in G(3)$. Consequently, the set of elements comprising the transformations $\{g_1, g_2, g_3\} \in G(3)$ closes under the binary operation of composition. The Galilean transformations also possess an identity element $e : eg_i = g_i = g_i e$, $i = 1, 2, 3$, and each element g possesses a unique inverse g^{-1}, so that $gg^{-1} = e = g^{-1}g$.

These properties, plus associativity, define a group. The smooth dependence of the group of Galilean transformations on its ten parameters means that the ***Galilean group*** $G(3)$ is a Lie group (except for the reflections, which are discrete, not smooth). ■

Remark 1.2.1 Compositions of Galilean boosts and translations commute. That is,

$$g_1 g_3 = g_3 g_1 \,.$$

However, the order of composition does matter in Galilean transformations when rotations and reflections are involved. For example, the action of the Galilean group composition $g_1 g_3 g_2$ on $(\mathbf{r}, t)$ from the left is given by

$$g(\mathbf{r}, t) \quad = \quad (O\mathbf{r} + t\mathbf{v}_0 + \mathbf{r}_0, t + t_0) \,,$$

for

$$g = g_1(\mathbf{r}_0, t_0) g_3(\mathbf{v}_0) g_2(O) =: g_1 g_3 g_2 \,.$$

However, the result for another composition, say $g_1 g_2 g_3$, would in general be different. □

Exercise. Write the corresponding transformations for $g_1 g_2 g_3$, $g_1 g_3 g_2$, $g_2 g_1 g_3$ and $g_3 g_2 g_1$, showing how they depend on the order in which the rotations, boosts and translations are composed. Write the inverse transformation for each of these compositions of left actions. ★

Answer. The various compositions of translations $g_1(\mathbf{r}_0, t_0)$, rotations $g_2(O)$ and boosts $g_3(\mathbf{v}_0)$ in general yield different results, as

$$
\begin{aligned}
g_1g_2g_3(\mathbf{r},t) &= \Big(O(\mathbf{r}+t\mathbf{v}_0)+\mathbf{r}_0,\ t+t_0\Big)\,,\\
g_1g_3g_2(\mathbf{r},t) &= \Big(O\mathbf{r}+t\mathbf{v}_0+\mathbf{r}_0, t+t_0\Big)\,,\\
g_2g_1g_3(\mathbf{r},t) &= \Big(O(\mathbf{r}+t\mathbf{v}_0+\mathbf{r}_0),\ t+t_0\Big)\,,\\
g_3g_2g_1(\mathbf{r},t) &= \Big(O(\mathbf{r}+\mathbf{r}_0)+t\mathbf{v}_0,\ t+t_0\Big)\,.
\end{aligned}
$$

The inverses are $(g_1g_2g_3)^{-1} = g_3^{-1}g_2^{-1}g_1^{-1}$, etc. ▲

Remark 1.2.2 (Decomposition of the Galilean group) Because the rotations take vectors into vectors, any element of the transformations $g_1g_2g_3$, $g_2g_1g_3$ and $g_3g_2g_1$ in the Galilean group may be written uniquely in the simplest form, as $g_1g_3g_2$.

Thus, any element of the Galilean group may be written uniquely as a rotation, followed by a space translation, a Galilean boost and a time translation. The latter three may be composed in any order, because they commute with each other. □

Exercise. What properties are preserved by the Galilean group? ★

Answer. The Galilean group $G(3)$ preserves the results of measuring length and time intervals, and relative orientation in different frames of motion related to each other by Galilean transformations. ▲

1.2.1 Admissible force laws for an N-particle system

For a system of N interacting particles, Newton's second law of motion (the law of acceleration) determines the motion resulting from the force $\mathbf{F}_j$ exerted on the jth particle by the other $N-1$ particles as

$$m_j\ddot{\mathbf{r}}_j = \mathbf{F}_j(\mathbf{r}_k - \mathbf{r}_l, \dot{\mathbf{r}}_k - \dot{\mathbf{r}}_l)\,, \quad \text{with } j,k,l = 1,2,\ldots,N \text{ (no sum)}\,.$$

This force law is independent of reference location, time or state of uniform translation at constant velocity. It will also be independent of reference orientation and thus will be ***Galilean-invariant***, provided the forces $\mathbf{F}_j$ transform under rotations and parity reflections as ***vectors***

$$m_j O\ddot{\mathbf{r}}_j = O\mathbf{F}_j = \mathbf{F}_j\Big(O(\mathbf{r}_k - \mathbf{r}_l), O(\dot{\mathbf{r}}_k - \dot{\mathbf{r}}_l)\Big)\,, \tag{1.2.1}$$

for any orthogonal transformation O.

This requirement for Galilean invariance that the force in Newton's law of acceleration transforms as a vector is the reason that vectors are so important in classical mechanics.

For example, Newton's law of gravitational motion is given by

$$m_j\ddot{\mathbf{r}}_j = \sum_{k\neq j} \mathbf{F}_{jk}\,, \tag{1.2.2}$$

in which the gravitational forces $\mathbf{F}_{jk}$ between $(j,\, k)$ particle pairs are given by

$$\mathbf{F}_{jk} = \frac{\gamma\, m_j m_k}{|\mathbf{r}_{jk}|^3}\mathbf{r}_{jk}\,, \quad \text{with} \quad \mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k\,, \tag{1.2.3}$$

and γ is the gravitational constant.

Exercise. Prove that Newton's law (1.2.2) for gravitational forces (1.2.3) is Galilean-invariant. That is, prove that Newton's law of gravitational motion takes the same form in any Galilean reference frame. ★

1.3 Subgroups of the Galilean transformations

Definition 1.3.1 (Subgroup) *A **subgroup** is a subset of a group whose elements also satisfy the defining properties of a group.*

Exercise. List the subgroups of the Galilean group that do not involve time. ★

Answer. The subgroups of the Galilean group that are independent of time consist of

- Spatial translations $g_1(\mathbf{r}_0)$ acting on $\mathbf{r}$ as $g_1(\mathbf{r}_0)\mathbf{r} = \mathbf{r} + \mathbf{r}_0$.
- Proper rotations $g_2(O)$ with $g_2(O)\mathbf{r} = O\mathbf{r}$ where $O^T = O^{-1}$ and $\det O = +1$. This subgroup is called $SO(3)$, the ***special orthogonal group*** in three dimensions.
- Rotations and reflections $g_2(O)$ with $O^T = O^{-1}$ and $\det O = \pm 1$. This subgroup is called $O(3)$, the ***orthogonal group*** in three dimensions.
- Spatial translations $g_1(\mathbf{r}_0)$ with $\mathbf{r}_0 \in \mathbb{R}^3$ compose with proper rotations $g_2(O) \in SO(3)$ acting on a vector $\mathbf{r} \in \mathbb{R}^3$ as

  $$E(O, \mathbf{r}_0)\mathbf{r} = g_1(\mathbf{r}_0)g_2(O)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0 \,,$$

  where $O^T = O^{-1}$ and $\det O = +1$. This subgroup is called $SE(3)$, the ***special Euclidean group*** in three dimensions. Its ***action*** on $\mathbb{R}^3$ is written abstractly as $SE(3) \times \mathbb{R}^3 \to \mathbb{R}^3$.
- Spatial translations $g_1(\mathbf{r}_0)$ compose with proper rotations and reflections $g_2(O)$, as $g_1(\mathbf{r}_0)g_2(O)$ acting on $\mathbf{r}$. This subgroup is called $E(3)$, the ***Euclidean group*** in three dimensions. ▲

Remark 1.3.1 Spatial translations and rotations do not commute in general. That is, $g_1 g_2 \neq g_2 g_1$, unless the direction of translation and axis of rotation are collinear. □

1.3.1 Matrix representation of $SE(3)$

As we have seen, the special Euclidean group in three dimensions $SE(3)$ acts on a position vector $\mathbf{r} \in \mathbb{R}^3$ by

$$E(O, \mathbf{r}_0)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0 \,.$$

A 4×4 ***matrix representation*** of this action may be found by noticing that its right-hand side arises in multiplying the matrix times the *extended* vector $(\mathbf{r}, 1)^T$ as

$$\begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix} = \begin{pmatrix} O\mathbf{r} + \mathbf{r}_0 \\ 1 \end{pmatrix} .$$

Therefore we may identify a group element of $SE(3)$ with a 4×4 matrix,

$$E(O, \mathbf{r}_0) = \begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} .$$

The group $SE(3)$ has six parameters. These are the angles of rotation about each of the three spatial axes by the orthogonal matrix $O \in SO(3)$ with $O^T = O^{-1}$ and the three components of the vector of translations $\mathbf{r}_0 \in \mathbb{R}^3$.

The ***group composition law*** for $SE(3)$ is expressed as

$$\begin{aligned} E(\tilde{O}, \tilde{\mathbf{r}}_0)E(O, \mathbf{r}_0)\mathbf{r} &= E(\tilde{O}, \tilde{\mathbf{r}}_0)(O\mathbf{r} + \mathbf{r}_0) \\ &= \tilde{O}(O\mathbf{r} + \mathbf{r}_0) + \tilde{\mathbf{r}}_0 \,, \end{aligned}$$

with $(O\,,\,\tilde{O}) \in SO(3)$ and $(\mathbf{r}\,,\,\tilde{\mathbf{r}}_0) \in \mathbb{R}^3$. This formula for group composition may be represented by matrix multiplication from the left as

$$E(\tilde{O},\tilde{\mathbf{r}}_0)E(O,\mathbf{r}_0) = \begin{pmatrix} \tilde{O} & \tilde{\mathbf{r}}_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{O}O & \tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0 \\ 0 & 1 \end{pmatrix},$$

which may also be expressed by simply writing the top row,

$$(\tilde{O}\,,\,\tilde{\mathbf{r}}_0)(O\,,\,\mathbf{r}_0) = (\tilde{O}O\,,\,\tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,.$$

The identity element (e) of $SE(3)$ is represented by

$$e = E(I,\mathbf{0}) = \begin{pmatrix} I & \mathbf{0} \\ 0 & 1 \end{pmatrix},$$

or simply $e = (I,\mathbf{0})$. The inverse element is represented by the matrix inverse

$$E(O,\mathbf{r}_0)^{-1} = \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{r}_0 \\ 0 & 1 \end{pmatrix}.$$

In this matrix representation of $SE(3)$, one checks directly that

$$E(O,\mathbf{r}_0)^{-1}E(O,\mathbf{r}_0) = \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{r}_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ 0 & 1 \end{pmatrix} = (I,\mathbf{0}) = e\,.$$

In the shorter notation, the inverse may be written as

$$(O\,,\,\mathbf{r}_0)^{-1} = (O^{-1}\,,\,-O^{-1}\mathbf{r}_0)$$

and $O^{-1} = O^T$ since the 3×3 matrix $O \in SO(3)$ is orthogonal.

Remark 1.3.2 The inverse operation of $SE(3)$ involves composition of the inverse for rotations with the inverse for translations. This entwining means that the group structure of $SE(3)$ is not simply a direct product of its two subgroups $\mathbb{R}^3$ and $SO(3)$. □

1.4 Lie group actions of $SE(3)$

Group multiplication in $SE(3)$ is denoted as

$$(\tilde{O}\,,\,\tilde{\mathbf{r}}_0)(O\,,\,\mathbf{r}_0) = (\tilde{O}O\,,\,\tilde{O}\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,. \tag{1.4.1}$$

This notation demonstrates the following group properties of $SE(3)$:

- Translations in the subgroup $\mathbb{R}^3 \subset SE(3)$ act on each other by vector addition,

$$\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 :\ (I\,,\,\tilde{\mathbf{r}}_0)(I\,,\,\mathbf{r}_0) = (I\,,\,\mathbf{r}_0 + \tilde{\mathbf{r}}_0)\,.$$

- Rotations in the subgroup $SO(3) \subset SE(3)$ act on each other by composition,

$$SO(3) \times SO(3) \mapsto SO(3) :\ (\tilde{O}\,,\,\mathbf{0})(O\,,\,\mathbf{0}) = (\tilde{O}O\,,\,\mathbf{0})\,.$$

- Rotations in the subgroup $SO(3) \subset SE(3)$ act ***homogeneously*** on the vector space of translations in the subgroup $\mathbb{R}^3 \subset SE(3)$,

$$SO(3) \times \mathbb{R}^3 \mapsto \mathbb{R}^3 :\ (\tilde{O}\,,\,\mathbf{0})(I\,,\,\mathbf{r}_0) = (\tilde{O}\,,\,\tilde{O}\mathbf{r}_0)\,.$$

  That is, the action of the subgroup $SO(3) \subset SE(3)$ on the subgroup $\mathbb{R}^3 \subset SE(3)$ maps $\mathbb{R}^3$ into itself. The translations $\mathbb{R}^3 \subset SE(3)$ are thus said to form a ***normal***, or ***invariant subgroup*** of the group $SE(3)$.

- Every element of $(O\,,\,\mathbf{r}_0)$ of $SE(3)$ may be represented uniquely by composing a translation acting from the left on a rotation. That is, each element may be decomposed into

$$(O\,,\,\mathbf{r}_0) = (I\,,\,\mathbf{r}_0)(O\,,\,\mathbf{0})\,,$$

  for a *unique* $\mathbf{r}_0 \in \mathbb{R}^3$ and $O \in SO(3)$. Conversely, one may uniquely represent

$$(O\,,\,\mathbf{r}_0) = (O\,,\,\mathbf{0})(I\,,\,O^{-1}\mathbf{r}_0)\,,$$

by composing a rotation acting from the left on a translation.

This equivalence endows the Lie group $SE(3)$ with a semidirect-product structure,

$$SE(3) = SO(3) \circledS \mathbb{R}^3 . \tag{1.4.2}$$

Definition 1.4.1 (Semidirect-product Lie group) *A Lie group G that may be decomposed uniquely into a normal subgroup N and a subgroup H such that every group element may be written as*

$$g = nh \quad \textit{or} \quad g = hn \quad \textit{(in either order)}, \tag{1.4.3}$$

for unique choices of $n \in N$ and $h \in H$, is called a ***semidirect product*** *of H and N, denoted here by $\circledS$, as in*

$$G = H \circledS N .$$

When the normal subgroup N is a vector space, the action of a semidirect-product group on itself is given as in formula (1.4.1) for $SE(3)$. If the normal subgroup N is not a vector space, then the operation of addition in formula (1.4.1) is replaced by the composition law for N.

1.5 Lie group actions of $G(3)$

The Galiliean group in three dimensions $G(3)$ has ten parameters $(O \in SO(3)\,,\ \mathbf{r}_0 \in \mathbb{R}^3\,,\ \mathbf{v}_0 \in \mathbb{R}^3\,,\ t_0 \in \mathbb{R}\,)$. The Galilean group is *also* a semidirect-product Lie group, which may be written as

$$G(3) = SE(3) \circledS \mathbb{R}^4 = \Big(SO(3) \circledS \mathbb{R}^3 \Big) \circledS \mathbb{R}^4 . \tag{1.5.1}$$

That is, the subgroup of Euclidean motions consisting of rotations and Galilean velocity boosts $(O, \mathbf{v}_0) \in SE(3)$ acts homogeneously on the subgroups of space and time translations $(\mathbf{r}_0, t_0) \in \mathbb{R}^4$ which commute with each other.

Exercise. Compute explicitly the *inverse* of the Galilean group element $g = g_1 g_3 g_2$ obtained by representing the action of the Galilean group as matrix multiplication $G(3) \times \mathbb{R}^4 \to \mathbb{R}^4$ on the extended vector $(\mathbf{r}, t, 1)^T \in \mathbb{R}^4$,

$$g_1 g_3 g_2 \begin{pmatrix} \mathbf{r} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} O & \mathbf{v}_0 & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} O\mathbf{r} + t\mathbf{v}_0 + \mathbf{r}_0 \\ t + t_0 \\ 1 \end{pmatrix}. \tag{1.5.2}$$

★

Answer. Write the product $g = g_1 g_3 g_2$ as

$$g = g_1 g_3 g_2 = \begin{pmatrix} I & \mathbf{0} & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{v}_0 & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} O & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}.$$

Then, the product $g^{-1} = (g_1 g_3 g_2)^{-1} = g_2^{-1} g_3^{-1} g_1^{-1}$ appears in matrix form as

$$\begin{aligned} g^{-1} &= \begin{pmatrix} O^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} I & -\mathbf{v}_0 & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & -\mathbf{r}_0 \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} O^{-1} & -O^{-1}\mathbf{v}_0 & -O^{-1}(\mathbf{r}_0 - t\mathbf{v}_0) \\ \mathbf{0} & 1 & -t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}. \end{aligned}$$

▲

Exercise. Write the corresponding matrices for the Galilean transformations for $g_1g_2g_3$, $g_2g_1g_3$ and $g_3g_2g_1$. ★

1.5.1 Matrix representation of $G(3)$

The formula for group composition $G(3) \times G(3) \to G(3)$ may be represented by matrix multiplication from the left as

$$\begin{pmatrix} \tilde{O} & \tilde{\mathbf{v}}_0 & \tilde{\mathbf{r}}_0 \\ \mathbf{0} & 1 & \tilde{t}_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} O & \mathbf{v}_0 & \mathbf{r}_0 \\ \mathbf{0} & 1 & t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{O}O & \tilde{O}\mathbf{v}_0 + \tilde{\mathbf{v}}_0 & \tilde{O}\mathbf{r}_0 + \tilde{\mathbf{v}}_0 t_0 + \tilde{\mathbf{r}}_0 \\ \mathbf{0} & 1 & \tilde{t}_0 + t_0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}, \tag{1.5.3}$$

which may be expressed more succinctly as

$$\begin{aligned} &(\tilde{O}, \tilde{\mathbf{v}}_0, \tilde{\mathbf{r}}_0, \tilde{t}_0)(O, \mathbf{v}_0, \mathbf{r}_0, t_0) \\ &\quad = (\tilde{O}O,\ \tilde{O}\mathbf{v}_0 + \tilde{\mathbf{v}}_0,\ \tilde{O}\mathbf{r}_0 + \tilde{\mathbf{v}}_0 t_0 + \tilde{\mathbf{r}}_0,\ \tilde{t}_0 + t_0). \end{aligned} \tag{1.5.4}$$

Exercise. Check the semidirect-product structure (1.5.1) for the Lie group $G(3) = SE(3) \circledS \mathbb{R}^4$, by writing explicit matrix expressions for $g = nh$ and $g = hn$ with $h = SE(3)$ and $n = \mathbb{R}^4$. ★

Answer. In verifying the semidirect-product structure condition (1.4.3) that $g = nh$ or $g = hn$ in either order, we write explicitly

$$\begin{aligned} (O, \mathbf{v}_0, \mathbf{r}_0, t_0) &= (I, 0, \mathbf{r}_0, t_0)\,(O, \mathbf{v}_0, 0, 0) \\ &= (O, \mathbf{v}_0, 0, 0)\,(I, 0, O^{-1}(\mathbf{r}_0 - \mathbf{v}_0 t_0), t_0)\,. \end{aligned} \tag{1.5.5}$$

▲

1.6 Lie algebra of $SE(3)$

A 4×4 matrix representation of tangent vectors for $SE(3)$ at the identity may be found by first computing the derivative of a general group element $(O(s), \mathbf{r}_0(s))$ along the group path with parameter s and bringing the result back to the identity at $s = 0$,

$$\left[\begin{pmatrix} O(s) & \mathbf{r}_0(s) \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} O'(s) & \mathbf{r}_0'(s) \\ 0 & 0 \end{pmatrix} \right]_{s=0}$$
$$= \begin{pmatrix} O^{-1}(0)O'(0) & O^{-1}(0)\mathbf{r}_0'(0) \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \widehat{\Xi} & \mathbf{r}_0 \\ 0 & 0 \end{pmatrix},$$

where in the last step we have dropped the unnecessary superscript prime ($'$). The quantity $\widehat{\Xi} = O^{-1}(s)O'(s)|_{s=0}$ is a 3×3 skew-symmetric matrix, since O is a 3×3 orthogonal matrix. Thus, $\widehat{\Xi}$ may be written using the ***hat map***, defined by

$$\widehat{\Xi} = \begin{pmatrix} 0 & -\Xi_3 & \Xi_2 \\ \Xi_3 & 0 & -\Xi_1 \\ -\Xi_2 & \Xi_1 & 0 \end{pmatrix}, \tag{1.6.1}$$

in terms of a vector $\boldsymbol{\Xi} \in \mathbb{R}^3$ with components Ξ_i, with $i = 1, 2, 3$. Infinitesimal rotations are expressed by the vector cross product,

$$\widehat{\Xi}\mathbf{r} = \boldsymbol{\Xi} \times \mathbf{r}\,. \tag{1.6.2}$$

The matrix components of $\widehat{\Xi}$ may also be written in terms of the components of the vector $\boldsymbol{\Xi}$ as

$$\widehat{\Xi}_{jk} = \left(O^{-1}\frac{dO}{ds} \right)_{jk}\bigg|_{s=0} = -\,\Xi_i \epsilon_{ijk}\,,$$

where ϵ_{ijk} with $i, j, k = 1, 2, 3$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$, $\epsilon_{213} = -1$, etc. One may compute directly, for a fixed vector $\mathbf{r}$,

$$\frac{d}{ds}e^{s\widehat{\Xi}}\mathbf{r} = \widehat{\Xi}e^{s\widehat{\Xi}}\mathbf{r} = \boldsymbol{\Xi} \times e^{s\widehat{\Xi}}\mathbf{r}\,.$$

Consequently, one may evaluate, at $s = 0$,

$$\frac{d}{ds} e^{s\widehat{\Xi}} \mathbf{r} \bigg|_{s=0} = \widehat{\Xi}\mathbf{r} = \boldsymbol{\Xi} \times \mathbf{r}\,.$$

This expression recovers the expected result in (1.6.2) in terms of the exponential notation. It means the quantity $\mathbf{r}(s) = \exp(s\widehat{\Xi})\mathbf{r}$ describes a finite, right-handed rotation of the initial vector $\mathbf{r} = \mathbf{r}(0)$ by the angle $s|\boldsymbol{\Xi}|$ around the axis pointing in the direction of $\boldsymbol{\Xi}$.

Remark 1.6.1 (Properties of the hat map) The hat map arises in the infinitesimal rotations

$$\widehat{\Xi}_{jk} = (O^{-1}dO/ds)_{jk}|_{s=0} = -\,\Xi_i \epsilon_{ijk}\,.$$

The hat map is an isomorphism:

$$(\mathbb{R}^3, \times) \mapsto (\mathfrak{so}(3),\, [\,\cdot\,,\,\cdot\,]\,)\,.$$

That is, the hat map identifies the composition of two vectors in $\mathbb{R}^3$ using the cross product with the commutator of two skew-symmetric 3×3 matrices. Specifically, we write for any two vectors $\mathbf{Q}, \boldsymbol{\Xi} \in \mathbb{R}^3$,

$$-\,(\mathbf{Q} \times \boldsymbol{\Xi})_k = \epsilon_{klm} \Xi^{\,l} Q^m = \widehat{\Xi}_{km}\, Q^m\,.$$

That is,

$$\boldsymbol{\Xi} \times \mathbf{Q} = \widehat{\Xi}\,\mathbf{Q} \quad \text{for all} \quad \boldsymbol{\Xi},\ \mathbf{Q} \in \mathbb{R}^3\,.$$

The following formulas may be easily verified for $\mathbf{P}, \mathbf{Q}, \boldsymbol{\Xi} \in \mathbb{R}^3$:

$$\begin{aligned} (\mathbf{P} \times \mathbf{Q})\widehat{\ } &= \left[\widehat{P}, \widehat{Q}\right], \\ \left[\widehat{P}, \widehat{Q}\right]\boldsymbol{\Xi} &= (\mathbf{P} \times \mathbf{Q}) \times \boldsymbol{\Xi}\,, \\ \mathbf{P} \cdot \mathbf{Q} &= -\frac{1}{2}\,\mathrm{trace}\left(\widehat{P}\widehat{Q}\right). \end{aligned}$$

□

Remark 1.6.2 The commutator of infinitesimal transformation matrices given by the formula

$$\left[\begin{pmatrix} \widehat{\Xi}_1 & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widehat{\Xi}_2 & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} \widehat{\Xi}_1\widehat{\Xi}_2 - \widehat{\Xi}_2\widehat{\Xi}_1 & \widehat{\Xi}_1\mathbf{r}_2 - \widehat{\Xi}_2\mathbf{r}_1 \\ 0 & 0 \end{pmatrix}$$

provides a matrix representation of $se(3)$, the Lie algebra of the Lie group $SE(3)$. In vector notation, this becomes

$$\left[\begin{pmatrix} \boldsymbol{\Xi}_1\times & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Xi}_2\times & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} (\boldsymbol{\Xi}_1\times\boldsymbol{\Xi}_2)\times & \boldsymbol{\Xi}_1\times\mathbf{r}_2 - \boldsymbol{\Xi}_2\times\mathbf{r}_1 \\ 0 & 0 \end{pmatrix}.$$

□

Remark 1.6.3 The $se(3)$ matrix commutator yields

$$\begin{aligned} \Big[(\widehat{\Xi}_1\,,\,\mathbf{r}_1)\,,\,(\widehat{\Xi}_2\,,\,\mathbf{r}_2)\Big] &= \Big(\widehat{\Xi}_1\widehat{\Xi}_2 - \widehat{\Xi}_2\widehat{\Xi}_1,\ \widehat{\Xi}_1\mathbf{r}_2 - \widehat{\Xi}_2\mathbf{r}_1\Big) \\ &= \Big(\Big[\widehat{\Xi}_1,\,\widehat{\Xi}_2\Big],\ \widehat{\Xi}_1\mathbf{r}_2 - \widehat{\Xi}_2\mathbf{r}_1\Big)\,, \end{aligned}$$

which is the classic expression for the Lie algebra of a semidirect-product Lie group. □

1.7 Lie algebra of $G(3)$

A 5×5 matrix representation of tangent vectors for $G(3)$ at the identity may be found by computing the derivative of a general group element $(O(s), \mathbf{v}_0(s), \mathbf{r}_0(s), t_0(s))$ along the group path with parameter s and bringing the result back to the identity at $s=0$,

$$\begin{aligned} &\left[\begin{pmatrix} O(s) & \mathbf{v}_0(s) & \mathbf{r}_0(s) \\ 0 & 1 & t_0(s) \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} O'(s) & \mathbf{v}_0'(s) & \mathbf{r}_0'(s) \\ 0 & 0 & t_0'(s) \\ 0 & 0 & 0 \end{pmatrix}\right]_{s=0} \\ &= \begin{pmatrix} O^{-1}(s)O'(s) & O^{-1}(s)\mathbf{v}_0'(s) & O^{-1}(s)(\mathbf{r}_0'(s) - \mathbf{v}_0'(s)t_0'(s)) \\ 0 & 0 & t_0'(s) \\ 0 & 0 & 0 \end{pmatrix}\Bigg|_{s=0} \\ &= \begin{pmatrix} \widehat{\Xi} & \mathbf{v}_0 & \mathbf{r}_0 - \mathbf{v}_0 t_0 \\ 0 & 0 & t_0 \\ 0 & 0 & 0 \end{pmatrix} =: \Big(\widehat{\Xi},\,\mathbf{v}_0,\,\mathbf{r}_0,\,t_0\Big)\,, \end{aligned}$$

in terms of the 3×3 skew-symmetric matrix $\widehat{\Xi} = O^{-1}(s)O'(s)|_{s=0}$. For notational convenience, the superscript primes that would have appeared on the tangents of the Galilean shift parameters $\mathbf{r}_0'(0)$, $\mathbf{v}_0'(0)$ and $t_0'(0)$ at the identity $s = 0$ have been dropped in the last line and replaced by the simpler forms $\mathbf{r}_0,\ \mathbf{v}_0,\ t_0$, respectively.

Exercise. (Galilean Lie algebra commutator) Verify the commutation relation

$$\Big[(\widehat{\Xi}_1,\, \mathbf{v}_1,\, \mathbf{r}_1,\, t_1),\, (\widehat{\Xi}_2,\, \mathbf{v}_2,\, \mathbf{r}_2,\, t_2)\Big]$$
$$= \Big(\Big[\widehat{\Xi}_1,\, \widehat{\Xi}_2\Big],\, \widehat{\Xi}_1\mathbf{v}_2 - \widehat{\Xi}_2\mathbf{v}_1,\, \widehat{\Xi}_1(\mathbf{r}_2, \mathbf{v}_2, t_2) - \widehat{\Xi}_2(\mathbf{r}_1, \mathbf{v}_1, t_1),\, 0\Big),$$

where

$$\begin{aligned}&\widehat{\Xi}_1(\mathbf{r}_2, \mathbf{v}_2, t_2) - \widehat{\Xi}_2(\mathbf{r}_1, \mathbf{v}_1, t_1)\\ &\quad = \Big(\widehat{\Xi}_1(\mathbf{r}_2 - \mathbf{v}_2 t_2) + \mathbf{v}_1 t_2\Big) - \Big(\widehat{\Xi}_2(\mathbf{r}_1 - \mathbf{v}_1 t_1) + \mathbf{v}_2 t_1\Big)\,.\end{aligned}$$

★

According to the principle of Galilean relativity, the laws of mechanics must take the same form in any uniformly moving reference frame. That is, the expressions of these laws must be invariant in form under Galilean transformations. In this chapter, we have introduced the Galilean transformations, shown that they comprise a Lie group, found its subgroups, endowed them with a matrix representation, and identified their group structure mathematically as a nested semidirect product.

Rigid motion in $\mathbb{R}^3$ corresponds to a smoothly varying sequence of changes of reference frame along a time-dependent path in the special Euclidean Lie group, $SE(3)$. This is the main subject of the text.